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Scalar Fields within Warped Extra Dimension

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ABSTRACT

In this thesis, we explored three different implications of scalar fields in warped extra dimension.

- First, scalar fields were employed to dynamically generate singular branes in Randall-Sundrum (RS)-like models by appropriate profiles — the smooth/thick-branes. In the context of thick-branes, we constructed four different setups: (*i*) a smooth generalization of RS2 where a scalar field dynamically generates a singular brane allowing symmetric or asymmetric warped geometries on either side of the brane; (*ii*) a double thick-brane scenario which mimics two positive tension branes and allows to address the hierarchy problem; (*iii*) a $\mathbb{Z}_2$ symmetric triple thick-brane; and (*iv*) a dilatonic thick-brane scenario. The stability of background solution is verified in all the above mentioned setups.
- Second, we considered a thick-brane cosmological model with warped fifth-dimension where dynamics of the 4D universe is driven by time-dependent five-dimensional (5D) background. Different scenarios were found for which the cosmic scale factor $a(t, y)$ and the scalar field $\phi(t, y)$ depend non-trivially on time t and fifth-dimension y.
- Third, we discussed a symmetric 5D model with three D3-branes (IR–UV–IR) where the Higgs doublet and the other Standard Model (SM) fields are embedded in the bulk. The $\mathbb{Z}_2$ geometric symmetry led to the warped KK-parity for all the bulk fields. Within this setup we investigated the low-energy effective theory for the bulk SM bosonic sector. It turned out that the zero-mode scalar sector contains an even scalar which mimics the SM Higgs boson and a second, stable odd scalar particle which is a dark matter candidate. The model that resulted from the $\mathbb{Z}_2$-symmetric background geometry resembles the Inert Two Higgs Doublet Model. Implications for dark matter were discussed within this model.

To the memory of my grandfather
Khuda-Bakhsh

CONTENTS

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Aqeel Ahmed August 2015, Warsaw

"Scientific thought and its creation are the common and shared heritage of mankind."

— Abdus Salam, *Nobel Laureate 1979*

PREFACE

The work presented in this dissertation is based on the following publications [1, 2, 3, 4, 5, 6, 7]:

1. *"Brane modeling in warped extra dimension",*
   Aqeel Ahmed and Bohdan Grzadkowski,
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2. *"Modeling branes in warped extra dimension",*
   Aqeel Ahmed, Lukasz Dulny and Bohdan Grzadkowski,
   Acta Phys.Polon. **B44** (2013) no. 11, 2381.
3. *"Generalized Randall-Sundrum model with a single thick-brane",*
   Aqeel Ahmed, Lukasz Dulny and Bohdan Grzadkowski,
   Eur. Phys. J. C **74** (2014) 2862 [arXiv:1312.3577].
4. *"Thick-Brane Cosmology",*
   Aqeel Ahmed, Bohdan Grzadkowski and Jose Wudka,
   JHEP **1404** (2014) 061, [arXiv:1312.3576].
5. *"Higgs dark matter from a warped extra dimension – the truncated-inert-doublet model",*
   Aqeel Ahmed, Bohdan Grzadkowski, John F. Gunion and Yun Jiang,
   JHEP **1510** (2015) 033, [arXiv:1504.03706].
6. *"Radius stabilization and dark matter with a bulk Higgs in warped extra dimension",*
   Aqeel Ahmed, Bohdan Grzadkowski, John F. Gunion and Yun Jiang,
   Acta Phys.Polon. **B46** (2015) no. 11, *in press*, [arXiv:1510.04116].
7. *"Higgs dark matter from a warped extra dimension",*
   Aqeel Ahmed, Bohdan Grzadkowski, John F. Gunion and Yun Jiang,
   PoS PLANCK **2015** (2015) 002, [arXiv:1510.05722].

CHAPTER 1

INTRODUCTION

The Standard Model (SM) of particle physics has been the most successful theory of elementary particles and their interactions, except gravity. Its predictions have been tested and verified in many experiments over the last few decades. The last missing piece of the SM was the Higgs boson, which has been found at the Large Hadron Collider (LHC) by ATLAS [8] and CMS [9] collaborations in 2012. The properties of the newly found particle – a Higgs boson – are very similar to that of the SM predictions, hence verifying the SM as the most accurate theory of elementary particles.

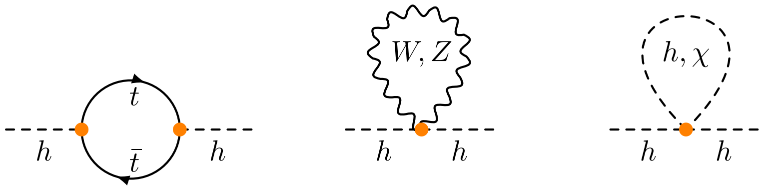
Apart from the enormous success of the SM, it is commonly believed that the SM is not the complete theory but is an effective theory of a more complete theory, since there are many unanswered puzzles that cannot be answered within the framework of the SM. Some of the puzzles are pure theoretical in nature, motivated by naturalness, e.g. gauge hierarchy problem, fermion mass hierarchy problem etc. and the others are observational in nature, e.g. dark matter, dark energy etc. There are many models beyond the SM which try to answer some of these questions, among them the most celebrated are the *supersymmetry* and *extra dimensions* [1]. In this dissertation we take the road through extra dimensions and see how one can answer some of these puzzles.

The simplest way to phrase the gauge hierarchy problem is to ask: why gravity is much weaker than the other fundamental forces of nature? In other words, why the gravity mass scale (Planck scale) is much bigger than the electroweak mass scale, i.e. $M_{\rm Pl} \simeq 10^{19}$ GeV $\gg m_{EW} \simeq 10^3$ GeV? The huge hierarchy between the gravity and electroweak mass scales is "bad" because it prevents fundamental scalars (like SM Higgs) to have small mass [2]. The reason lies in the fact that scalars get large contributions (proportional to the cutoff scale Λ) to their masses due to quantum corrections. For example, the main contribution to the

[1] For reviews on supersymmetry, see for example [10, 11] and for extra dimensions, see [12, 13, 14, 15, 16, 17, 18].

[2] The gauge bosons and the fermions masses are protected by the gauge and chiral symmetries, respectively, but there is no symmetry present in the SM which insures light scalars.

quantum corrections to the Higgs boson (h) mass, within the SM at the 1-loop level of the perturbative expansion, comes from the exchanges of the top quark (t), massive gauge bosons (W, Z) and Higgs boson (h), as is shown in the following Feynman diagrams:



The quantum corrections within the SM to the tree-level Higgs mass squared are:

$$\delta^{SM} m_h^2 = \frac{3\Lambda^2}{8\pi^2 v^2}\Big[4m_t^2 - 2m_W^2 - m_Z^2 - m_h^2\Big],$$

where v is vacuum expectation value (vev) of the SM, m_i are the masses for $i = t, W, Z, h$ (top quark, W, Z and Higgs bosons), and Λ is the cutoff scale. If the SM is valid up to the Planck scale then the cutoff scale $\Lambda \sim M_{\text{Pl}}$. The above equation implies that the Higgs mass should be of the order of Planck mass in contradiction with the observed Higgs mass $m_h \simeq 125$ GeV. This is only possible if there is large fine-tuning (cancellation) between the tree-level Higgs mass and the quantum corrections. To avoid this fine-tuning one must assume that there is some new physics beyond the SM (BSM) which gives quantum contributions to the Higgs mass such that it cancels the large quadratically divergent part of the SM. One of the example of such a BSM theory is supersymmetry, see for reviews [10, 11].

Arkani-Hamed, Dimopoulos and Dvali [19, 20, 21] proposed an alternative to supersymmetry for a solution to the hierarchy problem where compact flat extra dimensions were considered to show that the fundamental gravitational mass scale can be of the order of a few TeV with large volume of extra dimensions. Hence in such theories with large extra dimensions the hierarchy problem is alleviated by the virtue of large extra dimensions. In the models with flat extra dimensions the fine-tuning problem is not fully solved as these models require "large volume" of the extra dimensions which is another fine-tuning problem by itself. This pathology of the flat extra dimensions were resolved in the seminal work of Randall and Sundrum (RS) [22] which provided an elegant solution to the hierarchy problem without the requirement of large volume of extra dimension. Their proposal involves one extra dimension with a non-trivial warp factor due to the assumed anti-de Sitter (AdS) geometry along the extra dimension. Moreover, their model involves two D3-branes localized at the fixed points of an orbifold $S_1/\mathbb{Z}_2$; a "UV-brane" at $y = 0$ and an "IR-brane" at $y = L$. The solution for the RS geometry is [22, 23],

$$ds^2 = e^{-2k|y|}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2, \tag{1.1}$$

where k is the curvature of the AdS space. In RS1 model [22] it was assumed that the Standard Model (SM) is localized on the IR-brane, whereas the gravity is localized on the UV-brane and propagates through the bulk to the IR-brane. They famously showed that if the 5D fundamental theory involves only one mass scale M_* – the Planck mass in 5D – then, due to the presence of non-trivial warping along the extra dimension, the effective mass scale on the IR-brane is rescaled to $m_{KK} \sim ke^{-kL} \sim \mathcal{O}(\text{ TeV})$ and hence ameliorates the hierarchy problem for mild values of $kL \sim \mathcal{O}(35)$ with $k \simeq M_*$.

Usually the models with extra dimensions are required to be compactified as the gravity at large distances is almost perfectly described by 4D general relativity. It has been pointed out by Randall and Sundrum in their second seminal paper [23] that the extra dimension can be "infinite" and yet it can lead to nearly standard 4D gravity. The main idea in this second paper (RS2) is that a single D3 brane of *positive tension* is embedded in a 5D AdS geometry. They showed that the 4D graviton is localized on the brane and the low energy effective gravity is nearly 4D general relativity, hence RS2 is an alternative to compactification. However, the RS2 model having just one D3 brane is not addressing the hierarchy problem.

In spite of a great success of brane-world models in explaining some of the SM puzzles, a common pathology associated with these models is the presence of singular (Dirac delta) branes without any dynamical mechanism of generating them. One of the main goal of this dissertation is to provide smooth generalizations of RS-like models where the singular branes are dynamically generated. Some of the aspects of this issue were addressed and it was shown in Refs. [24, 25, 26], that a kink-like background scalar field configuration – the *smooth/thick* brane – could mimic a singular brane. Moreover, it was shown by [25, 26] that only a positive tension brane can be mimicked by a real scalar field configuration and hence the RS2 can be *smoothed* by a scalar field configuration. On the other hand RS1 cannot be smoothed by a scalar field configuration because of mainly two reasons:

- The negative tension brane is not possible to mimic with a real scalar field minimally coupled to gravity [25].
- Periodic solutions like RS1 are impossible to achieve with a smooth non-trivial scalar profile [27].

It is important to note that the above two conclusions are made for the case when the scalar field is minimally coupled to gravity, the generalization is explored in part of this dissertation with a non-minimal coupled scalar-gravity scenario. In this thesis we employ the scalar field with different profiles to dynamically generate different brane-world scenarios and provide a general prescription of modeling branes within warped extra dimensions. Another important question related to the thick-branes models is the 5D cosmological evolution of such models. In this dissertation we also explore this question, i.e. the 5D cosmological evolution of the thick-brane models, where we allow the 5D background to depend not only on fifth-dimension but also on time and analyze the 4D cosmological evolution.

Another outstanding puzzle of the SM is the lack of a candidate for dark matter (DM) which constitutes 83% of the observed matter density in the universe [28]. The most popular models of DM assume that the DM interaction strength is of the same order as that of the electroweak, hence known as weakly interacting massive particle (WIMP), see for reviews [29, 30, 31, 32].

In this dissertation we will consider a DM model involving an extra dimension in particular a warped extra dimension. Since the extra dimensional models have Kaluza-Klein (KK) modes corresponding to the bulk fields therefore it is natural to ask if the lightest stable KK-particle can be a candidate for DM.

In what follows we explore three different roles of scalar fields in warped extra dimensions:

- First, we consider smooth generalization of RS models with scalar fields. In the context of thick-branes, we construct a setup which can potentially solve the *hierarchy problem* due to non-trivial warping along the extra dimension, similar to that of RS1 model. Generalization of RS2 is also considered within thick-brane scenarios and it is shown that any scalar field profile can mimic the singular branes provided couple of mild properties are met.
- Second, a cosmological model is presented in the context of thick-branes in 5D warped extra dimension where dynamics of the 4D universe is driven by time-dependent 5D background with the bulk scalar field. Different scenarios are found for which the cosmic scale factor $a(t, y)$ and the scalar field $\phi(t, y)$ depend non-trivially on time t and fifth-dimension y.
- Third, we consider a bulk scalar (Higgs) field in a $\mathbb{Z}_2$ symmetric warped geometric setup which allows the *even* and *odd* KK-modes. We explore implications of the bulk Higgs field in the $\mathbb{Z}_2$ symmetric warped geometry. In the zero-mode effective theory the even zero-mode Higgs mimics the SM Higgs boson while the lowest odd KK-mode of the bulk Higgs field – the dark Higgs – is stable and hence can serve as a candidate for dark matter.

1.1. Structure of the dissertation

Chapter 2 of this thesis contains a brief review of RS models and their generalizations with singular branes. We consider two generalizations of RS-models with singular branes:

1. A $\mathbb{Z}_2$-symmetric generalization of RS1, where we extend the RS1-like warped geometry in such a way that the whole geometric setup becomes symmetric around a fixed point in the bulk. Two $\mathbb{Z}_2$ symmetric warped configurations are possible. In the first, two identical AdS patches are symmetrically glued together at a UV fixed point, while in the second, two identical AdS patches are symmetrically glued together at an IR fixed point. Our focus is on the geometric configuration when the two AdS copies are glued together at the UV fixed point, referred as "IR-UV-IR geometry".
2. Generalization of RS2 allowing different AdS geometries on either sides of the UV-brane, hence resulting to an asymmetric geometric setup.

The smooth generalizations of RS models with scalar field configurations are discussed in Chap. 3. We consider a thick-brane generalization of the RS2 with a single thick-brane such that a scalar field configuration can give a class of RS2-like models in a certain limit where the bulk-cosmological constants on each side of the brane can have different values. We present a thick-brane model which can address the hierarchy problem, like RS1 but it does not require compactification, like RS2. We employ a scalar field configuration which has a double kink-like profile and it mimics two positive tension branes. The distance between the two thick-branes is adjusted such that if the gravity is localized on one of the thick-branes and the Higgs field is localized on the other then the hierarchy problem can be addressed. We also consider a $\mathbb{Z}_2$ symmetric triple thick-brane model which mimics IR-UV-IR setup in the brane-limit, where all the branes have positive tension. The stability of the thick-brane background solutions is also the addressed in this chapter.

The cosmological implications of a thick-brane model are the subject of Chap. 4. We consider a 4D conformal time τ- and fifth-dimension y-dependent scale-factor $a(\tau, y)$ and the bulk scalar field $\phi(\tau, y)$ which constitutes the thick-brane. We discuss different scenarios where the cosmological evolution of the 5D geometric setup leads to different 4D cosmological solutions. Analytic and numerical analyses are presented for different scenarios for the thick-brane cosmology.

In Chap. 5 we place all the SM fields, including the Higgs doublet, in the bulk of the IR-UV-IR geometry. The geometric $\mathbb{Z}_2$ geometric symmetry ($y \to -y$ symmetry) leads to "warped KK-parity", i.e. there are towers of even and odd KK-modes corresponding to each bulk field. We focus on electroweak symmetry breaking (EWSB) induced by the bulk Higgs doublet and low energy aspects of the 4D effective theory for the even and odd zero-modes assuming the KK-mass scale is high enough $\sim \mathcal{O}(\text{few})$ TeV. In the zero-mode effective theory the even and odd Higgs doublets mimic a two-Higgs-doublet model (2HDM) scenario with the odd doublet similar to the inert doublet but without corresponding pseudoscalar and charged scalars — the "truncated" inert-doublet model. All the parameters of this *truncated* 2HDM are determined by the fundamental 5D parameters of the theory and the choice of boundary conditions for the fields at boundary branes. The symmetric setup yields an odd Higgs zero-mode that is a natural candidate for dark matter. We compute the one-loop quadratic (in cutoff) corrections to the two scalar zero modes within the effective theory and discuss their mass splitting. The dark matter candidate is a WIMP — we calculate its relic abundance in the cold dark matter paradigm.

Chapter 6 comprises the summary and conclusions. Moreover, we supplement this thesis with two Appendices:

- The linearized Einstein equations corresponding to a general 5D scalar-gravity warped geometries are presented in Appendix A, where we give a general treatment to the linearized scalar, vector and tensor perturbations. The results obtain in this Appendix are generic and are used in the main text especially to address the issue of localization of gravity and stability of the background solutions.
- In Appendix B we discuss spontaneous symmetry breaking (SSB) of a discrete symmetry with a real scalar in the bulk of our geometric setup. Many of the results obtained in this Appendix, especially the wave-functions of scalar fields in AdS geometries and the mass splitting between different KK-modes, are generic and are used in Chap. 5.

1.2. Conventions and notations

In this dissertation, we use the mostly plus metric signature, i.e. $\mathrm{diag}\big(-,+,+,+,+\big)$. In our conventions the capital roman indices represent five-dimensional (5D) objects, i.e. $M, N, \cdots = 0,1,2,3,5$, the Greek indices label four-dimensional (4D) objects, i.e. $\mu,\nu,\cdots = 0,1,2,3$, and the lowercase Roman indices $i,j,\cdots = 1,2,3$ represent the 3D spatial coordinates. In our notation a *prime* denotes a derivative w.r.t. the 5th coordinate y and an *overdot* will represent the derivative w.r.t. time t (or conformal time τ), unless otherwise said.

We use the following most general 4D Poincaré invariant metric ansatz throughout this thesis, unless otherwise stated:

$$ds^2 = e^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2, \tag{1.2}$$

where $A(y)$ is a general y-dependent warp-function and in different models it will have different forms, hence resulting in different geometries. Here and afterwards $\eta_{\mu\nu}$ will represent the 4D Minkowski metric. For the 5D metric we will use g_{MN}, which can be read from Eq. (1.2), i.e.

$$g_{MN} = \begin{pmatrix} e^{2A(y)}\eta_{\mu\nu} & 0 \\ 0 & 1 \end{pmatrix}, \tag{1.3}$$

and the 4D warped metric will be $g_{\mu\nu} \equiv \hat{g}_{\mu\nu} = e^{2A(y)}\eta_{\mu\nu}$. We use g as the determinant of the 5D metric g_{MN} and $\hat{g}$ as the determinant of the 4D metric $\hat{g}_{\mu\nu}$. The inverse metric g^{MN} is defined through: $g_{MA}g^{AN} = \delta^N_M$. The definition of a 5D covariant derivative acting on the contravariant and covariant vectors are $\nabla_M V^N = \partial_M V^N + \Gamma^N_{MA}V^A$ and $\nabla_M V_N = \partial_M V_N - \Gamma^A_{MN}V_A$, respectively. The 5D d'Alambertian operator ∇^2 is defined as,

$$\nabla^2 \equiv \nabla_M\nabla^M = \frac{1}{\sqrt{-g}}\partial_M\sqrt{-g}g^{MN}\partial_N. \tag{1.4}$$

The mass dimensionality of different objects in the 5D theory are as follows:

$$\text{5D Ricci scalar: } \big[R\big] = 2, \qquad \text{5D scalar field: } \big[\Phi\big] = \frac{3}{2}, \tag{1.5}$$

$$\text{5D fermion field: } \big[\Psi\big] = 2, \qquad \text{5D vector field: } \big[V_M\big] = \frac{3}{2}. \tag{1.6}$$

Due to the presence of singular branes in the warped extra dimensions there are discontinuities in the values of certain functions at the location of singular branes. We define a *discontinuity* or *jump* of a general function $f(y)$ across a singular brane located at y_α as:

$$\big[f(y)\big]_{y_\alpha} \equiv \lim_{\epsilon\to 0}\Big[f(y_\alpha+\epsilon) - f(y_\alpha-\epsilon)\Big]. \tag{1.7}$$

It is also useful to define an *average value* of a function $g(y)$ across the brane at y_α as:

$$\big\{g(y)\big\}_{y_\alpha} \equiv \lim_{\epsilon\to 0}\Big[\frac{g(y_\alpha+\epsilon) + g(y_\alpha-\epsilon)}{2}\Big]. \tag{1.8}$$

CHAPTER 2

RS MODELS AND THEIR GENERALIZATIONS

In this chapter we provide a brief review of RS models and their generalizations. The chapter is organized as follow: In Sec. 2.1 we review background solutions of RS1 – a model with two D3-branes compactified on $S_1/\mathbb{Z}_2$ orbifold – and show how it can potentially solve the hierarchy problem. We also briefly review RS2: a single brane model with non-compact extra dimension. Section 2.2 contains background solutions of a $\mathbb{Z}_2$ symmetric generalization of RS1 where three D3-branes are employed. An asymmetric generalization of RS2 is considered in Sec. 2.3. Section 2.4 is dedicated to address the issue of localization of gravity in RS2-like models with a non-compact extra dimension.

2.1. RS models: a brief review

The idea of extra dimensions offers a possibility of explaining the hierarchy between the Planck and the electroweak scales, therefore it has received a lot of attention during last decade or so, for reviews see e.g. [12, 13, 14, 15, 16, 17, 18]. Randall and Sundrum proposed a very elegant model (RS1) to solve the hierarchy problem [22] and also an attractive alternative (RS2) for a compactification of the extra dimension [23]. Below we briefly review the two models.

2.1.1. RS1: a solution to the hierarchy problem

Randall-Sundrum model-I (RS1) [22] employs an AdS geometry on an $S_1/\mathbb{Z}_2$ orbifold with two D3-branes localized at the fixed points of the orbifold, a "UV-brane" at $y = 0$ and an "IR-brane" at $y = L$, where y is the coordinate of the fifth-dimension and $L = \pi r_c$, with r_c being the radius of the circle in the fifth-dimension, see Fig. 2.1. The action for RS1 model can be written as,

$$S_{RS1} = \int d^5x\sqrt{-g}\Big\{2M_*^3 R - \Lambda_B - \lambda_{UV}\delta(y) - \lambda_{IR}\delta(y-L)\Big\}, \tag{2.1}$$

where R is the 5D Ricci scalar, M_* is the 5D Planck mass, Λ_B is the bulk cosmological constant and $\lambda_{UV}(\lambda_{IR})$ are the brane tensions at the UV(IR) fixed points. The metric ansatz considered

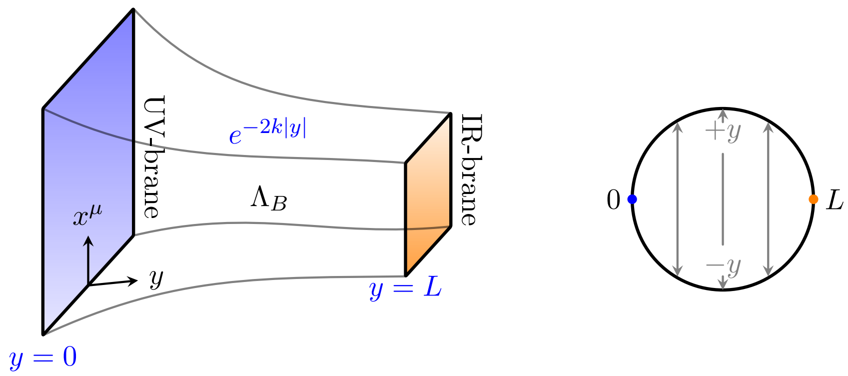


Figure 2.1.: Cartoon of RS1 geometry.

by Randall and Sundrum [22, 23] has the following form (1.2), i.e.,

$$ds^2 = e^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2, \tag{2.2}$$

where $e^{2A(y)}$ is so-called warped factor multiplied by the 4D part of the metric. The metric ansatz Eq. (2.2) has the non-factorizable form and is the most general 5D metric which preserves the 4D Poincaré invariance.

The Einstein equation resulting from the action (2.1) is

$$R_{MN} - \frac{1}{2}g_{MN}R = \frac{1}{4M_*^3}T_{MN}^{RS1}, \tag{2.3}$$

where R_{MN} is the 5D Ricci tensor and T_{MN}^{RS1} is the energy-momentum tensor corresponding to the RS1 setup:

$$T_{MN}^{RS1} = -\left[\Lambda_B g_{MN} + \frac{\sqrt{-g_{UV}}}{\sqrt{-g}}\lambda_{UV}g_{\mu\nu}^{UV}\delta_M^\mu\delta_N^\nu\delta(y) + \frac{\sqrt{-g_{IR}}}{\sqrt{-g}}\lambda_{IR}g_{\mu\nu}^{IR}\delta_M^\mu\delta_N^\nu\delta(y-L)\right], \tag{2.4}$$

where $g_{UV(IR)}$ is the determinant of the 4D induced metric $g_{\mu\nu}^{UV(IR)}$ on the UV (IR) brane. Following from the above Einstein equation (2.3) with the metric ansatz (2.2), one gets the $\mu\nu$ and 55 components as:

$$6A'^2 = -\frac{\Lambda_B}{4M_*^3}, \tag{2.5}$$

$$3A'' + 6A'^2 = -\frac{1}{4M_*^3}\big(\Lambda_B + \lambda_{UV}\delta(y) + \lambda_{IR}\delta(y-L)\big). \tag{2.6}$$

Equation (2.5) gives the following solution, consistent with the orbifold symmetry $S_1/\mathbb{Z}_2$,

$$A(y) = -k|y|, \qquad \text{where} \qquad k \equiv \sqrt{\frac{-\Lambda_B}{24M_*^3}}, \tag{2.7}$$

which implies that the bulk cosmological constant is negative, *i.e.* $\Lambda_B < 0$. As the RS geometry is periodic in y, therefore, the second derivative of warp function $A(y)$ that results from Eq. (2.7) as,

$$A'' = -2k\big(\delta(y) - \delta(y-L)\big). \tag{2.8}$$

Now it is straight forward to see that the Einstein equation (2.6) can only be satisfied if the following relation hold:

$$\lambda_{UV} = -\lambda_{IR} = 24 M_*^3 k. \tag{2.9}$$

A priori Λ_B, λ_{UV} and λ_{IR} are independent parameters but they all are interrelated through the above relations. These relations are necessary for the 4D Poincaré invariance. This is the so-called fine-tuning in the RS models.

With the above RS solution we can rewrite the metric (2.2) as

$$ds^2 = e^{-2k|y|}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2. \tag{2.10}$$

The solution to RS1 geometry Eq. (2.10) is a slice of Anti-de Sitter (AdS), where k is the inverse of the AdS radius. One can obtain the effective 4D theory of gravity by integrating-out the extra dimension. To calculate the 4D gravitational coupling we just focus on the curvature term:

$$\begin{aligned} S_{GR} &\supset 4M_*^3 \int d^4x \int_0^L dy e^{-2k|y|}\sqrt{-\hat{g}}\hat{R}, \\ &= 2M_{\rm Pl}^2 \int d^4x \sqrt{-\hat{g}}\hat{R}, \end{aligned} \tag{2.11}$$

where $\hat{g}$ and $\hat{R}$ are the determinant and the Ricci scalar corresponding to the 4D metric $\hat{g}_{\mu\nu}(x)$, respectively. Above $M_{\rm Pl}$ is the 4D Planck mass, given as

$$M_{\rm Pl}^2 = \frac{M_*^3}{k}\big[1 - e^{-2kL}\big]. \tag{2.12}$$

The above result implies that the 4D Planck mass depends weakly on the size of extra dimension L, i.e. one can get finite 4D effective theory of gravity with very large extra dimension, which is the main result of RS2 as will be described in the next section.

Below we show how RS1 gives a possible solution to the hierarchy problem. It is important to note that RS1 assumes the SM is localized on the IR-brane, whereas gravity is localized on the UV-brane and propagates through the bulk to the IR-brane. To understand how the hierarchy problem is addressed in RS1, let us consider the action for the SM and in particular for the Higgs field localized at the IR-brane with the following action:

$$S_{\rm Higgs} = -\int d^4x \sqrt{-g_{IR}}\left\{ g_{IR}^{\mu\nu}\partial_\mu H^\dagger \partial_\nu H - m^2|H|^2 + \lambda|H|^4 \right\}, \tag{2.13}$$

where g_{IR} is the determinant of the 4D induced metric on the IR-brane $g^{IR}_{\mu\nu} = e^{-2kL}\eta_{\mu\nu}$, with $L = \pi r_c$ and r_c being the radius of compactification. Above m is the Higgs mass parameter in the 5D theory. The above brane localized action for the Higgs field can be written as:

$$S_{\rm Higgs} = -\int d^4x \left\{ e^{-2kL}\eta^{\mu\nu}\partial_\mu H^\dagger \partial_\nu H - m^2 e^{-4kL}|H|^2 + \lambda e^{-4kL}|H|^4 \right\}. \tag{2.14}$$

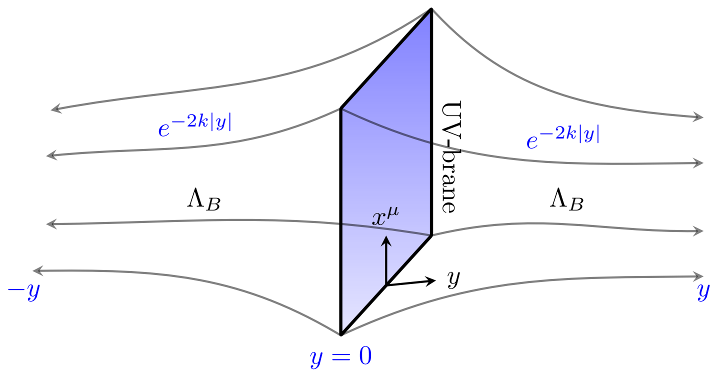


Figure 2.2.: The geometric configuration of RS2.

In order to obtain a canonically normalized Higgs field, we rescale, $H \to e^{kL}H$, such that,

$$S_{\text{Higgs}} = -\int d^4x \left\{\eta^{\mu\nu}\partial_\mu H^\dagger \partial_\nu H - m^2 e^{-2kL}|H|^2 + \lambda |H|^4\right\},$$
$$= -\int d^4x \left\{\eta^{\mu\nu}\partial_\mu H^\dagger \partial_\nu H - \mu^2|H|^2 + \lambda |H|^4\right\}, \tag{2.15}$$

where $\mu \equiv m e^{-kL}$ is the effective Higgs mass parameter at the IR-brane. If we assume that the 5D fundamental theory involves only one mass scale $M_* \simeq \mathcal{O}(10^{19})$ GeV – the Planck mass in 5D – then, due to the presence of non-trivial warping along the extra dimension, the effective mass scale on the IR-brane $\mu = me^{-kL} \simeq \mathcal{O}(10^3)$ GeV for mild values of $kL \sim \mathcal{O}(35)$. This is the geometric solution to the hierarchy problem due to Randall and Sundrum [22].

2.1.2. RS2: an alternative to compactification

Randall-Sundrum model-2 (RS2) [23] is a special case of the RS1 where the IR-brane is moved to infinity ($r_c \to \infty$ in RS1), i.e. it is no more present in the set up and the fifth-dimension y is infinite. In other words RS2 is a non-compact 5D AdS $\mathbb{Z}_2$ symmetric geometry with single D3-branes localized at the fixed point of the $\mathbb{Z}_2$ at $y = 0$, see Fig. 2.2. In the RS2 not only the gravity is localized on the brane at $y = 0$ as in the RS1 but also it is assumed that the SM is localized on this brane. The gravitational action for RS2 model can be written as,

$$S_{RS2} = \int d^5x \sqrt{-g}\Big\{2M_*^3 R - \Lambda_B - \lambda\delta(y)\Big\}, \tag{2.16}$$

and metric ansatz is the same as in RS1 Eq. (2.2).

The Einstein equation resulting from the action (2.16) is

$$R_{MN} - \frac{1}{2}g_{MN}R = \frac{1}{4M_*^3}T^{RS2}_{MN}, \tag{2.17}$$

where T^{RS2}_{MN} is the energy-momentum tensor corresponding to the RS2 setup:

$$T^{RS2}_{MN} = -\Big[\Lambda_B g_{MN} + g_{\mu\nu}\delta^\mu_M \delta^\nu_N \lambda\delta(y)\Big], \tag{2.18}$$

where the parameters are defined in the previous subsection. From the above Einstein equation

(2.17) with the metric ansatz (2.2), one gets:

$$6A'^2 = -\frac{\Lambda_B}{4M_*^3}, \tag{2.19}$$

$$3A'' + 6A'^2 = -\frac{1}{4M_*^3}\big(\Lambda_B + \lambda\delta(y)\big). \tag{2.20}$$

Equation (2.19) gives the same warp-function solution as in RS1 Eq. (2.7), i.e.

$$A(y) = -k|y|, \qquad \text{where} \qquad k \equiv \sqrt{\frac{-\Lambda_B}{24M_*^3}}, \tag{2.21}$$

which implies $\Lambda_B < 0$. The above solution is consistent with the geometric $\mathbb{Z}_2$ symmetry. The second derivative of warp-function $A(y)$ resulting from Eq. (2.21) is,

$$A'' = -2k\delta(y). \tag{2.22}$$

Now it is easy to see by comparing the Einstein equation (2.20) with the above equation that:

$$\lambda = 24M_*^3 k. \tag{2.23}$$

This is the fine-tuning required in the RS models in order to have zero 4D cosmological constant.

With the above RS solution we can rewrite the explicit form of the RS metric as

$$ds^2 = e^{-2k|y|}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2, \tag{2.24}$$

which implies that RS2 geometry Eq. (2.24) is a 5D AdS space with k being its curvature. It is straight forward to see from Eq. (2.12) that the 4D Planck mass $M_{\rm Pl}$ is (taking $r_c \to \infty$ in Eq. (2.12)):

$$M_{\rm Pl}^2 = \frac{M_*^3}{k}. \tag{2.25}$$

The above relation manifests that even for a non-compact geometry the 4D effective gravity can be recovered on the brane at $y = 0$ with $k \simeq M_* \simeq M_{\rm Pl}$. In Sec. 2.4 we present the localization of gravity in detail for the RS2 model and its generalization.

2.2. A $\mathbb{Z}_2$ symmetric generalization of RS1: the IR-UV-IR model

In this section we consider a $\mathbb{Z}_2$ symmetric generalization of RS1 with three D3-branes [5]. We consider an interval $y \in [-L, L]$ in the extra dimension, where on each end of the interval $y = \pm L$ there is a D3 brane with negative tension and at the center of the interval, $y = 0$, we place a positive tension brane where we assume that gravity is localized. We call the boundary branes "IR-branes" and the brane at $y = 0$ we term the "UV-brane", hence the resulting model is called IR-UV-IR model. Since the brane tensions of the two IR-branes are the same, this geometry is $\mathbb{Z}_2$ symmetric. Note that the end points of the interval at $y = \pm L$ are not the fixed points of the $\mathbb{Z}_2$, the only fixed point of the $\mathbb{Z}_2$ is at $y = 0$. This set up is different from the $S_1/\mathbb{Z}_2$ orbifold where $y = 0$ and $y = L$ are both fixed points of the $\mathbb{Z}_2$. The IR-UV-IR model

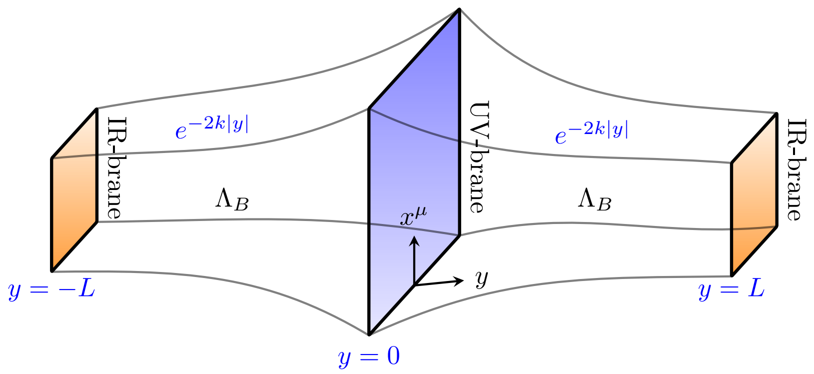


Figure 2.3.: The geometric configuration for IR-UV-IR setup.

leads to many interesting phenomenological implications which are the subject of Chap. 5.

The 5D gravity action for such a geometry can be written as,

$$S_G = \int d^5x\sqrt{-g}\Big\{2M_*^3 R - \Lambda_B - \lambda_{UV}\delta(y) - \lambda_{IR}\delta(y+L) - \lambda_{IR}\delta(y-L)\Big\} + S_{GH}, \tag{2.26}$$

where R is the Ricci scalar, Λ_B is the bulk cosmological constant and $\lambda_{UV}(\lambda_{IR})$ is the brane tension of the UV (IR)-brane . In this section the Dirac delta functions at $y = \pm L$ are defined in such a way that their integral is 1/2. Since our geometry is compact with boundaries, the action contains the Gibbons-Hawking boundary term, [1]

$$S_{GH} = -2M_*^3 \int_{\partial\mathcal{M}} d^4x\sqrt{-\hat{g}}\mathcal{K}, \tag{2.27}$$

where $\mathcal{K}$ is the intrinsic curvature of the surface of the boundary manifold $\partial\mathcal{M}$, given by

$$\mathcal{K} = -\hat{g}^{\mu\nu}\nabla_\mu n_\nu = \hat{g}^{\mu\nu}\Gamma^M_{\mu\nu}n_M, \tag{2.28}$$

with n_M being the unit normal vector to the surface of the boundary manifold $\partial\mathcal{M}$ and $\hat{g}_{\mu\nu}$ is the induced boundary metric. For the 5D manifold with 4D Poincaré invariance ($n^5 = 1$ and $n^\mu = 0$), the intrinsic curvature reduces to

$$\mathcal{K} = -\frac{1}{2}\hat{g}^{\mu\nu}\partial_5\hat{g}_{\mu\nu}. \tag{2.29}$$

The IR-UV-IR geometry and a pictorial description of such a geometric setup is shown in Fig. 2.3. The solution of the Einstein equations resulting from the above action is the RS metric (2.10), where the AdS curvature k is related to Λ_B by

$$\Lambda_B = -24M_*^3 k^2. \tag{2.30}$$

Since the above setup is compactified on an interval $y \in [-L, L]$, rather than on a circle as in RS1, one needs to be careful and show that the solution (2.10) is compatible with the boundaries and that the effective 4D cosmological constant is zero, see also [33]. We will see

[1] The Gibbons–Hawking boundary term is needed in order to cancel the variation of the Ricci tensor at the boundaries so that the RS metric (1.2) is indeed a solution of the Einstein equations of motion.

below that we need a *fine tuning* between the 5D cosmological constant Λ_B and the brane tensions $\lambda_{UV,IR}$ in order to get zero 4D cosmological constant. One can calculate the effective 4D cosmological constant $\bar{\Lambda}$ from the action (2.26) by integrating out the extra dimension,

$$\bar{\Lambda} = -\int_{-L}^{L} dy\sqrt{-g}\Big\{2M_*^3 R - \Lambda_B - \lambda_{UV}\delta(y) - \lambda_{IR}\big[\delta(y+L)+\delta(y-L)\big]\Big\} + 2M_*^3\sqrt{-\hat{g}}\mathcal{K}\Big|_{-L}^{L}, \quad (2.31)$$

where $R = -20A'^2 - 8A''$ and $\Lambda_B = -24M_*^3 A'^2$ corresponding to the solution (2.10). Using $A(y) = -k|y|$ we find,

$$\bar{\Lambda} = \big(\lambda_{UV} - 24M_*^3 k\big) + \big(\lambda_{IR} + 24M_*^3 k\big)\, e^{-4kL}, \quad (2.32)$$

which can only be zero if

$$\lambda_{UV} = -\lambda_{IR} = 24M_*^3 k. \quad (2.33)$$

This result explicitly shows that one needs a positive tension brane at $y = 0$ and two negative tension branes at $y = \pm L$ in order to obtain zero 4D cosmological constant. This is the usual fine tuning which appears in brane world scenarios [22, 34, 25]. Hence the resulting geometry is a 5D warped geometry (IR-UV-IR) with negative bulk cosmological constant, a positive tension brane in the middle and two equal negative tension branes at the end of the interval, see Fig. 2.3.

We would like to mention here that we are considering a rigid IR-UV-IR geometry where the distance L is tuned in order to solve the hierarchy problem. To stabilize the IR-UV-IR setup one needs to consider a stabilization method like Goldberger-Wise (GW) mechanism [34, 25] by introducing a bulk scalar field with appropriate brane potentials such that the minimum of this potential would set the size of the 5D interval and yield a compactification scale that would solve the hierarchy problem without fine-tuning of the parameters. The GW mechanism is beyond the scope of this thesis, hence we merely assume that such a mechanism exists.

2.3. Asymmetric generalization of RS2

In this section we generalized the RS2 such that the resulting geometry is asymmetric AdS space [2, 3, 35]. We consider the following action which is an extension of the Randall-Sundrum model with a single brane (RS2) [23],

$$S = \int d^5x\sqrt{-g}\Big\{2M_*^3 R - \Lambda_+\Theta(y-y_0) - \Lambda_-\Theta(-y+y_0) - \lambda\delta(y-y_0)\Big\}, \quad (2.34)$$

where Λ_+ and Λ_- are 5D cosmological constants for $y > y_0$ and $y < y_0$, respectively, whereas, y_0 is the brane location and λ represents the brane tension. In Eq. (2.34) Θ is the Heaviside theta function and δ is the Dirac delta function. For simplicity we will choose $y_0 = 0$.

We are going to look for solutions of the Einstein equations taking the 5D warped metric ansatz (1.2), i.e.

$$ds^2 = e^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2. \quad (2.35)$$

Then the Einstein equations following from the action (2.34) reduce to,

$$6A'^2 = -\frac{1}{4M_*^3}\Big[\Lambda_+\Theta(y) + \Lambda_-\Theta(-y)\Big], \tag{2.36}$$

$$3A'' + 6A'^2 = -\frac{1}{4M_*^3}\Big[\Lambda_+\Theta(y) + \Lambda_-\Theta(-y) + \lambda\delta(y)\Big], \tag{2.37}$$

The solution of Eq. (2.36) is given by,

$$A(y) = -|y|k_\pm \qquad \text{for} \qquad y \gtrless 0, \tag{2.38}$$

where $k_\pm \equiv \sqrt{\frac{-\Lambda_\pm}{24M_*^3}}$ are the AdS curvatures for $y \gtrless 0$. Now one can calculate the A' and A'' from the above expression as,

$$A'(y) = \mp k_\pm \;\text{ for }\; y \gtrless 0 \qquad \text{and} \qquad A''(y) = -(k_+ + k_-)\delta(y). \tag{2.39}$$

Discontinuity of $A'(y)$ at $y = 0$ results in the following jump

$$\big[A'\big]_0 = -\frac{\lambda}{12M_*^3}, \tag{2.40}$$

where $[A']_0$ is defined through Eq. (1.7). From Einstein equations (2.36) and (2.37), we have,

$$A''(y) = -\frac{\lambda}{12M_*^3}\delta(y). \tag{2.41}$$

Comparing (2.41) and the second equation of (2.39) yields,

$$\lambda = \sqrt{6M_*^3}\Big[\sqrt{-\Lambda_+} + \sqrt{-\Lambda_-}\Big], \tag{2.42}$$

which is an analogue of the Randall-Sundrum relation between the bulk cosmological constant and the brane tension, see Sec. 2.1.2. It is important to note that the relation (2.42) is necessary in order to recover the 4D Poincaré invariance on the brane. Note that for $k_+ = k_-$, we recover the standard RS2 results.

It will be important to see if the 4D effective gravity on the brane could be recovered and also to check if the background solution found above is stable or not. To answer these questions we will perturb the metric around the background solution and see how do the perturbations, especially the zero mode of tensor perturbations which corresponds to the 4D graviton, behave in our generalized RS2 case. This is the subject of next section.

2.4. Localization of gravity

In this section we employ general results from Appendix A and show how the gravity is localized on the brane in the warped extra dimensions. Here we will confine ourselves to only tensor perturbations as our goal is to investigate the properties of the localization of the gravity on the brane in warped extra dimension. As shown in the Appendix A the tensor perturbation of the warped metric (1.2) can be written as

$$ds^2 = e^{2A(y)}(\eta_{\mu\nu} + H_{\mu\nu})dx^\mu dx^\nu + dy^2, \tag{2.43}$$

where, $H_{\mu\nu} \equiv H_{\mu\nu}(x, y)$ is the transverse and traceless tensor fluctuation, i.e.

$$\partial^\mu H_{\mu\nu} = H^\mu_\mu = 0. \tag{2.44}$$

Following the generic results from Appendix A the linearized field equation for the tensor mode can be written as

$$\left(\partial_5^2 + 4A'\partial_5 + e^{-2A}\Box^{(4)}\right) H_{\mu\nu} = 0, \tag{2.45}$$

where $\partial_5 \equiv \partial/\partial y$ and $\Box^{(4)}$ is the 4D d'Alembertian operator. The zero-mode solution of the above equation represents the 4D graviton while the non-zero modes are the Kaluza-Klein (KK) graviton excitations.

In order to gain more intuition and understanding of the tensor mode equation of motion (2.45), it is convenient to change the variables such that we can get rid of the exponential factor in front of the d'Alembertian and the single derivative term with A', so that we can convert the above equation into the standard Schrödinger-like form. We can achieve this in two steps; first by changing coordinates such that the metric becomes conformally flat:

$$ds^2 = e^{2A(z)}\left(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2\right), \tag{2.46}$$

with z defined through the differential equation: $dz = e^{-A(y)}dy$. In the new coordinates Eq. (2.45) takes the form

$$\left(\partial_z^2 + 3\dot{A}(z)\partial_z + \Box^{(4)}\right) H_{\mu\nu} = 0, \tag{2.47}$$

where the *overdot* represents a derivative with respect to z coordinate. Now we can perform the second step; removing the single derivative term in (2.47) by the following redefinition of the tensor fluctuation

$$\tilde{H}_{\mu\nu}(x, z) = e^{3A(z)/2}H_{\mu\nu}(x, z). \tag{2.48}$$

Hence the Eq. (2.47) will take the form of the Schrödinger equation,

$$\left(\partial_z^2 - \frac{9}{4}\dot{A}^2(z) - \frac{3}{2}\ddot{A}(z) + \Box^{(4)}\right)\tilde{H}_{\mu\nu}(x, z) = 0. \tag{2.49}$$

We can KK-decompose the $\tilde{H}_{\mu\nu}(x, z)$ into the x and z dependent parts as:

$$\tilde{H}_{\mu\nu}(x, z) = \sum_n \hat{H}_{n\mu\nu}(x)\bar{H}_n(z), \tag{2.50}$$

where we consider the 4D plane wave solutions for $\hat{H}_{n\mu\nu}(x)$, i.e. $\hat{H}_{n\mu\nu}(x) \propto e^{ip_n x}$ such that

$\Box^{(4)}\hat{H}_{n\mu\nu}(x) = m_n^2\hat{H}_{n\mu\nu}(x)$, with $-p_n^2 = m_n^2$ being the 4D mass of the tensor KK-modes. Employing the plan-wave solutions for the 4D KK-modes we get the following equation of the bulk profiles for the KK-mode $\bar{H}_n(y)$:

$$\Big[-\partial_z^2 + \mathcal{V}(z)\Big]\bar{H}_n(z) = m_n^2\bar{H}_n(z), \tag{2.51}$$

where $\mathcal{V}(z)$ is the Schrödinger-like potential,

$$\mathcal{V}(z) = \frac{9}{4}\dot{A}^2(z) + \frac{3}{2}\ddot{A}(z). \tag{2.52}$$

Note that we can rewrite the Schrödinger-like equation (2.51) in supersymmetric quantum mechanics form as,

$$\mathcal{Q}^\dagger\mathcal{Q}\bar{H}_n(z) = \left(-\partial_z - \frac{3}{2}\dot{A}\right)\left(\partial_z - \frac{3}{2}\dot{A}\right)\bar{H}_n(z) = m_n^2\bar{H}_n(z). \tag{2.53}$$

The zero mode ($m_0^2 = 0$) profile, $\bar{H}_0(z)$, corresponds to the wave-function of the graviton in the 4D effective theory. The stability with respect to the tensor fluctuations of the background solution is guaranteed by the positivity of the operator $\mathcal{Q}^\dagger\mathcal{Q}$ in the supersymmetric quantum mechanics version of the equation of motion (2.53) as it forbids the existence of any tachyonic mode with negative mass square, $m_n^2 < 0$ [2]. So, in that case, the perturbation is not growing in time, hence the background solution is stable.

The zero-mode wave function $\bar{H}_0(z)$ can be obtained by noticing that

$$\mathcal{Q}\bar{H}_0 = \left(\partial_z - \frac{3}{2}\dot{A}\right)\bar{H}_0 = 0, \tag{2.54}$$

which implies that,

$$\bar{H}_0(z) = e^{\frac{3}{2}A(z)}. \tag{2.55}$$

For massive KK modes one should solve the Eq. (2.51) with $m_n^2 \neq 0$, which is presented at the end of this section.

From here on we employ the warp-function $A(y)$ of the generalized RS2 scenario, i.e. Eq. (2.38) (the standard RS2 results can be obtained for the special case of generalized RS2 with $k_+ = k_-$). We use the relation $e^{-A(y)}dy = dz$ to obtain $A(z)$ as:

$$A(z) \equiv A[y(z)] = -\ln(k_\pm|z| + 1) \qquad \text{for} \qquad z \gtrless 0, \tag{2.56}$$

where we used the initial condition: $A(0) = 0$. Now one can easily evaluate Schrödinger

[2] Since $\int dz(\mathcal{Q}\bar{H}_n)^2 + \bar{H}_n\mathcal{Q}\bar{H}_n\big|_{-\infty}^{+\infty} = m_n^2\int dz\bar{H}_n^2$ and the first term $\int dz(\mathcal{Q}\bar{H}_n)^2$ is definite non-negative, therefore in order to guarantee $m_n^2 \geq 0$ the boundary term (second term) must vanish or be positive.

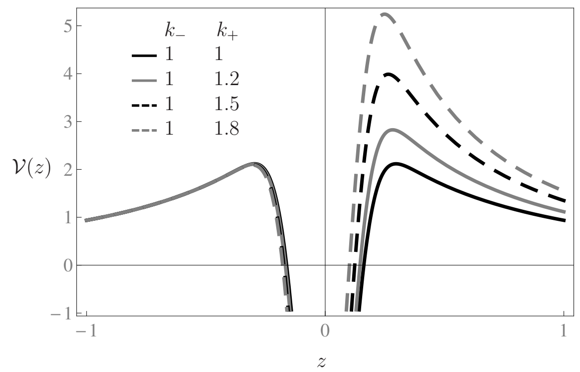


Figure 2.4.: The Schrödinger potential $\mathcal{V}(z)$ for different values of $k_\pm$.

potential $\mathcal{V}(z)$ (2.52) as,

$$\mathcal{V}(z) = \frac{15}{4}\left(\frac{k_+^2}{(1+k_+|z|)^2}\Theta(z) + \frac{k_-^2}{(1+k_-|z|)^2}\Theta(-z)\right) - \frac{3}{2}\left(\frac{k_+}{1+k_+|z|} + \frac{k_-}{1+k_-|z|}\right)\delta(z), \tag{2.57}$$

The potential $\mathcal{V}(z)$ is plotted in Fig. 2.4 for different values of $k_\pm$.

To calculate the 4D Plank mass let us focus on the kinetic term corresponding to the 4D graviton $\hat{H}_{0\mu\nu}(x)$:

$$S_0 = 2M_*^3 \int dz \bar{H}_0^2(z) \int d^4x \partial_\alpha \hat{H}_{0\mu\nu}(x) \partial^\alpha \hat{H}_0^{\mu\nu}(x). \tag{2.58}$$

Now we are able to identify the 4D Planck mass as a coefficient in front of the kinetic term for the 4D graviton (which is the standard 4D general relativity normalization of graviton kinetic term):

$$M_{\rm Pl}^2 \equiv M_*^3 \int dz \bar{H}_0^2(z) = \frac{M_*^3}{2k_+} + \frac{M_*^3}{2k_-}, \tag{2.59}$$

where $\bar{H}_0(z) = 1/(k_\pm|z|+1)^{3/2}$ satisfies the supersymmetric quantum mechanic equation (2.53) for $m_0^2 = 0$. Equation (2.59) implies that $M_{\rm Pl}^2$ is finite (so $\hat{H}_{\mu\nu}(x)$ is normalizable) for our generalized RS2 model. Therefore we conclude that the 4D General Relativity on the brane can be recovered, like in the standard RS2 scenario, even if the AdS space is not symmetric on both sides of the brane.

To complete the discussion of the generalized RS2 case we will discuss some properties of KK modes. There have been many studies on KK modes within the context of different non-compact 5D models, see for example [23, 35, 36, 37, 38, 39, 40, 41].

Before performing any calculations one can make the following comments resulting from the shape of the Schrödinger-like potential $\mathcal{V}(z)$ shown in Fig. 2.4:

- As $\mathcal{V}(z) \to 0$ for $|z| \to \infty$, therefore the KK-mass spectrum is continuous without a gap and it starts from $m_0 = 0$.
- The (asymmetric) volcano-like shape of $\mathcal{V}(z)$ in Fig. 2.4 suggests that at large z the wave function $\bar{H}_n(z)$ should have a plane wave behaviour.

- The presence of the large barriers near the brane implies that corrections to the Newton's law due to continuum spectrum of the KK modes will not be large [23].

