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Chequered Surfaces and Complex Matrices

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Abstract

We investigate a large N matrix model involving general complex matrices. It can be reinterpreted as a model of two hermitian matrices with specific couplings, and as a model of positive definite hermitian matrices. Large N perturbation theory generates dynamical triangulations in which the triangles can be chequered (i.e. coloured so that neighbours are opposite colours). On a sphere there is a simple relation between such triangulations and those generated by the single hermitian matrix model. For the torus (and a quartic potential) we solve the counting problem for the number of triangulations that cannot be chequered. The critical physics of chequered triangulations is the same as that of the hermitian matrix model. We show this explicitly by solving non-perturbatively pure 2 dimensional "chequered" gravity. The interpretative framework given here applies to a number of other generalisations of the hermitian matrix model.

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1. Introduction

Recent progress in understanding the partition function for non-perturbative (oriented) string theory in one dimension or less [1] –[3] has generated considerable interest in matrix models. These authors used $N \times N$ hermitian matrices H as the dynamical variables and an action whose potential is just a trace over some polynomial of H: The perturbation expansion for such a model sums over triangulations of arbitrary genus two dimensional surfaces. The triangulation can be seen directly by forming the dual diagram to the Feynman diagram using 't Hooft's large N counting rules to define loops (faces) in the Feynman diagram[4]. (Actually polygonations would be a better word since in general not just 3-sided polygons are used. Since we will wish to distinguish between triangles and polygons we will henceforth use this term.)

There is no reason why one cannot use other forms of matrices provided that they also have a natural interpretation in terms of triangulations. For example symmetric real matrices are readily interpreted as triangulating non-orientable surfaces.[5] Here we show that complex matrices (with no hermitian constraint) also have a natural interpretation. They can be understood as summing over chequered triangulations of orientable surfaces. Chequered triangulations, rather polygonations, are polygonations where the triangles[1] are chequered, that is to say where the triangles are coloured black and white in such a way that neighbouring triangles are always opposite colours (as in fig.2). This is a restricted set of polygonations because not all polygonations can be chequered. For example only even sided polygons can be used since odd sided polygons cannot be chequered. However some even sided polygonations cannot be chequered either; the simplest polygonation of the torus which is a square with opposite sides identified cannot be chequered (fig.3).

Our original motivation for looking at complex matrices as a method for describing non-perturbative strings is that such a formulation has some advantages over hermitian matrix models in dealing with aspects of non-perturbative strings in

[1] The polygons are divided into triangles in the obvious way, see fig. 1.

one dimension and greater. That however is another story which will have to await a future publication. Here we study the model in its own right.

In section 2 we formulate the model and demonstrate its interpretation in terms of chequered surfaces. We prove geometrically that all even sided polygonations of a sphere can be chequered. This leads to a simple relation between the counting problem for the number of polygonations (or its dual which are planar closed graphs[6] [7]) and the number of chequered polygonations on the sphere and hence to a simple relation between the two models for the partition function of the sphere. Indeed, provided that the hermitian matrix model potential is even, a simple scaling of the couplings relates the two partition functions. We also point out that the model may be interpreted as a model of two hermitian matrices with specific couplings.

In section 3 we analyze the model analytically. By integrating out certain angular modes we reduce the problem to one of orthogonal polynomials defined on the positive real line. We note here that a model of hermitian positive definite matrices is precisely equivalent to this complex matrix model. We derive recurrence relations for the coefficients in these polynomials. Solving for these coefficients using perturbation theory in $1/N$ allows us to derive (*away* from criticality) a number of relations. In particular we derive the same relation as proved geometrically for the sphere. We also solve the counting problem for the number of chequered quadrilations of the torus. By borrowing the hermitian matrix results[6] one can derive an expression that solves the counting problem for the number quadrilations of the torus that cannot be chequered.

In section 4 we take the continuum limit by tuning to a critical point. The equivalence of the partition function for hermitian and complex matrix models on the sphere implies identical critical behaviour for the sphere. Since the chequered polygonations only differ from those generated by hermitian matrix models by short distance details, the continuum physics we expect is the same to all orders in the genus. We confirm this explicitly by solving the model in the double scaling limit, deriving differential equations for the string susceptibility, for the simplest case of chequered quadrilations and showing that it gives pure 2D gravity (index $k = 2$) as in refs.[1], up to a rescaling of the string coupling. This also allows a number of non-trivial checks on the analysis of the preceding section. We also note the existence of another critical point in the torus contribution, suggesting a continuum limit with index $k = 1$, which does not necessarily have an interpretation in terms of chequered triangulations.

In section 5 we summarise and draw our conclusions. In particular we note that the interpretative framework we have developed here may readily be applied to matrix models where the matrices are arbitrary real, or arbritrary quaternionic, even when the matrices are not square: Thus the triangulations again only differ by short distance details and the models will be in the universality class of real symmetric or quaternion real self-dual matrices[5] giving non-orientable surfaces, or (for non-square complex matrices) give physics in the same universality class as hermitian models.

2. Chequered Polygonations and The Model.

Define the following partition function:

$$Z(\gamma, g_p) = 1/Z_0 \int [dM] \exp\{-N/\gamma \; U(M^\dagger M)\} \tag{2.1}$$

where M is a complex $N \times N$ matrix with elements $M_{\alpha\beta}$, $[dM]$ stands for

$$\prod_{\alpha\beta} dM_{\alpha\beta} d\overline{M}_{\alpha\beta} \quad , \tag{2.2}$$

$U(M^\dagger M) = U(MM^\dagger)$ is a polynomial of the form

$$U(M^\dagger M) = \mathrm{tr}\{\gamma M^\dagger M + \sum_{p\geq 2} g_p (M^\dagger M)^p\} \tag{2.3}$$

and γ, g_p are arbitrary parameters. Z_0 is a normalisation constant adjusted so that $Z = 1$ when $g_2, g_3, \cdots = 0$. The form of U as a function of M and $M^\dagger$ is fixed by the requirement that it be polynomial and that (2.1) have the invariance:

$$M \to V^\dagger M W \tag{2.4}$$

where V and W are any unitary matrices. We will use this invariance to solve the model, in sect. 3. (The reader will note that we have over-parameterized. This will be convenient later. In the usual fashion the parameters are restricted to ensure that (2.1) is well-defined but, in the triangulation interpretation to ensure that all terms in the perturbation series are positive and to reach the critical point, may be analytically continued).

We can reformulate (2.1) as an integral over two hermitian matrices H_1, H_2 by writing

$$M = H_1 + iH_2$$

to give

$$Z = 1/Z_0 \int [dH_1][dH_2] \, e^{-N/\gamma \, U}$$

where

$$[dH] = \prod_{\alpha} dH_{\alpha\alpha} \prod_{\alpha<\beta} dH_{\alpha\beta} d\overline{H}_{\alpha\beta} \tag{2.5}$$

and the action now appears as

$$U = \mathrm{tr}\{\gamma(H_1^2 + H_2^2) + g_2(H_1^4 + H_2^4 + 4H_1^2H_2^2 - 2H_1H_2H_1H_2) + \cdots\}$$

Viewing this formulation one might expect that such a model can access physics related to that of two hermitian matrices[2] (although the action used to obtain the Ising model is very different). However it is not possible to tune the relative couplings between H_1 and H_2 at each order separately from their self couplings, because these are fixed by the invariance (2.4), and such a tuning would be necessary in general to access all critical points. In any case it is shown later in the paper and in ref.[8] that the physics described by this model is that of one hermitian matrix.

