WEIGHT MULTIPLICITIES FOR THE CLASSICAL GROUPS.

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1. Introduction.

If G is a semi-simple compact Lie group of rank k, then the maximal toroidal subgroup, T_G, of G is isomorphic to the group $T_k = U(1)\times U(1)\times \ldots \times U(1)$, which consists of a direct product of k groups $U(1)$. A group element of T_k takes the form $(e^{i\phi_1}, e^{i\phi_2}, \ldots e^{i\phi_k})$ where ϕ_j, for $j=1,2 \ldots , k$, is a real parameter. An arbitrary irreducible representation of T_k, and thus of T_G, is specified by $\{w_1\}\times\{w_2\}\times \ldots\times\{w_k\}$, and this representation is defined by the mapping:

$$(e^{i\phi_1}, e^{i\phi_2}, \ldots e^{i\phi_k}) \to e^{i(w_1\phi_1 + w_2\phi_2 + \ldots + w_k\phi_k)} . \tag{1.1}$$

If a representation λ_G of G decomposes on restriction of the group elements to those of the subgroup T_G in accordance with the branching rule:

$$G \downarrow T_G \qquad \lambda_G\downarrow \sum_{\underset{\sim}{w}} m_{\lambda_G}^{\underset{\sim}{w}} \quad \{w_1\}\times\{w_2\}\times\ldots\times\{w_k\} , \tag{1.2}$$

then $\underset{\sim}{w} = (w_1, w_2, \ldots ,w_k)$ is said to be a weight vector of the representation λ_G, and its multiplicity is the coefficient $m_{\lambda_G}^{\underset{\sim}{w}}$.

To determine the weight vectors and their multiplicities it is therefore only necessary to evaluate certain branching rules. It is shown that in the case of covariant tensor irreducible representations of the group $U(k)$ this leads in a natural way to the use of both Gelfand patterns and Young tableaux. The generalisation to mixed tensor representations of $U(k)$ is also made and the group $Sp(2k)$ is treated in detail. Some comments are made on the tensor and spinor representations of $O(2k)$ and $O(2k+1)$, and some concluding remarks on the results obtained are presented.

2. Covariant Tensor Representations of $U(k)$.

The irreducible covariant tensor representations of $U(k)$ are specified by $\{\lambda\}$ where $(\lambda) = (\lambda_1,\lambda_2, \ldots ,\lambda_a)$ is a partition of ℓ into a non-vanishing parts with $a\leq k$. The branching rule appropriate to the restriction to the subgroup $U(k-1)\times U(1)$ takes the form:[(1)]

$$U(k)\downarrow U(k-1)\times U(1) \qquad \{\lambda\} \downarrow \sum_{\sigma,w_k} \{\sigma\} \times \{w_k\} , \tag{2.1}$$

with $\lambda_i \geq \sigma_i \geq \lambda_{i+1}$ and $w_k=\ell - s$, where (σ) is a partition of s into c non-vanishing parts with $c\leq k -1$.

It follows from the rules appropriate to S-function division enunciated by Littlewood[2] that the branching rule (2.1) may also be written in the form[3]:

$$U(k) \downarrow U(k-1)\times U(1) \qquad \{\lambda\} \downarrow \sum_{w_k} \{\lambda/w_k\} \times \{w_k\} \quad . \tag{2.2}$$

The repeated application of this rule to the chain

$$U(k) \downarrow U(k-1)\times U(1) \downarrow U(k-2)\times U(1)\times U(1) \downarrow \ldots \downarrow T_k \ , \tag{2.3}$$

yields the branching rules

$$\{\lambda\} \downarrow \sum_{w_k} \{\lambda/w_k\}\times\{w_k\} \downarrow \sum_{w_{k-1},w_k} \{\lambda/w_{k-1}w_k\}\times\{w_{k-1}\}\times\{w_k\}$$

$$\ldots \downarrow \sum_{\tilde{w}} \{\lambda/w_1 w_2 \ldots w_k\} \ \{w_1\}\times\{w_2\}\times\ldots\times\{w_k\} \ , \tag{2.4}$$

where $\ell = \sum_{c=1}^{k} w_i$. Thus the weight multiplicities may be evaluated using the formula

$$m_{\{\lambda\}}^{\tilde{w}} = \{\lambda/w_1 w_2 \ldots w_k\} \quad . \tag{2.5}$$