For the complete analysis one needs to solve the Schrödinger-like equation (2.51) for $m^2 \neq 0$ along with the following jump condition across the brane at $y = 0$:

$$\bar{H}'_n(0) = -\frac{3}{2}\bar{k}\bar{H}_n(0), \tag{2.60}$$

where we define:

$$\bar{k} \equiv \frac{1}{2}(k_+ + k_-), \qquad \Delta k \equiv \frac{1}{2}(k_+ - k_-). \tag{2.61}$$

The solution of the Schrödinger-like equation (2.51) can be written as a linear combination of Bessel functions as follows

$$\bar{H}_n(z) = \begin{cases} a_n(|z| + \frac{1}{\bar{k}+\Delta k})^{1/2} J_2\left(m_n(|z| + \frac{1}{\bar{k}+\Delta k})\right) & \\ \qquad +b_n(|z| + \frac{1}{\bar{k}+\Delta k})^{1/2} Y_2\left(m_n(|z| + \frac{1}{\bar{k}+\Delta k})\right) & z > 0 \\ c_n(|z| + \frac{1}{\bar{k}-\Delta k})^{1/2} J_2\left(m_n(|z| + \frac{1}{\bar{k}-\Delta k})\right) & \\ \qquad +d_n(|z| + \frac{1}{\bar{k}-\Delta k})^{1/2} Y_2\left(m_n(|z| + \frac{1}{\bar{k}-\Delta k})\right) & z < 0 \end{cases}, \tag{2.62}$$

where a_n, b_n, c_n and d_n are integration constants and they can be fixed by the boundary (jump) conditions. For instance, we can find a_n in terms of b_n adopting (2.60), for $z \geq 0$, the result reads,

$$a_n = \left(-\frac{96(\bar{k}+\Delta k)^4 \Delta k}{(8\bar{k}+5\Delta k)\pi m_n^4} + \frac{32(\bar{k}+\Delta k)^2(8\bar{k}^2 + 4\bar{k}\Delta k - \Delta k^2)}{(8\bar{k}+5\Delta k)^2 \pi m_n^2} \right) b_n. \tag{2.63}$$

Similarly we can find c_n in terms of d_n using the boundary condition (2.60) for $z \leq 0$, the result reads

$$c_n = \left(\frac{96(\bar{k}-\Delta k)^4 \Delta k}{(8\bar{k}-5\Delta k)\pi m_n^4} + \frac{32(\bar{k}-\Delta k)^2(8\bar{k}^2 - 4\bar{k}\Delta k - \Delta k^2)}{(8\bar{k}-5\Delta k)^2 \pi m_n^2} \right) d_n. \tag{2.64}$$

Now we are left with two unknown constants b_n and d_n, one of them can be found in terms of other using the fact that both branches for $z \gtrless 0$ of the solution $\bar{H}_n(z)$ must match at $z = 0$, that will fix say b_n in terms of d_n. Using the above relations for a_n and c_n in Eq. (2.62) and applying the boundary condition $\bar{H}_n(z > 0) = \bar{H}_n(z < 0)$ at $z = 0$, one obtains b_n in terms of d_n as follows

$$b_n = \left(\frac{\bar{k}-\Delta k}{\bar{k}+\Delta k} \right)^{5/2} \left(\frac{8\bar{k}+5\Delta k}{8\bar{k}-5\Delta k} \right) d_n. \tag{2.65}$$

Now we can write Eq. (2.62) as,

$$\bar{H}_n(z) \approx N_n \begin{cases} (|z| + \frac{1}{\bar{k}+\Delta k})^{1/2} \left(\frac{\bar{k}-\Delta k}{\bar{k}+\Delta k}\right)^{5/2} \left(\frac{8\bar{k}+5\Delta k}{8\bar{k}-5\Delta k}\right) \Big[Y_2 \left(m_n(|z| + \frac{1}{\bar{k}+\Delta k})\right) + & \\ \left(-\frac{96(\bar{k}+\Delta k)^4 \Delta k}{(8\bar{k}+5\Delta k)\pi m_n^4} + \frac{32(\bar{k}+\Delta k)^2(8\bar{k}^2+4\bar{k}\Delta k-\Delta k^2)}{(8\bar{k}+5\Delta k)^2 \pi m_n^2}\right) J_2 \left(m_n(|z| + \frac{1}{\bar{k}+\Delta k})\right) \Big] & z>0 \\ (|z| + \frac{1}{\bar{k}-\Delta k})^{1/2} \Big[Y_2 \left(m_n(|z| + \frac{1}{\bar{k}-\Delta k})\right) + & \\ \left(\frac{96(\bar{k}-\Delta k)^4 \Delta k}{(8\bar{k}-5\Delta k)\pi m_n^4} + \frac{32(\bar{k}-\Delta k)^2(8\bar{k}^2-4\bar{k}\Delta k-\Delta k^2)}{(8\bar{k}-5\Delta k)^2 \pi m_n^2}\right) J_2 \left(m_n(|z| + \frac{1}{\bar{k}-\Delta k})\right) \Big] & z<0 \end{cases} \tag{2.66}$$

The normalization constant $N_n \equiv d_n$ can be obtained by the delta function normalization of the $\bar{H}_n(z)$. It turns out that the dominant contribution to N_n is not sensitive to the splitting between k_+ and k_- with the result

$$N_n \sim \frac{\pi m^{5/2}}{8\bar{k}^2} + \mathcal{O}(\Delta k). \tag{2.67}$$

For large values of z ($z \gg 1/\bar{k}$) one can neglect $1/(\bar{k} \pm \Delta k)$ in the argument of the Bessel functions and the KK massive modes will indeed asymptote plane waves:

$$\bar{H}_n(z) \approx N_n \sqrt{\frac{2}{\pi m_n}} \begin{cases} \left(1 - \frac{15}{4}\frac{\Delta k}{k}\right) \sin\left(m_n|z| - \frac{5}{4}\pi\right) + & \\ \left(\frac{4\bar{k}^2}{m_n^2 \pi} - \frac{12\bar{k}^4+10\bar{k}^2 m_n^2}{m^4 \pi}\frac{\Delta k}{k}\right) \cos\left(m_n|z| - \frac{5}{4}\pi\right) & z>0 \\ \sin\left(m_n|z| - \frac{5}{4}\pi\right) + & \\ \left(\frac{4\bar{k}^2}{m_n^2 \pi} + \frac{12\bar{k}^4-5\bar{k}^2 m_n^2}{m^4 \pi}\frac{\Delta k}{k}\right) \cos\left(m_n|z| - \frac{5}{4}\pi\right) & z<0 \end{cases}, \tag{2.68}$$

where terms $\mathcal{O}(\Delta k)^2$ were neglected.

After obtaining the KK modes of the effective theory in 4D, we can now calculate the non-relativistic gravitational potential between two test masses m_1 and m_2 separated by a distance r at the location of the brane ($z = 0$) as

$$V(r) \simeq G_N \frac{m_1 m_2}{r} + \frac{1}{2M_*^3} \sum_{n=1}^{\infty} \int_0^{\infty} dm_n \frac{m_1 m_2 e^{-m_n r}}{r} |\bar{H}_n(0)|^2, \tag{2.69}$$

where the first term comes from the exchange of zero-mode (the 4D graviton), whereas, the second term is generated by exchanges of the continuum of massive KK modes. In the above formula $G_N \equiv (2M_{\rm Pl}^2)^{-1}$ is the 4D Newton's gravitational constant. The strength of KK modes at $z = 0$, $\bar{H}_n(0)$, can be easily calculated since the Bessel function of the first kind $J_2\left[m_n\big(|z| + 1/(\bar{k} \pm \Delta k)\big)\right] \to 0$ for small arguments, whereas the Bessel function of the second kind, $Y_2\left[m_n\big(|z| + 1/(\bar{k} \pm \Delta k)\big)\right]$ for small arguments can be expanded as follows

$$\begin{aligned} Y_2\left[m_n\big(|z| + 1/(\bar{k} \pm \Delta k)\big)\right] &\simeq -\frac{4}{\pi m_n^2 \left(|z| + \frac{1}{\bar{k}\pm\Delta k}\right)^2} - \frac{1}{\pi}, \\ &\simeq -\frac{4\bar{k}^2}{\pi m_n^2 \left[(\bar{k} \pm \Delta k)|z| + 1\right]^2} - \frac{1}{\pi} + \mathcal{O}(\Delta k). \end{aligned} \tag{2.70}$$

Now it is easy to find the approximate value of $|\bar{H}_n(0)|^2$ from Eq. (2.66) as

$$|\bar{H}_n(0)|^2 \simeq C\frac{m_n}{\bar{k}} + \mathcal{O}\left(\frac{\Delta k}{\bar{k}}\right)^2, \tag{2.71}$$

where C is a constant of the order of unity. Hence, neglecting $\mathcal{O}(\Delta k/\bar{k})^2$ contributions, we can estimate the Newton's gravitational potential as,

$$V(r) \simeq G_N\frac{m_1 m_2}{r} + \frac{1}{2M_*^3}\sum_{n=1}^{\infty}\int_0^{\infty} dm_n \frac{m_1 m_2 e^{-m_n r}}{r} C\frac{m_n}{\bar{k}}. \tag{2.72}$$

Since $\sum_n \int_0^\infty dm_n m_n e^{-m_n r} = 1/r^2$, therefore the above equation will read as,

$$\begin{aligned} V(r) &\simeq G_N\frac{m_1 m_2}{r} + \frac{1}{2M_*^3}\frac{C}{\bar{k}}\frac{m_1 m_2}{r^3}, \\ &= G_N\frac{m_1 m_2}{r}\left(1 + \frac{C}{r^2\bar{k}^2}\right), \end{aligned} \tag{2.73}$$

where we have used the equation (3.168) which relates the 4D Planck mass with that of the 5D and we have adopted the approximation of small Δk. This result shows, that the correction due to the KK modes is small for distances larger than the AdS curvature $1/k_\pm$. Since in our case $k_\pm$ is of the order of the Planck mass therefore this implies that for distances above the Planck length one would effectively reproduce the 4D Newton's gravitational potential.

2.5. Summary

To summarize this chapter, in Sec. 2.1 we review the RS models: RS1 provides an elegant solution to the hierarchy problem, while RS2 gives an alternative to compactification in warped extra dimension. Section 2.2 contains a $\mathbb{Z}_2$ symmetric generalization of RS1 presented in our paper [5]; see Chap. 5 for detailed phenomenological implications due to this background geometry. An asymmetric warped geometry is considered in Sec. 2.3 which is a generalization of RS2 allowing different AdS geometries on either side of the brane. In the next chapter we consider a smooth/thick-brane version of this asymmetric warped model, see also [2, 3]. The issue of localization of gravity on a brane in our asymmetric warped extra dimensional model is addressed in detail in Sec. 2.4.

CHAPTER 3

BRANE MODELING IN WARPED EXTRA DIMENSION

RS models and its generalizations discussed in the previous chapter employ the singular D3 branes without any dynamical mechanism. In this chapter our goal is to provide smooth generalizations of the RS models and their extensions by some dynamical fields. As we have pointed out in the Introduction that there were many attempts to avoid the presence of singular branes. It has been shown in Refs. [24, 25, 26, 39, 42, 43, 44, 45, 46, 47, 48] that the positive tension brane could be smoothed by a background scalar field configuration which we term here as the *smooth*- or *thick-brane*. As shown in the previous chapter the hierarchy problem is addressed within RS1 scenario which requires two D3-branes of opposite tension and periodicity due to the $S_1/\mathbb{Z}_2$ orbifold. There is no satisfactory simple strategy to model a negative tension brane at least for real scalar fields minimally coupled to gravity. For examples of existing attempts to generate negative tension branes see [49]. Moreover, it has been shown by Gibbons, Kallosh and Linde [27], periodicity of set-ups like RS1 are generically in conflict with the idea of a smooth non-trivial scalar profile. To conclude, smoothing the RS1 scenario is severely limited by mainly two reasons: (*i*) impossibility of generating a negative tension brane by a real scalar field configuration and (*ii*) periodicity. Note that this conclusion holds at least for the case when the scalar field is minimally coupled to gravity, the generalization will be considered below.

On the other hand, as we discussed in Chap. 2, RS2 model having just one D3 brane is not sufficient to address the hierarchy problem. In the following the hierarchy problem is one of the main motivations, therefore we will try to improve the scenario by introducing a second brane. The main purposes of this chapter:

- First, to see if one can overcome the above mentioned obstacles (periodicity and positivity of brane tension) to achieve a smooth version of RS1 in modified gravity with the scalar field non-minimally coupled to the Ricci scalar.
- Second, to verify if one can address the hierarchy problem with two thick-branes (which in a certain "brane limit" mimic two positive tension singular branes) with non-compact

warped extra dimension.

- Third, to give a smooth generalization of the RS2 which would allow, in the "brane limit", different bulk cosmological constants on each side of the positive tension brane.
- Fourth, to consider a $\mathbb{Z}_2$ symmetric triple brane model which could mimic three positive tension branes in the brane limit.
- Fifth, to see if a scalar field can be localized on the thick-brane.

As far as the first task is concerned, we will show, by generalizing the Gibbons-Kallosh-Linde sum rules [27], that it is not possible to achieve the periodicity for scalar field and metric solutions even with the scalar field non-minimally coupled to the Ricci scalar. The consistency conditions have been discussed following the strategy of Gibbons et al. [27] in the modified gravity set-up [50], however the authors consider the scalar field in the bulk with *singular branes*, this is exactly what we want to avoid for our set-up, i.e. we wish to have smooth branes instead of singular branes. Another attempt to overcome the problem of periodicity was discussed in [51]. Concerning the issue of the positivity of a brane tension generated by a scalar profile we also find that even with non-minimal scalar couplings there is no way to generate a negative tension brane. Therefore we turn our attention to models with only positive tension branes (e.g. the RS2) and non-compact (to avoid restrictions imposed by periodicity a'la [27]) with real scalar field non-minimally coupled to gravity.

In order to be able to address the hierarchy problem we will propose a model with two thick (smooth) branes, which, in an appropriately defined limit (so called "brane limit") approaches two singular branes. The limiting version of the model was discussed earlier by Lykken and Randall (LR) in [36]. As we will show, in our set-up of two thick-branes, different possible solutions for the warped factor can emerge, for instance, we can have the AdS or Minkowski geometry in different regions along the extra dimension. We will discuss three such configurations, (*i*) the two thick-branes between the AdS vacua so that we have warped geometry and hierarchy problem could be addressed in this set-up (this is the thick-brane version of the Lykken-Randall model [36]), (*ii*) the case when we can have the Minkowski background in between the two branes and the AdS geometry to the right and left of both branes and, (*iii*) with the Minkowski geometry to right or left of both branes and the AdS in the other regions along the extra dimension, which gives the thick-brane version of the Gregory, Rubakov, Sibiryakov (GRS) model [37] except that we have the second brane also with positive tension instead of the negative tension as in the original GRS model with singular branes.

For the third point above, we will give a smooth or thick-brane generalization of the RS2 model which allows departure from the $\mathbb{Z}_2$ symmetric case by allowing different bulk cosmological constants on each side of the brane. We are going to prove that under certain mild assumptions, the relation between the brane tension and the cosmological constants obtained in the brane limit of the thick-brane scenario does not depend on detailed shape of the scalar field profile. Hence a class of thick-models can be constructed with a scalar field configuration with certain generic properties. The fourth point raised above can be addressed by considering a scalar field profile like three kinks and we show that the resulting geometry has $\mathbb{Z}_2$ geometric symmetry and in the brane limit it mimics IR-UV-IR model with all positive tension branes. The fifth task of localization of a scalar field on a thick-brane is addressed by allowing the localized interactions of the scalar field with the field generating the thick-brane..

In all the above mentioned cases we study stability of the background solutions and also existence and localization of the zero-modes of the scalar, vector and tensor (SVT) perturbations of the solutions. The issue of stability and localization of the zero-modes have been extensively discussed in the past in the context of thin as well as thick-brane scenarios [25, 26, 39, 42, 43, 52, 53, 54, 38, 55, 56, 57, 58, 59, 60, 61, 62, 63]. We find that the SVT perturbation equations could be transformed into a supersymmetric quantum mechanics form so that they guarantee stability of these perturbations in all the configurations considered above and therefore the absence of tachyonic modes. It turns out that the zero-mode of the tensor perturbation wave function (that corresponds to the 4D gravitons) can be localized on the desired brane by using certain boundary conditions. on the other hand the zero-modes corresponding to the scalar and vector perturbations are not localized, as they are not normalizable modes, consequently they do not affect the 4D physics.

3.1. Thick brane generalization of RS1 in modified gravity

Our first goal is to mimic (regularize) D3-branes which appear in various five-dimensional (5D) scenarios that solve the hierarchy problem by warping the metric along the extra dimension in the spirit of [22]. The most natural approach is to introduce a 5D scalar field ϕ with a non-trivial profile (that satisfies equations of motion) that in certain limit could mimic a brane by approaching a delta-like energy distribution along extra dimension. However, as it was shown in [27], in the case of compact extra dimensions the idea of a non-trivial scalar profile (a thick-brane) is severely restricted by the requirement of periodicity. Arguments adopted in [27] apply for a scalar that is minimally coupled to gravity. Therefore here, we are going to discuss first a class of models allowing for non-minimal scalar-gravity coupling:

$$S_{MG} = \int dx^5 \sqrt{-g} \left\{ f(\phi) R - \frac{1}{2} g^{MN} \nabla_M \phi \nabla_N \phi - V(\phi) \right\}, \tag{3.1}$$

where $f(\phi)$ is a general smooth positive definite function of the scalar field $\phi(y)$ which is supposed to compose the D3-branes that are present in the RS1 scenario [22]. In other words, branes would be made of the scalar field while other fields could be dynamically localized in certain regions of the 5D space, see for instance Sec. 3.3 and Refs. [64, 65, 66, 67]. Thick branes in the presence of non-minimally coupled scalar was discussed earlier by [68, 69].

We will look for a solution of the Einstein equations with the 5D warped metric ansatz (1.2), i.e.

$$ds^2 = e^{2A(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2, \tag{3.2}$$

where $A(y)$ is a general warp-function. The Einstein's equation and scalar field equation of motion resulting from the action (3.1) are

$$R_{MN} - \frac{1}{2} g_{MN} R = \frac{1}{f(\phi)} \left\{ \frac{1}{2} T_{MN} + \nabla_M \nabla_N f(\phi) - g_{MN} \nabla^2 f(\phi) \right\}, \tag{3.3}$$

$$\nabla^2 \phi - \frac{dV}{d\phi} + R \frac{df}{d\phi} = 0, \tag{3.4}$$

with the energy-momentum tensor for the scalar field ϕ as

$$T_{MN} = \nabla_M \phi \nabla_N \phi - g_{MN} \left(\frac{1}{2} (\nabla \phi)^2 + V(\phi) \right). \tag{3.5}$$

From the Einstein equations (3.3) and (3.4), one can get the equations of motion for the metric ansatz (1.2) as,

$$6(A')^2 = \frac{1}{f} \left\{ \frac{1}{4} (\phi')^2 - \frac{1}{2} V - 4A'f' \right\}, \tag{3.6}$$

$$3A'' + 6(A')^2 = \frac{1}{f} \left\{ -\frac{1}{4} (\phi')^2 - \frac{1}{2} V - 3A'f' - f'' \right\}, \tag{3.7}$$

$$\phi'' + 4A'\phi' - \frac{dV}{d\phi} - \left(8A'' + 20(A')^2\right) \frac{df}{d\phi} = 0, \tag{3.8}$$

where it is understood that f and V are functions of the scalar field $\phi(y)$.

3.1.1. Thick branes with periodic extra dimensions

Here we would like to verify if the scenario with a non-trivial profile of a bulk scalar could be consistent with periodicity in the case of compact extra dimension. The authors of [27] derived elegant, simple and powerful sum rules that severely restrict thick-brane scenarios with periodic extra dimensions. From our perspective the most relevant result obtained there is the following condition that must be satisfied for periodic extra dimensions with a bulk scalar ϕ when singular branes are absent:

$$\oint dy \, \phi' \cdot \phi' = 0. \tag{3.9}$$

The above result implies that non-trivial scalar profiles are inconsistent with periodicity, the only allowed configuration is ϕ =const.. The sum rule (3.9) was obtained assuming minimal scalar-gravity coupling. In the following we are going to generalize the result for the case of non-minimal coupling described by the action (3.1).

It is easy (subtracting Eqs. (3.6) and (3.7)) to derive an equation of motion that contains only the warp function $A(y)$ and the input profile $\phi(y)$:

$$3fA'' = f'A' - f'' - \frac{1}{2} (\phi')^2. \tag{3.10}$$

It is useful to rewrite the above equation by the change of variables $X(y) = A'(y)$:

$$X'(y) = F(y) X(y) + G(y), \tag{3.11}$$

where,

$$F(y) \equiv \frac{f'(y)}{3f(y)}, \tag{3.12}$$

$$G(y) \equiv -\frac{f''(y) + \frac{1}{2}\phi'(y)^2}{3f(y)}. \tag{3.13}$$

We assume that the profile is periodic with a period L:

$$\phi(y+L) = \phi(y),$$

then $f(\phi)$ and consequently, $F(y)$ and $G(y)$ are also periodic with the same period L. Since

$$\oint dy\ F(y) = 0 \tag{3.14}$$

it is straightforward to notice that the solution of the homogeneous part of Eq. (3.11)

$$X(y) = X_0 e^{\int_{y_0}^{y} F(s)ds} \tag{3.15}$$

is periodic as well.

The inhomogeneous equation (3.11) could be rewritten in the following form

$$[Z(y)X(y)]' = Z(y)G(y), \tag{3.16}$$

where $Z(y) \equiv Z(0)e^{-\int_0^t F(s)ds}$ is a solution of the following homogeneous equation,

$$Z'(y) = -F(y)Z(y). \tag{3.17}$$

Integrating (3.16) over the period we obtain the following condition:

$$\oint dy\ G(y)Z(y) = 0, \tag{3.18}$$

which constitutes the proper generalization of the Gibbons-Kallosh-Linde sum rule (3.9). For $F(y)$ defined in (3.12) one obtains $Z(y)$ explicitly

$$Z(y) = Z(0)\left[\frac{f(0)}{f(y)}\right]^{1/3} \tag{3.19}$$

Then, after integrating by parts, the sum rule (3.18) reads

$$\oint dy\ \left[\frac{4}{3}\frac{1}{f^{1/3}}\left(\frac{f'}{f}\right)^2 + \frac{1}{2}\left(\frac{\phi'}{f^{2/3}}\right)^2\right] = 0 \tag{3.20}$$

The above sum rule again implies that even in the presence of non-minimal couplings, $f(\phi)R$, only the trivial profile, $\phi =$const., is consistent with periodicity.

Note that (3.20) holds also for multicomponent scalar fields, therefore even in that case non-trivial profiles in the absence of singular branes are excluded by periodicity. It is also worth mentioning that the result (3.20) could be obtained by writing the action (3.1) in the Einstein frame where the scalar field is minimally coupled to gravity. In the Einstein frame the standard GKL sum-rule (3.9) holds, and when the sum rule is rewritten back in the Jordan frame defined by (3.1), the result (3.20) is reproduced.

3.1.2. Negative tension brane in modified gravity

In the RS1 set-up one of the two 3-branes must have a negative tension. Therefore in this subsection we turn to the question weather a negative tension brane can be constructed out of a real scalar field in the modified gravity scenario when the scalar field is non-minimally coupled to gravity as in (3.1). By the virtue of the result of the previous subsection we discard the possibility of periodic extra dimensions. Let us assume, without loosing any generality, that the scalar filed $\phi(y)$ has a kink-like profile,

$$\phi(y) = \frac{\kappa}{\sqrt{\beta}} \tanh(\beta y), \tag{3.21}$$

where β is a brane-thickness controlling parameter. For $\beta \to \infty$ (the brane limit), as it will be discussed in details in Sec. 3.2.2, the profile $\phi(y)$ generates singular energy density localized at $y = 0$ that could mimic a D3-brane.

The action for the kink configuration (3.21) can be written as,

$$S_\phi = -\int d^5x\sqrt{-g}\left[\frac{1}{2}(\phi')^2 + V(\phi)\right], \tag{3.22}$$

while the action for a brane localized at $y = 0$ with a negative tension ($\lambda > 0$) reads

$$S_{3\text{-}brane} = \int d^5x\sqrt{-g}\lambda\delta(y). \tag{3.23}$$

Using the equations of motion (3.6) and (3.7) we can rewrite the action (3.22) as follows

$$-\int dy\left[\frac{1}{2}(\phi')^2 + V\right] = \int dy\left[-(\phi')^2 + 12(A')^2 f + 8A'f'\right]. \tag{3.24}$$

As it will be clear from the next section, the only interesting set-up is such that the warp function $A(y)$ reaches its maximum at the brane location (so $A'(0) = 0$), therefore among the above terms only the very first one contributes to the brane tension. However, as it is seen from (3.23) there is no possibility to reproduce the sign required by the negative tension. Therefore we conclude that a single kink-like profile can generate only a positive tension brane even in the case of modified gravity.

3.1.3. Conclusions on thick-brane generalization of the RS1 model

As we have shown in the proceeding subsections there is a conflict between the RS1 scenario and the idea of branes generated by bulk scalar profiles:

- As shown in Sec.3.1.1, even in the presence of the non-minimal scalar-gravity coupling $f(\phi)R$, periodicity in the extra coordinate can not be reconciled with a non-trivial profile.

- One of the branes in the RS1 scenario must have negative tension, however as we have shown in Sec.3.1.2 even if scalars interact non-minimally with gravity there is no way to generate a brane with negative tension.

The above observations prompt to give up compactness and therefore to discuss a possibility of mimicking the RS2 model with non-compact extra dimension. Since we would like to allow for solution of the hierarchy problem by the virtue of warping the metric along extra dimensions, we will introduce a scalar field, the profile of which could mimic a scenario with two branes of positive tension with warped metric in between them. This is what we are going to discuss in Sec. 3.2.2 limiting ourselves to the case of minimal scalar-gravity coupling, however the analysis could be easily extended to non-minimal scenarios as well.

3.2. Modeling branes with scalar fields minimally coupled to gravity

In this section our goal is to construct thick-brane models with scalar fields minimally coupled to gravity, which mimic positive tension branes, in a non-compact warped extra dimension. From now onwards in this chapter we will adopt the following action for a 5D scalar field minimally coupled to the Einstein-Hilbert gravity

$$S = \int dx^5 \sqrt{-g} \left\{ 2M_*^3 R - \frac{1}{2} g^{MN} \nabla_M \phi \nabla_N \phi - V(\phi) \right\}. \tag{3.25}$$

The Einstein's equation and the equation of motion for ϕ, resulting from the action (3.25) are

$$R_{MN} - \frac{1}{2} g_{MN} R = \frac{1}{4M_*^3} T_{MN}, \tag{3.26}$$

$$\nabla^2 \phi - \frac{dV}{d\phi} = 0, \tag{3.27}$$

where the energy-momentum tensor T_{MN} for the scalar field $\phi(y)$ is,

$$T_{MN} = \nabla_M \phi \nabla_N \phi - g_{MN} \left[\frac{1}{2} (\nabla \phi)^2 + V(\phi) \right]. \tag{3.28}$$

From the Eqs. (3.26) and (3.27), one can get the following equations of motion for the metric ansatz (1.2)

$$24M_*^3 (A')^2 = \frac{1}{2} (\phi')^2 - V(\phi), \tag{3.29}$$

$$12M_*^3 A'' + 24M_*^3 (A')^2 = -\frac{1}{2} (\phi')^2 - V(\phi), \tag{3.30}$$

$$\phi'' + 4A'\phi' - \frac{dV}{d\phi} = 0. \tag{3.31}$$

Superpotential method: In the following we will layout the so-called *superpotential method* for solving the above set of coupled scalar-gravity equations [25]. Although the use of this method is motivated by supersymmetry, no supersymmetry is involved in our set-up. The method is elegant and very efficient, in particular it applies to the system of second order differential equations (3.29)-(3.31) and reduces them to a set of first order ordinary differential equations which are much easier to deal with. It is assumed that the scalar potential $V(\phi)$ could be expressed in terms of the superpotential $W(\phi)$ as [25],

$$V(\phi) = \frac{1}{2}\left(\frac{\partial W(\phi)}{\partial \phi}\right)^2 - \frac{1}{6M_*^3}W(\phi)^2, \tag{3.32}$$

where $W(\phi)$ satisfies the following relations,

$$\phi' = \frac{\partial W(\phi)}{\partial \phi}, \qquad\qquad A' = -\frac{1}{12M_*^3}W(\phi). \tag{3.33}$$

It is worth to mention that the standard (and straightforward) application of the superpotential method is limited to the single scalar-field case since with multi-scalar fields it becomes difficult to handle analytically. However in Sec. 3.3 we extend the superpotential method to two field.

We are interested in the case where the scalar field $\phi(y)$ is given by kink-like profiles [1], i.e.

$$\phi_\alpha(y) = \sum_\alpha \frac{\kappa_\alpha}{\sqrt{\beta_\alpha}} \tanh\big(\beta_\alpha(y - y_\alpha)\big), \tag{3.34}$$

where β_α are the thickness regulators and κ_α parameterize tensions of the branes in the so called *brane limit* $\beta_\alpha \to \infty$ (from here on we will consider $\beta_\alpha = \beta$, i.e. all the branes have equal thickness, although that could be relaxed). Above α numbers kinks (anti-kinks) which correspond to the number of thick-branes. As it will be shown in the next subsections the profile (3.34) in the brane limit corresponds to 3-branes with brane-tensions given by

$$\lambda_\alpha = \frac{4}{3}\kappa_\alpha^2. \tag{3.35}$$

It is important to note that this set-up implies that only positive brane tensions could be mimicked by scalar filed configurations, as was also pointed out by DeWolfe et al. [25]. Therefore the scalar field can not reproduce the RS1 scenario where the IR brane has a negative tension.

3.2.1. Single asymmetric thick-brane model

In this subsection we will extend the solution found in Sec. 2.3 for a singular D3-brane to a thick-brane scenario. The action for a 5D scalar field minimally coupled to the Einstein-Hilbert gravity is given in Eq. (3.25) and we employ the superpotential method described above to get the asymmetric thick-brane. For a single thick-brane we consider the scalar field profile as

$$\phi_1(y) = \frac{\kappa}{\sqrt{\beta}} \tanh(\beta y), \tag{3.36}$$

where, as mentioned above, β is the thickness regulator and κ parameterizes tension of the brane in the so called *brane limit*: $\beta \to \infty$. We can find the superpotential $W(\phi_1)$ in such a way that it allows a solution of the scalar field $\phi_1(y)$ as in Eq. (3.36). This can be obtained

[1] The profile of the scalar field could be different from the standard kink but the essential concept holds for any profile which is monotonic and satisfies equations of motion, see Sec. 3.2.5 below.

from Eq. (3.33) as,

$$\phi_1'(y) = \frac{\partial W(\phi_1)}{\partial \phi_1} = \frac{\partial W[\phi_1(y)]}{\partial y}\frac{\partial y}{\partial \phi_1(y)} = \frac{W'(y)}{\phi_1'(y)}, \tag{3.37}$$

$$W(y) = \int_{y_0}^{y} \left(\phi_1'(y)\right)^2 dy + W_0, \tag{3.38}$$

where W_0 is some constant of integration. Deriving the above relation it is assumed that $\phi_1(y)$ is an invertible function of y, therefore superpotential could be explicitly written as a function of y:

$$W(y) = \kappa^2 \left\{\tanh(\beta y) - \frac{1}{3}\tanh^3(\beta y)\right\} + W_0. \tag{3.39}$$

The integration constant W_0 can be fixed by initial conditions imposed upon $A'(y)$, e.g. such that $A'(y_{\max}) = 0$) for a given $y_{\max}$. The non-zero value of W_0 turns out to be essential to reproduce, in the brane limit, the generalized RS2 model presented in Sec. 2.3, whereas for $W_0 = 0$ the solution for $A(y)$ is symmetric under $y \leftrightarrow -y$ and it corresponds to the standard RS2 in the brane limit, see Sec. 2.1.2. It is instructive to write down explicitly the brane-limit results for the thick-brane scenario in order to determine necessary relations that must be satisfied to reproduce the RS2 relations (2.42) in the brane limit. As we will show below there is a direct relation between $W_0 \neq 0$ and the fact that $\Lambda_+ \neq \Lambda_-$, where $\Lambda_\pm$ are the bulk cosmological constants for $y \gtrless 0$ [2].

Let us consider only the scalar field part of the action:

$$\begin{aligned} S_{\phi_1} &= -\int dx^5 \sqrt{-g}\left\{\frac{1}{2}g^{MN}\nabla_M\phi_1\nabla_N\phi_1 + V(\phi_1)\right\} \\ &= -\int dx^5 \sqrt{-g}\left\{\left(\frac{\partial W(\phi_1)}{\partial \phi_1}\right)^2 - \frac{1}{6M_*^3}W^2(\phi_1)\right\} \\ &= -\int dx^5 \sqrt{-g}\left\{\frac{\beta\kappa^2}{\cosh^4(\beta y)} - \frac{1}{6M_*^3}\left[\kappa^2\left(\tanh(\beta y) - \frac{1}{3}\tanh^3(\beta y)\right) + W_0\right]^2\right\}. \end{aligned} \tag{3.40}$$

In the brane limit, i.e. $\beta \to \infty$ we have,

$$\lim_{\beta\to\infty}\left\{\frac{\beta}{\cosh^4(\beta y)}\right\} = \frac{4}{3}\delta(y),$$

such that the scalar action (3.40) can be written as,

$$S_{\phi_1} = -\int dx^5 \sqrt{-g}\left\{\frac{4}{3}\kappa^2\delta(y) + \Lambda_+\Theta(y) + \Lambda_-\Theta(-y)\right\}. \tag{3.41}$$

[2] After our work, presented in this subsection, has appeared in Ref. [3], another interesting study on asymmetric thick-brane has been publicized in Ref. [70].

The values of the bulk cosmological constants $\Lambda_\pm$ are

$$\begin{aligned}\Lambda_\pm &= \lim_{\beta\to\infty}\left\{-\frac{1}{6M_*^3}\left[\pm\kappa^2\left(\tanh(\beta y)-\frac{1}{3}\tanh^3(\beta y)\right)+W_0\right]^2\right\},\\ &= -\frac{1}{6M_*^3}\left(\frac{\lambda}{2}\pm W_0\right)^2, \qquad\qquad y \gtrless 0, \end{aligned}\tag{3.42}$$

and $\lambda \equiv 4\kappa^2/3$ corresponds to the brane tension. Hereafter, we will consider the case $-\Lambda_+ > -\Lambda_-$, that implies $W_0 > 0$. It is also important to note that Eq. (3.42) implies that the bulk cosmological constants $\Lambda_\pm$ are negative on either side leading to anti-de Sitter vacua or in the case with $W_0 = \lambda/2$ corresponding to a Minkowski geometry in that region of space. Equation (3.42) implies that in order to reproduce the generalized RS2 scenario defined by a given M_*, λ and $\Lambda_\pm$, the following constraints on the parameters (κ, W_0) of the thick-brane model must hold:

$$\kappa^2 = \frac{3}{4}\lambda, \qquad\qquad W_0 = \sqrt{\frac{3}{2}M_*^3}\left(\sqrt{-\Lambda_+}-\sqrt{-\Lambda_-}\right). \tag{3.43}$$

For consistency of the above choice for W_0, the following inequality must hold:

$$0 < W_0 < \frac{\lambda}{2}. \tag{3.44}$$

Therefore, only scenarios with limited splitting between cosmological constants could be realized:

$$\sqrt{6M_*^3}\left(\sqrt{-\Lambda_+}-\sqrt{-\Lambda_-}\right) < \lambda. \tag{3.45}$$

Then, for W_0 within the limit (3.44), Eq. (3.42) implies that

$$\lambda = \sqrt{6M_*^3}\left(\sqrt{-\Lambda_+}+\sqrt{-\Lambda_-}\right), \tag{3.46}$$

which is identical as the generalized RS2 relation (2.42). Note that for the $\mathbb{Z}_2$ symmetric case (the standard RS2 model) for which $\Lambda_+ = \Lambda_- = \Lambda_B$, we recover the RS2 relation between the brane tension and bulk cosmological constant, i.e. $\lambda = \sqrt{-24M_*^3\Lambda_B}$ (see Secs. 2.1.1 and 2.1.2) and $W_0 = 0$.

It is straightforward to calculate the warp function $A(y)$ by integrating the second equation in Eq. (3.33) w.r.t. y. The result reads,

$$A(y) = -\frac{\kappa^2}{72M_*^3\beta}\left(\tanh^2(\beta y)+\ln\cosh^4(\beta y)\right)-\frac{W_0}{12M_*^3}y. \tag{3.47}$$

The integration constant above was fixed by the condition $A(0) = 0$. As we have shown in (3.43) W_0 is fixed uniquely to a non-zero value, then as a consequence, in the smooth case the warp function $A(y)$ will not have maxima on the brane location, i.e. $y = 0$ but it will be shifted to a position $y_{\max}$, for instance for $M_* = 1$, $\kappa = 1$ and $W_0 = 0.5M_*^4$,

$$y_{\max} \sim -\frac{0.6}{\beta}. \tag{3.48}$$

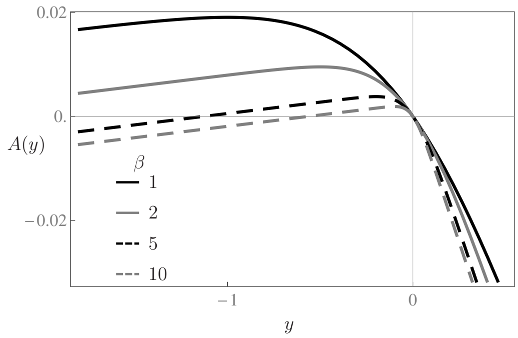


Figure 3.1.: This graph shows the behavior of $A(y)$ for different values of β showing the location of maxima for $A(y)$ for $W_0 = 0.5M_*^4$ and $M_* = \kappa = 1$.

It is worth noticing that even though $A'(0) \neq 0$, nevertheless the maxima of $A(y)$ approaches the brane location, i.e. $y_{\max} \to 0$ as $\beta \to \infty$, which is manifested from the above equation and is illustrated in Fig. 3.1. Note that far away from the thick-brane the warp function asymptotically approaches the generalized RS2 form as presented in Sec. 2.3,

$$A(y) \approx -k_\pm |y|, \qquad\qquad |y| \to \infty, \tag{3.49}$$

where

$$k_\pm = \frac{1}{24M_*^3}\lambda \pm \frac{W_0}{12M_*^3},$$

It is also important to note that one obtains the same behavior of $A(y)$ (3.49), for all values of y in the brane limit when $\beta \to \infty$, i.e.

$$A(y) \approx -k_\pm |y|, \qquad\qquad \beta \to \infty \quad \text{for} \quad y \gtrless 0.$$

Since $\phi_1(y)$ is invertible therefore we can write the superpotential $W(\phi_1)$ and the scalar potential $V(\phi_1)$ as follows:

$$W(\phi_1) = \kappa\sqrt{\beta}\phi_1 \left(1 - \frac{\beta}{3\kappa^2}\phi_1^2\right) + W_0, \tag{3.50}$$

$$\begin{aligned} V(\phi_1) =& \frac{\beta^3}{2\kappa^2}\left(\phi_1^2 - \frac{\kappa^2}{\beta}\right)^2 - \frac{1}{54M_*^3}\frac{\beta^3}{\kappa^2}\phi_1^2\left(\phi_1^2 - 3\frac{\kappa^2}{\beta}\right)^2 \\ &+ \frac{1}{9M_*^3}\frac{\beta^{3/2}}{\kappa}\phi_1\left(\phi_1^2 - 3\frac{\kappa^2}{\beta}\right)W_0 - \frac{1}{6M_*^3}W_0^2. \end{aligned} \tag{3.51}$$

Note that the constant term of superpotential W_0, in Eq. (3.50), plays the most crucial role in producing the asymmetry in the bulk cosmological constants and then in the warp function $A(y)$ on the left and the right of (thick) brane. In the left panel of Fig. 3.2 we have shown y-dependent shapes of $A(y)$ and $\phi_1(y)$, while in the right one $W(\phi_1)$ and $V(\phi_1)$ are plotted as functions of ϕ_1.

3.2.2. Double thick-brane model

In this subsection we consider two kinks corresponding to two thick-branes at locations $y = y_1$ and $y = y_2$. They are supposed to mimic two positive-tension branes in the brane limit, so the

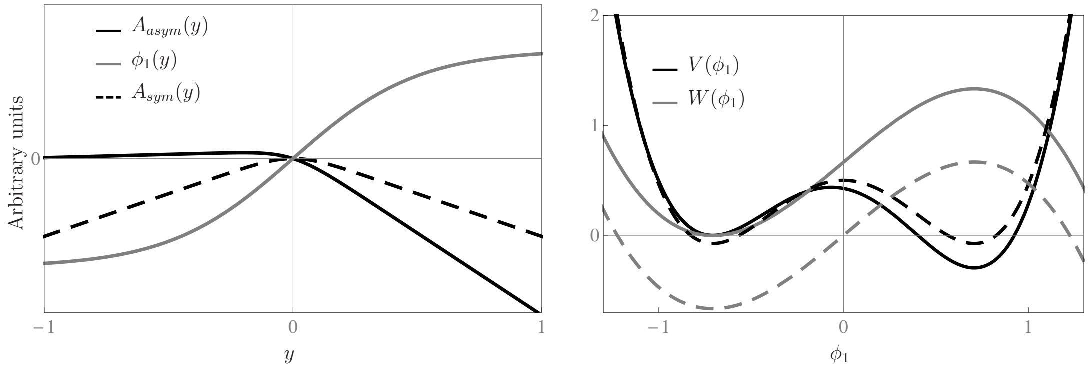


Figure 3.2.: This left graph shows the behavior of $A(y)$ and $\phi_1(y)$ as a function of y, whereas, the right graph presents the superpotential $W(\phi_1)$ and the potential $V(\phi_1)$ as a function of the scalar field ϕ_1. The solid curves correspond to $W_0 = 0.5M_*^4$ (asymmetric RS2 smooth model), whereas the dashed curves represent $W_0 = 0$ (standard RS2 smooth model), for $\beta = 2$ and $M_* = \kappa = 1$.

scalar profile $\phi_2(y)$ could be chosen as follows,

$$\phi_2(y) = \frac{\kappa_1}{\sqrt{\beta}} \tanh\big(\beta(y-y_1)\big) + \frac{\kappa_2}{\sqrt{\beta}} \tanh\big(\beta(y-y_2)\big), \tag{3.52}$$

where β is the thickness controlling parameter and $\kappa_{1,2}$ are the brane tension (strength) parameters. Using the superpotential method as described above the superpotential $W(\phi)$ could be written as a function of y as follows:

$$\begin{aligned} W(y) =& \kappa_1^2 \left(\tanh[\beta(y-y_1)] - \frac{1}{3}\tanh^3[\beta(y-y_1)]\right) \\ &+ \kappa_2^2 \left(\tanh[\beta(y-y_2)] - \frac{1}{3}\tanh^3[\beta(y-y_2)]\right) + W_0, \end{aligned} \tag{3.53}$$

where in deriving Eq. (3.53) we assume that the cross term is negligible as far as β is large and/or the separation "$y_2 - y_1$" between the two thick-branes is large such that,

$$\int dy \frac{2\beta\kappa_1\kappa_2}{\cosh^2\big(\beta(y-y_1)\big)\cosh^2\big(\beta(y-y_2)\big)} \approx 0$$

After obtaining the superpotential $W(y)$ we find from Eq. (3.33),

$$\begin{aligned} A'(y) = -\frac{1}{12M_*^3} \Bigg[& \kappa_1^2 \left(\tanh\big(\beta(y-y_1)\big) - \frac{1}{3}\tanh^3\big(\beta(y-y_1)\big)\right) \\ & + \kappa_2^2 \left(\tanh\big(\beta(y-y_2)\big) - \frac{1}{3}\tanh^3\big(\beta(y-y_2)\big)\right) + W_0 \Bigg]. \end{aligned} \tag{3.54}$$

The integration constant W_0 can be fixed by the requirement that $A(y)$ has a maximum at $y = y_0$. The location of maximum with respect to $y_{1,2}$ will correspond to different 5D geometric configurations that we will discuss in Sec.3.2.2. Therefore, we choose the integration constant W_0 as,

$$W_0 = -\Bigg[\kappa_1^2 \left(\tanh\big(\beta(y_{\max}-y_1)\big) - \frac{1}{3}\tanh^3\big(\beta(y_{\max}-y_1)\big)\right)$$

$$+\kappa_2^2\left(\tanh\left(\beta(y_{\max}-y_2)\right)-\frac{1}{3}\tanh^3\left(\beta(y_{\max}-y_2)\right)\right)\Bigg], \tag{3.55}$$

such that $A'(y_{\max}) = 0$. Now it is straightforward to find the warp factor $A(y)$ by integrating Eq. (3.54) w.r.t. y. The result reads,

$$\begin{aligned}A(y) =&\frac{1}{72M_*^3\beta}\left[\kappa_1^2\left(\frac{1}{\cosh^2\left(\beta(y-y_1)\right)}-\ln\cosh^4\left(\beta(y-y_1)\right)\right)\right.\\&\left.+\kappa_2^2\left(\frac{1}{\cosh^2\left(\beta(y-y_2)\right)}-\ln\cosh^4\left(\beta(y-y_2)\right)\right)\right]+\frac{1}{12M_*^3}W_0 y+A_0,\end{aligned} \tag{3.56}$$

where A_0 is a constant of integration which can be fixed by the requirement such that $A(y_{\max}) = 0$. Note that far away from the thick-branes the warp function is asymptotically AdS, i.e. of the RS form [22, 23]

$$A(y\to\infty)\sim -\kappa|y|, \qquad\qquad |y|\gg|y_1-y_2|, \tag{3.57}$$

where $\kappa = \frac{1}{24M_*^3}\left(\frac{4}{3}\kappa_1^2+\frac{4}{3}\kappa_2^2-W_0\right)$. It is easy to see that we get the same behavior, i.e. the RS form of $A(y)$, for all values of y in the brane limit when $\beta\to\infty$, which will be discussed in detail in the next section.

It should be mentioned that this set-up reduces to the usual single thick-brane discussed in the previous subsection if we assume that one of the branes is far away from the other brane, say $y_1 = 0$ and $y_2 \to \infty$ such that the second brane can be removed from the set-up and in that case $\kappa = \frac{1}{24M_*^3}\left(\frac{4}{3}\kappa_1^2\right)$. Unfortunately in the one-brane case the hierarchy problem remains unsolved, that is why we focus on the two-brane scenario as the hierarchy problem is one of the main motivations of this subsection. In our case the second thick-brane is located in the AdS space at suitable location y_2 such that the hierarchy problem could be addressed as in a similar way as it was done in the original RS1 set-up [22] or in the LR set-up [36].

Before closing this section it is instructive to discuss the shape of the scalar potential, that is determined by our requirement of having (3.52) together with the ansatz (1.2) as solutions of the equations of motion. Having the superpotential W determined, one can, using (3.32), find the scalar potential as a function of y. However, since $\phi(y)$ is an invertible function, therefore it is also possible to plot the potential $V(\phi)$ (3.32), as a function of ϕ. However, in order to develop some intuition, let us first consider the presence of just one-kink profile $\phi_1(y)$ (3.36) with parameters κ and β. Then the potential $V(\phi_1)$ could be determined analytically since one can easily solve equations of motion for $A(y)$, from the invertible profile one can find $y = y(\phi_1)$ and then adopt it, for instance, in (3.29). The results reads from Eq. (3.51) for $\mathbb{Z}_2$-symmetric case with $W_0 = 0$:

$$V(\phi_1)=\frac{\beta^3}{2\kappa^2}\left(\phi_1^2-\frac{\kappa^2}{\beta}\right)^2-\frac{1}{54M_*^3}\frac{\beta^3}{\kappa^2}\phi_1^2\left(\phi_1^2-3\frac{\kappa^2}{\beta}\right)^2. \tag{3.58}$$

Note that the above form of the potential applies, strictly speaking, only for $-\kappa/\sqrt{\beta} < \phi <$

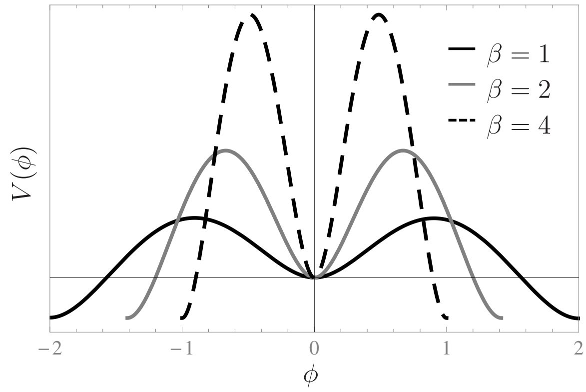


Figure 3.3.: The potential $V(\phi_2)$ plotted as a function of the scalar field ϕ_2 for different values of the thickness parameter β with the same brane tensions $\kappa_1 = \kappa_2 = 1$. Hereafter we assume $M_* = 1$, therefore the field strength is expressed in unites of $M_*^{3/2}$.

$\kappa/\sqrt{\beta}$ since the profile that is required to fulfill equations of motion varies in that range. For small field strengths, $\phi \lesssim M_*^{3/2}$, gravitational effects (the second term) are small, so that the dominant contribution is just the Mexican-hat potential (the first term). In the present case the scalar profile is a sum of two kinks (3.52), therefore its range of variation is roughly a "sum" of ranges for two separate kinks. In the absence of gravity each kink is a solution of equation of motion for a Mexican-hat type potential, again if a field strength is small comparing to the 5D Planck mass$^{3/2}$ the dominant contribution to the potential is the Mexican-hat component. Since kink profiles vary in between field strengths corresponding to the two minima therefore the shapes which we observe in Fig. 3.3 are roughly "sums" of inner parts (only the inner part could be determined) of two Mexican-hat like potentials; one centered around $-\kappa_1/(2\sqrt{\beta})$ and the other one around $+\kappa_2/(2\sqrt{\beta})$. In reality (with gravity) the picture is slightly distorted by the gravity effects that become relevant around the external ends of region of variation where $\phi/M_*^{3/2} \sim 1$.

In the case of double kink it is not possible to find the potential analytically, so $V(\phi)$ determined numerically is shown for several choices of the parameters β in Fig. 3.3. Since the strength of the profile field varies between $-(\kappa_1 + \kappa_2)/\sqrt{\beta}$ and $+(\kappa_1 + \kappa_2)/\sqrt{\beta}$ therefore the potential $V(\phi_2)$ can also be determined in that region only, which is manifest in Fig. 3.3. Note that in order to trust classical field theory results, the scalar field strength ϕ must be limited by the 5D Planck mass $M_*^{3/2}$, therefore we conclude that our results are consistent if $\beta \gtrsim k_{1,2}^2/M_*^3$. Since in Fig. 3.3 we assumed $\kappa_1 = \kappa_2 = 1$ and $M = 1$ therefore we are limited by $\beta \gtrsim 1$, so those cases were plotted and then the range of variation of ϕ_2 is appropriate.

The brane limit and the hierarchy problem

In this subsection we will consider different possible scenarios that could be realized with two thick-branes and then we will discuss limiting (the brane limit) solutions corresponding to thin (singular) branes. To show how scalar field ϕ_2 is making the two branes, we start by looking at its action calculated for the profile (3.52),

$$S_{\phi_2} = -\int dx^5 \sqrt{-g} \left\{ \frac{1}{2} g^{MN} \nabla_M \phi_2 \nabla_N \phi_2 + V(\phi_2) \right\},$$

$$
\begin{aligned}
= -\int dx^5 \sqrt{-g} \Bigg\{ & \frac{\beta\kappa_1^2}{\cosh^4\big(\beta(y-y_1)\big)} + \frac{\beta\kappa_2^2}{\cosh^4\big(\beta(y-y_2)\big)} \\
& -\frac{1}{6M_*^3}\Bigg[\kappa_1^2\Bigg(\tanh\big(\beta(y-y_1)\big) - \frac{1}{3}\tanh^3\big(\beta(y-y_1)\big)\Bigg) \\
& +\kappa_2^2\Bigg(\tanh\big(\beta(y-y_2)\big) - \frac{1}{3}\tanh^3\big(\beta(y-y_2)\big)\Bigg) + W_0\Bigg]^2\Bigg\}. \qquad (3.59)
\end{aligned}
$$

In the brane limit we have,

$$
\lim_{\beta\to\infty}\left\{\frac{\beta}{\cosh^4\big(\beta(y-y_i)\big)}\right\} = \frac{4}{3}\delta(y-y_i) \qquad \text{for} \qquad i=1,2
$$

such that the scalar action (3.59) can be written as,

$$
S_{\phi_2} = -\int dx^5\sqrt{-g}\left\{\frac{4}{3}\kappa_1^2\delta(y-y_1) + \frac{4}{3}\kappa_2^2\delta(y-y_2) + \Lambda_B(y)\right\}, \tag{3.60}
$$

where $\Lambda_B(y)$ is a function that generates cosmological constants in various regions of the bulk:

$$
\begin{aligned}
\Lambda_B(y) = \lim_{\beta\to\infty}\Bigg\{ & -\frac{1}{6M_*^3}\Bigg[\kappa_1^2\Bigg(\tanh\big(\beta(y-y_1)\big) - \frac{1}{3}\tanh^3\big(\beta(y-y_1)\big)\Bigg) \\
& +\kappa_2^2\Bigg(\tanh\big(\beta(y-y_2)\big) - \frac{1}{3}\tanh^3\big(\beta(y-y_2)\big)\Bigg) + W_0\Bigg]^2\Bigg\}, \\
= -\frac{1}{6M_*^3} & \left[\frac{2}{3}\kappa_1^2\,\mathrm{sgn}(y-y_1)) + \frac{2}{3}\kappa_2^2\,\mathrm{sgn}(y-y_2)) + W_0\right]^2. \qquad (3.61)
\end{aligned}
$$

Therefore, depending on the choice of the extremum location $y_{\max}$, different values of cosmological constant to the left, in between and to the right of the two branes could be generated.

In what follows we will find analytic solutions for the two positive tension branes in the brane limit $(\beta\to\infty)$ and numerical results for the corresponding thick-brane scenarios. In the brane limit, we can write the total action as,

$$
S_{\phi_2} = \int dx^5\sqrt{-g}\left\{2M_*^3 R - \lambda_1\delta(y-y_1) - \lambda_2\delta(y-y_2) - \Lambda_B(y)\right\}, \tag{3.62}
$$

where $\lambda_{1,2} = \frac{4}{3}\kappa_{1,2}^2$ are the respective brane tensions at each brane located at $y=y_1$ and $y=y_2$ and $\Lambda_B(y)$ is the bulk cosmological constant, defined in Eq. (3.61). In the brane limit we can obtain the equations of motion from action (3.62) as,

$$
24M_*^3\left(A'\right)^2 = -\Lambda_B, \tag{3.63}
$$

$$
12M_*^3A'' + 24M_*^3\left(A'\right)^2 = -\Lambda_B - \lambda_1\delta(y-y_1) - \lambda_2\delta(y-y_2), \tag{3.64}
$$

In the brane limit the smooth solution of $A'(y)$ (3.54) will take the following form,

$$
A'(y) = -\frac{1}{12M_*^3}\left[\frac{2}{3}\kappa_1^2\,\mathrm{sgn}(y-y_1) + \frac{2}{3}\kappa_2^2\,\mathrm{sgn}(y-y_2) + W_0\right]. \tag{3.65}
$$

From Eqs. (3.63) and (3.64), one gets (which is also manifested from Eq. (3.65)),

$$12M_*^3 A'' = -\lambda_1 \delta(y - y_1) - \lambda_2 \delta(y - y_2), \tag{3.66}$$

which implies that $A'' < 0$ as $\lambda_{1,2} > 0$, thus allowing for maxima of $A(y)$. Let us denote the location of the maxima by $y_{\max}$, that can be chosen anywhere along the extra dimension, such that,

$$A'(y_{\max}) = 0. \tag{3.67}$$

In the brane limit W_0 (3.55) takes the form,

$$W_0 = -\left[\frac{2}{3}\kappa_1^2 \,\mathrm{sgn}(y_{\max} - y_1) + \frac{2}{3}\kappa_2^2 \,\mathrm{sgn}(y_{\max} - y_2)\right]. \tag{3.68}$$

Therefore we obtain (in the brane limit) for the bulk cosmological constant, the following result

$$\Lambda_B(y) = \frac{-1}{6M_*^3}\left[\frac{2}{3}\kappa_1^2\Big(\mathrm{sgn}(y - y_1) - \mathrm{sgn}(y_{\max} - y_1)\Big) + \frac{2}{3}\kappa_2^2\Big(\mathrm{sgn}(y - y_2) - \mathrm{sgn}(y_{\max} - y_2)\Big)\right]^2. \tag{3.69}$$

It is also important to note that the equation of motion (3.63) implies that the bulk cosmological constant is negative leading to anti-de Sitter vacua or, in the case where it is zero, that the corresponding geometry will be Minkowski in that region of space.

In the following we will consider different cases depending on the location of the extremum point $y_{\max}$ along the extra dimension.

Case-I: We consider the case when the extremum location is on one of the branes, say at $y_{\max} = y_1$, we get the analytic results for $A'(y)$ as,

$$A'(y) = \begin{cases} \frac{1}{24M_*^3}\lambda_1 & y < y_1 \\ -\frac{1}{24M_*^3}\lambda_1 & y_1 < y < y_2 \\ -\frac{1}{24M_*^3}(\lambda_1 + 2\lambda_2) & y > y_2 \end{cases} \tag{3.70}$$

The corresponding bulk cosmological constant Λ_B, in different regions along the extra dimension, reads as,

$$\Lambda_B = \begin{cases} -\frac{1}{24M_*^3}\lambda_1^2 & y < y_1 \\ -\frac{1}{24M_*^3}\lambda_1^2 & y_1 < y < y_2 \\ -\frac{1}{24M_*^3}(\lambda_1 + 2\lambda_2)^2 & y > y_2 \end{cases}, \tag{3.71}$$

and the values of bulk cosmological constant at the brane locations are $\Lambda_B(y_1) = 0$ and $\Lambda_B(y_2) = -\frac{1}{12M_*^3}\left(2\lambda_2^2 + \lambda_1\lambda_2\right)$. Note that (3.71) implies in the brane limit correlations between the brane tensions λ_i and the bulk cosmological constant Λ_B. It is worth to write down relations between Λ_B, the slope of the warp function A' and the brane tensions λ_i in the asymptotic

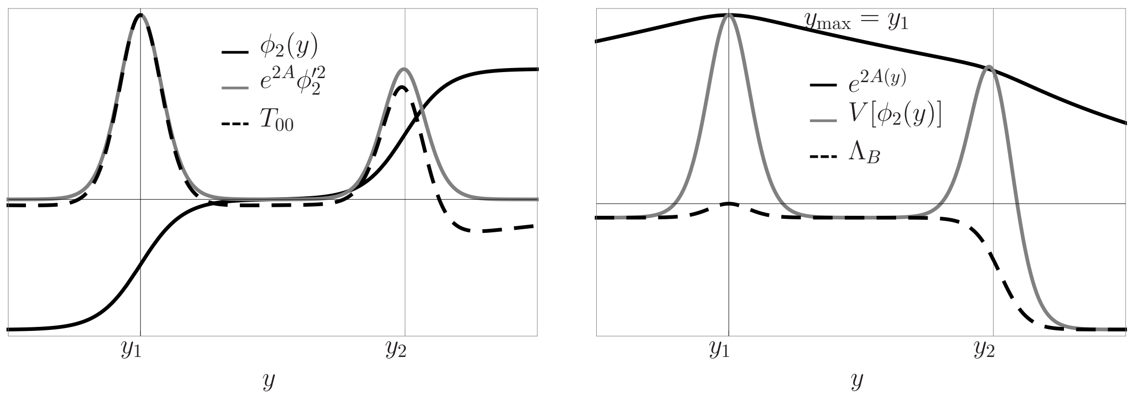


Figure 3.4.: The left-graph shows the shape of the scalar field profile, the corresponding energy density T_{00} and also the localized energy density $e^{2A}\phi'^2$ for the Case-I with $\beta = 2$ and $\kappa_1 = \kappa_2 = 1$, whereas, the right-graph illustrates the shape of the warped factor e^{2A}, the bulk cosmological constant Λ_B and the potential $V[\phi(y)]$ for the case-I when the maxima of the warp factor is located on the thick-brane, i.e. $y_{\max} = y_1$.

regions:

$$\frac{\Lambda_B}{A'} = \begin{cases} -\lambda_1 & y \to -\infty \\ \lambda_1 + 2\lambda_2 & y \to +\infty \end{cases} \tag{3.72}$$

In the RS2 (with one brane of positive tension λ) the corresponding relation is $\Lambda_B/k = \pm\lambda$, where k corresponds to $\pm A'$.