To obtain the interpretation in terms of polygonations we use perturbation theory about $g_2, g_3, \cdots = 0$. The first term in (2.3) acts as an inverse propagator giving

$$\langle M_{\alpha\beta} M^{\dagger}_{\gamma\delta} \rangle = 1/N \; \delta_{\gamma\beta}\delta_{\alpha\delta} \tag{2.6}$$

Using large N diagrammatics[4][6] we represent this as a double line, and with an arrow pointing towards the $M^{\dagger}$ (see fig.4). The vertices are adorned with arrows in a similar manner. As an illustration we list the diagrams to second order in the quartic (g_2) interaction in fig.5. In the standard fashion the closed lines count a factor of N and identify loops which are dual to vertices in the polygonation. The vertices each carry a factor of N/γ and are dual to faces (polygons) in the polygonation. The propagators carry the factor of $1/N$ shown above, and are dual to edges in the polygonation. Let E be the number of edges, F the number of faces, and V the number of vertices in the polygonation. Then the Euler number $\chi = F + V - E$ and it follows that each polygonation is accompanied by a factor

$$N^{-E}(N/\gamma)^F N^V = N^\chi \, e^{-F \ln\gamma} \tag{2.7}$$

Thus, as in refs.[1]–[7], the power of N counts genus and is the inverse bare string coupling; $\ln\gamma$ multiplies the area (equal to the number of polygons) and is the bare cosmological constant. One can also count the number of quadrilaterals, hexagons, etc. by counting powers of $g_2, g_3, \cdots$ respectively. To chequer the surface, shade the triangles in the polygon if their outer edge is dual to a propagator with an incoming arrow, otherwise leave them white: see fig.6. In other words the triangles are directly associated with the matrices: black for $M^\dagger$, white for M. The chequered property of the polygonations immediately follows from the structure of the vertices and the fact that the propagator only connects M's with $M^\dagger$'s.

As a final example, using fig.6, one can see that fig.5(i) corresponds to a chequered sphere made by folding a square along a diagonal and gluing the sides, whereas fig.5(ii) uses two squares to form a chequered ravioli.

It is thus clear that if it were not for the restriction of chequering (arrows), the counting of polygonations (closed graphs) would be precisely the same as for hermitian matrix models[6]. Indeed computing the diagrams in fig.5 we find

$$\begin{array}{llll} (i) \quad 2(-g)N^2 & (ii) \quad g^2N^2 & & \\ (iii) \quad 4g^2N^2 & (iv) \quad 4g^2N^2 & (v) \quad g^2N^0 & \end{array} \tag{2.8}$$

where we have introduced $g = g_2/\gamma$ so that it can be compared directly with fig.8 of Bessis et al[6]. Recall that the coefficient (ignoring the sign) of the $g^m N^\chi$ term multiplied by $m!$ is the number of ways of making a closed surface with Euler number χ from m elements: squares for fig.8 ref.[6], chequered squares for us. The result of (i) is the same in both cases because the number of ways of folding a square along a diagonal is the same whether or not it is chequered, and the result of (ii) is half that of Bessis et al because half of the orientations of the two squares correctly chequer the ravioli and half do not.

These observations immediately generalise. One sees that for any chequered polygonation using n polygons, there are just two correctly chequered surfaces (the original and its photographic negative) out of the total 2^n ways of chequering the individual polygons. We will shortly prove that on a sphere all polygonations can be chequered, provided only that the polygons be even sided. It follows that a simple relation exists between the N^2 contribution for this model and that of the hermitian model with an even potential: the coefficient of the γ^{-n} term here is just $2(1/2)^n$ times that of the corresponding hermitian contribution. Thus by rescaling N and γ we obtain precisely the same contribution *off criticality* for the two models at the spherical level. We will rederive these results analytically in the next section. Here it only remains for us to show geometrically that all even sided polygonations of a sphere can be chequered.

In an even sided polygonation of the sphere, a vertex of an individual triangle is either at the centre of a polygon (as in fig.1), in which case the vertex is shared by an even number of triangles, or else at a vertex of a polygon. But this latter vertex is shared with triangles from other polygons (see fig.2) and each polygon contributes two triangles. *Thus all vertices are shared by an even number of triangles.*

Now colour the sphere by starting at some triangle and colouring outwards, chequering triangles with common edges consistently, and colouring all the triangles one can this way until either the sphere is completely chequered or one is left with a mass of chequered triangles *connected* through their edges, and some obstructions. We will proceed by showing that on a sphere there cannot be such obstructions since if there were they must break the mass of chequered triangles into disconnected pieces, violating our just stated colouring procedure.

Clearly such obstructions will be of the form of triangles with one side bordered by a triangle already coloured black and another side bordered by a triangle already coloured white. Let us mark such triangles with crosses as in fig.7. The other triangles (apart from the dotted one) have not been drawn, however we know from above that there are an odd number of undrawn triangles sharing the vertex a. But an *odd* number of further triangles meeting at a cannot be correctly chequered to match the two coloured triangles in the figure. It follows that there must be at least one other crossed triangle meeting at a. Similarly we show that there is at least one other crossed triangle meeting at the bottom (possibly the dotted triangle), since otherwise moving anticlockwise around vertex b from the black we deduce that the dotted triangle is black, while moving clockwise from the white at c we deduce that the dotted triangle is white.

Now we have a situation in which every crossed triangle must be connected by its edges to at least two coloured triangles and at least joined to two crossed triangles at its vertices. The presence of the coloured triangles on their borders prevents the crossed triangles from forming a connected mass, so the only way they can satisfy these rules is by forming closed loops. But since a sphere is simply connected, this would separate the coloured mass into disconnected pieces □.

3. The Sphere and The Torus.

In this section we will derive the previous result analytically as well as others. Following Itzykson and Zuber[9] we note that the essential trick is to be able to integrate out order N^2 "angular" variables to be left with a much smaller set of invariants (N in fact) still to be integrated over. We can do this by using our invariance (2.4). It is a well known fact that one can always bring a complex matrix to diagonal form with such a transformation. By appropriate choice of phases we can arrange that the diagonal elements, which we write as $\sqrt{y_\alpha}$ $\alpha = 1, \cdots, N$, are real and positive. We give a simple proof of these statements in the appendix. Note that it follows that the y's are the eigenvalues of $M^\dagger M$, which are also the eigenvalues of $MM^\dagger$. Thus change variables to

$$M = V\sqrt{y}\, W^\dagger \tag{3.1}$$

(where $\sqrt{y}$ is the diagonal matrix with entries $\sqrt{y_\alpha}$). The measure changes to

$$[dM] = d^N y\ [dV][dW]\ \Delta^2(y) \tag{3.2}$$

up to a numerical constant which does not concern us since it is cancelled by Z_0 (in

(2.1)). Here we have introduced the Vandermonde determinant

$$\Delta(y) = \prod_{\alpha<\beta} (y_\alpha - y_\beta) \ .$$

$[dV]$ and $[dW]$ are the Haar measures for the $U(N)$ integrals. Equation (3.2) is derived in the appendix.

At this stage it is appropriate to make some comments on the book by Mehta[10]. In it, Mehta gives an analysis of integration over a general complex matrix. However his interest in the statistics of eigenvalues (guided by applications to random hamiltonian models of nuclei) led him to consider a change of variables to the eigenvalues of the complex matrix (and a similarity transformation). These eigenvalues are not related to the y_α of course because in general V and W are unrelated (in (3.1)). It turns out that even the analysis with just a $\mathrm{tr}(M^\dagger M)$ action is quite involved and it would not appear to generalise to our case (2.3). In any case this parameterization is not as convenient for our purposes as the one we use.