The relationship between S-function quotients and outer products of S-functions is such that

$$\{w_1\}.\ \{w_2\}.\ldots.\ \{w_k\} = \sum_{\lambda} \ m_{\{\lambda\}}^{\tilde{w}} \ \{\lambda\} \tag{2.6}$$

It follows from the fact that S-function multiplication is commutative that the symmetry group of the weight diagrams is the symmetric group associated with the permutations of the components of the weight vectors $\tilde{w}$. Furthermore, since the coefficients in (2.6) are known to be independent of k, the weight multiplicities of the covariant tensor representations of U(k) are k-independent.

This method of determining weight multiplicities, involving as it does the step by step reduction of a representation of U(k) into a set of one dimensional irreducible representations of the Abelian group T_k, yields two equivalent labelling schemes for the basis states of such a representation $\{\lambda\}$ of U(k). The repeated application of (2.1) gives rise to Gelfand patterns[5] in accordance with the extension of labels defined by

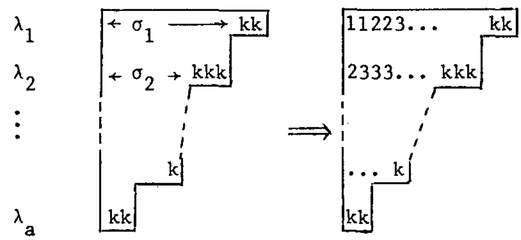
$$\left\{\begin{matrix} \lambda_1 & \lambda_2 & & \lambda_k \\ \sigma_1 & \sigma_2\cdots\sigma_{k-1} & & \\ & \vdots & & \\ & & & \end{matrix}\right\} \Longrightarrow \left\{\begin{matrix} m_{1k} & m_{2k} & \cdots\cdots & m_{kk} \\ m_{1k-1} & m_{2k-1} & \cdots\cdot & m_{k-1k-1} \\ & \vdots & & \\ & m_{12} \quad m_{22} & & \\ & m_{11} & & \end{matrix}\right\}$$

The constraints applying to (2.1) are such that m_{ij} is a non-negative integer, that $m_{ij+1} \geq m_{ij} \geq m_{i+1j+1}$ and that

$$w_j = \sum_{i=1}^{j} m_{ij} - \sum_{i=1}^{j-1} m_{ij-1} \qquad \text{for } j = 1,2,\ldots,k. \tag{2.7}$$

Similarly the repeated application of (2.2) to the Young diagrams specified by S-functions gives rise to Young tableaux(6):

Row lengths



In this case the constraints are such that the numbers in the tableau are non-decreasing across each row from left to right and are strictly increasing down each column from top to bottom, and

$$w_j = \text{the number of } j\text{'s in the tableau.} \tag{2.8}$$

The multiplicity of each weight is the number of distinct Gelfand patterns, or equivalently the number of distinct Young tableaux, whose entries satisfy the given constraints.

For example in the case of the group $U(5)$, for which $k = 5$, the multiplicity of the weight $\underset{\sim}{w} = (1,2,0,2,0)$ in the irreducible representation $\{\lambda\} = \{3,2\}$ is 2, corresponding to the existence of the two Young tableaux $\begin{array}{|c|c|c|}\hline 1&2&2\\ \hline 4&4 \\ \cline{1-2}\end{array}$ $\begin{array}{|c|c|c|}\hline 1&2&4\\ \hline 2&4 \\ \cline{1-2}\end{array}$ and the two

Gelfand patterns $\left\{\begin{array}{ccccccccc} 3&&2&&0&&0&&0\\ &3&&2&&0&&0&\\ &&3&&0&&0&&\\ &&&3&&0&&&\\ &&&&1&&&& \end{array}\right\}$ $\left\{\begin{array}{ccccccccc} 3&&2&&0&&0&&0\\ &3&&2&&0&&0&\\ &&2&&1&&0&&\\ &&&2&&1&&&\\ &&&&1&&&& \end{array}\right\}$

