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Edited by I. Oda

PREFACE

The workshop on Quantum Gravity and Topology was held at INS on February 21-23, 1991. Several introductory lectures and more than 15 talks were delivered for about 100 people paticipants. The main subjects discussed were

i) Topological quantum field theories and topological gravity
ii) Low dimensional and four dimensional gravity
iii) Topology change
iv) Superstring theories
etc.

We wish to thank the speakers and participants for their successful efforts to provide a stimulating and friendly atmosphere. We hope that someone who participated in this workshop will become second Einstein in the near future.

Ichiro Oda

CONTENTS

Two-dimensional Topological Gravity*

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ABSTRACT

A formulation of two-dimensional topological gravity due to Verlinde and Verlinde is reviewed. It is equivalent to $ISO(2)$ gauge theory with flat gauge connections that is the topological gauge theory in two dimension. The theory reduces conformally invariant free field theory and the equivariant cohomology can be investigated. The Feynmann rules for calculating amplitudes are also derived.

★ Talk presented at the Workshop on ***Quantum Gravity and Topology***, INS, Tokyo, Japan (February 1991).

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1. Introduction

Topological field theory[1] had been originally fascinating interest from a mathematical stand point as a quantum field theoretical realization of the Donaldson's theory.[2] The Donaldson invariants can be obtained via the path integral formulation and the Hamiltonian formulation leads naturally the Floer groups of three manifolds. Recent results about two-dimensional topological gravity, however, give much insight to two-dimensional quantum gravity and has been attracting much interest from a physical view point. The aim of this talk is present a review of these results of two-dimensional topological gravity.*

The recent remarkable progress in the random matrix formulation of two-dimensional quantum gravity[4] gives us a new way of studying the non-perturbative properties of string theory and an example of interesting phenomena come from the summation of the space-time topology. Their generating function of correlation functions are determined by Schwinger-Dyson equations which are written in the form of Virasoro constraints.[5] While two-dimensional topological gravity can be considered as its continuum theory and seems to lead deeper understanding of two-dimensional quantum gravity. This is the first example of topological field theory which is related to the interesting *physics* but not mathematics. Correlation functions on Riemann surface with different topology are intrinsically related and it is governed by recursion relations which coincide with the Schwinger-Dyson equations in the matrix model.

We will see, in this review, how the two-dimensional topological gravity are formulated and correlation functions are calculated. The theory reduces a free conformal field theory and the equivariant cohomology can be investigated explicitly in the operator formulation language. All the physical observables except for the puncture operator are almost BRST trivial and it can contribute amplitudes through singularities on the boundary of the moduli space. Any correlation functions can be obtained by means of conformal field theoretical technique and the stable compactification theory of moduli space in principle.

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* This is based on the work by Verlinde and Verlinde[3] and I recommend reading the original paper for detail.

2. The lagrangian of topological gravity

Two-dimensional topological gravity is a field theory which is associated with the cohomology of the moduli space of Riemann surfaces. The moduli space $\mathcal{M}$ of Riemann surfaces can be represented as the space of all zwei-beins, modulo diffeomorphisms, local-Lorentz and Weyl-transformations. However, here we constraint the Weyl-transformation by restricting curvature $R(x) = d\omega(e(x))$ to a particular value $R_0(x)$ such that the moduli space $\mathcal{M}$ is represented as

$$\mathcal{M} = \{(e^+, e^-);\ d\omega(e) = R_0\}/Diff \otimes lL \tag{2.1}$$

If we choose almost everywhere vanishing R_0, the theory will reduce a conformally invariant free field theory. We replace the constant-curvature condition into zero-curvature one and the delta-function singularities of curvature can be recovered by inserting curvature creation operator in correlation functions.

The theory can be constructed by means of general procedure due to Witten.[6]

The topological field theory has ghost number U, which is related to the dimension of moduli space and violated by an anomaly. We start with fields of $U = 0$; spin connection ω and zwei-bein $e^\pm$ both of which are one form, where we take the first order formulation and add torsion free conditions as equations of motion. We also introduce ghosts of $U = 1$, sometimes called topological ghost, which are fermionic one forms: ψ^0 and $\psi^\pm$. Ghost fields in $U = 2$ have the quantum numbers of gauge parameters and in our case they are bosonic zero forms: γ^0 and $\gamma^\pm$. These three types of fields (ω, $e^\pm$; ψ^0, $\psi^\pm$; γ^0, $\gamma^\pm$) form the fundamental multiplet in the topological field theory.

Next one introduces an anti-commuting symmetry δ_S with transformation laws

$$\begin{aligned}
&\delta_S \omega = \psi^0, \qquad \delta_S \psi^0_\mu = \partial_\mu \gamma^\lambda \psi^0_\lambda + \gamma^\lambda \partial_\lambda \psi^0_\mu, \\
&\delta_S e^\pm = \psi^\pm, \qquad \delta_S \psi^\pm_\mu = \pm \gamma^0 \psi^\pm_\mu + \partial_\mu \gamma^\lambda \psi^\pm_\lambda + \gamma^\lambda \partial_\lambda \psi^\pm_\mu, \\
&\delta_S \gamma^0 = 0, \qquad \delta_S \gamma^\pm = 0.
\end{aligned} \tag{2.2}$$

This is nilpotent up to local-Lorentz transformations and diffeomorphisms by construction.

If we introduce anti-ghost multiplet $(\chi_0, \chi_\pm; \pi_0, \pi_\pm)$ and define its δ_S- transformation as

$$\begin{aligned}
&\delta_S\chi_0 = \pi_0, \qquad \delta_S\pi_0 = \gamma^\lambda\partial_\lambda\pi_0, \\
&\delta_S\chi_\pm = \pi_\pm, \qquad \delta_S\pi_\pm = \mp\gamma^0\pi_\pm + \gamma^\lambda\partial_\lambda\pi_\pm,
\end{aligned} \tag{2.3}$$

the δ_S-invariant action can be obtained by

$$\begin{aligned}
S &= \delta_S\int(\chi_0 d\omega + \chi_+ De^+ + \chi_- De^-) \\
&= \int(\pi_0 d\omega + \pi_+ De^+ + \pi_- De^- - \chi_0 d\psi^0 - \chi_+ D\psi^+ - \chi_- D\psi^-),
\end{aligned} \tag{2.4}$$

which has field equations the zero-curvature and the torsion free conditions. Here we introduce (super) covariant derivatives for ghost fields as

$$D\psi^\pm = d\psi^\pm \mp \omega\wedge\psi^\pm \pm e^\pm\wedge\psi^0. \tag{2.5}$$

We can start from the action (2.4) and it can be quantized by means of conventional BRST method.

The action (2.4) has two types of symmetries. One is local-Lorentz transformations and diffeomorphisms:

$$\begin{aligned}
&\delta\omega_\mu = \partial_\mu\xi^\lambda\omega_\lambda + \xi^\lambda\partial_\lambda\omega_\mu, \\
&\delta e^\pm_\mu = \pm\alpha e^\pm_\mu + \partial_\mu\xi^\lambda e^\pm_\lambda + \xi^\lambda\partial_\lambda e^\pm_\mu, \\
&\delta\psi^0_\mu = \partial_\mu\xi^\lambda\psi^0_\lambda + \xi^\lambda\partial_\lambda\psi^0_\mu, \\
&\delta\psi^\pm_\mu = \pm\alpha\psi^\pm_\mu + \partial_\mu\xi^\lambda\psi^\pm_\lambda + \xi^\lambda\partial_\lambda\psi^\pm_\mu, \\
&\delta\chi_0 = \xi^\lambda\partial_\lambda\chi_0, \\
&\delta\chi_\pm = \mp\alpha\chi_\pm + \xi^\lambda\partial_\lambda\chi_\pm, \\
&\delta\pi_0 = \xi^\lambda\partial_\lambda\pi_0, \\
&\delta\pi_\pm = \mp\alpha\pi_\pm + \xi^\lambda\partial_\lambda\pi_\pm.
\end{aligned} \tag{2.6}$$

The other is its fermionic analogue

$$
\begin{aligned}
\hat{\delta}\psi^0_\mu &= \partial_\mu\hat{\xi}^\lambda\psi^0_\lambda + \hat{\xi}^\lambda\partial_\lambda\psi^0_\mu, \\
\hat{\delta}\psi^\pm_\mu &= \pm\hat{\alpha}\psi^\pm_\mu + \partial_\mu\hat{\xi}^\lambda\psi^\pm_\lambda + \hat{\xi}^\lambda\partial_\lambda\psi^\pm_\mu, \\
\hat{\delta}\pi_0 &= \hat{\xi}^\lambda\partial_\lambda\pi_0, \\
\hat{\delta}\pi_\pm &= \mp\hat{\alpha}\pi_\pm + \hat{\xi}^\lambda\partial_\lambda\pi_\pm.
\end{aligned}
\tag{2.7}
$$

These fermionic symmetries are the reason of being introduced bosonic ghosts γ^0 and $\gamma^\pm$.

These symmetries are equivalent to following "super" $ISO(2)$ gauge transformation when we impose the field equations.

$$
\begin{aligned}
\delta\omega &= d\rho^0, \\
\delta e^\pm &= \pm\rho^0 + D\rho^\pm, \\
\delta\psi^0 &= d\hat{\rho}^0, \\
\delta\psi^\pm &= \pm\rho^0\psi^\pm \pm \hat{\rho}^0 e^\pm + D\hat{\rho}^\pm, \\
\delta\chi_0 &= \pm\rho^\pm\chi_\pm, \\
\delta\chi_\pm &= \mp\rho^0\chi_\pm, \\
\delta\pi_0 &= \pm\rho^\pm\pi_\pm \pm \hat{\rho}^\pm\chi_\pm, \\
\delta\pi_\pm &= \mp\rho^0\pi_\pm \mp \hat{\rho}^0\chi_\pm,
\end{aligned}
\tag{2.8}
$$

where

$$
\begin{aligned}
\rho^0 &= \alpha + \xi\cdot\omega, & \rho^\pm &= \xi\cdot e^\pm, \\
\hat{\rho}^0 &= \hat{\alpha} + \hat{\xi}\cdot\omega + \xi\cdot\psi^0, & \hat{\rho}^\pm &= \xi\cdot\psi^\pm + \hat{\xi}\cdot e^\pm.
\end{aligned}
\tag{2.9}
$$

We can define the BRST transformation by adding δ_S and the part come from these gauge symmetries. One can see the resultant BRST transformation is nilpotent off shell but we do not give its explicit form here.

3. Topological gravity as a free conformal field theory

We will now discuss the quantization of our theory. First we choose the conformal gauge for fixing diffeomorphisms:

$$
\begin{aligned}
e^{+} &= e^{\phi_+} dz, \\
e^{-} &= e^{\phi_-} d\bar{z}.
\end{aligned}
\tag{3.1}
$$

The local-Lorentz transformations are fixed by imposing

$$
\phi_+ = \phi_- . \tag{3.2}
$$

For the gauge fixing of the local fermionic symmetries, it is convenient to take the gauge conditions which are *super-symmetric* to conformal gauge conditions:

$$
\begin{aligned}
\psi^{+} &= e^{\phi_+} \psi_+ dz, \\
\psi^{-} &= e^{\phi_-} \psi_- d\bar{z}, \\
\psi_+ &= \psi_- .
\end{aligned}
\tag{3.3}
$$

The complete gauge fixed lagrangian can be obtained by means of conventional BRST method.[7] After integrating out non-dynamical fields, the lagrangian reduces a following free field action.

$$
\mathcal{L} = \pi \partial\bar{\partial}\phi + \chi \partial\bar{\partial}\psi + b\bar{\partial}c + \beta\bar{\partial}\gamma + \bar{b}\partial\bar{c} + \bar{\beta}\partial\bar{\gamma}, \tag{3.4}
$$

where $\phi = \phi_+ + \phi_-$ and $\psi = \psi_+ + \psi_-$. Field c (γ) is the reparametrization ghost (its fermionic partner). Ghost field for local-Lorentz transformations and its partner are non-dynamical and determined by the equations

$$
\begin{aligned}
c^0 &= \frac{1}{2}(\partial c + c\partial\phi - \bar{\partial}\bar{c} + \bar{c}\bar{\partial}\phi), \\
\gamma^0 &= \frac{1}{2}(\partial\gamma + \gamma\partial\phi + c\partial\psi - \bar{\partial}\bar{\gamma} - \bar{\gamma}\bar{\partial}\phi - \bar{c}\bar{\partial}\psi).
\end{aligned}
\tag{3.5}
$$

This free field theory has "super" conformal symmetry generated by

$$
\begin{aligned}
T_L &= \partial\pi\partial\phi + \partial^2\pi + \partial\chi\partial\psi, \\
T_{gh} &= c\partial b + 2\partial c b + \gamma\partial\beta + 2\partial\gamma\beta, \\
G_L &= \partial\chi\partial\phi + \partial^2\chi, \\
G_{gh} &= c\partial\beta + 2\partial c\beta,
\end{aligned}
\tag{3.6}
$$

where T_L and T_{gh} are independently satisfy the Virasoro algebra with vanishing central charge and G_L and G_{gh} are their "super" partner with dimension two. The BRST charge is given by

$$
\begin{aligned}
Q_B &= Q + \bar{Q}, \\
Q &= Q_S + Q_G, \\
Q_S &= \oint(\partial\pi\psi + b\gamma), \\
Q_G &= \oint(c(T_L + \frac{1}{2}T_{gh}) + \gamma(G_L + \frac{1}{2}G_{gh})).
\end{aligned}
\tag{3.7}
$$

Q_S and Q_G are commutative and nilpotent respectively.

4. Equivariant cohomology and observables

Observables in the topological field theory is obtained by considering equivariant cohomology of δ_S, which means the δ_S- cohomology in the space of gauge-invariant quantities. We can construct physical observables by solving some descent equations.[1] In the topological gravity, however, a difficulty occur since gauge group contains diffeomorphisms. There is no local operator which is invariant under diffeomorphisms. Only the top-form observable integrated over whole manifold can be constructed in the usual sense. Nevertheless, two-dimensional topological gravity is rather special and the non-top-form observables can be constructed as follows.

The key point for obtaining local observables is changing our moduli space to the one of the *punctured* Riemann surfaces. Since diffeomorphisms map a puncture into itself we can consider local observables on the puncture. We should only consider δ_S- cohomology in the space of *the local-Lorentz invariant* operators. In this sense, we can get zero-form observable in the similar way to the topological Yang-Mills theory:

$$\sigma_n^{(0)} = \gamma_0^n, \tag{4.1}$$

where γ^0 is defined by (3.5). Higher form observables can be seen from descent equations:

$$\begin{aligned} \delta_S \sigma_n^{(0)} &= 0, \\ d\sigma_n^{(0)} &= \delta_S \sigma_n^{(1)}, \\ d\sigma_n^{(1)} &= \delta_S \sigma_n^{(2)}, \\ d\sigma_n^{(2)} &= 0. \end{aligned} \tag{4.2}$$

One finds

$$\begin{aligned} \sigma_n^{(1)} &= n\psi^0 \gamma_0^{n-1}, \\ \sigma_n^{(2)} &= nd\omega\gamma_0^{n-1} + \frac{1}{2}(n-1)\psi^0 \wedge \psi^0 \gamma_0^{n-2}, \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} (\omega, \ \bar{\omega}) &= \frac{1}{2}(\partial\phi, \ -\bar{\partial}\phi), \\ (\psi^0, \ \bar{\psi}^0) &= \frac{1}{2}(\partial\psi, \ -\bar{\partial}\psi). \end{aligned} \tag{4.4}$$

For *real* observables which are well worked in our theory, we must associate the puncture operator to σ's, which is necessary to kill diffeomorphisms at the point on where observables sit:

$$P = c\bar{c}\delta(\gamma)\delta(\bar{\gamma}).$$

The puncture operator is invariant under the BRST transformation.

5. Description of the amplitude

We begin with a remark concerning the curvature creation operation. As was explained in Chapter 2, we have been considering the Riemann surfaces with singular curvature. This is realized at the correlation function level as follows. We first note that the dimension of ϕ is anomalous

$$T(z)\phi(w) \sim \frac{1}{(z-w)^2} + \frac{1}{z-w}\partial\phi(w), \tag{5.1}$$

since ϕ is not a scalar but Liouville mode. This means that in the usual conformal gauge with constant-curvature Riemann surfaces

$$g_{ab} = e^{\phi}\hat{g}_{ab}, \tag{5.2}$$

the action has the form

$$S^{(cov)} = \int \sqrt{\hat{g}}(\hat{g}^{ab}\partial_a\pi\partial_b\phi + \frac{1}{2}\pi\hat{R}). \tag{5.3}$$

A consequence of this is that the singular curvature is obtained by inserting curvature creation operator

$$V_{q_i}(x_i) = e^{q_i\pi(x_i)}, \tag{5.4}$$

or its BRST invariant version obtained by replacing π with $\hat{\pi} = \pi + c\partial\chi + \bar{c}\bar{\partial}\chi$. The strength of singularities q_i are constrained by the condition that the integrated curvature must be equal to the Euler number

$$\sum_i q_i = 2g - 2. \tag{5.5}$$

The independence of the point where we insert the curvature creation operators is guaranteed by

$$de^{q\hat{\pi}} = q\{Q_B,\ d\chi e^{q\hat{\pi}}\}. \tag{5.6}$$

Furthermore, as in ordinary string theory, the amplitudes of topological strings can be written as integrals over the moduli space $\mathcal{M}_g$ of Riemann surfaces. In addition, we now have "super" moduli come from the fermionic field $(\psi^+,\ \psi^-)$. It should be noted that this super moduli due to *scalar* supersymmetry δ_S thus its dimension is the same as the bosonic moduli space $6g-6$ which is different from the well-known *super* Riemann surfaces.

Integration of super moduli leads insertion of "super current" as in the superstring. The super current in this case is the dimension two current G:

$$\frac{\partial S}{\partial \hat{m}_i} = G_i, \qquad G_i = \int_{\Sigma} d^2x \mu_i(x,\bar{x}) G(x), \tag{5.7}$$

where μ_i is the Beltrami differential corresponding to m_i.

Finally we get the rule for obtaining the physical amplitude as

$$\begin{aligned}\langle \sigma_{n_1}\sigma_{n_2}\cdots\sigma_{n_s}\rangle &= \int_{\mathcal{M}_g} \langle \prod_{i=1}^{3g-3} (ghosts) G_i \bar{G}_i \prod_j e^{q_j \hat{\pi}(x_j)} \prod_{k=1}^{s} \int_{\Sigma} \sigma^{(2)}_{n_k} \rangle_0 \\ &= \int_{\mathcal{M}_{g,n}} \langle \prod_{i=1}^{3g-3+n} (ghosts) G_i \bar{G}_i \prod_j e^{q_j \hat{\pi}(x_j)} \prod_{k=1}^{s} \sigma^{(0)}_{n_k} \rangle_0,\end{aligned} \tag{5.8}$$

where $\langle\cdots\rangle_0$ means the functional integration over all fields with fixed moduli. Ghost insertion $(ghosts)$ means

$$(ghosts) = b_i \bar{b}_i \delta(\beta_i) \delta(\bar{\beta}_i), \tag{5.9}$$

which come from the ghost path-integral as in the super string.

The recursion relations can be obtained by studying the amplitudes in detail. We only give their result here with a comment to obtain them:

$$\begin{aligned}\langle \sigma_{n+1} \prod_{i\in S} \sigma_{n_i} \rangle &= \sum_j \frac{1}{3}(2n_j+1) \langle \sigma_{n+n_j} \prod_{i\neq j} \sigma_{n_i} \rangle + \\ &\frac{1}{9} \sum_{k=1}^{n} \{ \langle \sigma_{k-1}\sigma_{n-k} \prod_{i\in S} \sigma_{n_i} \rangle + \sum_{S=X\cup Y} \langle \sigma_{k-1} \prod_{i\in X} \sigma_{n_i} \rangle \langle \sigma_{n-k} \prod_{j\in Y} \sigma_{n_j} \rangle \}.\end{aligned} \tag{5.10}$$

This can be rewritten in the form of the Virasoro constraints on the generating function of correlation functions.

The important relation to get this recursion relation is

$$\sigma_{n+1}^{(0)} = \frac{1}{2}\{Q_B, (\psi - \bar{\psi})\sigma_n^{(0)}\}, \qquad \text{for } n \geq 0. \tag{5.11}$$

This means that observables, except for puncture operator σ_0, are almost BRST trivial and it can contributes amplitudes only by picking up singularities on the boundary of moduli space. There are two types of boundaries which contribute amplitudes that is coincidence limit of two-operators and pinching limit of non-trivial cycle of Riemann surfaces. Each contribution generate each term in the recursion relation (5.10), which we do not discuss here. See original paper[3] for detail.

6. Discussions

It has been shown how the two-dimensional topological gravity has been constructed. It reduces a free conformal field theory in conformal gauge and the Feynmann rules for obtaining amplitudes are led in the operator formulation as in the string theory. The recursion relation which determine correlation functions in the theory coincide with those of one-matrix model. This means that correlation functions of topological gravity with appropriate finite perturbation reproduce the result of one-matrix model.

This suggests that there is some topological theory can be formulated as a continuum theory corresponding to the multi-matrix model also. Such a topological field theory has been constructed recently,[8] which is a theory of "topological matter system" coupling with the topological gravity. It is expected that the results of multi-matrix model can be obtained by means of topological formulation with deeper understanding of two-dimensional quantum gravity.

The topological W-gravity is also constructed recently[9] and it governed by W-algebra. The physical properties of this type of theories should be clarified and the geometrical meanings of W-gravity may be obtained from such investigations. The mathematical understanding, in particular, the stable compactification theory of this extended moduli space is needed.

The understanding of recursion relations and its geometrical meanings are still unclear and it is remaining problem to understand them. It seems to increase the physical importance of topological gravity in future.

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Chiral Ring and Topological Conformal Field Theory [1]

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We study the topological quantum field theories constructed from a twisted class of the N=3 and 4 superconformal field theories (SCFT's) and their chiral ring structures.

1. Introduction.

Recently two and three dimensional quantum gravities are understood as the topological quantum field theories. Both theories are expanded around a topological phase (unbroken phase) and exactly soluble. To understand the notion of the topological phase will be much more important in studying the quantization of gravity. Since the two dimensional quantum gravity naturally arise from non-critical strings, the topological conformal field theory (CFT) will be important to study the non-critical string and the string at unbroken phase.

Another motivation of our work is to extract some geometrical information of a conformal field theory. Our image of a conformal field theory is something like a nonlinear sigma model which classically gives a mapping from two dimensional surface to a target manifold. In supersymmetric nonlinear sigma models a relation between the extended supersymmetry and the target manifold is known. In N=2 and 4 supersymmetric conformal field theories an interesting relation between the modular invariant partition function and the Kähler geometry, *e.g.* K3 surface or Calabi-Yau manifold, are reported [2]. Here we are interested in the N=3 SCFT, since we have no geometrical information on it.

Most of topological field theories are constructed from the supersymmetric system. One will understand the role of supersymmetry in discussing the topological theory as follows. Consider an N-supersymmetric system with supercharge algebra

$$\begin{aligned} Q_i^2 &= H\,, \qquad i = 1, \cdots, N \\ Q_iQ_j + Q_jQ_i &= 0\,, \text{for } i \neq j\,. \end{aligned} \tag{1}$$

[1] This talk is based on the work [1] .
[2] JSPS fellow.

One may think of it as a quantum mechanical system or the one restricted to a zero-momemtum subspace. Supersymmetry is unbroken iff the energy of the ground state vanishes, which is equivalent to say that the Witten index does not vanish, $\mathrm{Tr}(-1)^F \neq 0$. Hence the system with unbroken SUSY has a nontrivial topological invariant. Since the Witten index counts the number of bosonic ground states minus fermionic ones

$$\mathrm{Tr}(-1)^F = n_B^{E=0} - n_F^{E=0}\,, \tag{2}$$

the topological information is in the supersymmetric ground states. In $N = 2$ case the ground states of the system have another topological invariant. Let us rewrite the above algebra (1) using $Q_\pm = Q_1 \pm iQ_2$

$$Q_+^2 = Q_-^2 = 0\,, \qquad Q_+Q_- + Q_-Q_+ = 2H\,. \tag{3}$$

It is known that the Q_+ cohomology exactly gives the supersymmetric ground state, dim(Ker Q_+/Im Q_+)= $n_B^{E=0} + n_F^{E=0}$. In topological quantum field theory the Q_+-cohomology plays an important role.