Substituting (3.1) and (3.2) into (2.1) one can trivially integrate over the angular variables to obtain

$$Z(\gamma, g_p) = 1/Z_0 \int_0^\infty d^N y \ \Delta^2(y) \ \exp\{-N/\gamma \ U(y)\} \tag{3.3}$$

The integration region is restricted to $y_\alpha \in (0,\infty)$ as indicated.

Precisely the same integral is obtained from hermitian matrix models[6][7] except for this restriction. But this restriction is trivially incorporated by integrating only over positive definite hermitian matrices. Thus this model may also be viewed as a matrix model of positive definite hermitian matrices H with action (from (2.3)):

$$U(H) = \mathrm{tr}\{\gamma H + \sum_{p\geq 2} g_p H^p\} \tag{3.4}$$

Following the methods of ref. [6], we introduce for a *single* real variable y

$$d\mu(y) = dy \ \mathrm{e}^{-N/\gamma \ U(y)} \tag{3.5}$$

where now
$$U(y) = \gamma y + \sum_{p\geq 2} g_p y^p$$

and define orthogonal polynomials by

$$\begin{aligned} P_n(y) = y^n + \text{lower powers} \qquad n = 1, 2, \cdots \\ \int_0^\infty d\mu(y) P_n(y) P_m(y) = h_n \delta_{nm} \, . \end{aligned} \tag{3.6}$$

Since there is no $y \to -y$ symmetry[1] we also need to define

$$\int_0^\infty d\mu(y) P_n^2(y) y = A_n h_n \, , \tag{3.7}$$

then following ref.[6] one shows that

$$y P_n(y) = P_{n+1}(y) + A_n P_n(y) + R_n P_{n-1}(y) \tag{3.8}$$

where $R_n = h_n / h_{n-1}$. Z is given by

$$\begin{aligned} Z &= \mathrm{e}^{-N^2 E} \\ E &= -1/N \ln h_0 - 1/N^2 \sum_{k=1}^{N} (N-k) \ln R_k \end{aligned} \tag{3.9}$$

where E is the vacuum energy. From (2.1) the numerical constant Z_0 simply serves to ensure that $E(g_2, g_3, \cdots = 0) = 0$. With this in mind we will ignore additive numerical constant contributions to E henceforth and adjust the zero accordingly.

Now we need two recurrence relations for R_n and A_n. Following Gross and Migdal[1], it is helpful to introduce bra-ket notation with normalised basis $|n> = P_n/\sqrt{h_n}$, and operators S and $\hat{n}$ such that

$$\begin{aligned} \hat{n}|n> &= n|n> \\ S^\dagger|n> &= |n+1> \\ S|n> &= |n-1> \\ S|0> &= 0 \end{aligned} \tag{3.10}$$

then

$$y = \sqrt{R_{\hat{n}}} \, S^\dagger + S\sqrt{R_{\hat{n}}} + A_{\hat{n}} \tag{3.11}$$

From the definition (3.6) it follows that $dP_n/dy = nP_{n-1} +$ lower powers. Thus $n = < n|yd/dy|n >$ and integrating by parts we get our first relation:

$$< n|yU'(y)|n > = \frac{\gamma}{N}(2n+1) \tag{3.12}$$

where U' is dU/dy.

Given the restriction of y to the positive real line, and viewing the problem formally[11], one might be tempted to consider only a differential operator of the form yd/dy since this is the operator that is essentially self-adjoint on $\mathbb{R}_+$ and takes the place of the conjugate momentum in such situations[12]. However the only other relation that can be obtained this way is as follows: Writing equation (3.8) as $P_{n+1} = (y - A_n)P_n - R_n P_{n-1}$ and iterating, one obtains

$$P_{n+1} = y^{n+1} - y^n \sum_{k=0}^{n} A_n + \text{lower powers}$$

thus
$$h_n \sum_{k=0}^{n} A_k = < n|y\frac{d}{dy}|n+1 >$$

and by parts
$$= N/\gamma \sqrt{R_{n+1}} < n+1|yU'(y)|n >$$

This equation, being non-local, is of no use. (Apart from the inherent difficulty in using it, it must be localizable near the "Fermi level" N in order to scale correctly in the continuum limit). Taking finite differences by subtracting the same equation with $n \to n-1$ gives an equation that can be used for the all orders calculation (sect. 4) but yields only the differential of the string equation. To do better one has to involve the $y = 0$ boundary, thus considering $< n|d/dy|n >$ one obtains

$$\frac{\gamma}{Nh_n} P_n^2(0) = < n|U'(y)|n >$$

and considering $< n-1|d/dy|n >$ one obtains

$$\frac{\gamma}{N\sqrt{h_n h_{n-1}}} P_n(0)P_{n-1}(0) = < n|U'(y)|n-1 > - \frac{n\gamma}{N\sqrt{R_n}} \ .$$

Combining these two equations yields our second equation:

$$< n|U'(y)|n >< n-1|U'(y)|n-1 > = \left(< n|U'(y)|n-1 > - \frac{n\gamma}{N\sqrt{R_n}} \right)^2 \qquad (3.13)$$

As an example, keeping only the g_2 interaction, one obtains from eqns. (3.12) and (3.13) respectively:

$$\begin{aligned} \frac{2n+1}{N} &= 2g(R_{n+1} + R_n + A_n^2) + A_n \\ (n/N - 2gR_n)^2 &= R_n(1 + 2gA_n)(1 + 2gA_{n-1}) \end{aligned} \qquad (3.14)$$

where as in (2.8), $g = g_2/\gamma$. It is easy to see that these equations determine all the A_n and R_n recursively once A_0 is known. R_0 is consistently taken to be zero as follows from eqns (3.10) and (3.11). A_0 follows from eqns. (3.6) and (3.7):

$$\begin{aligned} h_0 &= \int_0^\infty dy \ \mathrm{e}^{-N(y+gy^2)} \\ &= 1/N \sum_{m=0}^{\infty} (2m-1)!! \, (-2g/N)^m \end{aligned} \tag{3.15}$$

$$\begin{aligned} A_0 h_0 &= \int_0^\infty dy \ y \ \mathrm{e}^{-N(y+gy^2)} \\ &= 1/N^2 \sum_{m=0}^{\infty} (2m+1)!! \, (-2g/N)^m \end{aligned} \tag{3.16}$$

In fact the structure of the equations are such that they are readily solved perturbatively (in g) and no boundary conditions are needed for this[6]:

$$\begin{aligned} R_n &= (n/N)^2 - 12g(n/N)^3 + 20g^2/N^4(9n^4+n^2) + \cdots \\ A_n &= \frac{2n+1}{N} - 4g/N^2(3n^2+3n+1) + 8g^2/N^3(18n^3+27n^2+19n+5) + \cdots \end{aligned} \tag{3.17}$$

As an intermediate check on consistency, substituting the above and (3.15) into eqn. (3.9) one obtains to order g^2:

$$\begin{aligned} E &= 2g/N^2 - 10g^2/N^3 + 1/N^2 \sum_{k=1}^{N} (N-k)(12gk/N - 108g^2k^2/N^2 - 20g^2/N^2) \\ &= 2g - 9g^2 - g^2/N^2 \end{aligned} \tag{3.18}$$

in agreement with (2.8).