Numerical solutions for the thick-branes in the Case I are shown in Fig. 3.7. This configuration is such that the warping function $A(y)$ is positive (negative) to the left (right) of the branes, we will see that this scenario will have the normalizable zero-modes of the metric tensor perturbations (around the background solution) which correspond to the 4D graviton. In the brane limit this set-up is similar to the two positive D3-brane model considered by Lykken and Randall in [36].

In order to illustrate how are the branes generated, in Fig. 3.7 (left panel) we show the energy density $T_{00} = e^{2A}\big(\phi'^2/2 + V(\phi)\big)$ corresponding to the profile. Using equations of motion T_{00} could be rewritten as $e^{2A}(\phi'^2 - 24M_*^3A'^2)$. This form separates two contributions to T_{00}: local (one that "creates" the branes) $e^{2A}\phi'^2$ and non-local $\propto e^{2A}A'^2$ (one that generates bulk cosmological constants).

As it is seen from Fig. 3.7 (right panel) to the left and to the right of the branes the warp factor is quickly vanishing, that has been already observed in (3.57). If the branes are sufficiently thin (or well separated) then in between them the warping is also nearly exponential so that the hierarchy problem could be addressed. We will call the brane located at y_1 and y_2 as UV and IR branes, respectively. To illustrate consequences of the warped background geometry lets assume that the Higgs field is bounded at the IR brane and its action can be written as,

$$S_{\text{Higgs}} = -\int d^4x\sqrt{-g_{IR}}\left\{g_{IR}^{\mu\nu}\partial_\mu H^\dagger\partial_\nu H - m^2|H|^2 + \lambda|H|^4\right\}, \tag{3.73}$$

where $g^{\mu\nu}_{IR}$ is the 4D metric induced on the IR brane, $g^{\mu\nu}_{IR} = e^{-2A(y_2)}\eta^{\mu\nu}$, with $A(y_2)$ being the value of warped factor at the IR brane and m is the 5D Higgs mass parameter (of the order of 5D Planck mass). Now the effective 4D action for the Higgs field can be written as,

$$S_{\rm Higgs} = -\int d^4x \left\{ e^{2A(y_2)}\eta^{\mu\nu}\partial_\mu H^\dagger \partial_\nu H - m^2 e^{4A(y_2)}|H|^2 + \lambda e^{4A(y_2)}|H|^4 \right\}, \tag{3.74}$$

where we used the fact that, $\sqrt{-g_{IR}} = e^{4A(y_2)}$. In order to obtain canonically normalized Higgs field, we rescale, $H \to e^{-A}H$, such that,

$$\begin{aligned} S_{\rm Higgs} &= -\int d^4x \left\{ \eta^{\mu\nu}\partial_\mu H^\dagger \partial_\nu H - m^2 e^{2A(y_2)}|H|^2 + \lambda |H|^4 \right\}, \\ &= -\int d^4x \left\{ \eta^{\mu\nu}\partial_\mu H^\dagger \partial_\nu H - \mu^2 |H|^2 + \lambda |H|^4 \right\}, \end{aligned} \tag{3.75}$$

where $\mu = m e^{A(y_2)}$ is the effective Higgs mass parameter as viewed on the IR brane. If we assume that the fundamental mass scale of the 5D theory is the Planck mass then we can require the value of warped factor at IR brane such that we get the effective 4D Higgs mass parameter $\mu \sim$ TeV. From Eq. (3.70) we have

$$A(y_2) = -\frac{1}{24M_*^3}\left(\lambda_1 + \lambda_2\right) y_2, \tag{3.76}$$

therefore if $\frac{1}{24M_*^3}\left(\lambda_1 + \lambda_2\right) y_2 \sim 30$ then the hierarchy problem could be solved. Furthermore, as it will be shown in Sec. 3.4.3, in this scenario (i.e. $y_{\rm max} = y_1$) there exits a normalizable zero-mode which corresponds to the 4D graviton.

It should be emphasized that the scenario of solving the hierarchy problem described above *assumes* that the Higgs field H could be localized on the IR brane. The issue of localization of a generic scalar field on thick-brane will be discussed in Sec. 3.3, see also e.g. [42, 69, 71, 72, 73, 74, 75].

Case-II: Lets now consider the case when the extremum position is in between the two thick-branes such that $y_1 < y_{\rm max} < y_2$, in that case corresponding values of $A'(y)$ (3.65) in the brane limit are given by,

$$A'(y) = \begin{cases} \frac{1}{12M_*^3}\lambda_1 & y < y_1 \\ 0 & y_1 < y < y_2 \\ -\frac{1}{12M_*^3}\lambda_2 & y > y_2 \end{cases} \tag{3.77}$$

This situation is interesting since the 4D graviton is now normalizable and it is localized in between the two positive branes. However in this scenario the hierarchy problem can not be solved since there is no warping in between the two branes which is manifest in Fig. 3.5-(left panel). The two positive branes set-up is now similar to the single brane RS-2 [23].

Case-III: Now we consider the scenario with extremum located to the left of the left brane or to the right of the right brane, so $y_{\rm max} < y_1$ or $y_{\rm max} > y_2$. For the case when the extremum

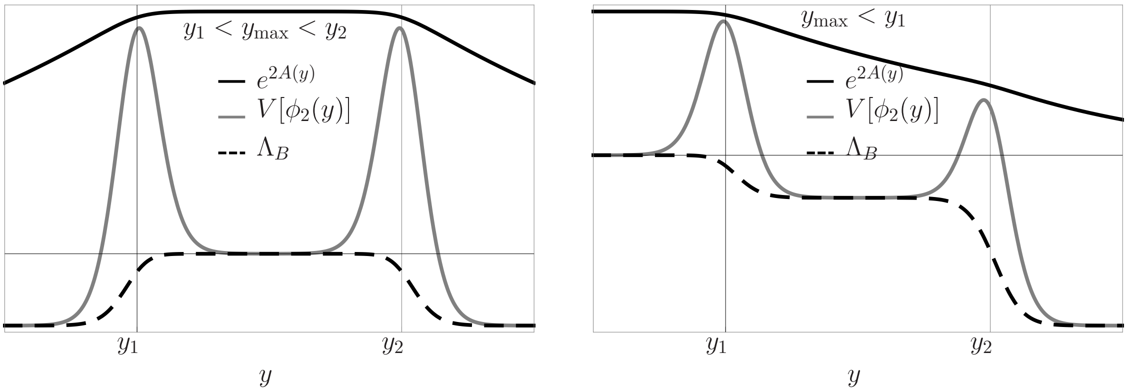


Figure 3.5.: The left-graph shows the warped factor e^{2A}, the bulk cosmological constant Λ_B and the potential $V[\phi(y)]$ for $y_1 < y_{\max} < y_2$ while the right-graph for $y_{\max} < y_1$. Parameters chosen: $\beta = 2$ and $\kappa_1 = \kappa_2 = 1$.

lies to the left of y_1, in the brane limit, $A'(y)$ (3.65) is given by,

$$A'(y) = \begin{cases} 0 & y < y_1 \\ -\frac{1}{12M_*^3}\lambda_1 & y_1 < y < y_2 \\ -\frac{1}{12M_*^3}(\lambda_1 + \lambda_2) & y > y_2 \end{cases}, \tag{3.78}$$

In this case we have Minkowski background to left of the brane located at y_1 which could be called the UV brane, however to the right of this brane we have the warped geometry, so that the hierarchy problem could be approached in the same way as it was discussed for the case-I. It is worth mentioning that similar geometrical configuration was considered by Gregory, Rubakov, Sibiryakov (GRS) [37] with singular branes. The important difference is that GRS model have one positive and one negative tension D3-brane while in our case the both branes are made out of scalar field which mimics two positive tension branes in the brane limit. Numerical results for corresponding thick-branes are shown in Fig. 3.5-(right panel). We would like to comment here that even though this scenario addresses the hierarchy problem, it does not have the normalizable 4D graviton, for details see Sec. 3.4.3. Similarly the other possibility could be considered when the extremum position is to the right of the brane located at y_2, i.e. with $y_{\max} > y_2$, then in the brane limit the $A'(y)$ is given by,

$$A'(y) = \begin{cases} \frac{1}{12M_*^3}(\lambda_1 + \lambda_2) & y < y_1 \\ \frac{1}{12M_*^3}\lambda_2 & y_1 < y < y_2 \\ 0 & y > y_2 \end{cases}, \tag{3.79}$$

So in this case geometry to the right of the brane at y_2 is Minkowski.

In Table 3.1 we summarize results for the cosmological constant in the brane limit, the regions I, II and III are defined as $y < y_1$, $y_1 < y < y_2$ and $y > y_2$, respectively.

Location of $y_{\max}$	Region-I	Region-II	Region-III
$y_1 < y_{\max} < y_2$	$-\frac{1}{6M_*^3}\lambda_1^2$	0	$-\frac{1}{6M_*^3}\lambda_2^2$
$y_{\max} < y_1$	0	$-\frac{1}{6M_*^3}\lambda_1^2$	$-\frac{1}{6M_*^3}\left(\lambda_1+\lambda_2\right)^2$
$y_{\max} > y_2$	$-\frac{1}{6M_*^3}\left(\lambda_1+\lambda_2\right)^2$	$-\frac{1}{6M_*^3}\lambda_2^2$	0

Table 3.1.: The bulk cosmological constant Λ_B when $y_{\max} \neq y_{1,2}$.

In all the above cases, in the brane limit, the $A'(y)$ have discontinuities at the brane locations $y = y_1$ and $y = y_2$. The discontinuities (or *jump*) are as follows

$$[A']_{y_1} = -\frac{1}{9M_*^3}\kappa_1^2 = -\frac{1}{12M_*^3}\lambda_1 \qquad y = y_1, \tag{3.80}$$

$$[A']_{Y_2} = -\frac{1}{9M_*^3}\kappa_2^2 = -\frac{1}{12M_*^3}\lambda_2 \qquad y = y_2. \tag{3.81}$$

A jump in $A'(y)$ implies that $A''(y)$ have delta-like singularity, which is consistent with the equation of motion (3.64), in fact one can also obtain the above jump conditions by integrating Eq. (3.64) from $y_i - \epsilon$ to $y_i + \epsilon$ and then matching the coefficients of delta functions.

3.2.3. Triple $\mathbb{Z}_2$-symmetric thick-brane model

In this subsection we consider a scalar field profile such that it corresponds to three $\mathbb{Z}_2$ symmetric thick-branes at locations $y = y_{UV}$ and $y = \pm y_{IR}$. They are supposed to mimic three positive-tension branes in the brane limit and the gravity would be localized at $y = y_{UV}$, whereas if we assume that the SM fields in particular Higgs field is localized at $y = \pm y_{IR}$ such that the distance is tuned in such a way that one can solve the hierarchy problem as we have shown in the two thick-brane case. This $\mathbb{Z}_2$ symmetric thick-brane setup mimics the IR-UV-IR model presented in Sec. 2.2 but there are some key differences which we enlist at the end of this subsection. To construct a triple thick-brane scenario the scalar profile $\phi_3(y)$ could be chosen as,

$$\phi_3(y) = \frac{\kappa_{UV}}{\sqrt{\beta}}\tanh\left(\beta(y-y_{UV})\right) + \frac{\kappa_{IR}}{\sqrt{\beta}}\tanh\left(\beta(y-y_{IR})\right) + \frac{\kappa_{IR}}{\sqrt{\beta}}\tanh\left(\beta(y+y_{IR})\right), \tag{3.82}$$

where β is the thickness controlling parameter and $\kappa_{UV,IR}$ are the brane tension (strength) parameters. Using the superpotential method as described above the superpotential $W(\phi)$ could be written as a function of y as follows:

$$\begin{aligned} W[\phi_3(y)] = &\ \kappa_{UV}^2\left(\tanh[\beta(y-y_{UV})] - \frac{1}{3}\tanh^3[\beta(y-y_{UV})]\right) \\ &+ \kappa_{IR}^2\left(\tanh[\beta(y-y_{IR})] - \frac{1}{3}\tanh^3[\beta(y-y_{IR})]\right) \\ &+ \kappa_{IR}^2\left(\tanh[\beta(y+y_{IR})] - \frac{1}{3}\tanh^3[\beta(y+y_{IR})]\right), \end{aligned} \tag{3.83}$$

where in deriving Eq. (3.83) we assume that the cross terms are negligible as far as β is large and/or the separation "$L \equiv |y_{IR} - y_{UV}|$" between the two thick-branes is large such that,

$$\int dy \frac{2\beta\kappa_{UV}\kappa_{IR}}{\cosh^2\left(\beta(y-y_{UV})\right)\cosh^2\left(\beta(y\pm y_{IR})\right)} \approx 0$$

After obtaining the superpotential $W(y)$ we have $A'(y)$ from Eq. (3.33), i.e.

$$A'(y) = -\frac{1}{12M_*^3} W[\phi_3(y)]. \tag{3.84}$$

The integration constant is fixed by the requirement that $W(y_{UV}) = 0$, i.e. $A(y)$ has a maximum at $y = y_{UV}$, which implies that the graviton zero-mode is localized at $y = y_{UV}$. The warp factor $A(y)$ can be found from the above equation by integrating Eq. (3.84) w.r.t. y, i.e.

$$\begin{aligned} A(y) = &\frac{1}{72M_*^3\beta}\left[\kappa_{UV}^2\left(\frac{1}{\cosh^2\left(\beta(y-y_{UV})\right)} - \ln\cosh^4\left(\beta(y-y_{UV})\right)\right)\right. \\ &+\kappa_{IR}^2\left(\frac{1}{\cosh^2\left(\beta(y-y_{IR})\right)} - \ln\cosh^4\left(\beta(y-y_{IR})\right)\right) \\ &\left.+\kappa_{IR}^2\left(\frac{1}{\cosh^2\left(\beta(y+y_{IR})\right)} - \ln\cosh^4\left(\beta(y+y_{IR})\right)\right)\right], \end{aligned} \tag{3.85}$$

where the integration constant is fixed by the requiring that $A(y_{UV}) = 0$. Note that far away from the thick-branes the warp function is asymptotically AdS, i.e. like RS form [22, 23]

$$A(y\to\infty) \sim -\kappa|y|, \qquad\qquad |y| \gg L, \tag{3.86}$$

where $\kappa = \frac{1}{24M_*^3}\left(\frac{4}{3}\kappa_{UV}^2 + \frac{8}{3}\kappa_{IR}^2\right)$. It is easy to see that we get the same behavior, i.e. the RS form of $A(y)$, for all values of y in the brane limit when $\beta\to\infty$, which discuss below.

To understand how the above scalar field $\phi_3(y)$ profile is dynamically generating three positive tension branes, let us take the brane limit of the scalar field action, i.e.

$$\begin{aligned} S_{\phi_3} &= -\lim_{\beta\to\infty}\int dx^5\sqrt{-g}\left\{\frac{1}{2}g^{MN}\nabla_M\phi_3\nabla_N\phi_3 + V(\phi_3)\right\}, \\ &= -\int dx^5\sqrt{-g}\left\{\lambda_{UV}\delta(y-y_{UV}) + \lambda_{IR}\delta(y-y_{IR}) + \lambda_{IR}\delta(y+y_{IR}) + \Lambda_B(y)\right\}, \end{aligned} \tag{3.87}$$

where $\lambda_{UV,IR} = \frac{4}{3}\kappa_{UV,IR}^2$ are the respective brane tensions at each brane located at $y = y_{UV}$ and $y = y_{IR}$ and $\Lambda_B(y)$ is the bulk cosmological constant given by:

$$\Lambda_B(y) = \frac{-1}{24M_*^3}\Big[\lambda_{UV}\,\mathrm{sgn}(y-y_{UV})) + \lambda_{IR}\,\mathrm{sgn}(y-y_{IR})) + \lambda_{IR}\,\mathrm{sgn}(y+y_{IR}))\Big]^2. \tag{3.88}$$

It is worth showing how the thick-branes are generated, in left panel of Fig. 3.6 we have plotted, scalar field profile $\phi_3(y)$, the energy density $T_{00} = e^{2A}\left(\phi_3'^2/2 + V(\phi)\right)$ corresponding to the profile and the localized energy density $e^{2A}\phi_3'^2$. The right panel of Fig. 3.6 illustrates the

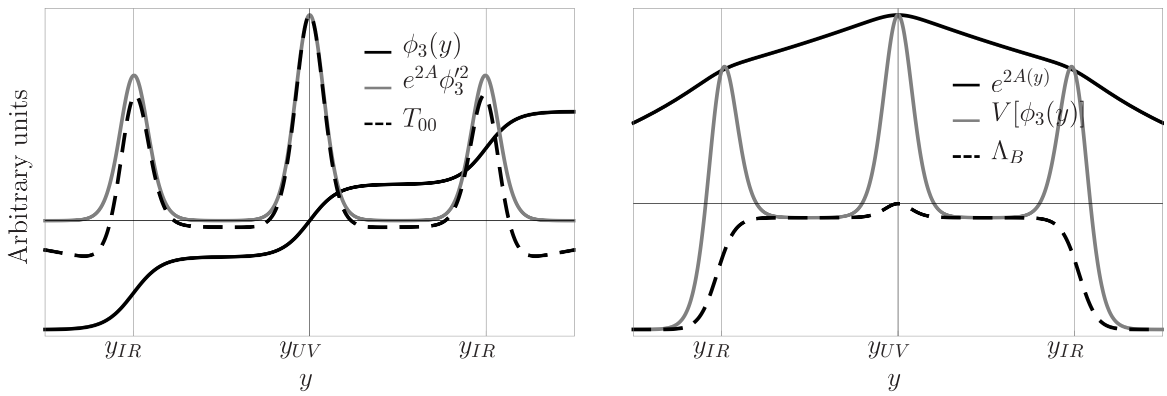


Figure 3.6.: The left-graph shows the shape of the scalar field profile, the corresponding energy density T_{00} and also the localized energy density $e^{2A}\phi_3'^2$ for the Case-I with $\beta = 2$ and $\kappa_1 = \kappa_2 = 1$, whereas, the right-graph illustrates the shape of the warped factor e^{2A}, the bulk cosmological constant Λ_B and the potential $V[\phi_3(y)]$ for the case-I when the maxima of the warp factor is located on the thick-brane, i.e. $y_{\max} = y_1$.

shape of the warp factor e^{2A}, scalar potential $V[\phi_3(y)]$ and the shape of bulk energy density (cosmological constant) $\Lambda_B(y)$.

In what follows we will find analytic solutions for the three positive tension branes in the brane limit $\big(\beta \to \infty\big)$ and we take $y_{UV} = 0$. In the brane limit the smooth solution of $A'(y)$ (3.84) will take the following form,

$$\begin{aligned} A'(y) &= -\frac{1}{24M_*^3}\Big[\lambda_{UV}\,\mathrm{sgn}(y) + \lambda_{IR}\,\mathrm{sgn}(y - y_{IR}) + \lambda_{IR}\,\mathrm{sgn}(y + y_{IR})\Big], \\ &= -\frac{1}{24M_*^3}\begin{cases} \lambda_{UV}\,\mathrm{sgn}(y) & |y| \le L \\ (\lambda_{UV} + 2\lambda_{IR})\,\mathrm{sgn}(y) & |y| > L \end{cases}, \end{aligned} \tag{3.89}$$

where L is the distance between the UV and IR branes. The explicit form of the warp function $A(y)$ in the brane limit:

$$A(y) = -\frac{1}{24M_*^3}\begin{cases} \lambda_{UV}|y| & |y| \le L \\ (\lambda_{UV} + 2\lambda_{IR})|y| & |y| > L \end{cases}, \tag{3.90}$$

It is important comment here that the above $\mathbb{Z}_2$ symmetric triple thick-brane model resembles the IR-UV-IR model presented in Sec. 2.2 but there are some key differences between the two scenarios:

- First, the IR-UV-IR model presented in Sec. 2.2 is an interval from $-L \le y \le L$, whereas the $\mathbb{Z}_2$ symmetric thick-brane has infinite extra dimension, i.e. $y \to \pm\infty$.
- Second, the IR-UV-IR model presented in Sec. 2.2 has negative tensions branes located at end points of the interval, whereas in the brane limit the thick-brane model mimics the positive tension branes.
- Third, the IR-UV-IR model of Sec. 2.2 has a *mass gap* as it is compactified, whereas the thick-brane counterpart presented here has no mass gap so it is not clear how to construct a viable low energy effective theory in this setup.

3.2.4. Dilatonic thick-brane

In the previous subsections we have adopted the strategy of constructing a thick-brane model by considering a particular kink-like profile of the scalar field and then using the superpotential method to find the warp-function and the scalar potential. In this subsection we will briefly present an example, which will be relevant for the next chapter in the context of thick-brane cosmology, where we start with a particular form of the superpotential $W(\phi)$ and then using Eqs. (3.32) and (3.33) we find the scalar field profile, warp-function and the scalar potential. We focus on the following form of *dilatonic superpotential* $W(\phi)$:

$$W(\phi) = W_0 \exp\left[\frac{-\gamma}{4M*^{3/2}}\phi\right], \tag{3.91}$$

where W_0 and γ are constants and, as we will see, the different values of dimensionless constant γ will correspond to classes of different solutions. The resulting scalar potential $V(\phi)$ is,

$$V(\phi) = \frac{W_0^2}{4M_*^3}\left(\frac{\gamma^2}{8} - \frac{2}{3}\right)\exp\left[\frac{-\gamma}{2M*^{3/2}}\phi\right]. \tag{3.92}$$

The scalar field $\phi(y)$ and the warped function $A(y)$ obtained from Eq. (3.33) are

$$\phi(y) = -\frac{4M_*^{3/2}}{\gamma}\ln\left(1 + \frac{\gamma^2}{16M_*^3}W_0 y\right), \tag{3.93}$$

$$A(y) = \frac{16M_*^3}{3\gamma^2}\ln\left(1 + \frac{\gamma^2}{16M_*^3}W_0 y\right). \tag{3.94}$$

The above result (3.92)-(3.93) and (3.94) represents a class of solutions parameterized by γ. For $\gamma = \pm\sqrt{\frac{8}{3}}$ along with $4M_*^3 = 1$, we recover our results found in Eqs. (4.82)-(4.83) and (4.84) for $W_0 = 3b_0$. Whereas, for $\gamma = \pm\sqrt{\frac{4}{3}}$ we recover the linear dilaton solution discussed by Antoniadis et al. [76]. It is instructive to notice that the metric given by Eq. (11) of Ref. [76] coincides with (3.94) for $\alpha = W_0$.

3.2.5. Generalized thick-branes

In this subsection we consider a general case for the background scalar field. We are going to show that even without a priori defined shape of the scalar field profile, the thin brane generalized RS2 relation (2.42) between the brane tension λ and the bulk cosmological constants $\Lambda_\pm$ is reproduced in the brane limit under certain mild assumptions. In other words the relation is independent of the function adopted to regularize (smooth) a thin brane. For this purpose we consider the following general form of the scalar background field,

$$\phi(y) = \frac{\phi_0(\beta y)}{\sqrt{\beta}}, \tag{3.95}$$

where β will turn out to be the thickness controlling parameter. *We assume that $\phi_0(\beta y)$ is monotonic, and $\left(\sqrt{\beta}\phi_0'(\beta y)\right)^2$ is an integrable function of y* [3]. We employ the superpotential

[3] It is interesting to notice that this condition is equivalent to the normalizability of one of the two scalar zero modes (spin zero fluctuations around the background solution (3.34) and (3.56)) related to the shift along the

method described above.

Let us consider the scalar field action

$$\begin{aligned} S_\phi &= -\int dx^5\sqrt{-g}\left\{\frac{1}{2}g^{MN}\nabla_M\phi\nabla_N\phi + V(\phi)\right\} \\ &= -\int dx^5\sqrt{-g}\left\{\left(\frac{\partial W(\phi)}{\partial\phi}\right)^2 - \frac{1}{6M_*^3}W^2(\phi)\right\} \\ &= -\int d^5x\sqrt{-g}\left\{\phi'^2 - \frac{1}{6M_*^3}\left(\int_0^y \left(\phi'(\bar{y})\right)^2 d\bar{y} + W_0\right)^2\right\}, \end{aligned} \tag{3.96}$$

where $V(\phi)$ and $W(\phi)$ are obtained from Eqs. (3.32) and (3.33), respectively. Since W_0 is an arbitrary integration constant the lower integration limit could be chosen at $\bar{y} = 0$ without compromising generality. After using equation (3.95) and changing variables from $\tilde{y} \to \beta\bar{y}$ one gets

$$S_\phi = -\int d^5x\sqrt{-g}\left\{\frac{1}{\beta}\left(\phi_0'(\beta y)\right)^2 - \frac{1}{6M_*^3}\left(\int_0^{\beta y}\left(\phi_0'(\tilde{y})\right)^2 d\tilde{y} + W_0\right)^2\right\}. \tag{3.97}$$

From the above scalar field action, one finds that in the brane limit, i.e. $\beta \to \infty$:

- The integrand $\left(\phi_0'(\beta y)\right)^2/\beta$ converges to zero everywhere except $y = 0$ (as the function is integrable) therefore the first term above approaches $-\lambda\delta(y)$, with

  $$\lambda = \int_{-\infty}^{+\infty}\left(\phi_0'(\tilde{y})\right)^2 d\tilde{y},$$

  where $\delta(y)$ is the Dirac delta function.

- The second term converges to a sum of contributions to bulk cosmological constants $-\Lambda_+\Theta(y) - \Lambda_-\Theta(-y)$, where

$$\Lambda_+ = -\frac{1}{6M_*^3}\left(\int_0^{+\infty}\left(\phi_0'(\tilde{y})\right)^2 d\tilde{y} + W_0\right)^2 \tag{3.98}$$

$$\Lambda_- = -\frac{1}{6M_*^3}\left(-\int_{-\infty}^{0}\left(\phi_0'(\tilde{y})\right)^2 d\tilde{y} + W_0\right)^2. \tag{3.99}$$

Equations (3.98)-(3.99) imply that in order to reproduce the generalized RS2 relation (2.42) the following inequality must hold

$$\sqrt{6M_*^3}\left(\sqrt{-\Lambda_+} - \sqrt{-\Lambda_-}\right) < \lambda. \tag{3.100}$$

Note that this is the same condition that was limiting the splitting between the cosmological constants which was obtained in Sec. 3.2.1. Therefore we conclude that regardless what is the choice of the scalars profile, only those thin brane models could be obtained in the brane limit

extra dimension $y \to y + \text{const.}$, for more details see [1].

for which (3.100) is satisfied. It is easy to see that if W_0 is chosen as

$$W_0 = \sqrt{\frac{3}{2}M_*^3}\left(\sqrt{-\Lambda_+} - \sqrt{-\Lambda_-}\right) + \frac{1}{2}\left(\int_{-\infty}^{0}\left(\phi_0'(y)\right)^2 dy - \int_{0}^{+\infty}\left(\phi_0'(y)\right)^2 dy\right), \tag{3.101}$$

then indeed

$$\lambda = \sqrt{6M_*^3}\left(\sqrt{-\Lambda_+} + \sqrt{-\Lambda_-}\right). \tag{3.102}$$

Thus we recover the result (2.42) for our generalized RS2 model. It is worth to rephrase the above result as follows. For any given thin brane model to be reproduced in the brane limit and any profile of the scalar field $\phi_0(y)$ (monotonic with $\left(\phi_0'(y)\right)^2$ integrable), the Eq. (3.101) provides the choice of the integration constant W_0 which guaranties that the condition (2.42) holds.

In the case of the kink-like profile considered in Sec. 3.2.1, $\left(\phi_0'(y)\right)^2$ was an even function of y therefore W_0 reduces to the value adopted in (3.43). Of course, if we limit ourself to the Z_2-symmetric case, W_0 must vanish as in order to mimic standard RS2.

3.3. Localization of a scalar field on a thick-brane

As we have seen in the previous section in order to solve the hierarchy problem within thick-brane scenarios one needs to localized the Higgs field on the thick-brane. In this section we will investigate the issue of localization of a scalar field $\xi(y)$ on top of a bulk scalar field (kink-like profile) $\phi(y)$ which mimics the branes. We consider a scalar field localization on the thick-brane through the localized interactions with the scalar field which mimics the branes. Since the scalar field $\xi(y)$ can also contribute to the background solution so we must take into account the back reaction of the scalar field $\xi(y)$. The action for the proposed setup can be written as,

$$\mathcal{S} = \int dx^5\sqrt{-g}\left\{2M_*^3 R - \frac{1}{2}f(\xi)g^{MN}\nabla_M\phi\nabla_N\phi - \frac{1}{2}g^{MN}\nabla_M\xi\nabla_N\xi - V(\phi,\xi)\right\}, \tag{3.103}$$

where $f(\xi)$ is a general function of ξ, later we will limit ourselves to the particular form this function. The Einstein's equations and the equation of motion for ϕ and ξ, resulting from the action (3.103) are

$$R_{MN} - \frac{1}{2}g_{MN}R = \frac{1}{4M_*^3}T_{MN}, \tag{3.104}$$

$$f(\xi)\nabla^2\phi + \frac{\partial f(\xi)}{\partial\xi}\xi'\phi' - \frac{\partial V}{\partial\phi} = 0, \tag{3.105}$$

$$\nabla^2\xi - \frac{1}{2}\left(\nabla\phi\right)^2\frac{\partial f(\xi)}{\partial\xi} - \frac{\partial V}{\partial\xi} = 0, \tag{3.106}$$

where ∇^2 is 5D covariant d'Alambertion operator and the energy-momentum tensor T_{MN} for the scalar field $\phi(y)$ is,

$$T_{MN} = f(\xi)\nabla_M\phi\nabla_N\phi + \nabla_M\xi\nabla_N\xi - g_{MN}\left[\frac{1}{2}f(\xi)(\nabla\phi)^2 + \frac{1}{2}(\nabla\xi)^2 + V(\phi,\xi)\right]. \tag{3.107}$$

From the Einstein equations (3.104), (3.105) and (3.106), one can get the following equations of motion for the metric ansatz (1.2)

$$24M_*^3(A')^2 = \frac{1}{2}f(\xi)(\phi')^2 + \frac{1}{2}(\xi')^2 - V(\phi,\xi), \tag{3.108}$$

$$12M_*^3A'' + 24M_*^3(A')^2 = -\frac{1}{2}f(\xi)(\phi')^2 - \frac{1}{2}(\xi')^2 - V(\phi,\xi), \tag{3.109}$$

$$f(\xi)\left(\phi'' + 4A'\phi'\right) + \frac{\partial f(\xi)}{\partial \xi}\xi'\phi' - \frac{\partial V}{\partial \phi} = 0. \tag{3.110}$$

$$\xi'' + 4A'\xi' - \frac{1}{2}\phi'^2\frac{\partial f(\xi)}{\partial \xi} - \frac{\partial V}{\partial \xi} = 0. \tag{3.111}$$

We assume that the scalar potential $V(\phi,\xi)$ could be expressed in terms of the superpotential $W(\phi,\xi)$ as [25],

$$V(\phi,\xi) = \frac{1}{2}\frac{1}{f(\xi)}\left(\frac{\partial W(\phi,\xi)}{\partial\phi}\right)^2 + \frac{1}{2}\left(\frac{\partial W(\phi,\xi)}{\partial\xi}\right)^2 - \frac{1}{6M_*^3}W(\phi,\xi)^2, \tag{3.112}$$

where $W(\phi,\xi)$ satisfies the following relations,

$$\phi' = \frac{1}{f(\xi)}\frac{\partial W(\phi,\xi)}{\partial\phi}, \qquad \xi' = \frac{\partial W(\phi,\xi)}{\partial\xi}, \quad \text{and} \quad A' = -\frac{1}{12M_*^3}W(\phi,\xi). \tag{3.113}$$

To see how this superpotential method or the choice of the potential $V(\phi,\xi)$ (3.112) satisfies the set of the equations of motion (3.108)-(3.111), let us consider these equations one by one. The first equation (3.108) is just the definition of the potential (3.112) and to see how the second equation (3.109) is satisfied one can subtract the Eq. (3.108) from Eq. (3.109) and get,

$$12M_*^3A'' = -f(\xi)(\phi')^2 - (\xi')^2. \tag{3.114}$$

The above equation is identically satisfied by the virtue of Eq. (3.113). Now to see if the above potential and superpotential choices can satisfy the equation of motion for the scalar field $\phi(y)$ (3.110), we take the partial derivative of the $V(\phi,\xi)$ (3.112) w.r.t. $\phi(y)$ as,

$$\begin{aligned}\frac{\partial V}{\partial\phi} &= \frac{\partial}{\partial\phi}\left(\frac{1}{2}\frac{1}{f(\xi)}\left(\frac{\partial W(\phi,\xi)}{\partial\phi}\right)^2 + \frac{1}{2}\left(\frac{\partial W(\phi,\xi)}{\partial\xi}\right)^2 - \frac{1}{6M_*^3}W(\phi,\xi)^2\right),\\ &= \frac{1}{f(\xi)}\frac{\partial W}{\partial\phi}\frac{\partial^2 W}{\partial\phi^2} + \frac{\partial W}{\partial\xi}\frac{\partial^2 W}{\partial\phi\partial\xi} - \frac{1}{3M_*^3}W\frac{\partial W}{\partial\phi},\\ &= f(\xi)\left(\phi'' + 4A'\phi'\right) + \frac{\partial f(\xi)}{\partial\xi}\xi'\phi',\end{aligned} \tag{3.115}$$

where in the last line of the above equation we have used constraints that superpotential $W(\phi,\xi)$ have to satisfy from Eq. (3.113) and also the fact that,

$$\begin{aligned}\phi'' &= \frac{d}{dy}\left(\frac{1}{f(\xi)}\frac{\partial W(\phi,\xi)}{\partial\phi}\right),\\ &= \left[\phi'\frac{\partial}{\partial\phi} + \xi'\frac{\partial}{\partial\xi}\right]\left(\frac{1}{f(\xi)}\frac{\partial W(\phi,\xi)}{\partial\phi}\right),\end{aligned}$$

$$= \frac{1}{f(\xi)}\frac{\partial^2 W(\phi,\xi)}{\partial\phi^2}\phi' + \frac{1}{f(\xi)}\frac{\partial^2 W(\phi,\xi)}{\partial\xi\partial\phi}\xi' - \frac{1}{f(\xi)}\frac{\partial f(\xi)}{\partial\xi}\xi', \tag{3.116}$$

such that the first two terms of second line in Eq. (3.115) are,

$$\frac{1}{f(\xi)}\frac{\partial W}{\partial\phi}\frac{\partial^2 W}{\partial\phi^2} + \frac{\partial W}{\partial\xi}\frac{\partial^2 W}{\partial\phi\partial\xi} = f(\xi)\phi'' + \frac{\partial f(\xi)}{\partial\xi}\xi'. \tag{3.117}$$

Equation (3.115) shows that the choice of potential $V(\phi,\xi)$ (3.112) and superpotential $W(\phi,\xi)$, satisfying the constraints in Eq. (3.113), satisfies the equation of motion for the scalar field ϕ. Similarly below we will repeat the same procedure to show that the potential $V(\phi,\xi)$ (3.112) and superpotential $W(\phi,\xi)$ satisfies the equation of motion for the scalar field $\xi(y)$ (3.111). To show that we take the partial derivative of the $V(\phi,\xi)$ (3.112) w.r.t. $\xi(y)$ as,

$$\begin{aligned}
\frac{\partial V}{\partial\xi} &= \frac{\partial}{\partial\xi}\left(\frac{1}{2}\frac{1}{f(\xi)}\left(\frac{\partial W(\phi,\xi)}{\partial\phi}\right)^2 + \frac{1}{2}\left(\frac{\partial W(\phi,\xi)}{\partial\xi}\right)^2 - \frac{1}{6M_*^3}W(\phi,\xi)^2\right), \\
&= \frac{1}{f(\xi)}\frac{\partial W}{\partial\phi}\frac{\partial^2 W}{\partial\phi\partial\xi} - \frac{1}{2}\frac{1}{f^2(\xi)}\frac{\partial f(\xi)}{\partial\xi}\left(\frac{\partial W}{\partial\phi}\right)^2 + \frac{\partial W}{\partial\xi}\frac{\partial^2 W}{\partial\xi^2} - \frac{1}{3M_*^3}W\frac{\partial W}{\partial\xi}, \\
&= \xi'' + 4A'\xi' - \frac{1}{2}\frac{\partial f(\xi)}{\partial\xi}\phi'^2,
\end{aligned} \tag{3.118}$$

where,

$$\begin{aligned}
\xi'' &= \frac{d}{dy}(\xi') = \frac{d}{dy}\left(\frac{\partial W(\phi,\xi)}{\partial\xi}\right) = \left[\phi'\frac{\partial}{\partial\phi} + \xi'\frac{\partial}{\partial\xi}\right]\left(\frac{\partial W(\phi,\xi)}{\partial\phi}\right), \\
&= \frac{\partial^2 W(\phi,\xi)}{\partial\xi\partial\phi}\phi' + \frac{\partial^2 W(\phi,\xi)}{\partial\xi^2}\xi', \\
&= \frac{1}{f(\xi)}\frac{\partial W}{\partial\phi}\frac{\partial^2 W}{\partial\phi\partial\xi} + \frac{\partial W}{\partial\xi}\frac{\partial^2 W}{\partial\xi^2},
\end{aligned} \tag{3.119}$$

such that we get the last line in Eq. (3.118). Hence we have shown that the superpotential method works elegantly and it solves all the equations of motion as long as the superpotential $W(\phi,\xi)$ fulfills the conditions defined in Eq. (3.113).

Now we can make an ansatz for the superpotential $W(\phi,\xi)$ such that our scalar field $\phi(y)$ can have a kink-like solutions. For this purpose we can make the following ansatz for the $W(\phi,\xi)$,

$$W(\phi,\xi) = W(\phi)f(\xi), \tag{3.120}$$

where,

$$W(\phi) = \kappa\sqrt{\beta}\phi\left(1 - \frac{\beta}{3\kappa^2}\phi^2\right). \tag{3.121}$$

For simplicity, we assume that $f(\xi)$ is quadratic in ξ field, such that,

$$f(\xi) = 1 - \frac{1}{2}\alpha\xi^2, \tag{3.122}$$

where α is a constant $\alpha \in [0,1]$. It is important to note that $\alpha = 0$ is the case when there is

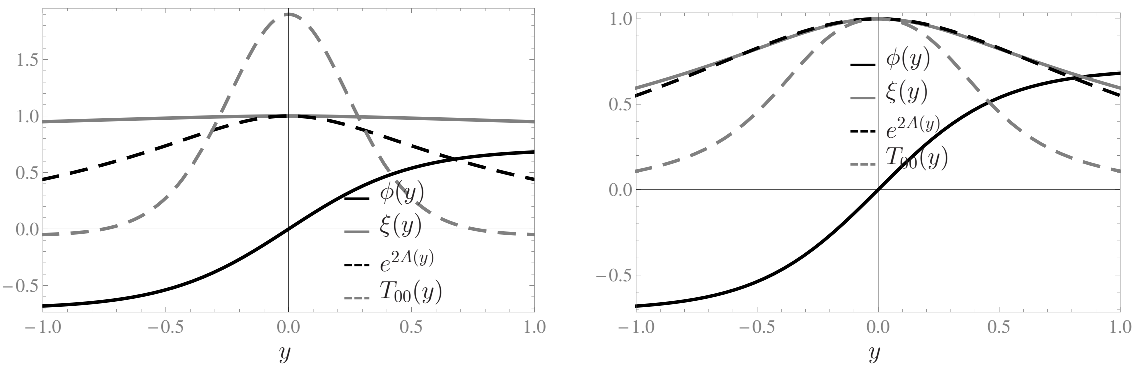


Figure 3.7.: The left graph shows the profiles of the scalar fields $\phi(y)$, $\xi(y)$, superpotential $W(\phi,\xi)$ and potential $V(\phi,\xi)$ as a function of y for $\alpha = 0$ (weak localization), whereas, the right graph shows the same but for $\alpha = 1$ (strong localization) with $\kappa = M_* = 1$ and $\beta = 2$.

no direct coupling of the scalar field $\xi(y)$ with the kink $\phi(y)$, where as the $\alpha = 1$ sets a strong coupling between the two scalar fields. Hence the superpotential is given by,

$$W(\phi) = \kappa\sqrt{\beta}\phi\left(1 - \frac{\beta}{3\kappa^2}\phi^2\right)\left(1 - \frac{1}{2}\alpha\xi^2\right), \tag{3.123}$$

Now we can determine the forms of the scalar fields $\phi(y)$ and $\xi(y)$ from Eq. (3.113) as:

$$\phi(y) = \frac{\kappa}{\sqrt{\beta}}\tanh(\beta y), \tag{3.124}$$

$$\xi(y) = \xi_0 e^{-\alpha\int dy W(\phi)} = \xi_0 e^{-\alpha\frac{\kappa^2}{6\beta}\left(\ln\cosh^4(\beta y) + \tanh^2(\beta y)\right)}, \tag{3.125}$$

where ξ_0 is the value of $\xi(y)$ at $y = 0$. For the superpotential defined in Eq. (3.123) one can write the explicit form of the warped function $A(y)$ after integrating Eq. (3.123) (we fixed the integration constant such that $A(0) = 0$),

$$A(y) = -\frac{\kappa^2}{72M_*^3\beta}\left(\tanh^2(\beta y) + \ln\cosh^4(\beta y)\right) - \frac{1}{48}\left[e^{-\frac{\alpha\kappa^2}{3\beta}\left(\tanh^2(\beta y) + \ln\cosh^4(\beta y)\right)} - 1\right]. \tag{3.126}$$

The profiles of the scalar fields $\phi(y)$ and $\xi(y)$ and shapes of the superpotential $W(\phi,\xi)$ and the potential $V(\phi,\xi)$ are plotted as a function of y shown in Fig. 3.7 for $\alpha = 0, 1$. The form of potential $V(\phi,\xi)$ for the given superpotential $W(\phi,\xi)$ (3.123) is given by,

$$\begin{aligned} V(\phi,\xi) = {} & \frac{\beta^3}{2\kappa^2}\left(\frac{\kappa^2}{\beta} - \phi^2\right)^2\left(1 - \frac{1}{2}\alpha\xi^2\right) + \frac{1}{9}\frac{\alpha^2\beta^3}{\kappa^2}\phi^2\xi^2\left(3\frac{\kappa^2}{\beta} - \phi^2\right)^2 \\ & - \frac{1}{54M_*^3}\frac{\beta^3}{\kappa^2}\phi^2\left(3\frac{\kappa^2}{\beta} - \phi^2\right)^2\left(1 - \frac{1}{2}\alpha\xi^2\right)^2. \end{aligned} \tag{3.127}$$

3.4. Stability of the background solutions

In this section our focus is to show that the scalar, vector and tensor perturbations in the context of thick-brane models do not lead to destabilization of the setup. We will show that the 4D effective gravity on the thick-brane could be recovered and the background solutions found earlier for different thick-brane models are stable. We will adopt most the general results from Appendix A in this section.

3.4.1. Scalar perturbations

The linearized field equations corresponding to the scalar modes of the perturbation are given by Eqs. (A.56), (A.58) and (A.59), i.e.

$$6A'\partial_\mu\psi + 3\partial_\mu\psi' = \frac{1}{4M_*^3}\phi'\partial_\mu\varphi, \tag{3.128}$$

$$e^{-2A}\Box^{(4)}\varphi + \varphi'' + 4A'\varphi' - \frac{\partial^2 V(\phi)}{\partial\phi^2}\varphi - 6\phi'\psi' - 4\frac{\partial V(\phi)}{\partial\phi}\psi = 0, \tag{3.129}$$

$$\psi'' + 2A'\psi' - e^{-2A}\Box^{(4)}\psi = \frac{1}{6M_*^3}\phi'\varphi'. \tag{3.130}$$

One can integrate Eq. (3.128) over x-coordinates and get the following equation,

$$6A'\psi + 3\psi' = \frac{1}{4M_*^3}\phi'\varphi, \tag{3.131}$$

where we have put the y-dependent integration constant to zero by the requirement that the perturbations vanish at 4D infinities. It is more convenient to use the conformal frame where the metric can be written as in Eq. (2.46), i.e.

$$ds^2 = e^{2A(z)}\left(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2\right), \tag{3.132}$$

with z defined through $dz = e^{-A(y)}dy$. Hence, in the new coordinates our equations of motion (3.129)-(3.131) take the following form,

$$\Box^{(4)}\varphi + \ddot{\varphi} + 3\dot{A}\dot{\varphi} - e^{2A}\frac{\partial^2 V(\phi)}{\partial\phi^2}\varphi - 6\dot{\phi}\dot{\psi} - 4e^{2A}\frac{\partial V(\phi)}{\partial\phi}\psi = 0, \tag{3.133}$$

$$\ddot{\psi} + \dot{A}\dot{\psi} - \Box^{(4)}\psi = \frac{1}{6M_*^3}\dot{\phi}\dot{\varphi}, \tag{3.134}$$

$$2\dot{A}\psi + \dot{\psi} = \frac{1}{12M_*^3}\dot{\phi}\varphi, \tag{3.135}$$

where the *overdot* represents derivative w.r.t. z-coordinate. First we solve Eq. (3.135) with respect to φ and calculate $\dot{\varphi}$ as,

$$\dot{\varphi} = \frac{12M_*^3}{\dot{\phi}^2}\left[\left(2\ddot{A}\psi + 2\dot{A}\dot{\psi} + \ddot{\psi}\right)\dot{\phi} - \left(2\dot{A}\psi + \dot{\psi}\right)\ddot{\phi}\right], \tag{3.136}$$

and then use it in (3.134), so that we obtain an equation only for ψ,

$$\ddot{\psi} + \left(3\dot{A} - 2\frac{\ddot{\phi}}{\dot{\phi}}\right)\dot{\psi} + \left(4\ddot{A} - 4\dot{A}\frac{\ddot{\phi}}{\dot{\phi}} + \Box^{(4)}\right)\psi = 0. \tag{3.137}$$

To convert this equation into the Schrödinger form it is instructive to remove the first derivative terms of the perturbation ψ, to do so we redefine the scalar perturbations as,

$$\psi(x,z) = e^{-\frac{3}{2}A(z)}\dot{\phi}\tilde{\psi}(x,z). \tag{3.138}$$

Then the linearized field equation for the scalar perturbation ψ takes the following form,

$$-\ddot{\tilde{\psi}} + \left[\frac{9}{4}\dot{A}^2 - \frac{5}{2}\ddot{A} + \dot{A}\frac{\ddot{\phi}}{\dot{\phi}} + 2\left(\frac{\ddot{\phi}}{\dot{\phi}}\right)^2 - \frac{\dddot{\phi}}{\dot{\phi}}\right]\tilde{\psi} = \Box^{(4)}\tilde{\psi}. \tag{3.139}$$

We can further decompose the $\tilde{\psi}(x,z)$ into $\tilde{\psi}(x,z) = \hat{\psi}(x)\bar{\psi}(z)$, where $\hat{\psi}(x) = e^{ipx}$ is a z-independent plane wave such that $\Box^{(4)}\hat{\psi}(x) = m^2\hat{\psi}(x)$, with $-p^2 = m^2$ being the 4D KK mass of the fluctuation. So, with this field decomposition Eq. (3.139) can be written as,

$$-\ddot{\bar{\psi}}(z) + \left[\frac{9}{4}\dot{A}^2 - \frac{5}{2}\ddot{A} + \dot{A}\frac{\ddot{\phi}}{\dot{\phi}} + 2\left(\frac{\ddot{\phi}}{\dot{\phi}}\right)^2 - \frac{\dddot{\phi}}{\dot{\phi}}\right]\bar{\psi}(z) = m^2\bar{\psi}(z). \tag{3.140}$$

The properties of this equation has also been explored in the past in the context of stability and dynamics of radion in [43, 56, 57] and [58]. To develop some intuition concerning this equations it is convenient to rewrite it in supersymmetric quantum mechanics form. For this purpose we introduce an auxiliary function $\alpha(z)$ defined by

$$\alpha(z) \equiv \frac{e^{\frac{3}{2}A(z)}\dot{\phi}(z)}{\dot{A}(z)}. \tag{3.141}$$

Now we can write the potential of the above equation in the following form,

$$U_\psi(z) = \left[\frac{9}{4}\dot{A}^2 - \frac{5}{2}\ddot{A} + \dot{A}\frac{\ddot{\phi}}{\dot{\phi}} + 2\left(\frac{\ddot{\phi}}{\dot{\phi}}\right)^2 - \frac{\dddot{\phi}}{\dot{\phi}}\right] = \alpha(z)\partial_z^2\left(\frac{1}{\alpha(z)}\right) = \omega^2(z) - \dot{\omega}(z), \tag{3.142}$$

where $\omega(z) \equiv \frac{\dot{\alpha}(z)}{\alpha(z)}$. Then we can rewrite the Eq. (3.140) in a supersymmetric quantum mechanics form as,

$$\begin{aligned} -\partial_z^2\bar{\psi} + \left(\omega^2(z) - \dot{\omega}(z)\right)\bar{\psi} &= m^2\bar{\psi} \\ \mathcal{A}^\dagger\mathcal{A}\bar{\psi} &= m^2\bar{\psi}, \end{aligned} \tag{3.143}$$

where the operator $\mathcal{A}^\dagger$ and $\mathcal{A}$ are defined as,

$$\mathcal{A}^\dagger = \left(-\partial(z) + \omega(z)\right), \qquad \mathcal{A} = \left(\partial(z) + \omega(z)\right). \tag{3.144}$$

The above supersymmetric form of the scalar perturbation equation (3.143) guarantee that there is no solution for $\bar{\psi}$ with $m^2 < 0$, hence the fluctuation ψ can not destabilize the background solution. The zero-mode for the scalar perturbation $\bar{\psi}(z)$ can be obtained from (3.143) as,

$$\bar{\psi}_0(z) = \frac{1}{\alpha(z)} = \frac{\dot{A}(z)}{e^{\frac{3}{2}A(z)}\dot{\phi}(z)}. \tag{3.145}$$

It is important to note that the zero-mode for the scalar perturbation $\bar{\psi}(z)$ is not normalizable. As one can easily see from the above expression, $\bar{\psi}_0$ diverges when $z \to \infty$ (then $\dot{\phi}(z) \to 0$ and also $e^{\frac{3}{2}A(z)} \to 0$ for the scenarios considered in Sec. 3.2). Hence $\psi_0(x,z) = e^{-\frac{3}{2}A(z)}\dot{\phi}\hat{\psi}(x)\bar{\psi}_0(z) = \hat{\psi}(x)e^{-3A(z)}\dot{A(z)}$ is not normalizable.

Although our main concern here is to verify the stability, nevertheless it is worth to check the behavior of the potential at $z \to \pm\infty$ in order to see whether there is a mass gap in the spectrum of scalar modes. From (3.142) one can easily find the explicit form of the potential as a function of z

$$U_\psi(z) = e^{2A[y(z)]}\left[-\frac{7}{2}A'' + \frac{3}{4}A'^2 + 2A'\frac{\phi''}{\phi'} + 2\left(\frac{\phi''}{\phi'}\right)^2 - \frac{\phi'''}{\phi'}\right]_{y=y(z)}, \tag{3.146}$$

where y as a function of z could be determined from

$$\int_{y_{\max}}^{y} e^{-A(y')}dy' = z(y) - z(y_{\max}). \tag{3.147}$$

If we limit ourself to the large y region and the integration constant $y_{\max}$ is large enough, we can use the asymptotic behavior of $A = A(y)$ as in (3.57) then we find

$$y(z) \sim \frac{1}{\kappa}\ln(\kappa z + \text{const.}) \tag{3.148}$$

where $\kappa = \frac{1}{24M_*^3}\left(\frac{4}{3}\kappa_1^2 + \frac{4}{3}\kappa_2^2 - W_0\right)$. From (3.146) we find that $\lim_{z\to\pm\infty} U_\psi(z) = 0$ [4], therefore we conclude that the spectrum is continuous starting at $m^2 = 0$.

It is worth to comment on another possible zero mode solution. The theory that we are discussing here is invariant with respect to a shift along the extra dimension: $y \to y + \epsilon$, therefore if a given metric $g_{MN}(x,y)$ and a scalar field $\phi(x,y)$ are solutions of equations of motion, then so are $g_{MN}(x, y+\epsilon)$ and $\phi(x, y+\epsilon)$. Expanding them around $\epsilon = 0$ one obtains

$$\begin{aligned} g_{MN}(x,y+\epsilon) &= g_{MN}(x,y) + g_{MN}(x,y)'\epsilon + \cdots \\ \phi(x,y+\epsilon) &= \phi(x,y) + \phi'(x,y)\epsilon + \cdots, \end{aligned} \tag{3.149}$$

where ellipsis stand for higher powers in ϵ. Since $g_{MN}(x,y+\epsilon)$ and $\phi(x,y+\epsilon)$ and also $g_{MN}(x,y)$ and $\phi(x,y)$ satisfy the equations of motion, therefore $g_{MN}(x,y)'$ and $\phi'(y)$ satisfy linearized equations of motion. In our parameterizations of the perturbations, (A.17)-(A.19), that corresponds to

$$\psi(x,y) = -A'(y), \qquad \varphi(x,y) = \phi'(y) \qquad \text{and} \qquad B = E = \chi = 0 \tag{3.150}$$

[4] Of course, $\lim_{y\to\pm\infty} U_\psi[z(y)] = 0$, as well.

As ψ and φ given by (3.150) correspond to modifications of the field configuration (that satisfies the equations of motion) along the symmetry directions therefore it is supposed to be a zero mode. Indeed, it could be verified explicitly that ψ and φ given by (3.150) satisfy linearized Einstein equations (A.55)-(A.57) together with the scalar field equation of motion (A.16). It should be emphasized that in this case the relation $\partial_\mu\partial_\nu\left(2\psi-\chi\right)=0$ does not hold by the virtue of $2\psi-\chi=0$, but by the fact that $\psi(x,y)$ is x-independent while $\chi=0$. We will not consider those modes any more since they do not depend on x and therefore can not be localized in 4D.

3.4.2. Vector perturbations

The field equation obtained for the transverse vector mode of the perturbation, after integrating Eq. (A.63) w.r.t. x-coordinate, is,

$$\Box^{(4)}C_\mu=0, \qquad C'_\mu+2A'C_\mu=0, \tag{3.151}$$

where we have set the integration constant to zero by using the fact that perturbations should be localized in 4D so that they do vanish far away from sources. It is more intuitive to write the vector perturbation in the conformal coordinates so that the results can be interpreted easily. Therefore, in the conformal frame the equations of motion for the vector modes of the perturbation take the form,

$$\Box^{(4)}C_\mu=0, \qquad \dot{C}_\mu+3\dot{A}C_\mu=0. \tag{3.152}$$

One can immediately notice from Eqs. (3.152) that the vector modes of perturbations are massless.