A topological field theory is a general covariant theory and independent of choice of the metric. In terms of BRST quantization this statement is equivalent to say that the stress tensor is given by the BRST transformation of some operator

$$T_{\mu\nu} = \{Q_{BRST}, \Lambda_{\mu\nu}\}\,. \tag{4}$$

In two dimensions the above condition (4) naïvely implies the vanishing central charge. If the system enjoys the conformal invariance the BRST operator will be written as a sum of left- and right-movers:

$$Q_{BRST} = Q_L + Q_R\,. \tag{5}$$

Throughout this article we work on the conformally invariant system and restrict ourselves to the left-moving sector, where we will denote that $Q_{BRST} = Q_L$.

The topological structure of the N=2 superconformal theory is studied through the *chiral* ring structure [3]. The N=2 extended superconformal algebra (SCA) consists of a stress tensor, L, two spin $\frac{3}{2}$ currents, $G^\pm$, and a spin 1 current, J. The N=2 superconformal algebra [3] is given by

$$\begin{aligned}
\{G_r^+, G_s^-\} &= 2L_{r+s} + (r-s)J_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}\,,\\
[J_m, G_r^\pm] &= \pm G_{m+r}^\pm\,,\\
[J_m, J_n] &= \frac{c}{3} m\delta_{r+s,0}\,.
\end{aligned} \tag{6}$$

[3] The algebra including the Virasoro operator, L_n, are understood.

This algebra has an automorphism (spectral flow)

$$
\begin{aligned}
u_\theta L_n u_\theta^{-1} &= L_n + \theta J_n + \frac{c}{6}\theta^2 \delta_{n,0}\,, \\
u_\theta G_r^{\pm} u_\theta^{-1} &= G_{r\pm\theta}^{\pm}\,, \\
u_\theta J_n u_\theta^{-1} &= J_n + \frac{c}{3}\theta \delta_{n,0}\,.
\end{aligned} \tag{7}
$$

The $\theta \in \mathbf{Z} + \frac{1}{2}$ spectral flow interpolates between NS and R sectors.

One will find the algebra (3) in the R-sector of the algebra (6)

$$\{G_0^+, G_0^-\} = 2(L_0 - \frac{c}{24}) = 2H\,. \tag{8}$$

Supersymmetry is unbroken iff the ground state in the R sector has conformal dimension $h = \frac{c}{24}$, which is equivalent to impose the following condition on the ground states

$$G_0^+|0\rangle_R = G_0^-|0\rangle_R = 0\,. \tag{9}$$

The $\theta = -\frac{1}{2}$ spectral flow of the above condition (9) gives the *chiral* state condition

$$G_{-\frac{1}{2}}^+|\phi\rangle_{NS} = 0\,, \tag{10}$$

where $|\phi\rangle_{NS} = u_{-\frac{1}{2}}|0\rangle_R$. Primary states satisfying eq. (10) form a closed ring called *chiral* ring. The spectral flow of the hamiltonian operator is given by

$$u_{-\frac{1}{2}} H u_{-\frac{1}{2}}^{-1} = L_0 - \frac{1}{2}J_0 \equiv \hat{L}_0\,, \tag{11}$$

which is the zero mode of the following operator

$$\hat{L} = L + \frac{1}{2}\partial J\,. \tag{12}$$

The new operator $\hat{L}$ satisfies the Virasoro algebra with vanishing central charge. Eguchi and Yang [4] observed that the G^+ coincides with one of the charge screening operators with respect to the twisted stress tensor (12). The supercurrent G^+ is a nilpotent operator and its contour integral is interpreted as a BRST operator in Feigin-Fuks-Felder construction. The *chiral* state condition (10) reads as the BRST invariant condition

$$Q_{BRST}|\phi\rangle = \oint d\zeta G^+(\zeta)|\phi\rangle = 0\,. \tag{13}$$

From the first algebra of eq. (6), one finds that the twisted stress tensor (12) is given by the BRST transformation of the operator, G^-

$$\hat{L} = \frac{1}{2}\{Q_{BRST}, G^-\} = \frac{1}{2}\oint d\zeta G^+(\zeta)G^-(z)\,. \tag{14}$$

Eguchi and Yang showed a correspondence between the twisted class of the N=2 unitary discrete series with central charge, $c = 3k/k+2$ ($k \in \mathbf{Z}$) and the $c=0$ subclass of the A_1 coset models, $\hat{su}(2)_k \otimes \hat{su}(2)_0/\hat{su}(2)_{k+0}$. They construct BRST invariant observables, using primary *chiral* [4] fields [3] of the N=2 models.

2. N=3.

The $N=3$ SCA consists of a stress tensor T, $SU(2)$ triplet supercurrents G^α ($\alpha = 0,+,-$), $SU(2)$ currents J^α ($\alpha = 0,+,-$) and a free fermion ψ, which have spins 2, $\frac{3}{2}$, 1 and $\frac{1}{2}$, respectively. By imposing the super-Jacobi identity, a relation between the central charge of the Virasoro algebra, c, and the level of the $SU(2)$ current algebra, k, determines to be $c = \frac{3}{2}k$. The associative N=3 operator algebra has a following form

$$G^\alpha(z)G^\beta(w) = \frac{k\eta^{\alpha\beta}}{(z-w)^3} + \frac{2i\epsilon^{\alpha\beta}_{\gamma}J^\gamma(w)}{(z-w)^2} + \frac{2\eta^{\alpha\beta}T(w) + i\epsilon^{\alpha\beta}_{\gamma}\partial J^\gamma(w)}{z-w}, \tag{15a}$$

$$G^\alpha(z)J^\beta(w) = \frac{\eta^{\alpha\beta}\psi(w)}{(z-w)^2} + \frac{i\epsilon^{\alpha\delta}_{\gamma}[\eta^\beta_\delta G^\gamma(w) + \frac{1}{2}i\epsilon^{\beta\gamma}_{\delta}\partial\psi(w)]}{z-w}, \tag{15b}$$

$$G^\alpha(z)\psi(w) = \frac{J^\alpha(w)}{z-w}, \tag{15c}$$

$$J^\alpha(z)J^\beta(w) = \frac{k\eta^{\alpha\beta}}{2(z-w)^2} + \frac{i\epsilon^{\alpha\beta}_{\gamma}J^\gamma(w)}{z-w}, \tag{15d}$$

$$\psi(z)\psi(w) = \frac{k}{z-w}, \tag{15e}$$

where

$$\pm 2i\epsilon^{0\pm}_{\pm} = i\epsilon^{+-}_{0} = 2, \quad 2\eta^{\alpha\beta} = -\epsilon^{\alpha\gamma}_{\delta}\epsilon^{\beta\delta}_{\gamma}. \tag{16}$$

Let

$$\begin{aligned} \tilde{T}^B_\pm &= T \pm \frac{1}{2}\partial J^0, \\ \tilde{T}^F_\pm &= \frac{1}{2}G^0 \pm \frac{1}{2}\partial\psi, \end{aligned} \tag{17}$$

then $\tilde{T}^B_\pm$ and $\tilde{T}^F_\pm$ satisfy the N=1 SCA with vanishing central charge. Here we choose $\tilde{T}^B_+$ and $\tilde{T}^F_+$. The deformation of the (super-)stress tensor (17) retains superconformal properties of the operators, $\mathbf{G}^\pm = J^\pm \pm \theta G^\pm$, which acquire the new conformal dimension $1 \mp \frac{1}{2}$. We should note that the bosonic operators $J^\pm$ acquire the half odd integral spins $1 \mp \frac{1}{2}$, whereas the fermionic ones $G^\pm$ acquire the integral spins, $\frac{3}{2} \mp \frac{1}{2}$. Let us assume G^+

[4] As we concentrate on the holomorphic part of the CFT, we use the terms *chiral* and *anti-chiral* in the sense of N=2, 3, 4 supersymmetries.

to be a BRST current. From eqs. (15a)-(15c), one can read off the BRST transformation properties of the N=3 generators. We summarize the result in the following diagram

$$\begin{array}{ccccccc} & & G^{+} & & J^{+} & & \\ & & & \nwarrow & & \nwarrow & \\ \tilde{T}^{B}_{+} & & \tilde{T}^{F}_{+} & & J^{3} & & \psi \\ & \nwarrow & & \nwarrow & & & \\ & & G^{-} & & J^{-} & & \end{array} \tag{18}$$

where an arrow between two operators denotes that the one at the tip of the arrow is given by the BRST transformation of the one at the end of the arrow.

To discuss the topological structure of the N=3 conformal field theory we restrict ourselves to the R-sector and give several features of the algebra. An irreducible representation is labeled by the eigenvalues of the elements of the Cartan subalgebra, which are given by $(k,\ L_0,\ J_0^0,\ R)$. The fourth element of Cartan subalgebra, R, is defined by

$$R \equiv i[G_0^0, \psi_0]\,. \tag{19}$$

[R is defined so as to be a hermitean operator, $R^\dagger = R$, under the hermitean conjugation given by $(L_n)^\dagger = L_{-n}$, $(J_n^\pm)^\dagger = J_{-n}^\mp$, $(J_n^0)^\dagger = J_{-n}^0$, $(\psi_n)^\dagger = \psi_{-n}$, $(G_n^\pm)^\dagger = G_{-n}^\mp$, $(G_n^0)^\dagger = G_{-n}^0$.]

An irreducible highest weight representation of the R-algebra is defined by

$$\begin{gathered} J_0^+|h,l,r\rangle = G_0^+|h,l,r\rangle = G_1^-|h,l,r\rangle = J_1^-|h,l,r\rangle = 0\,, \\ L_0|h,l,r\rangle = h|h,l,r\rangle\,, \quad J_0^0|h,l,r\rangle = l|h,l,r\rangle\,, \quad R|h,l,r\rangle = r|h,l,r\rangle. \end{gathered} \tag{20}$$

A simple calculation shows that the eigenvalue of the operator R is only doubly generated;

$$r^\pm = \pm\sqrt{k(h-\frac{k}{16}) - l^2}\,. \tag{21}$$

With this fact the inner product of Verma module in the R sector is hermitean iff k, h and l are real and $k(h-\frac{k}{16}) \geq l^2$.

The algebra has an automorphism (spectral flow) given by

$$\begin{aligned} u_\theta L_n u_\theta^{-1} &= L_n + \theta J_n^0 + \frac{k}{4}\theta^2\delta_{n,0}\,, \\ u_\theta G_n^\pm u_\theta^{-1} &= G_{n\pm\theta}^\pm\,, \quad u_\theta G_n^0 u_\theta^{-1} = G_n^0 + \theta\psi_n\,, \\ u_\theta J_n^\pm u_\theta^{-1} &= J_{n\pm\theta}^\pm\,, \quad u_\theta J_n^0 u_\theta^{-1} = J_n^0 + \frac{k}{2}\theta\delta_{n,0}\,, \\ u_\theta \psi_n u_\theta^{-1} &= \psi_n\,. \end{aligned} \tag{22}$$

The spectral flow of the N=3 SCA never interpolates between the NS and R sectors. Hence the situation is not as simple as the N=2 case.

At first we consider the $\theta = \frac{1}{2}$ spectral flow of the highest weight state condition (20);

$$J^{\pm}_{\frac{1}{2}}|\tilde{h},\ \tilde{l},\ \tilde{r}\rangle = G^{\pm}_{\frac{1}{2}}|\tilde{h},\ \tilde{l},\ \tilde{r}\rangle = 0\,, \quad L_0|\tilde{h},\ \tilde{l},\ \tilde{r}\rangle = \tilde{h}|\tilde{h},\ \tilde{l},\ \tilde{r}\rangle\,, \quad J^0_0|\tilde{h},\ \tilde{l},\ \tilde{r}\rangle = \tilde{l}|\tilde{h},\ \tilde{l},\ \tilde{r}\rangle\,, \tag{23}$$

where $|\tilde{h},\ \tilde{l},\ \tilde{r}\rangle = u_{\frac{1}{2}}|h,l,r\rangle$ and

$$\tilde{h} = h - \frac{l}{2} + \frac{k}{16}\,, \quad \tilde{l} = l - \frac{k}{4}\,.$$

The BRST invariant state condition is given by

$$\oint d\zeta G^{+}(\zeta)|\psi\rangle = G^{+}_{-\frac{1}{2}}|\psi\rangle = 0\,. \tag{24}$$

We impose this condition on the states flowed from the R-ground states by $\theta = \frac{1}{2}$. Then one obtains two additional conditions:

$$\begin{aligned} \{G^{+}_{-\frac{1}{2}}, G^{-}_{\frac{1}{2}}\}|\tilde{h},\ \tilde{l},\ \tilde{r}\rangle &= (4L_0 - 2J^0_0)|\tilde{h},\ \tilde{l},\ \tilde{r}\rangle = 0\,, \\ [J^{-}_{\frac{1}{2}}\,,\ G^{+}_{-\frac{1}{2}}]|\tilde{h},\ \tilde{l},\ \tilde{r}\rangle &= (\psi_0 - 2G^0_0)|\tilde{h},\ \tilde{l},\ \tilde{r}\rangle = 0\,. \end{aligned} \tag{25}$$

The first equation implies a relation between two eigenvalues of the Cartan components, $\tilde{h} = \tilde{l}/2$. Since R is a hermitean operator, the second equation of (25) implies that the eigenvalue of the operator R must vanish. Hence one obtains the unique state $|\frac{k}{8}\rangle$ with $\tilde{h} = \frac{1}{2}\tilde{l} = \frac{k}{8}$. We conclude that the geometrical structure of the N=3 SCA is trivial.

3. N=4.

To present topological natures of the N=4 SCFT's, we should give some general properties of the N=4 SCA [5]. The algebra consists of a stress tensor T, two pairs of SU(2) doublet supercurrents $(G^a, \overline{G}^a)_{a=1,2}$ and SU(2) currents J^i $(i = 1,2,3)$. By imposing the super-Jacobi identity, the central charge is related to the level of the SU(2) current algebra by $c = 6k$. The N=4 associative operator algebra is given by

$$J^i(z)G^a(w) = \frac{-\frac{1}{2}(\sigma^i)_{ab}G^b(w)}{z-w}\,, \tag{26a}$$

$$J^i(z)\overline{G}^a(w) = \frac{\frac{1}{2}(\sigma^i)^*_{ba}\overline{G}^b(w)}{z-w}\,, \tag{26b}$$

$$G^a(z)\overline{G}^b(w) = \frac{4k\delta^{ab}}{(z-w)^3} - \frac{4(\sigma^i)_{ab}J^i(w)}{(z-w)^2} + \frac{2[\delta^{ab}T(w) - (\sigma^i)_{ab}\partial J^i(w)]}{z-w}\,, \tag{26c}$$

where σ^i's are Pauli matrices and the basis for the $SU(2)$ algebra is taken so as the totally antisymmetric structure constant being $\epsilon^{123} = 1$. In the NS sector highest weight states are defined by

$$J_0^+|h,l\rangle = G^1_{\frac{1}{2}}|h,l\rangle = \bar{G}^2_{\frac{1}{2}}|h,l\rangle = 0\,, \qquad L_0|h,l\rangle = h|h,l\rangle\,, \quad J_0^3|h,l\rangle = l|h,l\rangle\,. \tag{27}$$

which is characterized by a set of three numbers (k,h,l). In the N=4 SCA there exists a spectral flow which interpolates the NS and R sector. The above condition flows to the highest weight condition for the R sector.

$$J_0^-|\tilde{h},\tilde{l}\rangle = J_1^+|\tilde{h},\tilde{l}\rangle = G^1_0|\tilde{h},\tilde{l}\rangle = \bar{G}^2_0|\tilde{h},\tilde{l}\rangle = 0\,, \qquad L_0|\tilde{h},\tilde{l}\rangle = \tilde{h}|\tilde{h},\tilde{l}\rangle\,, \quad J_0^3|\tilde{h},\tilde{l}\rangle = \tilde{l}|\tilde{h},\tilde{l}\rangle\,, \tag{28}$$

where $|\tilde{h},\tilde{l}\rangle$ belongs to the R sector and the relations to the one of the NS sector are given by

$$\tilde{h} = h - l + \frac{c}{24}\,, \qquad \tilde{l} = l - \frac{k}{2}\,. \tag{29}$$

Hence one can restrict oneself to the NS sector. Two classes of unitary representations are known.

(A) Massless representations.

$$h = l, \qquad l = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots, \tfrac{k}{2} \tag{30a}$$

(B) Massive representations.

$$h > l, \qquad l = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots, \tfrac{1}{2}(k-1) \tag{30b}$$

Next we introduce the *chiral* states for the N=4 SCFT's. *Left-chiral* states are states in the NS sector satisfying

$$G^2_{-\frac{1}{2}}|\phi\rangle = \bar{G}^1_{-\frac{1}{2}}|\phi\rangle = 0\,. \tag{31}$$

Primary *chiral* states are the left-chiral states satisfying the highest weight state conditions of N=4 algebra (27). For such states, the N=4 algebra (26c) implies that

$$\{\bar{G}^2_{\frac{1}{2}}\,,\ G^2_{-\frac{1}{2}}\}|h,l\rangle = 2(L_0 - J_0^3)|h,l\rangle = 0\,. \tag{32}$$

Therefore the dimension h of a primary *chiral* state equals to its isospin, *i.e.*, $h = l$. The *chiral* state condition (31) excludes the massive representations. One can easily show that the primary chiral states flow to the ground state of the R sector. If one replace G^2 ($\bar{G}^1$)

with G^1 $(\bar{G}^2)$, one obtains the *anti-chiral* states. A similar calculation shows the dimension of a primary *anti-chiral* state being $h = -l$.

In N=4 model, since $c = 6k$, we have two twisted stress tensors with vanishing central charge for any $k \in \mathbf{C}$:

$$\tilde{T}_{\pm} = T \pm \partial J^3 . \tag{33}$$

Let us work with $\tilde{T}_+$. The supercurrents G^1 and $\bar{G}^2$ acquire the new dimension 2, whereas G^2 and $\bar{G}^1$ acquire the dimension 1. If one think of G^2 being the BRST current, eqs. (26a) and (26c) give the BRST transformations of the N=4 generators. We can summarize them in the following two diagrams:

$$\begin{array}{ccccc} & G^1 & & & \\ & & \nwarrow & & \\ \tilde{T}_+ & & & J^- & \\ & \nwarrow & & & \\ & \bar{G}^2 & & & \end{array} \qquad \begin{array}{ccccc} & G^2 & & & \\ & & \nwarrow & & \\ \partial J^+ & & & J^3 & \\ & \nwarrow & & & \\ & \bar{G}^1 & & & \end{array} \tag{34}$$

where $J^{\pm} = J^1 \pm iJ^2$ have dimensions 1 ∓ 1 and an arrow between two operators denotes that the one at the tip of the arrow is the BRST descendent of the one at the end of the arrow. The left diagram in (34) is the one for the operators with dimension 2 and the right one is for the dimension 1 operators. In the twisted N=4 theory the original superconformal symmetries are completely broken down [5].

The condition of *left-chiral* states (31) is read as the BRST invariance of the states. As a result of twisting the stress tensor (33), an N=4 primary field labeled by (k, h, l) acquires a new conformal dimension $\tilde{h} = h - l$, which vanishes for the BRST invariant observables. Let $\sigma^{(0)}$ be a primary *chiral* operator. If we define that

$$\sigma^{(1)} = \frac{1}{2} \oint d\xi \bar{G}^2(\xi) \sigma^{(0)} , \tag{35}$$

then $\sigma^{(1)}$ satisfy the descendent equation.

$$\begin{aligned} d\sigma^{(0)} &= \{Q_{BRST}, \sigma^{(1)}\} , \\ d\sigma^{(1)} &= 0 . \end{aligned} \tag{36}$$

From (34), one immediately finds a candidate for the physical operator:

$$\begin{aligned} \sigma^{(0)} &= J^+ , \\ \sigma^{(1)} &= -\frac{1}{2} \bar{G}^1 . \end{aligned} \tag{37}$$

[5] Recently, Nojiri [6] showed that a twisting class of the N=4 SCFT's with $so(4)$ Kac-Moody algebra gives a topological CFT with N=2 superconformal symmetry.

There is another dimension one operator, $\bar{G}^1$. If one think of $\oint d\xi \bar{G}^1(\xi)$ as a BRST operator, the above diagrams change as follows;

$$\begin{array}{ccccccccccc} & \swarrow & G^1 & & & & & \swarrow & G^2 & & \\ \tilde{T}_+ & & & & J^- & & \partial J^+ & & & & J^3 \\ & & \bar{G}^2 & \swarrow & & & & & \bar{G}^1 & \swarrow & \end{array} \tag{38}$$

Note that, the zero mode of G^2 and $\bar{G}^1$ anti-commute to each other. If we take $\tilde{T}_-$ as the stress tensor of the theory, G^1 and $\bar{G}^2$ become the candidates for the BRST current.

The N=4 chiral ring has been examined through a study of irreducible massless representations of the N=4 SCA[5]. The Witten index for an irreducible representation, $(k,\ l_R)$, $l_R < 0$, is $-2l_R + 1$.

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Topological Pregauge-Pregeometry

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The pregauge-pregeometric action, i.e. the fundamental matter action whose quantum fluctuations give rise to the Einstein-Hilbert and the Yang-Mills actions is investigated from the viewpoint of the topological field theory. We show that the scalar pregauge-pregeometric action is a topological invariant for appropriate choices of the internal gauge group. This model realizes the picture that the gravitational and internal gauge theory at the low energy scale is induced as the quantum effects of the topological field theory at the Planck scale.

1. *Introduction* In general relativity, we usually start with the Einstein-Hilbert action written in terms of the metric or vierbein field, and describe various physical phenomena as its consequences. Pregeometry, first proposed by Sakharov, is based on the eminent idea that the very Einstein-Hilbert action itself may not be a fundamental one, but rather an effective action induced by quantum fluctuations of elementary matter fields in the vacuum.[1,2] Many authors pursued this interesting possibility.[3-7] For example, the pregeometric actions were written down in terms only of the matter fields without using metric fields,[4,6] and the pregeometric phase at Planck scale was suggested.[8] Prior to them, Bjorken[9] proposed a composite model of the photon of the Nambu-Jona-Lasinio type,[10] where the kinetic action of the gauge theory is induced through quantum fluctuations of matter fields. Since then, much progress has been made on this line, including that on the induced Yang-Mills action. We call them pregauge theories.

Recently Witten introduced the topological quantum field theory (TQFT).[11] He has shown how to use the path integral methods of field theory to construct certain topological invariants which are of interest to mathematicians. For instance, in four dimensions these topological invariants are known as Donaldson invariants.[12] At first sight, TQFT's may seem to be physically irrelevant, since there is no local dynamics and their observables are only topological invariants. Nevertheless, there is some expectation that TQFT's may describe a phase of unbroken diffeomorphism invariance in quantum gravity.[11,13] This is very appealing in light of arguments that the space time metric should be a derived quantity in quantum gravity.[14]

The pregeometry and TQFT share the important feature that they realize space-time diffeomorphism without a metric. In the pregeometric phase, however, matters still exist, while the topological phase involves no local physical motion. We expect that the pregeometry, in some extreme case, becomes a kind of TQFT, and TQFT may, in the broken phase, exhibit its physical implications through the pregeometry. In fact, recently it is shown that the above speculation really works in the two models; scalar pregeometry where the number of the scalar fields coincides with that of the space-time dimensions,[15] and two-dimensional spinor pregeometry with Weyl invariance.[16] In this talk we will show that a pregauge-pregeometric model[3,17–20] becomes topological under a specific choice of the internal gauge group.[21] It is an extension of the model in Ref. 15 to include gauge symmetries. The quantum fluctuations of the present model induce not only the Einstein-Hilbert action, but also the Yang-Mills action. Our approach may shed light on spontaneous symmetry breakdown of the topological quantum field theories (TQFT).