The symmetry of the weight diagram is exemplified by the fact that

$$m_{\{32\}}^{(22100)} = m_{\{32\}}^{(12020)} = m_{\{32\}}^{(20102)} = \ldots = 2 \quad ,$$

and the k-independence of the weight multiplicities by the fact that

$$m_{\{32\}}^{(22100\ldots0)} = 2 \quad .$$

3. Mixed Tensor Representations of U(k).

The irreducible mixed tensor representations of $U(k)$ are specified(7) by $\{\bar{\mu};\lambda\}$ where (λ) and (μ) are partitions of ℓ and m respectively, into a and b non-vanishing parts such that $a + b \leq k$. The generalisation of the branching rule (2.1) takes the form:

$$U(k) \downarrow U(k-1)\times U(1) \qquad \{\bar{\mu};\lambda\} \downarrow \sum_{\tau,\sigma,p_k,q_k} \{\bar{\tau};\sigma\} \times \{p_k - q_k\} \tag{3.1}$$

with $\lambda_i \geq \sigma_i \geq \lambda_{i+1}$, $\mu_j \geq \tau_j \geq \mu_{j+1}$ and $w_k = p_k - q_k$ where (σ) and (τ) are partitions of s and t into c and d non-vanishing parts such that $p_k = \ell - s$, $q_k = m - t$ and $c + d \leq k-1$. This result corresponds to the fact that the appropriate generalisation of (2.2) is [3]

$$U(k) \downarrow U(k-1)\times U(1) \qquad \{\bar{\mu};\lambda\} \downarrow \sum_{p_k,q_k} \{\overline{\mu/q_k};\lambda/p_k\} \times \{p_k - q_k\} . \tag{3.2}$$

The repeated application of this rule to the chain (2.3) gives the result:

$$U(k) \downarrow T_k \quad \{\bar{\mu};\lambda\} \downarrow \sum_{\underset{\sim}{p},\underset{\sim}{q}} \{\overline{\mu/q_1 q_2 \cdots q_k}\ ;\lambda/p_1 p_2 \cdots p_k\}\{p_1-q_1\}\times\{p_2-q_2\}\times\ldots\times\{p_k-q_k\}. \tag{3.3}$$

It then follows from the definition, (1.2), of weights that

$$m^{\underset{\sim}{w}}_{\{\bar{\mu};\lambda\}} = \sum_{\underset{\sim}{p},\underset{\sim}{q}} \{\overline{\mu/q_1 q_2 \cdots q_k}\ ;\ \lambda/p_1 p_2 \cdots p_k\} \prod_{i=1}^{k} \delta^{w_i}_{p_i - q_i} \quad . \tag{3.4}$$

The corresponding generalisation of the Gelfand patterns arises as a result of the extension of labels defined by

$$\left\{\begin{matrix} \lambda_1\ \lambda_2 \cdots\ \lambda_a 0 \ldots 0 -\mu_b \cdots -\mu_2\ -\mu_1 \\ \sigma_1\ \sigma_2 \cdots \sigma_c 0 \ldots 0 -\tau_d \cdots -\tau_2 -\tau_1 \\ \vdots \\ \\ \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} m_{1k} \quad m_{2k} \quad \cdots\cdots \quad m_{kk} \\ m_{1k-1}\ m_{2k-1} \cdots m_{k-1k-1} \\ \vdots \\ m_{12}\ m_{22} \\ m_{11} \end{matrix}\right\}$$

The constraints applying to (3.1) are such that, once again, $m_{ij+1} \geq m_{ij} \geq m_{i+1j+1}$ and

$$w_j = \sum_{c=1}^{j} m_{ij} - \sum_{i=1}^{j-1} m_{ij-1} \qquad \text{for } j = 1,2,\ldots,k, \tag{3.5}$$

but now m_{ij} may be any integer: positive, negative or zero. Furthermore a generalisation of Young tableaux following immediately from (3.2) takes the form

Row lengths.