Since the Eq. (3.152) is first order in z-derivatives so it can not be put into an elegant Schrödinger like form as for the case of tensor and scalar modes. Therefore to see if these modes are localized or not we have to find canonical normal modes of these perturbations from the second order perturbation of the action [56], the result reads:

$$\delta^2\mathcal{S}_V=\int d^5x\frac{1}{2}\left(\eta^{\mu\nu}\partial_\mu\tilde{C}^\alpha\partial_\nu\tilde{C}_\alpha\right), \tag{3.153}$$

where, $\tilde{C}_\mu=e^{\frac{3}{2}A}C_\mu$ corresponds to the canonical normal mode. From Eq. (3.152), one finds that $C_\mu(x,z)=\hat{C}(x)e^{-3A}$. So the canonical normal zero-mode of the vector perturbation can be given as,

$$\tilde{C}_\mu=e^{-\frac{3}{2}A}\hat{C}_\mu(x), \tag{3.154}$$

where $\hat{C}_\mu$ satisfies the equation $\Box^{(4)}\hat{C}_\mu(x)=0$. As we will show in the next subsection, the requirement of reproducing the General Relativity at low energies we had

$$M_{\rm Pl}^2=M_*^3\int dze^{3A(z)}. \tag{3.155}$$

Therefore the canonical normal vector modes can not be localized since the integral $\int dze^{-3A(z)}$ must be divergent (as a consequence of the finiteness of the 4D Planck mass). Hence, the vector modes of the perturbation are not localized and therefore they do not affect issue of stability.

3.4.3. Tensor perturbations and localization of gravity

In order to illustrate stability for tensor modes of our background solutions discussed in the previous section, we will use most of the general results obtained in Sec. 2.4. The only difference will be that here we will use the warp-function $A(y)$ obtained for the thick-brane models.

We employ the tensor perturbations of the form given by Eq. (2.43) and then after changing to the conformal coordinates the tensor perturbation follow equation of motion (2.47), i.e.

$$\left(\partial_z^2 + 3\dot{A}(z)\partial_z + \Box^{(4)}\right) H_{\mu\nu}(z) = 0. \tag{3.156}$$

To put the above equation in the Schrödinger equation form we redefine of the tensor fluctuation as

$$\tilde{H}_{\mu\nu}(x,z) = e^{3A(z)/2} H_{\mu\nu}(x,z), \tag{3.157}$$

which transforms the Eq. (3.156) into the form of the Schrödinger equation,

$$\left(\partial_z^2 - \frac{9}{4}\dot{A}^2(z) - \frac{3}{2}\ddot{A}(z) + \Box^{(4)}\right) \tilde{H}_{\mu\nu}(x,z) = 0. \tag{3.158}$$

Similarly to Sec. 2.4 we can KK-decompose the $\tilde{H}_{\mu\nu}(x,z)$ into the x and z dependent parts as:

$$\tilde{H}_{\mu\nu}(x,z) = \sum_n \hat{H}_{n\mu\nu}(x)\bar{H}_n(z), \tag{3.159}$$

where we consider the 4D plane wave solutions for $\hat{H}_{n\mu\nu}(x)$, i.e. $\hat{H}_{n\mu\nu}(x) \propto e^{ip_n x}$ such that $\Box^{(4)}\hat{H}_{n\mu\nu}(x) = m_n^2 \hat{H}_{n\mu\nu}(x)$, with $-p_n^2 = m_n^2$ being the 4D KK mass of the tensor mode.

Recalling the results from Sec. 2.4, the zero-mode wave-function for tensor perturbation $\bar{H}_0(z)$ reads:

$$\bar{H}_0(z) = e^{\frac{3}{2}A(z)}. \tag{3.160}$$

In Fig. 3.8 we have plotted the zero-mode wave-function of tensor perturbations (3.160) and the Schrödinger-like potential $\mathcal{V}(z)$ Eq. (2.52), i.e.

$$\mathcal{V}(z) = \frac{9}{4}\dot{A}^2(z) + \frac{3}{2}\ddot{A}(z). \tag{3.161}$$

for the single thick-brane (a)symmetric warp-function $A[y(z)]$ Eq. (3.47) for $W_0 = 0$ and $W_0 = 0.5M_*^4$. Figure 3.9 shows the plots of the zero-mode wave-function of tensor perturbations $\bar{H}_0(z)$ and the Schrödinger-like potential $\mathcal{V}(z)$ for the double thick-brane with warp-function $A[y(z)]$ given by Eq. (3.56) for different cases of Sec. 3.2.2.

For massive KK modes we need to solve Eq. (2.51) with $m^2 \neq 0$. For large z the potential

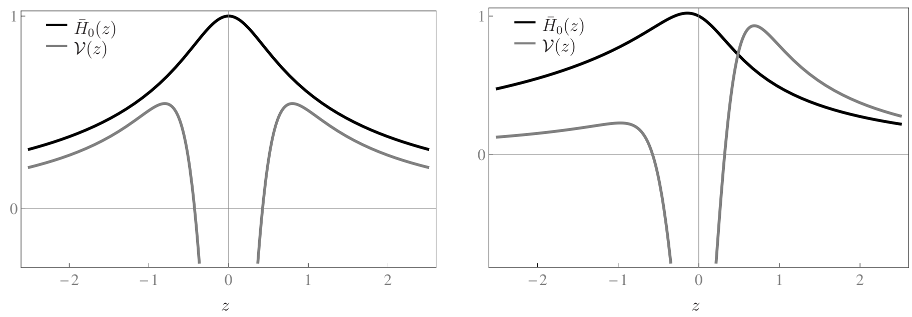


Figure 3.8.: The left graph shows the behavior of the zero-mode for tensor perturbations $\bar{H}_0(z)$ and the Schrödinger-like potential $\mathcal{V}(z)$ as a function of z for the $\mathbb{Z}_2$ symmetric case $W_0 = 0$, whereas, the right graph shows the same for asymmetric case with $W_0 = 0.5M_*^4$ for $M_* = 1$, $\beta = 2$ and $\kappa = 3$.

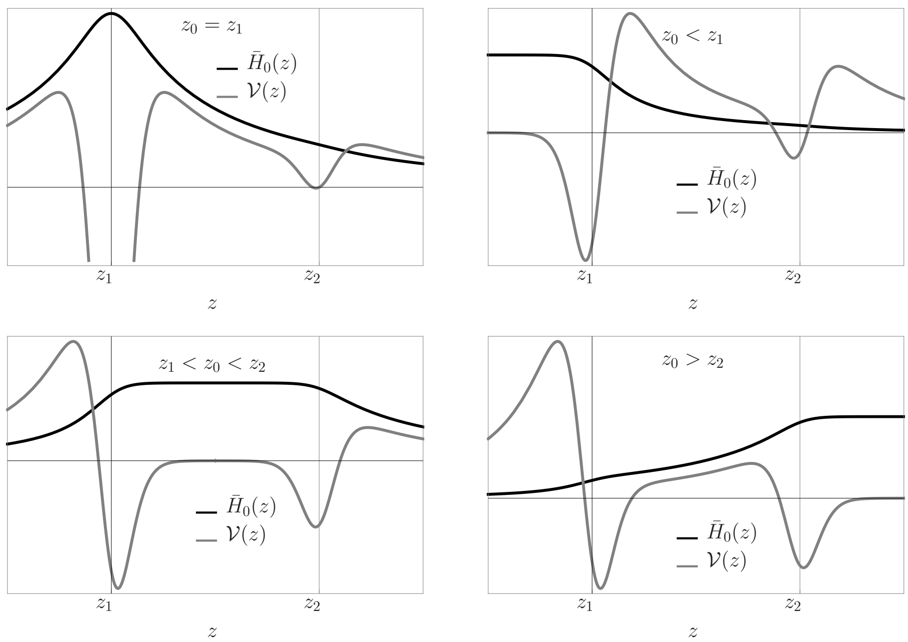


Figure 3.9.: These graphs illustrate the shape of the quantum mechanics potential $\mathcal{V}(z)$ in gray for all the scenarios that we have considered in Sec. 3.2.2 and the corresponding shape of the zero-mode (4D graviton) in black curve. Parameters chosen: $\beta = 2$, $\kappa_1 = 3$ and $\kappa_2 = 1$.

$U(z)$ goes to zero for the case (i) and (ii) as shown in Fig. 3.9 (upper left) and (upper right), so Eq. (2.51) reduces to one dimensional Klein-Gordon (KG) equation, i.e.

$$\left(\partial_z^2 + m_n^2\right) \bar{H}_n(z) = 0. \tag{3.162}$$

Therefore in the large z limit, we expect,

$$\bar{H}_n(z) \approx c_1 \cos(m_n z) + c_2 \sin(m_n z), \tag{3.163}$$

where c_1 and c_2 are constants. Therefore the massive KK modes are plane wave normalizable and we have a continuum spectrum of KK states for the case-I and case-II discussed in Sec. 3.2.2.

One can make the following comments resulting from the profile of the zero-mode for tensor perturbations $\bar{H}_0(z)$ and the Schrödinger-like potential $\mathcal{V}(z)$ shown in Fig. 3.8:

- The zero-mode $\bar{H}_0(z)$ implies that

$$\int dz \bar{H}_0^2(z) = \int dz e^{3A(z)} = \int dy e^{2A(y)} < \infty, \tag{3.164}$$

  therefore $\bar{H}_0(z)$ is normalizable and it turns out that the effective 4D Planck mass $M_{\rm Pl}^2$ is finite, hence the effective 4D gravity can be reproduced for the thick-brane case.

- As $\mathcal{V}(z) \to 0$ as $|z| \to \infty$, therefore the KK-mass spectrum is continuous without a gap and it starts from $m_0 = 0$.

- The (asymmetric) volcano-like shape of $\mathcal{V}(z)$ in Fig. 3.8 suggests that at large z the wave function of massive KK modes should have a plane wave behaviour.

- The presence of the large barriers near the thick-brane (z=0) implies that corrections to the Newton's law due to continuum spectrum of the KK modes will not be large [39, 40].

Before closing this subsection we will briefly comment on the effective 4D gravity. We are going to estimate the effective 4D Plank mass and discuss the localization of the zero-mode of the perturbation and then corrections to the Newton's potential due to the massive KK modes. To calculate the 4D Plank mass it is important to note that Eq. (3.158) only involves 2nd derivatives of the metric perturbation $\tilde{H}_{\mu\nu}(x,z)$. This is related to the fact that in the action these fluctuations have the following canonical kinetic term

$$S \supset M_*^3 \int d^4x dz \partial_M \tilde{H}_{\mu\nu}(x,z) \partial^M \tilde{H}^{\mu\nu}(x,z), \tag{3.165}$$

where the indices are contracted with the 5D Minkowski metric η_{MN}. Using the KK-decomposition of $\tilde{H}_{\mu\nu}(x,z)$, Eq. (3.159), the above equation takes the form

$$S \supset M_*^3 \int dz \bar{H}_0^2(z) \int d^4x \partial_\alpha \hat{H}_{0\mu\nu}(x) \partial^\alpha \hat{H}_0^{\mu\nu}(x) + \cdots, \tag{3.166}$$

where the ellipses represent the non-zero KK-modes. Now we can read out the effective 4D linearized gravity as,

$$S \supset M_{\rm Pl}^2 \int d^4x \partial_\alpha \hat{H}_{\mu\nu}(x) \partial^\alpha \hat{H}^{\mu\nu}(x) + \cdots, \tag{3.167}$$

where $M_{\rm Pl}$ is the effective 4D Planck mass, i.e.

$$M_{\rm Pl}^2 = M_*^3 \int dz \bar{H}_0^2(z), \tag{3.168}$$

where $\bar{H}_0(z)$ satisfies the supersymmetric quantum mechanic equation (2.53) for $m_0^2 = 0$. In order to reproduce the standard 4D General Relativity, $M_{\rm Pl}^2$ must be finite, in other words $\bar{H}_0(z)$ must be normalizable. It is easy to see from (2.55) that indeed $\bar{H}_0(z)$ is normalizable for different scenarios considered in Sec. 3.2 for which the warp function $A(y)$ posses the following asymptotic behavior (see Eqs. (3.70) and (3.79)),

$$A'(y) < 0 \quad \text{as} \quad y \to \infty, \tag{3.169}$$

$$A'(y) > 0 \quad \text{as} \quad y \to -\infty. \tag{3.170}$$

The above implies that

$$\int dz \bar{H}_0^2(z) = \int dz e^{3A(z)} = \int dy e^{2A(y)} < \infty, \tag{3.171}$$

therefore $\bar{H}_0(z)$ is normalizable [see also Fig. 3.9 (upper left) and (upper right)] and $M_{\rm Pl}^2$ is finite for the case-I and case-II of Sec. 3.2.2. The situation for the case-III of Sec. 3.2.2 is far more complicated, as there neither we have the finite 4D effect Planck mass nor we have a normalizable zero-mode, see Fig. 3.9 (lower left) and (lower right). However, as it was pointed out for similar singular brane set-up (GRS [37]), the effective 4D gravity on the brane can be reproduced and we could have the quasi-localized gravity [38, 40, 77, 41].

3.5. Summary

In this chapter we consider the dynamical mechanism of generating singular branes in warped extra dimensions [1, 2, 3, 4]. A possibility of periodic extra dimension is discussed in the presence of non-minimal scalar-gravity coupling and a generalized Gibbons-Kallosh-Linde sum rule is found in Sec. 3.1. In order to avoid constraints imposed by periodicity, a non-compact spatial extra dimension is considered in Sec. 3.2 and different thick-brane models have been constructed with scalar fields. In particular, we considered four scenario: (*i*) an asymmetric thick-brane model which mimics generalized RS2 model presented in Sec. 2.3 in the brane limit [2, 3]; (*ii*) a double thick-brane model which allows the possibility to address the hierarchy problem within the context of thick-branes [1]; (*iii*) a $\mathbb{Z}_2$ symmetric thick-brane setup which, in the brane limit, mimics three positive tension branes; and (*iv*) a dilatonic thick-brane [4]. In Sec. 3.3 an issue of localization of a scalar field on thick-brane is addressed. Stability of the background solutions was discussed and verified in the presence of the most general perturbations of the metric and the scalar field in Sec. 3.4.

CHAPTER 4

THICK-BRANE COSMOLOGY

The cosmological implications of the extra dimensions, and in particular to that of RS models, have been studied in detail by many groups [78, 79, 80, 81, 82, 83, 84, 85, 86]. We will briefly review the brane-world cosmology within the context of RS models in Sec. 4.1. The main purpose of this chapter is to study the cosmology of models where singular branes are replaced by regularized counterparts, the thick-branes which were discussed in detail in the previous chapter. Here we will confine ourself to the cosmology of RS2 and its thick-brane version. There have been only few studies in this direction, where the *smooth/thick* brane cosmological implications have been discussed [87, 88, 89]. In Sec. 4.2 we employ a scalar field, which constitutes the thick-brane, with and without time-dependence in the presence of 5D gravity and study the cosmological evolution of the 5D background solutions.

4.1. Brane-world cosmology: a brief review

In this section we review some of the results from brane-world cosmology in particular we focus on RS2 cosmology [78, 79, 80, 81, 82]. The goal of this section is to show that RS2 model gives the cosmological evolution of scale factor nearly same as in the standard 4D FRW cosmology.

The most general 5D metric, with 3-spatial dimensions, homogeneous and isotropic can be written as

$$ds^2 = -n^2(t,y)dt^2 + a^2(t,y)g_{ij}dx^i dx^j + b^2(t,y)dy^2, \tag{4.1}$$

where g_{ij} is the 3-dimensional spatial metric. The Einstein equations are given by [1],

$$R_{MN} - \frac{1}{2}g_{MN}R = T_{MN}, \tag{4.2}$$

where T_{MN} is the energy-momentum tensor, R_{MN} is the 5D Ricci tensor and R is the corre-

[1] In this chapter we will use the unit system such that $4M_*^3 = 1$.

sponding Ricci scalar. The energy momentum tensor can be decomposed into two parts,

$$T^M_N = \tilde{T}^M_N + \hat{T}^M_N, \tag{4.3}$$

where the $\tilde{T}^M_N$ is the bulk energy-momentum tensor and $\hat{T}^M_N$ is the brane localized energy momentum tensor. The bulk and brane localized energy-momentum tensors in RS2 have the following forms:

$$\tilde{T}^M_N = -\left(\Lambda_B,\ \Lambda_B,\ \Lambda_B,\ \Lambda_B,\ \Lambda_B\right), \tag{4.4}$$

$$\hat{T}^M_N = \frac{\delta(y)}{b}\left(-\lambda-\rho,\ \lambda+p,\ \lambda+p,\ \lambda+p,\ 0\right). \tag{4.5}$$

Above Λ_B is the bulk cosmological constant, λ is the brane tension, ρ is the energy density due to matter source on the brane and p is the corresponding pressure on the brane due to the matter source. We consider the equation of state as $\rho = \omega p$, with constant ω. Components of the Einstein equation (4.2) corresponding to the metric (4.1) can be written as,

$$00: \quad \frac{3}{n^2}\left[\frac{\dot{a}}{a}\left(\frac{\dot{a}}{a}+\frac{\dot{b}}{b}\right) - \frac{n^2}{b^2}\left\{\frac{a''}{a}+\frac{a'}{a}\left(\frac{a'}{a}-\frac{b'}{b}\right)\right\}\right] = \Lambda_B + \frac{\delta(y)}{b}(\lambda+\rho), \tag{4.6}$$

$$\begin{aligned} ij: \quad & \frac{1}{b^2}\left\{\frac{a'}{a}\left(\frac{a'}{a}+2\frac{n'}{n}\right) - \frac{b'}{b}\left(\frac{n'}{n}+2\frac{a'}{a}\right) + 2\frac{a''}{a}+\frac{n''}{n}\right\} \\ & +\frac{1}{n^2}\left\{\frac{\dot{a}}{a}\left(\frac{-\dot{a}}{a}+2\frac{\dot{n}}{n}\right)+\frac{\dot{b}}{b}\left(\frac{\dot{n}}{n}-2\frac{\dot{a}}{a}\right)-2\frac{\ddot{a}}{a}-\frac{\ddot{b}}{b}\right\} = -\Lambda_B + \frac{\delta(y)}{b}(\lambda+p), \end{aligned} \tag{4.7}$$

$$05: \quad 3\left(\frac{a'}{a}\frac{\dot{b}}{b}+\frac{n'}{n}\frac{\dot{a}}{a}-\frac{\dot{a}'}{a}\right) = 0, \tag{4.8}$$

$$55: \quad \frac{3}{b^2}\left[\frac{a'}{a}\left(\frac{a'}{a}+\frac{n'}{n}\right) - \frac{b^2}{n^2}\left\{\frac{\ddot{a}}{a}+\frac{\dot{a}}{a}\left(\frac{\dot{a}}{a}-\frac{\dot{n}}{n}\right)\right\}\right] = -\Lambda_B. \tag{4.9}$$

One can recover the effective 4D equation of energy momentum conservation on the brane from the Bianchi identity $\nabla_M G^M_N = 0$ and the 05-component of Einstein equation as

$$\dot{\rho} + 3(\rho+p)\frac{\dot{a}_0}{a_0} = 0, \tag{4.10}$$

where a_0 is the value of $a(t,y)$ at $y = 0$. Matching the delta functions in the 00 and ij components of the Einstein equations one gets the following *jumps* in $a'(t,y)$ and $n'(t,y)$,

$$\frac{[a']_0}{a_0 b_0} = -\frac{1}{3}(\lambda+\rho), \qquad \frac{[n']_0}{n_0 b_0} = \frac{2}{3}(\lambda+\rho)+(\lambda+p). \tag{4.11}$$

Now we take the *jump* of the 55 component of the Einstein equations to get

$$\frac{[a']_0}{a_0}\frac{\{a'\}_0}{a_0} + \frac{1}{2}\frac{[a']_0}{a_0}\frac{\{n'\}_0}{n_0} + \frac{1}{2}\frac{[n']_0}{n_0}\frac{\{a'\}_0}{a_0} = 0. \tag{4.12}$$

After using the values of the jumps from Eq. (4.11), the above expression gives,

$$\frac{\{a'\}_0}{a_0}(\lambda + p) = \frac{1}{3}(\lambda + \rho)\frac{\{n'\}_0}{n_0}. \tag{4.13}$$

By using the fact that there is no energy-momentum flow along the extra dimension, i.e. $T_{05} = 0$, one can get the 00 and 55 components of Einstein equation (4.6) and (4.9) in the bulk as (using $G_{05} = 0$):

$$F' = \frac{2a'a^3}{3}T^0_0, \tag{4.14}$$

$$\dot{F} = \frac{2\dot{a}a^3}{3}T^5_5, \tag{4.15}$$

where F is a function of t and y, given by

$$F(t, y) \equiv \frac{(a'a)^2}{b^2} - \frac{(\dot{a}a)^2}{n^2}. \tag{4.16}$$

where $T^0_0 = -\left(\Lambda_B + \frac{\delta(y)}{b}(\lambda + \rho)\right)$ and $T^5_5 = -\Lambda_B$. We can integrate Einstein equation (4.15) w.r.t. t to get,

$$F(t, y) = -\frac{a^4}{6}\Lambda_B + \mathcal{C}(y), \tag{4.17}$$

where $\mathcal{C}(y)$ is constant w.r.t. t but in general it is a function of y. To figure out the exact form of $\mathcal{C}(y)$, we plugin the solution of $F(t, y)$ in Eq. (4.14) and we get,

$$\mathcal{C}'(y) = -\frac{2a'a^3}{3b}\rho_0\delta(y), \tag{4.18}$$

which implies that $\mathcal{C}'(y) = 0$ for $y \neq 0$, such that $\mathcal{C}$ is a constant and at $y = 0$, $\mathcal{C}(y)$ has the following jump:

$$\left[\mathcal{C}(y)\right]_0 = -\frac{2a_0^3}{3b_0}\rho_0\{a'\}_0, \tag{4.19}$$

and if we use the fact that the scale factor is Z_2 symmetric then $\{a'\}_0 = 0$, implying that $\mathcal{C}$ is a constant everywhere both w.r.t. t and y. Hence the Eq. (4.17) can be written explicitly in more familiar form of the Friedmann type equation,

$$\left(\frac{\dot{a}}{na}\right)^2 = \frac{\Lambda_B}{6} + \left(\frac{a'}{ba}\right)^2 - \frac{\mathcal{C}}{a^4}. \tag{4.20}$$

Since we are interested in cosmological evolution of the scale factor on the brane therefore we take the average of the Friedmann-like equation (4.20) to get,

$$\left(\frac{\dot{a}_0}{n_0 a_0}\right)^2 = \frac{\Lambda_B}{6} + \frac{1}{4}\left(\frac{[a']_0}{b_0 a_0}\right)^2 + \left(\frac{\{a'\}_0}{b_0 a_0}\right)^2 - \frac{\mathcal{C}}{a^4}. \tag{4.21}$$

After using the jump $\left[a'\right]_0$ from Eq. (4.11) and average value of $\left\{a'(y)\right\}_0 = 0$ (due to $\mathbb{Z}_2$

symmetry in scale factor) the Friedmann equation (4.21) on the brane can be written as,

$$\left(\frac{\dot{a}_0}{a_0}\right)^2 = \frac{\Lambda_B}{6} + \frac{1}{36}(\lambda+\rho)^2 - \frac{\mathcal{C}}{a_0^4}. \tag{4.22}$$

Above we set $n_0 = 1$ (value of n on the brane) and impose the Z_2 symmetry, i.e. $a(t, y)$ is a function of $|y|$. Now by using the RS2 relation between the bulk cosmological constant Λ_B and the brane tension λ, i.e. $\lambda = \sqrt{-6\Lambda_B}$, and the fact that $\rho \ll \lambda$, one recovers the usual behavior of the Hubble parameter $H \equiv \dot{a}_0/a_0$ on the brane, i.e.

$$\left(\frac{\dot{a}_0}{a_0}\right)^2 = \frac{1}{18}\lambda\rho + \frac{1}{36}\rho^2 - \frac{\mathcal{C}}{a_0^4}. \tag{4.23}$$

Hence the cosmology resulted from the RS2 model is nearly 4D FRW cosmology with small corrections when the matter energy density ρ is much smaller than the brane tension λ.

4.2. Thick brane cosmological solutions

In this section our goal is to study cosmological evolution of the scale-factor and Hubble parameter with a dynamical background scalar field.

We will consider 5D space-time for which the metric takes the following (4D conformal) form,

$$ds^2 = a^2(\tau, y) g_{\mu\nu} dx^\mu dx^\nu + dy^2, \tag{4.24}$$

where x^μ are 4D coordinates while $g_{\mu\nu}$ is the 4D metric that we take as the usual Robertson-Walker metric. The function $a(\tau, y)$ is a scale factor which depends on the 4D conformal time τ and 5th dimension y; we will also refer to it as a warp factor because of its y-dependence. The action for scalar field in the presence of 5D gravity reads,

$$S = \int dx^5 \sqrt{-g} \left\{ \frac{R}{2} - \frac{1}{2} g^{MN} \nabla_M \phi \nabla_N \phi - V(\phi) \right\}. \tag{4.25}$$

We assume that the scalar field ϕ depends exclusively on conformal time τ and the extra coordinate y; $V(\phi)$ is the potential for the scalar field.

The Einstein equation and the equation of motion for ϕ resulting from the above action (4.25) are

$$R_{MN} - \frac{1}{2} g_{MN} R = T_{MN}, \tag{4.26}$$

$$\nabla^2 \phi - \frac{dV}{d\phi} = 0, \tag{4.27}$$

where energy-momentum tensor T_{MN} for the scalar field $\phi(\tau, y)$ is given by Eq. (3.28), i.e.

$$T_{MN} = \nabla_M \phi \nabla_N \phi - g_{MN} \left(\frac{1}{2} (\nabla\phi)^2 + V(\phi) \right). \tag{4.28}$$

The explicit form of the components of the Einstein equation for the metric ansatz (4.24) can be written as,

$$00: \qquad 3\left[\frac{1}{a^2}\frac{\dot{a}^2}{a^2}-\left(\frac{a''}{a}+\frac{a'^2}{a^2}\right)+\frac{k}{a^2}\right]=\frac{1}{2}\phi'^2+\frac{1}{2}\frac{1}{a^2}\dot{\phi}^2+V(\phi), \tag{4.29}$$

$$ij: \qquad \frac{1}{a^2}\left(2\frac{\ddot{a}}{a}-\frac{\dot{a}^2}{a^2}\right)-3\left(\frac{a''}{a}+\frac{a'^2}{a^2}\right)+\frac{k}{a^2}=\frac{1}{2}\phi'^2-\frac{1}{2}\frac{1}{a^2}\dot{\phi}^2+V(\phi), \tag{4.30}$$

$$05: \qquad \frac{a'}{a}\frac{\dot{a}}{a}-\frac{\dot{a}'}{a}=\frac{1}{3}\phi'\dot{\phi}, \tag{4.31}$$

$$55: \qquad 3\left[2\frac{a'^2}{a^2}-\frac{1}{a^2}\frac{\ddot{a}}{a}-\frac{k}{a^2}\right]=\frac{1}{2}\phi'^2+\frac{1}{2}\frac{1}{a^2}\dot{\phi}^2-V(\phi). \tag{4.32}$$

where $k=0,\ \pm 1$ denotes the spatial curvature of the 4D homogeneous and isotropic spacetime for Minkowski, de Sitter and anti-de Sitter space, respectively. The scalar field equation of motion can be written as,

$$\phi''-\frac{1}{a^2}\ddot{\phi}+4\frac{a'}{a}\phi'-\frac{2}{a^2}\frac{\dot{a}}{a}\dot{\phi}-\frac{dV}{d\phi}=0. \tag{4.33}$$

In the following two subsections we will consider two cases, one with time-independent (static) scalar field and the other with time-dependent scalar field.

4.2.1. Static thick-brane solutions

In this subsection we will consider a static scalar field scenario, in other words we assume $\phi(\tau,y)=\phi(y)$, but still allow $a(\tau,y)$ to be time-dependent. In this case the Einstein equations (4.29)-(4.32) simplify as follows,

$$00: \qquad 3\left[\frac{1}{a^2}\frac{\dot{a}^2}{a^2}-\left(\frac{a''}{a}+\frac{a'^2}{a^2}\right)+\frac{k}{a^2}\right]=\frac{1}{2}\phi'^2+V(\phi), \tag{4.34}$$

$$ij: \qquad \frac{1}{a^2}\left(2\frac{\ddot{a}}{a}-\frac{\dot{a}^2}{a^2}\right)-3\left(\frac{a''}{a}+\frac{a'^2}{a^2}\right)+\frac{k}{a^2}=\frac{1}{2}\phi'^2+V(\phi), \tag{4.35}$$

$$05: \qquad \frac{a'}{a}\frac{\dot{a}}{a}-\frac{\dot{a}'}{a}=0, \tag{4.36}$$

$$55: \qquad 3\left[2\frac{a'^2}{a^2}-\frac{1}{a^2}\frac{\ddot{a}}{a}-\frac{k}{a^2}\right]=\frac{1}{2}\phi'^2-V(\phi). \tag{4.37}$$

The equation of motion of the scalar field (4.27) reduces to,

$$\phi''+4\frac{a'}{a}\phi'-\frac{dV}{d\phi}=0. \tag{4.38}$$

Evolution of the scale factor

The assumption that scalar field ϕ is time independent implies that $T_{05}=0$, consequentially $G_{05}=0$, i.e. Eq. (4.36). This requires $\partial_t\partial_y \ln a=0$, which implies that a is separable:

$a(\tau, y) = \hat{a}(\tau)\bar{a}(y)$. Using this, the remaining Einstein equations become

$$00: \qquad \frac{1}{\hat{a}^2}\frac{\dot{\hat{a}}^2}{\hat{a}^2} + \frac{k}{\hat{a}^2} = \frac{\bar{a}^2}{3}\left[3\left(\frac{\bar{a}''}{\bar{a}} + \frac{\bar{a}'^2}{\bar{a}^2}\right) + \frac{1}{2}\phi'^2 + V(\phi)\right], \tag{4.39}$$

$$ij: \qquad \frac{1}{\hat{a}^2}\left(2\frac{\ddot{\hat{a}}}{\hat{a}} - \frac{\dot{\hat{a}}^2}{\hat{a}^2}\right) + \frac{k}{\hat{a}^2} = \bar{a}^2\left[3\left(\frac{\bar{a}''}{\bar{a}} + \frac{\bar{a}'^2}{\bar{a}^2}\right) + \frac{1}{2}\phi'^2 + V(\phi)\right], \tag{4.40}$$

$$55: \qquad \frac{1}{\hat{a}^2}\frac{\ddot{\hat{a}}}{\hat{a}} + \frac{k}{\hat{a}^2} = \frac{\bar{a}^2}{3}\left[6\frac{\bar{a}'^2}{\bar{a}^2} - \frac{1}{2}\phi'^2 + V(\phi)\right]. \tag{4.41}$$

where the left (right) hand sides depend only on τ (y). We then obtain the following set of equations for $\hat{a}(\tau)$:

$$00: \qquad \frac{1}{\hat{a}^2}\frac{\dot{\hat{a}}^2}{\hat{a}^2} + \frac{k}{\hat{a}^2} = C_\tau, \tag{4.42}$$

$$ij: \qquad \frac{1}{\hat{a}^2}\left(2\frac{\ddot{\hat{a}}}{\hat{a}} - \frac{\dot{\hat{a}}^2}{\hat{a}^2}\right) + \frac{k}{\hat{a}^2} = C_x, \tag{4.43}$$

$$55: \qquad \frac{1}{\hat{a}^2}\frac{\ddot{\hat{a}}}{\hat{a}} + \frac{k}{\hat{a}^2} = C_y, \tag{4.44}$$

where $C_{\tau,x,y}$ are constants. It is easy to see that in order for the first two equations to be consistent with the third one it is necessary that

$$C_y = \frac{C_\tau + C_x}{2}. \tag{4.45}$$

On the other hand, form the right hand sides of (4.39)-(4.41) one obtains for the y-dependent functions the following equations

$$00: \qquad C_\tau = \frac{\bar{a}^2}{3}\left[3\left(\frac{\bar{a}''}{\bar{a}} + \frac{\bar{a}'^2}{\bar{a}^2}\right) + \frac{1}{2}\phi'^2 + V(\phi)\right], \tag{4.46}$$

$$ij: \qquad C_x = \bar{a}^2\left[3\left(\frac{\bar{a}''}{\bar{a}} + \frac{\bar{a}'^2}{\bar{a}^2}\right) + \frac{1}{2}\phi'^2 + V(\phi)\right], \tag{4.47}$$

$$55: \qquad C_y = \frac{\bar{a}^2}{3}\left[6\frac{\bar{a}'^2}{\bar{a}^2} - \frac{1}{2}\phi'^2 + V(\phi)\right]. \tag{4.48}$$

Then (4.46)-(4.47) immediately imply that

$$C_x = 3C_\tau, \tag{4.49}$$

so that all the constants can be expressed in terms C_y that will be denoted by $\bar{\Lambda}$:

$$C_y \equiv \bar{\Lambda}, \qquad C_\tau = \frac{1}{2}\bar{\Lambda}, \qquad C_x = \frac{3}{2}\bar{\Lambda}. \tag{4.50}$$

As we will see below, the constant $\bar{\Lambda}$ is related to the 4D cosmological constant and its different values will correspond to the flat, de Sitter or anti-de Sitter space-time and it has non-trivial consequences on the evolution of the scale factor. Then one finds the following equations that must be satisfied by $\hat{a}(\tau)$:

$$\frac{\ddot{\hat{a}}}{\hat{a}} - 2\frac{\dot{\hat{a}}^2}{\hat{a}^2} - k = 0, \tag{4.51}$$

$$\frac{\dot{\hat{a}}^2}{\hat{a}^2} - \frac{\bar{\Lambda}}{2}\hat{a}^2 + k = 0. \tag{4.52}$$

As it will be shown below, even though the above equations look independent, solutions that satisfy both of them do exist for each possible $k = 0, \pm 1$. Before we proceed, it is worth discussing the relations between (4.51) and (4.52). First note that Eq. (4.52) can be rewritten as

$$\frac{\dot{\hat{a}}^2}{\hat{a}^4} - \frac{\bar{\Lambda}}{2} + \frac{k}{\hat{a}^2} = 0, \tag{4.53}$$

which is a first integral of Eq. (4.51): a derivative of (4.53) reproduces Eq. (4.51). So, one can recognize in (4.51) and (4.52) the analogs of classical equation of motion and energy conservation, respectively. Therefore the role of Eq. (4.52) is just to adjust "velocity" at the initial moment such that "energy" is properly matched.

Note also that (4.52) is identical with the standard Friedman equation written in terms of the conformal time, thus one finds that $\bar{\Lambda}$ has the interpretation of 4D cosmological constant.

The solutions of Eq. (4.51) are

$$\hat{a}(\tau) = \sqrt{\frac{2}{|\bar{\Lambda}|}} \begin{cases} \operatorname{sech}(\tau) & k = -1 & (\bar{\Lambda} < 0) \\ \big(-\tau\big)^{-1} & k = 0 & (\bar{\Lambda} > 0) \\ \sec(\tau) & k = +1 & (\bar{\Lambda} > 0) \end{cases} \tag{4.54}$$

Note that for $k = 0, 1$ the metric is singular at a finite time $\tau_{\text{sing}} = (s + 1/2)\pi k$, where s is an integer. It is instructive to write the scale factor $\hat{a}$ as a function of cosmological time t instead of the conformal time τ, as

$$\hat{a}(t) = \sqrt{\frac{2}{|\bar{\Lambda}|}} \begin{cases} \operatorname{sech}\left[2\,\operatorname{arctanh}\left[\tan\left(\sqrt{\frac{|\bar{\Lambda}|}{8}}t\right)\right]\right] & k = -1 & (\bar{\Lambda} < 0) \\ \exp\left(\sqrt{\frac{|\bar{\Lambda}|}{2}}t\right) & k = 0 & (\bar{\Lambda} > 0) \\ \cosh\left(\sqrt{\frac{|\bar{\Lambda}|}{2}}t\right) & k = 1 & (\bar{\Lambda} > 0) \end{cases} \tag{4.55}$$

Using Eq. (4.55), the evolution of the Hubble parameter $H \equiv \dot{\hat{a}}(t)/\hat{a}(t)$ is

$$H(t) = \sqrt{\frac{|\bar{\Lambda}|}{2}} \begin{cases} \sec\left(\sqrt{\frac{|\bar{\Lambda}|}{2}}t\right)\tanh\left[2\,\operatorname{arctanh}\left[\tan\left(\sqrt{\frac{|\bar{\Lambda}|}{8}}t\right)\right]\right] & k = -1 & (\bar{\Lambda} < 0) \\ 1 & k = 0 & (\bar{\Lambda} > 0) \\ \tanh\left(\sqrt{\frac{|\bar{\Lambda}|}{2}}t\right) & k = 1 & (\bar{\Lambda} > 0) \end{cases} \tag{4.56}$$

Note that when $k = -1$, $\bar{\Lambda}$ must be negative, which corresponds to anti-de Sitter geometry; for $k = 0, +1$, $\bar{\Lambda}$ must be positive, thus representing de Sitter space-time. As one can notice from the Eq. (4.55), for the de Sitter space-time we have exponentially growing scale factor while

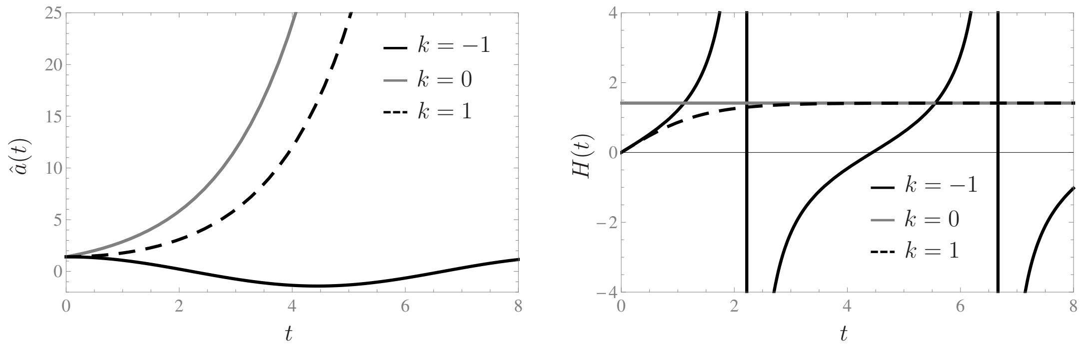


Figure 4.1.: The left graph shows the behavior of the 4D conformal scale factor $\hat{a}(t)$ as a function of t, whereas, the right graph presents the Hubble parameter $H(t)$ as a function of t for different values of spatial curvature k. We choose the value of constants $\alpha_0 = 0$ and $\alpha_1 = \sqrt{2}$.

for the case of anti-de Sitter space-time the scale factor has bouncing (oscillatory) behavior. In Fig. 4.1 we have plotted the scale factor $\hat{a}(t)$ and the Hubble parameter $H(t)$ for $k = 0, \ \pm 1$.

When $\bar{\Lambda} = 0$ there are no (real) solutions when $k = +1$. When $k = -1$, known as the Milne universe in the conventional cosmology, Eqs. (4.47)-(4.48) reduce to the standard static equations considered e.g. in [25, 1]. In this case time-dependent part of the scale factor is determined by Eqs. (4.51)-(4.52), whose general solution is a linear combination of the following functions

$$\hat{a}(\tau) = \alpha_0 e^{\sqrt{-k}\tau}, \qquad k = -1, 0; \qquad \bar{\Lambda} = 0, \tag{4.57}$$

or in terms of the cosmological time t the scalar factor can be written as,

$$\hat{a}(t) = \sqrt{-k}t + \alpha_1, \qquad k = -1, 0; \qquad \bar{\Lambda} = 0, \tag{4.58}$$

where $\alpha_{0,1}$ are some integration constants. Note that the static solution requires $k = 0$ and $\bar{\Lambda} = 0$.

In the following section we focus on the y-dependent solutions of Eqs. (4.47) and (4.48).

Extra-dimensional profiles

In this section we will determine y-dependent part of solutions that are governed by Eqs. (4.47)-(4.48). For this purpose it is useful to define $\bar{a}(y) \equiv e^{A(y)}$, such that our y-dependent Einstein equations Eqs. (4.47)-(4.48) and the scalar field equation (4.38) can be written as,

$$3A'' + \frac{3}{2}\bar{\Lambda}e^{-2A} = -\phi'^2, \tag{4.59}$$

$$6A'^2 - 3\bar{\Lambda}e^{-2A} = \frac{1}{2}\phi'^2 - V(\phi), \tag{4.60}$$

$$\phi'' + 4A'\phi' - \frac{dV}{d\phi} = 0. \tag{4.61}$$

The procedure we follow begins by assuming $A(y)$ is a known function, so the above conditions are to be considered as equations to determine $\phi(y)$ and $V(\phi)$. [2].

[2] In the literature there are few known analytic de Sitter and anti-de Sitter solutions of the system (4.59)-(4.60), see for example [43, 90, 91, 92, 93].

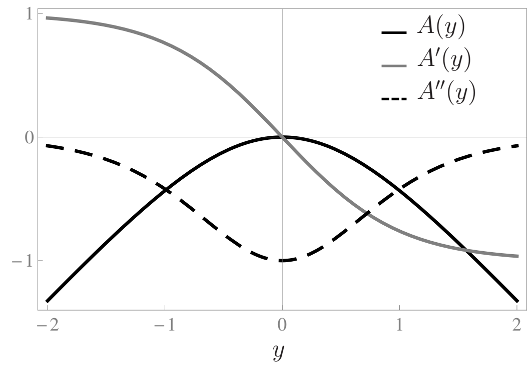


Figure 4.2.: The warp function $A(y)$ and its derivatives $A'(y)$ and $A''(y)$ as a function of y for $\beta = 1$.

Specifically, we will consider the following form of the warp function $A(y)$,

$$A(y) = -\ln\cosh(\beta y), \tag{4.62}$$

where β is a parameter. The above choice is dictated by simplicity and by a desire to have a warp factor which behaves as $\sim \exp(-|y|)$ at large y, so that it mimics RS solutions and the hierarchy problem can be in principle approached. This choice of $A(y)$ approximates well the static solution obtained in Sec. 3.2, see also [25, 1], for a kink profile of the scalar field. Figure 4.2 shows the warp function and its derivatives. It is difficult to find an exact analytical solution for $\phi(y)$ from Eq. (4.59), however if one considers separately regions of small ($|y| \lesssim \beta^{-1}$) and large ($|y| \gtrsim \beta^{-1}$) y then approximate analytical solutions are easy to obtain. When β is large (which is the case of our interest) then for small values of y, one can ignore [3] the exponential term in Eq. (4.59), i.e.

$$3A'' = -\phi'^2, \qquad (y \to 0) \tag{4.63}$$

with the following solution

$$\phi_s(y) = 2\sqrt{3}\arctan[\tanh(\beta y/2)], \qquad (y \to 0) \tag{4.64}$$

where $\phi_s(y)$ denotes the solution for small y. It is important to note that dropping the exponential term in Eq. (4.59) is a reasonable assumption in the vicinity of $y = 0$ if β is larger than $\bar{\Lambda}$, as illustrated in Fig. 4.3 for $\beta = 5$ and $\bar{\Lambda} = -1$. On the other hand, for large values of y, the exponential term dominates in Eq. (4.59) and we can ignore $A''(y)$, i.e.

$$\frac{3}{2}\bar{\Lambda}e^{-2A} = -\phi'^2, \tag{4.65}$$

with $\bar{\Lambda} < 0$. The solution of above equation reads

$$\phi_l(y) = \sqrt{-\frac{3\bar{\Lambda}}{2}\frac{1}{\beta}}\sinh(\beta y), \qquad (|y| \to \infty) \tag{4.66}$$

[3] When $y \to 0$ then $A(y) \to 0$ and $A'' \to -\beta^2$, therefore in the vicinity of $y = 0$ the Eq. (4.59) behaves as $-3\beta^2 + \frac{3}{2}\bar{\Lambda} = -\phi'^2$, implying $\phi_s(y)$ linear in y. For values of β adopted here the $\bar{\Lambda}$ term is negligible.

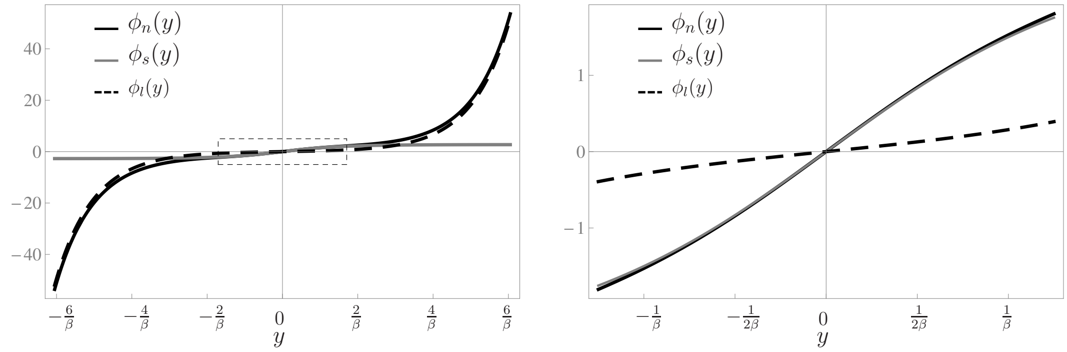


Figure 4.3.: These graphs show the exact numerical solution for the scalar field $\phi_n(y)$, the approximate analytic solution for small ($\phi_s(y)$) and large ($\phi_l(y)$) values of y as a function of y in units of β^{-1} for $\beta = 5$ and $\bar{\Lambda} = -1$. The right graph shows the zoomed central region of the left graph.

where $\phi_l(y)$ denotes the solution valid for large values of y.

In Fig. 4.3 we have plotted the approximate analytic solutions $\phi_{s,l}(y)$ and the exact numerical one $\phi_n(y)$. For large y the quality of the approximation can be easily estimated from the figure; one finds that for $|y| \gtrsim 5\beta^{-1}$, $\phi_n \simeq \phi_l$. For small y the right panel of the figure shows that for $|y| \lesssim \beta^{-1}$, $\phi_n \simeq \phi_s$. In the intermediate region $\beta^{-1} \lesssim |y| \lesssim 5\beta^{-1}$ the approximations $\phi_{n,s}$ are less accurate. It is also worth to mention that as β grows the region of applicability of ϕ_s shrinks, and ϕ_l converges to the exact numerical solution ϕ_n.

There is a comment here in order. If, instead of (4.62), we had used the solution obtained in the static case of Sec. 3.2, then for $\bar{\Lambda} = 0$ we would reproduce exactly the kink profile for the scalar field and the corresponding potential as in Sec. 3.2. In that case (with $k = -1$), the time evolution of the scale factor would be governed by (4.58) while the scalar profile would preserve its shape. In this special case the time evolution in 4D and source (the scalar field profile) along the extra dimension fully decouple, so that the scalar profile is retained while non-trivial time evolution of the scale factor has purely 4D nature.

We then substitute the solutions we obtained for $A(y)$ and $\phi(y)$ in (4.60) to obtain the scalar potential $V(\phi)$, which we plot as a function of ϕ in Fig. 4.4. To get approximate analytic results for the scalar potential $V(\phi)$ corresponding to small and large values of $\phi(y)$, we use the Einstein equation (4.60) along with the analytic solutions of scalar field $\phi_s(y)$ and $\phi_l(y)$. For small values of scalar field $\phi_s(y)$ the scalar potential is [4],

$$V_s(\phi) = \left(\frac{3}{2}\beta^2 + \frac{9}{4}\bar{\Lambda}\right) + \left(-\frac{5}{2}\beta^2 + \frac{3}{4}\bar{\Lambda}\right)\phi^2 + \mathcal{O}\left(\phi^4\right), \qquad (\phi \to 0). \tag{4.67}$$

For large values of scalar field $\phi_l(y)$ we can write the potential as,

$$V_l(\phi) = -\frac{3}{2}\beta^2\phi^2 + \left(-6\beta^2 + \frac{9}{4}\bar{\Lambda}\right) + \mathcal{O}\left(\phi^{-2}\right), \qquad (|\phi| \to \infty). \tag{4.68}$$

Note that the potential is unbounded from below. In fact this is a generic consequence of the

[4]For $\bar{\Lambda} = 0$ one would reproduce the standard bottom of a wine bottle potential as in the previous chapter.

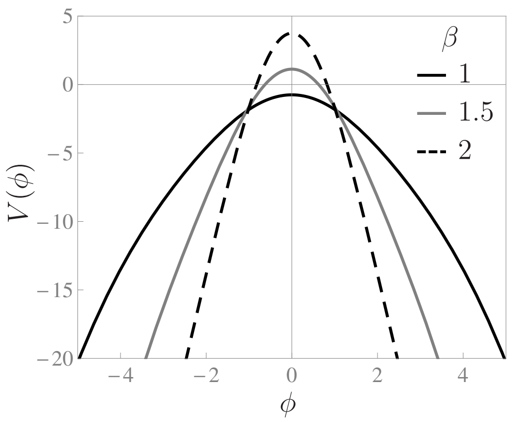


Figure 4.4.: The scalar potential $V(\phi)$ as a function of ϕ for different values of β and $\bar{\Lambda} = -1$.

requirement that the warp function $A(y)$ has a linear dependence on y as $|y| \to \infty$. Indeed, as seen from (4.59)-(4.60), for large y, $V(y) \sim \frac{9}{4}\bar{\Lambda}e^{-2A} \sim \frac{9}{4}\bar{\Lambda}e^{2\beta|y|}$. Since at large $|y|$, $\phi \propto e^{\beta|y|}$ we find $V(\phi) \propto -\phi^2$, so the potential is always unbounded from below.

An alternative approach to solve Eqs. (4.59)-(4.60) is to reduce these equations into the following first order equations by the use of the superpotential method for non-zero 4D cosmological constant [25],

$$A' = -\frac{1}{3}\mathcal{W}\gamma(y), \tag{4.69}$$

$$\phi' = \frac{1}{\gamma(y)}\frac{\partial \mathcal{W}}{\partial \phi}, \tag{4.70}$$

$$V = \frac{1}{\gamma(y)^2}\left(\frac{\partial \mathcal{W}}{\partial \phi}\right)^2 - \frac{2}{3}\mathcal{W}^2, \tag{4.71}$$

where $\gamma(y)$ is defined by,

$$\gamma(y) = \left(1 + \frac{9}{2}\frac{\bar{\Lambda}}{\mathcal{W}^2}e^{-2A(y)}\right)^{1/2}. \tag{4.72}$$

The superpotential method reduces the second order non-linear differential equations (4.59)-(4.61) to system of first order nonlinear differential equations (4.69)-(4.71). However, obtaining the solution are less straightforward when $\bar{\Lambda} \neq 0$ (for $\bar{\Lambda} = 0$ see Sec. 3.2 and Refs. [25, 1]). Unfortunately, for the present case, one can not start with a desired shape for the scalar profile, our strategy is instead to solve a system of first order nonlinear equations by first choosing the warp function $A(y)$. Next we solve Eq. (4.69) algebraically for $\mathcal{W}(y)$, and then solve the following equation for $\phi(y)$

$$\phi'(y) = \sqrt{\frac{\mathcal{W}'(y)}{\gamma(y)}}. \tag{4.73}$$

Then Eq. (4.71) gives the potential $V(y)$ which can be written as $V(\phi)$ after inverting $\phi(y)$ as $y = y(\phi)$. We will not further investigate these solutions for $\phi(y)$, we only note that choosing $A(y)$ as in (4.62) one would reproduce the result obtained earlier in this subsection.

4.2.2. Time-dependent thick-brane solutions

In this subsection we will look for solutions of the Einstein equations (4.29)-(4.32) allowing for time-dependence of the scalar field.

Boosted solutions

In this subsection we show how the a static solution for the warp factor, $a(y)$ and scalar field $\phi(y)$ can be promoted to a time-dependent solution through a boost along the extra dimension: $y \to y' = \gamma(vt + y)$, where $\gamma = 1/\sqrt{1-v^2}$ and v is a relative velocity. It proves to be more convenient first to redefine the fifth coordinate y so that the length element (4.24) is written as

$$ds^2 = a^2(z)\left(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2\right), \tag{4.74}$$

Let us consider a Lorentz transformation $t' = \gamma(t + vz)$ and $z' = \gamma(vt + z)$. It is easy to check that $a'(t', z') = a(t, z)$, since ϕ is a scalar field $\phi'(t', z') = \phi(t, z)$. By general covariance $a'(t', z')$ and $\phi'(t', z')$ are also solutions on the Einstein equations. Therefore we conclude that for any given stationary solution $a(y)$ and $\phi(y)$, the functions $a\big[\gamma\big(-vt+z(y)\big)\big]$ and $\phi\big[\gamma\big(-vt+z(y)\big)\big]$ also satisfy the Einstein equations. This strategy could be applied to any stationary solution, e.g. to the kink solution discussed in Sec. 3.2.

Twisted solutions

In this subsection we return to the fifth dimensional coordinate y. We will to show that one can obtain a class of interesting solutions assuming that a and ϕ depend on y and t only through the combination $\eta \equiv c\tau + dy$ [5] where c and d are non-zero constants. In the next subsection we will show that if the superpotential method is used, the 05 component of the Einstein equations in fact implies such a dependence on η for ϕ. With this assumption the Einstein equations (4.29)-(4.32) become,

$$00: \quad 3\frac{c^2}{a^2}\frac{a'^2}{a^2} - 3d^2\left(\frac{a''}{a} + \frac{a'^2}{a^2}\right) + \frac{3k}{a^2} = \frac{d^2}{2}\phi'^2 + \frac{1}{2}\frac{c^2}{a^2}\phi'^2 + V(\phi), \tag{4.75}$$

$$ij: \quad \frac{c^2}{a^2}\left(2\frac{a''}{a} - \frac{a'^2}{a^2}\right) - 3d^2\left(\frac{a''}{a} + \frac{a'^2}{a^2}\right) + \frac{k}{a^2} = \frac{d^2}{2}\phi'^2 - \frac{1}{2}\frac{c^2}{a^2}\phi'^2 + V(\phi), \tag{4.76}$$

$$05: \quad \frac{a''}{a} - \frac{a'^2}{a^2} = -\frac{1}{3}\phi'^2, \tag{4.77}$$

$$55: \quad 6d^2\frac{a'^2}{a^2} - 3\frac{c^2}{a^2}\frac{a''}{a} - \frac{3k}{a^2} = \frac{d^2}{2}\phi'^2 + \frac{1}{2}\frac{c^2}{a^2}\phi'^2 - V(\phi). \tag{4.78}$$

Note that in this section a *prime* denotes a derivative w.r.t. η. If now we add Eqs. (4.76)-(4.78), and then the use of Eq. (4.77) gives

$$\frac{a''}{a} + \frac{a'^2}{a^2} + \frac{2k}{c^2} = 0. \tag{4.79}$$

[5] Note that this is not a boost of a stationary solution.

On the other hand, subtracting Eqs. (4.75)-(4.76) and using (4.79) gives

$$\frac{a'^2}{a^2} + \frac{k}{c^2} = \frac{1}{6}\phi'^2. \tag{4.80}$$

Using the above two relations in Eq. (4.78), we obtain the following form for the scalar potential $V(\phi)$

$$V(\phi) = d^2 \left(-\frac{1}{2}\phi'^2 + \frac{6k}{c^2} \right). \tag{4.81}$$

At this point the strategy is clear, one first solves (4.79) for $a(\eta)$, then ϕ is easily determined from (4.80). If $\phi(\eta)$ is an invertible function of η then $V(\phi)$ can be found from (4.81). In the following we will find such solutions for each possible value of k.

<u>$k = 0$</u>**:** In this case the warp factor $a(\eta)$ is obtained by integrating Eq. (4.79):

$$a(\eta) = a_0 \left(1 + 2b_0\eta\right)^{1/2}, \tag{4.82}$$

where a_0 and b_0 are integration constants. It is important to note that the above solution is only valid in the region of the space-time where $\eta > -1/2b_0$, and we will see below that the scalar field is singular at $\eta \to -1/2b_0$. With this explicit expression for $a(\eta)$, we use Eq. (4.80) to find the scalar field $\phi(\eta)$:

$$\phi(\eta) = \pm\sqrt{\frac{3}{2}} \ln\left(1 + 2b_0\eta\right) + \phi_0, \tag{4.83}$$

where ϕ_0 is an integration constant. Then the scalar potential takes the form

$$V(\phi) = -3b_0^2 e^{-\sqrt{\frac{8}{3}}(\phi-\phi_0)}. \tag{4.84}$$

It is noteworthy that the above potential is similar to the dilaton potential $V_{\text{dilaton}} = -|\Lambda| e^{\sqrt{4/3}\phi}$ studied in a 5D context in Sec. 3.2.4, see also Antoniadis et al. [76]. In our case, however, the argument of the exponent is $\sqrt{8/3}\,\phi$, while in [76] it is $\sqrt{(4/3)}\phi$, see Sec. 3.2.4. The left panel of Fig. 4.5 shows the behavior of the scalar field $\phi(\eta)$ and the warp factor $a(\eta)$ for $k = 0$ case.