2. *Pregauge-Pregeometry* We begin by considering the pregauge-pregeometry with the internal local $\prod_{J=1}^{M} SO(N_J)$ symmetry in the d dimensional space-time. We assume that the fundamental matter fields are real scalar fields $\phi_J = (\phi_J^1, \cdots, \phi_J^{N_J})$ $(J = 1, 2, \cdots, M)$, where ϕ_J forms an N_J-plet of the group $SO(N_J)$ and a singlet of the other group $SO(N_K)$ $(K \neq J)$. The starting pregauge-pregeometric action for ϕ_J is written by using the auxiliary metric fields $g_{\mu\nu}$ and the auxiliary gauge fields $A_{J\mu}^{ab}$ $(J = 1, \cdots, M;\ a, b = 1, \cdots, N_J;\ \mu, \nu = 0, \cdots, d-1)$ as

$$S^{gA} = \int \mathrm{d}^d x \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} \sum_J \eta_J (D_\mu \phi_J \cdot D_\nu \phi_J) - \frac{d-2}{2} F \right), \tag{1}$$

where $g = \det g_{\mu\nu}$, $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$, and the $SO(N_J)$ covariant derivative $D_\mu \phi_J$ is written in terms of $A^{ab}_{J\mu}$ as

$$D_\mu \phi_J = (\partial_\mu - i\lambda_{Jab} A^{ab}_{J\mu}) \phi_J \tag{2}$$

with the group generator matrix

$$(\lambda_{Jab})^{ij} = i(\delta^i_a \delta^j_b - \delta^i_b \delta^j_a). \tag{3}$$

In Eq. (1) F is an arbitrary function of $(\phi_J \cdot \phi_J)$, the inner product $(X \cdot Y)$ stands for $\sum_{i=1}^{N_J} X^i Y^i$, $\eta_J = \pm 1$ is a signature factor, at least one of which should be negative. The action S^{gA} is invariant under space-time diffeomorphism, and internal gauge transformations.

It is known that this type of action gives rise to the Einstein-Hilbert and the Yang-Mills actions as its quantum fluctuations.[19,20] In fact, the path integral over ϕ_J in the partition function

$$Z = \int [\mathrm{d}g_{\mu\nu}][\mathrm{d}A^{ab}_{J\mu}][\mathrm{d}\phi_J] \exp(iS^{gA}) \tag{4}$$

can be performed to yield $Z = \int [\mathrm{d}g_{\mu\nu}][\mathrm{d}A^{ab}_{J\mu}] \exp(iS_{\text{eff}})$ with the effective action

$$\begin{aligned} S_{\text{eff}} = \int \mathrm{d}^d x \sqrt{-g} \left(\lambda + \frac{1}{16\pi G_{\mathrm{N}}} R - \sum_J \frac{1}{4} A^{ab}_{J\mu\nu} A^{\mu\nu}_{Jab} \right) \\ + \text{higher order terms in } g_{\mu\nu} \text{ and } A^{ab}_{J\mu}, \end{aligned} \tag{5}$$

where R is the scalar curvature written in terms of $g_{\mu\nu}$, $A^{ab}_{J\mu\nu}$ is the field strength of $A^{ab}_{J\mu}$, and the cosmological constant λ, the Newtonian gravitational constant G_{N} and the gauge coupling constants g_J are given in terms of the fundamental length scale Λ which serves as the ultraviolet cutoff. Their values depend on the precise mechanism of the cutoff. For example, in four dimensions, the one-loop approximation with the Pauli-Villars regularization gives $\lambda = N\Lambda^4/8(4\pi)^2 - F(\phi_J = 0)$, $G_N = 24\pi/N\Lambda^2$ with $N = \sum_J N_J$, [4] and $g_J = 4\pi\sqrt{6/\ln\Lambda^2}$. [18] The terms explicitly written in the Eq. (5) are those which dominate in the low energy limit.

3. *Action without the Metric and Gauge Fields* To see topological structure of the pregauge-pregeometric action we derive here the equivalent action written without the auxiliary metric and gauge fields. The equations of motion derived from the action S^{gA} yield

$$D_\mu(\sqrt{-g}g^{\mu\nu}D_\nu\phi_J)\eta_J = -\frac{d-2}{2}\sqrt{-g}\frac{\partial F}{\partial\phi_J}, \tag{6}$$

$$g_{\mu\nu} = \sum_J \eta_J(D_\mu\phi_J \cdot D_\nu\phi_J)F^{-1}, \tag{7}$$

$$\mathcal{P}_J^{abcd}\left(A_{J\mu}^{cd} - \frac{\phi_J^c \overleftrightarrow{\partial}_\mu \phi_J^d}{2(\phi_J\cdot\phi_J)}\right) = 0, \tag{8}$$

where $\mathcal{P}_J^{abcd}$ is the projection operator

$$\mathcal{P}_J^{abcd} = (\delta^{ac}\phi_J^b\phi_J^d + \delta^{bd}\phi_J^a\phi_J^c - \delta^{ad}\phi_J^b\phi_J^c - \delta^{bc}\phi_J^a\phi_J^d)/2(\phi_J\cdot\phi_J). \tag{9}$$

Among them, (6) is a net equation of motion for ϕ_J, but (7) and (8) are constraints determining the auxiliary fields $g_{\mu\nu}$ and $A_{J\mu}^{ab}$ in terms of ϕ_J.

Eliminating $g_{\mu\nu}$ and $A_{J\mu}^{ab}$ from S^{gA} by (7) and (8), we obtain the pregauge-pregeometric action written in terms only of ϕ_J as

$$S^\phi = \int \mathrm{d}^d x \sqrt{-\det_{\mu\nu}\sum_J \eta_J\,(\mathcal{D}_\mu\phi_J\cdot\mathcal{D}_\nu\phi_J)} \times F^{\frac{2-d}{2}}, \tag{10}$$

where the covariant derivative $\mathcal{D}_\mu\phi_J$ is written in terms only of ϕ_J as

$$\mathcal{D}_\mu\phi_J = (\partial_\mu - i\lambda_{Jab}\omega_{J\mu}^{ab})\phi_J \quad \text{with} \quad \omega_{J\mu}^{ab} = \frac{\phi_J^a \overleftrightarrow{\partial}_\mu \phi_J^b}{2(\phi_J\cdot\phi_J)}. \tag{11}$$

The equation of motion from the action S^ϕ in (10) reads

$$\mathcal{D}_\mu(\sqrt{-W}\bar{W}^{\mu\nu}\mathcal{D}_\nu\phi_J F^{\frac{2-d}{2}})\eta_J = \sqrt{-W}\frac{\partial}{\partial\phi_J}F^{\frac{2-d}{2}}. \tag{12}$$

where $W = \det W_{\mu\nu}$ and $\bar{W}^{\mu\nu} = (W^{-1})^{\mu\nu}$ with $W_{\mu\nu} = \sum_J \eta_J \left(\mathcal{D}_\mu\phi_J \cdot \mathcal{D}_\nu\phi_J\right)$. The equation of motion (12) coincides with (6) if we eliminate the auxiliary fields $g_{\mu\nu}$ and $A^{ab}_{J\mu}$ by (7) and (8). It establishes effective equivalence of the actions S^{gA} and S^ϕ at classical level.

4. *Quantum Mechanical Equivalence* Here we prove quantum mechanical equivalence of the actions S^{gA} and S^ϕ. For this purpose, we first perform the path integral over $A^{ab}_{J\mu}$ in the partition function Z in (4) to get $Z = \int[\mathrm{d}g_{\mu\nu}][\mathrm{d}\phi_J]\exp(iS^g)$ with

$$S^g = \int \mathrm{d}^d x\sqrt{-g}\left(1/2g^{\mu\nu}\sum_J \eta_J(\mathcal{D}_\mu\phi_J \cdot \mathcal{D}_\nu\phi_J) - \frac{d-2}{2}F\right), \tag{13}$$

where the Lee-Yang term $\prod_{x,J}|\phi|^{(d-1)(N_J-1)}$ is absorbed into the canonically invariant measure. Notice that in S^g in (13) the covariant derivative D_μ is replaced by $\mathcal{D}_\mu$ defined without $A^{ab}_{J\mu}$ in (11).

Then we show that the commutator (or Dirac bracket) algebra of S^g coincides with that of S^ϕ. It is almost parallel to that for the scalar pregeometry in Ref. 22. The system described by the action S^ϕ has $d + \sum_J(N_J - 1)$ independent first class constraints:

$$T_0 = \sum_J(\pi_J \cdot \pi_J)/\eta_J + F^{2-d}\det_{mn} W_{mn} \approx 0, \quad (m,n = 1,\cdots,d-1) \tag{14}$$

$$T_m = \sum_J(\pi_J \cdot \partial_m\phi_J) \approx 0, \qquad (m = 1,\cdots,d-1) \tag{15}$$

$$G^{ab}_J = (\pi_J \cdot \lambda^{ab}_J\phi_J) = i(\pi^a_J\phi^b_J - \pi^b_J\phi^a_J) \approx 0, \tag{16}$$

where π_J is the canonical conjugate variable of ϕ_J. Note that among G^{ab}_J only $(N_J - 1)$ operators are independent. The T_0 and T_m are the generator of diffeomorphism, while G^{ab}_J is that of $SO(N_J)$ transformations. The Hamiltonian is given by a linear combination the constraints.

On the other hand, the system with the action S^g has $d(d-1)/2+\sum_J(N_J-1)$ independent primary constraints $p^{\mu\nu} \approx 0$ and $G_J^{ab} \approx 0$, where $p^{\mu\nu}$ is the canonical conjugate of $g_{\mu\nu}$ and G_J^{ab} is defined in (16), and $d(d-1)/2$ independent secondary constraints

$$\Phi_{\mu\nu} = g_{\mu\nu} - \sum_J \eta_J(\mathcal{D}_\mu\phi_J \cdot \mathcal{D}_\nu\phi_J)F^{-1} \approx 0, \tag{17}$$

where the time derivative $\partial_0\phi_J$ in $\mathcal{D}_0\phi_J$ is written in terms of the canonical conjugate π_J of ϕ_J and other variables according to

$$\pi_J = \sqrt{-g}g^{0\nu}\mathcal{D}_\nu\phi_J. \tag{18}$$

We can show that T_μ defined in (14) and (15) are linear combinations of $\Phi_{\mu\nu}$, so that they are again constraints here. Their Poisson bracket algebra shows that there are $2d+\sum_J(N_J-1)$ independent first class constraints $p^{0\mu} \approx 0$, $\widehat{T}_\mu \approx 0$, and $G_J^{ab} \approx 0$, where

$$\widehat{T}_\mu(x) = T_\mu(x) - \int d^dy\, p^{mn}(y)[\Phi_{mn}(y), T_\mu(x)]. \quad (m,n = 1,\cdots,d-1) \tag{19}$$

The rest $(d-1)(d-2)$ independent constraints $p^{mn} \approx 0$ and $\Phi_{mn} \approx 0$ $(m,n = 1,\cdots,d-1)$ belong to the second class. Hamiltonian is a linear combination of the first class constraints. Now we partially fix the gauge by the condition

$$\chi_{0\mu} = g_{0\mu} - f_\mu(\phi_J, \pi_J), \tag{20}$$

where $f_\mu(\phi_J, \pi_J)$ are first class operators. Using only a part of the constraints, we define the intermediate Dirac bracket $[\ ,\ ]^*$ by

$$\begin{aligned}[X(x), Y(y)]^* = [X(x), Y(y)] &- \int d^dz[X(x), p^{\mu\nu}(z)][\chi_{\mu\nu}(z), Y(y)] \\ &+ \int d^dz[X(x), \chi_{\mu\nu}(z)][p^{\mu\nu}(z), Y(y)]\end{aligned} \tag{21}$$

with $\chi_{mn} = \Phi_{mn}$ $(m,n = 1,\cdots,d-1)$. Then the algebra with respect to $[\ ,\ ]^*$ entirely coincides with the Poisson bracket algebra of the system with the action

S^ϕ. Accordingly the Dirac bracket algebras derived from them in each system coincide with each other. If we adopt the same ordering prescriptions in the both systems, the quantum commutator algebras coincide with each other, even when some anomalies exist. This establishes the quantum mechanical equivalence of S^g and S^ϕ, and, hence, of S^{gA} and S^ϕ.

5. *Topological Invariance* Now we show that the pregauge-pregeometric action (10) is a topological invariant if the number of the gauge groups is equal to the number of the space-time dimensions (i. e. $M = d$). By (11) and (3), the covariant derivative becomes

$$\mathcal{D}_\mu \phi_J = \frac{\phi_J(\phi_J \cdot \partial_\mu \phi_J)}{(\phi_J \cdot \phi_J)} = \frac{\phi_J}{|\phi_J|}\partial_\mu |\phi_J| \tag{22}$$

where $|\phi_J| = \sqrt{(\phi_J \cdot \phi_J)}$. Then S^ϕ in (10) can be rewritten into the form

$$S^\phi = -\int \mathrm{d}^d x \sqrt{-\det_{\mu\nu} \sum_J \eta_J \partial_\mu |\phi_J| \partial_\nu |\phi_J|} \times F^{\frac{2-d}{2}}, \tag{23}$$

In particular for $M = d$, S^ϕ in (23) becomes

$$S^\phi(M = d) = \int_{\mathcal{R}} \mathrm{d}^d x \det_{\mu J} \partial_\mu |\phi_J| F^{\frac{2-d}{2}} \tag{24}$$

where the integration domain $\mathcal{R}$ which is so far suppressed is explicitly shown. We denote by $\tilde{\mathcal{R}}$ the image of $\mathcal{R}$ by the mapping ϕ_J. Then we have

$$S^\phi(M = d) = n S_0 \quad \text{with} \quad S_0 = \int_{\tilde{\mathcal{R}}} \mathrm{d}^d |\phi_J| F^{\frac{2-d}{2}}, \tag{25}$$

where n is the winding number of the mapping ϕ_J, and S_0 is a definite integral on the domain $\tilde{\mathcal{R}}$ and is not affected by changes of the mapping ϕ_J. (If the domain $\mathcal{R}$ has a boundary, we fix the field ϕ_J on the boundary in variations of ϕ_J.)

Thus we have shown that S^ϕ with $M = d$ is a topological invariant in the sense that it is invariant under any continuous variations of the mapping ϕ_J. Then any continuous functions ϕ_J are solutions to the equation of motion derived by varying S^ϕ. Furthermore, the topological phase has no physical specification such as the number and sizes of the internal gauge groups. We expect that they are spontaneously chosen in the course of the symmetry breakdown due to quantum effects.

Let us see that the topological case is continuously connected with some non-topological cases. For this purpose we explicitly break the internal gauge symmetries by adding to the action S^{gA} in (1) the mass terms of the gauge field $A^{ab}_{J\mu}$ with the mass M_J. Then $\omega^{ab}_{J\mu}$ in Eq. (8) is replaced by

$$\omega^{ab}_{J\mu} = \frac{\phi^a_J \overleftrightarrow{\partial}_\mu \phi^b_J}{\left(2(\phi_J \cdot \phi_J) + M_J^2\right)}, \tag{26}$$

and, hence, $\partial_\mu|\phi_J|\partial_\nu|\phi_J|$ in (23) is replaced by

$$\frac{\partial_\mu|\phi_J|\partial_\nu|\phi_J|\left(1 + \frac{M_J^2}{(\phi_J\cdot\phi_J)}\right) + \frac{M_J^4(\partial_\mu\phi_J\cdot\partial_\nu\phi_J)}{(\phi\cdot\phi_J)^2}}{\left(1 + \frac{M_J^2}{2(\phi_J\cdot\phi_J)}\right)^2}. \tag{27}$$

Non-vanishing M_J prevents S^ϕ from being rewritten into the form like (24) even for $M = d$, so that S^ϕ is not a topological invariant. The proof of classical and quantum mechanical equivalence between S^{gA} and S^ϕ remains valid for $M_J \neq 0$, though the constraints $G^{ab}_J \approx 0$ in (16) disappear here. Thus the topological case ($M_J = 0$) is connected with the non-topological case ($M_J \neq 0$) by the continuous parameters M_J. In the limit $M_J \to 0$ the action $S^\phi(M = d)$ restores the topological invariance. On the other hand, the quantum effect $S_{\rm eff}$ in (5) remains non-vanishing in this limit, and violates topological invariance. From the technical point of view, it is the ultraviolet cutoff that breaks down the topological symmetry and enables the topological system to give rise to physical effects. It is, however, not necessarily an artifact since it is smoothly connected with the non-topological cases. The quantum effects by themselves can intrinsically involve such an fundamental length scale, though we do not yet know the precise mechanism.

In conclusion, we have shown that the scalar pregauge-pregeometry becomes topological if the number of the internal gauge groups is equal to the number of the spacetime dimensions. The gravitational and gauge theories at low energy scales are induced by the quantum fluctuations from a topological action, violating the original topological symmetry. We expect that the number and sizes of the internal gauge groups are spontaneously chosen in the course of the symmetry breakdown. It will be quite interesting for us to examine the dynamical symmetry breakdown mechanism of TQFT's in further details. If this mechanism is effective, we would be able to obtain a new splendid understanding from mathematics to physics.

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Global Strings in Five-dimensional Supergravity[1]

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In this talk, we show the existence of solitonic solutions of five-dimensional supergravity [1][2], which can be interpreted as global cosmic strings in our universe. They possess the same mathematical structure as the stringy cosmic strings studied by Greene, Shapere, Vafa and Yau [3].

Dabholkar et al. [4] and Strominger [5] studied another class of solitonic solutions in string theory and discussed supersymmetry in the background of the topological object. We also show that supersymmetry is partially broken in the presence of the global strings in our model.

"Strings" in our five-dimensional model belong to the same type as studied by Greene et al. [3]. A simple model they offered in their paper is a six-dimensional model. We examine some classical solutions of the

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N=2, D=5 supergravity theory in the present talk. One of the aims of this work is to provide the simplest and pedagogical model which realizes similar solutions. Another aim is to discuss supersymmetry in the presence of the string in the specified model. While the analysis is very similar to ref. [4] and [5], only holomorphicity of "moduli" is needed in the present analysis.

We begin with the five-dimensional N=2 supergravity theory. The supermultiplet consists of fünfbein $e_M{}^A(x^N)$, gravitino $\psi_M{}^a(x^N)$ (where a =1, 2) , and the gauge field $A_M(x^N)$. Our notation is almost the same as ref. [2].

The supersymmetric action is

$$S^{(5)} = \int d^5x \, [- \frac{1}{4} e e_A^M e_B^N R_{MN}^{AB} - \frac{1}{4} e F_{MN} F^{MN} - \frac{1}{6\sqrt{3}} \varepsilon^{MNPSL} F_{MN} F_{PS} A_L$$

$$- \frac{i}{2} e \overline{\psi}_M^a \Gamma^{MNP} D_N(\omega + \widehat{\omega}) \psi_P^a - \frac{i\sqrt{3}}{32} e (F_{MN} + \widehat{F}_{MN}) \overline{\psi}^{Pa} (\gamma_P \Gamma^{MN} \gamma_S - \gamma_S \Gamma^{MN} \gamma_P) \psi_a^S \,] \tag{1}$$

To construct string-like classical solution, we give the following vacuum configurations of fields:

$$\langle e_\mu^\alpha \rangle = \frac{1}{\sqrt{b}} \bar{e}_\mu^\alpha (x^\mu) , \tag{2a}$$

$$\langle e_5^5 \rangle = b \, (x^\mu) , \tag{2b}$$

$$\langle A_5 \rangle = A_5 \, (x^\mu) , \tag{2c}$$

$$\langle \text{all other fields} \rangle = 0, \tag{2d}$$

where b means the radius of S^1, the extra space.

To analyze such configurations, we look into the relevant part of the four-dimensional action originating from Einstein-Maxwell system in five dimensions through dimensional reduction;

$$S_{\rm eff}^{(4)} = 2\pi \int d^4x \, \frac{-\bar{e}}{4}\left[\bar{R}^{(4)} + \frac{3}{8} \frac{\bar{\nabla}_\mu \tau \bar{\nabla}^\mu \bar{\tau}}{(\tau-\bar{\tau})^2} \right], \qquad (3)$$

where

$$\tau = \tau_1 + i\,\tau_2, \text{ where } \tau_1 = A_5 \text{ and } \tau_2 = \frac{\sqrt{3}}{2} b\,. \qquad (4)$$

This action resembles one discussed by Greene et al.[3], up to a coefficient in the kinetic term of τ. They took an ansatz for the complex scalar field τ (moduli of 2-torus)

$$\tau = \tau(x^2, x^3)\,, \qquad (5)$$

and that for the metric of the four-dimensional theory

$$\overline{ds}^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = (dx^1)^2 - e^\phi \left\{ (dx^2)^2 + (dx^3)^2 \right\} - (dx^4)^2\,, \qquad (6)$$

where $\phi = \phi(x^2, x^3)$. Now we adopt their assumptions (5) and (6) in our model. The equation of motion for τ takes the same form as theirs. Any holomorphic (or anti-holomorphic) function τ is a solution to the eq. of motion, that is,

$$\tau = \tau(z) \ (\text{or } \tau = \tau(\bar{z}))\ , \qquad (7)$$

where $z = x^2 + ix^3$ and $\bar{z} = x^2 - ix^3$.

We concentrate on the holomorphic solution of this type here.

The solution to the Einstein equations is obtained as

$$\phi\,(z,\bar{z}) = \ln \tau_2^3(z,\bar{z}) + \ln f(z) + \ln f(\bar{z})\,, \tag{8}$$

where f(z) is some regular function.

From observations so far, we can conclude that the string-like solution, which is very similar to the "stringy string" obtained in ref. [3], can be constructed in terms of τ and ϕ in five-dimensional supergravity. The only difference is the powers of τ_2 in eq. (8).

We should note that $\tau+1$ is equivalent to τ because only local gauge equivalence permitted in the fifth dimension is the identification $A_5 \approx A_5 + 1$ in the periodic dimension [6].

Let us discuss the supersymmetric structure of the string-like configuration in five-dimensional supergravity. Infinitesimal transformation of the supersymmetry on the gravitino field is as follows [2]:

$$\Delta\psi_M = \partial_M\varepsilon + \frac{1}{4}\hat{\omega}_{MAC}\Gamma^{AC}\varepsilon + \frac{1}{4\sqrt{3}}\left(\Gamma^{PQ}{}_M + 4\gamma^P\delta^Q{}_M\right)\hat{F}_{PQ}\varepsilon\,, \tag{9}$$

where indices a (=1, 2) of ε and ψ are implicit.

In the dimensionally reduced theory, we wish to concentrate our attention on zero modes corresponding to unbroken symmetries. We must investigate what form of ε satisfies the equation $\langle\Delta\psi_M\rangle = 0$ as in refs. [4] and [5].

Note that we can treat ε as a complex spinor instead of two "generalized Majorana" spinors. Hereafter we forget the label on ε.

By substituting eqs. (2), (4) and (6) into eq. (9) and using eqs. (7) (holomorphicity of τ, i. e., Cauchy-Riemann equation on τ) and (8), the solution to the equation $\langle\Delta\psi_M\rangle = 0$ is given by a linear combination of $\hat{\varepsilon}^+$

and $\hat{\varepsilon}^-$

$$\hat{\varepsilon}^{\pm} = \tau_2^{-1/4}\left(\frac{f}{\bar{f}}\right)^{\pm 1/4}\varepsilon_0^{\pm}\,, \tag{10}$$

where $\varepsilon_0^{\pm}$ are constant spinors satisfying $(1-\Gamma^{14})\varepsilon_0^{\pm} = 0$ and $(-1 \pm i\Gamma^{23})\varepsilon_0^{\pm} = 0$.

Now we are led to a result that the background given by the cosmic string in our model has partially broken supersymmetry. (i. e., supersymmetry associated with the restricted form of ε (10) remains unbroken.) This conclusion is independent of explicit functional form of solution, since we have used only the holomorphicity of τ and the Einstein equations.

We can see this symmetry breaking from the point of view of supersymmetry algebra. Generally speaking, extended supersymmetry algebra has central charge which is to give rise to partial symmetry breaking if background has non-trivial charge (see refs. [4], [5] and references there in). If we concentrate on string-like solutions, we should define supercharge per unit length and study the relation between the central charge (per unit length) and supersymmetry breaking. For the details, please see ref. [0].