In preparation for the limit $N \to \infty$ it is helpful now to define $x = n/N$ and $\epsilon = 1/N$. We write $|n> = |x>$ (the state is still normalised to 1), $r \equiv r(x) = R_n$ and $a \equiv a(x) = A_n$ (to be understood as operators when between bra-kets). Define

$$\begin{aligned} \Omega(x) &= < n|U'(y)|n> \\ &= < x|U'(S^\dagger + Sr + a)|x> , \end{aligned} \tag{3.19}$$

$$\begin{aligned} \bar{\Omega}(x) &= n\gamma/N - \sqrt{R_n} < n-1|U'(y)|n> \\ &= \gamma x - < x|S^\dagger U'(S^\dagger + Sr + a)|x> \end{aligned} \tag{3.20}$$

where use has been made of the identity

$$y\sqrt{h_{\hat{n}}} = \sqrt{h_{\hat{n}}}[S^\dagger + Sr(\hat{x}) + a(\hat{x})] \, .$$

Now equations (3.12) and (3.13) can be succinctly written:

$$\tilde{\Omega}(x) + \tilde{\Omega}(x+\epsilon) = a(x)\Omega(x) \tag{3.21}$$

$$r(x)\Omega(x)\Omega(x-\epsilon) = \tilde{\Omega}^2(x) \tag{3.22}$$

Following ref.[6] we note that these equations define r and a as a power series in ϵ:

$$r \equiv r_\epsilon(x) = \sum_{k=0}^{\infty} r_k(x)\epsilon^k \tag{3.23}$$

and similarly a. To zeroth order in ϵ, the spherical contribution, we deduce immediately from (3.21) and (3.22):

$$4r_0 = a_0^2 \, . \tag{3.24}$$

At this order S commutes with a, thus substituting this into eqn. (3.11) and using (3.12), one obtains

$$\gamma x = w(a_0) \tag{3.25}$$

where $$w(a) = 1/2 < x|(a/2[S + S^\dagger] + a)U'(a/2[S + S^\dagger] + a)|x> \, .$$

Remembering that U is the polynomial $U = \sum_{p\geq 1} g_p y^p$ $(g_1 = \gamma)$ we have

$$\begin{aligned} w(a) &= 1/2 \sum_{p\geq 1} p g_p < x|(a/2[S + S^\dagger] + a)^p|x> \\ &= 1/2 \sum_{p\geq 1} p g_p a^p(x) \int_0^{2\pi} \frac{d\theta}{2\pi}(1+\cos\theta)^p \\ &= 1/2 \sum_{p\geq 1} g_p (a/2)^p \frac{(2p)!}{p!(p-1)!} \qquad , \end{aligned} \tag{3.26}$$

or rearranging slightly

$$w(a)/\gamma = a/2 + 1/2 \sum_{p\geq 2} g_p/\gamma \frac{(2p)!}{p!(p-1)!}(a/2)^p \, . \tag{3.27}$$

The intermediate expression arises by noting that the coefficient of S^0 ($S^\dagger = S^{-1}$) is obtained by the substitution $S = e^{i\theta}$ and averaging over θ. Equations (3.25) and (3.27) serve to determine a_0 in terms of x. For the simplest case where only $g = g_2/\gamma$ is non-zero we have

$$w(a_0)/\gamma = a_0/2 + 3ga_0^2/2 = x \ . \tag{3.28}$$

From eqns. (3.9) and (3.24) we have to zeroth order:

$$E_0 = -2\int_0^1 dx(1-x)\ln(a_0/2x) \, . \tag{3.29}$$

The subscript 0 is to indicate the spherical contribution. The $2x$ in the log adds a numerical constant and is required to normalise the vacuum energy (cf. comments below eqn.(3.9)) as follows from eqn. (3.17). Equations (3.25), (3.27) and (3.29) define the spherical contribution and can be compared directly with the classic paper of Bessis et al.[6]. Indeed starting from an action for the Hermitian matrix H (eqn. (4.1) of ref.[6] up to a $\sqrt{N}$ rescaling of the matrix, and $g \to g/\gamma$):

$$\text{Action} = N\{\tfrac{1}{2}H^2 + \sum_{p\geq 2} g_p/\gamma H^{2p}\}$$

one obtains the spherical contribution to the vacuum energy

$$e_0(g_p/\gamma) = -\int_0^1 dx(1-x)\ln(r_0^B(x)/x)$$

where r_0^B is determined from the equation $w^B(r_0^B) = x$ and

$$w^B(r) = r + \sum_{p\geq 2} g_p/\gamma \frac{(2p)!}{p!(p-1)!} r^p \ .$$

(We used equations (5.5-7) of ref.[6]. Note that the relative factor of 1/2 in the actions quadratic term is necessary for equality of propagators).

Comparing with equations (3.25) and (3.27) respectively we see that w/γ is to be identified with w^B, $a_0/2$ is to be identified with r_0^B and the couplings g_p $p \geq 2$ must be scaled by a factor of 2. Finally comparison with equation (3.29) implies:

$$E_0(g_p/\gamma) = 2e_0(\frac{g_p}{2\gamma}) \ .$$

This is the statement we derived geometrically in the previous section. It implies, up to some simple rescalings, complete equivalence even *off criticality* of the spherical contribution to the vacuum partition function for the two models.

Using our identifications above and lifting two further formulae from ref.[6] we deduce for the simplest case where only $g = g_2/\gamma$ is non-zero:

$$\begin{aligned} E_0 &= -\ln\left(\frac{A}{2}\right) + \frac{1}{48}(A-2)(18-A) \\ &= -2\sum_{p=1}^{\infty}(-6g)^p \frac{(2p-1)!}{p!(p+2)!} \\ &= 2g - 9g^2 + 72g^3 - \cdots \end{aligned} \tag{3.30}$$

in agreement with eqn. (3.18). Here A is the solution for a_0 of eqn.(3.28) at $x = 1$ that matches on to perturbation theory (identified as $A \equiv 2a^2$ in ref.[6]):

$$A = \frac{-1+\sqrt{1+24g}}{6g} \ . \tag{3.31}$$

For this simplest case, we now derive the off-critical contribution from the torus. As a generating function, it solves the counting problem for the number of chequered quadrilations of the torus, as does the above for the sphere. It also allows us to find out (for this case) whether the critical point(s) for the torus are just those deduced from the sphere.

Taylor expanding eqns. (3.14) to order ϵ, using eqn. (3.23) and the material before eqn. (3.19) and eqns. (3.24) and (3.28) one finds eqns. for r_1 and a_1:

$$1 = 4gr_1 + 2gr_0' + 4ga_0a_1 + a_1$$

$$2gr_0a_0' = (1+4ga_0)r_1 + 4gr_0a_1$$

which when solved yield

$$r_1 = 0 \qquad\qquad a_1 = \frac{1}{1+6ga_0} \tag{3.32}$$

Expanding equation (3.14) to order ϵ^2 and using the above, one similarly obtains a pair of linear equations for a_2 and r_2 which when solved give

$$\begin{aligned} r_2 &= g^2a_0^2/\Delta\{5 + 30ga_0 + 48g^2a_0^2\} \\ a_2 &= -4g/\Delta\{g^2a_0^2(1+6ga_0) + (1+2ga_0)(1+3ga_0)(1+4ga_0)\} \\ \text{where} \qquad \Delta &= (1+2ga_0)^2(1+6ga_0)^4 \end{aligned} \tag{3.33}$$

The above equations may be readily expanded for small g and agree with the expansion (3.17). Expanding eqn.(3.9) as

$$E = \sum_{g=0}^{\infty} N^{-2g} E_g \tag{3.34}$$

and using the Euler-Maclaurin formula in the form

$$\begin{aligned} E = &-\int_0^1 dx(1-x)\ln r_\epsilon(x) - 1/N\{\ln h_0 - 1/2\ln r_\epsilon(0)\} - \frac{1}{12N^2}[(1-x)\ln r_\epsilon(x)]'\Big|_0^1 \\ &+ \frac{1}{6!N^4}[(1-x)\ln r_\epsilon(x)]'''\Big|_0^1 + O([(1-x)\ln r_\epsilon(x)]^{(6)}/N^6) \end{aligned} \tag{3.35}$$

we obtain

$$E_1 = -\int_0^1 dx(1-x)r_2/r_0 \ + 2g + 1/12\{\ln r_0 - (1-x)r_0'/r_0\}\Big|_0^1 .$$