<u>$k = 1$</u>**:** In this case the warp factor $a(\eta)$ found from (4.79) reads

$$a(\eta) = a_0 \sqrt{\cos(2\eta/c + c_0)}, \tag{4.85}$$

where a_0 and c_0 are integration constants. The above solution is applicable in the region $|\eta + cc_0/2| < c\pi/4$. With this expression for $a(\eta)$, we use Eq. (4.80) to find the scalar field $\phi(\eta)$:

$$\phi(\eta) = \sqrt{\frac{3}{2}} \ln\left(\frac{1 + \tan(\eta/c + c_0/2)}{1 - \tan(\eta/c + c_0/2)}\right) + \phi_0, \tag{4.86}$$

where ϕ_0 is an integration constant. Then the scalar potential is given by

$$V(\phi) = \frac{3d^2}{2c^2}\left[3 - \cosh\left(\sqrt{\frac{8}{3}}(\phi - \phi_0)\right)\right]. \tag{4.87}$$

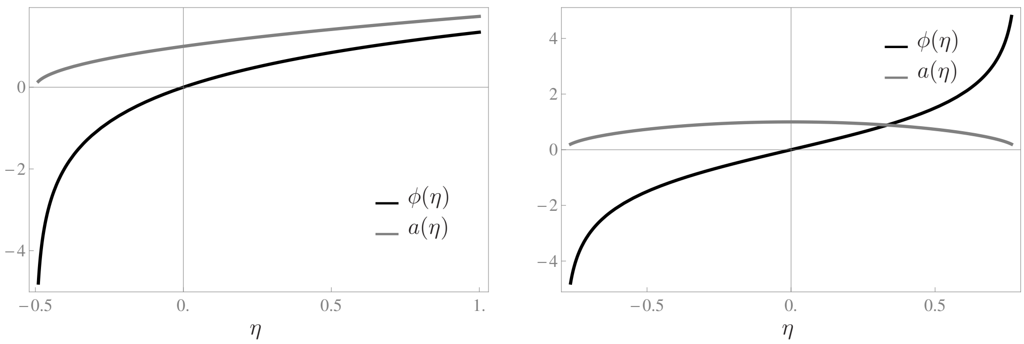


Figure 4.5.: The above graphs shows the behavior of $\phi(\eta)$ (4.86) and $a(\eta)$ as a function of η for $k=0$ (left panel) and $k=1$ (right panel) with parameters: $a_0=b_0=c=d=1$ and $\phi_0=c_0=0$.

It is important to note that these solutions for $k=1$ case have a singularity at $\eta=(\pm\pi-2c_0)c/4$. Right panel of Fig. 4.5 illustrates the behavior of the scalar field $\phi(\eta)$ and the warp factor $a(\eta)$ for $k=1$ case.

$\underline{k=-1}$**:** In this case (4.79) yields

$$a(\eta)=a_0\sqrt{\cosh(2\eta/c+c_0)}, \tag{4.88}$$

where a_0 and c_0 are integration constants. In this case, however, there are no real solutions for $\phi'(\eta)$, so we will not consider this possibility further.

4.2.3. Generalized superpotential method

It is instructive to develop an analogue of the superpotential method for the time dependent scalar field in 5D warped space-time. We define the following quantities,

$$\frac{a'}{a}\equiv-\frac{1}{3}W(\phi), \qquad \frac{\dot a}{a}\equiv-\frac{1}{3}H(\phi), \tag{4.89}$$

where $W(\phi)$ and $H(\phi)$ are functions of $\phi(\tau,y)$. In the above equation and in the following, unless otherwise stated, we return to τ and y derivatives by a dot and a prime respectively. With the above definitions we find from the 05 component of the Einstein equations (4.31),

$$\frac{\partial W(\phi)}{\partial\phi}=\phi', \qquad \frac{\partial H(\phi)}{\partial\phi}=\dot\phi. \tag{4.90}$$

Now, if we re-express the 55 (or 00) component of the Einstein equations in terms of the superpotential variables $W(\phi)$ and $H(\phi)$ through Eqs. (4.89) and (4.90) we get the potential $V(\phi)$ as,

$$V(\phi)=\frac{1}{2}\left(\frac{\partial W(\phi)}{\partial\phi}\right)^2-\frac{2}{3}W(\phi)^2-\frac{1}{a^2}\left(\frac{1}{2}\left(\frac{\partial H(\phi)}{\partial\phi}\right)^2-\frac{1}{3}H(\phi)^2-3k\right). \tag{4.91}$$

The ij components of the Einstein equation (4.30) then give

$$\frac{1}{2}\left(\frac{\partial H(\phi)}{\partial\phi}\right)^2-\frac{1}{3}H(\phi)^2-3k=0. \tag{4.92}$$

For the $k=0$ case this gives
$$\frac{\partial H(\phi)}{\partial \phi}\frac{1}{H(\phi)} = \pm\sqrt{\frac{2}{3}}. \tag{4.93}$$
with solution
$$H(\phi) = H_0 e^{\pm\sqrt{\frac{2}{3}}\phi}. \tag{4.94}$$
where, $H_0 \equiv H(0)$ is a constant of integration. It is important to note that the superpotentials $W(\phi)$ and $H(\phi)$ are related to each other since from (4.90) one obtains
$$\frac{\partial^2 W(\phi)}{\partial \phi^2}\dot{\phi} = \dot{\phi}', \qquad \frac{\partial^2 H(\phi)}{\partial \phi^2}\phi' = \dot{\phi}', \tag{4.95}$$
which implies, along with Eq. (4.90), that,
$$\frac{\partial^2 W(\phi)}{\partial \phi^2}\frac{\partial H(\phi)}{\partial \phi} = \frac{\partial^2 H(\phi)}{\partial \phi^2}\frac{\partial W(\phi)}{\partial \phi}. \tag{4.96}$$
Hence,
$$W(\phi) = A_0 H(\phi) + W_0, \tag{4.97}$$
where A_0 and W_0 are constants on integration.

In order to determine ϕ we use (4.92) together with (4.90) to obtain,
$$\dot{\phi} = \pm\sqrt{\frac{2}{3}} H_0 e^{\pm\sqrt{\frac{2}{3}}\phi}. \tag{4.98}$$
On the other hand, from (4.90) and (4.97) we find,
$$\phi' = \pm\sqrt{\frac{2}{3}} A_0 H_0 e^{\pm\sqrt{\frac{2}{3}}\phi}. \tag{4.99}$$
Therefore from Eqs. (4.98) and (4.99), we have,
$$\dot{\phi}(\tau, y) = \frac{1}{A_0}\phi'(\tau, y), \tag{4.100}$$
which implies that, as claimed previously, ϕ can depend on t and y only through η (with $d = A_0\, c$). For simplicity, hereafter we choose $c = d = 1$. Then from Eq. (4.98) we obtain
$$\frac{d\phi(\eta)}{d\eta} = \pm\sqrt{\frac{2}{3}} H_0 e^{\pm\sqrt{\frac{2}{3}}\phi(\eta)}. \tag{4.101}$$
with solution
$$\phi(\eta) = \mp\sqrt{\frac{3}{2}} \ln\left(-\frac{2}{3}H_0\eta + e^{\mp\sqrt{\frac{2}{3}}\phi_0}\right), \tag{4.102}$$
where ϕ_0 is an integration constant. Note that the above solution is valid only for $-\frac{2}{3}H_0\eta + e^{\sqrt{\frac{2}{3}}\phi_0} > 0$ and therefore is singular at $-\frac{2}{3}H_0\eta + e^{\sqrt{\frac{2}{3}}\phi_0} = 0$. Also one can see that for the choice $H_0 = -3b_0$ and $\phi_0 = 0$ the above result for $\phi(\eta)$ matches the one obtained in (4.83).

In order to determine the warp factor a we use Eq. (4.89):

$$\frac{\dot{a}}{a} \equiv -\frac{1}{3}H(\phi) = \mp\sqrt{\frac{1}{6}}\frac{\partial H(\phi)}{\partial\phi} = \mp\sqrt{\frac{1}{6}}\dot{\phi}, \tag{4.103}$$

which can be solved to obtain $a(\tau, y)$ as,

$$a(\tau, y) = a(\tau_0, y)e^{\mp\sqrt{\frac{1}{6}}\left(\phi(\tau,y)-\phi(\tau_0,y)\right)}, \tag{4.104}$$

where $a(\tau_0, y)$ and $\phi(\tau_0, y)$ are functions of y at the constant time slice τ_0. $a(\tau_0, y)$ can be found by substituting the above expression for $a(\tau, y)$ into the first equation in Eq. (4.89); we then find

$$a(\tau_0, y) = a(\tau_0, y_0)e^{\mp\sqrt{\frac{1}{6}}\left(\phi(\tau_0,y)-\phi(\tau_0,y_0)\right)-\frac{1}{3}W_0 y}, \tag{4.105}$$

inserting this in Eq. (4.104) we find

$$a(\tau, y) = a(\tau_0, y_0)e^{\mp\sqrt{\frac{1}{6}}\left(\phi(\tau,y)-\phi(\tau_0,y_0)\right)-\frac{1}{3}W_0 y}, \tag{4.106}$$

where $a(\tau_0, y_0)$ and $\phi(\tau_0, y_0)$ are constants. Since $\phi(\tau, y) = \phi(\eta)$; then, for $\phi(\tau_0, y_0) \equiv \phi_0 = 0$, $W_0 = 0$ and $H_0 = -3b_0$, we recover the result that $a(\tau, y) \equiv a(\eta)$ as in Eq. (4.82).

$$a(\eta) = a_0\left(1 + 2b_0\eta\right)^{1/2}. \tag{4.107}$$

Since $W(\phi)$ has been found we can determine the potential $V(\phi)$ directly from (4.91)

$$V(\phi) = -\frac{1}{3}\left(A_0 H_0 e^{\pm\sqrt{\frac{2}{3}}\phi} + 2W_0\right)^2 + \frac{2}{3}W_0^2. \tag{4.108}$$

We recover the result for $V(\phi)$ as in Eq. (4.84) with $H_0 = -3b_0$, $A_0 = 1$, $W_0 = 0$ and the lower sign (minus sign) in the exponent. Similarly one can reproduce all the results obtained in Sec. 4.2.2 adopting the superpotential method for non-zero k values. Hence, we conclude that the superpotential method is equivalent to the assumption that ϕ and a depend on τ and y only through $\eta = c\tau + dy$. The advantage of this method this that it reduces the second order differential equations into to first order equations which are much easier to solve analytically.

4.3. Summary

In this chapter we have presented cosmological solutions of 5D warped extra dimensional models with smooth/thick-branes. In the case of static thick-brane cosmology, time-independent scalar field configurations are employed and the evolution of the scale factor is discussed and determined. Whereas for the case of dynamical thick-branes, time-dependent scalar field and the scale factor are considered and various cosmological solutions are discussed. It is found that for the case of time-dependent scalar fields $\phi(\tau, y)$ there exist class of solutions that depend on 4D conformal time τ and the 5D coordinate y only through the combination $\eta = c\tau + dy$, where c, d are constants. A generalized superpotential method is developed which allows time-dependent warp factor.

CHAPTER 5

WARPED HIGGS DARK MATTER

As we have reviewed in Chap. 2 the RS1 [22] provides an elegant solution to the *hierarchy problem*, soon after the RS proposal, many important improvements to the model were considered. First, a stabilization mechanism for the RS1 setup was proposed by Goldberger and Wise [34]; it employs a real scalar field in the bulk of AdS geometry with localized potentials on both of the branes, see also [25]. A second interesting observation, which could potentially solve the fermion mass hierarchy problem within the SM, was made by many groups [64, 66, 67, 94, 95]. The core idea of these works was to allow all the SM fields to propagate in the RS1 bulk, except the Higgs field which was kept localized on the IR-brane. In this way, the zero-modes of these bulk fields correspond to the SM fields and the overlap of y-dependent profiles of fermionic fields with the Higgs field could generate the required fermion mass hierarchy. To suppress the EW precision observables, the symmetry of the gauge group was enhanced by introducing custodial symmetry in Ref. [96]. The common lore, in the RS1 model and its extensions, was to keep the Higgs field localized on the IR-brane in order to solve the hierarchy problem. The first attempt to consider the Higgs field in the bulk of RS1 was made by Luty and Okui [97]. They employed AdS/CFT duality [1] to argue that a *bulk Higgs* scenario can address the hierarchy problem by making the Higgs mass operator marginal in the dual CFT.

A study of electroweak symmetry breaking (EWSB) within the *bulk Higgs* scenario was first performed in the RS1 setup by Davoudiasl et al. [100]; they showed that the zero-mode of the bulk Higgs is tachyonic and hence could lead to a vacuum expectation value (vev) at the TeV scale. Recently there have been many studies where a *bulk Higgs* scenario was considered from different perspectives — see for example: a study with custodial symmetry in the Higgs sector[101]; models with soft wall setup [102]; bulk Higgs mediated flavor changing neutral currents (FCNCs) [103]; suppression of electroweak precision observables by modifying the warped metric near the IR-brane [104, 105, 106]; and, a bulk Higgs as the modulus stabilization field (Higgs–radion unification) [107]. Different phenomenological aspects after the Higgs discovery

[1]For the phenomenological applications of AdS/CFT with RS1 geometry, see for example [98, 99].

were explored in [108, 109, 110, 111, 112, 113, 114]. These phenomenological studies show that the RS1 model with bulk SM fields and its descendants with modified geometry (RS-like warped geometries in general) are consistent with the current experimental bounds and EW precision data.

As we discussed above, RS-like warped geometries, being consistent with the experimental data, offer an attractive solution to many of the fundamental puzzles of the SM, mostly through geometric means. In the same spirit, one can ask if RS-like warped extra dimensions can shed some light on another outstanding puzzle of SM, the lack of a candidate for dark matter (DM) which constitutes 83% of the observed matter in the universe [28]. It appears that unlike (flat) universal extra dimensions (UED), where the KK-modes of the bulk fields can be even and odd under KK-parity (implying that the lowest KK-odd particle (LKP) could be a natural dark matter candidate [115, 116]), RS1-like models (involving two branes and warped bulk) are unable to offer an analogue of KK-parity. The reason lies in the fact that the RS1 geometry is just a single slice of AdS space and, since warped, cannot be symmetric around any point along the extra dimension and hence does not allow a *KK-parity*. As a result it cannot accommodate a realistic dark matter candidate. To cure this problem in the warped geometries, usually extra discrete symmetries are introduced such that the SM fields are even while the DM is odd under such discrete symmetries in order to make it stable [117, 118, 119, 120]. Another way to mend this problem in warped geometries is to introduce an additional *hidden sector* with some local gauge symmetries such that only DM is charged under the hidden sector gauge symmetries and it couples to the SM very weakly [121, 122], (see also [123]).

An alternative to introducing additional symmetries, is to extend the RS1-like warped geometry in such a way that the whole geometric setup becomes symmetric around a fixed point in the bulk. Two $\mathbb{Z}_2$ symmetric warped configurations are possible. In the first, two identical AdS patches are symmetrically glued together at a UV fixed point, while in the second two identical AdS pathes are symmetrically glued together at an IR fixed point. The geometric configuration when the two AdS copies are glued together at the UV fixed point will be referred as "IR-UV-IR geometry", whereas the geometry corresponding to the setup when two AdS copies are glued at the IR fixed point is called "UV-IR-UV geometry". We confine ourselves to the IR-UV-IR geometric setup presented in Sec. 2.2 — it is straight forward to extend our analysis to the UV-IR-UV geometries. (A common pathology associated with this latter type of geometry is the appearance of ghosts.) We are aware of only two earlier attempts to construct a similar setup. The first [124] treated the lowest odd KK gauge mode as the DM candidate. The second employed a kink-like UV thick-brane [125] and the corresponding dark-matter was the first odd KK-radion [126].

In this chapter, we place all the SM fields, including the Higgs doublet, in the bulk of the IR-UV-IR geometry. The geometric $\mathbb{Z}_2$ parity ($y \to -y$ symmetry) leads to "warped KK-parity", i.e. there are towers of even and odd KK-modes corresponding to each bulk field. We focus on EWSB induced by the bulk Higgs doublet and low energy aspects of the 4D effective theory for the even and odd zero-modes assuming the KK-mass scale is high enough

$\sim \mathcal{O}$(few) TeV. In the effective theory the even and odd Higgs doublets mimic a two-Higgs-doublet model (2HDM) scenario with the odd doublet similar to the inert doublet but without corresponding pseudoscalar and charged scalars — *the truncated inert-doublet model*. All the parameters of this *truncated* 2HDM are determined by the fundamental 5D parameters of the theory and the choice of boundary conditions for the fields at $\pm L$. (Note that the boundary or "jump" conditions at $y = 0$ follow from the bulk equations of motion in the case of even modes, whereas odd modes are required to be zero by symmetry.) There are many possible alternative choices for the b.c. at $\pm L$. We allow the y-derivative of a field to have an arbitrary value at $\pm L$ as opposed to requiring that the field value itself be zero, i.e. we employ Neumann or mixed b.c. rather than Dirichlet b.c. at $\pm L$, see also [127] for general discussion on the choice of b.c.. Only the former yields a non-trivial theory allowing spontaneous symmetry breaking (SSB), whereas the latter leads to an explicit symmetry breaking scenario in which there are no Goldstone modes and the gauge bosons do not acquire mass. With these choices, the symmetric setup yields an odd Higgs zero-mode that is a natural candidate for dark matter. We compute the one-loop quadratic (in cutoff) corrections to the two scalar zero modes within the effective theory and discuss their mass splitting. The dark matter candidate is a WIMP — we calculate its relic abundance in the cold dark matter paradigm.

This chapter is organized as follows. In Sec. 5.1, we discuss the manifestation of KK-parity due the $\mathbb{Z}_2$ geometric setup presented in Sec. 2.2. An Abelian Higgs mechanism, with a complex scalar field and a gauge field, is studied in our background geometry in Sec. 5.2. In the Abelian case we lay down the foundation for SSB due to bulk Higgs, which is later useful for the case of EWSB in the SM. Two apparently different approaches are considered to study SSB in the Abelian case: (*i*) SSB by vacuum expectation values of the KK modes; and, (*ii*) SSB via a vacuum expectation value of the 5D Higgs field. Low energy (zero-mode) 4D effective theories are obtained within the two approaches and we find that the effective theories are identical up to corrections of order $\mathcal{O}\big(m_0^2/m_{KK}^2\big)$, where m_0 and m_{KK} are the zero-mode mass and KK-mass scale, respectively. Section 5.3 contains the main part of our work. There, we focus on EWSB for the SM gauge sector due to the bulk Higgs doublet in our $\mathbb{Z}_2$ symmetric geometry and obtain a low-energy 4D effective theory containing all the SM fields plus a real scalar – a dark matter candidate – which is odd under the discrete $\mathbb{Z}_2$ symmetry. In the subsequent two subsections of Sec. 5.3, we consider quantum corrections to scalar masses below the KK-scale $\sim \mathcal{O}$(few) TeV and explore possible implications of the dark-matter candidate by calculating its relic abundance. For a warmup, Appendix B discusses SSB of a discrete symmetry with a real scalar in the bulk of our geometric setup.

5.1. Warped KK-parity

In this section we employ the background solution for the $\mathbb{Z}_2$ symmetric background (IR-UV-IR) geometry considered in Sec. 2.2 and show how KK-parity is manifested within this geometric setup. The IR-UV-IR geometry of Sec. 2.2 is $\mathbb{Z}_2$-symmetric and we will consider this symmetry to be exact for our 5D theory. If the 5D theory has this $\mathbb{Z}_2$-parity (symmetry) then the Schrödinger-like potential for all the fields is symmetric, resulting in even (symmetric) and odd (antisymmetric) eigenmodes under this parity. Thus, a general field $\Phi(x,y)$ can be KK decomposed,

$$\Phi(x,y) = \sum_n \phi_n(x) f_n(y),$$

and, due to the $\mathbb{Z}_2$ geometry, the wave functions $f_n(y)$ are either even or odd. As a result:

$$\Phi(x,y) \equiv \Phi^{(\pm)}(x,y), \tag{5.1}$$

where

$$\Phi^{(+)}(x,y) = \sum_n \phi_n^{(+)}(x) f_n^{(+)}(y) \xrightarrow{y\to -y} +\Phi^{(+)}(x,y),$$
$$\Phi^{(-)}(x,y) = \sum_n \phi_n^{(-)}(x) f_n^{(-)}(y) \xrightarrow{y\to -y} -\Phi^{(-)}(x,y).$$

Due to the geometric $\mathbb{Z}_2$ symmetry, a single odd KK-mode cannot couple to two even KK-modes in the 4D effective theory, which will ensure that the lowest odd KK-mode will be stable. To understand this point better let us consider the following interaction term in the action:

$$\begin{aligned} S_{\rm odd} &= -\frac{1}{\sqrt{L}} \int d^4x \int_{-L}^{+L} dy \sqrt{-g}\, \Phi^{(-)}(x,y) \Phi^{(+)2}(x,y) + \cdots , \\ &= -\frac{1}{\sqrt{L}} \sum_{l,m,n} \int_{-L}^{+L} dy \sqrt{-g} f_l^{(-)}(y) f_m^{(+)}(y) f_n^{(+)}(y) \int d^4x\, \phi_l^{(-)}(x) \Phi_m^{(+)}(x) \Phi_n^{(+)}(x) + \cdots , \\ &= 0, \end{aligned} \tag{5.2}$$

where the ellipses denote other possible odd terms with odd number of odd fields. Above in the second line we used the KK-decomposition and last line follows from the fact that the y integration of an odd function vanishes, i.e.

$$\int_{-L}^{+L} dy \sqrt{-g} f_l^{(-)}(y) f_m^{(+)}(y) f_n^{(+)}(y) = 0. \tag{5.3}$$

Above we assumed the $\mathbb{Z}_2$ symmetric background geometry, i.e. the warp-function $A(y)$ is symmetric. This is the geometric manifestation of *warped KK-parity*, i.e. any term with odd number of odd KK-modes must vanish. Hence, the lowest of the odd KK-modes would be a stable particle. Furthermore, as the geometry is $\mathbb{Z}_2$ symmetric in $y \in [-L, L]$, the continuity conditions for odd and even modes at $y = 0$ strongly impact the physics scenario. Our choice will be that the odd (even) modes satisfy Dirichlet (Neumann or mixed) boundary (jump) conditions (b.c.) at $y = 0$, respectively. As for the odd modes, continuity implies that they must be zero at $y = 0$, but we could also have demanded the Neumann conditions that their y derivative be zero at $y = 0$. We choose not to impose this additional b.c. in this work. As regards the even modes, one cannot choose Dirichlet b.c. at $y = 0$ because of the presence of the UV-brane and associated “jump” conditions following from the equations of motion.

5.2. SSB in the IR-UV-IR model: the Abelian Higgs mechanism

In this section we will discuss the mechanism of spontaneous symmetry breaking (SSB) for an Abelian case with the Higgs field (a complex scalar) in the IR-UV-IR geometry of Sec. 2.2. The metric is given by Eq. (1.1), we will neglect the back reaction of the bulk fields on the geometry. We will borrow most of our results from Appendix B, and focus here on gauge-symmetric aspects of the model. We start by specifying the 5D Abelian action,

$$\begin{aligned}S_{Ab} = -\int d^5x\sqrt{-g}\Big\{&\frac{1}{4}F_{MN}F^{MN} + |D_M H|^2 + \mu_B^2 H^* H \\ &+ V_{IR}(H)\delta(y+L) + V_{UV}(H)\delta(y) + V_{IR}(H)\delta(y-L)\Big\},\end{aligned} \tag{5.4}$$

where $D_M \equiv \partial_M - ig_5 A_M$ with the 5D $U(1)$ coupling constant g_5 [2] and $F_{MN} \equiv \partial_M A_N - \partial_N A_M$. We require that the bulk potential and the UV-brane potential have only quadratic terms whereas the IR-brane potential is allowed to have a quartic term:

$$V_{UV}(H) = \frac{m_{UV}^2}{k} H^* H, \qquad V_{IR}(H) = -\frac{m_{IR}^2}{k} H^* H + \frac{\lambda_{IR}}{k^2}\left(H^* H\right)^2. \tag{5.5}$$

In this way EWSB is mainly triggered by the IR-brane. Above, H is a complex scalar field and the parametrization is such that m_{UV} and m_{IR} have mass dimensions while λ_{IR} is dimensionless. The gauge transformations can be written as

$$H(x,y) \to H'(x,y) = e^{i\Lambda(x,y)} H(x,y), \tag{5.6}$$

$$A_M(x,y) \to A'_M(x,y) = A_M(x,y) + \frac{1}{g_5}\partial_M \Lambda(x,y), \tag{5.7}$$

where $\Lambda(x,y)$ is the gauge parameter.

As one can see from the toy model discussed in Appendix B, the fields in the IR-UV-IR setup have even and odd bulk wave functions implied by the geometric $\mathbb{Z}_2$ symmetry. Hence, in our Abelian model, it is convenient to decompose the generic Higgs and the gauge field into fields of definite parity as follows

$$H(x,y) = H^{(+)}(x,y) + H^{(-)}(x,y), \qquad A_M(x,y) = A_M^{(+)}(x,y) + A_M^{(-)}(x,y), \tag{5.8}$$

where $\pm$ denotes the even and odd states. The gauge transformations for the even and odd parity modes are,

$$A_\mu^{(\pm)}(x,y) \to A_\mu^{\prime(\pm)}(x,y) = A_\mu^{(\pm)}(x,y) + \frac{1}{g_5}\partial_\mu \Lambda^{(\pm)}(x,y), \tag{5.9}$$

$$A_5^{(\pm)}(x,y) \to A_5^{\prime(\pm)}(x,y) = A_5^{(\pm)}(x,y) + \frac{1}{g_5}\partial_5 \Lambda^{(\mp)}(x,y). \tag{5.10}$$

$$\begin{pmatrix} H^{(+)} \\ H^{(-)} \end{pmatrix} \to \begin{pmatrix} H'^{(+)} \\ H'^{(-)} \end{pmatrix} = e^{i\Lambda^{(+)}\mathbb{1}} e^{i\Lambda^{(-)}\tau_1} \begin{pmatrix} H^{(+)} \\ H^{(-)} \end{pmatrix}, \tag{5.11}$$

[2]The 5D coupling constant g_5 has mass dimension $-1/2$.

where $\mathbb{1}$ is a 2×2 unit matrix, whereas $\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the Pauli matrix.

With this decomposition the above action can be written as

$$\begin{aligned} S_{Ab} = -\int d^5x\sqrt{-g}\Big\{ & \frac{1}{4}F^{(+)}_{\mu\nu}F^{\mu\nu}_{(+)} + \frac{1}{2}F^{(+)}_{\mu 5}F^{\mu 5}_{(+)} + D_M H^{(+)*}D^M H^{(+)} + \mu_B^2 H^{(+)*}H^{(+)} \\ & + \frac{1}{4}F^{(-)}_{\mu\nu}F^{\mu\nu}_{(-)} + \frac{1}{2}F^{(-)}_{\mu 5}F^{\mu 5}_{(-)} + D_M H^{(-)*}D^M H^{(-)} + \mu_B^2 H^{(-)*}H^{(-)} \\ & + V_{IR}(H^{(\pm)})\delta(y+L) + V_{UV}(H^{(+)})\delta(y) + V_{IR}(H^{(\pm)})\delta(y-L)\Big\}, \end{aligned} \tag{5.12}$$

where,

$$F^{(\pm)}_{\mu\nu} \equiv \partial_\mu A^{(\pm)}_\nu - \partial_\mu A^{(\pm)}_\nu, \qquad F^{(\pm)}_{\mu 5} \equiv \partial_\mu A^{(\pm)}_5 - \partial_5 A^{(\mp)}_\mu. \tag{5.13}$$

The brane localized potentials for the Higgs field, $V_{UV}(H)$ and $V_{IR}(H)$, can be written in terms of even and odd parity modes as

$$V_{UV}(H^{(+)}) = \frac{m^2_{UV}}{k}|H^{(+)}|^2, \tag{5.14}$$

$$\begin{aligned} V_{IR}(H^{(\pm)}) = & -\frac{m^2_{IR}}{k}|H^{(+)}|^2 - \frac{m^2_{IR}}{k}|H^{(-)}|^2 + \frac{\lambda_{IR}}{k^2}|H^{(+)}|^4 + \frac{\lambda_{IR}}{k^2}|H^{(-)}|^4 \\ & + \frac{4\lambda_{IR}}{k^2}|H^{(+)}|^2|H^{(-)}|^2 + \frac{\lambda_{IR}}{k^2}\left((H^{(+)*}H^{(-)})^2 + h.c.\right). \end{aligned} \tag{5.15}$$

In the above, we have not written $H^{(-)}$ terms in V_{UV} since $H^{(-)}(0) = 0$. Moreover, we have not written terms which are odd as they will not contribute after integration over the y-coordinate. One can easily check that the above brane potentials are invariant under the gauge transformations defined above. Also note that $F^{(\pm)}_{\mu\nu}$ and $F^{(\pm)}_{\mu 5}$ are gauge invariant under the gauge transformations (5.9) and (5.10). In the even/odd basis, the covariant derivatives D_μ and D_5 following from $D_M \equiv \partial_M - ig_5 A_M$, take the form

$$D_\mu \begin{pmatrix} H^{(+)} \\ H^{(-)} \end{pmatrix} \equiv \left[\partial_\mu - ig_5 \begin{pmatrix} A^{(+)}_\mu & A^{(-)}_\mu \\ A^{(-)}_\mu & A^{(+)}_\mu \end{pmatrix}\right] \begin{pmatrix} H^{(+)} \\ H^{(-)} \end{pmatrix}, \tag{5.16}$$

$$D_5 \begin{pmatrix} H^{(+)} \\ H^{(-)} \end{pmatrix} \equiv \left[\partial_5 - ig_5 \begin{pmatrix} A^{(-)}_5 & A^{(+)}_5 \\ A^{(+)}_5 & A^{(-)}_5 \end{pmatrix}\right] \begin{pmatrix} H^{(+)} \\ H^{(-)} \end{pmatrix}. \tag{5.17}$$

Under the gauge transformations the covariant derivative transforms as

$$D_M \begin{pmatrix} H^{(+)} \\ H^{(-)} \end{pmatrix} \to D'_M \begin{pmatrix} H'^{(+)} \\ H'^{(-)} \end{pmatrix} = e^{i\Lambda^{(+)}\mathbb{1}} e^{i\Lambda^{(-)}\tau_1} D_M \begin{pmatrix} H^{(+)} \\ H^{(-)} \end{pmatrix}, \tag{5.18}$$

i.e. it transforms the same way as complex scalar field transforms (5.11). It is important to note that the above action is manifestly gauge invariant under the gauge group $U(1)' \times U(1)$, where the corresponding gauge functions are $\Lambda^{(+)}(x,y)$ and $\Lambda^{(-)}(x,y)$.

The next two subsections are devoted to two possible strategies for implementing spontaneous gauge symmetry breaking for this Abelian $U(1)$ symmetric case. We are going to describe and compare: (*i*) SSB by vacuum expectation values of the KK modes and (*ii*) SSB by a y-dependent vacuum expectation value of the 5D Higgs field. Readers can either follow and continue, or, as we would advise, one may consider warming up within a toy model of a real scalar field and spontaneous symmetry breaking which is discussed in Appendix B and then return to the following subsections. In Appendix B we also consider the above two possible approaches to SSB.

5.2.1. SSB by vacuum expectation values of KK modes

In this case we will choose the 5D axial gauge, $A_5^{(\pm)} = 0$. This gauge is realized by choosing the gauge parameter such that,

$$\Lambda^{(\pm)}(x,y) = -g_5 \int dy A_5^{(\mp)}(x,y) + \hat{\Lambda}^{(\pm)}(x), \tag{5.19}$$

where $\hat{\Lambda}^{(\pm)}(x)$ is the integration constant (residual gauge freedom) and only depends on x^μ. Note that the $\hat{\Lambda}^{(-)}(x)$, being an odd function of y, must vanish. Consequently, we are left with only one 4D gauge function, $\hat{\Lambda}^{(+)}(x)$. In this gauge, the Abelian action reduces to,

$$\begin{aligned} S_{Ab} = -\int d^5x\sqrt{-g}\Big\{ &\frac{1}{4}F^{(+)}_{\mu\nu}F^{\mu\nu}_{(+)} + \frac{1}{2}\partial_5 A^{(+)}_\mu \partial^5 A^\mu_{(+)} + D_M H^{(+)*} D^M H^{(+)} + \mu_B^2 |H^{(+)}|^2 \\ &+ \frac{1}{4}F^{(-)}_{\mu\nu}F^{\mu\nu}_{(-)} + \frac{1}{2}\partial_5 A^{(-)}_\mu \partial^5 A^\mu_{(-)} + D_M H^{(-)*} D^M H^{(-)} + \mu_B^2 |H^{(-)}|^2 \\ &+ V_{IR}(H^{(\pm)})\delta(y+L) + V_{UV}(H^{(+)})\delta(y) + V_{IR}(H^{(\pm)})\delta(y-L)\Big\}, \end{aligned} \tag{5.20}$$

where the brane potentials are given by Eqs. (5.14) and (5.15). It is convenient to parameterize the complex scalar field $H^{(\pm)}(x,y)$ in the following form,

$$\begin{pmatrix} H^{(+)} \\ H^{(-)} \end{pmatrix} \equiv e^{ig_5(\pi^{(+)}\mathbb{1} + \pi^{(-)}\tau_1)} \begin{pmatrix} \Phi^{(+)} \\ \Phi^{(-)} \end{pmatrix}, \tag{5.21}$$

where $\Phi^{(\pm)}(x,y)$ and $\pi^{(\pm)}(x,y)$ are real scalar fields. We KK-decompose the scalar fields $\Phi^{(\pm)}(x,y)$, $\pi^{(\pm)}(x,y)$ and the gauge fields $A_\mu^{(\pm)}(x,y)$ as

$$\Phi^{(\pm)}(x,y) = \sum_n \Phi_n^{(\pm)}(x) f_n^{(\pm)}(y), \tag{5.22}$$

$$\pi^{(\pm)}(x,y) = \sum_n \pi_n^{(\pm)}(x) a_n^{(\pm)}(y), \tag{5.23}$$

$$A_\mu^{(\pm)}(x,y) = \sum_n A_{\mu n}^{(\pm)}(x) a_n^{(\pm)}(y), \tag{5.24}$$

where the wave-functions $f_n^{(\pm)}(y)$ satisfy Eq. (B.8) from Appendix B.1. (We borrow the results for the wave-functions $f_n^{(\pm)}(y)$ from Appendix B.1.) We choose gauge wave-functions $a_n^{(\pm)}(y)$ to satisfy

$$-\partial_5\big(e^{2A(y)}\partial_5 a_n^{(\pm)}(y)\big) = m^2_{A_n^{(\pm)}} a_n^{(\pm)}(y). \tag{5.25}$$

The wave-functions $a_n^{(\pm)}$ satisfy the following orthonormality conditions,

$$\int_{-L}^{+L} dy a_m^{(\pm)}(y) a_n^{(\pm)}(y) = \delta_{mn}. \tag{5.26}$$

It is worth commenting here that the gauge field $A_\mu^{(\pm)}(x,y)$ and the scalar field $\pi^{(\pm)}(x,y)$ share the same y-dependent KK-eigen bases $a_n^{(\pm)}(y)$, this is a convenient choice. The KK-modes satisfy $\Box^{(4)} A_{\mu n}^{(\pm)}(x) = m^2_{A_n^{(\pm)}} A_{\mu n}^{(\pm)}(x)$. The boundary (jump) conditions for $a_n^{(\pm)}(y)$ at $y=0$ and $y=\pm L$ are,

$$\partial_5 a_n^{(+)}(y)\Big|_{0^+} = 0, \qquad a_n^{(-)}(y)\Big|_{0^+} = 0, \qquad \partial_5 a_n^{(\pm)}(y)\Big|_{\pm L^{\mp}} = 0. \tag{5.27}$$

We choose the Neumann b.c. for $a_n^{(+)}(y)$ at $y=0,\pm L$ in order to insure that we get non-zero even zero-mode gauge profiles. With regard to the odd modes, we have chosen the Neumann b.c. of $\partial_5 a_n^{(-)}(\pm L)=0$, as the other choice of $a_n^{(-)}(\pm L)=0$ would lead to a trivial theory with $a_n^{(-)}(y)=0$ everywhere.

With the above KK-decomposition we can write the effective 4D action for the Abelian case, with the Higgs brane localized potentials, as

$$\begin{aligned} S_{Ab} = - \int d^4x \Big\{ & \frac{1}{4} F^{n(+)}_{\mu\nu} F^{\mu\nu}_{n(+)} + \frac{1}{2} m^2_{A_n^{(+)}} A^{n(+)}_\mu A^\mu_{n(+)} + \frac{1}{4} F^{n(-)}_{\mu\nu} F^{\mu\nu}_{n(-)} + \frac{1}{2} m^2_{A_n^{(-)}} A^{n(-)}_\mu A^\mu_{n(-)} \\ & + \partial_\mu \Phi_n^{(+)} \partial^\mu \Phi_n^{(+)} + m_n^{2(+)} \Phi_n^{(+)2} + \partial_\mu \Phi_n^{(-)} \partial^\mu \Phi_n^{(-)} + m_n^{2(-)} \Phi_n^{(-)2} \\ & + \left(g^{(+)2}_{klmn} \Phi_m^{(+)} \Phi_n^{(+)} + \bar{g}^{(+)2}_{klmn} \Phi_m^{(-)} \Phi_n^{(-)} \right) \left(A^{(+)}_{k\mu} - \partial_\mu \pi_k^{(+)} \right) \left(A_l^{(+)\mu} - \partial^\mu \pi_l^{(+)} \right) \\ & + \left(g^{(-)2}_{klmn} \Phi_m^{(+)} \Phi_n^{(+)} + \bar{g}^{(-)2}_{klmn} \Phi_m^{(-)} \Phi_n^{(-)} \right) \left(A^{(-)}_{k\mu} - \partial_\mu \pi_k^{(-)} \right) \left(A_l^{(-)\mu} - \partial^\mu \pi_l^{(-)} \right) \\ & + 4 g^2_{klmn} \Phi_m^{(+)} \Phi_n^{(-)} \left(A^{(+)}_{k\mu} - \partial_\mu \pi_k^{(+)} \right) \left(A_l^{(-)\mu} - \partial^\mu \pi_l^{(-)} \right) + 6 \lambda_{klmn} \Phi_k^{(+)} \Phi_l^{(+)} \Phi_m^{(-)} \Phi_n^{(-)} \\ & + \lambda^{(+)}_{klmn} \Phi_k^{(+)} \Phi_l^{(+)} \Phi_m^{(+)} \Phi_n^{(+)} + \lambda^{(-)}_{klmn} \Phi_k^{(-)} \Phi_l^{(-)} \Phi_m^{(-)} \Phi_n^{(-)} \Big\}, \end{aligned} \tag{5.28}$$

where the indices in this action are raised and lowered by Minkowski metric and the coupling constants are given as

$$\lambda^{(\pm)}_{klmn} = e^{4A(L)} \frac{\lambda_{IR}}{k^2} f_k^{(\pm)} f_l^{(\pm)} f_m^{(\pm)} f_n^{(\pm)}\Big|_L, \qquad \lambda_{klmn} = e^{4A(L)} \frac{\lambda_{IR}}{k^2} f_k^{(+)} f_l^{(+)} f_m^{(-)} f_n^{(-)}\Big|_L, \tag{5.29}$$

$$g^{(\pm)2}_{klmn} = g_5^2 \int_{-L}^{L} dy e^{2A} a_k^{(\pm)} a_l^{(\pm)} f_m^{(+)} f_n^{(+)}, \qquad \bar{g}^{(\pm)2}_{klmn} = g_5^2 \int_{-L}^{L} dy e^{2A} a_k^{(\pm)} a_l^{(\pm)} f_m^{(-)} f_n^{(-)}, \tag{5.30}$$

$$g^2_{klmn} = g_5^2 \int_{-L}^{L} dy e^{2A} a_k^{(+)} a_l^{(-)} f_m^{(+)} f_n^{(-)}, \tag{5.31}$$

where the superscripts $\pm$ on the coupling constants are just for notational purposes and do not refer to the parity.

The above action is valid for all KK-modes. Assuming that the KK-scale is high enough, i.e. $m_{KK} \sim \mathcal{O}(\text{few})$ TeV, we can derive effective theory where only the lowest modes (zero-modes with masses much below m_{KK}) are considered. Equation (5.25) along with the b.c. (5.27) imply that the odd zero-mode wave-function of the gauge boson is zero, i.e. $a_0^{(-)} = 0$. As a result, in the effective theory the odd zero-mode gauge boson $A_{0\mu}^{(-)}$ and odd parity Goldstone mode $\pi_0^{(-)}$ are not present. In contrast, the even zero-mode wave-function for the gauge boson has a constant profile in the bulk, i.e. $a_0^{(+)} = 1/\sqrt{2L}$, implying that the couplings of the even zero-mode gauge boson g_{00mn}^{+} and $\bar{g}_{00mn}^{+}$ are equal (see Eq. (5.30)), which, in turn, implies $g_{00mn}^{(+)} = \bar{g}_{00mn}^{(+)} \equiv g_4\delta_{mn}$, with $g_4 \equiv g_5/\sqrt{2L}$. The forms of the scalar zero-mode wave functions $f_0^{(\pm)}(y)$ are given by Eq. (B.24). We can now write down the low-energy effective action for the zero-modes:

$$\begin{aligned} S_{Ab}^{eff} = -\int d^4x \Big\{ & \frac{1}{4} F_{\mu\nu}^{0(+)} F_{0(+)}^{\mu\nu} + \partial_\mu \Phi_0^{(+)} \partial^\mu \Phi_0^{(+)} - \mu^2 \Phi_0^{(+)2} + \partial_\mu \Phi_0^{(-)} \partial^\mu \Phi_0^{(-)} - \mu^2 \Phi_0^{(-)2} \\ & + \left(g_{0000}^{(+)2} \Phi_0^{(+)2} + \bar{g}_{0000}^{(+)2} \Phi_0^{(-)2} \right) \left(A_{0\mu}^{(+)} - \partial_\mu \pi_0^{(+)} \right)^2 \\ & + \lambda_{0000}^{(+)} \Phi_0^{(+)4} + \lambda_{0000}^{(-)} \Phi_0^{(-)4} + 6\lambda_{0000} \Phi_0^{(+)2} \Phi_0^{(-)2} \Big\}, \end{aligned} \tag{5.32}$$

where the couplings can be read from Eqs. (5.29) and (5.30) and the mass parameter μ is defined as $\mu^2 \equiv (1+\beta) m_{KK}^2 \delta_{IR}$, with the parameters defined as

$$\delta_{IR} \equiv \frac{m_{IR}^2}{k^2} - 2(2+\beta), \qquad m_{KK} \equiv k e^{-kL} \quad \text{and} \quad \beta \equiv \sqrt{4 + \mu_B^2/k^2}. \tag{5.33}$$

By using the results from Appendix B, we get the following couplings in terms of the parameters of the fundamental theory:

$$\lambda_{0000}^{(\pm)} = \lambda_{0000} \simeq \lambda \equiv \lambda_{IR}(1+\beta)^2, \qquad g_{0000}^{(+)} = \bar{g}_{0000}^{(+)} = g_4 \equiv \frac{g_5}{\sqrt{2L}}. \tag{5.34}$$

Our effective theory could also be described by redefining $A_{0\mu}^{(+)}(x) \equiv A_\mu(x)$, $\pi_0^{(+)}(x) \equiv \pi(x)$ and

$$H_1(x) \equiv e^{ig_4\pi(x)} \Phi_0^{(+)}(x), \qquad H_2(x) \equiv e^{ig_4\pi(x)} \Phi_0^{(-)}(x), \tag{5.35}$$

in which case the above effective action can be written in a nice gauge covariant form as

$$S_{Ab}^{eff} = -\int d^4x \left\{ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \mathcal{D}_\mu H_1^* \mathcal{D}^\mu H_1 + \mathcal{D}_\mu H_2^* \mathcal{D}^\mu H_2 + V(H_1, H_2) \right\}, \tag{5.36}$$

where the covariant derivative is defined as

$$\mathcal{D}_\mu \equiv \partial_\mu - i g_4 A_\mu, \tag{5.37}$$

and the scalar potential can be written as

$$V(H_1, H_2) = -\mu^2 |H_1|^2 - \mu^2 |H_2|^2 + \lambda |H_1|^4 + \lambda |H_2|^4 + 6\lambda |H_1|^2 |H_2|^2.$$

Note that the action (5.36) is symmetric under $\mathbb{Z}_2' \times \mathbb{Z}_2$ under which $H_1 \to -H_1$ and $H_2 \to -H_2$, respectively.

It is important to note that, after choosing the gauge $A_5(x,y) = 0$, we are left with a residual gauge freedom with a single purely 4D gauge parameter $\hat{\Lambda}^{(+)}(x)$ such that the above Lagrangian is invariant under the $U(1)$ gauge transformation,

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \frac{1}{g_4}\partial_\mu \hat{\Lambda}^{(+)}(x), \tag{5.38}$$

$$H_1(x) \to H'_1(x) = e^{ig_4\hat{\Lambda}^{(+)}} H_1(x), \qquad H_2(x) \to H'_2(x) = e^{ig_4\hat{\Lambda}^{(+)}} H_2(x). \tag{5.39}$$

Thus, besides $\mathbb{Z}'_2 \times \mathbb{Z}_2$ symmetry, the above potential is invariant under $U(1)' \times U(1)$. One $U(1)$ has been gauged while the other is a global remnant of unbroken symmetry associated with the odd gauge transformation ($\Lambda^{(-)}$) defined in Eqs. (5.10)-(5.11).

As illustrated in the toy model Appendix B.1, we choose the vacuum such that the even parity Higgs H_1 acquires a vev, whereas the odd parity Higgs H_2 does not. That choice of vacuum implies values of v_1 and v_2 given by,

$$v_1^2 = \frac{\mu^2}{\lambda}, \qquad v_2 = 0. \tag{5.40}$$

Now let us consider fluctuations around our choice of the vacuum,

$$H_1(x) = \frac{1}{\sqrt{2}}\Big(v_1 + h\Big)e^{ig_4\pi(x)}, \qquad H_2(x) = \frac{1}{\sqrt{2}}\chi e^{ig_4\pi(x)}. \tag{5.41}$$

We rewrite our effective action (5.36) only up to the quadratic order in fluctuations as

$$\begin{aligned} S^{(2)}_{Ab} = -\int d^4x \Big\{ &\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{g_4^2 v_1^2}{2}\Big(A_\mu - \partial_\mu \pi\Big)^2 \\ &+ \frac{1}{2}\partial_\mu h \partial^\mu h + \frac{1}{2}m_h^2 h^2 + \frac{1}{2}\partial_\mu \chi \partial^\mu \chi + \frac{1}{2}m_\chi^2 \chi^2 \Big\}. \end{aligned} \tag{5.42}$$

The mixing between A_μ and π in the above action can be removed by an appropriate 4D gauge choice. Here we will choose the 4D unitary gauge such that $\pi = 0$ and the gauge field acquires mass. The remaining scalars are h and χ with masses

$$m_h^2 = m_\chi^2 = 2\mu^2. \tag{5.43}$$

Hence, the full effective Abelian action can be written in the 4D unitary gauge as

$$\begin{aligned} S^{eff}_{Ab} = -\int d^4x \Big\{ &\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m_A^2 A_\mu A^\mu + \frac{1}{2}\partial_\mu h \partial^\mu h + \frac{1}{2}m_h^2 h^2 + \frac{1}{2}\partial_\mu \chi \partial^\mu \chi + \frac{1}{2}m_\chi^2 \chi^2 \\ &+ \lambda v_1 h\Big(h^2 + 3\chi^2\Big) + \frac{1}{4}\lambda h^4 + \frac{1}{4}\lambda \chi^4 + \frac{3}{2}\lambda h^2 \chi^2 \\ &+ g_4^2 v_1 h A_\mu A^\mu + \frac{1}{2}g_4^2\Big(h^2 + \chi^2\Big)A_\mu A^\mu \Big\}, \end{aligned} \tag{5.44}$$

where

$$m_A^2 \equiv g_4^2 v_1^2 = g_4^2 \frac{\mu^2}{\lambda}. \tag{5.45}$$

To summarize, the zero-mode effective theory for the Abelian case has two real scalars with equal mass and a massive gauge boson. The above action is invariant under a $\mathbb{Z}_2$ symmetry: $h \to h$ and $\chi \to -\chi$.

5.2.2. SSB by a vacuum expectation value of the 5D Higgs field

In this subsection, we write the complex scalar fields $H^{(\pm)}$ as

$$\begin{pmatrix} H^{(+)}(x,y) \\ H^{(-)}(x,y) \end{pmatrix} = \frac{1}{\sqrt{2}} e^{ig_5(\pi^{(+)}(x,y)\mathbb{1}+\pi^{(-)}(x,y)\tau_1)} \begin{pmatrix} v(y) + h^{(+)}(x,y) \\ h^{(-)}(x,y) \end{pmatrix}. \tag{5.46}$$

As mentioned above, the vev $v(y)$ is only associated with the even Higgs field $H^{(+)}$. The fluctuations $h^{(+)}(x,y)$ and $\pi^{(+)}(x,y)$ are even, whereas the fluctuations $h^{(-)}(x,y)$ and $\pi^{(-)}(x,y)$ are odd under the warped KK-parity.

We can write the action Eq. (5.12) up to quadratic order in fields as

$$\begin{aligned} S^{(2)}_{Ab} = -\int d^5x \Big\{ & \frac{1}{4} F^{(+)}_{\mu\nu} F^{(+)\mu\nu} + \frac{1}{2} e^{2A(y)} \Big((\partial_\mu A_5^{(+)})^2 + (\partial_5 A_\mu^{(+)})^2 + g_5^2 v^2 A_\mu^{(+)} A^{(+)\mu} \Big) \\ & + \Big(e^{2A(y)} g_5^2 v^2 \pi^{(+)} - \partial_5 (e^{2A(y)} A_5^{(-)}) \Big) \partial_\mu A^{(+)\mu} + \frac{1}{2} e^{2A(y)} \Big((\partial_\mu h^{(+)})^2 + g_5^2 v^2 (\partial_\mu \pi^{(+)})^2 \Big) \\ & + \frac{1}{2} e^{4A(y)} \Big((\partial_5 v + \partial_5 h^{(+)})^2 + g_5^2 v^2 \Big(A_5^{(-)} - \partial_5 \pi^{(+)} \Big)^2 + \mu_B^2 (v + h^{(+)})^2 \Big) \\ & + \frac{1}{4} F^{(-)}_{\mu\nu} F^{(-)\mu\nu} + \frac{1}{2} e^{2A(y)} \Big((\partial_\mu A_5^{(-)})^2 + (\partial_5 A_\mu^{(-)})^2 + g_5^2 v^2 A_\mu^{(-)} A^{(-)\mu} \Big) \\ & + \Big(e^{2A(y)} g_5^2 v^2 \pi^{(-)} - \partial_5 (e^{2A(y)} A_5^{(+)}) \Big) \partial_\mu A^{(-)\mu} + \frac{1}{2} e^{2A(y)} \Big((\partial_\mu h^{(-)})^2 + g_5^2 v^2 (\partial_\mu \pi^{(-)})^2 \Big) \\ & + \frac{1}{2} e^{4A(y)} \Big((\partial_5 h^{(-)})^2 + g_5^2 v^2 \Big(A_5^{(+)} - \partial_5 \pi^{(-)} \Big)^2 + \mu_B^2 h^{(-)2} \Big) \Big\}. \end{aligned} \tag{5.47}$$

where the indices are raised and lowered by the Minkowski metric. The bulk equation of motion for the background Higgs vev corresponding to the above action is

$$\Big(-\frac{1}{2} \partial_5 \left(e^{4A(y)} \partial_5 \right) + \frac{1}{2} \mu_B^2 e^{4A(y)} \Big) v(y) = 0, \tag{5.48}$$

and the bulk equations of motion for all the fluctuations are

$$\Big(-\frac{1}{2} e^{2A(y)} \Box^{(4)} - \frac{1}{2} \partial_5 \left(e^{4A(y)} \partial_5 \right) + \frac{1}{2} \mu_B^2 e^{4A(y)} \Big) h^{(\pm)}(x,y) = 0, \tag{5.49}$$

$$\Box^{(4)} A_\mu^{(\pm)} + \partial_5 \Big(M_A^2 \partial_5 A_\mu^{(\pm)} \Big) - M_A^2 A_\mu^{(\pm)} - \partial_\mu \Big(\partial^\nu A_\nu^{(\pm)} + \partial_5 (e^{2A(y)} A_5^{(\mp)}) - M_A^2 \pi^{(\pm)} \Big) = 0, \tag{5.50}$$

$$\Box^{(4)} A_5^{(\pm)} - \partial_5 \Big(\partial^\nu A_\nu^{(\mp)} \Big) - M_A^2 \Big(A_5^{(\pm)} - \partial_5 \pi^{(\mp)} \Big) = 0, \tag{5.51}$$

$$\Box^{(4)} \pi^{(\pm)} - \partial^\nu A_\nu^{(\pm)} - M_A^{-2} \partial_5 \Big(M_A^2 e^{2A(y)} (A_5^{(\mp)} - \partial_5 \pi^{(\pm)}) \Big) = 0, \tag{5.52}$$

where $M_A^2 \equiv g_5^2 v^2(y) e^{2A(y)}$. The jump conditions at the UV-brane following from the equations of motion above are:

$$\left(\partial_5 - \frac{\partial V_{UV}(v)}{\partial v} \right) v(y) \Big|_{0^+} = 0, \qquad \left(\partial_5 - \frac{\partial^2 V_{UV}(v)}{\partial v^2} \right) h^{(+)}(x,y) \Big|_{0^+} = 0, \tag{5.53}$$

whereas the odd fields must vanish at $y = 0$. In addition, we choose the boundary conditions at $\pm L$ following the logic of our earlier discussion (see Eq. (5.27) and Appendix B.2):

$$\left(\pm\partial_5 + \frac{\partial V_{IR}(v)}{\partial v}\right) v(y)\Big|_{\pm L^{\mp}} = 0, \qquad \left(\pm\partial_5 + \frac{\partial^2 V_{IR}(v)}{\partial v^2}\right) h^{(\pm)}(x,y)\Big|_{\pm L^{\mp}} = 0, \tag{5.54}$$

$$\partial_\mu A_5^{(\pm)}(x,y) - \partial_5 A_\mu^{(\mp)}(x,y)\Big|_{\pm L^{\mp}} = 0, \qquad A_5^{(\mp)}(x,y) - \partial_5 \pi^{(\pm)}(x,y)\Big|_{\pm L^{\mp}} = 0. \tag{5.55}$$

In the above action there are mixing terms of $\partial_\mu A^{(\pm)\mu}$ with the scalars $\pi^{(\pm)}$ and $A_5^{(\pm)}$, which can be canceled by adding the following gauge fixing Lagrangian to the above action,

$$\begin{aligned} S_{GF} = -\int d^5x \Bigg\{ & \frac{1}{2\xi}\left[\partial_\mu A^{\mu(+)} - \xi\left(M_A^2 \pi^{(+)} - \partial_5\left(e^{2A(y)} A_5^{(-)}\right)\right)\right]^2 \\ & + \frac{1}{2\xi}\left[\partial_\mu A^{\mu(-)} - \xi\left(M_A^2 \pi^{(-)} - \partial_5\left(e^{2A(y)} A_5^{(+)}\right)\right)\right]^2 \Bigg\}. \end{aligned} \tag{5.56}$$

One can identify the Goldstone modes from the above two Eqs. (5.47) and (5.56):

$$\Pi^{(\pm)}(x,y) \equiv M_A^2 \pi^{(\pm)} - \partial_5\big(e^{2A(y)} A_5^{(\mp)}\big), \tag{5.57}$$

along with the two pseudoscalars $\mathcal{A}_5^{(\pm)}(x,y)$ given as

$$\mathcal{A}_5^{(\pm)}(x,y) \equiv A_5^{(\pm)} - \partial_5 \pi^{(\mp)}. \tag{5.58}$$

The resulting four pseudoscalars above along with the two $h^{(\pm)}$ scalar fields agrees with the naive counting before SSB of three even-parity scalars ($h^{(+)}(x,y)$, $\pi^{(+)}(x,y)$ and $A_5^{(+)}(x,y)$) and three odd-parity scalars ($h^{(-)}(x,y)$, $\pi^{(-)}(x,y)$ and $A_5^{(-)}(x,y)$). It is seen from the Eq. (5.47) that both the even and odd gauge bosons $A_\mu^{(\pm)}(x,y)$ acquire mass from the Higgs mechanism, whereas the two Goldstone bosons are eaten up by these gauge bosons.