In future works, we will clarify the symmetry among the Kaluza-Klein excited modes, using a similar technique as ref. [2].

Recently we have been informed of the string solution which involves non-zero torsion [7].

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The Cosmic Strings as The Dislocations

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Abstract

The effective action from the heterotic string compactification is studied on the manifolds with absolute parallelism. The cosmic string solutions resulting from the effective action are discussed in analogy with the dislocations in three-dimensional crystalline solid. The cosmic string density is concluded to be given by the torsion tensor of space-time, which gives rise a compactification at least in one-dimensional direction in space-time. It is shown also the contorsion tensor defines the deficit angle. We have found a stringy solution in a six-dimensional model on $M^4 \times T^2$ which coincides with that of Greene et al..

The cosmic strings which are the topological defects given rise from the symmetry breaking on the manifold with nontrivial π_1, have been extensively investigated as a candidate of the seeds for galaxy and large scale structure formation[1]. In this paper we will consider the background gauge fields of the heterotic strings to search for some solutions of the cosmic strings. In order to be able to treat the space-time symmetry and gauge symmetry on the same footing, we will formulate the theory in terms of vielbein formalism, though we exclusively concentrate on the solutions of the translational gauge fields.

As pointed out in ref.[2],the gauge theory of the space-time translation connects with the absolute parallelism. The background geometry of the theory might be the Weitzenböck space-time. We can see the cosmic string as the dislocation with absolute parallelism. We will give some interpretations on the case where a contour integral $\oint h^i_\mu dx^\mu$ is nontrivial, where h^i_μ is the vielbein field. In this case the translational group is broken into some multiply connected group. The S^1 (T^2)is the simplest one(two)-dimensional compact manifold with fundamental group $\pi_1(S^1) = \mathbf{Z}$ and $\pi_1(T^2) = \mathbf{Z} \oplus \mathbf{Z}$ where $\mathbf{Z}$ is integer, but the second homotopy group $\pi_2 = 0$. Such S^1 or T^2 structure is obtained from the coset group $T(D)/\mathbf{Z}^f$ for $f = 1$, $f = 2$ respectively. In these cases the following formula

$$\oint h^i_\mu dx^\mu = \int_S (\partial_\mu h^i_\nu - \partial_\nu h^i_\mu) dx^\mu \wedge dx^\nu = \int_S T^i_{\mu\nu} dx^\mu \wedge dx^\nu = 2\pi N, N \in \mathbf{Z}, \qquad (1)$$

assures the existence of cosmic strings and give their analogy with dislocations, where $T^i_{\mu\nu}$ is the field strength of translational gauge fields. We assume the vielbeins are static hereafter. As founded by Kröner in ref.[3], $T^i_{\mu\nu} dx^\mu \wedge dx^\nu$ is the closure failure which is interpreted as the resulting (infinitesimal) Burgers vector db^i of the dislocation density flux through the area element $dx^\mu \wedge dx^\nu$ and the dislocation density is identified with torsion tensor. Global coordinates x^μ can not exist in the presence of torsion tensors,though they can still have a local meaning with dx^μ (anholonomic coordinates). For an observer on this coordinate space the contour is seen closed by the definition of the contour integral in eq.(1). But for a observer on the orthogonal frame the closure failure is disclosed. Therefore the two end points of the circuit must be identified, which is nothing but a compactification of the space-time symmetry (translational group) in one dimensional direction into S^1.

The contorsion tensor defined by

$$S_{\rho\mu\nu} = S^{ij}_\rho h_{\mu i} h_{\nu j} = \frac{1}{2}(T_{\rho\mu\nu} + T_{\mu\nu\rho} + T_{\nu\rho\mu}), \qquad (2)$$

with the torsion tensor

$$T_{\rho\mu\nu} = h_{\rho i}(\partial_\mu h^i_\nu - \partial_\nu h^i_\mu) = h_{\rho i} T^i_{\mu\nu}, \qquad (3)$$

is called Nye's lattice curvature in theories of three dimensional crystalline solids. A contour integral

$$\oint d\theta_{ij} = \oint S_{\mu ij} dx^\mu = \int_S (\partial_\mu S_{\nu ij} - \partial_\nu S_{\mu ij}) dx^\mu \wedge dx^\nu, \qquad (4)$$

formally measures the "rotation" angle on the ij plane. In three dimensional space the connection between dislocation density and contorsion is given as satisfying Nye's law[3]

$$S_{ij} = T_{ji} - \frac{1}{2}T_{kk}\delta_{ij}, \tag{5}$$

where S_{ij} and T_{ij} are defined by

$$\begin{aligned} S_{ij} &= \varepsilon_{ikl}S_j^{kl} \\ T_{ij} &= \varepsilon_{ikl}T_j^{kl}. \end{aligned} \tag{6}$$

The above relations show that dislocations and contorsions create an incompatible situation and make possible to calculate misfit angle for some grain boundary arrangements of dislocations. The analogy to see cosmic strings as dislocations assures us to interprete the misfit angle as the deficit angle for a cosmic string configulation.

It has already been seen that the problem of cosmic strings reduces analogically to that of the dislocations in three-dimensional space[3] and the space-time seems also compactified as well as the extra dimensions. It is therefore apparent that the cosmic strings have much similarity with the dislocations in their structures and evolutions.

The action for the heterotic string in conformal gauge is given by,

$$\begin{aligned} S_{N=\frac{1}{2}} = \frac{1}{4\pi\alpha'}\int dz d\bar{z} d\theta e^{(2-d)\sigma}\{&\eta_{ij}h^i_\mu DX^\mu h^j_\nu \bar{\partial} X^\nu + \alpha'(-4D\bar{\partial}\sigma)\Phi \\ &+\psi^I[\delta_{IJ}D + A^a_\mu DX^\mu T^a_{IJ}]\psi^J\}, \end{aligned} \tag{7}$$

where T^a is a generator of the gauge group, $z, \bar{z}$ label the string world sheet, and

$$D = \partial_\theta + \theta\partial, \quad D^2 = \partial. \tag{8}$$

It is easily seen that the gravity and the internal gauge interactions are treated in the same ways, since the vielbeins and the gauge fields appear as $h^i_\mu DX^\mu$ and $A^a_\mu DX^\mu$.

When the space-time has the absolute parallelism, the vielbein gauge fields h^μ_i of the translational group are satisfying the covariant constant condition

$$D_\nu h^\lambda_i = \partial_\nu h^\lambda_i + \Gamma^\lambda_{\mu\nu}h^\mu_i = 0. \tag{9}$$

Then the affine connection of this space is derived as

$$\Gamma^\rho_{\mu\nu} = h^\rho_k \partial_\nu h^k_\mu = -h^k_\mu \partial_\nu h^\rho_k, \tag{10}$$

and the curvature tensor is identically vanishing in this case.

The second order effects of quantum fluctuations $\xi(z, \bar{z}, \theta)$ of strings X^μ is calculated by the standard method for the vielbein field as

$$S_{II} = \int d^2 z d\theta[\eta_{ij}\nabla^i_\mu\xi^\mu\bar{\nabla}^j_\nu\xi^\nu + \frac{1}{2}\nabla_\rho T_{\sigma\mu\nu}DX^\mu\bar{\partial}X^\sigma\xi^\nu\xi^\rho], \tag{11}$$

where

$$\nabla^i_\mu = h^i_\mu D + \partial_\mu h^i_\rho D X^\rho,$$
$$\bar{\nabla}^j_\nu = h^j_\nu \bar{\partial} + \partial_\nu h^j_\rho \bar{\partial} X^\rho. \tag{12}$$

Then the conformal invariance of the background vielbein fields gives a condition that the β function of the renormarization group equations should be identically vanishing, and we get the effective equations of motion for the field strength of translational gauge fields $T^i_{\mu\nu}$,

$$\nabla^\mu T^i_{\mu\nu} = 0, \tag{13}$$

where we have neglected the contributions from the dilaton field. The effective action of the vielbein fields in D-dimensions may then be given in the following form

$$S^{(D)}_{eff} = \int d^D x h[-\frac{1}{4} T^{\mu\nu}_i T^i_{\mu\nu}], \tag{14}$$

where $h = det(h^i_\mu)$.

This type of theories have been investigated by two methods (dimensional reduction and spontaneous compactification), whose mutual connections are observed[4]. Recently it has been proposed for a superstring vacuum to view the noncompact Minkowski space and the internal manifold on the same footing and to consider the more general situation of string propagation in a nontrivial ten-dimensional space[5]. Adding to the two observations above the four dimensional general relativity constructed on Weitzenböck geometry agrees with all the experiments so far carried out and can have the same classical solutions as Einstein theory[2]. It is therefore natural to start from the geometrical structure with absolute parallelism. We will find compactified solutions by imposing a parallelizability condition on the vielbein gauge fields.

Now we will investigate some problems derived from dimensional reduction, which give rise to string like objects. We will concentrate on the two solutions which are connected directly to the dislocations in three-dimensional crystalline solid. Bais et.al. have reviewed the dimensional reduction procedure of pure gauge theory in ref.[4,5], to obtain four-dimensional theories where scalar fields and a symmetry breaking potential appear naturally In our case the gauge group is the D-dimensional translations $T(D)$ in $n+(D-n)$ dimensional manifolds whose coordinates are denoted by (x^A, y^α). Assuming the coset space of the group defined by $T(D)/H$ where H is an Abelian discrete group, the vielbein fields are redefined as

$$h^i_A = h^i_A(x^A),\ A = 0 \ldots n-1$$
$$h^i_\alpha = \Phi^i_\alpha(x^A),\ \alpha = n \ldots D-1. \tag{15}$$

Let us consider a model, for an example, which we start from the six-dimensional effective action and which we take the internal space to be a two-dimensional torus, i.e. $M^6 \longrightarrow M^4 \times T^2$ [5]. The six-dimensional effective action in our model is

$$S^{(6)}_{eff} = \int d^4 x d^2 y h[-\frac{1}{4} T^{\mu\nu}_i T^i_{\mu\nu}]. \tag{16}$$

Assuming that the six-dimensional metric is block diagonal

$$G^{(6)}_{\mu\nu}(x) = \begin{pmatrix} g^{(4)}_{AB}(x) & 0 \\ 0 & g^{(2)}_{\alpha\beta}(\tau) \end{pmatrix}, \tag{17}$$

where $A, B(\alpha,\beta)$ index the uncompactified (compactified) directions and

$$g^{(2)}_{\alpha\beta}(x) = \frac{1}{\tau_2}\begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau| \end{pmatrix} \tag{18}$$

is the metric on the internal two-torus where $\tau(x) = \tau_1 + i\tau_2, detg^{(2)} = 1$. After dimensional reduction, we find the following form for the four-dimensional effective action

$$\begin{aligned} S^{(4)}_{eff} &= \Omega \int d^4xh[-\frac{1}{4}T^{AB}_i T^i_{AB} + \frac{1}{8}\partial_A(\phi^{i\alpha}\phi^\beta_i)\partial^A(\phi^i_\alpha\phi_{i\beta})] \\ &= \Omega \int d^4xh[-\frac{1}{4}T^{AB}_i T^i_{AB} - \frac{1}{4}\frac{\partial_A\tau\partial^A\bar{\tau}}{\tau_2^2}], \end{aligned} \tag{19}$$

where Ω is the volume of the compactified space. The contribution from the compactified space in equation (19) becomes the same form with the case that starts from the six-dimensional Einstein-Hilbert action[5,6]. Thus the equation with respect to $\bar{\tau}$ is

$$\partial\bar{\partial}\tau + \frac{2\partial\tau\bar{\partial}\tau}{\bar{\tau}-\tau} = 0, \tag{20}$$

where we have neglected the gravitational effects. Therefore the solitonic solutions are assured to exist in this model, when we assume the moduli depend only on the two-space coordinate (the stringy cosmic string solutions studied by Greene et al.[5]).

Finally the heterotic compactificatins on a multiply-connected six-manifold previously investigated in ref.[7] where the symmetry breaking of the internal gauge group and the index was calculated, may be more clarified by means of the method here partially studied. One problem is the reason why we should start from the Weitzenböck space. It is sure that we can also start from more general Riemann-Cartan geometry and compactify it into four-dimensional Weitzenböck space-time (instead of Minkowski space) times some internal manifolds. The main observation of this letter is the cosmic strings can be derived naturally in this formalism.

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O(2) Chern-Simons theory

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1. A (3-dimensional pure) Chern-Simons gauge theory (CSGT) is topological. As it is locally trivial, serious study of the theory requires a precise global formulation. With the CSGT of a general gauge group G, possible contributions from various G bundles must be considered. Not only this is necessary for logical completeness. But also in the relationship with 2 dimensional conformal field theory (CFT), the sectors of non-trivial G bundles are counted as an essential part of the theory to be related to certain CFTs[2,4].

In this talk we discuss the O(2) CSGT of gauge group O(2). This provides a simple, fully calculable example in which non-trivial bundles are relevant to CSGT. As regard to the relationship with CFT, we will see that the consideration of non-trivial bundles is indispensable in understanding why the O(2) CSGT relates to rational $\mathbf{Z}_2$ orbifold models (RZOMs).

2. We take the space-time manifold to be $\mathcal{M} = \Sigma \times \mathbf{R}$, where space Σ is an arbitrary closed orientable surface. The O(2) CSGT is the theory of a gauge field A. With the matrix $\mathbf{i} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ as a basis for the Lie algebra of O(2), $\mathbf{i}A$ is defined to represent a connection in an O(2) bundle over $\mathcal{M}$. To write down the action, we pick an arbitrary gauge field configuration $A_\triangle$ as a reference. Then the CSGT action can be written in the form

$$S[A] = -\frac{k}{4\pi}\int_{\mathcal{M}} (A - A_\triangle)\mathrm{d}A + S_\triangle, \qquad (1)$$

where $S_\triangle = S[A_\triangle]$ is constant for the fixed $A_\triangle$.†

† Eq.(1) is valid for any bundle and for any choice of local sections. This formula can be derived from eq.(1.5) of ref.[4].

We take the temporal gauge $A_0 = 0$. There is no obstruction to doing so, for the bundle cannot twist in the temporal direction. Now $\mathbf{i}A$ is a connection in an O(2) bundle over Σ. The action (1) dictates the 'flat connection constraint' $\mathrm{d}A = 0$ as well as the Poisson brackets of the (spatial) components of A.

3. Let us recall the notion of a (principal) G bundle. Let $\mathcal{P}$ denote a G bundle over the base manifold Σ. If $\mathcal{P}$ is 'trivial,' it is essentially the direct product $G \times \Sigma$. The point $u = (g, x) \in G \times \Sigma \approx \mathcal{P}$ is said to be 'over' $x \in \Sigma$, and the 'canonical projection' $\pi : \mathcal{P} \to \Sigma$ is given by the map $(g, x) \mapsto x$. The set of points over $x \in \Sigma$, or the inverse image $\pi^{-1}(x)$ is called the fibre over x. The structure group G operates on $\mathcal{P}$ on the right as follows: for $h \in G$ and $u = (g, x) \in \mathcal{P}$, $uh = (gh, x)$. There is no canonical way of identifying $\mathcal{P}$ with $G \times \Sigma$. But rather the identification is made by specifying a global section $\sigma : \Sigma \to G, \pi \circ \sigma(x) = x$. Then every point of $\mathcal{P}$ takes the form $u = \sigma(x)g$, which is identified with $(g, x) \in G \times \Sigma$.

So much for the case when $\mathcal{P}$ is trivial. If $\mathcal{P}$ is not a trivial bundle, it is locally trivial in the following sense. $\mathcal{P}$ is equipped with the canonical projection $\pi : \mathcal{P} \to \Sigma$, and we can find an open covering of Σ, $\{U_\alpha\}$ such that each $\pi^{-1}(U_\alpha)$ is a trivial G bundle over U_α with the restriction to U_α of π as its canonical projection. Thus $\mathcal{P}$ admits a system of local sections $\{\sigma_\alpha\}$, $\sigma_\alpha : U_\alpha \to \mathcal{P}$. The local sections define the transition functions $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \to G$, given by $\sigma_\beta(x) = \sigma_\alpha(x)\psi_{\alpha\beta}(x)$. We can think of $\mathcal{P}$ as $\bigcup_\alpha (G \times U_\alpha)$, where we identify $(\psi_{\alpha\beta} g_\beta, x) \in G \times U_\alpha$ with $(g_\beta, x) \in G \times U_\beta$ for all $g_\beta \in G$, $x \in U_\alpha \cap U_\beta$. Roughly $\mathcal{P}$ is a collection of trivial bundles $G \times U_\alpha$ pasted together by the transition functions $\{\psi_{\alpha\beta}\}$.

A gauge transformation in $\mathcal{P}$ is a map of $\mathcal{P}$ onto itself that leaves fibres invariant and satisfies $f(ua) = f(u)a$ at any $u \in \mathcal{P}$ for all $a \in G$. Given local sections $\{\sigma_\alpha\}$, we prefer to represent the gauge transformation by a set of G valued functions $\{f_\alpha\}$, each $f_\alpha : U_\alpha \to G$ given by the relation $f(\sigma_\alpha(x)) = \sigma_\alpha(x) f_\alpha(x)$. Then we can rephrase the definition: A gauge transformation f is given by a set of G valued functions f_α that commute with the transition functions on $U_\alpha \cap U_\beta$ according to the rule

$$f_\alpha(x)\ \psi_{\alpha\beta}(x) \ = \psi_{\alpha\beta}(x)\ f_\beta(x)\ . \tag{2}$$

We also represent a connection by local objects dependent on the choice of local sections. When restricted to the G =O(2) case, a connection in $\mathcal{P}$ is given by a set of local real 1-forms $\{\mathbf{i}A_\alpha\}$, defined on U_α and satisfying on overlaps $U_\alpha \cap U_\beta$ the consistency condition

$$\mathbf{i}A_\beta = {\psi_{\alpha\beta}}^{-1}\mathbf{i}A_\alpha\psi_{\alpha\beta} + {\psi_{\alpha\beta}}^{-1}d\psi_{\alpha\beta}. \tag{3}$$

A gauge transformation $f = \{f_\alpha\}$ acts on the connection as follows.

$$\mathbf{i}A_\alpha \longrightarrow \mathbf{i}A'_\alpha = f_\alpha^{-1}\mathbf{i}A_\alpha f_\alpha + f_\alpha^{-1}\mathrm{d}f_\alpha. \tag{4}$$

A 'flat structure' in $\mathcal{P}$, if it exists, is given by an open covering $\{U_\alpha\}$ equipped with local sections $\{\sigma_\alpha\}$ such that all of the transition functions are constant maps. $\mathcal{P}$ is said to be 'flat' if it admits a flat structure. Obviously trivial bundles are flat. If $\{\sigma_\alpha\}$ gives a flat structure in $\mathcal{P}$, we can define a flat connection by setting $A_\alpha = 0$ on every U_α. Thus $\mathcal{P}$ admits a flat connection. The converse is also true: A G bundle is flat if and only if it admits a flat connection.

4 O(2) bundles on Σ can be constructed by the following operations♠:
(a) SO(2) twisting: Pick a directed simple loop on Σ and identify the fibres across the loop with an SO(2) twist that winds m times in SO(2)$\approx$ S^1 as it goes around the loop. That is, we use transition functions of the form $\psi_{\alpha\beta} = e^{\mathbf{i}m\theta}$ where α denotes a patch from the right side of the directed loop and β an overlapping patch from the left side, and θ is some angular variable around the loop.

♠ Our construction seems to exhaust all O(2) bundles on Σ but we have no formal proof for this.

(b) $\mathbf{Z}_2$ twisting: Pick a simple non-contractible loop on Σ and identify the fibres across it with a twist by a reflection ($\varrho \in$O(2), $\det\varrho = -1$) that is constant along the loop. So we use transition functions of the form $\psi_{\alpha\beta} = \varrho$ where the α and β denote patches overlapping from different sides.

(c) a combination of the above two: Apply (a) and (b) on non-intersecting loops.

In (a) the location of the loop is immaterial to the bundle structure. In (b) only the homology class of the loop counts. In this, two classes that differ only by an even multiple of a class should be identified as they give the same bundle structure. In (c) it suffices to consider only the combinations of an SO(2) twist ($m = 1$) and a $\mathbf{Z}_2$ twist. In fact applying (b) an even number of times produces no effects unobtainable from the application of (a) for any m.

Thus we obtain three types of O(2) bundles, types a, b and c, according to the type of the operation involved. For type a bundles different $m \in \mathbf{Z}$ give different bundles and only the trivial one ($m = 0$) is flat. To type b belong $2^{2g} - 1$ distinct bundles and they are all flat by definition. Type c also contains $2^{2g} - 1$ distinct bundles. They also turn out to be flat.

Only flat O(2) bundles can concern us here because of the flat connection condition. In dealing with flat bundles, it is particularly convenient to work with local sections such that the transition functions are all constants and the whole system of local sections can be organized into a multi valued section over Σ. This choice in a way 'trivializes' the bundle. In this trivialization the O(2) gauge field A becomes at most double valued on Σ. Here we adopt the following explicit choice of trivializations for the flat bundles. In the trivial bundle we choose an ordinary trivialization. In the non-trivial cases (types b and c) we choose a canonical homology basis $\{\alpha_1, \cdots, \alpha_g, \beta_1, \cdots \beta_g\}$ on Σ such that α_g corresponds to the $\mathbf{Z}_2$ twist cut. In type c we draw an additional cut along β_g.[1] Discrepancies in sections one experiences in round trips across the cuts are described by the transition functions between adjacent sheets,

$$\psi = \varrho \quad (\text{cut } \alpha_g), \tag{5b}$$

$$\psi = \varrho \quad (\text{cut } \alpha_g) \quad \text{and} \quad \psi = -1 \quad (\text{cut } \beta_g) \tag{5c}$$

for types b and c respectively. An O(2) gauge field A is single valued on Σ if it is in the trivial bundle, and is double valued with the cut α_g if it is in a bundle of type b or c. This follows from eqs.(3), (5), and the fact that $g^{-1}\,\mathbf{i}\,g = \pm\mathbf{i}$ for $g\in$O(2) depending on whether $\det g = \pm 1$. A gauge transformation in the trivial bundle is given by just a function on Σ that takes values in O(2). In a non-trivial flat bundle, a gauge transformation is given by a double valued function. It is either an SO(2) valued function f which undergoes the change $f(x) \to f(x)^{-1}$ (inverse in O(2)) with respect to sheet exchange associated with the $\mathbf{Z}_2$ twist cut α_g, or a function of the form ϱf with f a function of the previous sort.

In our representation, it looks as if there is no distinction between type b and type c bundles with a common $\mathbf{Z}_2$ twist cut in terms of gauge fields and gauge transformations. Distinctions do emerge however, when we think of a parallel transport in $\mathcal{P}$ over a closed path in Σ. Taking a trace of the holonomy around a closed path C, we can construct observable (gauge invariant) $\mathcal{T}_C$, given by

$$\mathcal{T}_C^{(b)} = \begin{cases} \mathrm{Tr}(\ e^{-\mathrm{i}\int_C A}\ ) & \text{if } \#(C,\alpha_g) = \text{even} \\ 0 & \text{if } \#(C,\alpha_g) = \text{odd} \end{cases} \tag{6}$$

for type b, and $\mathcal{T}_C^{(c)} = (-1)^{\#(C,\beta_g)}\,\mathcal{T}_C^{(b)}$ for type (c). For flat connections $\mathcal{T}_C^{(b)}$ is independent of $\#(C,\,\beta_g)^\star$ but $\mathcal{T}_C^{(c)}$ alters its sign depending on that intersection number.

5 The canonical quantization of the O(2) CSGT in the temporal gauge can be performed by applying to each flat O(2) bundle the functional coherent state method of Bos and Nair[3,1].

$\star$ In this case we have $\int_{\alpha_g} A = 0$, because the homology class of a lift of α_g on the double covering associated with the cut α_g does not change under sheet exchange while the gauge field acquires a minus sign.