$\mu_1;\lambda_1$

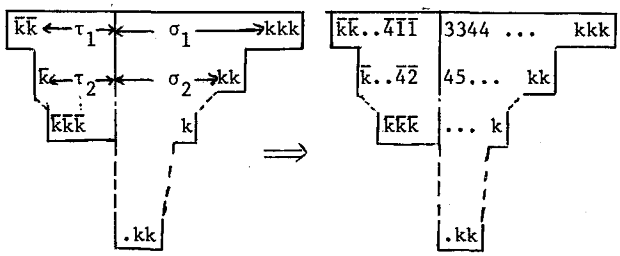
$\mu_2;\lambda_2$

$\vdots$

$\mu_b;\lambda_b$

$\vdots$

λ_a



The numbers in the tableau are non-decreasing across each row from left to right and are strictly increasing in magnitude down each column from top to bottom where an entry $\bar{j}$ is to be interpreted as $-j$. In addition if the lowest rows in which j and $\bar{j}$ appear are the x-th and y-th then $x + y \leqslant j$. Finally:

$$w_j = \text{the number of } j\text{'s} - \text{the number of } \bar{j}\text{'s in the tableau.} \qquad (3.6)$$

Once more the multiplicity of each weight is the number of distinct patterns, or equivalently the number of distinct tableaux, whose entries satisfy the given constraints.

For example in the case of the group $U(5)$, for which $k=5$, the multiplicity of the weight $\underset{\sim}{w} = (0,1,\bar{1},1,\bar{1})$, where $\bar{1}= -1$, in the irreducible representation $\{\bar{\mu};\lambda\} = \{\bar{1}^3;21\}$ is 2, corresponding to the existence of the tableaux
$$\begin{array}{|c|cc|}\hline \bar{2} & 2 & 2 \\ \bar{3} & 4 & \\ \bar{5} & & \\ \hline\end{array} \qquad \begin{array}{|c|cc|}\hline \bar{3} & 2 & 4 \\ \bar{4} & 4 & \\ \bar{5} & & \\ \hline\end{array}$$
and the patterns
$$\left\{\begin{array}{c} 2\ 1\ \bar{1}\ \bar{1}\ \bar{1} \\ 2\ 1\ \bar{1}\ \bar{1} \\ 2\ \bar{1}\ \bar{1} \\ 2\ \bar{1} \\ 0 \end{array}\right\} \qquad \left\{\begin{array}{c} 2\ 1\ \bar{1}\ \bar{1}\ \bar{1} \\ 2\ 1\ \bar{1}\ \bar{1} \\ 1\ 0\ \bar{1} \\ 1\ 0 \\ 0 \end{array}\right\}$$

The symmetry group of the weight diagram is once again S_k since the multiplicities are invariant under permutations of the components of the weight vectors as illustrated by the fact that

$$m^{(01\bar{1}1\bar{1})}_{\{\bar{1}^3;21\}} = m^{(\bar{1}\bar{1}011)}_{\{\bar{1}^3;21\}} = m^{(11\bar{1}\bar{1}0)}_{\{\bar{1}^3;21\}} = \ldots = 2 .$$

Now however, due to the cancellations that take place between p_j and q_j in defining w_j for $j = 1,2,\ldots,k$, the multiplicities are no longer k-independent. Indeed if the same example is considered for the group $U(k)$ the appropriate tableaux corresponding to the weight vector $(\bar{1}\bar{1}1100\ldots0)$

are $\begin{array}{|c|c|}\hline \bar{1} & 34 \\ \hline \bar{2} & j \\ \hline \bar{j} & \\ \hline \end{array}$ $\quad$ $\begin{array}{|c|c|}\hline \bar{1} & 3j \\ \hline \bar{2} & 4 \\ \hline \bar{j} & \\ \hline \end{array}$ with $j = 5,6,\ldots,k$. Hence $m^{(\bar{1}^2;1^2)}_{\{\bar{1}^3;21\}} = 2k - 8$.