In order to obtain an effective 4D Lagrangian we need to integrate the above quadratic Lagrangian over the y-coordinate. The first step to achieve this is to decompose all the fields in KK-modes. We will use the following decomposition,

$$A_\mu^{(\pm)}(x,y) = \sum_n A_{\mu n}^{(\pm)}(x)\tilde{a}_n^{(\pm)}(y), \qquad h^{(\pm)}(x,y) = \sum_n h_n^{(\pm)} \tilde{f}_n^{(\pm)}(y), \tag{5.59}$$

$$\Pi^{(\pm)}(x,y) = \sum_n \Pi_n^{(\pm)}(x)\tilde{a}_n^{(\pm)}(y)\tilde{m}_{A_n}^{(\pm)}, \qquad \mathcal{A}_5^{(\pm)}(x,y) = \sum_n \mathcal{A}_n^{(\pm)}(x)\eta_n^{(\pm)}(y), \tag{5.60}$$

where $\tilde{a}_n^{(\pm)}(y)$, $\eta_n^{(\pm)}(y)$ and $\tilde{f}_n^{(\pm)}(y)$ are the 5D profiles for the vector fields (the same for the Goldstone fields), the pseudoscalars and the Higgs bosons, respectively. The e.o.m. for the wave-functions $\tilde{f}_n^{(\pm)}(y)$, $\tilde{a}_n^{(\pm)}(y)$ and $\eta_n^{(\pm)}(y)$ are

$$-\partial_5(e^{4A(y)}\partial_5 \tilde{f}_n^{(\pm)}(y)) + \mu_B^2 e^{4A(y)} \tilde{f}_n^{(\pm)}(y) = \tilde{m}_n^{(\pm)2} e^{2A(y)} \tilde{f}_n^{(\pm)}(y), \tag{5.61}$$

$$-\partial_5(e^{2A(y)}\partial_5 \tilde{a}_n^{(\pm)}(y)) + M_A^2 \tilde{a}_n^{(\pm)}(y) = \tilde{m}_{A_n}^{(\pm)2} \tilde{a}_n^{(\pm)}(y), \tag{5.62}$$

$$-\partial_5\left(M_A^{-2}\partial_5(M_A^2 e^{2A(y)}\eta_n^{(\pm)}(y))\right) + M_A^2 \eta_n^{(\pm)}(y) = m_{\mathcal{A}_n}^{(\pm)2}\eta_n^{(\pm)}(y), \tag{5.63}$$

where $\tilde{m}_n^{(\pm)}$, $\tilde{m}_{\mathcal{A}_n}^{(\pm)}$ and $m_{\mathcal{A}_n}^{(\pm)}$ are KK-masses for $h_n^{(\pm)}$, $A_{\mu n}^{(\pm)}(x)$ and $\phi_n^{(\pm)}(x)$. The Higgs profiles $\tilde{f}_n^{(\pm)}(y)$ are exactly the same as in Appendix B.2 since they follow the same e.o.m and b.c.; thus, we borrow the results here. The normalization conditions for the wave-functions $\tilde{a}_n(y)$ and $\eta_n(y)$ are

$$\int_{-L}^{+L} dy \tilde{a}_m^{(\pm)}(y)\tilde{a}_n^{(\pm)}(y) = \delta_{mn}, \qquad \int_{-L}^{+L} dy \frac{M_A^2 e^{2A(y)}}{m_{\mathcal{A}_m}^{(\pm)} m_{\mathcal{A}_n}^{(\pm)}} \eta_m^{(\pm)}(y)\eta_n^{(\pm)}(y) = \delta_{mn}. \tag{5.64}$$

Following the general strategy mentioned in Sec. 5.1 and Appendix B.2, we choose the $y = 0$ b.c. for the even wave functions as Neumann (or mixed) b.c., whereas *all* the odd-mode wave functions satisfy Dirichlet b.c. at $y = 0$:

$$\partial_5 \tilde{a}_n^{(+)}(y)\Big|_0 = 0, \qquad \tilde{a}_n^{(-)}(y)\Big|_0 = 0, \qquad \partial_5 \eta_n^{(+)}(y)\Big|_0 = 0, \qquad \eta_n^{(-)}(y)\Big|_0 = 0. \tag{5.65}$$

The b.c. for wave-functions $\tilde{a}_n^{(\pm)}$ and $\eta_n^{(\pm)}$ at $y = \pm L$ follow from Eqs. (5.54)-(5.55),

$$\left(\pm\partial_5 - \frac{m_{IR}^2}{2k} + \frac{3\lambda_{IR}}{2k^2}v^2(y)\right)\tilde{f}_n^{(\pm)}(y)\Big|_{\pm L^\mp} = 0, \quad \partial_5 \tilde{a}_n^{(\pm)}(y)\Big|_{\pm L} = 0, \quad \eta_n^{(\pm)}(y)\Big|_{\pm L} = 0. \tag{5.66}$$

One can also easily find the KK-decomposition of the fluctuation fields $A_5^{(\pm)}(x,y)$ and $\pi^{(\pm)}(x,y)$ in terms of Goldstone bosons $\Pi^{(\pm)}$ and the physical scalars $\mathcal{A}_5^{(\pm)}$ from Eqs. (5.57)-(5.58):

$$A_5^{(\pm)}(x,y) = \sum_n \left(\frac{\Pi_n^{(\mp)}(x)}{\tilde{m}_{\mathcal{A}_n}^{(\mp)}}\partial_5 \tilde{a}_n^{(\mp)}(y) - \frac{M_A^2}{(m_{\mathcal{A}_n}^{(\pm)})^2}\mathcal{A}_n^{(\pm)}(x)\eta_n^{(\pm)}(y)\right), \tag{5.67}$$

$$\pi^{(\pm)}(x,y) = \sum_n \left(\frac{\Pi_n^{(\pm)}(x)}{\tilde{m}_{\mathcal{A}_n}^{(\pm)}}\tilde{a}_n^{(\pm)}(y) - \frac{M_A^{-2}}{(m_{\mathcal{A}_n}^{(\mp)})^2}\partial_5\left(M_A^2 e^{2A(y)}\eta_n^{(\mp)}(y)\right)\mathcal{A}_n^{(\mp)}(x)\right), \tag{5.68}$$

Now we consider the low-energy effective theory obtained by assuming the KK-mass scale is high enough so that we can integrate out all the heavier KK modes and keep only the zero-modes of the theory. From here on, we choose the unitary gauge such that $\xi \to \infty$ which implies $\Pi_n^{(\pm)}(x) \to 0$. Moreover, with our choice of boundary conditions for $a_0^{(-)}(y)$ and $\eta_0^{(\pm)}(y)$ in Eqs. (5.65) and (5.66) one can see that the corresponding wave-functions for zero-modes are vanishing, i.e. there will be no zero-modes $A_{0\mu}^{(-)}(x)$ and $\mathcal{A}_0^{(\pm)}(x)$ in our effective theory. The y-dependent vev and zero-mode profiles for even and odd Higgs are (see Appendix B.2):

$$v(y) \equiv v_4 f_v(y), \quad \text{with} \quad v_4 \equiv \mu/\sqrt{\lambda}, \qquad f_v(y) \equiv \sqrt{k(1+\beta)}e^{kL}e^{(2+\beta)k(|y|-L)}, \tag{5.69}$$

$$\tilde{f}_0^{(\pm)}(|y|) \approx \sqrt{k(1+\beta)}e^{kL}e^{(2+\beta)k(|y|-L)}, \qquad \tilde{f}_0^{(-)}(y) = \epsilon(y)\tilde{f}_0^{(-)}(|y|), \tag{5.70}$$

where $\mu^2 \equiv (1+\beta)m_{KK}^2\delta_{IR}$ and $\lambda \equiv \lambda_{IR}(1+\beta)^2$. It is important to comment here that at the leading order the vev profile and zero-mode profiles are the same. However, there are finite corrections which are suppressed by $\mathcal{O}\left(m_h^2/m_{KK}^2\right)$ as given below and also depicted in Fig. 5.1,

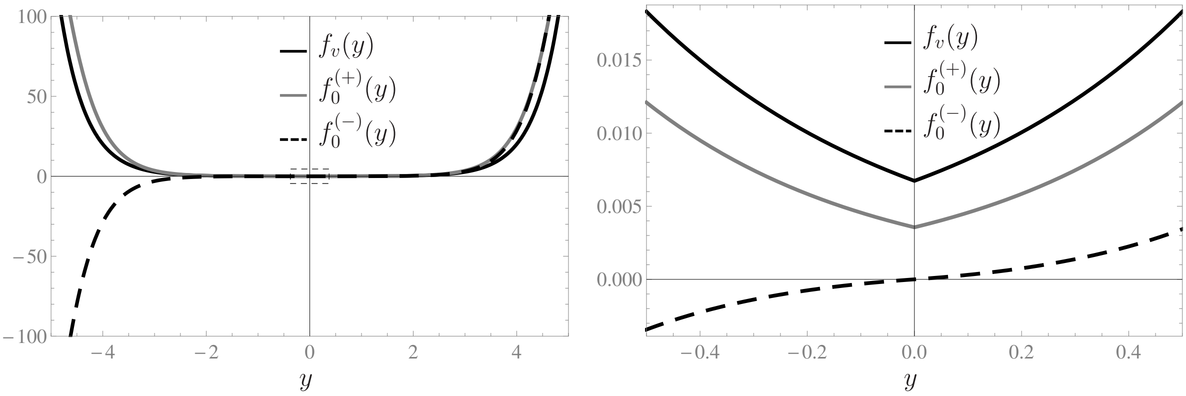


Figure 5.1.: The left graph shows the profile of the vev $f_v(y)$ and the zero-mode profiles $\tilde{f}_0^{(\pm)}(y)$ as functions of y with all the fundamental parameters of order one and $kL = 5$. (We use a mild value of kL to show the small differences near the IR-brane which will be hard to see if we use $kL \simeq 35$, this latter value being required (see the main text) to solve the hierarchy problem.) The right plot is the same as the left but focused near the origin.

$$\frac{\tilde{f}_0^{(\pm)}(|y|)}{f_v(|y|)} = 1 + \frac{m_h^2}{m_{KK}^2}\left(\frac{1-e^{2k(|y|-L)}}{4(1+\beta)} + \mathcal{O}\Big(\frac{m_h^2}{m_{KK}^2}\Big)\right). \tag{5.71}$$

We can now write down the effective theory for the zero-modes in the unitary gauge:

$$\begin{aligned} S_{eff} = -\int d^4x \Big\{ & \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\tilde{m}_A^2 A_\mu A^\mu + \frac{1}{2}\partial_\mu h \partial^\mu h + \frac{1}{2}\tilde{m}_h^2 h^2 + \frac{1}{2}\partial_\mu \chi \partial^\mu \chi + \frac{1}{2}\tilde{m}_\chi^2 \chi^2 \\ & + \frac{1}{4}\lambda h^4 + \frac{1}{4}\lambda \chi^4 + \frac{3}{2}\lambda h^2 \chi^2 + \lambda v_4 h\Big(h^2 + 3\chi^2\Big) \\ & + \tilde{g}_4^2 v_4 h A_\mu A^\mu + \frac{1}{2}\tilde{g}_4^2\Big(h^2 + \chi^2\Big)A_\mu A^\mu \Big\}, \end{aligned} \tag{5.72}$$

where we have denoted $A_{0\mu}^{(+)}(x) \equiv A_\mu(x)$ and we have suppressed the zero-mode subscript '0' for all modes. After some algebra, using the boundary conditions, one can find the masses of the zero-mode scalars and gauge boson at the leading order:

$$\tilde{m}_h^2 = \tilde{m}_\chi^2 \simeq 2\mu^2, \qquad\qquad \tilde{m}_A^2 \simeq \frac{1}{2L}\int_{-L}^{L} dy M_A^2 = \tilde{g}_4^2 v_4^2, \tag{5.73}$$

where $\mu^2 \equiv (1+\beta)m_{KK}^2\delta_{IR}$, $v_4 \equiv \mu/\lambda$ and $\tilde{g}_4 \equiv g_5/\sqrt{2L}$.

Comparison: In order to facilitate comparison between the two approaches, we collect information concerning all the low-energy degrees of freedom for both pictures in Table 5.1. Comparing the effective theories obtained within EWSB induced by the Higgs KK-mode vev and by a 5D-Higgs vev in (5.44) and (5.72) one finds that both approaches give exactly the *same* zero-mode effective theory up to $\mathcal{O}(m_h^2/m_{KK}^2 \sim 10^{-3})$ corrections. We have checked that the scalar masses are exactly same to all orders in the expansion parameter m_h^2/m_{KK}^2. In contrast, the gauge boson masses and the couplings can have subleading differences of order $\mathcal{O}(m_h^2/m_{KK}^2)$. Note that we have neglected all the effects due to the non-zero KK-modes, such effects being suppressed by their masses, i.e. $\mathcal{O}(m_h^2/m_n^2)$. Hence we conclude that the two approaches to EWSB discussed above give the same low-energy (zero-mode) effective theory

	EWSB by KK mode vev			EWSB by 5D Higgs vev		
5D fields	KK-modes	$n=0$	$n\neq 0$	KK-modes	$n=0$	$n\neq 0$
Re$H^{(+)}$	$\phi_n^{(+)}(x)$	✓	✓	$h_n(x)$	✓	✓
Re$H^{(-)}$	$\phi_n^{(-)}(x)$	✓	✓	$\chi_n(x)$	✓	✓
Im$H^{(+)}$	$\pi_n^{(+)}(x)$	✗(4D g.c.)	✓	$\Pi_n^{(+)}(x)$	✗(4D g.c.)	✗(4D g.c.)
Im$H^{(-)}$	$\pi_n^{(-)}(x)$	✗(b.c.)	✓	$\Pi_n^{(-)}(x)$	✗(4D g.c.)	✗(4D g.c.)
$A_5^{(+)}$	$A_{5n}^{(+)}(x)$	✗(5D g.c.)	✗(5D g.c.)	$\mathcal{A}_n^{(+)}(x)$	✗(b.c.)	✓
$A_5^{(-)}$	$A_{5n}^{(-)}(x)$	✗(5D g.c.)	✗(5D g.c.)	$\mathcal{A}_n^{(-)}(x)$	✗(b.c.)	✓
$A_\mu^{(+)}$	$A_{\mu n}^{(+)}(x)$	✓	✓	$A_{\mu n}^{(+)}(x)$	✓	✓
$A_\mu^{(-)}$	$A_{\mu n}^{(-)}(x)$	✗(b.c.)	✓	$A_{\mu n}^{(-)}(x)$	✗(b.c.)	✓

Table 5.1.: Comparison of dynamical d.o.f. between KK-mode-vev and 5D-Higgs-vev EWSB. The b.c. (boundary condition) and g.c. (gauge choice) show why a given mode is not present in the corresponding effective theory. Note that $\Pi_n^{(\pm)}$ is a mixture of $\pi_n^{(\pm)}$ and $A_{5n}^{(\pm)}$, see Eq. (5.57).

EWSB by KK mode vev	EWSB by 5D Higgs vev	Comment
$f_0(y)\simeq\sqrt{k(1+\beta)}e^{kL}e^{(2+\beta)k(\lvert y\rvert-L)}$	$\tilde f_0(y)\simeq\sqrt{k(1+\beta)}e^{kL}e^{(2+\beta)k(\lvert y\rvert-L)}$	same
$v_1^2=\frac{\mu^2}{\lambda}\left(1-\mathcal{O}\left(\frac{m_h^2}{m_{KK}^2}\right)\right)$	$v_4^2=\frac{\mu^2}{\lambda}$ and $f_v(y)=\tilde f_0(y)$	$\mathcal{O}\left(\frac{m_h^2}{m_{KK}^2}\right)$
$m_h^2=m_\chi^2=2\mu^2\left(1-\mathcal{O}\left(\frac{m_h^2}{m_{KK}^2}\right)\right)$	$\tilde m_h^2=\tilde m_\chi^2=2\mu^2\left(1-\mathcal{O}\left(\frac{m_h^2}{m_{KK}^2}\right)\right)$	same
$a_0(y)=\frac{1}{\sqrt{2L}}$	$\tilde a_0(y)=\frac{1}{\sqrt{2L}}\left(1+\mathcal{O}\left(\frac{m_h^2}{m_{KK}^2}\right)\right)$	$\mathcal{O}\left(\frac{m_h^2}{m_{KK}^2}\right)$
$g_4=\frac{g_5}{\sqrt{2L}}$	$\tilde g_4=\frac{g_5}{\sqrt{2L}}\left(1-\mathcal{O}\left(\frac{m_h^2}{m_{KK}^2}\right)\right)$	$\mathcal{O}\left(\frac{m_h^2}{m_{KK}^2}\right)$
$m_A^2=\frac{g_5^2}{2L}\frac{\mu^2}{\lambda}\left(1-\mathcal{O}\left(\frac{m_h^2}{m_{KK}^2}\right)\right)$	$\tilde m_A^2=\frac{g_5^2}{2L}\frac{\mu^2}{\lambda}\left(1-\mathcal{O}\left(\frac{m_h^2}{m_{KK}^2}\right)\right)$	$\mathcal{O}\left(\frac{m_h^2}{m_{KK}^2}\right)$

Table 5.2.: Comparison of the effective parameters in terms of the fundamental parameters of the 5D theory, for $\mu^2\equiv(1+\beta)m_{KK}^2\delta_{IR}$ and $\lambda=(1+\beta)^2\lambda_{IR}$. Here we explicitly show the presence of corrections of order of the expansion parameter, $m_h^2/m_{KK}^2\sim 10^{-3}$, and we neglect effects $\mathcal{O}(e^{-2\beta kL})$; the latter ones are extremely small for $\beta>0$ and $kL\sim 35$.

aside from small deviations of order $\mathcal{O}(m_h^2/m_{KK}^2)$. To make this comparison more transparent we summarize the parameters of both effective theories in terms of the fundamental parameters of the 5D theory in Table 5.2. The observed agreement is a non-trivial verification of the results obtained here.

5.3. SM EWSB by a bulk Higgs doublet

In this section we consider all the SM fields in the bulk and study phenomenological implications of our symmetric geometry. Note that vev's of effective 4D scalar fields are of the electroweak scale which is much smaller than the gravity (Planck mass) scale, therefore their back-reaction on the background geometry would be negligible, see e.g. [128]. Therefore we employ the background gravitational solution (1.1) throughout the manuscript.

The 5D action for the electroweak sector of the SM can be written as

$$S=-\int d^5x\sqrt{-g}\bigg\{\frac{1}{4}F^a_{MN}F^{aMN}+\frac{1}{4}B_{MN}B^{MN}+|D_MH|^2+\mu_B^2|H|^2 + V_{IR}(H)\delta(y+L)+V_{UV}(H)\delta(y)+V_{IR}(H)\delta(y-L)\bigg\}, \tag{5.74}$$

where F^a_{MN} and B_{MN} are the 5D field strength tensors for $SU(2)$ and $U(1)_Y$, respectively with a numbering generators of $SU(2)$. Above, H is the $SU(2)$ doublet and its brane potentials are

$$V_{UV}(H) = \frac{m^2_{UV}}{k}|H|^2, \qquad V_{IR}(H) = -\frac{m^2_{IR}}{k}|H|^2 + \frac{\lambda_{IR}}{k^2}|H|^4. \tag{5.75}$$

In our approach, we do not put the Higgs quartic terms in the bulk nor on the UV-brane since we want EWSB to take place near the IR-brane [3]. The covariant derivative D_M is defined as follows:

$$D_M = \partial_M - i\frac{g_5}{2}\tau^a A^a_M - i\frac{g'_5}{2}B_M, \tag{5.76}$$

where τ^a are Pauli matrices and $g_5(g'_5)$ is the coupling constant for the $A^a_M(B_M)$ fields.

It is instructive to make the usual redefinition of the gauge fields,

$$W^{\pm}_M \equiv \frac{1}{\sqrt{2}}\Big(A^1_M \mp iA^2_M\Big), \tag{5.77}$$

$$Z_M \equiv \frac{1}{\sqrt{g_5^2 + g'^2_5}}\Big(g_5 A^3_M - g'_5 B_M\Big), \tag{5.78}$$

$$A_M \equiv \frac{1}{\sqrt{g_5^2 + g'^2_5}}\Big(g'_5 A^3_M + g_5 B_M\Big). \tag{5.79}$$

Analogous to the 4D procedure, we define the 5D Weinberg angle θ as follows:

$$\cos\theta \equiv \frac{g_5}{\sqrt{g_5^2 + g'^2_5}}, \qquad \sin\theta \equiv \frac{g'_5}{\sqrt{g_5^2 + g'^2_5}}. \tag{5.80}$$

The 5D gauge fields corresponding to the gauge group $SU(2)\times U(1)_Y$ are then

$$\mathbb{A}_M(x,y) \equiv \begin{pmatrix} \sin\theta A_M + \frac{\cos^2\theta - \sin^2\theta}{2\cos\theta} Z_M & \frac{1}{\sqrt{2}}W^+_M \\ \frac{1}{\sqrt{2}}W^-_M & -\frac{1}{2\cos\theta}Z_M \end{pmatrix}. \tag{5.81}$$

The gauge transformations for the Higgs doublet $H(x,y)$ and gauge fields $\mathbb{A}_M$ under the gauge group $SU(2)\times U(1)_Y$ can be written as

$$H(x,y) \to H'(x,y) = U(x,y)H(x,y), \tag{5.82}$$

$$\mathbb{A}_M(x,y) \to \mathbb{A}'_M(x,y) = U(x,y)\mathbb{A}_M(x,y)U^{-1}(x,y) - \frac{i}{g_5}\big(\partial_M U(x,y)\big)U^{-1}(x,y), \tag{5.83}$$

where $U(x,y)$ is the unitary matrix corresponding to the fundamental representation of $SU(2)\times U(1)_Y$ gauge transformations.

We will choose the 5D axial gauge analogous to the Abelian case by taking $\mathbb{A}_5(x,y) = 0$. Note that we can always find $U(x,y)$ such that the axial gauge is manifest, i.e. $\mathbb{A}_5(x,y) = 0$.

[3] The UV Higgs quartic operator, i.e. $V_{UV}(H) \supset \lambda_{UV}/k^2|H|^4$ is highly suppressed as $\lambda_{UV}/k^2 \sim \mathcal{O}(M_{\rm Pl}^{-2})$. Whereas, for the IR Higgs quartic operator suppression in the λ_{IR}/k^2 is reduced to $\sim \mathcal{O}(m_{KK}^{-2})$ due to the non-trivial warp factor at the IR-brane, see also [111]. Similarly the bulk quartic term would also be suppressed by some intermediate scale. For simplicity we ignore those options.

We employ an axial gauge choice for the non-Abelian case of the form

$$U(x,y) = \widehat{U}(x)\mathcal{P}e^{-ig_5\int_0^y dy'\mathbb{A}_5(x,y')}, \tag{5.84}$$

where $\widehat{U}(x)$ is the residual 4D gauge transformation and $\mathcal{P}$ denotes path-ordering of the exponential. Another key point for the later discussion is that this 4D residual gauge transformation $\widehat{U}(x)$ is independent of y and thus automatically even under the geometric parity. As we have demonstrated in Sec. 5.2, due to the symmetric geometry the background fields in the IR-UV-IR setup separate into even and odd bulk wave functions. Hence, it is straightforward to generalize the results obtained in Sec. 5.2 for the Abelian model to the electroweak sector of the SM. Let us start by decomposing the Higgs doublet and gauge fields into components of definite parity as follows:

$$H(x,y) = H^{(+)}(x,y) + H^{(-)}(x,y), \qquad V_M(x,y) = V_M^{(+)}(x,y) + V_M^{(-)}(x,y), \tag{5.85}$$

where $V_M \equiv (A_M, W_M^{\pm}, Z_M)$. We can write the action (5.74) up to quadratic level in the $\mathbb{A}_5(x,y) = 0$ gauge as

$$\begin{aligned} S^{(2)} = -\int d^5x\sqrt{-g}\Big\{ & \frac{1}{2}\mathcal{W}^{+}_{(+)\mu\nu}\mathcal{W}^{-\mu\nu}_{(+)} + \partial_5 W^{+}_{(+)\mu}\partial^5 W^{-\mu}_{(+)} + \frac{1}{4}\mathcal{Z}^{(+)}_{\mu\nu}\mathcal{Z}^{\mu\nu}_{(+)} + \frac{1}{2}\partial_5 Z^{(+)}_{\mu}\partial^5 Z^{\mu}_{(+)} \\ & + \frac{1}{2}\mathcal{W}^{+}_{(-)\mu\nu}\mathcal{W}^{-\mu\nu}_{(-)} + \partial_5 W^{+}_{(-)\mu}\partial^5 W^{-\mu}_{(-)} + \frac{1}{4}\mathcal{Z}^{(-)}_{\mu\nu}\mathcal{Z}^{\mu\nu}_{(-)} + \frac{1}{2}\partial_5 Z^{(-)}_{\mu}\partial^5 Z^{\mu}_{(-)} \\ & + \frac{1}{4}\mathcal{F}^{(+)}_{\mu\nu}\mathcal{F}^{\mu\nu}_{(+)} + \frac{1}{2}\partial_5 A^{(+)}_{\mu}\partial_5 A^{\mu}_{(+)} + \frac{1}{4}\mathcal{F}^{(-)}_{\mu\nu}\mathcal{F}^{\mu\nu}_{(-)} + \frac{1}{2}\partial_5 A^{(-)}_{\mu}\partial^5 A^{\mu}_{(-)} \\ & + \mathbb{D}_M H^{(+)\dagger}\mathbb{D}^M H^{(+)} + \mu_B^2|H^{(+)}|^2 + \mathbb{D}_M H^{(-)\dagger}\mathbb{D}^M H^{(-)} + \mu_B^2|H^{(-)}|^2 \\ & + \frac{m_{UV}^2}{k}|H^{(+)}|^2\delta(y) - \frac{m_{IR}^2}{k}\left(|H^{(+)}|^2 + |H^{(-)}|^2\right)\left[\delta(y+L) + \delta(y-L)\right]\Big\}, \end{aligned} \tag{5.86}$$

where we have adopted the following definitions:

$$\tilde{\mathcal{V}}^{(\pm)}_{\mu\nu} \equiv \partial_\mu \tilde{V}^{(\pm)}_\nu - \partial_\nu \tilde{V}^{(\pm)}_\mu, \qquad \mathcal{F}^{(\pm)}_{\mu\nu} \equiv \partial_\mu A^{(\pm)}_\nu - \partial_\nu A^{(\pm)}_\mu, \tag{5.87}$$

$$\mathbb{D}_\mu \begin{pmatrix} H^{(+)} \\ H^{(-)} \end{pmatrix} \equiv \left[\partial_\mu - ig_5\begin{pmatrix} \mathbb{A}^{(+)}_\mu & \mathbb{A}^{(-)}_\mu \\ \mathbb{A}^{(-)}_\mu & \mathbb{A}^{(+)}_\mu \end{pmatrix}\right]\begin{pmatrix} H^{(+)} \\ H^{(-)} \end{pmatrix}, \tag{5.88}$$

$$\mathbb{D}_5 \begin{pmatrix} H^{(+)} \\ H^{(-)} \end{pmatrix} \equiv \left[\partial_5 - ig_5\begin{pmatrix} \mathbb{A}^{(-)}_5 & \mathbb{A}^{(+)}_5 \\ \mathbb{A}^{(+)}_5 & \mathbb{A}^{(-)}_5 \end{pmatrix}\right]\begin{pmatrix} H^{(+)} \\ H^{(-)} \end{pmatrix}, \tag{5.89}$$

where $\tilde{V}_\mu \equiv (W^{\pm}_\mu, Z_\mu)$ and $\mathbb{A}^{(\pm)}_M$ was defined in (5.81). It is convenient to write the Higgs doublets in the following form:

$$\begin{pmatrix} H^{(+)} \\ H^{(-)} \end{pmatrix} = e^{ig_5(\Pi^{(+)}\mathbb{1} + \Pi^{(-)}\tau_1)}\begin{pmatrix} \mathcal{H}^{(+)} \\ \mathcal{H}^{(-)} \end{pmatrix}, \tag{5.90}$$

where $\mathcal{H}$ and Π are defined as (the parity indices are suppressed)

$$\mathcal{H}(x,y) \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ h(x,y) \end{pmatrix}, \qquad \Pi(x,y) \equiv \begin{pmatrix} \frac{\cos^2\theta - \sin^2\theta}{2\cos\theta}\pi_Z & \frac{1}{\sqrt{2}}\pi_W^+ \\ \frac{1}{\sqrt{2}}\pi_W^- & -\frac{1}{2\cos\theta}\pi_Z \end{pmatrix}. \tag{5.91}$$

We KK-decompose the Higgs doublets $H^{(\pm)}(x,y)$ and the gauge fields $V_\mu^{(\pm)}(x,y)$ as

$$\mathcal{H}^{(\pm)}(x,y) = \sum_n \mathcal{H}_n^{(\pm)}(x) f_n^{(\pm)}(y), \tag{5.92}$$

$$\pi_{\tilde{V}}^{(\pm)}(x,y) = \sum_n \pi_{\tilde{V}n}^{(\pm)}(x) a_{\tilde{V}n}^{(\pm)}(y), \qquad V_\mu^{(\pm)}(x,y) = \sum_n V_{\mu n}^{(\pm)}(x) a_{V_n}^{(\pm)}(y), \tag{5.93}$$

where the wave-functions $f_n^{(\pm)}(y)$ and $a_{V_n}^{(\pm)}(y)$ satisfy

$$-\partial_5\left(e^{4A(y)}\partial_5 f_n^{(\pm)}(y)\right) + \mu_B^2 e^{4A(y)} f_n^{(\pm)}(y) = m_n^{2(\pm)} e^{2A(y)} f_n^{(\pm)}(y), \tag{5.94}$$

$$-\partial_5\left(e^{2A(y)}\partial_5 a_{V_n}^{(\pm)}(y)\right) = m^2_{V_n^{(\pm)}} a_{V_n}^{(\pm)}(y), \tag{5.95}$$

and, for our background geometry, $A(y) = -k|y|$. The y-dependent wave functions $f_n^{(\pm)}(y)$ and $a_{V_n}^{(\pm)}(y)$ satisfy the following orthonormality conditions:

$$\int_{-L}^{+L} dy e^{2A(y)} f_m^{(\pm)}(y) f_n^{(\pm)}(y) = \delta_{mn}, \qquad \int_{-L}^{+L} dy a_{V_m}^{(\pm)}(y) a_{V_n}^{(\pm)}(y) = \delta_{mn}. \tag{5.96}$$

The even modes are subject to jump conditions at $y=0$ while the odd modes are required to vanish by continuity at $y=0$, resulting in the following boundary conditions:

$$\left(\partial_5 - \frac{m_{UV}^2}{k}\right) f_n^{(+)}(y)\Big|_0 = 0, \quad f_n^{(-)}(y)\Big|_0 = 0, \quad \partial_5 a_{V_n}^{(+)}(y)\Big|_0 = 0, \quad a_{V_n}^{(-)}(y)\Big|_0 = 0. \tag{5.97}$$

The b.c. at $y = \pm L$ are:

$$\left(\pm\partial_5 - \frac{m_{IR}^2}{k}\right) f_n^{(\pm)}(y)\Big|_{\pm L^{\mp}} = 0, \qquad \partial_5 a_{V_n}^{(\pm)}(y)\Big|_{\pm L^{\mp}} = 0. \tag{5.98}$$

As pointed out in the Abelian case, the choices of b.c. for $a_n^{(+)}(y)$ at $y = 0, \pm L$ are motivated by the requirement that the even zero-mode profiles for gauge boson be non-zero.

It is worth mentioning here that the choice of writing the Higgs doublets $H^{(\pm)}$ in the form of Eq. (5.90) and using the KK decomposition for the pseudoscalars $\pi_{\tilde{V}}^{(\pm)}$ as given in Eq. (5.93) are both motivated by model-building considerations discussed below. The other possibility is to choose different KK bases and b.c. for the pseudoscalars $\pi_{\tilde{V}}^{(\pm)}$ such that after SSB these pseudoscalars become Nambu-Goldstone bosons (NGB). The even zero-mode gauge bosons would then acquire masses by eating up the even-parity NGB, whereas the odd-parity NGB would remain in the spectrum (the odd zero-mode gauge boson fields being zero, see below). An effective potential for the odd-parity NGB would be generated through their interactions with gauge bosons, hence making them pseudo-NGB. We don't follow this approach here but it is an interesting possibility in which the neutral odd pseudo-NGB would be a composite dark Higgs in the dual CFT description.[4]

We assume that the KK-scale is high enough, i.e. $m_{KK} \sim \mathcal{O}(\text{few})$ TeV, that we can consider an effective theory where only the lowest modes (zero-modes with masses much below m_{KK}) are kept. It is important to note that the odd zero-mode wave functions obey $a_{V_0}^{(-)}(y) = 0$, as can be easily seen from Eq. (5.95) along with the b.c. (5.97) and (5.98). As a consequence of $a_{V_0}^{(-)}(y) = 0$, the odd zero-mode gauge fields $V_{0\mu}^{(-)}(x)$ and the odd Goldstone modes $\pi_{\tilde{V}_0}^{(-)}(x)$ will not be present in the effective 4D theory. Moreover, the even zero-mode gauge profile is constant, i.e. $a_{V_0}^{(+)}(y) = 1/\sqrt{2L}$. Using the results from Sec. 5.2, we can determine values of the couplings and mass parameters in the effective 4D theory in terms of the parameters of the fundamental 5D theory. The result is that we can write down the effective 4D action for the zero-modes as

$$
\begin{aligned}
S_{eff}^{(2)} = -\int d^4x \Big\{ & \frac{1}{4}\mathcal{F}_{\mu\nu}^{0(+)}\mathcal{F}_{0(+)}^{\mu\nu} + \frac{1}{4}\mathcal{Z}_{\mu\nu}^{0(+)}\mathcal{Z}_{0(+)}^{\mu\nu} + \frac{1}{2}\mathcal{W}_{\mu\nu}^{+0(+)}\mathcal{W}_{0(+)}^{-\mu\nu} + \partial_\mu \mathcal{H}_0^{(+)\dagger}\partial^\mu \mathcal{H}_0^{(+)} \\
& + \partial_\mu \mathcal{H}_0^{(-)\dagger}\partial^\mu \mathcal{H}_0^{(-)} + m_0^{2(+)}|\mathcal{H}_0^{(+)}|^2 + m_0^{2(-)}|\mathcal{H}_0^{(-)}|^2 - ig_4 \partial_\mu \mathcal{H}_0^{(+)\dagger}\mathbb{M}^\mu \mathcal{H}_0^{(+)} \\
& + ig_4 \mathcal{H}_0^{(+)\dagger}\mathbb{M}_\mu^\dagger \partial^\mu \mathcal{H}_0^{(+)} + g_4^2 \mathcal{H}_0^{(+)\dagger}\mathbb{M}_\mu^\dagger \mathbb{M}^\mu \mathcal{H}_0^{(+)} + g_4^2 \mathcal{H}_0^{(-)\dagger}\mathbb{M}_\mu^\dagger \mathbb{M}^\mu \mathcal{H}_0^{(-)} \Big\},
\end{aligned} \tag{5.99}
$$

where $\mathbb{M}_\mu$ is defined as

$$
\mathbb{M}_\mu \equiv \mathbb{U}^\dagger \hat{\mathbb{A}}_{0\mu}^{(+)}\mathbb{U} + \frac{i}{g_4}\mathbb{U}^\dagger \partial_\mu \mathbb{U}, \tag{5.100}
$$

with $\mathbb{U} \equiv e^{ig_4 \widehat{\Pi}_0^{(+)}}$ and $g_4 \equiv g_5/\sqrt{2L}$. In the above action $\mathcal{H}_0^{(\pm)}$ are real doublets defined in Eq. (5.91), implying that $\mathcal{H}_0^{(\pm)\dagger} = \mathcal{H}_0^{(\pm)\intercal}$, whereas $\hat{\mathbb{A}}_{0\mu}^{(+)}$ and $\widehat{\Pi}_0^{(+)}$ are defined as (below we suppress the parity indices and zero-mode index):

$$
\hat{\mathbb{A}}_\mu(x) \equiv \begin{pmatrix} \sin\theta A_\mu + \frac{\cos^2\theta - \sin^2\theta}{2\cos\theta} Z_\mu & \frac{1}{\sqrt{2}} W_\mu^+ \\ \frac{1}{\sqrt{2}} W_\mu^- & -\frac{1}{2\cos\theta} Z_\mu \end{pmatrix}, \tag{5.101}
$$

$$
\widehat{\Pi}(x) \equiv \begin{pmatrix} \frac{\cos^2\theta - \sin^2\theta}{2\cos\theta}\pi_Z & \frac{1}{\sqrt{2}}\pi_W^+ \\ \frac{1}{\sqrt{2}}\pi_W^- & -\frac{1}{2\cos\theta}\pi_Z \end{pmatrix}. \tag{5.102}
$$

It is important to comment here that the above action is manifestly gauge invariant under the following $SU(2) \times U(1)_Y$ gauge transformation,

$$
\hat{\mathbb{A}}_\mu^{(+)} \to \widehat{U}\hat{\mathbb{A}}_\mu^{(+)}\widehat{U}^\dagger - \frac{i}{g_4}(\partial_\mu \widehat{U})\widehat{U}^\dagger, \qquad \mathbb{U} \to \widehat{U} e^{ig_4 \widehat{\Pi}^{(+)}}, \tag{5.103}
$$

whereas the $\mathcal{H}_0^{(\pm)}$ are gauge invariant under the 4D residual gauge transformation $\widehat{U}$. Equation (5.99) is a non-Abelian analog of the Abelian zero-mode action given by (5.32).

We introduce a convenient notion for our effective theory by redefining $V_{0\mu}^{(+)}(x) \equiv V_\mu(x)$,

[4]At the final stages of the present work, Ref. [129] appeared where the authors considered composite dark sectors. A similar construction can be naturally realized as a CFT dual to the model considered here.

$\pi^{(+)}_{\tilde{V}0}(x) \equiv \pi_{\tilde{V}}(x)$, $\widehat{\Pi}^{(+)}_0(x) \equiv \widehat{\Pi}(x)$ and

$$H_1(x) \equiv e^{ig_4\widehat{\Pi}(x)}\mathcal{H}^{(+)}_0(x), \qquad H_2(x) \equiv e^{ig_4\widehat{\Pi}(x)}\mathcal{H}^{(-)}_0(x). \tag{5.104}$$

Now the above effective action, including the scalar interaction terms, can be written in a nice gauge covariant form as[5]

$$S_{eff} = -\int d^4x\Big\{\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + \frac{1}{4}\mathcal{Z}_{\mu\nu}\mathcal{Z}^{\mu\nu} + \frac{1}{2}\mathcal{W}^{+}_{\mu\nu}\mathcal{W}^{-\mu\nu} + \big(\mathcal{D}_\mu H_1\big)^\dagger\mathcal{D}^\mu H_1 + \big(\mathcal{D}_\mu H_2\big)^\dagger\mathcal{D}^\mu H_2 + V(H_1,H_2)\Big\}, \tag{5.105}$$

where the scalar potential can be written as

$$V(H_1,H_2) = -\mu^2|H_1|^2 - \mu^2|H_2|^2 + \lambda|H_1|^4 + \lambda|H_2|^4 + 6\lambda|H_1|^2|H_2|^2. \tag{5.106}$$

The covariant derivative $\mathcal{D}_\mu$ is defined as

$$\mathcal{D}_\mu = \partial_\mu - ig_4\hat{\mathbb{A}}^{(+)}_\mu, \tag{5.107}$$

where $\hat{\mathbb{A}}_\mu$ is defined in Eq. (5.101). In the above scalar potential the mass parameter μ is defined as (see Sec. B.1 and Appendix B.1),

$$\mu^2 \equiv -m_0^{2(\pm)} = (1+\beta)m^2_{KK}\delta_{IR},$$

where δ_{IR}, m_{KK} and β are defined in Eq. (5.33).

Concerning the symmetries of the above potential, one can notice that $V(H_1,H_2)$ is invariant under $[SU(2)\times U(1)_Y]'\times[SU(2)\times U(1)_Y]$, where one of the blocks has been gauged while the other one survived as a global symmetry. The zero-modes of the four odd vector bosons ($W^{(-)\pm}_{0\mu}, Z^{(-)}_{0\mu}$ and $A^{(-)}_{0\mu}$) and the three would-be-Goldstone bosons $\Pi^{(-)}_0$ have been removed by appropriate b.c., implying that the corresponding gauge symmetry has been broken explicitly. What remains is *the truncated inert doublet model*, that contains $H_{1,2}$, and the corresponding residual $SU(2)\times U(1)_Y$ global symmetry of the action. Symmetry under the above mentioned $U(1)'\times U(1)$ implies in particular that $V(H_1,H_2)$ is also invariant under various $\mathbb{Z}_2$'s, for example $H_1\to -H_1$, $H_2\to -H_2$ and $H_1\to \pm H_2$.

As explained in the Abelian case, we choose the vacuum such that the even parity Higgs field H_1 acquires a vev, whereas the odd parity Higgs field H_2 does not, i.e.

$$v_1^2 \equiv v^2 = \frac{\mu^2}{\lambda}, \qquad v_2 = 0. \tag{5.108}$$

Let us now consider fluctuations around the vacuum of our choice

$$H_1(x) = \frac{1}{\sqrt{2}}e^{ig_4\widehat{\Pi}}\begin{pmatrix}0\\ v+h\end{pmatrix}, \qquad H_2(x) = \frac{1}{\sqrt{2}}e^{ig_4\widehat{\Pi}}\begin{pmatrix}0\\ \chi\end{pmatrix}, \tag{5.109}$$

[5]Note that the action of Eq. (5.105) is a non-Abelian version of the Abelian zero-mode action (5.36).

where $\widehat{\Pi}$ (defined in Eq. (5.102)) contains the pseudoscalar Goldstone bosons $\pi_{W^\pm,Z}$. We choose the unitary gauge in which $\pi_{W^\pm,Z}$ are gauged away, that is they are eaten up by the massive gauge bosons $W^\pm_\mu$ and Z_μ. Hence in the unitary gauge our effective action up to the quadratic order in fluctuations is

$$S^{(2)}_{eff} = -\int d^4x \Big\{ \frac{1}{2}\mathcal{W}^+_{\mu\nu}\mathcal{W}^{-\mu\nu} + \frac{1}{4}\mathcal{Z}_{\mu\nu}\mathcal{Z}^{\mu\nu} + \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + m_W^2 W^+_\mu W^{-\mu} + \frac{1}{2}m_Z^2 Z_\mu Z^\mu + \frac{1}{2}\partial_\mu h \partial^\mu h + \frac{1}{2}m_h^2 h^2 + \frac{1}{2}\partial_\mu \chi \partial^\mu \chi + \frac{1}{2}m_\chi^2 \chi^2 \Big\}, \tag{5.110}$$

where the masses are,

$$m_h^2 = m_\chi^2 = 2\mu^2, \qquad m_W^2 = \frac{1}{4}g_4^2\frac{\mu^2}{\lambda}, \qquad m_Z^2 = \frac{1}{4}\Big(g_4^2 + g_4'^2\Big)\frac{\mu^2}{\lambda} = \frac{m_W^2}{\cos^2\theta_W}. \tag{5.111}$$

It is worth noticing here that the Higgs mass m_h and the dark scalar mass m_χ are degenerate at the tree level. However, as we demonstrate below, this degeneracy is lifted by the quantum corrections predicted by the effective theory below the KK-mass scale. The interaction terms for effective theory can be written as

$$S_{int} = -\int d^4x \Big\{ \lambda v h^3 + \frac{\lambda}{4}h^4 + \frac{\lambda}{4}\chi^4 + 3\lambda v h \chi^2 + \frac{3}{2}\lambda h^2\chi^2 + \frac{g_4^2}{2} v W^+_\mu W^{-\mu} h + \frac{g_4^2}{4}W^+_\mu W^{-\mu}(h^2+\chi^2) + \frac{1}{4}(g_4^2 + g_4'^2) v h Z_\mu Z^\mu + \frac{1}{8}(g_4^2 + g_4'^2) Z_\mu Z^\mu (h^2+\chi^2) \Big\}, \tag{5.112}$$

where we have omitted terms involving gauge interactions alone as they are irrelevant to our discussion below.

5.3.1. Quantum corrections to scalar masses

In this subsection we will study quantum corrections to the tree-level scalar masses of the Higgs boson h and the dark matter candidate χ.

Before proceeding further, we want to point out here that in this work we have not studied fermions in our geometric setup since our focus is on the bosonic sector of the SM and EWSB. For the sake of self-consistency, we mention here three possibilities for fermion localization and their implications in our geometric setup:

1. In the first scenario, one takes heavy (top) quarks to be localized towards the IR-brane, while the light quarks and leptons are localized towards the UV-brane. Through this geometric localization one can address the fermion mass hierarchy problem. In this scenario the even and odd zero-modes corresponding to the heavy quarks will be almost degenerate in our symmetric geometry, whereas the odd zero-modes corresponding to the light quarks could be much heavier than their corresponding even zero-modes [124, 125].
2. In the second scenario, *all* the fermions have flat zero-mode profiles. This can be achieved by the choice of appropriate bulk mass parameters for the fermions. As a consequence of flat profiles the odd fermion zero-modes have to disappear and the even zero-modes will correspond to the SM fermions (in this case the fermion mass hierarchy problem is

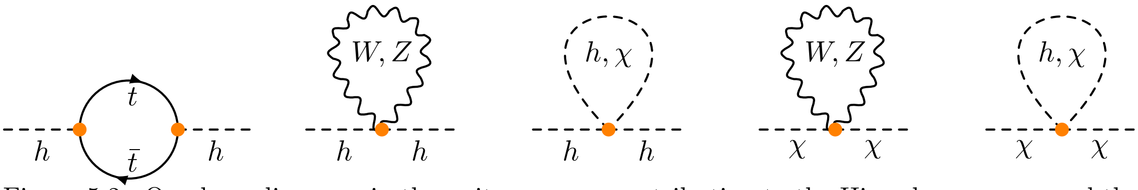


Figure 5.2.: One-loop diagrams in the unitary gauge contributing to the Higgs boson mass and the DM scalar mass.

reintroduced).

3. In the third scenario *all* the fermions are localized towards UV-brane. In this case the masses of *all* odd zero-modes of the fermions could be heavier than their corresponding even zero-modes.

In this study we implicitly limit ourselves to the last two cases in order that the dark Higgs be the lightest odd particle and all the other odd zero-modes are either not present in our low-energy effective theory or they are much heavier that the dark Higgs, which will therefore be the only relevant dark matter candidate. For either of the choices 2. or 3. above, the top Yukawa coupling y_t in the low-energy effective theory will be the same as in the SM and the top-quark loop correction to the SM Higgs boson mass will be exactly as in the SM up to the KK cutoff. In case 2., the $n \neq 0$ fermion KK-modes are all much heavier than the KK cutoff, m_{KK}, and will not significantly influence the radiative corrections to the SM Higgs mass. We leave the study of the complete fermionic sector associated with our geometric setup for future studies.

The quantum corrections to the Higgs boson (h) mass and the dark-Higgs (χ) mass within our effective theory below the KK-scale are quite essential for breaking the mass degeneracy of Eq. (5.111). For instance, at the 1-loop level of the perturbative expansion, the main contributions (quadratically divergent) to the masses of the SM Higgs and the dark-Higgs come from the exchanges of the top quark (t), massive gauge bosons ($W,\ Z$), Higgs boson (h) and the dark-Higgs (χ), see Fig. 5.2.[6] It is instructive to write the general 1-loop effective scalar potential $V_{eff}(H_1, H_2)$ for our effective theory, described in the previous section, as [7]

$$V_{eff}(H_1, H_2) = V_0(H_1, H_2) + V_1(H_1, H_2), \tag{5.113}$$

where $V_0(H_1, H_2)$ is the tree level scalar potential given by Eq. (5.106) and $V_1(H_1, H_2)$ is the 1-loop effective potential, given by (see for example Refs. [130, 131, 132])

$$V_1(H_1, H_2) = \frac{\Lambda^2}{32\pi^2}\left[3\Big(g_4^2 + \frac{1}{2}(g_4^2 + g_4'^2) + 8\lambda\Big)(|H_1|^2 + |H_2|^2) - 12y_t^2|H_1|^2\right] + \cdots , \tag{5.114}$$

[6] Another scalar which could be potentially present in our effective theory is the *radion*, which is responsible for the stabilization of the set-up. The stabilization mechanism is beyond the scope of this thesis, as here we assume a rigid geometrical background with no fluctuations of the 5D metric. However, we want to comment here that if the radion were present in our effective theory, because of it bosonic nature it would likely reduce the fine-tuning much in the manner that the χ does.

[7] Note that in this section we are considering the Higgs doublets $H_{1,2}$ in the unitary gauge, such that $H_1(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v_1 + h \end{pmatrix}$ and $H_2(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v_2 + \chi \end{pmatrix}$, where at tree level our choice was $v_1 = v$ and $v_2 = 0$.

where y_t is the top Yukawa coupling, related to top mass through $m_t^2 = y_t^2 v^2/2$. We use the momentum cut-off regularization. Also it is important to comment here that H_2 is odd under the geometric $\mathbb{Z}_2$ parity, implying that it does not couple to the even zero-mode fermions. Moreover, we consider only the quadratically divergent part of the effective scalar potential and the ellipses in the above equation represent terms which are not quadratically divergent.

The minimization of the effective potential $V_{eff}(H_1, H_2)$, i.e.

$$\frac{\partial V_{eff}}{\partial H_i}\Big|_{H_i=\langle H_i\rangle} = 0, \qquad \text{where} \quad \langle H_i \rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v_i \end{pmatrix} \qquad i = 1,2 \tag{5.115}$$

gives the following set of conditions for the global minimum,

$$\lambda v_1^2 = \mu^2 - \delta\mu^2 - 3\lambda v_2^2, \qquad \text{or} \qquad v_1 = 0, \tag{5.116}$$

and

$$\lambda v_2^2 = \mu^2 - \delta\mu^2 + \frac{3}{8}\frac{\Lambda^2}{\pi^2}y_t^2 - 3\lambda v_1^2, \qquad \text{or} \qquad v_2 = 0, \tag{5.117}$$

where $\delta\mu^2$ is given by

$$\delta\mu^2 = \frac{3\Lambda^2}{32\pi^2}\Big[g_4^2 + \frac{1}{2}(g_4^2 + g_4'^2) + 8\lambda - 4y_t^2\Big]. \tag{5.118}$$

Of the four possible global minima of Eqs. (5.116) and (5.117), we will choose the vacuum such that H_1 acquires the vev, whereas H_2 does not:

$$v_1 = v, \qquad\qquad v_2 = 0,$$

where $v \simeq 246$ GeV is the vacuum expectation value of the SM Higgs doublet. With this choice of vacuum, the 1-loop corrected masses for the fluctuations around the vacuum are

$$m_h^2 = \frac{\partial^2 V_{eff}(H_1,H_2)}{\partial H_1^2}\Big|_{H_1=v,H_2=0} = \Big(-\mu^2 + \delta\mu^2\Big) + 3\lambda v^2 = 2\lambda v^2, \tag{5.119}$$

$$\begin{aligned} m_\chi^2 = \frac{\partial^2 V_{eff}(H_1,H_2)}{\partial H_2^2}\Big|_{H_1=v,H_2=0} &= \Big(-\mu^2 + \delta\mu^2\Big) + 3\lambda v^2 + \frac{3}{8}\frac{\Lambda^2}{\pi^2}y_t^2, \\ &= 2\lambda v^2 + \frac{3}{4}\frac{\Lambda^2}{\pi^2 v^2}m_t^2. \end{aligned} \tag{5.120}$$

To get $m_h = 125$ GeV, equivalent to $v \simeq 246$ GeV, we need to fine-tune the parameters of the theory. To quantify the level of fine-tuning, we employ the Barbieri–Giudice fine-tuning measure Δ_{m_h} [133, 131, 132]:

$$\Delta_{m_h} \equiv \left|\frac{\delta\mu^2}{\mu^2}\right| = \left|\frac{\delta m_h^2}{m_h^2}\right|. \tag{5.121}$$

We plot the fine-tuning measure Δ_{m_h} as a function of the effective cutoff scale Λ in Fig. 5.3. If one allows fine-tuning of about 10%, i.e. $\Delta_{m_h} = 10$, then the effective cutoff scale is $\Lambda \simeq 2$ TeV. The most stringent bounds on the KK-scale m_{KK} in RS1 geometry with a bulk Higgs come from electroweak precision tests (EWPT) by fitting the S, T and U parameters

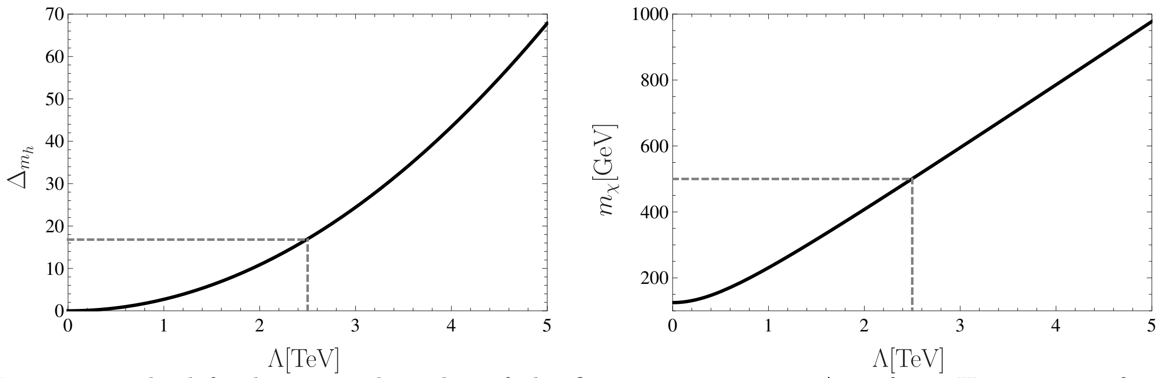


Figure 5.3.: The left plot gives the value of the fine-tuning measure Δ_{m_h} for a Higgs mass of 125 GeV as a function of the cutoff Λ. The right plot shows the dependence of m_χ on Λ for $m_h = 125$ GeV. In our model $\Lambda = m_{KK}$. The vertical dashed gray line indicates the current lower bound on the KK mass scale coming from EWPT.

[111]. The lower bound on the KK mass scale in our model (AdS geometry, i.e. $A(y) = -k|y|$) is $m_{KK} \gtrsim 2.5$ TeV for $\beta = 0$ and $m_{KK} \gtrsim 4.3$ TeV for $\beta = 10$ at 95% C.L. [111]. This implies a tension between fine-tuning (naturalness) and the lower bound on the KK mass scale m_{KK}. The region within the dashed gray lines in Fig. 5.3 shows the current bounds on the KK mass scale for our geometry and the associated fine-tuning. It is important to comment here that a slight modification to the AdS geometry (for example, models with soft wall or thick-branes) leads to a considerable relaxation of the above mentioned lower bound on KK mass scale [104, 105, 106]. For instance, a mild modification to the AdS metric in the vicinity of the IR-brane can relax the KK mass scale to $m_{KK} \gtrsim 1$ TeV [104, 105, 106, 114, 134]. Needless to say, the generalization of our model to modified AdS geometries with soft walls or thick-branes is possible.