We introduce a complex structure on Σ. In analytic coordinates the gauge field decomposes into the sum $A = A_z \mathrm{d}z + A_{\bar{z}} \mathrm{d}\bar{z}$. We choose a representation in which a state vector is a ('holomorphic') functional of $A_{\bar{z}}$, $\mathbf{A}_{\bar{z}}$ acts on states as multiplication by $A_{\bar{z}}$ (the $\bar{z}$ component of the classical gauge connection), and $\mathbf{A}_z$ acts as the functional differential operator $\frac{\pi}{k}\frac{\delta}{\delta A_{\bar{z}}}$. Then the components of the operator gauge field satisfy correct commutation relations corresponding to the Poisson brackets. As for the flat connection constraint, we impose it as a physical condition on states along with other symmetry requirements. To secure the reality condition $\mathbf{A}_z = \mathbf{A}_{\bar{z}}^\dagger$ we take the inner product to be given by the functional integral

$$(\Psi, \Psi') = \int \mathcal{D}(A_z, A_{\bar{z}}) \exp\left(-\frac{k}{\pi}\int_\Sigma \mathrm{d}^2 z A_z A_{\bar{z}}\right) \overline{\Psi[A_{\bar{z}}]}\Psi'[A_{\bar{z}}], \tag{7}$$

where $\mathrm{d}^2 z = \mathrm{i}\mathrm{d}z \wedge \mathrm{d}\bar{z}/2$. The integral measure $\mathcal{D}(A_z, A_{\bar{z}})$, formally induced from the functional inner product $< A\,, A >= \int A_{\bar{z}} A_z \mathrm{d}^2 z$ as usual, is gauge invariant and the integrand in (7) for physical states is required to be gauge invariant. [Here by gauge invariance we mean residual gauge invariance, *i.e.*, invariance under time independent gauge transformations.] Thus physical states in our scheme are given by the functionals $\Psi[A_{\bar{z}}]$ that satisfy (*i*) the flat connection condition $\mathrm{d}\mathbf{A}\Psi = 0$, (*ii*) gauge invariance and (*iii*) reparametrization invariance. A functional which satisfies condition (*i*) is invariant under small gauge transformations, *i.e.*, those generated by infinitesimal transformations. In condition (*ii*), we further demand that the state space be invariant under large gauge transformations. [We do not mind if the states are shuffled so long as they stay within the state space.] By condition (*iii*), we impose invariance under the action of diffeomorphisms from Σ onto itself that leave the bundle structures unchanged. In a non-trivial bundle we exclude those which send the $\mathbf{Z}_2$ twist cut into a non-equivalent one. [Another tricky point is that diffeomorphism acts on the complex structure. When we consider the action on states of a reparametrization $\phi : \Sigma \to \Sigma$, we have to pick a (unique) new complex structure for the target surface so that ϕ becomes a conformal map between Riemann surfaces (with the same base manifold Σ). This is necessary to keep a state functional ($\Psi(A_{\bar z})$) a state functional ($\Psi'(A_{\bar z'})$, with respect to the new complex structure) under the action of ϕ. Thus to impose the reparametrization invariance we have to somehow identify different state spaces obtained by assuming different complex structures. This can be done, successfully enough to render condition (*iii*) meaningful, by comparing the action of observables on the state spaces.]

For the details of the procedure for finding the physical space see ref.[1]. Here we just quote the results. The coefficient k in the action (1) must be integral and even in order that Σ of any genus admits a physical state space. We assume that k is a positive even integer. [For negative k we exchange the roles of $A_{\bar z}$ and A_z. Then the theory is the same with that of the positive coefficient $k' = -k$. We do not accept the $k = 0$ case as a CSGT.] The physical space $\mathcal{H}$ for the trivial bundle over Σ with genus g has an orthonormal basis $\{\Psi_r;\ r \in (\mathbf{Z_k})^g\}$ and decomposes into two eigenspaces $\mathcal{H}_\pm$ of gauge transformations spanned by states $\Psi_r^\pm = \Psi_r \pm \Psi_{-r}$. Gauge transformations f act trivially on $\mathcal{H}_+$: 'f': $\Psi_r^+ \mapsto \Psi_r^+$ while on $\mathcal{H}_-$ they are ± 1: 'f': $\Psi_r^- \mapsto \pm\Psi_r^-$ for $\det f = \pm 1$. The action of $\mathcal{T}_C^{triv.} = \mathrm{Tr}\, e^{-i\int_C \mathbf{A}}$ depends only on the homology class of C due to the flat connection condition. For $C = \sum_{j=1}^{g}(m_j\alpha_j + n_j\beta_j)$,

$$\mathcal{T}_C\ \Psi_r^\pm = e^{i\pi m\cdot n/k}\big(\ e^{i2\pi r\cdot m/k}\Psi_{r+n}^\pm + e^{-i2\pi r\cdot m/k}\Psi_{r-n}^\pm\ \big). \tag{8}$$

For type b bundles the results are the same except that r should run through $(\mathbf{Z}_k)^{g-1}$ and that eq.(8) is valid for $C \equiv \sum_{j=1}^{g-1}(m_j\alpha_j + n_j\beta_j) \pmod{\alpha_g, 2\beta_g}$ but $\mathcal{T}_C^{(b)} = 0$ for other C. The state functionals are the same for type c bundles and the action of $\mathcal{T}_C^{(c)}$ can be inferred from the relation $\mathcal{T}_C^{(c)} = (-1)^{\#(C,\beta_g)}\ \mathcal{T}_C^{(b)}$.

6. According to ref.[2] the O(2) CSGT relates to RZOMs. To see how this is possible, we identify $\mathbf{Z}_2$ orbifold field φ on Σ as a parameter for the SO(2) valued gauge transformations in (flat) bundles. We set

$$f = e^{i\varphi/R} \tag{9}$$

with R a real constant. The condition for a gauge transformation (2), or

$$\begin{aligned} e^{i\varphi_\beta/R} &= \psi_{\alpha\beta}^{-1} e^{i\varphi_\alpha/R} \psi_{\alpha\beta} \\ &= e^{\pm i\varphi_\alpha/R} \qquad (\det\psi_{\alpha\beta} = \pm 1) \end{aligned} \tag{10}$$

is consistent with the notion that the target space of the field φ is the orbifold $S^1/\mathbf{Z}_2$;*i.e.*, the circle $\varphi \equiv \varphi + 2\pi R$ divided by the actions of the group generated by $\varphi \to -\varphi$. In fact, any SO(2) valued gauge transformation f defines a configuration of the $\mathbf{Z}_2$ orbifold boson field φ by (9), and any configuration of φ can be obtained by this in some bundle. This correspondence is not one-to-one, however. By (10), φ does not register SO(2) twists in the bundle but only $\mathbf{Z}_2$ twists, and therefore cannot distinguish between type b and type c bundles with the same $\mathbf{Z}_2$ twisting. Type a bundles cannot be distinguished from one another for the same reason. It is remarkable that we can cover all configuration of φ by considering only flat bundles and yet the trivial bundle is not just enough.

We consider a manifestly gauge invariant partition function

$$\begin{aligned} Z[A] &= \int \mathcal{D}f\, e^{-S}, \\ S &= -\frac{R^2}{4\pi} \int_\Sigma \mathrm{Tr}\left[\{i(2q)A_z + (\partial f)f^{-1}\}\{i(2q)A_{\bar z} + (\bar\partial f)f^{-1}\}\right], \end{aligned} \tag{11}$$

where the 'charge' $2q$ is some integer, κ is a real constant, and the functional measure $\mathcal{D}f$ is defined by that for the orbifold field φ. The region of the functional integration could be taken to include non-SO(2) valued gauge transformations, but it would only double the value of $Z[A]$, for such transformations take the form $e^{i\varphi/R}\varrho$ and we impose the invariance of the measure under the right action by a gauge transformation ($f \to f\varrho$, in particular). We do not try to keep track of numerical factors. Nothing unconventional is involved in the functional integral as long as we work in our trivialization scheme. What we got here is the partition function for a $\mathbf{Z}_2$ orbifold model coupled to external O(2) gauge field A. The functional integral in the trivial bundle corresponds to that in the untwisted sector in the $\mathbf{Z}_2$ orbifold, and the functional integral in a type b (or c) bundle to that in the twisted sector with the same $\mathbf{Z}_2$ twist cut.[5] The partition function (11) is well-defined and invariant under gauge transformations of the external gauge field A.

We can obtain a concrete relationship of RZOMs with the O(2) CSGT if we set $k = 2pq$, $R^2 = \frac{2p}{q}$, $p, q \in \mathbf{Z}$. Then we get[1]

$$Z[A] \sim \exp\left(-\frac{k}{\pi}\int_\Sigma \mathrm{d}^2 z A_z A_{\bar{z}}\right) \sum \left(\Psi^+_{pr+qs}\overline{\Psi^+_{pr-qs}} + \Psi^-_{pr+qs}\overline{\Psi^-_{pr-qs}}\right), \qquad (12)$$

where the sum is over $r \in (\mathbf{Z}_{2q})^g, s \in (\mathbf{Z}_{2p})^g$ for the trivial bundle (or the untwisted sector of the RZOM), and over $r \in (\mathbf{Z}_{2q})^{g-1}, s \in (\mathbf{Z}_{2p})^{g-1}$ for the non-trivial flat bundles (twisted sectors of the RZOM).

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Continuum Approaches to Two-Dimensional Quantum Gravity

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Abstract

Aspects of the continuum Liouville field approach to two-dimensional quantum gravity are reviewed. It is explained how to compute correlation functions of physical operators by using an ansatz of David, Distler and Kawai. Conformal symmetry properties, a spectrum and a state-operator correspondence of the quantum Liouville theory are discussed.

1. Introduction

String theories in non-critical dimensions require the quantization of a metric on two-dimensional world-surfaces. Two-dimensional quantum gravity is also useful as a toy model to understand four-dimensional quantum gravity. There are various approaches to two-dimensional quantum gravity. They are complementary to one another. The random triangulation by means of matrix models [1] are successful to investigate non-perturbative issues [2]. The continuum Liouville field approach [3], on the other hand, is suited to see symmetries and physical observables of the theory. Here I would like to review some aspects of the continuum Liouville field approach to two-dimensional quantum gravity. We will use various techniques of the critical string theories such as path integrals on Riemann surfaces and conformal field theories (CFTs).

We consider two-dimensional matter CFTs with central charge c coupled to quantum gravity. Massive matters can be also treated but are more complicated. A typical examples of such conformal models is a string theory in a D-dimensional flat space ($c = D$), which has an action

$$S_{\text{matter}}[g, X] = \frac{T}{2} \int d^2\xi \sqrt{g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu, \tag{1.1}$$

where ξ^α is a coordinate of a two-dimensional surface, $g_{\alpha\beta}$ a two-dimensional metric and X^μ a string coordinate in the D-dimensional space-time. From a two-dimensional point of view X^μ are D scalar fields. Other examples are the pure gravity ($c = 0$) and a Majorana spinor field coupled to gravity ($c = \frac{1}{2}$). The classical actions of these models are invariant under both of the general coordinate transformation and the Weyl transformation. In general, the Weyl symmetry is broken by anomalies in the quantum theory and the quantum gauge symmetry is only the general coordinate one.

In the next section we explain an approach by David, Distler and Kawai (DDK) [4, 5] and show how to compute correlation functions of physical operators in the model. In this approach a consistency of the theory is examined by a free field theory obtained by setting the cosmological constant to be zero. In sect. 3 we consider a quantization of the Liouville theory with non-zero cosmological constant term. In particular, its conformal properties, spectrum and state-operator correspondence are discussed.

2. David-Distler-Kawai's Approach

The partition function Z of a two-dimensional CFT with a central charge c coupled to gravity is given by a sum over topologies (genera h) and geometries (metrics $g_{\alpha\beta}$) of Riemann surfaces. A contribution from a genus h surface is denoted as Z_h and the whole partition function is given by [3]

$$\begin{aligned} Z &= \sum_{h=0}^{\infty} Z_h, \\ Z_h &= g_{\mathrm{st}}^{2h-2} \int \frac{\mathcal{D}_g g_{\alpha\beta} \mathcal{D}_g \varphi}{V_{\mathrm{gauge}}} \, \mathrm{e}^{-\mu_0 \int d^2\xi \sqrt{g} - S_{\mathrm{matter}}[g,\varphi]}, \end{aligned} \tag{2.1}$$

where g_{st} and μ_0 are the genus expansion parameter and the bare cosmological constant respectively. We denote general matter fields as φ. The functional measures are defined by the general coordinate invariant norm on the functional spaces and V_{gauge} is the volume of the group of general coordinate transformations.

The general coordinate symmetry can be fixed by the conformal gauge [3]

$$g_{\alpha\beta}(\xi) = \mathrm{e}^{\phi(\xi)} \, \hat{g}_{\alpha\beta}(\xi;\tau), \tag{2.2}$$

where $\hat{g}_{\alpha\beta}(\xi;\tau)$ is a reference metric which depends on the moduli parameters τ of the Riemann surface. Introducing the Faddeev-Popov ghost fields $b_{\alpha\beta}$, c^{α}, the path integral (2.1) becomes [3]

$$\begin{aligned} Z_h = g_{\mathrm{st}}^{2h-2} &\int \frac{(d\tau)}{V_{\mathrm{CKV}}} \int \mathcal{D}_{\hat{g}} b \mathcal{D}_{\hat{g}} c \mathcal{D}_{\hat{g}} \varphi \, \mathrm{e}^{-S_{\mathrm{ghost}}[\hat{g},b,c] - S_{\mathrm{matter}}[\hat{g},\varphi]} \\ &\times \int \mathcal{D}_g \phi \, \mathrm{e}^{-\mu_0 \int d^2\xi \sqrt{\hat{g}} \, \mathrm{e}^{\phi} - \frac{26-c}{48\pi} S_{\mathrm{L}}[\hat{g},\phi]}, \end{aligned} \tag{2.3}$$

where $(d\tau)$ is the modular invariant Weil-Petersson measure and V_{CKV} is the volume of the group generated by conformal Killing vectors. We have changed the measures for the ghosts and matter fields from those defined by a metric $g_{\alpha\beta}$ to those defined by $\hat{g}_{\alpha\beta}$. This change has introduced a factor with the Liouville action

$$S_{\rm L}[\hat{g},\phi]=\int d^2\xi\sqrt{\hat{g}}\left(\frac{1}{2}\hat{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi+\hat{R}\phi+\mu e^{\phi}\right). \tag{2.4}$$

The functional measure for ϕ is induced from the measure for $g_{\alpha\beta}$ and is defined by

$$||\delta\phi||_g^2=\int d^2\xi\sqrt{\hat{g}}\,e^{\phi}(\delta\phi)^2. \tag{2.5}$$

There are two difficult points in the quantization of ϕ. First, the exponential interaction cannot be treated as a perturbation, since a magnitude of the parameter μ can be changed by a constant shift of ϕ and therefore cannot be regarded as small. Secondly, the functional measure contains a factor e^{ϕ} and it is not obvious how to perform the integral. It is nice if one can relate it to $\mathcal{D}_{\hat{g}}\phi$, which has no ϕ dependent factor. This relation is difficult to obtain in contrast to the case of the ghost and matter fields, for which ϕ can be treated as a background field.

In refs. [4, 5] the relation between $\mathcal{D}_g\phi$ and $\mathcal{D}_{\hat{g}}\phi$ is given as an ansatz. According to them the difference of two measures is a factor which is an exponential of a local functional of $\hat{g}_{\alpha\beta}$ and ϕ. The form of this functional is assumed to be the same form as eq. (2.4) except normalizations of each term. Using this ansatz the Liouville part of the partition function (2.3) becomes

$$Z_h^{\phi}=\int\mathcal{D}_{\hat{g}}\phi\;e^{-S_{\rm eff}[\hat{g},\phi]}, \tag{2.6}$$

where the new action is given by

$$S_{\rm eff}[\hat{g},\phi]=\frac{1}{8\pi}\int d^2\xi\sqrt{\hat{g}}\left(\hat{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi-Q\hat{R}\phi+4\mu_1 e^{\alpha\phi}\right). \tag{2.7}$$

The field ϕ has been rescaled such that the normalization of its kinetic term is $\frac{1}{8\pi}$. There are three parameters μ_1, Q and α. In refs. [4, 5] μ_1 is chosen to be zero. Then the Liouville field is described by a free CFT with a central charge $1+3Q^2$. The parameters Q and α are fixed by requiring that the theory does not depend on a gauge choice. The theory should be independent of a choice of the reference metric $\hat{g}_{\alpha\beta}$ and therefore should be invariant under $\hat{g}_{\alpha\beta}(\xi) \to e^{\Lambda}(\xi)\hat{g}_{\alpha\beta}(\xi)$. This requirement leads to a condition that the total central charge of the Liouville, ghosts and matter fields should vanish. It also leads to a condition that $e^{\alpha\phi}$ should be a primary field with a conformal weight one. These conditions determines the parameters as

$$Q = \sqrt{\frac{25-c}{3}}, \quad \alpha = -\frac{1}{2\sqrt{3}}(\sqrt{25-c} \mp \sqrt{1-c}). \tag{2.8}$$

The upper sign in α is consistent with a semiclassical analysis [6]. According to a value of c, the values of the parameters are classified into three cases. For $c < 1$, both of Q and α are real, while for $c > 25$ they are pure imaginary. In the latter case a redefinition $\phi \to i\phi$ makes the action and the metric $g_{\alpha\beta}$ real. This redefinition changes the sign of a kinetic term in the action. For $1 < c < 25$, Q is real but α is complex. The complex α indicates an instability of the theory as we will discuss later.

Having determined the parameters we can compute the partition function and correlation functions of physical operators using the functional measure in eq. (2.6). For each spinless primary field $\Phi_\Delta(\xi)$ with a conformal weight Δ in the matter CFT, there exists a physical operator

$$O_\Delta = \int d^2\xi \sqrt{\hat{g}}\, \Phi_\Delta\, e^{\beta\phi}. \tag{2.9}$$

The parameter β is determined by a requirement that O_Δ is independent of the gauge choice $\hat{g}_{\alpha\beta}$. This requirement is equivalent to a condition that the operator in the integrand is a primary field with conformal weight one. It determines β as

$$\beta = -\frac{1}{2\sqrt{3}}(\sqrt{25-c} \mp \sqrt{1-c+24\Delta}). \tag{2.10}$$

The correlation functions of these operators can be factored into a matter part and a Liouville part. The Liouville part is [7]

$$\begin{aligned}&\int \mathcal{D}_{\hat{g}}\phi\ \mathrm{e}^{-S_{\rm eff}[\hat{g},\phi]-\frac{\mu}{2\pi}\int d^2\xi\sqrt{\hat{g}}\mathrm{e}^{\alpha\phi}}\mathrm{e}^{\beta_1\phi(\xi_1)}\dots\mathrm{e}^{\beta_N\phi(\xi_N)}\\&\quad=\frac{\Gamma(-s)}{|\alpha|}\int\mathcal{D}_{\hat{g}}\tilde{\phi}\ \mathrm{e}^{-S_{\rm eff}[\hat{g},\tilde{\phi}]+\sum_i\beta_i\tilde{\phi}(\xi_i)}\left(\frac{\mu}{2\pi}\int d^2\xi\sqrt{\hat{g}}\mathrm{e}^{\alpha\tilde{\phi}}\right)^s,\end{aligned}\tag{2.11}$$

where $s=-\frac{Q}{\alpha}(1-h)-\sum_{i=1}^N\frac{\beta_i}{\alpha}$. In the second line we have performed the integration of the zero mode ϕ_0 ($\phi=\phi_0+\tilde{\phi}$). We have introduced the cosmological constant term again in order to make the ϕ_0 integration finite. The exponent of μ is consistent with a value obtained for $h=0$ in ref. [8] using the light-cone gauge. It is difficult to perform the integral of the non-zero modes $\tilde{\phi}$ exactly unless s is a non-negative integer.

The $h=1$ partition function has $s=0$ and was computed for a compactified one-dimensional string ($c=1$) [9, 10] and for the $c<1$ minimal CFTs [10]. The $h=0$ three point functions in the $c<1$ minimal CFTs were computed in ref. [11]. They have fractional s in general and were computed by using an 'analytic continuation' in s. All of these results are consistent with those of the matrix models.

3. Quantum Liouville Theory

In the DDK approach the cosmological constant is first chosen to be zero and ϕ becomes free. Then, the gauge independence of the theory is examined in order to determine the parameters Q and α. When one computes the partition function and the correlation functions, the cosmological constant term is again introduced. It is not obvious whether the gauge independence is preserved in such a procedure. Moreover, the Liouville theory with non-zero cosmological constant has properties quite different from those of the free theory. Here we shall consider the Liouville theory with the cosmological constant term directly.

3.1 Canonical quantization

We begin with the action (2.7) and consider its canonical quantization [12]. For that purpose, the two-dimensional surface is chosen to have a topology of $S^1 \times \mathbf{R}$ and a flat Lorentzian metric $\hat{g}_{\alpha\beta} = \eta_{\alpha\beta}$. The canonical variables are expanded as

$$\begin{aligned} \phi(\sigma,t) &= \phi_0(t) + \sum_{n\neq 0} \frac{i}{n}\left(a_n(t)\, e^{-in\sigma} + b_n(t)\, e^{in\sigma}\right), \\ P(\sigma,t) &= \frac{1}{2\pi}\, p_0(t) + \frac{1}{4\pi}\sum_{n\neq 0}\left(a_n(t)\, e^{-in\sigma} + b_n(t)\, e^{in\sigma}\right), \end{aligned} \tag{3.1}$$

where P is the conjugate momentum of ϕ. The expansion coefficients satisfy the equal time commutation relations:

$$[a_m, a_n] = m\delta_{m+n,0} = [b_m, b_n], \quad [\phi_0, p_0] = i \tag{3.2}$$

and other commutators are zero. We use a normal ordering prescription for products of operators defined by

$$: p_0\, e^{\beta\phi_0} := e^{\frac{1}{2}\beta\phi_0}\, p_0\, e^{\frac{1}{2}\beta\phi_0}, \quad : a_{-n}a_n := a_{-n}a_n, \quad b_{-n}b_n := b_{-n}b_n, \quad (n > 0). \tag{3.3}$$

The Virasoro generators, which are Fourier transforms of the energy-momentum tensor, are given by

$$\begin{aligned} L_n &= \frac{1}{2}\sum_{m=-\infty}^{\infty} : a_{n-m}a_m : -\frac{1}{2} iQna_n + \frac{\mu}{4\pi}\int_0^{2\pi} d\sigma\, e^{in\sigma} : e^{\alpha\phi} : + \alpha_0\delta_{n,0}, \\ \bar{L}_n &= \frac{1}{2}\sum_{m=-\infty}^{\infty} : b_{n-m}b_m : -\frac{1}{2} iQnb_n + \frac{\mu}{4\pi}\int_0^{2\pi} d\sigma\, e^{-in\sigma} : e^{\alpha\phi} : + \bar{\alpha}_0\delta_{n,0}, \end{aligned} \tag{3.4}$$

where $\alpha_0, \bar{\alpha}_0$ are constants to be fixed and we have introduced $\mu = -\frac{1}{2}\alpha Q\mu_1$. Using the equal time commutators (3.2), it can be shown that they satisfy the Virasoro algebra with a central charge $1 + 3Q^2$ if $\alpha^2 + \alpha Q + 2 = 0$ and $\alpha_0 = \bar{\alpha}_0 = \frac{1}{8}Q^2$ [12]. Therefore, the gauge independence, *i.e.* the conformal invariance of the theory gives the same values of Q and α as in the DDK approach.