4. Representations of Sp(2k).

The irreducible representations of $Sp(2k)$ are specified by $\langle\lambda\rangle$ where (λ) is a partition of ℓ into a non-vanishing parts with $a \leq k$. Zhelobenko [8] has derived the branching rule

$$Sp(2k)\downarrow Sp(2k-2)\times U(1) \qquad \langle\lambda\rangle\downarrow \sum_{p_k,q_k} \langle\lambda/q_kp_k\rangle\times\{p_k-q_k\} , \tag{4.1}$$

with $\sigma_i \geq \rho_i \geq \sigma_{i+1}$, $\lambda_i \geq \sigma_i \geq \lambda_{i+1}$ and $w_k = p_k - q_k$ where (σ) and (ρ) are partitions of s and r into c and e non-vanishing parts such that $p_k = s - r$, $q_k = \ell - s$ and $c \leq k$, $e \leq k-1$. In the notation appropriate to S-functions this takes the form

$$Sp(2k)\downarrow Sp(2k-2)\times U(1) \qquad \langle\lambda\rangle\downarrow \sum_{p_k,q_k} \langle\lambda/q_kp_k\rangle\times\{p_k-q_k\} . \tag{4.2}$$

The repeated application of this rule to the chain

$$Sp(2k)\downarrow Sp(2k-2)\times U(1) \downarrow Sp(2k-4)\times U(1)\times U(1) \downarrow \ldots \downarrow T_k , \tag{4.3}$$

gives

$$\langle\lambda\rangle\downarrow \ldots \downarrow \sum_{\underset{\sim}{p},\underset{\sim}{q}} \langle\lambda/q_1p_1q_2p_2\cdots q_kp_k\rangle \{p_1-q_1\}\times\{p_2-q_2\}\times\ldots\times\{p_k-q_k\} , \tag{4.4}$$

so that the weight multiplicities are given by

$$m^{\underset{\sim}{w}}_{\langle\lambda\rangle} = \sum_{\underset{\sim}{p},\underset{\sim}{q}} \langle\lambda/q_1p_1q_2p_2\cdots q_kp_k\rangle \prod_{i=1}^{k} \delta^{w_i}_{p_i-q_i} . \tag{4.5}$$

It follows from (4.1) that the Gelfand pattern generalisation takes the form:

$$\left\langle \begin{array}{cccc} \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ \sigma_1 & \sigma_1 & \cdots & \sigma_k \\ & \rho_1 & \rho_2 \cdots & \rho_{k-1} \\ & & \vdots & \\ & & & \end{array} \right\rangle \Rightarrow \left\langle \begin{array}{cccc} m_{1k} & m_{2k} & \cdots\cdots & m_{kk} \\ m_{1\bar{k}} & m_{2\bar{k}} & \cdots\cdots & m_{k\bar{k}} \\ m_{1k-1} & m_{2k-1} & \cdots & m_{k-1k-1} \\ m_{1\overline{k-1}} & m_{2\overline{k-1}} & \cdots & m_{k-1\overline{k-1}} \\ & \vdots & & \\ & m_{12} & m_{22} & \\ & m_{1\bar{2}} & m_{2\bar{2}} & \\ & & m_{11} & \\ & & m_{1\bar{1}} & \end{array} \right\rangle$$

The constraints appropriate to (4.1) are such that m_{ij} and $m_{i\bar{j}}$ are both non-negative integers, that $m_{ij} \geq m_{i\bar{j}} \geq m_{i+1j}$, $m_{i\bar{j}} \geq m_{ij-1} \geq m_{i+1\bar{j}}$ and

$w_j = p_j - q_j$ where $p_j = \sum_{i=1}^{j} m_{ij} - \sum_{i=1}^{j} m_{i\bar{j}}$ and $q_j = \sum_{i=1}^{j} m_{i\bar{j}} - \sum_{i=1}^{j-1} m_{ij-1}$.

In the same way it follows from (4.2) that the Young tableaux generalise to yield tableaux of the form:

Row lengths.

$$
\begin{array}{l}
\lambda_1 \\
\lambda \\
\vdots \\
\lambda_a
\end{array}
\quad
\begin{array}{|l}
\leftarrow \sigma_1 \quad \rightarrow \bar{k}\bar{k}kkk \\
\leftarrow \sigma_2 \ \rightarrow \bar{k}kkk \\
\vdots \\
\bar{k}k
\end{array}
\quad \Longrightarrow \quad
\begin{array}{|l}
\bar{1}\bar{1}122\ldots\bar{k}\bar{k}kkk \\
\bar{2}23\ldots\bar{k}kkk \\
\vdots \\
..\bar{k}k
\end{array}
$$