Here (cf. eqn.(3.29)) we have not been so careful to subtract divergent numerical constants arising at $x = 0$: It is easy to take them into account at this stage. Expanding the solution a_0 of eqn.(3.28) for small x gives $a_0 = 2x - 12gx^2 + \cdots$ in accord with the perturbative result (3.17), thus

$$\begin{aligned} \ln r_0 &= \ln x + O(x) \\ r_0'/r_0 &= \frac{4}{a_0(1+6ga_0)} \\ &= 2/x - 1 - 12g + O(x) \end{aligned}$$

Using these results, the above result for r_2, and changing variables from x to a_0 one obtains

$$\begin{aligned} E_1 &= 1/6\ln A + g - g^2\int_0^A da\,\frac{(2-a-3ga^2)(5+30ga+48g^2a^2)}{(1+6ga)^3(1+2ga)^2} \\ &= \frac{1}{24}\ln\left(\frac{(4-A)(4+A)^3}{432}\right) \\ &= -g^2 + \frac{80}{3}g^3 - 614\,g^4 + \cdots \end{aligned} \tag{3.36}$$

As indicated below eqn.(3.9) we have subtracted numerical constants appropriately in eqn. (3.36). The first term in the expansion of E_1 agrees with the $1/N^2$ term in eqn.(3.18). E_1 is the generating function for the number of chequered quadrilations of the torus: the coefficient (ignoring the sign) of the g^n term in (3.36) multiplied by $n!$ is the number of ways of making a correctly chequered torus from n chequered squares. In view of our geometrical discussion in the previous section we know that letting $g \to 2g$ and dividing the torus result (3.36) by 2 gives the generating function for the number of quadrilations of a torus that can (in principle) be chequered: that is the number of ways of building torii out of n (colourless) squares, without regard to their relative orientations but such that the result could be consistently chequered. Thus this expression is

$$E_1^{chequer} = \frac{1}{48} \ln \left(\frac{(2-a^2)(2+a^2)^3}{27} \right) \tag{3.37}$$

where we have introduced $A = 2a^2$ so that

$$12ga^4 + a^2 - 1 = 0$$

to conform with the notation of ref.[6] (not to be confused with a above). Subtracting from equation (7.21) of that paper, which gives the number of quadrilations of the torus irrespective of questions of chequering, yields a generating function for the number of quadrilations of the torus that cannot be chequered:

$$\begin{aligned} E_1^{nochequer} &= 1/12 \ln(2-a^2) - E_1^{chequer} \\ &= \frac{1}{16} \ln \left(3\frac{2-a^2}{2+a^2} \right) \\ &= g - 28g^2 + \frac{2848}{3} g^3 - \dots \end{aligned} \tag{3.38}$$

The first term in this expansion corresponds to the single square (fig.3) mentioned in the introduction.

4. The Non-perturbative Physics.

To obtain the continuum limit of the matrix model we tune the couplings to a critical point. $\ln \gamma$, the cosmological constant, is tuned down from large values where perturbation theory in the couplings g_m/γ is valid and surfaces of large area (large number of polygons) are heavily suppressed (eqn.(2.7)) to a critical point at which the entropy of the surface exactly cancels the exponential suppression and the surface area begins to diverge. Equivalently, in physical units the polygons shrink to vanishing area.

The critical behaviour of the spherical contribution to the vacuum energy is the same as for hermitian matrix models since, as shown in the previous section, the contributions from each model are the same (up to simple rescalings) even off criticality. Beyond the sphere we expect the physics obtainable from chequered surfaces to be the same as that from hermitian matrices because the requirement of chequering is only a short distance detail (in physical units). We will show this here for the simplest case of quadrilations (only $g = g_2/\gamma \neq 0$). (A much more thorough investigation will be reported in ref.[8]). Our demonstration will also provide several non-trivial checks on the results of the previous section.

Given the equivalence of the spherical contribution in this model and the model of a single hermitian matrix[1], it is clear that a critical point for the sphere will also arise here from tuning γ to a point where $A = a_0(1)$ becomes non-analytic in the couplings. From equation (3.31) we see that this occurs at $g = -1/24$. At this point $a_0(x)$ and more generally $a(x)$ and $r(x)$ (e.g. from eqns.(3.23), (3.31)–(3.33)) will become non-analytic in x at $x = 1$.

Introduce a scaling parameter δ by defining

$$g = -\frac{1}{24}(1 - \mu\delta^2) \tag{4.1}$$

where μ is the renormalised cosmological constant, and $\delta \to 0$. Substituting the above into eqn. (3.31) one obtains

$$A = \frac{4}{1 + \delta\sqrt{\mu}} \; . \tag{4.2}$$

And substituting this into (3.30), yields the spherical contribution to E:

$$E_0 = \frac{8}{15}\mu^{5/2}\delta^5$$

which of course is the same behaviour as observed before in refs.[1]. (As in those refs. we have dropped the non-universal constant and linear term in μ). It is helpful to absorb the N^2 next to E in eqn.(3.9) by defining the effective action $\Gamma = N^2 E$:

It is this that must be finite in the continuum limit. Inspecting the above formula we see that the bare string coupling therefore scales as

$$\epsilon = 1/N = \nu\delta^{5/2} \tag{4.3}$$

where ν is the finite renormalised string coupling, and thus the spherical contribution to the effective action is

$$\Gamma_0 = \frac{8}{15\nu^2}\mu^{5/2}\ . \tag{4.4}$$

Now let us extract the critical torus behaviour from the previous section. Inspecting the expressions (3.36) and (3.31) allows us to check for any further critical points that might only show up at the torus level, and to obtain independently of the non-perturbative analysis the torus contribution at the $k = 2$ critical point. Thus from (3.31) we see that A is a 2-sheeted function of g if, for generality, we consider g to be complex. From (3.36) and the defining quadratic equation for A ((3.28) with $x = 1$) we learn in addition that E_1 has logarithmic singularities in A at $A = 4$ ($g = -1/24$ uniquely) and $A = -4$ ($g = 1/8$). From eqn. (3.31) we see that the latter point is found on A's second sheet. Thus E_1 is analytic in g and its perturbative expansion converges, for $|g| < 1/24$. E_1 diverges at the critical point $g = -1/24$ which corresponds to diverging surface area, or vanishing size for the dynamical triangulations in physical units. Substituting eqn. (4.2) one finds as $\delta \to 0$,

$$\Gamma_1 = E_1 = \frac{1}{48}\ln\mu + \text{const.} \tag{4.5}$$

(where the constant includes a term diverging as $\ln\delta$).

It is tempting to associate the non-analytic behaviour at $g = 1/8$ with some new phase for strings. This critical point does not exist for hermitian matrix models. It appears to have critical behaviour with $k = 1$ (i.e. critical index of string susceptibility $\gamma_0 = -1/k = -1$) because the sphere has no critical behaviour at this point. However the critical point is not accessible from perturbation theory in small g and therefore does not have a direct interpretation in terms of a continuum limit for dynamically triangulated surfaces. We will not consider it further in this paper.