To construct physical operators from primary fields of the matter CFT, we need gravitational dressing operators, which are primary fields made of the Liouville field. In the DDK approach they have the form $e^{\beta\phi}$. We have to examine whether they are primary fields in the case of non-zero cosmological constant. The commutator of those operators with the Virasoro generators are

$$\begin{aligned}[L_n, :e^{\beta\phi}:] &= e^{in\sigma}\left(nh_\beta :e^{\beta\phi}: -\frac{1}{2}i:(\partial_t+\partial_\sigma)e^{\beta\phi}:\right) \\ &= e^{in\sigma}\left(nh_\beta - \frac{1}{2}i(\partial_t+\partial_\sigma)\right):e^{\beta\phi}: \\ &\quad -\frac{\mu}{4\pi}e^{in\sigma}\int_0^{2\pi} d\sigma' g(\sigma-\sigma') :e^{\alpha\phi(\sigma')}::e^{\beta\phi(\sigma)}:,\end{aligned} \tag{3.5}$$

where

$$g(\sigma) \equiv \sum_{n=1}^{\infty}\left(e^{\frac{1}{n}\alpha\beta\, e^{-in\sigma}} - e^{-\frac{1}{n}\alpha\beta\, e^{in\sigma}} + e^{\frac{1}{n}\alpha\beta\, e^{in\sigma}} - e^{-\frac{1}{n}\alpha\beta\, e^{-in\sigma}}\right)e^{-\frac{1}{n}\alpha\beta\, e^{in\sigma}}. \tag{3.6}$$

Since our normal ordering prescription depends on time, the operations of normal ordering and time derivative do not commute. This is the origin of the last term in eq. (3.5). Because of this term, the operator $:e^{\beta\phi}:$ does not have a correct commutation relation as a primary field. However, it is a good approximation of a primary field for $\phi \to +\infty$, since the last term is exponentially small in this region. It is expected that for each value of β there is a primary field O_β which has an asymptotic form $:e^{\beta\phi}:$ for $\phi \to +\infty$.

3.2 Spectrum

Next we shall study the spectrum of the Hamiltonian

$$\begin{aligned}H &= L_0 + \bar{L}_0 \\ &= p_0^2 + \frac{Q^2}{4} + \sum_{n=1}^{\infty}(a_{-n}a_n + b_{-n}b_n) + \frac{\mu}{2\pi}\int_0^{2\pi} d\sigma :e^{\alpha\phi}:.\end{aligned} \tag{3.7}$$

Due to the exponential interaction it is not easy to obtain the spectrum exactly. We use the mini-superspace approximation [13], in which one makes a replacement

$$:\mathrm{e}^{\alpha\phi}: \longrightarrow \mathrm{e}^{\alpha\phi_0}. \tag{3.8}$$

A change in H caused by this replacement is of order α^2. Since $\alpha = O((-c)^{-1})$ for $c \to -\infty$, it is a good approximation for $-c \gg 1$. With the replacement (3.8) the zero mode and the non-zero modes of ϕ are decoupled. The Hamiltonian of the non-zero modes are quadratic in a_n and b_n. The eigenstates and the eigenvalues of the total Hamiltonian are

$$\psi(\phi_0)\ |\text{a Fock state of } a_n,\ b_n\rangle, \quad E = E_0 + N + \bar{N}, \tag{3.9}$$

where $\psi(\phi_0)$ and E_0 are an eigenfunction and its eigenvalue of the zero mode Hamiltonian, and $N, \bar{N}$ are non-negative integers.

Let us obtain the zero-mode eigenfunctions and eigenvalues. The Schrödinger equation for the zero mode wave function is

$$\left(-\frac{\partial^2}{\partial\phi_0^2} + \frac{Q^2}{4} + \mu\,\mathrm{e}^{\alpha\phi_0}\right)\psi(\phi_0) = E_0\psi(\phi_0). \tag{3.10}$$

This equation can be solved in terms of the modified Bessel functions $K_\nu(x), I_\nu(x)$. The normalizable solutions and their energy eigenvalues are [14, 13]

$$\psi_p(\phi_0) = c(p)K_{\frac{2i}{|\alpha|}p}\left(\frac{2\sqrt{\mu}}{|\alpha|}\,\mathrm{e}^{\frac{1}{2}\alpha\phi_0}\right), \quad E_0 = p^2 + \frac{1}{4}Q^2 \quad (p > 0), \tag{3.11}$$

where $c(p)$ is a normalization constant. Note that there is no normalizable solution for $p = 0$ and therefore there is no ground state in this system.

When we consider a state-operator correspondence later, we need to know general solutions of eq. (3.10), which are not normalizable in general. They are given by

$\left(x \equiv \frac{2\sqrt{\mu}}{|\alpha|} e^{\frac{1}{2}\alpha\phi_0}\right)$

$$\begin{aligned} \psi(\phi_0) &= c_1 I_\nu(x) + c_2 I_{-\nu}(x) \quad \text{for } E_0 \neq \frac{1}{4}Q^2, \\ \psi(\phi_0) &= c_1 I_0(x) + c_2 K_0(x) \quad \text{for } E_0 = \frac{1}{4}Q^2. \end{aligned} \tag{3.12}$$

The two modified Bessel functions are related as

$$K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\pi\nu)}. \tag{3.13}$$

The asymptotic behaviors of these functions are

$$I_\nu(x) \sim e^{\frac{1}{2}\alpha\nu\phi_0} \tag{3.14}$$

for $\phi_0 \to +\infty$ and

$$\begin{aligned} I_\nu(x) &\sim \exp\left(-\frac{1}{4}\alpha\phi_0 + \frac{2\sqrt{\mu}}{|\alpha|} e^{\frac{1}{2}\alpha\phi_0}\right) \to \infty, \\ K_\nu(x) &\sim \exp\left(-\frac{1}{4}\alpha\phi_0 - \frac{2\sqrt{\mu}}{|\alpha|} e^{\frac{1}{2}\alpha\phi_0}\right) \to 0 \end{aligned} \tag{3.15}$$

for $\phi_0 \to -\infty$. We see that the general solution has a very bad behavior for $\phi_0 \to -\infty$ except for a particular combination K_ν in eq. (3.13).

The spectrum in eq. (3.11) was also obtained using the exact operator solution of the Liouville theory [15, 13, 16]. At the classical level it is known that the general solution of the Liouville field equation can be represented by a free field. The relation between the Liouville field and the free field is known as the Bäcklund transformation. It can be generalized to the quantum theory. Some of the operators in the Liouville theory are expressed by a free field operator [15, 16]. This free field representation is possible only if the space of states in the free field theory is restricted to a half of the whole space as in eq. (3.11) [13].

3.3 State-operator correspondence

In ordinary CFTs there is a relation between states and operators. For each field operator $O(z,\bar{z})$ there exists a state $O(0,0)\,|0\rangle$, where $|0\rangle$ is the $SL(2,\mathbf{C})$ invariant vacuum state. In the Liouville theory there is no vacuum state as we have seen in the previous subsection. However, we can construct a state for each field operator using a path integral [17] in a similar way to the Hartle-Hawking wave functions [18]. A wave function corresponding to an operator $O(z,\bar{z})$ is given by a path integral on a disk $\mathrm{D}=\{z=\mathrm{e}^{\tau+i\sigma}|\,|z|\le 1\}$ with an insertion of O at $z=0$:

$$\Psi[\phi(\sigma)]=\int \mathcal{D}_{\hat{g}}\bar{\phi}\ \mathrm{e}^{-S_{\mathrm{eff}}[\bar{\phi}]}\,O(0,0),\quad \bar{\phi}(z,\bar{z})|_{\partial M}=\phi(\sigma). \tag{3.16}$$

Here, we have denoted the Liouville field on the disk as $\bar{\phi}$ in order to distinguish it from its boundary value ϕ. These wave functions are eigenfunctions of the Hamiltonian. Eigenvalues of $L_0=\oint_{\partial\mathrm{D}}\frac{dz}{2\pi i}\,z\,T_{zz}$ can be computed by deforming the contour to a small circle around $z=0$ and using the operator product expansion of T_{zz} and $O(0,0)$. We obtain the eigenvalue corresponding to a primary field $O_\beta\sim\mathrm{e}^{\beta\phi}$ $(\phi\to+\infty)$ as

$$E=-(\beta+\frac{1}{2}Q)^2+\frac{1}{4}Q^2+N+\bar{N}, \tag{3.17}$$

where $N,\bar{N}$ are non-negative integers.

In the mini-superspace approximation we can compute the asymptotic behavior of these wave functions for $\phi_0\to+\infty$. They are given by [17]

$$\psi(\phi_0)\ \sim\ \mathrm{e}^{(\beta+\frac{1}{2}Q)\phi_0}. \tag{3.18}$$

Comparing these asymptotic form with those of the wave functions obtained by solving the Schrödinger equation in the previous subsection, we obtain an exact form of the wave functions within the mini-superspace approximation. In this way we obtain a state-operator correspondence

$$O_\beta(z,\bar z) \quad \longleftrightarrow \quad \psi(\phi_0) = c_\nu I_\nu\left(\frac{2\sqrt{\mu}}{|\alpha|}e^{\frac{1}{2}\alpha\phi_0}\right), \quad \nu = \frac{2}{\alpha}\left(\beta + \frac{1}{2}Q\right). \tag{3.19}$$

As we have seen in the previous subsection, the function I_ν has a very bad behavior for $\phi_0 \to -\infty$ except for a particular combination K_ν in eq. (3.13) [17, 19]. We will consider only such a combination. Then, the corresponding operator is a particular combination of $e^{\beta\phi}$ and $e^{-(\beta+Q)\phi}$ for $\beta \neq -\frac{1}{2}Q$, and $\phi\, e^{-\frac{1}{2}Q\phi}$ and $e^{-\frac{1}{2}Q\phi}$ for $\beta = -\frac{1}{2}Q$. According to their asymptotic behaviors, the wave functions are classified into three cases:

$$\begin{aligned}
&\text{(i)} \quad \beta = -\frac{1}{2}Q \pm \beta' \ \ (\mathrm{Re}\beta' > 0), \quad \psi(\phi_0) \sim e^{\beta'\phi_0} + c(\beta')e^{-\beta'\phi_0}, \\
&\text{(ii)} \quad \beta = -\frac{1}{2}Q, \quad \psi(\phi_0) \sim \phi_0 + \text{constant}, \\
&\text{(iii)} \quad \beta = -\frac{1}{2}Q \pm ip \ \ (p > 0), \quad \psi(\phi_0) \sim \sin\left(p\phi_0 + \delta(p)\right).
\end{aligned} \tag{3.20}$$

An insertion of local operators of CFTs on two-dimensional surfaces can be regarded as creating an infinitesimally small hole with a specific boundary condition. In the present case of two-dimensional gravity, a hole created by the operator O_β is small as in ordinary CFTs when it is measured in the reference metric $\hat g_{\alpha\beta}$. However, a size of a hole measured in the physical metric $g_{\alpha\beta}$ depends on a behavior of the wave function corresponding to the operator, since the Weyl factor of the physical metric is an argument of the wave function. The hole can be regarded as small when the wave function has a large value only for the limit $\phi_0 \to +\infty$, which corresponds to a short distance limit ($g_{\alpha\beta} = e^{\alpha\phi}\hat g_{\alpha\beta} \to 0$) [17]. In this case a probability to find a small hole is dominant.

Let us look at behaviors of the wave functions in eq. (3.20). The wave functions in the case (i) diverge for $\phi_0 \to +\infty$. Therefore, the corresponding operators create small holes. These states are called 'microscopic'. On the other hand, the wave functions in the case (iii) do not have a particularly large value for $\phi_0 \to +\infty$ and represent 'macroscopic' states. They create holes with a finite size. The case (ii) is a critical one between (i) and (iii).

3.4 Coupling to matter CFTs

A physical operator corresponding to a matter primary field with a conformal weight Δ receives a gravitational dressing O_β with $\beta = -\frac{1}{2}Q \pm \sqrt{2(\Delta - \frac{c-1}{24})}$. (See eq. (2.9).) According to the classification of O_β in eq. (3.20), a physical operator is called [17] (i) massive for $\Delta > \frac{c-1}{24}$, (ii) massless for $\Delta = \frac{c-1}{24}$ and (iii) tachyonic for $\Delta < \frac{c-1}{24}$. The massive operators create microscopic states, while the tachyonic ones create macroscopic states.

The existence of tachyonic operators as a physical operator causes the following problems [17]. First, if the action is perturbed by tachyonic operators, a typical surface is full of large holes. They do not have an interpretation of ordinary two-dimensional surface. Furthermore, even if the coefficients of tachyonic operators in the action are fine tuned to zero, they cause divergences in higher genus correlation functions as in the critical bosonic string theory. Therefore, it is desirable to find theories without tachyonic operators.

For the minimal CFTs with $c < 1$ coupled to gravity, all physical operators are massive and there is no tachyon problem. For the one-dimensional string, a matter primary field e^{ipX} gives a massive state except that a case $p = 0$ gives a massless state. For CFTs with $1 < c < 25$, the cosmological constant operator is tachyonic. This is a problem which makes the analysis in the region $1 < c < 25$ difficult. However, the tachyon problem is also present in general CFTs with infinite numbers of primary fields even for $c < 1$ [17, 20].

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CLASSICAL SCATTERING IN (2+1)-DIMENSIONAL GRAVITY.

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ABSTRACT

The formalism for N-particle classical scattering solutions in 2+1-dimensional gravity is presented. The two particle case is given as a specific illustration.

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In this meeting we have allready heard talks on various aspects of 2+1 dimensional gravity from Professors Yahikozawa and Hosoya. In studying new theories it is allways good to ask physical questions, and Carlip's discussion of Quantum scattering based on braiding[1] is a good example. In this report I want to talk about a simpler situation: the classical scattering of particles. One purpose of studying the classical theory is to show how the concept of braiding enters the calculation even at that level, another is to allow us to clarify exactly what is meant by scattering in this context.

At the simplest level, the 2+1 dimensional version of the Schwartzchild solution is flat everywhere except at the source, so the spatial geometry is a cone with opening angle proportional to the mass (figure 1). We can consider the scattering situation in which a test particle moves on this background geometry. This is the approximation $m/M \ll 1$ in which we pretend that the test particle does not affect the geometry. Scattering is shown in figure 2; but it might be argued that this isn't real scattering since the particle always moves along a straight line, as is most apparent if the cut is moved, figure 3. (the position of the cut is like a gauge choice and it can be moved at will). But this is a misleading observation, clearly something non-trivial is happening because if two particles enter with parallel trajectories, and pass on either side of the source, then the trajectories will eventually cross (figure 4.).

It is worthwhile to compare this situation with the deflection of starlight by the sun in 3+1 dimensions. As is well known, there are two equal contributions to the deflection: one Newtonian, arising from the g_{00} component of the metric, and another from the spatial curvature of geodesics. In 2+1 dimensions, there is no Newtonian force, but because of global effects the other term still contributes. The deflection angle in 3+1 dimensions can be carefully defined by comparing trajectories with the fiducial geodesics that exist in the asymptotic Minkowski space. There is no such "straight-through" geodesic in 2+1 dimensions because space is not asymptotically Minkowski, and in fact changes as the interaction is turned on. However, alternative definitions of scattering can be given. One possibility is based on the observation above; if geodesics that pass on opposite sides of the scattering centre are compared, then the angle between them is independent of the position of the cut. This method is only simple for two particles, for more or to define the quantum mechanical differential cross section, a more sophisticated approach is needed.

Following this brief introduction and discussion, the general formalism for classical scattering will be introduced and then illustrated in the two particle case.[3]

Formalism for N Dynamical Particles.

The basic property of 2+1 dimensional gravity with pointlike sources is that spacetime is flat away from the sources, as follows from an identity relating $R_{\mu\nu\alpha\beta}$ to $R_{\mu\nu}$, and the Einstein equation. As has been well known for many years, this allows us to generate solutions by cutting pieces out of flat Minkowski space and suitably identifying the edges. For example, the Schwarzchild solution for a single spinless source that was introduced above, has the following description in this language. We choose a surface in the full Minkowski space, starting from the particle's worldline, $a_\mu(\tau)$ (in this case for a particle at rest at the origin), and stretching out to infinity. Each point, x_μ, on this surface is identified with a corresponding point, x'_μ, on the surface rotated by $\Omega = e^{-MJ^0}$, and the region between the surfaces is removed.

This kind of cut and paste operation can be generalised to a moving particle,[2] where the identifications are under Poincaré transformations rather than the simple rotations of the static case. The specific transformation is:

$$(x' - a) = L\Omega L^{-1}(x - a) = e^{-p.J}(x - a). \qquad (1)$$

Where p is the momentum after a boost of L from rest, and Ω is a rotation by the mass. Now it is much more difficult to draw pictures of the geometry because they are necessarily three dimensional since the identification relates points at different times. The position of the original surface is arbitrary and should be considered as a gauge choice.

For a collection of such particles, besides being able to deform the surfaces, there is also a discrete choice of what order they appear as one passes around asymptotically. In this case it is important to work in the centre of mass frame. This is defined to be the frame in which space is asymptotically conical, and is the only one in which asymptotic time translations are a good symmetry so a Hamiltonian exists. By appropriately choosing the surfaces it is possible to combine the single particle identifications asymptotically to find the centre of mass condition:

$$e^{-HJ_0} = e^{-p_N.J} \dots e^{-p_2.J} e^{-p_1.J}. \tag{2}$$

This consists of three equations; two restrictions on the momenta, analagous to the requirement that the sum of the spatial momenta vanish in the flat space centre of mass, and a definition of the Hamiltonian.

The form of solution we have been considering is only valid for a finite time interval. Solutions with different orderings must be patched together to obtain the full solution. As the particles evolve, the surfaces for adjacent particles may approach each other. In that situation, one particle must be moved through the excised region of the other particle. Two things result: the order on the right hand side of (2) changes, and the transferred particle suffers a Lorentz transformation corresponding to the identification needed for the cut. Bear in mind that the momentum is defined with respect to the complete Minkowski space, in the identified space there is no discontinuity. If particle 1 is transferred anticlockwise through the cut for particle 2, then:

$$p_1' = e^{-p_2 J} p_1 \tag{3}$$

and the centre of mass condition becomes:

$$\begin{aligned} e^{-H'J_0} =& e^{-p_N.J} \dots e^{-p_1'.J} e^{-p_2.J} \\ =& e^{-p_N.J} \dots \left(e^{-p_2.J} e^{-p_1.J} e^{p_2.J} \right) e^{-p_2.J} \\ =& e^{-HJ_0} \end{aligned} \tag{4}$$

So the Hamiltonian is conserved through this process. A similar analysis can be performed if instead, particle 2 is transferred through the cut due to particle 1. Which particle is in fact transferred, depends on the sign of the relative angular momentum of the particles.

This process should be interpreted as the scattering due to gravitational interactions. The full process starts from an initial state with widely separated incoming particles. Classical scattering data usually consists of the momentum and impact parameter for all particles. Here, instead of the set of continuous impact parameters, only a discrete set of parameters are needed. For two-particle interactions only the sign of the impact parameter, or equivalently the sign of the relative angular momentum is needed. N-particle processes are labelled by a braid that connects the initial and final ordering, and which tells us which particle is transferred in each of the two-particle interactions that together reorder the particles.

Two Particle Scattering.

The centre of mass condition (2), with p_i written as $(E_i, p_i \cos\theta_i, p_i \sin\theta_i)$, yields a sum form for the Hamiltonian $H = H_1 + H_2$. Each H_i only depends on the ith particle data as: $\tan H_i/2 = E_i/m_i \;\tan m_i/2$. For weak coupling, that is, $E\kappa^2 \ll 1$, the H_i's are simply the energies E_i.

When we consider scattering in the framework discussed above, only one particle changes its momentum. If we look at the solution with braid ( ) then it is p_1 that must be changed (3), $p_1 \to p_1'$.

$$
\begin{aligned}
e^{-p_1' J} &= e^{-p_2 J} e^{-p_1 J} e^{p_2 J} \\
&= e^{-p_2 J} e^{-p_1 J} e^{-p_1 J} e^{p_1 J} e^{p_2 J} \\
&= e^{-H J_0} e^{-p_1 J} e^{H J_0}
\end{aligned}
\tag{5}
$$

So p_1 merely suffers a rotation by angle H, $\Delta\theta_1 = \theta_1' - \theta_1 = H$ and $\Delta\theta_2 = 0$. The energy remains the same so $H_1' = H_1$, as it must to conserve the total Hamiltonian.

If we use the first procedure described above to define the scattering angle, then we must also consider the solution with the other braid ( ).

$$
\begin{aligned}
e^{-p_2' J} &= e^{p_1 J} e^{-p_2 J} e^{-p_1 J} \\
&= e^{H J_0} e^{-p_2 J} e^{-H J_0}
\end{aligned}
\tag{6}
$$

In that case, p_2 is rotated by $-H$. The difference in change of angle for the two trajectories is the same for each particle:

$$
\Delta\theta_1 \quad - \Delta\theta_1 \quad = \Delta\theta_2 \quad - \Delta\theta_2 \quad = H \tag{7}
$$

We can check that this is a sensible definition by calculating the same quantity in the alternative gauge where we start in a state with ordering 2-1 instead of 1-2. Also note that these angles make good sense in the intuitive limit discussed in the introduction, in which a light particle moves on the almost static geometry generated by the heavy one.

ACKNOWLEDGEMENTS

Much of this work was done in collaboration with N.Sasakura, and a more detailed report is given in reference [3]. I would like to thank the Japan Society for the Promotion of Science for support.

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FIGURES

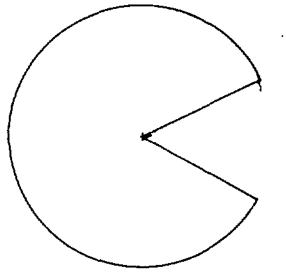

Figure 1.

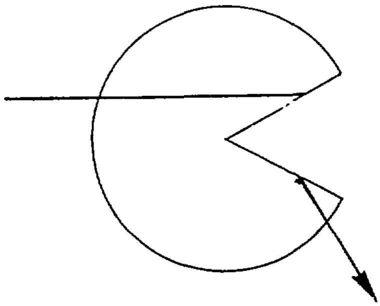

Figure 2.

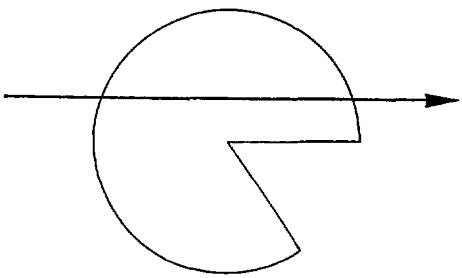

Figure 3.

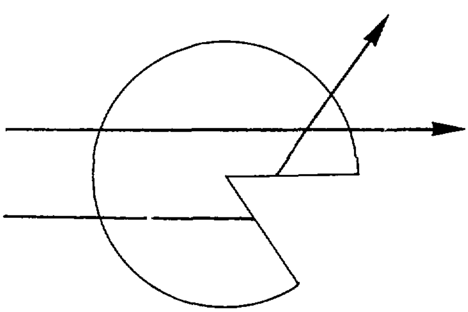

Figure 4.

Introduction to 2-form gravity and Ashtekar formalism*

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ABSTRACT

This is a talk aimed at an introduction to a formalism of gravity recently enlightened by Capovilla, Dell, Jacobson, and Mason, in which the basic gravitational variable is a self-dual 2-form rather than a tetrad 1-form. Although the gravitational theory is usually described making use of the metric as a basic variable after Einstein or is often described employing the tetrad and spin connection 1-forms in the Cartan formalism, it is pointed out that the self-dual 2-form can be employed as a basic variable instead of the tetrad 1-form. The formalism of self-dual 2-form naturally leads to the Ashtekar's constraints in terms of his new variables and it may play an interesting role to study a connection between the Yang-Mills gauge theory and the theory of gravity. We also comment on the relation between the Samuel's ansatz for the classical solution and the Kodama's solution for the quantum constraints.

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§1 Chiral theory of gravity

At first, let us consider the first-order Palatini action :

$$S[e,\omega] = -\int d^4x\ e\ e^{\mu}_{a} e^{\nu}_{b} R^{ab}{}_{\mu\nu}(\omega) \quad ,$$

which can be written as

$$\begin{aligned} S[e,\omega] &= \int \frac{1}{2}\epsilon_{abcd}\, R^{ab}(\omega)\wedge e^c \wedge e^d \\ &= \int {}^*R_{cd}(\omega)\wedge e^c \wedge e^d \end{aligned}$$

by the use of differential forms, where e^a is the tetrad 1-form, $\omega^{ab} = -\omega^{ba}$ is the spin connection 1-form and $R^a{}_b(\omega) := d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$ is the curvature 2-form of ω^{ab}.