The constraints are such that with the ordering of the entries defined by $\bar{1}<1<\bar{2}<2<\ldots<\bar{k}<k$, the entries are non-decreasing across each row from left to right and are strictly increasing down each column from top to bottom. If the lowest rows in which j and $\bar{j}$ appear are the x-th and y-th then $x \leq j$ and $y \leq j$. Finally

$$w_j = \text{the number of } j\text{'s} - \text{the number of } \bar{j}\text{'s in the tableau.} \tag{4.6}$$

As before the multiplicity of each weight is just the number of distinct patterns, or equivalently the number of distinct tableaux, whose entries satisfy the given constraints.

For example in the case of the group $Sp(6)$, for which $k=3$, the weight $\underset{\sim}{w} = (102)$ has multiplicity 5 in the irreducible representation $\langle\lambda\rangle = \langle 32\rangle$, corresponding to the existence of the tableaux

$$
\begin{array}{|ccc|}\hline \bar{1}&1&1\\ \cline{1-3} 3&3& \\ \cline{1-2}\end{array}
\quad
\begin{array}{|ccc|}\hline 1&\bar{2}&2\\ \cline{1-3} 3&3& \\ \cline{1-2}\end{array}
\quad
\begin{array}{|ccc|}\hline 1&\bar{2}&3\\ \cline{1-3} 2&3& \\ \cline{1-2}\end{array}
\quad
\begin{array}{|ccc|}\hline 1&2&3\\ \cline{1-3} \bar{2}&3& \\ \cline{1-2}\end{array}
\quad
\begin{array}{|ccc|}\hline \bar{2}&2&3\\ \cline{1-3} 2&3& \\ \cline{1-2}\end{array}
\quad ,
$$

and the patterns

$$
\left\langle\begin{array}{ccc} 3&2&0\\ 3&0&0\\ 3&0&\\ 3&0&\\ 3&&\\ 1&& \end{array}\right\rangle
\quad
\left\langle\begin{array}{ccc} 3&2&0\\ 3&0&0\\ 3&0&\\ 2&0&\\ 1&&\\ 0&& \end{array}\right\rangle
\quad
\left\langle\begin{array}{ccc} 3&2&0\\ 2&1&0\\ 2&1&\\ 2&0&\\ 1&&\\ 0&& \end{array}\right\rangle
\quad
\left\langle\begin{array}{ccc} 3&2&0\\ 2&1&0\\ 2&1&\\ 1&1&\\ 1&&\\ 0&& \end{array}\right\rangle
\quad
\left\langle\begin{array}{ccc} 3&2&0\\ 2&1&0\\ 2&1&\\ 1&0&\\ 0&&\\ 0&& \end{array}\right\rangle
$$

Hence $m_{\langle 32\rangle}^{(102)} = 5$.

The symmetry group of the weight diagrams of $Sp(2k)$ is the hyperoctohedral group generated by the permutations of the components of the weight vectors and changes of the signs of these components. For example

$$m^{(102)}_{<32>} = m^{(210)}_{<32>} = m^{(\bar{1}02)}_{<32>} = m^{(\bar{2}\bar{1}0)}_{<32>} = \ldots = 5.$$

Once more the fact that $w_j = p_j - q_j$ for $j=1,2,\ldots,k$ leads to the weights being k-dependent. Extending the example to the case of the group $Sp(2k)$ yields for the weight $\underset{\sim}{w} = (210\ldots0)$ the tableaux

$$\begin{array}{llll}
\begin{array}{|l}112\\ \bar{2}2\end{array} & \begin{array}{|l}11\bar{2}\\ 22\end{array} & & \\
\begin{array}{|l}112\\ \bar{3}3\end{array} & \begin{array}{|l}112\\ \bar{4}4\end{array} & \cdots & \begin{array}{|l}112\\ \bar{k}k\end{array} \\
\begin{array}{|l}11\bar{3}\\ 23\end{array} & \begin{array}{|l}11\bar{4}\\ 24\end{array} & \cdots & \begin{array}{|l}11\bar{k}\\ 2k\end{array} \\
\begin{array}{|l}113\\ 2\bar{3}\end{array} & \begin{array}{|l}114\\ 2\bar{4}\end{array} & \cdots & \begin{array}{|l}11k\\ 2\bar{k}\end{array}
\end{array}$$