The 1-loop quantum corrected dark matter squared mass m_χ^2 is:

$$m_\chi^2 = m_h^2 + \frac{3}{4}\frac{\Lambda^2}{\pi^2 v^2} m_t^2 \,. \tag{5.122}$$

Hence, m_χ is raised almost linearly with the large Λ. This is illustrated in Fig. 5.3. An interesting aspect of our model is that dark matter is predicted to be heavier than the SM Higgs boson. A natural value of the cutoff coincides with the mass of the first KK excitations, which are experimentally limited [135] to lie above a few TeV (depending on model details and KK mode sought). Requiring that the fine-tuning measure Δ_{m_h} be less than 100 implies that m_{KK} should be below about 6 TeV. Meanwhile, the strongest version of the EWPT bound requires $m_{KK} \gtrsim 2.5$ TeV, corresponding to $m_\chi \sim 500$ GeV, for which Δ_{m_h} is a very modest ~ 18. In short, our model is most consistent for 500 GeV $\lesssim m_\chi \lesssim$ 1200 GeV.

5.3.2. Dark matter relic abundance

In this subsection we calculate the dark matter relic abundance. The diagrams contributing to dark matter annihilation are shown in Fig. 5.4. The squared amplitudes $|\mathcal{M}|^2$ corresponding

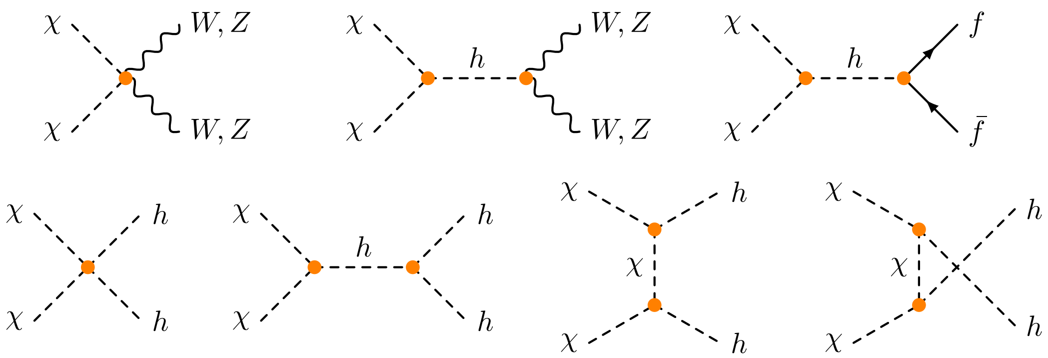


Figure 5.4.: Dark matter annihilation diagrams.

to the contribution of each final state to dark matter annihilation are:

$$\left|\mathcal{M}(\chi\chi \to \tilde{V}\tilde{V})\right|^2 = \frac{4m_{\tilde{V}}^4}{S_{\tilde{V}}v^4}\left(1+\frac{3m_h^2}{s-m_h^2}\right)^2\left[2+\left(1-\frac{s}{2m_{\tilde{V}}^2}\right)^2\right], \tag{5.123}$$

$$\left|\mathcal{M}(\chi\chi \to f\bar{f})\right|^2 = 18N_c\frac{m_f^2 m_h^4}{v^4}\frac{s-4m_f^2}{(s-m_h^2)^2}, \tag{5.124}$$

$$\left|\mathcal{M}(\chi\chi \to hh)\right|^2 = \frac{9m_h^4}{2v^4}\left[1+3m_h^2\left(\frac{1}{s-m_h^2}+\frac{1}{t-m_\chi^2}+\frac{1}{u-m_\chi^2}\right)\right]^2, \tag{5.125}$$

where $\tilde{V} = W, Z$ and $S_W = 1$ and $S_Z = 2$ are the symmetry factors accounting for the identical particles in the final state; N_c refers to the number of “color” degrees of freedom for the given fermion and s, t, u are the Mandelstam variables. Here, we ignore the loop-induced $\gamma\gamma$ and $Z\gamma$ final states, which are strongly suppressed. Note that the first term in the parenthesis in Eq. (5.123) and the first term in the square bracket in Eq. (5.125) arise from the $\chi\chi\tilde{V}\tilde{V}$ and the $\chi\chi hh$ contact interactions, respectively. The former channel is present in our model since χ is a component of the (truncated) odd $SU(2)$ doublet.

In the left panel of Fig. 5.5 we plot the annihilation cross-section for the contributing channels as a function of m_χ. (Note that the parameter Λ would only enter if we performed this calculation at the one-loop level.) As seen from the plot, the total cross section is dominated by WW and ZZ final states. The main contributions for these final states are those generated by the contact interactions $\chi\chi\tilde{V}\tilde{V}$. In fact, in our model, the $\tilde{V}\tilde{V}$ final states are additionally enhanced by a constructive interference of the contact $\chi\chi\tilde{V}\tilde{V}$ interaction with the s-channel Higgs-exchange diagram. In addition, for low m_χ, there is a comparable contribution from $\chi\chi$ annihilation into hh. (The dip at $m_\chi \sim 210$ GeV is caused by cancellation between the contact interaction and s, t, u-channel diagrams.) Fermionic final states are always irrelevant; even $\chi\chi \to t\bar{t}$ production is very small in comparison to $\chi\chi \to \tilde{V}\tilde{V}$. Then, adopting the standard cold dark matter approximation [136], we find the present χ abundance $\Omega_\chi h^2$ shown in the right panel of Fig. 5.5. We observe that $\Omega_\chi h^2 \lesssim 10^{-4}$ once the EWPT bound of $m_\chi \gtrsim 500$ GeV is imposed. Clearly, some other dark matter component is needed within this model to satisfy the Planck measurement, $\Omega_\chi h^2 \sim 0.1$ [28].

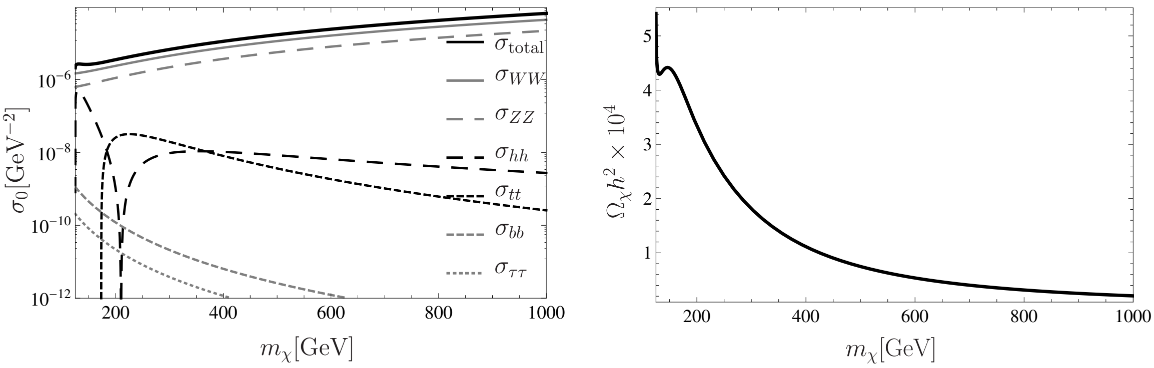


Figure 5.5.: The above graphs show the annihilation cross-section σ_0 for different final states (left) and the χ abundance $\Omega_\chi h^2 \times 10^4$ (right) as a function of dark matter mass m_χ.

5.4. Summary

In this chapter we have studied the phenomenological implications of the IR-UV-IR model presented in Sec. 2.2. Within this geometric setup we investigate the low-energy effective theory for the bulk SM bosonic sector. The $\mathbb{Z}_2$-even zero-modes correspond to known standard degrees of freedom, whereas the $\mathbb{Z}_2$-odd zero modes might serve as a dark sector. We discuss two scenarios for spontaneous breaking of the gauge symmetry, one based on expansion of the bulk Higgs field around an extra dimensional vev with non-trivial profile and the second in which the symmetry breaking is triggered by a vev of the Kaluza-Klein modes of the bulk Higgs field. It is shown that they lead to the same low-energy effective theory. The effective low-energy scalar sector contains a scalar which mimics the Standard Model (SM) Higgs boson and a second stable scalar particle (dark-Higgs) that is a dark matter candidate; the latter is a component of the zero-mode of the $\mathbb{Z}_2$-odd Higgs doublet. The model that results from the $\mathbb{Z}_2$-symmetric background geometry resembles the Inert Two Higgs Doublet Model. The effective theory turns out to have an extra residual $SU(2) \times U(1)$ global symmetry that is reminiscent of an underlying 5D gauge transformation for the odd degrees of freedom. At tree level the SM Higgs and the dark-Higgs have the same mass; however, when leading radiative corrections are taken into account the dark-Higgs turns out to be heavier than the SM Higgs. Implications for dark matter are discussed; it is found that the dark-Higgs can provide only a small fraction of the observed dark matter abundance.

CHAPTER 6

SUMMARY AND CONCLUSIONS

Five dimensional RS1-like models offer an elegant and simple solution to the hierarchy problem, whereas RS2-like models give an alternative to the compactification. In Chap. 2 we briefly reviewed the RS models and presented two generalizations of RS models:

- First scenario is a $\mathbb{Z}_2$ symmetric generalization of RS1 on an interval $y \in [-L, L]$ such that two identical AdS patches are glued together at $y = 0$, where y is the coordinate of the fifth dimension. We considered three D3-branes, one at $y = 0$ referred to as the UV-brane where gravity is assumed to be localized and two branes at $y = \pm L$ referred to as the IR-branes — the IR-UV-IR model, see Sec. 2.2.
- Second scenario is an asymmetric generalization of RS2 which allows different bulk cosmological constants on either sides of the brane such that it leads to asymmetric warp-function and different AdS geometries on each side of the brane, see Sec. 2.3.

To understand the issue of localization of gravity in the noncompact RS2-like models we worked out the 4D effective theory at linearized level in Sec. 2.4. It is shown that the zero-mode of tensor perturbation corresponds to the 4D graviton and is localized on the brane in generalized RS2 model. In the low energy limit it is shown that the corrections to Newton's gravitational potential due to non-zero KK-modes of graviton are suppressed by one higher power of the distance r, than the leading zero-mode contribution. Hence the 4D effective gravity on the brane is the standard 4D Einstein general relativity with small and calculable corrections due to non-zero KK-modes of gravitons.

The standard formulation of RS models and their generalizations assume the presence of infinitesimally thin branes embedded in 5D space-time. In order to avoid infinitesimally thin branes and instead to model them dynamically by physical objects, we presented a generic mechanism of smoothing the singular branes by background profiles of scalar fields — the thick-branes. In RS1 and IR-UV-IR model the IR-branes have negative tension, see Chap. 2. There is no clear mechanism to generate a negative tension smooth brane by a scalar field configuration, moreover, it was shown by Gibbons, Kallosh and Linde [27] that periodicity of

RS1 can not be achieved through a non-trivial scalar field minimally coupled to gravity. We have shown in Chap. 3 that even in the presence of non-minimal scalar-gravity coupling it is not possible to mimic a negative tension brane. In an attempt to construct a model that is periodic in the extra dimension we have derived a generalization of the Gibbons-Kallosh-Linde sum rule that holds also if the scalar field couples non-canonically to the Ricci scalar. It turns out that even for the case of non-minimal scalar-gravity coupling, periodicity forbids to have any non-trivial scalar field profile which could possibly mimic RS1 model. Therefore we have focused on non-compact extra dimensions to construct thick-brane models in Sec. 3.2. We have considered four different models of thick-branes, summarized below.

- First, a thick-brane version of an asymmetric generalization of RS2 in which we employ different cosmological constants on two sides of the brane. In this scenario coupled scalar-gravity equations have been solved through the so-called *superpotential method* and stability of the solution has been illustrated. The thin brane limit of the model have been discussed. Properties of the thick-brane solution have been considered in details. It has been shown that, under mild assumptions, the relation between cosmological constants and the brane tension of the generalized RS2 could be obtained in the brane limit of our model by an appropriate choice of an integrating constant (that defines the scalar potential) independently of particular profile of the scalar field.
- Second, in order to have a chance to address the hierarchy problem, a scalar field is employed with two kink-like profiles in Sec. 3.2.2. This set-up, in the brane limit, corresponds to a model with two thin branes with positive tensions. Various possible cases, depending on the location of the maximum of the warp function has been considered. The most attractive option turned out to be the one with the maximum located on top of one of the thick-branes, which implies that the gravity is localized on that brane and if one allows the localization of Higgs field on the other brane then one could potentially address the hierarchy problem within this thick-brane scenario.
- Third, we considered a $\mathbb{Z}_2$ symmetric triple thick-brane model in Sec. 3.2.3 which resembles the IR-UV-IR model in the brane limit. The key differences between the thick-brane model and the IR-UV-IR model (see Sec. 2.2) are the fact that the thick-brane model is non-compact and allows only positive tension brane, whereas the IR-UV-IR model is compact and it requires IR-branes to be negative.
- Fourth, we consider a class of the thick-brane models where a thick brane is generated through the dilaton-like scalar fields — the dilatonic thick-brane. It turns out that such thick-branes naturally appear in the cosmological thick-brane models.

The stability of the thick-brane background solutions has been discussed in details in Sec. 3.4 and was verified in the presence of the most general perturbations of the metric and the scalar fields.

The issue of localization of a scalar field on a thick-brane has been presented in Sec. 3.3. A generalized superpotential method has been presented which could solve the coupled two-scalar field and gravity equations. The scalar field is localized on the thick-brane though localized interactions with the thick-brane and we have investigated different properties of localization in this scenario.

In Chap. 4 we have analyzed a 5D scenario with a scalar field in the presence of gravity looking for non-stationary solutions. We have found solutions of the Einstein equations for the case of time-independent scalar field assuming a conformal form of the 4D metric. Both the evolution of the scale factor, its extra dimensional shape and the profile of scalar field were discussed and determined for different values of spatial curvature $k = 0, \pm 1$. Also for the time-dependent scalar field $\phi(\tau, y)$ and 4D conformal metric, analytic solutions were obtained in certain cases. We have also formulated a superpotential method for t- and y-dependent profiles of the scalar field. For the solution which has been found both the scalar filed ϕ and the scale factor a depends on the conformal time τ and y only through $\eta = c\tau + dy$, where c and d are constants.

Chapter 5 is dedicated to construct a warped extra dimensional model with singular branes, with SM fields in the bulk and the possibility of having a dark-matter candidate. The motivation of this work was twofold: (i) to analyze the situation where EWSB is due to the bulk Higgs in this $\mathbb{Z}_2$ symmetric geometry; and (ii) to discuss the lowest odd KK-mode as a dark matter candidate. Concerning EWSB, we discussed in detail many important aspects of SSB due to a bulk Higgs. We first considered an Abelian gauge group and then generalized to the SM gauge group. In the Abelian case, we followed two apparently different approaches for SSB. In one approach, the symmetry breaking is triggered by a vev of the KK zero-mode of the bulk Higgs field. The second approach is based on the expansion of the 5D bulk Higgs field around an extra dimensional vev with non-trivial y profile. The comparison between the two Abelian scenarios is summarized in Tables 5.1 and 5.2. The (zero-mode) effective theories obtained from the two approaches are identical and the most intriguing feature of the Abelian Higgs mechanism is that the even and odd Higgs zero-modes have degenerate mass at tree-level — a feature that is also present in the SM case.

To achieve SSB, choice of boundary conditions for the fields at $\pm L$ is critical. In both the above two approaches to the Abelian case, we allowed y-derivative of a field to have an arbitrary value at $\pm L$ as opposed to requiring that the field value itself be zero, i.e. we employed Neumann or mixed b.c. rather than Dirichlet b.c. at $\pm L$. The latter choice would have led to an explicit symmetry breaking scenario in which there are no Goldstone modes and the gauge bosons do not acquire mass. (Note that the boundary or "jump" conditions at $y = 0$ follow from the bulk equations of motion in the case of even modes, whereas odd modes are required to be zero by symmetry.)

Following this introductory material, we considered EWSB assuming that the SM gauge group is present in the bulk of our $\mathbb{Z}_2$ symmetric 5D warped model. The zero-mode effective theory appropriate at scales below the KK scale, m_{KK}, was obtained. For appropriate Higgs field potentials in the bulk and localized at the UV and IR-branes, SM-like EWSB is obtained when only the IR-branes have a quartic potential term. In contrast, quadratic mass-squared terms are allowed both on the branes and in the bulk. Of course, to achieve spontaneous EWSB, we employed the same boundary conditions as delineated above for the Abelian model. The resulting model has the following features.

- Due to geometric $\mathbb{Z}_2$ symmetry all fields develop even and odd towers of KK-modes in the 4D effective theory.
- Assuming that the KK-scale is high enough ($m_{KK} \sim \mathcal{O}(\text{few})$ TeV), we derived the low energy effective theory which includes only zero-modes of the theory.
- In the effective theory, the symmetry of the model is $[SU(2) \times U(1)_Y]' \times [SU(2) \times U(1)_Y]$, where the unprimed symmetry group is gauged while the other stays as a global symmetry. The zero-mode odd gauge fields and the corresponding Goldstone modes from the odd Higgs doublet are *eliminated* due to the b.c..
- In the low energy effective theory, we are left with all the SM fields plus a *dark-Higgs* – the odd zero-mode Higgs. This dark-Higgs and the SM Higgs (the even zero-mode) are degenerate at tree level.
- In order to get the SM Higgs mass of 125 GeV, we need to fine-tune the 5D fundamental parameters of theory to about $1\% - 5\%$, where the upper bound is determined by the lower bound on the KK scale coming from EWPT requirements.
- We computed the one-loop quantum corrections to the tree-level masses of the SM Higgs and the dark Higgs assuming that the cutoff scale of our effective theory is the KK-scale, m_{KK}. One finds that the dark-Higgs mass is necessarily larger than the SM Higgs mass, the difference being quadratically dependent on m_{KK}.
- Requiring that the fine-tuning measure Δ_{m_h} be less than 100 implies that m_{KK} should be below about 6 TeV. Meanwhile, the strongest version of the EWPT bound requires $m_{KK} \gtrsim 2.5$ TeV, corresponding to $m_\chi \sim 500$ GeV, for which Δ_{m_h} is a very modest ~ 18. In short, our model is most consistent for 500 GeV $\lesssim m_\chi \lesssim$ 1200 GeV.
- We calculated the relic abundance of the dark-Higgs in the cold dark matter approximation. For m_χ in the above preferred range, $\Omega_\chi h^2 \lesssim 10^{-4}$ as compared to the current experimental value of ~ 0.1. To obtain a more consistent dark matter density, one needs to either assume another DM particle or perform a more rigorous analysis of our model by considering the even and odd higher KK-modes in the effective theory.

APPENDIX A

LINEARIZED EINSTEIN EQUATIONS

In this Appendix we consider fluctuations around the background solutions discussed in the main text. We start by perturbing the 5D metric $g_{MN}(y)$ defined through Eq. (1.2) by:

$$g_{MN}(y) \to g_{MN}(x,y) = \bar{g}_{MN}(y) + h_{MN}(x,y), \tag{A.1}$$

where $\bar{g}_{MN}(y)$ is the unperturbed background metric, given as

$$\bar{g}_{\mu\nu} = e^{2A}\eta_{\mu\nu}, \qquad \bar{g}_{\mu 5} = 0, \qquad \bar{g}_{55} = 1. \tag{A.2}$$

It is convenient to adopt the Einstein equations in the Ricci form as,

$$R_{MN} = \frac{1}{4M_*^3}\tilde{T}_{MN}, \tag{A.3}$$

where

$$\tilde{T}_{MN} = T_{MN} - \frac{1}{3}g_{MN}T^A_{\ A}, \tag{A.4}$$

where T_{MN} is the energy-momentum tensor for scalar field ϕ given in Eq. (3.28), which leads to:

$$\tilde{T}_{MN} = \nabla_M\phi\nabla_N\phi + \frac{2}{3}g_{MN}V(\phi). \tag{A.5}$$

The perturbations in the $\tilde{T}_{MN}$ will correspond to fluctuations of the scalar field $\phi(y)$ as $\phi(x,y) = \phi(y) + \varphi(x,y)$ and of the metric $g_{MN}(x,y) = \bar{g}_{MN}(y) + h_{MN}(x,y)$. These perturbations can be calculated order by order in perturbation expansion as,

$$\tilde{T}_{MN} = \tilde{T}^{(0)}_{MN} + \tilde{T}^{(1)}_{MN} + \cdots ,$$

where ellipses correspond to the higher order fluctuations in $\varphi(x,y)$ and $h_{MN}(x,y)$. The zeroth and the first order terms are as follows,

$$\tilde{T}^{(0)}_{\mu\nu} = \frac{2}{3}e^{2A}\eta_{\mu\nu}V(\phi), \qquad \tilde{T}^{(0)}_{55} = \phi'^2 + \frac{2}{3}V(\phi), \qquad \tilde{T}^{(0)}_{\mu 5} = 0. \tag{A.6}$$

$$\tilde{T}^{(1)}_{\mu\nu} = \frac{2}{3}\left(e^{2A}\eta_{\mu\nu}\frac{\partial V(\phi)}{\partial\phi}\varphi + V(\phi)h_{\mu\nu}\right), \tag{A.7}$$

$$\tilde{T}^{(1)}_{55} = 2\phi'\varphi' + \frac{2}{3}\frac{\partial V(\phi)}{\partial\phi}\varphi + \frac{2}{3}h_{55}V(\phi), \qquad \tilde{T}^{(1)}_{\mu 5} = \phi'\partial_\mu\varphi + \frac{2}{3}h_{\mu 5}V(\phi). \tag{A.8}$$

The explicit forms of components of the Ricci tensor in the zeroth and first order are:

$$R^{(0)}_{\mu\nu} = -\,e^{2A}\left(A'' + 4A'^2\right)\eta_{\mu\nu}, \qquad R^{(0)}_{55} = -4\left(A'' + A'^2\right), \qquad R^{(0)}_{\mu 5} = 0, \tag{A.9}$$

$$\begin{aligned} R^{(1)}_{\mu\nu} = &-\frac{1}{2}\partial_\mu\partial_\nu h_{55} + e^{2A}\eta_{\mu\nu}\left(A'' + 4A'^2\right)h_{55} + \frac{1}{2}e^{2A}\eta_{\mu\nu}A'h'_{55} + \frac{1}{2}\left(\partial_\mu h'_{\nu 5} + \partial_\nu h'_{\mu 5}\right) \\ &+ A'\left(\partial_\mu h_{\nu 5} + \partial_\nu h_{\mu 5}\right) - \frac{1}{2}e^{-2A}\Box^{(4)}h_{\mu\nu} + \frac{1}{2}e^{-2A}\eta^{\rho\sigma}\left(\partial_\mu\partial_\rho h_{\nu\sigma} + \partial_\nu\partial_\rho h_{\mu\sigma} - \partial_\mu\partial_\nu h_{\rho\sigma}\right) \\ &- \frac{1}{2}h''_{\mu\nu} - \frac{1}{2}A'\eta_{\mu\nu}\eta^{\rho\sigma}h'_{\rho\sigma} - A'^2\left(2h_{\mu\nu} - \eta_{\mu\nu}\eta^{\rho\sigma}h_{\rho\sigma}\right) + A'\eta_{\mu\nu}\eta^{\rho\sigma}\partial_\rho h_{\sigma 5}, \end{aligned} \tag{A.10}$$

$$\begin{aligned} R^{(1)}_{\mu 5} = &\frac{1}{2}e^{-2A}\eta^{\rho\sigma}\left(\partial_\rho h'_{\mu\sigma} - \partial_\mu h'_{\rho\sigma}\right) - e^{-2A}A'\eta^{\rho\sigma}\left(\partial_\rho h_{\mu\sigma} - \partial_\mu h_{\rho\sigma}\right) \\ &+ \frac{3}{2}A'\partial_\mu h_{55} - \frac{1}{2}e^{-2A}\left(\Box^{(4)}h_{\mu 5} - \eta^{\rho\sigma}\partial_\rho\partial_\mu h_{\sigma 5}\right) - \left(A'' + 4A'^2\right)h_{\mu 5}, \end{aligned} \tag{A.11}$$

$$R^{(1)}_{55} = e^{-2A}\left(A'\eta^{\rho\sigma}h'_{\rho\sigma} + A''\eta^{\rho\sigma}h_{\rho\sigma} - \frac{1}{2}\eta^{\rho\sigma}h''_{\rho\sigma}\right) - \frac{1}{2}e^{-2A}\Box^{(4)}h_{55} + 2A'h'_{55} + e^{-2A}\eta^{\rho\sigma}\partial_\rho h'_{\sigma 5}. \tag{A.12}$$

Having all the components of the Ricci tensor (A.10)-(A.12) and $\tilde{T}_{MN}$ (A.7)-(A.8), one can write down the equations of motion for the metric fluctuations $h_{MN}(x,y)$:

$$\begin{aligned} (\mu\nu): \quad &-\frac{1}{2}\partial_\mu\partial_\nu h_{55} + e^{2A}\eta_{\mu\nu}\left(A'' + 4A'^2\right)h_{55} + \frac{1}{2}e^{2A}\eta_{\mu\nu}A'h'_{55} + \frac{1}{2}\left(\partial_\mu h'_{\nu 5} + \partial_\nu h'_{\mu 5}\right) \\ &+ A'\left(\partial_\mu h_{\nu 5} + \partial_\nu h_{\mu 5}\right) - \frac{1}{2}e^{-2A}\Box^{(4)}h_{\mu\nu} + \frac{1}{2}e^{-2A}\eta^{\rho\sigma}\left(\partial_\mu\partial_\rho h_{\nu\sigma} + \partial_\nu\partial_\rho h_{\mu\sigma} - \partial_\mu\partial_\nu h_{\rho\sigma}\right) \\ &- \frac{1}{2}h''_{\mu\nu} - \frac{1}{2}A'\eta_{\mu\nu}\eta^{\rho\sigma}h'_{\rho\sigma} - A'^2\left(2h_{\mu\nu} - \eta_{\mu\nu}\eta^{\rho\sigma}h_{\rho\sigma}\right) + A'\eta_{\mu\nu}\eta^{\rho\sigma}\partial_\rho h_{\sigma 5} \\ &= \frac{1}{4M_*^3}\frac{2}{3}\left(e^{2A}\eta_{\mu\nu}\frac{\partial V(\phi)}{\partial\phi}\varphi + V(\phi)h_{\mu\nu}\right), \end{aligned} \tag{A.13}$$

$$\begin{aligned} (\mu 5): \quad &\frac{1}{2}e^{-2A}\eta^{\rho\sigma}\left(\partial_\rho h'_{\mu\sigma} - \partial_\mu h'_{\rho\sigma}\right) - e^{-2A}A'\eta^{\rho\sigma}\left(\partial_\rho h_{\mu\sigma} - \partial_\mu h_{\rho\sigma}\right) + \frac{3}{2}A'\partial_\mu h_{55} \\ &- \frac{1}{2}e^{-2A}\left(\Box^{(4)}h_{\mu 5} - \eta^{\rho\sigma}\partial_\rho\partial_\mu h_{\sigma 5}\right) = \frac{1}{4M_*^3}\phi'\partial_\mu\varphi, \end{aligned} \tag{A.14}$$

$$\begin{aligned} (55): \quad &e^{-2A}\left(A'\eta^{\rho\sigma}h'_{\rho\sigma} + A''\eta^{\rho\sigma}h_{\rho\sigma} - \frac{1}{2}\eta^{\rho\sigma}h''_{\rho\sigma}\right) - \frac{1}{2}e^{-2A}\Box^{(4)}h_{55} + 2A'h'_{55} + e^{-2A}\eta^{\rho\sigma}\partial_\rho h'_{\sigma 5} \\ &= \frac{1}{4M_*^3}\left(2\phi'\varphi' + \frac{2}{3}\frac{\partial V(\phi)}{\partial\phi}\varphi + \frac{2}{3}h_{55}V(\phi)\right). \end{aligned} \tag{A.15}$$

Additionally, we also have the equation of motion of the scalar field ϕ (3.27) in the first order in the fluctuations $h_{MN}(x,y)$ and $\varphi(x,y)$ as,

$$e^{-2A}\Box^{(4)}\varphi+\varphi''+4A'\varphi'-\frac{\partial^2 V(\phi)}{\partial\phi^2}\varphi+\frac{1}{2}\phi'\left(e^{-2A}h\right)'-\frac{1}{2}\phi' h_5^{5\prime}-\left(\phi''+4A'\phi'\right)h_5^5=0, \quad (A.16)$$

where $h\equiv\eta^{\mu\nu}h_{\mu\nu}$.

In the remaining part of this Appendix we will derive equations of motion for perturbations of the metric and the scalar field. We are going to adopt a decomposition of the metric perturbation h_{MN} into *scalar*, *vector* and *tensor* (SVT) components.

A.1. SVT decomposition of perturbations and gauge choice

In this sub-appendix we review the decomposition of most general symmetric perturbation h_{MN} into *scalar*, *vector* and *tensor* (SVT) modes. The matter of gauge choice in the warped extra dimension in the presence of a scalar field is also discussed. These issues were studied in the literature, see for example, [43, 60, 55, 56, 57, 137].

Due to the symmetries (4D Poincáre invariance) of the background metric and the energy-momentum tensor, we can decompose the perturbations h_{MN} into scalars, vectors and tensors as follows,

$$h_{\mu\nu}=e^{2A}\left[-2\psi\eta_{\mu\nu}-2\partial_\mu\partial_\nu E+\partial_\mu G_\nu+\partial_\nu G_\mu+H_{\mu\nu}\right], \quad (A.17)$$

$$h_{\mu5}=\partial_\mu B+C_\mu, \quad (A.18)$$

$$h_{55}=2\chi, \quad (A.19)$$

where ψ, χ, B and E are scalars, whereas, C_μ and G_μ are divergenceless vectors and $H_{\mu\nu}$ is the transverse and traceless tensor, i.e.

$$\partial^\mu C_\mu=\partial^\mu G_\mu=0, \qquad \partial^\mu H_{\mu\nu}=H^\mu_\mu=0. \quad (A.20)$$

The perturbation modes are functions of x and y coordinates.

Let us discuss the uniqueness of the above decomposition. It is easy to see that B is determined by $h_{\mu5}$ as follows

$$\Box^{(4)}B=\partial^\mu h_{\mu5}. \quad (A.21)$$

Therefore shifting B by a solution the homogeneous equation $\Box^{(4)}\lambda=0$ leads to another allowed solution of (A.21) [1]. In order to specify the solution of $\Box^{(4)}\lambda=0$ one has to fix initial conditions, that can be done e.g. by specifying $\lambda(t,\vec{x},y)$ and $\partial_t\lambda(t,\vec{x},y)$ at a given time. Hereafter we are going to assume that at a certain time $t=t_0$ that is far enough in the past both $\lambda(t,\vec{x},y)=0$ and $\partial_t\lambda(t,\vec{x},y)=0$. That assumption is physically well motivated as there is no reason to observe any perturbations at the very beginning and implies that the

[1] Another way of seeing the same freedom in determining B and C_μ is to notice that a shift $B\to B+\lambda$ can be compensated by an appropriate change of C_μ, $C_\mu\to C_\mu-\partial_\mu\lambda$. Requiring $\Box^{(4)}\lambda=0$, guaranties that C_μ remains divergenceless.

only solution of $\Box^{(4)}\lambda = 0$ is in fact $\lambda = 0$. Therefore the decomposition (A.18) is unique. Similar strategy could be adopted to show uniqueness of the decomposition of $h_{\mu\nu}$ provided appropriate initial conditions are adopted. We start by determining E as a solution of the following equation that is implied by (A.17):

$$\Box^{(4)2}E = \frac{1}{3}e^{-2A}\left(\frac{1}{4}\Box^{(4)}h^\mu_\mu - \partial^\mu\partial^\nu h_{\mu\nu}\right). \tag{A.22}$$

Having E determined (with appropriate initial conditions that ensures uniqueness) one can find ψ solving

$$\psi = -\frac{1}{8}e^{-2A}h^\mu_\mu - \frac{1}{4}\Box^{(4)}E. \tag{A.23}$$

Then G_μ is a solution of

$$\Box^{(4)}G_\mu = e^{-2A}\partial^\nu h_{\mu\nu} + 2\partial_\mu(\psi + \Box^{(4)}E). \tag{A.24}$$

Now we can write down the first order Einstein equations in terms of the scalar, vector and tensor (SVT) components defined in (A.17)-(A.19) as,

$$\begin{aligned}(\mu\nu):\quad & e^{2A}\eta_{\mu\nu}\left[2\left(A'' + 4A'^2\right)\chi + A'\chi' + 2\left(A'' + 4A'^2 + \frac{1}{2}e^{-2A}\Box^{(4)}\right)\psi + 8A'\psi' + \psi''\right] \\ & + \partial_\mu\partial_\nu B' + 2A'\partial_\mu\partial_\nu B + A'\eta_{\mu\nu}\Box^{(4)}B + \frac{1}{2}\left(\partial_\mu C'_\nu + \partial_\nu C'_\mu\right) + A'\left(\partial_\mu C_\nu + \partial_\nu C_\mu\right) \\ & + e^{2A}\left[2\left(A'' + 4A'^2 + \Box^{(4)}\right)\partial_\mu\partial_\nu E + \eta_{\mu\nu}A'\Box^{(4)}E' + 4A'\partial_\mu\partial_\nu E' + \partial_\mu\partial_\nu E''\right] \\ & - \left(A'' + 4A'^2 + \frac{1}{2}\Box^{(4)}\right)\left(\partial_\mu G_\nu + \partial_\nu G_\mu\right) - \frac{1}{2}e^{2A}\left[\partial_\mu G''_\nu + \partial_\nu G''_\mu + 4A'\left(\partial_\mu G'_\nu + \partial_\nu G'_\mu\right)\right] \\ & + \partial_\mu\partial_\nu\left(2\psi - \chi\right) - \left(A'' + 4A'^2 + \frac{1}{2}\Box^{(4)}\right)H_{\mu\nu} - 2e^{2A}A'H'_{\mu\nu} - \frac{1}{2}e^{2A}H''_{\mu\nu} \\ & = \frac{1}{4M_*^3}\frac{2}{3}e^{2A}\left[\eta_{\mu\nu}\frac{\partial V(\phi)}{\partial\phi}\varphi + V(\phi)\left(-2\psi\eta_{\mu\nu} - 2\partial_\mu\partial_\nu E + \partial_\mu G_\nu + \partial_\nu G_\mu + H_{\mu\nu}\right)\right],\end{aligned} \tag{A.25}$$

$$(\mu 5):\quad 3\partial_\mu\psi' + 3A'\partial_\mu\chi - \frac{1}{2}e^{-2A}\Box^{(4)}C_\mu + \frac{1}{2}\Box^{(4)}G'_\mu = \frac{1}{4M_*^3}\phi'\partial_\mu\varphi, \tag{A.26}$$

$$\begin{aligned}(55):\quad & 4\left(\psi'' + 2A'\psi'\right) + 4A'\chi' - e^{-2A}\Box^{(4)}\chi + \Box^{(4)}\left(E'' + 2A'E'\right) + e^{-2A}\Box^{(4)}B' \\ & = \frac{1}{4M_*^3}\left[2\phi'\varphi' + \frac{2}{3}\frac{\partial V(\phi)}{\partial\phi}\varphi + \frac{4}{3}V(\phi)\chi\right].\end{aligned} \tag{A.27}$$

Adopting the background equations of motion the above equations could be simplified as follows:

$$\begin{aligned}(\mu\nu):\quad & e^{2A}\eta_{\mu\nu}\left[2\left(A'' + 4A'^2\right)\chi + A'\chi' + e^{-2A}\Box^{(4)}\psi + 8A'\psi' + \psi'' + e^{-2A}A'\Box^{(4)}B + A'\Box^{(4)}E'\right] \\ & + \partial_\mu\partial_\nu\left[2\psi - \chi + B' + 2A'B + e^{2A}\left(2\Box^{(4)}E + 4A'E' + E''\right)\right]\end{aligned}$$

$$+\frac{1}{2}\partial_\mu\left[C'_\nu + 2A'C_\nu - 4e^{2A}A'G'_\nu - e^{2A}G''_\nu\right] + \frac{1}{2}\partial_\nu\left[C'_\mu + 2A'C_\mu - 4e^{2A}A'G'_\mu - e^{2A}G''_\mu\right]$$
$$-\frac{1}{2}\left(\Box^{(4)}H_{\mu\nu} + 4e^{2A}A'H'_{\mu\nu} + e^{2A}H''_{\mu\nu}\right) = \frac{1}{4M_*^3}\frac{2}{3}e^{2A}\eta_{\mu\nu}\frac{\partial V(\phi)}{\partial\phi}\varphi, \tag{A.28}$$

$$(\mu 5): \quad 3\partial_\mu\psi' + 3A'\partial_\mu\chi - \frac{1}{2}e^{-2A}\Box^{(4)}C_\mu + \frac{1}{2}\Box^{(4)}G'_\mu = \frac{1}{4M_*^3}\phi'\partial_\mu\varphi, \tag{A.29}$$

$$(55): \quad 4\left(\psi'' + 2A'\psi'\right) + 4A'\chi' - e^{-2A}\Box^{(4)}\chi + \Box^{(4)}\left(E'' + 2A'E'\right) + e^{-2A}\Box^{(4)}B'$$
$$= \frac{1}{4M_*^3}\left[2\phi'\varphi' + \frac{2}{3}\frac{\partial V(\phi)}{\partial\phi}\varphi + \frac{4}{3}V(\phi)\chi\right]. \tag{A.30}$$

Now comparing the coefficients of $\eta_{\mu\nu}$, $\partial_\mu\partial_\nu$, ∂_ν and the tensors on both sides we get from $(\mu\nu)$ components the following equations of motion for the scalar, vector and tensor modes of the perturbations

$$2\left(A'' + 4A'^2\right)\chi + A'\chi' + e^{-2A}\Box^{(4)}\psi + 8A'\psi' + \psi'' + e^{-2A}A'\Box^{(4)}B + A'\Box^{(4)}E'$$
$$= \frac{1}{4M_*^3}\frac{2}{3}\frac{\partial V(\phi)}{\partial\phi}\varphi, \tag{A.31}$$

$$\partial_\mu\partial_\nu\left[2\psi - \chi + B' + 2A'B + e^{2A}\left(2\Box^{(4)}E + 4A'E' + E''\right)\right] = 0, \tag{A.32}$$

$$\partial_\nu\left[C'_\mu + 2A'C_\mu - 4e^{2A}A'G'_\mu - e^{2A}G''_\mu\right] = 0, \tag{A.33}$$

$$-\frac{1}{2}\left(\Box^{(4)}H_{\mu\nu} + 4e^{2A}A'H'_{\mu\nu} + e^{2A}H''_{\mu\nu}\right) = 0. \tag{A.34}$$

For $(\mu 5)$ and (55) we obtain the following equations:

$$\partial_\mu\left(3\psi' + 3A'\chi - \frac{1}{4M_*^3}\phi'\varphi\right) = 0, \tag{A.35}$$

$$e^{-2A}\Box^{(4)}C_\mu - \Box^{(4)}G'_\mu = 0, \tag{A.36}$$

$$4\left(\psi'' + 2A'\psi'\right) + 4A'\chi' - e^{-2A}\Box^{(4)}\chi + \Box^{(4)}\left(E'' + 2A'E'\right) + e^{-2A}\Box^{(4)}B'$$
$$= \frac{1}{4M_*^3}\left[2\phi'\varphi' + \frac{2}{3}\frac{\partial V(\phi)}{\partial\phi}\varphi + \frac{4}{3}V(\phi)\chi\right]. \tag{A.37}$$

The above equations of motion for the scalar, vector and tensor modes of perturbations are applicable for any gauge choice, in the main text we decided to choose the longitudinal gauge defined by the condition $B = E = G_\mu = 0$ as discussed below.

Now we will consider the coordinate/gauge transformations and then we will turn to the question of choosing the appropriate gauge in order to eliminate artifacts of the freedom of choosing a reference frame. Lets consider the following coordinate transformation,

$$\check{x}^M = x^M - \xi^M, \tag{A.38}$$

where the ξ^M is an infinitesimally small function of space time, i.e. $|\xi^M| << |x^M|$ and $\xi^M = (\xi^\mu, \xi^5)$ with ξ^μ being a 4D vector and ξ^5 a scalar change in the 5th coordinate y. In order to write down corresponding gauge transformations of the decomposed scalars, vector and tensor modes, it is useful to decompose also the 4D vector ξ^μ into the divergenceless vector $\xi^\mu_\perp$ and gradient of the scalar $\xi_\parallel$, i.e.

$$\xi^\mu = \xi^\mu_\perp + \partial_\mu \xi_\parallel, \qquad \partial_\mu \xi^\mu_\perp = 0. \tag{A.39}$$

It is easy to show that the change in metric perturbation h_{MN} corresponding to (A.38) reads

$$\check{h}_{MN} = h_{MN} + \delta h_{MN}, \qquad \text{with} \qquad \delta h_{MN} = \nabla_M \xi_N + \nabla_N \xi_M. \tag{A.40}$$

The explicit form of the components of δh_{MN} are given by,

$$\delta h_{\mu\nu} = \partial_\mu \xi_{\perp\nu} + \partial_\nu \xi_{\perp\mu} + 2\partial_\mu \partial_\nu \xi_\parallel + 2A' e^{2A} \eta_{\mu\nu} \xi_5, \tag{A.41}$$

$$\delta h_{\mu 5} = \partial_\mu \xi_5 + \partial_\mu \xi'_\parallel + \xi'_{\perp\mu} - 2A' \xi_{\perp\mu} - 2A' \partial_\mu \xi_\parallel, \tag{A.42}$$

$$\delta h_{55} = 2\xi'_5, \tag{A.43}$$

The above transformations of the metric perturbation h_{MN} induce corresponding transformations of the metric perturbation components defined by Eqs. (A.17)-(A.19) as,

$$\check{\psi} = \psi - A' \xi^5, \qquad \check{E} = E - e^{-2A} \xi_\parallel, \tag{A.44}$$

$$\check{\chi} = \chi + \xi'_5, \qquad \check{B} = B + \xi'_\parallel + \xi^5 - 2A' \xi_\parallel, \tag{A.45}$$

$$\check{C}_\mu = C_\mu + \xi'_{\perp\mu} - 2A' \xi_{\perp\mu}, \qquad \check{G}_\mu = G_\mu + e^{-2A} \xi_{\perp\mu}, \tag{A.46}$$

whereas, $H_{\mu\nu}$ is unaffected by the coordinate transformations. Similarly, the gauge transformation of the scalar field perturbation φ can be easily obtained as,

$$\check{\varphi} = \varphi + \delta\varphi = \varphi + \phi' \xi^5. \tag{A.47}$$

Similarly, the gauge transformations of the energy momentum tensor can be written as,

$$\check{\tilde{T}}^{(1)}_{MN} = \tilde{T}^{(1)}_{MN} + \delta \tilde{T}^{(1)}_{MN}, \tag{A.48}$$

where,

$$\delta \tilde{T}^{(1)}_{MN} = \tilde{T}^{(0)}_{MA} \nabla_N \xi^A + \tilde{T}^{(0)}_{NB} \nabla_M \xi^B + \nabla_C \tilde{T}^{(0)}_{MN} \xi^C. \tag{A.49}$$

Now we turn our attention towards the issue of choosing a gauge. It proves to be convenient to adopt the so-called longitudinal or Newtonian gauge defined by the conditions: $\check{B} = \check{E} = 0$ in the scalar and $\check{G}_\mu = 0$ in the vector sector. It is important to note that, indeed, one can always choose the gauge parameters such that the gauge conditions are satisfied, i.e.

$$\xi_\parallel(x, y) = e^{2A} E, \tag{A.50}$$

$$\xi^5(x, y) = -B - e^{2A} \left(e^{-2A} E\right)', \tag{A.51}$$

$$\xi_{\perp\mu}(x, y) = -e^{2A} G_\mu. \tag{A.52}$$

Here one can note that this choice of gauge fixing completely fixes the gauge and so that there is no residual gauge freedom left.

A.2. Scalar perturbations

Scalar perturbations contribute to the metric as follows

$$ds^2 = e^{2A}\left[(1-2\psi)\,\eta_{\mu\nu} - 2\partial_\mu\partial_\nu E\right]dx^\mu dx^\nu + \partial_\mu B dx^\mu dy + (1+2\chi)\,dy^2, \tag{A.53}$$

The scalar modes appearing here are not gauge invariant, i.e. their values are affected by the choice of different coordinates. It is therefore instructive to either work with the gauge invariant quantities or choose a suitable gauge such that the ambiguities related to the coordinate transformations can be removed. Here we choose the longitudinal gauge such that the gauge freedom is fixed completely, as discussed in Appendix A.1. Therefore, for the scalar modes of perturbation, we have $B = E = 0$ [2].

In the longitudinal gauge the perturbed metric (A.53) is of the form,

$$ds^2 = e^{2A}\,(1-2\psi)\,\eta_{\mu\nu}dx^\mu dx^\nu + (1+2\chi)\,dy^2. \tag{A.54}$$

Adopting the general results from the Appendix A.1 we find the following form of the linearized field equations for the scalar modes,

$$(\mu\nu): \qquad e^{2A}\eta_{\mu\nu}\left[2\left(A'' + 4A'^2\right)\chi + A'\chi' + e^{-2A}\Box^{(4)}\psi + 8A'\psi' + \psi''\right] + \partial_\mu\partial_\nu\left(2\psi - \chi\right) = \frac{1}{6M_*^3}e^{2A}\eta_{\mu\nu}\frac{\partial V(\phi)}{\partial\phi}\varphi, \tag{A.55}$$

$$(\mu 5): \qquad 3A'\partial_\mu\chi + 3\partial_\mu\psi' = \frac{1}{4M_*^3}\phi'\partial_\mu\varphi, \tag{A.56}$$

$$(55): \qquad 4\left(\psi'' + 2A'\psi'\right) + 4A'\chi' - e^{-2A}\Box^{(4)}\chi = \frac{1}{4M_*^3}\left[2\phi'\varphi' + \frac{2}{3}\frac{\partial V(\phi)}{\partial\phi}\varphi + \frac{4}{3}V(\phi)\chi\right]. \tag{A.57}$$

One can notice from Eq. (A.55) that the absence of the $\partial_\mu\partial_\nu$ term on the right hand side implies $\partial_\mu\partial_\nu\left(2\psi - \chi\right) = 0$ so that $\chi = 2\psi + c(y)$. Where $c(y)$ is a y-dependent constant of integration which can be fixed by the requirement that at 4D infinities $\chi, \psi \to 0$, therefore $c(y) = 0$.

When $\chi = 2\psi$ the equation of motion for the scalar field fluctuation (A.16) simplifies,

$$e^{-2A}\Box^{(4)}\varphi + \varphi'' + 4A'\varphi' - \frac{\partial^2 V(\phi)}{\partial\phi^2}\varphi - 6\phi'\psi' - 4\left(\phi'' + 4A'\phi'\right)\psi = 0. \tag{A.58}$$

It is important to note that, as usually in such cases, the equations of motions (A.55)-(A.58) are not independent. Adopting the relation $\chi = 2\psi$ and the background equations of motion

[2] We suppress the $\check{}$ signs hereafter, as it is clear that we are referring the modes in the new reference frame as discussed in Appendix A.1.

one derives the following equation that we will use instead of (A.55) and (A.57)

$$3\psi'' + 6A'\psi' - 3e^{-2A}\Box^{(4)}\psi = \frac{1}{2M_*^3}\phi'\varphi'. \tag{A.59}$$

Hence, Eqs. (A.56), (A.58) and (A.59) complete the set of linearized equations for the scalar modes. In the subsequent section we will use these equations to study stability of scalar field perturbations.

A.3. Vector perturbations

We can write down the metric for the vector perturbations as,

$$ds^2 = e^{2A}\left(\eta_{\mu\nu} + \partial_\mu G_\nu + \partial_\nu G_\mu\right) dx^\mu dx^\nu + C_\mu dx^\mu dy + dy^2, \tag{A.60}$$

where C_μ and G_μ are divergenceless vectors defined in Eqs. (A.17) and (A.18). Adopting the general results from the Appendix A.1 we find the following form of the linearized field equations for the vector modes

$$\partial_\mu\left[C'_\nu + 2A'C_\nu + -4e^{2A}A'G'_\nu - e^{2A}G''_\nu\right] = 0, \tag{A.61}$$

$$e^{-2A}\Box^{(4)}C_\nu - \Box^{(4)}G'_\mu = 0. \tag{A.62}$$

Since we are working in the gauge where $G_\mu = 0$ so the equations of motion for the vector modes of the metric perturbations read

$$\Box^{(4)}C_\nu = 0, \qquad \partial_\mu\left(C'_\nu + 2A'C_\nu\right) = 0. \tag{A.63}$$

A.4. Tensor perturbations

The tensor metric perturbation of metric (1.2) can be written as,

$$ds^2 = e^{2A(y)}(\eta_{\mu\nu} + H_{\mu\nu})dx^\mu dx^\nu + dy^2, \tag{A.64}$$

where, $H_{\mu\nu} = H_{\mu\nu}(x, y)$ is the tensor fluctuation as defined in (A.17). Adopting general results from the Appendix A.1 we find the following form of the linearized field equations for the tensor modes

$$\left(\partial_5^2 + 4A'\partial_5 + e^{-2A}\Box^{(4)}\right) H_{\mu\nu} = 0. \tag{A.65}$$

The zero-mode solution (corresponding to $\Box^{(4)}H_{\mu\nu} = 0$) of the above equation should represent the 4D graviton while the non-zero modes are the Kaluza-Klein (KK) graviton excitations.

APPENDIX B

SSB IN THE IR-UV-IR MODEL: REAL SCALAR CASE

In this Appendix, in order to gain some intuition concerning properties of models based on IR-UV-IR $\mathbb{Z}_2$-symmetric geometry, we consider a simple setup with a real scalar in the geometry defined in Sec. 2.2. The action for this toy model with a real scalar field $\Phi(x,y)$ is:

$$S_{toy} = -\int d^5x\sqrt{-g}\Big\{\frac{1}{2}g^{MN}\nabla_M\Phi\nabla_N\Phi + \frac{1}{2}\mu_B^2\Phi^2 + V_{IR}(\Phi)\delta(y+L) + V_{UV}(\Phi)\delta(y) + V_{IR}(\Phi)\delta(y-L)\Big\}, \tag{B.1}$$

where μ_B is bulk mass parameter and,

$$V_{UV}(\Phi) = \frac{m_{UV}^2}{2k}\Phi^2, \qquad V_{IR}(\Phi) = -\frac{m_{IR}^2}{2k}\Phi^2 + \frac{\lambda_{IR}}{4k^2}\Phi^4, \tag{B.2}$$

are the scalar field potentials localized on the UV and IR-branes, respectively. The background metric for the IR-UV-IR geometric setup is given by Eq. (1.2). It is important to note that the above action is invariant under $\Phi(x,y) \to -\Phi(x,y)$. In the following two sub-Appendices we consider two different strategies for spontaneous breaking (SSB) of the discrete symmetry: (i) SSB by vacuum expectation values of KK modes, and (ii) SSB by a vacuum expectation value of the 5D scalar field. Later we will compare the effective theories obtained within the two approaches.

B.1. SSB by vacuum expectation values of KK modes

Within this approach, we first KK-decompose the scalar field $\Phi(x,y)$ of the above action (B.1) as

$$\Phi(x,y) = \sum_n \Phi_n(x)f_n(y). \tag{B.3}$$

The wave-functions $f_n(y)$ are chosen to satisfy the following equation and orthonormality condition in the bulk,

$$-\partial_5\big(e^{4A(y)}\partial_5 f_n(y)\big) + \mu_B^2 e^{4A(y)} f_n(y) + e^{4A(y)}\frac{m_{UV}^2}{k} f_n(y)\delta(y) = m_n^2 e^{2A(y)} f_n(y), \tag{B.4}$$

$$\int_{-L}^{+L} dy e^{2A(y)} f_m(y) f_n(y) = \delta_{mn}. \tag{B.5}$$

In the presence of Dirac delta function $\delta(y)$ (UV-brane) at $y=0$ there is a discontinuity (jump) in the first derivatives of $f_n(y)$. The corresponding jump condition at $y=0$ and the boundary conditions at $y=\pm L$ are chosen to be:

$$\left(\partial_5 - \frac{m_{UV}^2}{2k}\right) f_n(y)\Big|_{0^+} = 0, \qquad \left(\pm\partial_5 - \frac{m_{IR}^2}{2k}\right) f_n(y)\Big|_{\pm L^\mp} = 0, \tag{B.6}$$

where $0^+ \equiv 0+\epsilon$ and $L^\pm \equiv L\pm\epsilon$ with $\epsilon \to 0$. Equation (B.4) together with the jump-boundary conditions (B.6) defines the basis for the KK decomposition. It is easy to see that the choice of (B.4) implied by the orthogonality relations (B.5) eliminates non-diagonal bulk mass terms (i.e. quadratic in $f_n(y)$) in the effective 4D action. The first condition in (B.6) is dictated by integrating (B.4) around $y=0$ (jump across the UV-brane), while the second one is imposed in order to eliminate non-diagonal terms $\propto \partial_5 f_n(y) f_m(y)|_{\pm L^\mp}$ at the ends of the interval.[1] This strategy will be often employed hereafter and in the main text to get rid of non-diagonal quadratic KK terms.

Since our background geometry is $\mathbb{Z}_2$-symmetric under $y \to -y$ therefore solutions of the Eq. (B.4) will have defined (*even* or *odd*) parity w.r.t. y. We denote the even and odd wave-functions as $f_n^{(+)}(y)$ and $f_n^{(-)}(y)$, respectively. It is instructive to write the KK-decomposition as

$$\begin{aligned}\Phi(x,y) &= \Phi^{(+)}(x,y) + \Phi^{(-)}(x,y),\\ &= \sum_n \Phi_n^{(+)}(x) f_n^{(+)}(|y|) + \epsilon(y)\sum_n \Phi_n^{(-)}(x) f_n^{(-)}(|y|),\end{aligned} \tag{B.7}$$

where $\epsilon(y)$ is ± 1 for $y \gtrless 0$. We rewrite Eq. (B.4) as

$$-\partial_5\big(e^{4A(y)}\partial_5 f_n^{(\pm)}(y)\big) + \mu_B^2 e^{4A(y)} f_n^{(\pm)}(y) + e^{4A(y)}\frac{m_{UV}^2}{k} f_n^{(\pm)}(y)\delta(y) = m_n^2 e^{2A(y)} f_n^{(\pm)}(y). \tag{B.8}$$

The jump and boundary conditions for the even and odd profiles follow from Eq. (B.6),

$$\left(\partial_5 - \frac{m_{UV}^2}{2k}\right) f_n^{(+)}(y)\Big|_{0^+} = 0, \qquad f_n^{(-)}(y)\Big|_0 = 0. \tag{B.9}$$

$$\left(\pm\partial_5 - \frac{m_{IR}^2}{k}\right) f_n^{(\pm)}(y)\Big|_{\pm L^\mp} = 0. \tag{B.10}$$

[1]The other choice $f_n(y)|_{\pm L} = 0$ eliminates all the IR-brane interactions, so it will not be considered.