The variational equation with respect to ω^{ab} is satisfied when the connection is equal to the usual Levi-Civita connection $\omega^{ab}(e)$ composed of tetrad, then substitution of $\omega^{ab}(e)$ for ω^{ab} turns the action $S[e,\omega]$ into the one which is equal to the usual Einstein-Hilbert action :

$$S[e,\omega(e)] = \int d^4x\ eR\big(\omega(e)\big) = \int d^4x\ \sqrt{-g}\ R(g) \quad .$$

The spin connection 1-form ω^{ab} can be decomposed into the self-dual and anti-self-dual parts with respect to its anti-symmetric local Lorentz indices :

$$\omega^{ab} = {}^{+}\omega^{ab} + {}^{-}\omega^{ab} \quad ,$$

where the (anti)-self-dual parts of ω^{ab} are defined by

$${}^*({}^{\pm}\omega)^{ab} = \pm i\,{}^{\pm}\omega^{ab}$$

in our convention ${}^*\omega^{ab} := \frac{1}{2}\epsilon^{abcd}\omega_{cd}$, which can be satisfied by the combinations

$${}^{\pm}\omega^{ab} := \frac{1}{2}(\omega^{ab} \mp i\,{}^*\omega^{ab}) \quad .$$

The curvature 2-form R^{ab} can be also decomposed additively according to this decomposition :

$$R^{ab}(\omega) = R^{ab}({}^+\omega) + R^{ab}({}^-\omega) = {}^+R^{ab}(\omega) + {}^-R^{ab}(\omega) \quad .$$

Accordingly $S[e,\omega]$ decomposes as[1]

$$S[e,\omega] = {}^+S[e,{}^+\omega] + {}^-S[e,{}^-\omega] \quad ,$$

$${}^\pm S[e,\omega] = \int {}^*({}^\pm R_{cd}(\omega)) \wedge e^c \wedge e^d \quad .$$

If we employ one of ${}^\pm S[e,{}^\pm\omega]$'s as an action instead of $S[e,\omega]$, we will see it is also equal to (half) the usual Einstein-Hilbert action when the equation of motion for ${}^\pm\omega^{ab}$ is satisfied. As the equation of motion $\frac{\delta}{\delta {}^\pm\omega} S[e,{}^\pm\omega] = 0$ implies ${}^\pm\omega^{ab} = {}^\pm\omega^{ab}(e)$, by virtue of this substitution and recalling the relation

$$R^{ab}({}^\pm\omega) = {}^\pm R^{ab}(\omega) = \frac{1}{2}\left(R^{ab}(\omega) \mp i{}^*R^{ab}(\omega)\right) \quad ,$$

we have

$$\begin{aligned}
{}^\pm S[e,{}^\pm\omega(e)] &= \int \frac{1}{2}[\ {}^*R_{ab}(\omega(e)) \mp i^{**}R_{ab}(\omega(e))\ ] \wedge e^a \wedge e^b \\
&= \int \frac{1}{2}[\ {}^*R_{ab}(\omega(e)) \pm iR_{ab}(\omega(e))\ ] \wedge e^a \wedge e^b \\
&= \frac{1}{2}S[e,\omega(e)] \ \pm\ i \int \frac{1}{2}\left[\ R_{ab}(\omega(e)) \wedge e^a \wedge e^b\ \right] \\
&= \frac{1}{2}S[e,\omega(e)] \\
&= \frac{1}{2}(\text{ Einstein-Hilbert action }) \quad ,
\end{aligned}$$

$$\left(\begin{array}{ll} \because \quad \text{1st Bianchi identity} & : \ R^a{}_b(\omega(e)) \wedge e^b \equiv 0 \\ & \qquad\qquad (\ R_{\mu[\nu\alpha\beta]} \equiv 0\ ) \end{array} \right) \quad .$$

Although the action ${}^\pm S[e,{}^\pm\omega]$ is complex, its imaginary part vanishes by use of the cyclic Bianchi identity when the spin connection is equal to the tetrad Levi-Civita connection using the equation of motion.

§2 Spinorial gravity

We shall use the $SL(2,C)$ spinor notation hereafter.[2] The translation from $SO(3,1)$ to $SL(2,C)$ can be made as follows (for example).

$$\begin{array}{ccc} SO(3,1) & \rightarrow & SL(2,C) \\ v^a & \rightarrow & v^{AA'} \\ (a=0,1,\cdots,3) & \rightarrow & (A=0,1\ ,\ A'=0,1) \end{array}$$

$$\left(\quad v^{AA'} := \tau_a^{AA'} v^a \quad ; \ \tau_a^{AA'} = \frac{1}{\sqrt{2}}(\ 1\ ,\ \sigma\ ) \quad \right)$$

We use the metrices η_{ab} , ϵ_{AB} and $\epsilon_{A'B'}$ for raising and lowering indices $a,b,\cdots$, $A,B,\cdots$ and $A',B',\cdots$:

$$\left(\begin{array}{l} \eta_{ab} = \text{diag}\ (1,-1,-1,-1,-1) \\ \epsilon_{AB} = \epsilon^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \epsilon_{A'B'} = \epsilon^{A'B'} \end{array} \right) .$$

The detailed explanations will be found in the reference[2].

Let us consider the irreducible decomposition of anti-symmetric tensor F_{ab} and its relation to the duality. An anti-symmetric tensor $F_{ab} = F_{AA'BB'} = F_{ABA'B'}$ would be decomposed as

$$F_{ABA'B'} = \epsilon_{A'B'}\ \phi_{AB} + \epsilon_{AB}\ \bar{\phi}_{A'B'} \quad ,$$

where

$$\left(\begin{array}{c} \phi_{AB} = \frac{1}{2} F_{ABC'}{}^{C'}\ ,\ \bar{\phi}_{A'B'} = \frac{1}{2} F_C{}^C{}_{B'A'} \\ \phi_{AB} = \phi_{(AB)}\ ,\ \bar{\phi}_{A'B'} = \bar{\phi}_{(A'B')} \end{array} \right) .$$

The dual tensor ${}^*F_{ab}$ for F_{ab} is given by

$${}^*F_{ABA'B'} = -i\ \epsilon_{A'B'}\ \phi_{AB} + i\ \epsilon_{AB}\ \bar{\phi}_{A'B'} \quad ,$$

because of the fact that the totally anti-symmetric Levi-Civita symbol is written as

$$\epsilon_{abcd} = i(\epsilon_{AC}\epsilon_{BD}\epsilon_{A'D'}\epsilon_{B'C'} - \epsilon_{AD}\epsilon_{BC}\epsilon_{A'C'}\epsilon_{B'D'})$$

in the spinor notation.

Here we see that the duality operation $*$ is equivalent to the operation

$$\begin{cases} \phi_{AB} \to -i\,\phi_{AB} \\ \bar{\phi}_{A'B'} \to \ i\,\bar{\phi}_{A'B'} \end{cases} ,$$

and that the each term in the above corresponds to the (anti)-self-dual part :

$$\begin{cases} {}^{-}F_{ABA'B'} = \epsilon_{A'B'}\,\phi_{AB} \ : \ \text{anti-self-dual part} \\ {}^{+}F_{ABA'B'} = \epsilon_{AB}\,\bar{\phi}_{A'B'} \ : \ \text{self-dual part} \end{cases} .$$

The previously mentioned chiral decomposition of the Palatini action would be re-derived in the spinor form as follows. The Palatini action

$$\begin{aligned} S[e,\omega] &= \int \frac{1}{2}\epsilon_{abcd}\,R^{ab}(\omega)\wedge e^c \wedge e^d \\ &= \int {}^{*}R_{cd}(\omega)\wedge e^c \wedge e^d \\ &= \int {}^{*}R_{cd}(\omega)\wedge \Sigma^{cd}(e) \quad , \end{aligned}$$

$\Big($ where we define $\Sigma^{ab}(e)$ by $\Sigma^{ab}(e) := e^a \wedge e^b$ for convenience, $\Big)$

is translated into the spinor form by means of replacing the local Lorentz indices by pairs of spinor indices :

$$e^a \to e^{AA'} \quad , \quad \omega^{ab} \to \omega^{AA'BB'} \quad .$$

ω^{ab} and therefore R^{ab} would be decomposed into chiral parts :

$$\omega^{ab} = \omega^{AB}\,\epsilon^{A'B'} + \omega^{A'B'}\,\epsilon^{AB} \quad ,$$

$$R^{ab} = R^{AB}\,\epsilon^{A'B'} + R^{A'B'}\,\epsilon^{AB} \quad .$$

$\Sigma^{ab} = e^a \wedge e^b$ would be also decomposed as

$$\Sigma^{ab} = \frac{1}{2}\,\Sigma^{AB}\epsilon^{A'B'} + \frac{1}{2}\,\Sigma^{A'B'}\epsilon^{AB} \quad ,$$

$$\begin{pmatrix} \Sigma^{AB} := \epsilon_{A'B'}(e^{AA'}\wedge e^{BB'}) \\ \Sigma^{A'B'} := \epsilon_{AB}(e^{AA'}\wedge e^{BB'}) \end{pmatrix} .$$

We have

$$\begin{aligned} S[e,\omega] &= \int {}^*R_{ab} \wedge \Sigma^{ab} \\ &= -i \int R_{AB} \wedge \Sigma^{AB} + i \int R_{A'B'} \wedge \Sigma^{A'B'} \end{aligned}$$

as a result, then we may employ

$$S[e^{AA'}, \omega_{AB}] = \int R_{AB} \wedge \Sigma^{AB}$$

as a chiral action of gravity.[4] (We will omit a factor $-i$.)

§3 2-form action without metric

Looking at the chiral action of gravity

$$S[e^{AA'}, \omega_{AB}] = \int R_{AB} \wedge \Sigma^{AB} , \quad \left(\Sigma^{AB} = \Sigma^{AB}(e) := \epsilon_{A'B'}(e^{AA'}\wedge e^{BB'}), \right)$$

we may incline to treat the (anti)-self-dual 2-form Σ^{AB} as a basic variable forgetting the fact that it is composed of tetrad 1-form $e^{AA'}$.[3,4] We will inquire what the condition to recover the relation $\Sigma^{AB} = \epsilon_{A'B'}(e^{AA'}\wedge e^{BB'})$ is. Although we can employ a method of constraint by Lagrange multiplier, we would like to use less trivial constraint than the simple equality and we also require a geometrical interpretation of the multiplier. It is proposed that the constraint term

$$\mathcal{L}_C := -\frac{1}{2} \Psi_{ABCD} \Sigma^{AB} \wedge \Sigma^{CD}$$

will meet the requirements, where

$$\begin{cases} \text{multiplier} : \Psi_{ABCD} = \Psi_{(ABCD)} \\ \text{constraint} : \Sigma^{(AB} \wedge \Sigma^{CD)} = 0 \end{cases} .$$

Solving this constraint, Σ^{AB} turns out to be written as

$$\Sigma^{AB} = \epsilon_{A'B'}(e^{AA'}\wedge e^{BB'}) \quad ,$$

though We will give a proof for this relation in the next section.

Accounting this fact, the chiral action in the above would be replaced by

$$S[\Sigma^{AB},\omega_{AB},\Psi_{ABCD}] = \int \Big[R_{AB}\wedge\Sigma^{AB} - \frac{1}{2}\Psi_{ABCD}\,\Sigma^{AB}\wedge\Sigma^{CD} \Big]$$

making use of the constraint incorporated by the multiplier Ψ_{ABCD}.[4]

§4 self-dual 2-form and tetrad 1-form

In this section we shall give a proof for the relation[4]

$$\Sigma^{(AB}\wedge\Sigma^{CD)} = 0 \iff \Sigma^{AB} = \epsilon_{A'B'}(e^{AA'}\wedge e^{BB'})$$

which is an important property used in this formalism.

The proof for the relation from right to left $\Leftarrow$ is almost evident. Assume $\Sigma^{AB} = \epsilon_{A'B'}(e^{AA'}\wedge e^{BB'})$, which is sufficient to

$$\Sigma^{(AB}\wedge\Sigma^{CD)} = \epsilon^{A'B'}\epsilon^{C'D'} \underbrace{(e^{(A}{}_{A'}\wedge e^{B}{}_{B'}\wedge e^{C}{}_{C'}\wedge e^{D)}{}_{D'})}_{\text{anti-symmetric in } [A'B'C'D']} = 0$$

because $A', B', \cdots$ have two components and anti-symmetrization of more than three indices will turns out to be 0.

The relation from left to right $\Rightarrow$ will be shown as follows.

As $\Sigma^{(AB}\wedge\Sigma^{CD)} = 0$ has 5 independent components

$$\begin{cases} \Sigma^{00}\wedge\Sigma^{00} = 0 \\ \Sigma^{00}\wedge\Sigma^{01} = 0 \\ \Sigma^{00}\wedge\Sigma^{11} + 2\Sigma^{01}\wedge\Sigma^{01} = 0 \\ \Sigma^{01}\wedge\Sigma^{11} = 0 \\ \Sigma^{11}\wedge\Sigma^{11} = 0 \qquad , \end{cases}$$

we can use the properties of the "simple" 2-form to solve these conditions.

The "simple" 2-form is a 2-form which is composed of a wedge product of a couple of 1-forms. The necessary and sufficient condition for a 2-form F to be simple is $F \wedge F = 0$,

$$F \wedge F = 0 \;\Leftrightarrow\; F = \alpha \wedge \beta \qquad ; \; (\; F \; : \; \text{2-form} \; , \; \alpha, \beta \; : \; \text{1-form} \;) \quad ,$$

and there remains an arbitrariness to replace α and β by

$$\begin{aligned} \alpha &\rightarrow a\alpha + b\beta \\ \beta &\rightarrow c\alpha + d\beta \quad , \end{aligned}$$

($ab - cd = 1$), which may be considered as a $SL(2, C)$ transformation.

Observing these properties, we will solve the constraints as

$$\Sigma^{00} \wedge \Sigma^{00} = 0 \;\Leftrightarrow\; \Sigma^{00} = 2\, \theta^{00'} \wedge \theta^{01'} \quad ,$$

$$\Sigma^{11} \wedge \Sigma^{11} = 0 \;\Leftrightarrow\; \Sigma^{11} = 2\, \check{\theta}^{10'} \wedge \check{\theta}^{11'} \quad ,$$

where we define two couples of 1-forms $(\theta^{00'}, \theta^{01'}) =: \theta^{0A'}$, $(\check{\theta}^{00'}, \check{\theta}^{01'}) =: \check{\theta}^{0A'}$ up to $SL(2, C)$ transformation with respect to A' and we may regard it as a primed spinor index.

To satisfy the remaining constraints

$$2\Sigma^{01} \wedge \Sigma^{01} = -\Sigma^{00} \wedge \Sigma^{11} \quad ,$$

$$\Sigma^{00} \wedge \Sigma^{01} = 0 \; , \; \Sigma^{11} \wedge \Sigma^{01} = 0 \quad ,$$

Σ^{01} is required to be the form of

$$\Sigma^{01} = \theta^{00'} \wedge (a\check{\theta}^{10'} + b\check{\theta}^{11'}) + \theta^{01'} \wedge (c\check{\theta}^{10'} + d\check{\theta}^{11'}) \quad ,$$

with a set of arbitrary coefficients a, b, c, d which obeys $ad - bc = 1$.

Making use of the arbitrariness in the description of a simple 2-form, we may redefine $\theta^{1A'}$ from $\breve{\theta}^{1A'}$:

$$\begin{aligned}\theta^{1A'} &:= (\theta^{10'}, \theta^{1,1'}) \\ &:= \Big(-(c\breve{\theta}^{10'} + d\breve{\theta}^{11'}) \, , \, (a\breve{\theta}^{10'} + b\breve{\theta}^{11'}) \Big) \quad ,\end{aligned}$$

which preserve the form of

$$\Sigma^{11} = 2\, \breve{\theta}^{10'} \wedge \breve{\theta}^{11'} = 2\, \theta^{10'} \wedge \theta^{11'}$$

because

$$\theta^{10'} \wedge \theta^{11'} = (ad - bc)\, \breve{\theta}^{10'} \wedge \breve{\theta}^{11'} = \breve{\theta}^{10'} \wedge \breve{\theta}^{11'} \quad .$$

On the other hand Σ^{01} turns to be

$$\Sigma^{01} = \theta^{00'} \wedge \theta^{11'} - \theta^{01'} \wedge \theta^{10'} \quad .$$

Here we have

$$\begin{cases} \Sigma^{00} = 2\, \theta^{00'} \wedge \theta^{01'} \\ \Sigma^{11} = 2\, \theta^{10'} \wedge \theta^{11'} \\ \Sigma^{01} = \theta^{00'} \wedge \theta^{11'} - \theta^{01'} \wedge \theta^{10'} \end{cases}$$

as a result, which can be arranged into the simple form

$$\Sigma^{AB} = \epsilon_{A'B'} (\theta^{AA'} \wedge \theta^{BB'}) \quad .$$

At this stage, we can regard $\theta^{AA'}$ as a tetrad 1-form $e^{AA'}$ and then we arrive at the result that the 2-form Σ^{AB} is composed of the tetrad 1-form :

$$\Sigma^{AB} = \epsilon_{A'B'} (e^{AA'} \wedge e^{BB'}) \quad .$$

§5 2-form gravity with cosmological term : comment on Samuel ansatz

The chiral action with the cosmological term is given by

$$S[\Sigma^{AB},\omega_{AB},\Psi_{ABCD}]$$
$$= \int \left[R_{AB} \wedge \Sigma^{AB} - \frac{\Lambda}{6}\Sigma_{AB}\wedge\Sigma^{AB} - \frac{1}{2}\Psi_{ABCD}\Sigma^{AB}\wedge\Sigma^{CD} \right] \quad .$$

The equations of motion derived from this action are

$$\begin{cases} \Sigma^{AB}\cdots(1) & R_{AB} - \frac{\Lambda}{3}\Sigma_{AB} - \Psi_{ABCD}\Sigma^{CD} = 0 \\ \omega^{AB}\cdots(2) & D\Sigma^{AB} = 0 \\ \Psi_{ABCD}\cdots(3) & \Sigma^{(AB}\wedge\Sigma^{CD)} = 0 \end{cases}$$

and we may solve these equation in order of $(3) \to (2) \to (1)$ to see that the Einstein equation appears and the multiplier Ψ_{ABCD} is determined to be equal to the anti-self-dual part of the Weyl curvature spinor which is the spinor form of the Weyl tensor:

$$C_{abcd} = {}^{-}C_{abcd} + {}^{+}C_{abcd}$$
$$= \Psi_{ABCD}\, \epsilon_{A'B'}\epsilon_{C'D'} + \bar{\Psi}_{A'B'C'D'}\, \epsilon_{AB}\epsilon_{CD} \quad .$$

It is easy to check that the ansatz

$$R^{AB} = \frac{\Lambda}{3}\, \Sigma^{AB}$$

which is proposed by Samuel presents a class of solutions for the equation of motion.[6,5] As the 2-form Σ^{AB} is not only anti-self-dual with respect to its local Lorentz indices but also anti-self-dual in the sense of the hodge duality with the use of metric defined by itself, $g_{\mu\nu} = (e_\mu \cdot e_\nu)$, this ansatz requires that the curvature 2-form R^{AB} is also anti-self-dual in the sense of the hodge duality. This means that the above ansatz produces anti-self-dual Yang-Mills instantons when we consider the case of the Euclidean signature. The consistency of the ansatz with the equation of motion requires

$$\Psi_{ABCD} = 0 \ ,$$

that is to say, the anti-self-dual Weyl tensor is equal to 0. This is nothing but the condition for the self-dual gravitational instantons. It is interesting that the above ansatz implies an interrelation between the Yang-Mills and gravitational instantons.

It is also remarkable that the WKB wave functional for the classical solution belonging to this ansatz is a solution for the quantum Ashtekar constraints. This functional is nothing but the solution which is proposed by Kodama.[7] Although Kodama has pointed out that the Chern-Simons functional solves the quantum Ashtekar constraints, starting from the solution for Bianchi IX model and generalizing it, his solution will be well understood as follows. We can evaluate the action functional for the classical solution according to the ansatz by substituting $\Sigma^{AB} = \frac{3}{\Lambda} R^{AB}$, then we have

$$S_{cl} = \frac{6}{\Lambda} \int_{\mathcal{M}} \left[R_{AB} \wedge R^{AB} \right] \quad ,$$

which is a surface integral, as is well known, and leaves

$$S_{cl} = \frac{6}{\Lambda} S_{\mathrm{CS}} = \frac{6}{\Lambda} \int_{\partial \mathcal{M}} \left[\omega_A{}^B \wedge d\omega_B{}^A + \frac{2}{3} \omega_A{}^B \wedge \omega_B{}^C \wedge \omega_C{}^A \right] \quad .$$

The WKB wave function obtained by exponentiating this classical action is

$$\Psi = \exp \left[\frac{6i}{\Lambda} S_{\mathrm{CS}} \right] \quad ,$$

which turns out to be equivalent to the functional given by Kodama after some rearrengements of factors. It is also a solution for the theory without the constraint term accompanied by Ψ_{ABCD} which is a kind of topological theories.[8]

We expect that the self-dual 2-form may play an important role and be useful to study a connection between Yang-Mills theory and gravity. It may also be useful for an investigation of a theory of quantum gravity.

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(2+1)Dimensional Gravity with Spinor Field

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1. Introduction

Canonical approach to the quantum gravity is an excellent and traditional method as nonperturbative approach. Based on the metric representation of the canonical quantum gravity in 3+1 dimensions, a great effort has been made to solve the Wheeler-DeWitt(WDW) equation but until now only approximated solutions (in mini-superspace approximation) are known due to complicated structure of the WDW equation. In Ashtekar's reformulation of General Relativity,[1] based on the self-dual representation, it has been shown that all constraints become polynomial. In this formulation a class of exact solutions of the WDW equation, which is related to loops in 3 dimensional manifold, has been presented by T.Jacobson and L.Smolin.[2] Using a new representation, called the loop representation, C.Rovelli and L.Smolin[3] have exhibited a large class of solutions of all the constraints. Although there are unsolved problems, e.g.the Hilbert structure on the physical states and so on, yet this approach may be powerful to study nonperturbative structure of the quantum gravity.

In this talk, we study the 2+1 dimensional quantum gravity including a spinor field following Ashtekar's formalism. There are no local degrees of freedom of the graviton in 2+1 dimensions and only global structure comes into question in the pure gravity. Witten[4] has shown that 2+1 dimensional pure quantum gravity based on a Chern-Simons gauge theory for the ISO(2,1) Poincare group is exactly soluble. A.Hosoya and K.Nakao[5] have also solved it based on a metric representation along by ADM formalism. On the other hand, Ashtekar et al.[6] have carried out it on a connection representation based on a Palatini formalism which accords with the Ashtekar formalism in 3+1 dimensions. When matter fields which have no relations with an extended Chern-Simons gauge group are coupled to gravity, it is very hard to solve because of the existence of local degrees of freedom of matter fields even in 2+1 dimensions. In spite of the existence of graviton's local degrees of freedom the 3+1 dimensional pure gravity has a large class of solutions of full constraints as mentioned in the above paragraph. Therefore we expect that it is possible to solve a matter coupled gravity system in 2+1 dimensions, at least in the same or inferior level as 3+1 dimensional pure gravity case.