Introduce the scaling ansatz:

$$x = \frac{1 - z\delta^2}{1 - \mu\delta^2} \tag{4.6}$$

The range of z is $\mu \leq z \leq 1/\delta^2$; the denominator factor is purely a convenience. At the spherical level one obtains from eqns. (3.28), (4.1) and (4.6):

$$\begin{aligned} a_0 &= 4(1 - \delta\sqrt{z}) \\ r_0 &= 4(1 - 2\delta\sqrt{z}) \end{aligned} \tag{4.7}$$

to lowest non-trivial order in the scaling parameter. Using this and eqn.(4.1) we can now extract the critical behaviour of r and a from the previous section. We find that (3.32) does not contribute to order δ and from (3.33) we obtain

$$\begin{aligned} a &= a_0 + \epsilon^2 a_2 \\ &= 4[1 - \delta(\sqrt{z} - \frac{\nu^2}{96z^2})] \end{aligned} \tag{4.8a}$$

$$\begin{aligned} r &= r_0 + \epsilon^2 r_2 \\ &= 4[1 - 2\delta(\sqrt{z} - \frac{\nu^2}{96z^2})] \end{aligned} \tag{4.8b}$$

This gives the behaviour of r and a to order ϵ^2 = order ν^2.

Note that the torus contributions (ν^2) scale in the same way as the spherical contributions in (4.8) as indeed they must if we are to derive contributions from all orders in genus. This observation justifies our scaling ansatz (4.6) since it is easy to see that this would not be the case for a different power of δ in (4.6). Thus in general we have scaling as

$$\begin{aligned} a(z) &= 4[1 - \delta\alpha(z)] \\ r(z) &= 4[1 - 2\delta\rho(z)] \ . \end{aligned} \tag{4.9}$$

One final preliminary is necessary before we derive a non-perturbative differential equation (string equation[1]) for ρ. From eqns. (4.1), (4.3), (4.6), (4.9) and (3.35) one finds, as in refs. [1], that only the integral in (3.35) contributes to Γ, and differentiating with respect to μ one derives

$$\frac{\partial^2\Gamma}{\partial\mu^2} = \frac{2}{\nu^2}\rho(\mu) \tag{4.10}$$

identifying ρ as the string susceptibility (with respect to the cosmological constant). Using eqns. (4.4), (4.5) and (4.8b) one verifies this relation to order ν^2. This is a non-trivial check on the analysis of the preceding section.

Writing equations (3.14) in continuum notation

$$\begin{aligned} 2x+\epsilon &= 2g(r(x+\epsilon)+r(x)+a^2(x))+a(x) \\ (x-2gr(x))^2 &= r(x)(1+2ga(x))(1+2ga(x-\epsilon)) \, , \end{aligned} \tag{4.11}$$

substituting (4.9), (4.1) and (4.3), and Taylor expanding $r(x+\epsilon)$ and $a(x-\epsilon)$ one obtains:

$$0 = 2\delta(\rho-\alpha)+\nu\delta^{3/2}\rho_z+\delta^2(\nu^2\rho_{zz}-2\alpha^2+3z)+\cdots \tag{4.12}$$

$$0 = 4\delta(\alpha-\rho)-2\nu\delta^{3/2}\alpha_z+\delta^2(\nu^2\alpha_{zz}+\alpha^2-8\alpha\rho-\rho^2+6z)+\cdots \tag{4.13}$$

where $\alpha \equiv \alpha(z)$ and similarly ρ. To order δ it is clear from these equations that $\rho = \alpha$ as indeed we had seen perturbatively in (4.8a, b). It is just as clear that setting $\alpha = \rho$ is not enough to solve the equations, in fact we need to determine their relative dependence as a power series. Solving (4.12) for α perturbatively (in $\delta^{1/2}$) we obtain

$$\alpha = \rho + \frac{\nu}{2}\delta^{1/2}\rho_z + \delta(\frac{\nu^2}{4}\rho_{zz} - \rho^2 + \frac{3}{2}z) + \cdots \tag{4.14}$$

and stuffing into (4.13) one finds that all terms of order less than δ^2 vanish and one obtains at order δ^2:

$$\frac{\nu^2}{12}\frac{d^2\rho}{dz^2} - \rho^2 + z = 0 \tag{4.15}$$

which under the trivial rescaling $\nu \to 2\nu$ is precisely the Painlevé I equation for 2D pure gravity found previously[1]. Solving (4.15) perturbatively in ν, one can compare with eqns.(4.9) and (4.8b) . One finds[2]

$$\rho = \sqrt{z}\sum_{g=0}^{\infty} P_g \left(\frac{\nu^2}{z^{5/2}}\right)^g$$

with $\qquad P_0^2 = 1 \qquad P_1 = -1/96$

and $\qquad P_{g+1}P_0 = \frac{1}{96}(25g^2-1)P_g - \frac{1}{2}\sum_{m=1}^{g} P_m P_{g+1-m} \, . \qquad (4.16)$

[2] This analysis is of course the same as that in refs.[1].

The first term in ρ from (4.8b) determines the sign for P_0 as $P_0 = 1$. The second term agrees with the expansion and acts as a non-trivial check on the analysis here and in sect. 3. Higher terms are readily obtained from the recurrence relation for P_g given above.

5. Discussion and Conclusions.

In this paper we have shown that complex matrices M can be used to generate dynamical polygonations. They differ from those generated by hermitian matrices only in the requirement that the triangles in a polygonation be chequered (as in fig.2). The chequering is a direct consequence of having both M and $M^\dagger$ in the perturbation theory and an action that respects the invariance

$$M \to V^\dagger M W \tag{5.1}$$

where V and W are unrelated unitary matrices. This invariance is a natural one to impose: in higher dimensional analogues of this zero-dimensional field theory it plays the rôle of a gauge invariance. Here it greatly reduces the dynamical degrees of freedom to just the eigenvalues of $M^\dagger M$, which is what allows the solution of the model. The chequered polygonations arise from associating white triangles with M and black triangles with $M^\dagger$.

This geometrical interpretation, in terms of chequered polygonations, is enough to show that the sphere contribution is related by simple rescalings to the sphere contribution obtained from the hermitian matrix model, even off criticality. No such simple relation exists for the higher genus surfaces, and indeed in sect. 3 we derive a generating function for the number of chequered quadrilations of a torus and contrast this with the generating function for the number of quadrilations irrespective of requirements of chequering.

Nevertheless intuitively one expects that when tuned to a critical point the continuum physics of the model will be insensitive to the short distance effects of chequering. We confirmed this, partially, in sect. 5. We will show much more rigorously in a later paper[8] that the physics of such triangulations is in the same universality class as the model of a single hermitian matrix[1]. Several points are worth noting however. Firstly, in section 3 we noted that one recovers precisely the

Painlevé I equation for 2D pure gravity found previously[1] if one rescales the string coupling $\nu \to 2\nu$. Recalling the formula (4.3) we see that the complex matrix model behaves in the continuum limit like a $2N \times 2N$ hermitian matrix model with the related potential.

Secondly, while it is true that the physics obtainable from chequered surfaces is the same as that from surfaces that are not chequered, it is not true that the requirement of chequering is irrelevant in the continuum limit: On a sphere we include all the polygonations (provided only that the polygons be even sided) because, as we showed in sect. 2, all the polygonations can be chequered. The torus however has critical behaviour both from quadrilations that can be chequered, and quadrilations that can not. Indeed, inspecting equations (3.37) and (3.38) and recalling that the continuum limit arises from $a^2 \to 2$, one sees that they contribute in the ratio 1:3. Roughly speaking, there are three times as many quadrilations of a torus that cannot be chequered as there are quadrilations that can be chequered in the continuum limit. Thus it is important to include only those polygonations that can be correctly chequered in order to obtain the correct contribution from the higher genus surfaces.