so that: $\quad m^{(21)}_{<32>} = 3k-4.$

5. Tensor and Spinor Representations of O(2k) and O(2k+1).

The irreducible tensor and spinor representations of $O(2k)$ and $O(2k+1)$ are specified by $[\lambda]$ and $[\Delta;\lambda]$ respectively, where (λ) is a partition of ℓ into a non-vanishing parts with $a \leqslant k$. The branching rules appropriate to these representations may be derived from those given elsewhere[3] and take the form:

$$O(2k)\downarrow O(2k-2)\times U(1) \qquad [\lambda] \downarrow \sum_{p_k q_k} [\lambda/q_k p_k] \times \{p_k - q_k\}\ , \tag{5.1}$$

$$[\Delta;\lambda] \downarrow \sum_{p_k q_k} [\Delta;\lambda/q_k p_k] \times (\{p_k - q_k + \tfrac{1}{2}\} + \{p_k - q_k - \tfrac{1}{2}\})\ . \tag{5.2}$$

The application of these results to the chain

$$O(2k)\downarrow O(2k-2)\times U(1)\downarrow O(2k-4)\times U(1)\times U(1)\downarrow \quad \ldots\downarrow T_k \tag{5.3}$$

leads to the formulae

$$O(2k) \qquad m^{\underset{\sim}{w}}_{[\lambda]} = \sum_{\underset{\sim}{p},\underset{\sim}{q}} [\lambda/q_1 p_1 q_2 p_2 \cdots q_k p_k] \prod_{i=1}^{k} \delta^{w_i}_{p_i - q_i}\ . \tag{5.4}$$

$$O(2k) \qquad m^{w}_{[\Delta;\lambda]} = \sum_{p,q,r} [\Delta;\lambda/q_1p_1q_2p_2\cdots q_kp_k] \quad \prod_{i=1}^{k} \delta^{w_i}_{p_i-q_i+r_i}\, \delta^{\frac{1}{2}}_{|r_i|} . \tag{5.5}$$

Corresponding formulae for $O(2k+1)$ may then be written down by noting the rules(3)

$$O(2k+1) \downarrow O(2k) \qquad [\lambda] \downarrow \sum_m [\lambda/m] , \tag{5.6}$$

$$[\Delta;\lambda] \downarrow \sum_m [\Delta;\lambda/m] . \tag{5.7}$$

It is then a straightforward task to draw up the appropriate generalisations of Gelfand patterns and Young tableaux. The former have of course been given Gelfand and Tseitlin(9). In defining the latter care has to be taken in interpreting the S-function quotients. It is necessary to use the modification rules(3,10,11) for both tensor and spinor representations of the group $O(n)$. If this is done the resulting tableaux provide an easy way of deriving results such as:

$$O(2k) \qquad m^{(21)}_{[32]} = 3k-5 \qquad m^{(\Delta;11)}_{[\Delta;31]} = (3k^2 - 7k + 4)/2$$

$$O(2k+1) \qquad m^{(21)}_{[32]} = 3k-3 \qquad m^{(\Delta;11)}_{[\Delta;31]} = (3k^2 - k + 2)/2 .$$

6. Conclusions.

The branching rule associated with the subgroup chain leading from $U(k)$ to T_k was first used to calculate weight multiplicities by Delaney and Gruber(4), who exploited the connection with patterns and tableaux described here. The generalisation to the other classical groups was suggested by Gilmore(12,13) who did not however arrive at the generalised Young tableaux which are seen here to provide the most efficient means of calculating weight multiplicities. The great merit of using these tableaux is that results are readily obtained for groups of arbitrary rank. Considerable tables of results for the tensor and spinor representations of the groups $O(n)$ have been compiled by Plunkett(14). It is hoped to publish these, and similar results for $U(k)$ and $Sp(2k)$, elsewhere.

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