We find the following general solutions of Eq. (B.8):

$$f_n^{(\pm)}(|y|) = \frac{e^{2k|y|}}{N_n^{(\pm)}} \left[J_\beta \left(m_n^{(\pm)} e^{k|y|}/k \right) + b_n^{(\pm)} Y_\beta \left(m_n^{(\pm)} e^{k|y|}/k \right) \right], \tag{B.11}$$

with $f_n^{(-)}(y) \equiv \epsilon(y) f_n^{(-)}(|y|)$. Above J_β and Y_β are the Bessel functions with weight $\beta \equiv \sqrt{4 + \mu_B^2/k^2}$, which parameterizes the bulk mass. Above $N_n^{(\pm)}$ and $b_n^{(\pm)}$ are the two integration constants for each parity mode. $N_n^{(\pm)}$ are fixed by the orthogonality condition (B.5). The coefficients $b_n^{(+)}$ and $b_n^{(-)}$ are implied by the jump conditions at $y = 0$ for even and odd wave functions:

$$b_n^{(+)} = -\frac{k^2 \delta_{UV} J_\beta \left(\frac{m_n^{(+)}}{k} \right) + k m_n^{(+)} J_{\beta+1} \left(\frac{m_n^{(+)}}{k} \right)}{k^2 \delta_{UV} Y_\beta \left(\frac{m_n^{(+)}}{k} \right) + k m_n^{(+)} Y_{\beta+1} \left(\frac{m_n^{(+)}}{k} \right)}, \qquad b_n^{(-)} = -\frac{J_\beta \left(\frac{m_n^{(-)}}{k} \right)}{Y_\beta \left(\frac{m_n^{(-)}}{k} \right)}, \tag{B.12}$$

where δ_{UV} is the UV-brane dimensionless parameter defined as

$$\delta_{UV} \equiv \frac{m_{UV}^2}{k^2} - 2(2 + \beta). \tag{B.13}$$

Now the boundary condition at $y = \pm L$ implies the following equation whose roots determine the mass spectrum of KK-modes,

$$\begin{aligned} &\frac{m_n^{(\pm)}}{m_{KK}} \left[J_{\beta+1} \left(\frac{e^{kL} m_n^{(\pm)}}{k} \right) + b_n^{(\pm)} Y_{\beta+1} \left(\frac{e^{kL} m_n^{(\pm)}}{k} \right) \right] \\ &\qquad = -\frac{1}{2} \delta_{IR} \left[J_\beta \left(\frac{e^{kL} m_n^{(\pm)}}{k} \right) + b_n^{(\pm)} Y_\beta \left(\frac{e^{kL} m_n^{(\pm)}}{k} \right) \right], \end{aligned} \tag{B.14}$$

where m_{KK} and the IR-brane dimensionless parameter δ_{IR} are defined in Eq. (5.33). We solve the above equation for the KK mass eigenvalues in the approximation $kL \gg 1$ and $m_n^{(\pm)} \ll k$, such that one can set $b_n^{(\pm)} \approx 0$. With this simplification we get the following equation for the zero-mode mass $m_0^{(\pm)}$:

$$\frac{m_0^{(\pm)}}{m_{KK}} \frac{J_{\beta+1} \left(\frac{m_0^{(\pm)}}{m_{KK}} \right)}{J_\beta \left(\frac{m_0^{(\pm)}}{m_{KK}} \right)} \simeq -\frac{1}{2} \delta_{IR}. \tag{B.15}$$

Expanding around $m_0^{(\pm)} \sim 0$, we find the following mass for the zero-mode,

$$m_0^{2(\pm)} \simeq -(1 + \beta) m_{KK}^2 \delta_{IR} \left[1 - \frac{\delta_{IR}}{2 + \beta} + \frac{2\delta_{IR}^2}{(2 + \beta)^2 (3 + \beta)} + \mathcal{O}(\delta_{IR}^3) \right]. \tag{B.16}$$

Note that the above zero-mode mass must be negative in order to trigger the spontaneous symmetry breaking. Therefore we assume at this point that $\delta_{IR} > 0$. The above solution implies that for $\delta_{IR} \sim \mathcal{O}(1)$ the zero-mode mass is of the order of KK mass scale m_{KK}. In order to have a light zero-mode (of order of the electroweak scale), we need to fine-tune δ_{IR}

such that $|m_0^{(\pm)}| \ll m_{KK}$ which implies that $\delta_{IR} \sim 10^{-3}$ for $m_{KK} \sim \mathcal{O}(\text{few})$ TeV (as required in a more realistic context by the EWPT, see Sec. 5.3.1). For non-zero KK mode masses (excited KK states with $n \neq 0$) we assume that $m_n^{(\pm)} \geq m_{KK}$ so that we can use $\delta_{IR} \approx 0$.[2] Hence the non-zero KK-mode masses $m_n^{(\pm)}$ read:

$$m_n^{(\pm)} \simeq \left(n + \frac{\beta}{2} - \frac{3}{4}\right) \pi m_{KK}, \tag{B.17}$$

which implies that masses of even and odd excited KK-modes are degenerate.

Now we can write the 4D effective action for the toy model (B.1) by using the above KK-decomposition and integrating over the extra dimension y, as:

$$\begin{aligned} S_{toy} = -\int d^4x \Big\{ & \frac{1}{2}\partial_\mu \Phi_n^{(+)} \partial^\mu \Phi_n^{(+)} + \frac{1}{2}\partial_\mu \Phi_n^{(-)} \partial^\mu \Phi_n^{(-)} + \frac{1}{2} m_n^{2(+)} \Phi_n^{2(+)} + \frac{1}{2} m_n^{2(-)} \Phi_n^{2(-)} \\ & + \frac{\lambda_{klmn}^{(+)}}{4} \Phi_k^{(+)} \Phi_l^{(+)} \Phi_m^{(+)} \Phi_n^{(+)} + \frac{\lambda_{klmn}^{(-)}}{4} \Phi_k^{(-)} \Phi_l^{(-)} \Phi_m^{(-)} \Phi_n^{(-)} + \frac{3}{2} \lambda_{klmn} \Phi_k^{(+)} \Phi_l^{(+)} \Phi_m^{(-)} \Phi_n^{(-)} \Big\}, \end{aligned} \tag{B.18}$$

where $\lambda_{klmn}^{(\pm)}$ and λ_{klmn} are quartic couplings given by,

$$\lambda_{klmn}^{(\pm)} = e^{4A(L)} \frac{\lambda_{IR}}{k^2} f_k^{(\pm)} f_l^{(\pm)} f_m^{(\pm)} f_n^{(\pm)} \Big|_L, \qquad \lambda_{klmn} = e^{4A(L)} \frac{\lambda_{IR}}{k^2} f_k^{(+)} f_l^{(+)} f_m^{(-)} f_n^{(-)} \Big|_L. \tag{B.19}$$

The above action (B.18) is symmetric under $\mathbb{Z}_2' \times \mathbb{Z}_2$ under which $\Phi_n^{(+)} \to -\Phi_n^{(+)}$ and $\Phi_n^{(-)} \to -\Phi_n^{(-)}$, respectively. Note that it has been taken into account that the integration of the IR-brane delta functions will provide a factor of 1/2 instead of 1, as our geometry is an interval, assuming that there is nothing outside $[-L, +L]$. The above effective action is valid for all the KK-modes. In order to obtain the low-energy effective action we limit ourself to zero-modes only. The low energy (zero-mode) effective action for our toy model is:

$$\begin{aligned} S_{toy}^{eff} = -\int d^4x \Big\{ & \frac{1}{2}\partial_\mu \Phi_0^{(+)} \partial^\mu \Phi_0^{(+)} + \frac{1}{2}\partial_\mu \Phi_0^{(-)} \partial^\mu \Phi_0^{(-)} - \frac{1}{2}\mu^2 \Phi_0^{2(+)} - \frac{1}{2}\mu^2 \Phi_0^{2(-)} \\ & + \frac{\lambda_{0000}^{(+)}}{4} \Phi_0^{4(+)} + \frac{\lambda_{0000}^{(-)}}{4} \Phi_0^{4(-)} + \frac{3}{2} \lambda_{0000} \Phi_0^{2(+)} \Phi_0^{2(-)} \Big\}, \end{aligned} \tag{B.20}$$

where $\lambda_{0000}^{(\pm)}$ and λ_{0000} are given by Eq. (B.19), whereas μ is the zero-mode mass parameter defined as

$$-m_0^{(\pm)2} \simeq \mu^2 \equiv (1+\beta) m_{KK}^2 \delta_{IR}, \tag{B.21}$$

In order to get more insight into the above results we find also the wave-function for the zero-modes $f_0^{(\pm)}(y)$.[3] Following the above mentioned approximation, $kL \gg 1$ and $m_0^{(\pm)} \ll k$

[2] For the zero-mode mass this approximation does not hold, as $m_0^{(\pm)}/m_{KK}$ is of the same order as $\delta_{IR} \sim 10^{-3}$.
[3] One can also get the solutions for the zero-mode wave functions $f_0^{(\pm)}(y)$ by solving the Eq. (B.4) for $m_0^{(\pm)} = 0$.

(such that $b_0^{(\pm)} \approx 0$), the wave-function for zero-modes $f_0^{(\pm)}(y)$ has the following simple form

$$f_0^{(\pm)}(|y|) \simeq \frac{e^{2k|y|}}{N_0^{(\pm)}} J_\beta \left(\frac{m_0^{(\pm)}}{k} e^{k|y|} \right), \tag{B.22}$$

with $f_0^{(-)}(y) \equiv \epsilon(y) f_0^{(-)}(|y|)$. The normalization $N_0^{(\pm)}$ can be fixed by Eq. (B.5):

$$N_0^{2(\pm)} = 2 \in \tau_0^L dy \left(e^{2ky} J_\beta \left(\frac{m_0^{(\pm)}}{k} e^{ky} \right)^2 \right) \simeq \frac{e^{2kL}}{k(1+\beta)} J_\beta \left(\frac{m_0^{(\pm)}}{k} e^{kL} \right)^2. \tag{B.23}$$

We get the zero-mode wave-functions $f_0^{(\pm)}(y)$, after expanding the Bessel functions around zero ($m_0^{(\pm)} \approx 0$), as

$$f_0^{(\pm)}(|y|) \simeq \sqrt{k(1+\beta)} e^{kL} e^{(2+\beta)k(|y|-L)}. \tag{B.24}$$

Hence for $f_0^{(\pm)}(\pm L) \simeq \sqrt{k(1+\beta)} e^{kL}$, the quartic couplings are:

$$\lambda_{0000}^{(\pm)} = \lambda_{0000} \simeq \lambda, \qquad \text{where} \qquad \lambda \equiv \lambda_{IR}(1+\beta)^2. \tag{B.25}$$

After calculating all the parameters of the effective action (B.20) in terms of the fundamental 5D parameters, we can proceed further to find vevs of the scalar fields $\Phi^{(\pm)}$ (from hereon in the Appendix we drop the subscripts 0 from zero-modes). We can write the scalar potentials for even and odd fields as

$$V(\Phi^{(\pm)}) = -\frac{1}{2}\mu^2 \Phi^{(+)2} - \frac{1}{2}\mu^2 \Phi^{(-)2} + \frac{\lambda}{4}\Phi^{(+)4} + \frac{\lambda}{4}\Phi^{(-)4} + \frac{3}{2}\lambda \Phi^{(+)2}\Phi^{(-)2}. \tag{B.26}$$

Note that the above scalar potential has $\mathbb{Z}_2' \times \mathbb{Z}_2$ symmetry as pointed out earlier. One of the discrete symmetry factors will be spontaneously broken when $\Phi^{(\pm)}$ develop a non-zero vev. We find the following conditions for global minima of the potential:

$$v^{(+)2} = \left(\frac{\mu^2}{\lambda} - 3v^{(-)2} \right), \qquad \text{or} \qquad v^{(+)} = 0, \tag{B.27}$$

and

$$v^{(-)2} = \left(\frac{\mu^2}{\lambda} - 3v^{(+)2} \right), \qquad \text{or} \qquad v^{(-)} = 0. \tag{B.28}$$

One can easily see from Fig. B.1 that the scalar potential $V(\Phi^{(+)}, \Phi^{(-)})$ has four degenerate global minima at $(\pm v^{(+)}, 0)$ and $(0, \pm v^{(-)})$. One can choose any of these global vacua. We select the vacuum where the even mode $\Phi^{(+)}$ acquires a vev, whereas $\Phi^{(-)}$ has zero vev. In this case the above minimization condition is,

$$v^{(+)} = \frac{\mu}{\sqrt{\lambda}}, \qquad v^{(-)} = 0. \tag{B.29}$$

This choice of the vacuum breaks $\mathbb{Z}_2'$ spontaneously. Now we perturb the even and odd modes

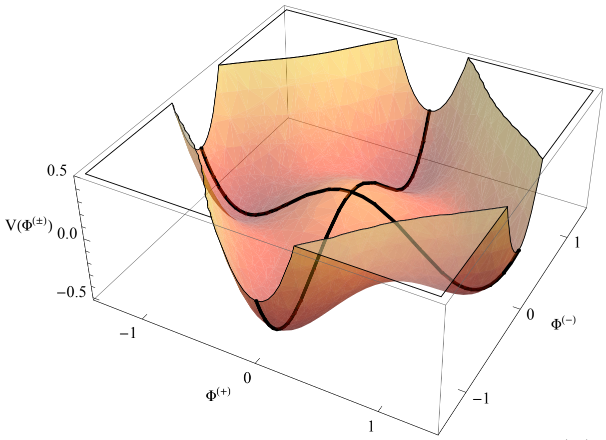


Figure B.1.: This graph illustrates the shape of the scalar potential $V(\Phi^{(\pm)})$ as a function of the fields $\Phi^{(\pm)}$ for $\mu = 1$, and $\lambda = 1$.

around the vacuum of our choice as:

$$\Phi^{(+)}(x) = v^{(+)} + \phi(x), \qquad \Phi^{(-)}(x) = \chi(x). \tag{B.30}$$

The effective toy action in terms of the even and odd fluctuations can be written as

$$\begin{aligned} S_{toy}^{eff} = -\int d^4x \Big\{ & \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\partial_\mu\chi\partial^\mu\chi + \frac{1}{2}m^2\phi^2 + \frac{1}{2}m^2\chi^2 + \frac{\lambda}{4}\phi^4 + \frac{\lambda}{4}\chi^4 \\ & + \sqrt{\lambda}\mu\phi\Big(\phi^2 + 3\chi^2\Big) + \frac{3}{2}\lambda\phi^2\chi^2 \Big\}, \end{aligned} \tag{B.31}$$

where $m^2 \equiv 2\mu^2$. It is important to note that both even and odd fluctuations have the same mass m and the same quartic coupling λ. Moreover, the above action has an unbroken discrete $\mathbb{Z}_2$ symmetry, under which the fields transform as $\phi \to +\phi$ and $\chi \to -\chi$.

B.2. SSB by a vacuum expectation value of 5D scalar field

In this approach we perturb the 5D scalar field $\Phi(x, y)$ around a y-dependent vev $v(y)$ as

$$\Phi(x, y) = v(y) + \phi(x, y), \tag{B.32}$$

where $v(y)$ is the background solution for the toy action (B.1) and $\phi(x, y)$ is a perturbation around the background. The e.o.m. for $v(y)$ and $\phi(x, y)$ read as:

$$\begin{aligned} & \left[-\partial_5\left(e^{4A(y)}\partial_5\right) + \mu_B^2 e^{4A(y)} \right] v(y) \\ & = -e^{4A(y)} \left[\frac{\partial V_{IR}(v)}{\partial v}\delta(y+L) + \frac{\partial V_{UV}(v)}{\partial v}\delta(y) + \frac{\partial V_{IR}(v)}{\partial v}\delta(y-L) \right], \end{aligned} \tag{B.33}$$

$$\begin{aligned} & \left[-e^{2A(y)}\Box^{(4)} - \partial_5\left(e^{4A(y)}\partial_5\right) + \mu_B^2 e^{4A(y)} \right] \phi(x,y) \\ & = -e^{4A(y)} \left[\frac{\partial^2 V_{IR}(v)}{\partial v^2}\phi\delta(y+L) + \frac{\partial^2 V_{UV}(v)}{\partial v^2}\phi\delta(y) + \frac{\partial^2 V_{IR}(v)}{\partial v^2}\phi\delta(y-L) \right]. \end{aligned} \tag{B.34}$$

Note that the above e.o.m. for the perturbation $\phi(x,y)$ (B.34) is obtained by Taylor series expansion of the brane potentials around the background $v(y)$ and only quadratic terms in field $\phi(x,y)$ are kept such that the KK-mass matrix for the field $\phi(x,y)$ will be diagonal. The following jump and boundary conditions for $v(y)$ and $\phi(x,y)$ are implied by the general strategy adopted in the main text,

$$\left(\partial_5 - \frac{m_{UV}^2}{2k}\right) v(y)\Big|_{0^+} = 0, \qquad \left(\pm\partial_5 - \frac{m_{IR}^2}{2k} + \frac{\lambda_{IR}}{2k^2}v^2(y)\right) v(y)\Big|_{\pm L^\mp} = 0, \tag{B.35}$$

$$\left(\partial_5 - \frac{m_{UV}^2}{2k}\right) \phi(x,y)\Big|_{0^+} = 0, \qquad \left(\pm\partial_5 - \frac{m_{IR}^2}{2k} + \frac{3\lambda_{IR}}{2k^2}v^2(y)\right) \phi(x,y)\Big|_{\pm L^\mp} = 0. \tag{B.36}$$

Our geometric setup is $\mathbb{Z}_2$ symmetric under $y \to -y$, therefore we can look for solutions possessing a definite parity; the even $v^{(+)}(y)$ and the odd $v^{(-)}(y)$. Since our geometric setup is symmetric, therefore we choose the even vacuum solution $v^{(+)}(y)$ for the scalar field. Note that the choice of odd vacuum solution could lead to the breaking of geometric $\mathbb{Z}_2$ symmetry. As discussed in the main text, the jump (boundary) conditions for the even solutions at $y = 0$ and $y = \pm L$ are:

$$\left(\partial_5 - \frac{m_{UV}^2}{2k}\right) v^{(+)}(y)\Big|_{0^+} = 0, \qquad \left(\pm\partial_5 - \frac{m_{IR}^2}{2k} + \frac{\lambda_{IR}}{2k^2}v^{2(\pm)}(y)\right) v^{(\pm)}(y)\Big|_{\pm L^\mp} = 0. \tag{B.37}$$

We find the following solutions for the even background vev $v^{(+)}(y)$:

$$v^{(+)}(y) = C_1 e^{(2+\beta)k|y|} + C_2 e^{(2-\beta)k|y|}, \qquad -L \leq y \leq L, \tag{B.38}$$

where C_1 and C_2 are the integration constants. We apply the jump condition (5.53) at $y = 0$ and the boundary condition at $y = \pm L$ (5.54) to fix the two integration constants as,

$$C_2 = -\frac{\delta_{UV}}{\delta_{UV} + 4\beta} C_1, \tag{B.39}$$

$$C_1 = \sqrt{\frac{k^3}{\lambda_{IR}}\left(\delta_{IR} - \frac{\delta_{UV}(\delta_{IR}+4\beta)}{\delta_{UV}+4\beta} e^{-2\beta kL}\right) e^{-(2+\beta)kL}} \left(1 - \frac{e^{-2\beta kL}\delta_{UV}}{\delta_{UV}+4\beta}\right)^{-3/2}, \tag{B.40}$$

where $\delta_{UV} \equiv m_{UV}^2/k^2 - 2(2+\beta)$ and δ_{IR} is defined in Eq. (5.33). For $kL \gg 1$ and $\beta > 0$, the terms proportional to $e^{-2\beta kL}$ are negligible in the region of interest (near the IR-brane). Hence the vacuum solution for the scalar field can be written as:

$$v^{(+)}(y) \simeq \sqrt{\frac{k^3\delta_{IR}}{\lambda_{IR}}} e^{(2+\beta)k(|y|-L)} \equiv v_4 f_v(y), \tag{B.41}$$

where the constant vev v_4 and the y-dependent vev profile $f_v(y)$ are:

$$v_4 \equiv \sqrt{\frac{m_{KK}^2\delta_{IR}}{\lambda_{IR}(1+\beta)}}, \qquad f_v(y) \equiv \sqrt{k(1+\beta)} e^{kL} e^{(2+\beta)k(|y|-L)}. \tag{B.42}$$

The y-dependent vev profile satisfies the orthonormality condition

$$\int_{-L}^{L} dy e^{2A(y)} f_v^2(y) = 1. \tag{B.43}$$

From the above solution we conclude that for $\lambda_{IR} > 0$ (as required by the positivity of the tree-level potential) one needs $\delta_{IR} > 0$, i.e. $m_{IR}^2/k^2 > 2(2+\beta)$. It is worth mentioning here that the quartic term in the IR-brane potential is crucial for a non-trivial vev profile $v^{(\pm)}(y)$. If the quartic term would have been absent in the V_{IR}, i.e. $\lambda_{IR} = 0$, then the b.c. (5.53) and (5.54) would have implied $v(y) = 0$. Even though the quartic term is not in the bulk (only localized at the IR-brane), nevertheless, the b.c. imply the non-zero profile in the bulk.

Next we deal with the fluctuation field $\phi(x,y)$ by KK-decomposing it as

$$\phi(x,y) = \sum_n \phi_n(x)\tilde{f}_n(y), \tag{B.44}$$

such that the e.o.m. for the wave-function $\tilde{f}_n(y)$ that follows from Eq. (B.34) reads

$$-\partial_5\big(e^{4A(y)}\partial_5\tilde{f}_n(y)\big) + \mu_B^2 e^{4A(y)}\tilde{f}_n(y) + e^{4A(y)}\frac{m_{UV}^2}{k}\tilde{f}_n(y)\delta(y) = \tilde{m}_n^2 e^{2A(y)}\tilde{f}_n(y), \tag{B.45}$$

while the KK-modes satisfy $\square^{(4)}\phi_n(x) = m_n^2\phi_n(x)$. Our symmetric $\mathbb{Z}_2$ geometry implies that the solutions of the Eq. (B.45) are *even* and *odd* under $y \to -y$. Since the wave functions have definite parity therefore it is instructive to rewrite the KK-decomposition for the fluctuation field $\phi(x,y)$ (B.44) as:

$$\phi(x,y) = \sum_n \phi_n^{(+)}(x)\tilde{f}_n^{(+)}(|y|) + \epsilon(y)\sum_n \phi_n^{(-)}(x)\tilde{f}_n^{(-)}(|y|). \tag{B.46}$$

The even and odd solutions are subject to different jump conditions at $y = 0$ that follow from Eq. (B.36):

$$\left(\partial_5 - \frac{m_{UV}^2}{2k}\right)\tilde{f}_n^{(+)}(y)\Big|_{0^+} = 0, \qquad \tilde{f}_n^{(-)}(y)\Big|_{0} = 0, \tag{B.47}$$

$$\left(\pm\partial_5 - \frac{m_{IR}^2}{2k} + \frac{3\lambda_{IR}}{2k^2}v^2(y)\right)\tilde{f}_n^{(\pm)}(y)\Big|_{\pm L^{\mp}} = 0. \tag{B.48}$$

Now we write the 4D effective action for the toy model (B.1) by using the above KK-decomposition and integrating over the extra dimension y, as

$$\begin{aligned} S_{toy}^{eff} = -\int d^4x \Big\{ & \frac{1}{2}\partial_\mu\phi_n^{(+)}\partial^\mu\phi_n^{(+)} + \frac{1}{2}\partial_\mu\phi_n^{(-)}\partial^\mu\phi_n^{(-)} + \frac{1}{2}\tilde{m}_n^{2(+)}\phi_n^{2(+)} + \frac{1}{2}\tilde{m}_n^{2(-)}\phi_n^{2(-)} \\ & + \tilde{\lambda}_{lmn}^{(+)}\phi_l^{(+)}\phi_m^{(+)}\phi_n^{(+)} + 3\tilde{\lambda}_{lmn}^{(-)}\phi_l^{(+)}\phi_m^{(-)}\phi_n^{(-)} + \frac{\tilde{\lambda}_{klmn}^{(+)}}{4}\phi_k^{(+)}\phi_l^{(+)}\phi_m^{(+)}\phi_n^{(+)} \\ & + \frac{\tilde{\lambda}_{klmn}^{(-)}}{4}\phi_k^{(-)}\phi_l^{(-)}\phi_m^{(+)}\phi_n^{(+)} + \frac{3}{2}\tilde{\lambda}_{klmn}\phi_k^{(+)}\phi_l^{(+)}\phi_m^{(-)}\phi_n^{(-)} \Big\}, \end{aligned} \tag{B.49}$$

where,

$$\tilde{\lambda}^{(\pm)}_{lmn} = \frac{\lambda_{IR}}{k^2} v_4 e^{4A(y)} f_v(y) \tilde{f}^{(+)}_l(y) \tilde{f}^{(\pm)}_m(y) \tilde{f}^{(\pm)}_n(y)\Big|_L, \tag{B.50}$$

$$\tilde{\lambda}^{(\pm)}_{klmn} = \frac{\lambda_{IR}}{k^2} e^{4A(y)} \tilde{f}^{(\pm)}_k(y) \tilde{f}^{(\pm)}_l(y) \tilde{f}^{(\pm)}_m(y) \tilde{f}^{(\pm)}_n(y)\Big|_L, \tag{B.51}$$

$$\tilde{\lambda}_{klmn} = \frac{\lambda_{IR}}{k^2} e^{4A(y)} \tilde{f}^{(+)}_k(y) \tilde{f}^{(+)}_l(y) \tilde{f}^{(-)}_m(y) \tilde{f}^{(-)}_n(y)\Big|_L. \tag{B.52}$$

For the warped (AdS) geometry the general solutions for Eq. (B.45) corresponding to the even and odd $\tilde{f}^{(\pm)}_n(y)$ are the same as in Eq. (B.11). From the b.c. at $y = \pm L$ we get the following equation whose roots will determine the mass spectrum of the KK-modes,

$$\begin{aligned} \frac{\tilde{m}^{(\pm)}_n}{m_{KK}} &\left[J_{\beta+1}\left(\frac{e^{kL}\tilde{m}^{(\pm)}_n}{k}\right) + b^{(\pm)}_n Y_{\beta+1}\left(\frac{e^{kL}\tilde{m}^{(\pm)}_n}{k}\right)\right] \\ &= \delta_{IR}\left[J_\beta\left(\frac{e^{kL}\tilde{m}^{(\pm)}_n}{k}\right) + b^{(\pm)}_n Y_\beta\left(\frac{e^{kL}\tilde{m}^{(\pm)}_n}{k}\right)\right], \end{aligned} \tag{B.53}$$

where m_{KK} and δ_{IR} are defined in Eq. (5.33). Assuming that $kL \gg 1$ and $\tilde{m}^{(\pm)}_0 \ll k$, we find $b^{(\pm)}_n \approx 0$. Hence the above mass eigenvalue equation takes the following form for the zero-mode masses $\tilde{m}^{(\pm)}_0$,

$$\frac{\tilde{m}^{(\pm)}_0}{m_{KK}} J_{\beta+1}\left(\frac{\tilde{m}^{(\pm)}_0}{m_{KK}}\right) \simeq \delta_{IR} J_\beta\left(\frac{\tilde{m}^{(\pm)}_0}{m_{KK}}\right). \tag{B.54}$$

We expand the above expression around $\tilde{m}^{(\pm)}_0 \sim 0$ to get the following masses for the zero-modes,

$$\tilde{m}^{2(\pm)}_0 \simeq 2(1+\beta) m^2_{KK} \delta_{IR}\left[1 - \frac{\delta_{IR}}{2+\beta} + \frac{2\delta^2_{IR}}{(2+\beta)^2(3+\beta)} + \mathcal{O}(\delta^3_{IR})\right]. \tag{B.55}$$

As explained in the previous sub-Appendix, in order to have the light zero-mode mass $\sim \mathcal{O}(100)$ GeV, we need to fine-tune $\delta_{IR} \sim 10^{-3}$. The above result also implies that the odd zero-mode is degenerate in the mass with the even zero-mode. For the non-zero modes (excited KK-modes) we assume that $\tilde{m}^{(\pm)}_n / m_{KK} \gg \delta_{IR}$, hence the $\tilde{m}^{(\pm)}_n$ for the non-zero KK-modes ($n \neq 0$) are:

$$\tilde{m}^{(\pm)}_n \simeq \left(n + \frac{\beta}{2} - \frac{3}{4}\right)\pi m_{KK}, \tag{B.56}$$

which implies that masses of the even and odd excited KK-modes are the same.

Assuming the KK-scale m_{KK} is large enough, we can write down the low-energy effective action for lowest even and odd modes $\phi^{(\pm)}_0$ from the action (B.49) as:

$$\begin{aligned} S^{eff}_{toy} = -\int d^4x \Big\{ &\frac{1}{2}\partial_\mu \phi^{(+)}_0 \partial^\mu \phi^{(+)}_0 + \frac{1}{2}\partial_\mu \phi^{(-)}_0 \partial^\mu \phi^{(-)}_0 + \frac{1}{2}\tilde{m}^{(+)2}_0 \phi^{(+)2}_0 + \frac{1}{2}\tilde{m}^{(-)2}_0 \phi^{(-)2}_0 \\ &+ \tilde{\lambda}^{(+)}_{000}\phi^{(+)3}_0 + 3\tilde{\lambda}^{(-)}_{000}\phi^{(+)}_0\phi^{(-)2}_0 + \frac{\tilde{\lambda}^{(+)}_{0000}}{4}\phi^{(+)4}_0 + \frac{\tilde{\lambda}^{(-)}_{0000}}{4}\phi^{(-)4}_0 + \frac{3\tilde{\lambda}_{0000}}{2}\phi^{(+)2}_0\phi^{(-)2}_0\Big\}, \end{aligned} \tag{B.57}$$

where couplings take the following values implied by Eqs. (B.50) and (B.52):

$$\tilde{\lambda}^{(\pm)}_{000} \simeq \sqrt{\lambda}\mu, \qquad \tilde{\lambda}^{(\pm)}_{0000} = \tilde{\lambda}_{0000} \simeq \lambda, \tag{B.58}$$

with $\mu^2 \equiv (1+\beta)m^2_{KK}\delta_{IR}$ and $\lambda \equiv \lambda_{IR}(1+\beta)^2$. Simplifying our notation ($\phi^{(+)}_0 = \phi$, $\phi^{(-)}_0 = \chi$ and $\tilde{m}_0 = \tilde{m}$), we can write the above effective action (B.57) in the following form:

$$\begin{aligned} S_{eff} = -\int d^4x \Big\{ & \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\partial_\mu\chi\partial^\mu\chi + \frac{1}{2}\tilde{m}^2\phi^2 + \frac{1}{2}\tilde{m}^2\chi^2 + \frac{\lambda}{4}\phi^4 + \frac{\lambda}{4}\chi^4 \\ & + \sqrt{\lambda}\mu\phi\Big(\phi^2 + 3\chi^2\Big) + \frac{3}{2}\lambda\phi^2\chi^2 \Big\}, \end{aligned} \tag{B.59}$$

where $\tilde{m}_0$ is the mass of the zero-modes given by Eq. (B.55); the leading term is, $\tilde{m}_0^2 = 2\mu^2$.

Let us conclude this Appendix by comparing the effective theories obtained in the previous sub-Appendix (B.31) and here (B.59). It is straight forward to see that the low energy (zero-mode) d.o.f. in the both actions are same. Moreover, the effective theories are identical, for example, all the masses and coupling constants are same in terms of 5D fundamental parameters of the theory. This is a non-trivial matching and the core of this matching lies in the fact that the y-dependent vev $v(y)$ can be written as constat v_4 times the normalized y-dependent profile $f_v(y)$ such that $v_4 = v_1 = \frac{\mu}{\sqrt{\lambda}}$ and $f_0(y) \simeq f_v(y)$, where v_1 and $f_0(y)$ are the vev and the zero-mode profile, respectively, obtained in the previous sub-Appendix. Once this matching is realized then masses and coupling constants in both low-energy effective theories are exactly identical. In other words, consistently with our expectations, operations of KK-expansion and the expansion around y-dependent vev do commute (at least for the zero modes).

BIBLIOGRAPHY

[1] A. Ahmed and B. Grzadkowski, *Brane modeling in warped extra-dimension*, *JHEP* **1301** (2013) 177, [arXiv:1210.6708].

[2] A. Ahmed, L. Dulny, and B. Grzadkowski, *Modeling Branes in Warped Extra-Dimension*, *Acta Phys.Polon.* **B44** (2013), no. 11 2381–2387.

[3] A. Ahmed, L. Dulny, and B. Grzadkowski, *Generalized Randall-Sundrum model with a single thick brane*, *Eur.Phys.J.* **C74** (2013) 2862, [arXiv:1312.3577].

[4] A. Ahmed, B. Grzadkowski, and J. Wudka, *Thick-Brane Cosmology*, *JHEP* **1404** (2014) 061, [arXiv:1312.3576].

[5] A. Ahmed, B. Grzadkowski, J. F. Gunion, and Y. Jiang, *Higgs dark matter from a warped extra dimension the truncated-inert-doublet model*, *JHEP* **10** (2015) 033, [arXiv:1504.03706].

[6] A. Ahmed, B. Grzadkowski, J. F. Gunion, and Y. Jiang, *Radius stabilization and dark matter with a bulk Higgs in warped extra dimension*, *Acta Phys.Polon.* **B46** (2015), no. 11 in press, [arXiv:1510.04116].

[7] A. Ahmed, B. Grzadkowski, J. F. Gunion, and Y. Jiang, *Higgs dark matter from a warped extra dimension*, *PoS* **PLANCK2015** (2015) 002, [arXiv:1510.05722].

[8] **ATLAS** Collaboration, G. Aad et al., *Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC*, *Phys. Lett.* **B716** (2012) 1–29, [arXiv:1207.7214].

[9] **CMS** Collaboration, S. Chatrchyan et al., *Observation of a new boson at a mass of 125 GeV with the CMS experiment at the LHC*, *Phys. Lett.* **B716** (2012) 30–61, [arXiv:1207.7235].

[10] S. P. Martin, *A Supersymmetry primer*, hep-ph/9709356. [Adv. Ser. Direct. High Energy Phys.18,1(1998)].

[11] A. Djouadi, *The Anatomy of electro-weak symmetry breaking. II. The Higgs bosons in the minimal supersymmetric model*, *Phys. Rept.* **459** (2008) 1–241, [hep-ph/0503173].

[12] V. A. Rubakov, *Large and infinite extra dimensions: An Introduction*, *Phys. Usp.* **44** (2001) 871–893, [hep-ph/0104152]. [Usp. Fiz. Nauk171,913(2001)].

[13] G. Gabadadze, *ICTP lectures on large extra dimensions*, hep-ph/0308112.

[14] C. Csaki, *TASI lectures on extra dimensions and branes*, hep-ph/0404096.

[15] R. Sundrum, *Tasi 2004 lectures: To the fifth dimension and back*, hep-th/0508134.

[16] R. Rattazzi, *Cargese lectures on extra-dimensions*, hep-ph/0607055.

[17] H.-C. Cheng, *Introduction to Extra Dimensions*, arXiv:1003.1162.

[18] E. Ponton, *TASI 2011: Four Lectures on TeV Scale Extra Dimensions*, arXiv:1207.3827.

[19] N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, *The Hierarchy problem and new dimensions at a millimeter*, *Phys.Lett.* **B429** (1998) 263–272, [hep-ph/9803315].

[20] N. Arkani-Hamed, S. Dimopoulos, and G. R. Dvali, *Phenomenology, astrophysics and cosmology of theories with submillimeter dimensions and TeV scale quantum gravity*, *Phys. Rev.* **D59** (1999) 086004, [hep-ph/9807344].

[21] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, *New dimensions at a millimeter to a Fermi and superstrings at a TeV*, *Phys.Lett.* **B436** (1998) 257–263, [hep-ph/9804398].

[22] L. Randall and R. Sundrum, *A Large mass hierarchy from a small extra dimension*, *Phys.Rev.Lett.* **83** (1999) 3370–3373, [hep-ph/9905221].

[23] L. Randall and R. Sundrum, *An Alternative to compactification*, *Phys.Rev.Lett.* **83** (1999) 4690–4693, [hep-th/9906064].

[24] V. Rubakov and M. Shaposhnikov, *Do We Live Inside a Domain Wall?*, *Phys.Lett.* **B125** (1983) 136–138.

[25] O. DeWolfe, D. Freedman, S. Gubser, and A. Karch, *Modeling the fifth-dimension with scalars and gravity*, *Phys.Rev.* **D62** (2000) 046008, [hep-th/9909134].

[26] M. Gremm, *Four-dimensional gravity on a thick domain wall*, *Phys.Lett.* **B478** (2000) 434–438, [hep-th/9912060].

[27] G. W. Gibbons, R. Kallosh, and A. D. Linde, *Brane world sum rules*, *JHEP* **0101** (2001) 022, [hep-th/0011225].

[28] **Planck** Collaboration, P. Ade et al., *Planck 2015 results. XIII. Cosmological parameters*, arXiv:1502.01589.

[29] G. Bertone, D. Hooper, and J. Silk, *Particle dark matter: Evidence, candidates and constraints*, *Phys. Rept.* **405** (2005) 279–390, [hep-ph/0404175].

[30] V. Sahni, *Dark matter and dark energy*, *Lect. Notes Phys.* **653** (2004) 141–180, [astro-ph/0403324]. [,141(2004)].

[31] D. Hooper and S. Profumo, *Dark matter and collider phenomenology of universal extra dimensions*, *Phys. Rept.* **453** (2007) 29–115, [hep-ph/0701197].

[32] J. L. Feng, *Dark Matter Candidates from Particle Physics and Methods of Detection*, *Ann. Rev. Astron. Astrophys.* **48** (2010) 495–545, [arXiv:1003.0904].

[33] Z. Lalak and R. Matyszkiewicz, *Boundary terms in brane worlds*, *JHEP* **0111** (2001) 027, [hep-th/0110141].

[34] W. D. Goldberger and M. B. Wise, *Modulus stabilization with bulk fields*, *Phys.Rev.Lett.* **83** (1999) 4922–4925, [hep-ph/9907447].

[35] G. Gabadadze, L. Grisa, and Y. Shang, *Resonance in asymmetric warped geometry*, *JHEP* **0608** (2006) 033, [hep-th/0604218].

[36] J. D. Lykken and L. Randall, *The Shape of gravity*, *JHEP* **0006** (2000) 014, [hep-th/9908076].

[37] R. Gregory, V. Rubakov, and S. M. Sibiryakov, *Opening up extra dimensions at ultra large scales*, *Phys.Rev.Lett.* **84** (2000) 5928–5931, [hep-th/0002072].

[38] C. Csaki, J. Erlich, and T. J. Hollowood, *Graviton propagators, brane bending and bending of light in theories with quasilocalized gravity*, *Phys.Lett.* **B481** (2000) 107–113, [hep-th/0003020].

[39] C. Csaki, J. Erlich, T. J. Hollowood, and Y. Shirman, *Universal aspects of gravity localized on thick branes*, *Nucl.Phys.* **B581** (2000) 309–338, [hep-th/0001033].

[40] C. Csaki, J. Erlich, and T. J. Hollowood, *Quasilocalization of gravity by resonant modes*, *Phys.Rev.Lett.* **84** (2000) 5932–5935, [hep-th/0002161].

[41] G. Dvali, G. Gabadadze, and M. Porrati, *Metastable gravitons and infinite volume extra dimensions*, *Phys.Lett.* **B484** (2000) 112–118, [hep-th/0002190].

[42] A. Kehagias and K. Tamvakis, *Localized gravitons, gauge bosons and chiral fermions in smooth spaces generated by a bounce*, *Phys.Lett.* **B504** (2001) 38–46, [hep-th/0010112].

[43] S. Kobayashi, K. Koyama, and J. Soda, *Thick brane worlds and their stability*, *Phys.Rev.* **D65** (2002) 064014, [hep-th/0107025].

[44] K. A. Bronnikov and B. E. Meierovich, *A General thick brane supported by a scalar field*, *Grav.Cosmol.* **9** (2003) 313–318, [gr-qc/0402030].

[45] D. Bazeia, A. Gomes, L. Losano, and R. Menezes, *Braneworld Models of Scalar Fields with Generalized Dynamics*, *Phys.Lett.* **B671** (2009) 402–410, [arXiv:0808.1815].

[46] D. Bazeia, A. Gomes, and L. Losano, *Gravity localization on thick branes: A Numerical approach*, *Int.J.Mod.Phys.* **A24** (2009) 1135–1160, [arXiv:0708.3530].

[47] A. Melfo, N. Pantoja, and A. Skirzewski, *Thick domain wall space-times with and without reflection symmetry*, *Phys.Rev.* **D67** (2003) 105003, [gr-qc/0211081].

[48] V. Dzhunushaliev, V. Folomeev, and M. Minamitsuji, *Thick brane solutions*, *Rept.Prog.Phys.* **73** (2010) 066901, [arXiv:0904.1775].

[49] K.-y. Oda, T. Suyama, and N. Yokoi, *Smoothing out Negative Tension Brane*, *Phys.Lett.* **B675** (2009) 455–459, [arXiv:0811.3459].

[50] M. Abdalla, M. Guimaraes, and J. Hoff da Silva, *Positive tension 3-branes in an AdS_5 bulk*, *JHEP* **1009** (2010) 051, [arXiv:1001.1075].

[51] J. Wudka, *Periodic thick-brane configurations and their stability*, *J.Phys.* **G38** (2011) 075003, [arXiv:1107.0944].

[52] J. Garriga and T. Tanaka, *Gravity in the brane world*, *Phys.Rev.Lett.* **84** (2000) 2778–2781, [hep-th/9911055].

[53] S. B. Giddings, E. Katz, and L. Randall, *Linearized gravity in brane backgrounds*, *JHEP* **0003** (2000) 023, [hep-th/0002091].

[54] A. Karch and L. Randall, *Locally localized gravity*, *JHEP* **0105** (2001) 008, [hep-th/0011156].

[55] O. DeWolfe and D. Z. Freedman, *Notes on fluctuations and correlation functions in holographic renormalization group flows*, hep-th/0002226.

[56] M. Giovannini, *Gauge invariant fluctuations of scalar branes*, *Phys.Rev.* **D64** (2001) 064023, [hep-th/0106041].

[57] M. Giovannini, *Localization of metric fluctuations on scalar branes*, *Phys.Rev.* **D65** (2002) 064008, [hep-th/0106131].

[58] C. Csaki, M. L. Graesser, and G. D. Kribs, *Radion dynamics and electroweak physics*, *Phys. Rev.* **D63** (2001) 065002, [hep-th/0008151].

[59] M. Cvetic and M. Robnik, *Gravity Trapping on a Finite Thickness Domain Wall: An Analytic Study*, *Phys.Rev.* **D77** (2008) 124003, [arXiv:0801.0801].

[60] Z. Kakushadze, *Localized (super)gravity and cosmological constant*, *Nucl.Phys.* **B589** (2000) 75–118, [hep-th/0005217].

[61] A. Brandhuber and K. Sfetsos, *Nonstandard compactifications with mass gaps and Newton's law*, *JHEP* **9910** (1999) 013, [hep-th/9908116].

[62] S. M. Aybat and D. P. George, *Stability of Scalar Fields in Warped Extra Dimensions*, *JHEP* **1009** (2010) 010, [arXiv:1006.2827].

[63] A. A. Andrianov, V. A. Andrianov, and O. O. Novikov, *Localization of scalar fields on self-gravitating thick branes*, *Phys.Part.Nucl.* **44** (2013) 190–203, [arXiv:1210.3698].

[64] H. Davoudiasl, J. Hewett, and T. Rizzo, *Bulk gauge fields in the Randall-Sundrum model*, *Phys.Lett.* **B473** (2000) 43–49, [hep-ph/9911262].

[65] A. Pomarol, *Gauge bosons in a five-dimensional theory with localized gravity*, *Phys.Lett.* **B486** (2000) 153–157, [hep-ph/9911294].

[66] Y. Grossman and M. Neubert, *Neutrino masses and mixings in nonfactorizable geometry*, *Phys.Lett.* **B474** (2000) 361–371, [hep-ph/9912408].

[67] S. Chang, J. Hisano, H. Nakano, N. Okada, and M. Yamaguchi, *Bulk standard model in the Randall-Sundrum background*, *Phys.Rev.* **D62** (2000) 084025, [hep-ph/9912498].

[68] C. Bogdanos, A. Dimitriadis, and K. Tamvakis, *Brane models with a Ricci-coupled scalar field*, *Phys.Rev.* **D74** (2006) 045003, [hep-th/0604182].

[69] H. Guo, Y.-X. Liu, Z.-H. Zhao, and F.-W. Chen, *Thick branes with a non-minimally coupled bulk-scalar field*, *Phys.Rev.* **D85** (2012) 124033, [`arXiv:1106.5216`].

[70] D. Bazeia, R. Menezes, and R. da Rocha, *A Note on Asymmetric Thick Branes*, *Adv.High Energy Phys.* **2014** (2014) 276729, [`arXiv:1312.3864`].

[71] B. Bajc and G. Gabadadze, *Localization of matter and cosmological constant on a brane in anti-de Sitter space*, *Phys.Lett.* **B474** (2000) 282–291, [`hep-th/9912232`].

[72] Y.-X. Liu, H. Guo, C.-E. Fu, and H.-T. Li, *Localization of Bulk Matters on a Thick Anti-de Sitter Brane*, *Phys.Rev.* **D84** (2011) 044033, [`arXiv:1101.4145`].

[73] J. Liang and Y.-S. Duan, *Localization of matters on thick branes*, *Phys.Lett.* **B678** (2009) 491–496.

[74] A. Melfo, N. Pantoja, and J. D. Tempo, *Fermion localization on thick branes*, *Phys.Rev.* **D73** (2006) 044033, [`hep-th/0601161`].

[75] R. Koley and S. Kar, *Scalar kinks and fermion localisation in warped spacetimes*, *Class.Quant.Grav.* **22** (2005) 753–768, [`hep-th/0407158`].

[76] I. Antoniadis, A. Arvanitaki, S. Dimopoulos, and A. Giveon, *Phenomenology of TeV Little String Theory from Holography*, *Phys.Rev.Lett.* **108** (2012) 081602, [`arXiv:1102.4043`].

[77] R. Gregory, V. Rubakov, and S. M. Sibiryakov, *Gravity and antigravity in a brane world with metastable gravitons*, *Phys.Lett.* **B489** (2000) 203–206, [`hep-th/0003045`].

[78] P. Binetruy, C. Deffayet, U. Ellwanger, and D. Langlois, *Brane cosmological evolution in a bulk with cosmological constant*, *Phys.Lett.* **B477** (2000) 285–291, [`hep-th/9910219`].

[79] P. Binetruy, C. Deffayet, and D. Langlois, *Nonconventional cosmology from a brane universe*, *Nucl.Phys.* **B565** (2000) 269–287, [`hep-th/9905012`].

[80] J. M. Cline, C. Grojean, and G. Servant, *Cosmological expansion in the presence of extra dimensions*, *Phys.Rev.Lett.* **83** (1999) 4245, [`hep-ph/9906523`].

[81] C. Csaki, M. Graesser, C. F. Kolda, and J. Terning, *Cosmology of one extra dimension with localized gravity*, *Phys.Lett.* **B462** (1999) 34–40, [`hep-ph/9906513`].

[82] C. Csaki, M. Graesser, L. Randall, and J. Terning, *Cosmology of brane models with radion stabilization*, *Phys.Rev.* **D62** (2000) 045015, [`hep-ph/9911406`].

[83] E. E. Flanagan, S. H. Tye, and I. Wasserman, *Cosmological expansion in the Randall-Sundrum brane world scenario*, *Phys.Rev.* **D62** (2000) 044039, [`hep-ph/9910498`].

[84] P. Kanti, I. I. Kogan, K. A. Olive, and M. Pospelov, *Single brane cosmological solutions with a stable compact extra dimension*, *Phys.Rev.* **D61** (2000) 106004, [`hep-ph/9912266`].

[85] P. Kanti, I. I. Kogan, K. A. Olive, and M. Pospelov, *Cosmological three-brane solutions*, *Phys.Lett.* **B468** (1999) 31–39, [`hep-ph/9909481`].

[86] D. Bazeia, F. Brito, and F. Costa, *First-order framework and domain-wall/brane-cosmology correspondence*, *Phys.Lett.* **B661** (2008) 179–185, [arXiv:0707.0680].

[87] M. Giovannini, *Time-dependent gravitating solitons in five dimensional warped space-times*, *Phys.Rev.* **D76** (2007) 124017, [arXiv:0708.1830].

[88] D. P. George, M. Trodden, and R. R. Volkas, *Extra-dimensional cosmology with domain-wall branes*, *JHEP* **0902** (2009) 035, [arXiv:0810.3746].

[89] A. Kadosh, A. Davidson, and E. Pallante, *Slinky evolution of domain wall brane cosmology*, *Phys.Rev.* **D86** (2012) 124015, [arXiv:1202.5255].

[90] M. Gremm, *Thick domain walls and singular spaces*, *Phys.Rev.* **D62** (2000) 044017, [hep-th/0002040].

[91] A. Wang, *Thick de Sitter 3 branes, dynamic black holes and localization of gravity*, *Phys.Rev.* **D66** (2002) 024024, [hep-th/0201051].

[92] N. Sasakura, *A de Sitter thick domain wall solution by elliptic functions*, *JHEP* **0202** (2002) 026, [hep-th/0201130].

[93] V. Afonso, D. Bazeia, and L. Losano, *First-order formalism for bent brane*, *Phys.Lett.* **B634** (2006) 526–530, [hep-th/0601069].

[94] T. Gherghetta and A. Pomarol, *Bulk fields and supersymmetry in a slice of AdS*, *Nucl.Phys.* **B586** (2000) 141–162, [hep-ph/0003129].

[95] S. J. Huber and Q. Shafi, *Fermion masses, mixings and proton decay in a Randall-Sundrum model*, *Phys.Lett.* **B498** (2001) 256–262, [hep-ph/0010195].

[96] K. Agashe, A. Delgado, M. J. May, and R. Sundrum, *RS1, custodial isospin and precision tests*, *JHEP* **0308** (2003) 050, [hep-ph/0308036].

[97] M. A. Luty and T. Okui, *Conformal technicolor*, *JHEP* **0609** (2006) 070, [hep-ph/0409274].

[98] N. Arkani-Hamed, M. Porrati, and L. Randall, *Holography and phenomenology*, *JHEP* **0108** (2001) 017, [hep-th/0012148].

[99] R. Rattazzi and A. Zaffaroni, *Comments on the holographic picture of the Randall-Sundrum model*, *JHEP* **0104** (2001) 021, [hep-th/0012248].

[100] H. Davoudiasl, B. Lillie, and T. G. Rizzo, *Off-the-wall Higgs in the universal Randall-Sundrum model*, *JHEP* **0608** (2006) 042, [hep-ph/0508279].

[101] G. Cacciapaglia, C. Csaki, G. Marandella, and J. Terning, *The Gaugephobic Higgs*, *JHEP* **0702** (2007) 036, [hep-ph/0611358].

[102] A. Falkowski and M. Perez-Victoria, *Electroweak Breaking on a Soft Wall*, *JHEP* **0812** (2008) 107, [arXiv:0806.1737].

[103] A. Azatov, M. Toharia, and L. Zhu, *Higgs Mediated FCNC's in Warped Extra Dimensions*, *Phys.Rev.* **D80** (2009) 035016, [arXiv:0906.1990].

[104] J. A. Cabrer, G. von Gersdorff, and M. Quiros, *Warped Electroweak Breaking Without Custodial Symmetry*, *Phys.Lett.* **B697** (2011) 208–214, [`arXiv:1011.2205`].

[105] J. A. Cabrer, G. von Gersdorff, and M. Quiros, *Suppressing Electroweak Precision Observables in 5D Warped Models*, *JHEP* **1105** (2011) 083, [`arXiv:1103.1388`].

[106] J. A. Cabrer, G. von Gersdorff, and M. Quiros, *Improving Naturalness in Warped Models with a Heavy Bulk Higgs Boson*, *Phys.Rev.* **D84** (2011) 035024, [`arXiv:1104.3149`].

[107] M. Geller, S. Bar-Shalom, and A. Soni, *Higgs-radion unification: radius stabilization by an SU(2) bulk doublet and the 126 GeV scalar*, *Phys.Rev.* **D89** (2014) 095015, [`arXiv:1312.3331`].

[108] P. R. Archer, *The Fermion Mass Hierarchy in Models with Warped Extra Dimensions and a Bulk Higgs*, *JHEP* **1209** (2012) 095, [`arXiv:1204.4730`].

[109] M. Frank, N. Pourtolami, and M. Toharia, *Higgs Bosons in Warped Space, from the Bulk to the Brane*, *Phys.Rev.* **D87** (2013), no. 9 096003, [`arXiv:1301.7692`].

[110] R. Malm, M. Neubert, K. Novotny, and C. Schmell, *5D perspective on Higgs production at the boundary of a warped extra dimension*, *JHEP* **1401** (2014) 173, [`arXiv:1303.5702`].

[111] P. R. Archer, M. Carena, A. Carmona, and M. Neubert, *Higgs Production and Decay in Models of a Warped Extra Dimension with a Bulk Higgs*, *JHEP* **1501** (2015) 060, [`arXiv:1408.5406`].

[112] B. M. Dillon and S. J. Huber, *Non-Custodial Warped Extra Dimensions at the LHC?*, *JHEP* **06** (2015) 066, [`arXiv:1410.7345`].

[113] K. Agashe, A. Azatov, Y. Cui, L. Randall, and M. Son, *Warped Dipole Completed, with a Tower of Higgs Bosons*, `arXiv:1412.6468`.

[114] A. M. Iyer, K. Sridhar, and S. K. Vempati, *Bulk RS models, Electroweak Precision tests and the 125 GeV Higgs*, `arXiv:1502.06206`.

[115] G. Servant and T. M. Tait, *Is the lightest Kaluza-Klein particle a viable dark matter candidate?*, *Nucl.Phys.* **B650** (2003) 391–419, [`hep-ph/0206071`].

[116] H.-C. Cheng, J. L. Feng, and K. T. Matchev, *Kaluza-Klein dark matter*, *Phys.Rev.Lett.* **89** (2002) 211301, [`hep-ph/0207125`].

[117] K. Agashe and G. Servant, *Warped unification, proton stability and dark matter*, *Phys.Rev.Lett.* **93** (2004) 231805, [`hep-ph/0403143`].

[118] G. Panico, E. Ponton, J. Santiago, and M. Serone, *Dark Matter and Electroweak Symmetry Breaking in Models with Warped Extra Dimensions*, *Phys.Rev.* **D77** (2008) 115012, [`arXiv:0801.1645`].

[119] E. Ponton and L. Randall, *TeV Scale Singlet Dark Matter*, *JHEP* **0904** (2009) 080, [`arXiv:0811.1029`].

[120] L. Vecchi, *Majorana dark matter in warped extra dimensions*, *Phys.Rev.* **D90** (2014), no. 2 025017, [arXiv:1310.7862].

[121] T. Gherghetta and B. von Harling, *A Warped Model of Dark Matter*, *JHEP* **1004** (2010) 039, [arXiv:1002.2967].

[122] B. von Harling and K. L. McDonald, *Secluded Dark Matter Coupled to a Hidden CFT*, *JHEP* **1208** (2012) 048, [arXiv:1203.6646].

[123] A. R. Frey, R. J. Danos, and J. M. Cline, *Warped Kaluza-Klein Dark Matter*, *JHEP* **0911** (2009) 102, [arXiv:0908.1387].

[124] K. Agashe, A. Falkowski, I. Low, and G. Servant, *KK Parity in Warped Extra Dimension*, *JHEP* **0804** (2008) 027, [arXiv:0712.2455].

[125] A. D. Medina and E. Ponton, *Warped Universal Extra Dimensions*, *JHEP* **1106** (2011) 009, [arXiv:1012.5298].

[126] A. D. Medina and E. Ponton, *Warped Radion Dark Matter*, *JHEP* **1109** (2011) 016, [arXiv:1104.4124].

[127] C. Csaki, J. Hubisz, and P. Meade, *TASI lectures on electroweak symmetry breaking from extra dimensions*, hep-ph/0510275.

[128] P. Cox, A. D. Medina, T. S. Ray, and A. Spray, *Radion/Dilaton-Higgs Mixing Phenomenology in Light of the LHC*, *JHEP* **02** (2014) 032, [arXiv:1311.3663].

[129] A. Carmona and M. Chala, *Composite Dark Sectors*, *JHEP* **06** (2015) 105, [arXiv:1504.00332].

[130] B. Grzadkowski and J. Wudka, *Pragmatic approach to the little hierarchy problem: the case for Dark Matter and neutrino physics*, *Phys.Rev.Lett.* **103** (2009) 091802, [arXiv:0902.0628].

[131] C. F. Kolda and H. Murayama, *The Higgs mass and new physics scales in the minimal standard model*, *JHEP* **0007** (2000) 035, [hep-ph/0003170].

[132] J. Casas, J. Espinosa, and I. Hidalgo, *Implications for new physics from fine-tuning arguments. 1. Application to SUSY and seesaw cases*, *JHEP* **0411** (2004) 057, [hep-ph/0410298].

[133] R. Barbieri and G. Giudice, *Upper Bounds on Supersymmetric Particle Masses*, *Nucl.Phys.* **B306** (1988) 63.

[134] A. Carmona, E. Ponton, and J. Santiago, *Phenomenology of Non-Custodial Warped Models*, *JHEP* **1110** (2011) 137, [arXiv:1107.1500].

[135] **Particle Data Group** Collaboration, K. Olive et al., *Review of Particle Physics*, *Chin.Phys.* **C38** (2014) 090001.

[136] E. W. Kolb and M. S. Turner, *The Early Universe*, *Front.Phys.* **69** (1990) 1–547.

[137] A. Riazuelo, F. Vernizzi, D. A. Steer, and R. Durrer, *Gauge invariant cosmological perturbation theory for brane worlds*, *Phys.Rev.D* (2002) [hep-th/0205220].