Thirdly, we noted in sect. 4 the existence of another critical point for the torus. This point exists beyond the radius of convergence for perturbation theory in the couplings and thus does not necessarily have an interpretation in terms of diverging area of triangulated surfaces. Nevertheless it is intriguing because it appears to have $k = 1$ ($\gamma_0 = -1$) critical behaviour. Certainly this and its multicritical analogues deserve further investigation.

Finally we note that the framework, in particular the triangulation interpretation, developed here for complex matrices applies little changed to several other generalisations of hermitian matrix models. Thus for example real non-symmetric matrices would have a natural "kinetic term" $M^T M$. (T stands for transpose). It is natural to impose the invariance $M \to V^T M W$ where V and W are unrelated orthogonal matrices, which then allows the solution of the model. From the propagator

$$\langle M_{\alpha\beta} M^T_{\gamma\delta} \rangle \sim \delta_{\gamma\beta}\delta_{\alpha\delta}$$

(cf. eqn(2.6)) and the fact that the invariance enforces the potential to be a trace of a polynomial of $M^T M$ we see that one can use the same interpretation as before for the propagator and vertices as in figures 4 and 6. Here we are associating an incoming arrow with M^T and an outgoing arrow with M or what is the same: black triangles with M^T and white with M. However because the above equation also implies:

$$\langle M_{\alpha\beta} M_{\delta\gamma} \rangle = \langle M^T_{\beta\alpha} M^T_{\gamma\delta} \rangle \sim \ \delta_{\gamma\beta}\delta_{\alpha\delta}$$

we see that it is possible to alter the direction of the arrow by twisting the propagator. In the triangulation interpretation this corresponds to changing the colour of the triangle while flipping over the triangle (plaquette) to display its back! Thus, as a little thought confirms, arbitrary real matrices generate dynamical chequered polygonations in which the polygons should be thought of as having two sides (front and back), and their constituent triangles are chequered so that in addition the back of the polygon carries the photographic negative colouration. Similarly more general versions of quaternionic matrices than quaternionic self-dual can be expected to show interpretations in terms of chequering. Intuitively one expects from this interpretation, as found with the complex matrix case, that the chequering is a short distance detail which will not alter the continuum physics seen in the case of unchequered triangulations. Thus we predict this model of arbritrary real matrices to be in same universality class reported in ref.[5].

Viewing eqn. (5.1) one notes that effectively, since V and W are unrelated, the indices on the matrix live in different spaces; there is no reason, from the point of view of perturbation theory, why these spaces need be the same dimension. Thus we have yet another generalisation to non-square matrices which we discuss for the complex matrix case, but may be readily given for real, or quaternionic. Once again we have the interpretations displayed by figures 4 and 6. We can represent the fact that the matrices indices live in different spaces by colouring the lines of the propagator say blue for the left line, looking along the direction of the arrow, and red for the right. In terms of the polygonation this corresponds to colouring the vertices of the polygon similarly. The contractions of the red and blue lines identify common vertices in the polygonation, thus we see that the polygonation must be built such that only vertices of the same colour meet. Of course this is no further restriction than that of chequering, since as we have already noted the indices of a square complex matrix also live in different spaces. Setting the dimension of the matrix to be $aN \times bN$, where a and b are independent[3]of N so that N still counts genus, we expect once again that the continuum physics obtainable from these polygonations is just that obtained previously[1].

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Appendix A. Diagonalization.

We first prove that any $N \times N$ complex matrix can be written as

$$\begin{aligned} M = VxW^{\dagger} \qquad & V, W \in U(N) \\ x = \mathrm{diag}(x_\alpha) \qquad & x_\alpha \text{ real and} \geq 0 \end{aligned} \tag{A.1}$$

We can always find a unitary matrix W such that the substitution

$$M \rightarrow MW^{\dagger} \tag{A.2}$$

diagonalizes $M^{\dagger}M$ and places any zero eigenvalues at the bottom of the diagonal ((A.4)):

$$\sum_{\gamma} \overline{M}_{\gamma\beta} M_{\gamma\alpha} = y_\alpha \delta_{\alpha\beta} \,. \tag{A.3}$$

When $\alpha = \beta$ the left hand side is a sum of non-negative terms, thus we have for some n:

$$\begin{aligned} y_\alpha \geq 0 \quad & \text{for } \alpha = 1, \cdots, n \\ y_\alpha = 0 \quad \text{and} \quad M_{\gamma\alpha} = 0 \quad & \text{for } \alpha = n+1, \cdots, N \quad (\forall \gamma) \,. \end{aligned} \tag{A.4}$$

Let $\{v(\alpha)\}$ be the set of vectors such that

$$v_\gamma(\alpha) = M_{\gamma\alpha}/x_\alpha \quad \text{for } \alpha = 1, \cdots, n \tag{A.5}$$

[3] Other possibilities have recently been considered by Anderson, Myers and Periwal[13].

where $x_\alpha = \sqrt{y_\alpha}$ $\alpha = 1, \cdots, N$. These vectors are orthonormal by (A.3). Define further $N - n$ orthonormal vectors $v(\alpha)$ $\alpha = n+1, \cdots, N$ to make an orthonormal basis, then $V_{\gamma\alpha} = v_\gamma(\alpha)$ is a unitary matrix and from (A.4) and (A.5):

$$M_{\gamma\alpha} = V_{\gamma\alpha} x_\alpha \quad \text{for } \gamma, \alpha = 1, \cdots, N$$

Now doing the inverse transformation to (A.2) proves the assertion $\square$.

The change of variables to y, V and W as in equation (3.2) is most simply performed by borrowing a standard trick from quantum gravity. (This is true for the hermitian model also). We write the infinitessimal variation of (A.1) as

$$V^\dagger \delta M W = \delta S x - x \delta T + \delta x \tag{A.6}$$

where $\delta S = V^\dagger \delta V$ and $\delta T = W^\dagger \delta W$, and then write an invariant integral on the tangent space to the fields:

$$\begin{aligned} \text{Const.} &= \int [d(\delta M)] \exp -\text{tr}\delta M^\dagger \delta M \\ &= \int [d(\delta S)][d(\delta T)] d^N(\delta x) J \exp \text{tr}\{(\delta S x - x\delta T + \delta x)(\delta S x - x \delta T + \delta x)\} \end{aligned} \tag{A.7}$$

where the measures for δM and the anti-hermitian δS, δT are just the flat measures (2.2) and (2.5) respectively. The measure for δS and δT is nothing but the tangent space version of the Haar measure for V and W . J is the jacobian

$$J = \left| \frac{\partial M}{\partial x}; \frac{\partial M}{\partial S}; \frac{\partial M}{\partial T} \right|$$

Making the linear substitution

$$\delta S_{\alpha\beta} \to \delta S_{\alpha\beta} + \frac{2\delta T_{\alpha\beta} x_\alpha x_\beta}{x_\alpha^2 + x_\beta^2}$$

and rearranging we have

$$\text{tr}\delta M^\dagger \delta M = \sum_{\alpha\beta} \{(x_\alpha^2 + x_\beta^2)\delta S_{\alpha\beta}\delta \overline{S}_{\alpha\beta} + \delta T_{\alpha\beta}\delta \overline{T}_{\alpha\beta} \frac{(x_\alpha^2 - x_\beta^2)^2}{(x_\alpha^2 + x_\beta^2)}\} + \sum_\alpha \delta x_\alpha^2$$

Thus performing (A.7) we obtain

$$\text{Const.} = J \prod_\alpha \frac{1}{x_\alpha} \prod_{\alpha<\beta} \frac{1}{(x_\alpha^2 - x_\beta^2)^2}$$

which can be rearranged to give J and hence, by changing variables from x to y, the measure (3.2).

It is easy to compute the overall multiplicative constant involved in the change of variables either by taking some more care in the above analysis or by computing Z_0 (eqn.(2.1)) for each set of coordinates .

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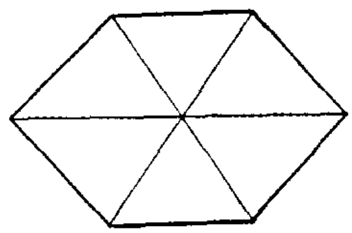

Fig. 1: A polygon divided into triangles. The triangles meet at its centre

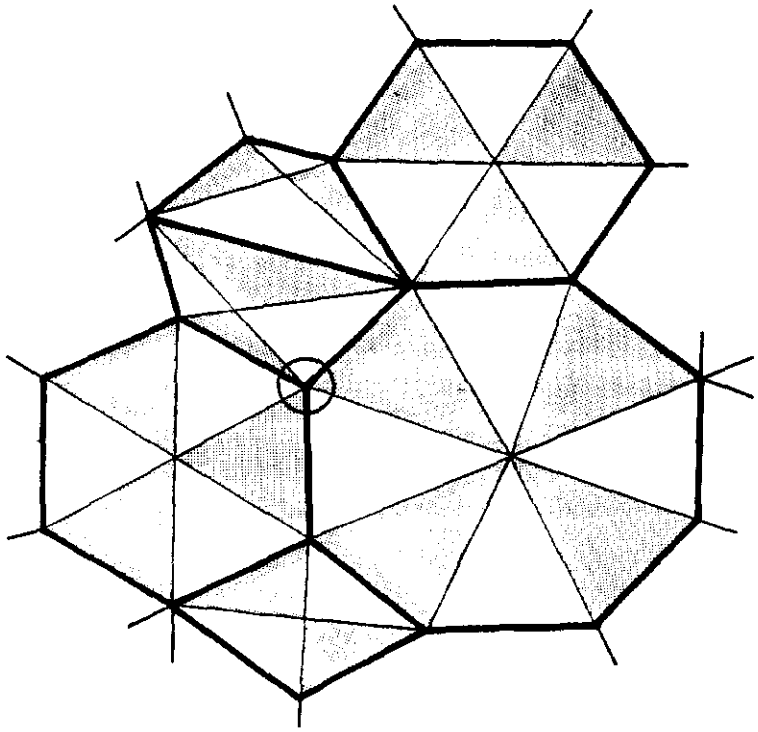

Fig. 2: A portion of a chequered polygonation. A common vertex is circled

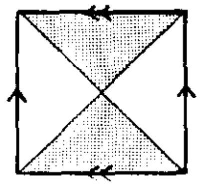

Fig. 3: A torus quadrilation that cannot be chequered. The attempt puts black next to black and white next to white when opposite sides are sewn together

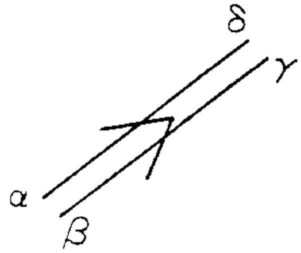

Fig. 4: The propagator $\langle M_{\alpha\beta} M^{\dagger}_{\gamma\delta} \rangle$

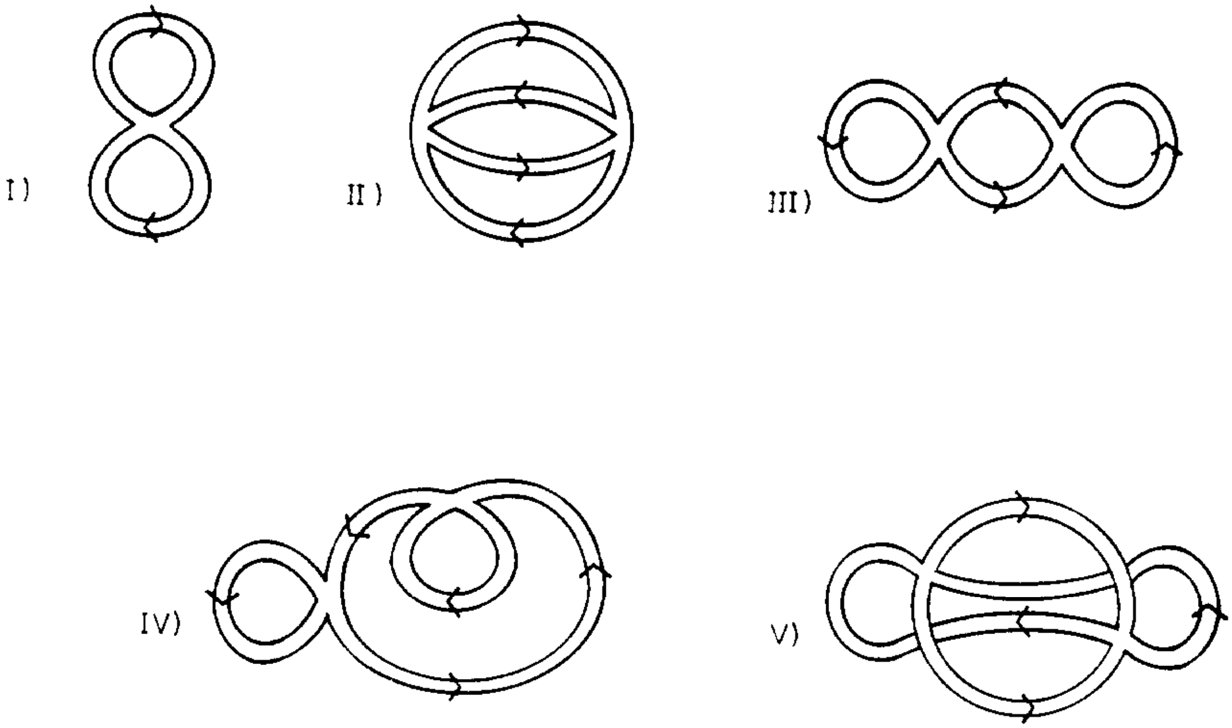

Fig. 5: Feynman diagrams to second order in the quartic interaction

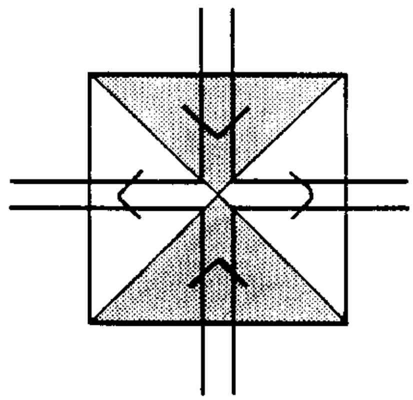

Fig. 6: A quartic interaction with the quadrilateral superimposed. The incoming propagators are associated with black triangles. Outgoing propagators with white

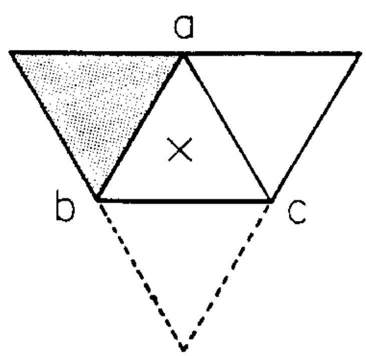

Fig. 7: The triangle marked with a × cannot be chequered because the triangle on the left is already coloured black, and the triangle on the right white.