2. Action of 2+1 dimensional gravity coupled to spinor field

For the action, we use the Palatini form as gravity part and the minimally coupled system as spinor field part but only the forward derivative part to avoid complexity, e.g. the appearance of second class constraints with respect to the spinor fields, as follows.

$$S_T = \int d^3x\ [\ L_g + L_m\ ], \qquad L_g = \tfrac{1}{2}\varepsilon^{\alpha\beta\gamma} e_\alpha{}^i F_{\beta\gamma i}\ , \qquad (1.a,b)$$

$$L_m = e\bar{\Psi}\not{D}\Psi = e\bar{\Psi}\Gamma^\alpha(\partial_\alpha + A_\alpha{}^i\tau_i)\Psi, \qquad (1.c)$$

where $\varepsilon^{\alpha\beta\gamma}$, $e_\alpha{}^i$, $F_{\alpha\beta}{}^i$, $\bar{\Psi}\equiv -i\Psi^\dagger\gamma_0$, τ_i and $\Gamma_\alpha\equiv e_\alpha{}^i\gamma_i$ are the Levi-Civita antisymmetric tensor density, a dreibein, the curvature tensor of the spin connection $A_\alpha{}^i$ ($F_{\alpha\beta}{}^i \equiv 2\partial_{[\alpha}A_{\beta}{}^i{}_{]} + f^i{}_{jk}A_\alpha{}^j A_\beta{}^k$), a conjugate two-component spinor field, generators of the SO(2,1) Lorentz group which satisfy $[\tau_i, \tau_j] = f_{ij}{}^k\tau_k$ ($f_{ij}{}^k$ are the structure constants) and the Dirac matrices in 2+1 dimensional curved space-time, respectively. We use the metric signature (-,+,+) and the convention $\varepsilon^{012} = +1 = -\varepsilon_{012}$. The Dirac matrices γ_i(i=0,1,2) are given by $\gamma_i = (i\sigma_3, \sigma_1, \sigma_2)$ with σ_i being the Pauli matrices which satisfy a Clifford algebra, $\{\gamma_i, \gamma_j\} = 2\eta_{ij}$. To see the relation between our action and the standard Einstein-Dirac action and whether our action is Hermite, we first take a variation of (1.a) with respect to $A_\alpha{}^i$. The equation of motion for $A_\alpha{}^i$ yields $\varepsilon^{\alpha\beta\gamma}D_{[\beta}e_{\gamma}{}^i{}_{]} = -e\bar{\psi}\Gamma^\alpha\tau^i\psi$, and therefore our theory has torsion. This is not surprising because we are adapting the Palatini formalism. If we introduce the torsion-free connection (covariant derivative) ∇ which is defined by $\nabla_\alpha e_\beta{}^i = 0$ and define C as a difference of D and ∇, then one can derive $D_{[\alpha}e_{\beta}{}^i{}_{]} = C_{[\alpha}e_{\beta}{}^i{}_{]}$ and $C_\alpha{}^i = -e_\alpha{}^i(\bar{\psi}\psi)/4 - \varepsilon^{ijk}e_{\alpha j}(\bar{\psi}\gamma_k\psi)/2$. Thus the connection $A_\alpha{}^i$ is uniquely determined by $e_\alpha{}^i$ and ψ. Substituting C into the action S_T and using the partial integration in the kinetic term of the spinor field, we obtain the reduced action S_T' as follows:

$$S_T' = \int d^3x \, [\, \tfrac{1}{2}\varepsilon^{\alpha\beta\gamma} e_{\alpha}{}^i \mathring{F}_{\beta\gamma i} + \tfrac{1}{2}e(\bar{\psi}\overrightarrow{\nabla\!\!\!/}\psi - \psi\overleftarrow{\nabla\!\!\!/}\psi) - \tfrac{1}{2}e\nabla_\alpha(\bar{\psi}\gamma^\alpha\psi) + 15e(\bar{\psi}\psi)^2/16 \,]. \qquad (2)$$

Here we have used the Fierz transformation $(\bar{\psi}\gamma^i\psi)^2 = -3(\bar{\psi}\psi)^2$, to obtain the last term in S_T'. The first two terms in (2) correspond to the action of a standard Einstein-Dirac theory because the first term is equal to $-\tfrac{1}{2}eR$, where R is the scalar curvature of the three-metric $g_{\alpha\beta} = e_\alpha{}^i e_\beta{}^j\eta_{ij}$. The third term is anti-Hermitian due to our convention but does not affect the equations of motion for dynamical variables since this term is a total divergence. Thus our theory is real, except for a surface term, and corresponds to the Einstein-Cartan theory because of the existence of a quartic term of the spinor field owing to the use of the Palatini formalism.

3. The classical canonical structure

3.1 The space-time decomposition

We first carry out a 2+1 decomposition of the action to pass on to the Hamiltonian formalism. We assume that the space-time manifold M has a topology $M = \Sigma\times R$ with Σ being a compact two dimensional manifold. We introduce a time coordinate t on M so that M is foliated by spacelike two dimensional surfaces Σ_t each with the topology of Σ. One can define a timelike unit vector n^α with $n^\alpha n^\beta g_{\alpha\beta}=-1$ which is normal to Σ_t and a smooth time vector field t^α which is chosen such that $t^\alpha\nabla_\alpha t=1$. We then define the spatial metric $q_{\alpha\beta}$ by $q_{\alpha\beta}\equiv g_{\alpha\beta}+n_\alpha n_\beta$; $n^\alpha q_{\alpha\beta}=0$, the timelike part n^i of $e_\alpha{}^i$ by $n^i\equiv n^\alpha e_\alpha{}^i$ and the projected part of $e_\alpha{}^i$ into Σ_t by $E_\alpha{}^i\equiv e_\beta{}^i(g^\beta{}_\alpha+n^\beta n_\alpha)$; $n^\alpha E_\alpha{}^i=0$, $q_{\alpha\beta}=E_\alpha{}^i E_\beta{}^j\eta_{ij}$. The time vector t^α is decomposed into the lapse and shift fields N and N^α as $t^\alpha=Nn^\alpha+N^\alpha$; $n_\alpha N^\alpha=0$. The Levi-Civita density $\varepsilon^{\alpha\beta\gamma}$ is related to a density on Σ_t, $\varepsilon^{\alpha\beta}$ by $\varepsilon^{\alpha\beta\gamma}=3Nn^{[\alpha}\varepsilon^{\beta\gamma]}$. Finally we have to define the $\gamma_0\equiv\Gamma_0$ in the curved space-time to decompose the action (1.c). To preserve the properties of γ_0 that it corresponds to the time component of γ_i and satisfies $\gamma_0{}^2=-1$, we define such that $\Gamma_0\equiv n^\alpha\Gamma_\alpha=n^i\Gamma_i$.

Now we can obtain a 2+1 decomposed version of the action S_T. The gravitational part L_g is decomposed as

$$\begin{aligned} L_g &= 3t^{[\alpha}\varepsilon^{\beta\gamma]}(E_\alpha{}^i-n_\alpha n^i)F_{\beta\gamma i}/2 = (N^\alpha E_{\alpha i})F^i + (t^\beta F_{\beta\gamma i})\widetilde{E}^{\gamma i} + NF^i n_i \\ &= \widetilde{E}^{\alpha i}(\mathcal{L}_t\Lambda_{\alpha i}) + \Lambda_{0i}(D_\alpha\widetilde{E}^{\alpha i}) + N(n^iF_i) + N^\alpha(E_{\alpha i}F^i) - \partial_\alpha[\widetilde{E}^{\alpha i}\Lambda_{0i}] \\ &= \widetilde{E}^{\alpha i}(\mathcal{L}_t\Lambda_{\alpha i}) + \Lambda_{0i}(D_\alpha\widetilde{E}^{\alpha i}) + \underset{\sim}{N}(\tfrac{1}{2}\varepsilon^{ijk}\widetilde{E}^\alpha{}_i\widetilde{E}^\beta{}_jF_{\alpha\beta k}) + N^\alpha(\widetilde{E}^{\beta i}F_{\beta\alpha i}) \\ &\quad + (\text{surface term}), \end{aligned} \tag{3.a}$$

where $\widetilde{E}^{\alpha i}, \Lambda_0{}^i, F^i$, $\underset{\sim}{N}$ and $\mathcal{L}_t$ are a vector density on Σ_t with $\widetilde{E}^{\alpha i}\equiv\varepsilon^{\alpha\beta}E_\beta{}^i$, a time component of $A_\alpha{}^i$ with $A_0{}^i \equiv t^\alpha A_\alpha{}^i$, a space-space component (density weight +1) of $F_{\alpha\beta}{}^i$ with $F^i \equiv \frac{1}{2}\varepsilon^{\alpha\beta}F_{\alpha\beta}{}^i$, a lapse with density weight -1 and the Lie derivative by t^α. We have used the identity $\mathcal{L}_t A_\alpha{}^i = D_\alpha A_0{}^i + t^\beta F_{\beta\alpha}{}^i$. The spinor field part L_m is decomposed as

$$\begin{aligned} L_m &= E[\ -i\Psi^\dagger(\mathcal{L}_t\Psi) - i\Psi^\dagger(t\cdot A)\Psi + i\Psi^\dagger N^\alpha D_\alpha\Psi - iN\Psi^\dagger(n^\alpha\Gamma_\alpha)\Gamma^\beta D_\beta\Psi\ ] \quad (3.b)\\ &= -iE\Psi^\dagger(\mathcal{L}_t\Psi) - iE(\Psi^\dagger\tau_i\Psi)A_0{}^i + N^\alpha(iE\Psi^\dagger D_\alpha\Psi) + \underset{\sim}{N}(2iE\widetilde{E}^{\alpha i}\Psi^\dagger\tau_i D_\alpha\Psi), \end{aligned}$$

where $E \equiv \det(E_\alpha{}^i)$, $\mathcal{L}_t\Psi \equiv t^\alpha\partial_\alpha\Psi$, $t\cdot A \equiv t^\alpha A_\alpha{}^i\tau_i$, and we have used the relations $\tau_i = -\frac{i}{2}\gamma_i$ and $n^\alpha\Gamma_\alpha\Gamma^\beta = n^i E^{\beta j}\gamma_i\gamma_j = -\varepsilon_{ijk}n^i E^{\beta j}\gamma^k = E^{-1}\widetilde{E}^{\beta k}\gamma_k$. The canonical conjugate momentum of the spinor field Π which is defined by $\Pi \equiv \delta^L L_m/\delta(\mathcal{L}_t\Psi) = iE\Psi^\dagger$, where δ^L denotes the left derivative. Using by Π and two dimensional indices a,b...=1,2, S_T is expressed as

$$\begin{aligned} S_T = \int d^3x[&\widetilde{E}^{ai}(\mathcal{L}_t A_{ai}) + A_0{}^i(D_a\widetilde{E}^a{}_i) + N^a(\widetilde{E}^{bi}F_{bai}) + \underset{\sim}{N}(\tfrac{1}{2}\varepsilon^{ijk}\widetilde{E}^a{}_i\widetilde{E}^b{}_j F_{abk})\\ &-\Pi(\mathcal{L}_t\Psi) - A_0{}^i(\Pi\tau_i\Psi) + N^a(\Pi D_a\Psi) + \underset{\sim}{N}(\widetilde{E}^{ai}\Pi\tau_i D_a\Psi)]. \qquad (3) \end{aligned}$$

From the final form of the decomposed action (3) we can directly describe the Hamiltonian formalism. The gravitational (spinor field) configuration and canonical conjugate momentum variables are a spin connection A_{ai} and a vector density $\widetilde{E}^{ai}$ (Ψ and Π) on Σ_t, respectively. $A_0{}^i$, N^a and $\underset{\sim}{N}$ play the role of Lagrange multipliers. By the variations of (3) with respect to Lagrange multipliers we obtain three kinds of constraints

$$G^i \equiv -\delta S_T/\delta A_{0i} = -[(D_a\widetilde{E}^{ai}) - (\Pi\tau^i\Psi)] \simeq 0, \qquad (4.a)$$

$$V_a \equiv -\delta S_T/\delta N^a = -[(\widetilde{E}^{bi}F_{bai}) + (\Pi D_a\Psi)] \simeq 0, \qquad (4.b)$$

$$S \equiv -\delta S_T/\delta \underset{\sim}{N} = -[\tfrac{1}{2}\varepsilon^{ijk}(\widetilde{E}^a{}_i\widetilde{E}^b{}_j F_{abk}) + 2\widetilde{E}^{ai}(\Pi\tau_i D_a\Psi)] \simeq 0. \qquad (4.c)$$

The total Hamiltonian H_T can be expressed by these constraints as

$$H_T \equiv \int d^2x\ [\ A_{0i}G^i + N^a V_a + \underset{\sim}{N}S\ ]. \qquad (5)$$

G^i, V_a and S are called the Gauss-law, vector and scalar constraints, respectively. They are all first class constraints and there is no second class constrain. Note that they are composed of phase space field variables on the two dimensional Σ_t and manifestly polynomial in the basic canonical variables. This polynomial character is a feature in the Ashtekar formalism.

Although this constraint structure is similar to the 3+1 dimensional gravity,[1),7)] it is quite different from the 2+1 pure gravity.[6)] In the pure 2+1 dimensional gravity, the vector and scalar constraints are combined into a constraint $F^i \approx 0$ which is obtained from (3.a) using the relation such that $N^a(E_{ai}F^i)+N(n_iF^i)=(N^aE_{ai}+Nn_i)F^i=t_iF^i$. The Lagrange multipliers t_i are time components of $e_{\alpha i}$ defined by $t_i \equiv t^\alpha e_{\alpha i}$. This constraint is the generator of the translation in the ISO(2,1) Poincare group. Thus one can conclude that by the Gauss-law constraint G^i and F^i, the SO(2,1) gauge invariant flat connections constitute the physical configuration space and the theory is exactly solvable. As mentioned above, when matter fields are included in the 2+1 dimentional gravity, the vector and scalar constraints appear separately.

3.2 The canonical structure: constraint algebra

In this subsection we shall discuss the geometrical meaning and algebra of constraints. The fundamental nonvanishing Poisson brackets are

$$\{\tilde{E}^{ai}(x), \Lambda_{bj}(y)\} = \eta^i{}_j q^a{}_b \delta^2(x-y), \quad \{\Pi^A(x), \Psi_B(y)\} = \varepsilon^A{}_B \delta^2(x-y), \quad (6)$$

We define the Poisson brackets for spinor fields such that $\{f,g\} \equiv \int d^2z \, [(\delta^R f/\delta\Pi^A(z))(\delta^L g/\delta\Psi_A(z)) - (-1)^{fg}(\delta^R g/\delta\Pi^A(z))(\delta^L f/\delta\Psi_A(z))]$ for some functionals f and g. We use the two component spinor notation with antisymmetric metric ε_{AB} defined by $\varepsilon_{12}=+1$ such that $\Psi^A \equiv \varepsilon^{AB}\Psi_B$ and $\varepsilon^A{}_B \equiv \varepsilon^{AC}\varepsilon_{BC}$. To make expression in the calculation finite, we use the following constraints smeared with suitable well-defind fields on Σ,

$$G_N \equiv \int d^2x\, N^i G_i = -\int d^2x\, N^i [\, D_a \tilde{E}^a{}_i - \Pi \tau_i \Psi \,], \tag{7.a}$$

$$V_{\vec{N}} \equiv \int d^2x\, N^a V_a = -\int d^2x\, N^a [\, \tilde{E}^{bi} F_{bai} + \Pi D_a \Psi \,], \tag{7.b}$$

$$S_{\underset{\sim}{N}} \equiv \int d^2x\, \underset{\sim}{N} S = -\int d^2x\, \underset{\sim}{N} [\, \tfrac{1}{2}\epsilon^{ij k} \tilde{E}^a{}_i \tilde{E}^b{}_j F_{abk} + 2\tilde{E}^{ai}(\Pi \tau_i D_a \Psi) \,], \tag{7.c}$$

where N^i, N^a and $\underset{\sim}{N}$ are the smearing fields.

Now we discuss the infinitesimal canonical transformation properties of the fundamental phase space variables generated by (7.a,b,c) and the constraint algebra. The infinitesimal transformations of the field variables generated by the Gauss-law constraint (7.a) are $\{G_N, A_a{}^i\} = D_a N^i$, $\{G_N, \tilde{E}^{ai}\} = -f^i{}_{jk} N^j \tilde{E}^{ak}$, $\{G_N, \Psi_A\} = -N^i(\tau_i \Psi)_A$ and $\{G_N, \Pi^A\} = N^i(\Pi \tau_i)^A$, which are precisely the infinitesimal SO(2,1) Lorentz transformations. The Poisson brackets of the constraints with the Gauss-law constraint become $\{G_N, G_M\} = G_{[N,M]}$, $\{G_N, V_{\vec{M}}\} = 0$ and $\{G_N, S_{\underset{\sim}{M}}\} = 0$ with $[N,M]^i \equiv f^i{}_{jk} N^j M^k$.

The vector constraint is related to the infinitesimal coordinate transformation in the two dimensional space Σ_t but does not exactly generate it. The transformation of $A_a{}^i$ by $V_{\vec{N}}$ generates an extra term other than the coordinate transformation of $A_a{}^i$, i.e. $\{V_{\vec{N}}, A_a{}^i\} = N^b F_{ba} = (\mathcal{L}_N A_a{}^i) - D_a(N^b A_b{}^i)$. The vector constraint $V_{\vec{N}}$ corresponds to the Lie derivative in the gauge with $N^a A_a{}^i = 0$ which is achieved by the local Lorentz transformation. Therefore we can construct the constraint $D_{\vec{N}}$, called the diffeomorphism constraint which generates the infinitesimal coordinate transformation, as $D_{\vec{N}} \equiv V_{\vec{N}} + G_{N \cdot A} = \int d^2x [\tilde{E}^{ai}(\mathcal{L}_N A_{ai}) - \Pi^A(\mathcal{L}_N \Psi_A) - (\text{surface term})]$. By this constraint, the phase space variables are satisfactorily transformed as $\{D_{\vec{N}}, A_a{}^i\} = \mathcal{L}_N A_a{}^i$, $\{D_{\vec{N}}, \tilde{E}^{ai}\} = \mathcal{L}_N \tilde{E}^{ai}$, $\{D_{\vec{N}}, \Psi_A\} = \mathcal{L}_N \Psi_A$ and $\{D_{\vec{N}}, \Pi^A\} = \mathcal{L}_N \Pi^A$. For the Poisson brackets between $V_{\vec{N}}$ or $D_{\vec{N}}$ and other constraints, we have $\{V_{\vec{N}}, V_{\vec{M}}\} = -V_{\vec{Q}} - G_Q$, $\{V_{\vec{N}}, S_{\underset{\sim}{M}}\} = -S(\mathcal{L}_N \underset{\sim}{M}) - G_L$, $\{G_N, D_{\vec{M}}\} = G(\mathcal{L}_M N)$, $\{D_{\vec{N}}, D_{\vec{M}}\} = -D_{\vec{Q}}$ and $\{D_{\vec{N}}, S_{\underset{\sim}{M}}\} = -S(\mathcal{L}_N \underset{\sim}{M})$ with $\vec{Q} \equiv Q^a \equiv \mathcal{L}_N M^a = N^b \partial_b M^a - M^b \partial_b N^a$, $Q^i \equiv N^a M^b F_{ab}{}^i$, $\mathcal{L}_N \underset{\sim}{M} = N^a \partial_a \underset{\sim}{M} - \underset{\sim}{M} \partial_a N^a$ and $L^i \equiv \underset{\sim}{M} N^a (\epsilon^i{}_{jk} \tilde{E}^{bj} F_{ab}{}^k + 2\Pi \tau_i D_a \Psi)$. Note that the smearing fields Q^i and L^i in the Gauss-law constraints appeared in the right hand side of the algebra are structure <u>functions</u> depending on the gravitational and spinor fields.

The remaining contraint $S_{\underset{\sim}{N}}$ generates "time evolution" and it is also called the hamiltonian constraint. The transformations of fundamental fields by $S_{\underset{\sim}{N}}$ and the Poisson bracket of itself become $\{S_{\underset{\sim}{N}},\Lambda_a{}^i\}=\underset{\sim}{N}(\varepsilon^{ijk}\tilde{E}^b{}_j F_{bak}-2\Pi\tau_i D_a\Psi)$, $\{S_{\underset{\sim}{N}},\tilde{E}^{ai}\}=-D_b(\varepsilon^{ijk}\underset{\sim}{N}\tilde{E}^b{}_j\tilde{E}^a{}_k)-\underset{\sim}{N}[\varepsilon^{ijk}\tilde{E}^a{}_j(\Pi\tau_k\Psi)-\frac{1}{2}\tilde{E}^{ai}(\Pi\Psi)]$, $\{S_{\underset{\sim}{N}},\Psi_A\}=2\underset{\sim}{N}(\tilde{E}^a D_a\Psi)_A$, $\{S_{\underset{\sim}{N}},\Pi^A\}=2\partial_a[\underset{\sim}{N}(\Pi\tilde{E}^a)^A]-2\underset{\sim}{N}(\Pi\tilde{E}^a\Lambda_a)^A$ and $\{S_{\underset{\sim}{N}},S_{\underset{\sim}{M}}\}=-V_{\vec{K}}=-D_{\vec{K}}+G_{K\cdot A}$ with the structure <u>functions</u> $\vec{K}\equiv K^a\equiv(\underset{\sim}{N}\partial_b\underset{\sim}{M}-\underset{\sim}{M}\partial_b\underset{\sim}{N})\tilde{E}^{ai}\tilde{E}^b{}_i=(\underset{\sim}{N}\partial_b\underset{\sim}{M}-\underset{\sim}{M}\partial_b\underset{\sim}{N})\tilde{\tilde{q}}^{ab}$ and $(K\cdot\Lambda)^i\equiv K^a\Lambda_a{}^i$. Finally we finish this section by enumerating the constraint algebra related to the Gauss-law G_N, diffeomorphism $D_{\vec{N}}$ and scalar constraints $S_{\underset{\sim}{N}}$;

$$\{G_N,G_M\}=G_{[N,M]},\quad \{G_N,S_{\underset{\sim}{M}}\}=0,\quad \{G_N,D_{\vec{M}}\}=G(\mathcal{L}_{\vec{M}}N),\qquad (8.a,b,c)$$

$$\{D_{\vec{N}},D_{\vec{M}}\}=-D_{\vec{O}},\qquad \{D_{\vec{N}},S_{\underset{\sim}{M}}\}=-S(\mathcal{L}_{\vec{N}}\underset{\sim}{M}),\qquad (8.d,e)$$

$$\{S_{\underset{\sim}{N}},S_{\underset{\sim}{M}}\}=-V_{\vec{K}}=-D_{\vec{K}}+G_{K\cdot A}.\qquad (8.f)$$

Note that the structure <u>functions</u> (K^a and $K^a\Lambda_a{}^i$) appear only in the right hand side of the Poisson bracket between the scalar constraints itself and only depend on the gravitational phase space variable Λ_{ai} and $\tilde{E}^{ai}$. This situation is similar to the 3+1 pure[1] and matter coupled[7] gravity theories.

4. The classical observable

A physical obsevable at the classical level is a function on the phase space which commutes with all constraints in the theory. In the 2+1 pure gravity there are known classical physical observables,[6] for example,

$$T^0[\alpha]\equiv \mathrm{Tr}P[\exp(\oint du\dot{\alpha}^a(u)\Lambda_a{}^i(\alpha(u))\tau_i)]\equiv \mathrm{Tr}H(s,s+1)=H(s,s+1)_A{}^A,\quad (9)$$

$$T^1[\alpha]\equiv \oint ds\dot{\alpha}^a(s)\varepsilon_{ab}T^1[\alpha]^b\equiv \oint ds\dot{\alpha}^a(s)\varepsilon_{ab}\mathrm{Tr}[\tilde{E}^a(\alpha(s))H(s,s+1)],\quad (10)$$

Here $\alpha(s)$ is a loop in Σ with a basepoint at an arbitrary point $p=\alpha(s)$

of Σ, which can be obtained from any smooth mapping of S^1 into Σ (s is a parameter defined such that $s\in[0,1]$ and $\alpha(s)=\alpha(s+1)$). $\oint ds$ denotes a loop integral. $H(s,s+1)_A{}^B$ is a holonomy with respect to the connection $A_{aA}{}^B$.

Although we are treating the 2+1 dimensional theory, the observables such as T^0 and T^1 are no longer the physical one in the matter coupled theory. The canonical structure of our theory is similar to the 3+1 dimensional one in which T^0 and T^{1a} commute with the Gauss-law constraint but not with other constraints.[3] At present, the physical observables are unknown for the case of the spacelike compact Σ in the pure 3+1 general relativity. Therefore we have to study based on the program which has been used on the 3+1 dimensional pure gravity by Rovelli and Smolin.[3] The program in ref.3) is as follows, 1) construct the phase space by means of the objects which commute with the Gauss-law constraint, which hereafter we call the Gauss observables, 2) organize the algebra, if exists, and represent all constraints in terms of the Gauss observables, 3) seek out exact solutions (wave function) for the remaining constraints. The process of 2) and 3) has been done in the new representation so called the loop representation. In this talk we only treat the program 1) in this section.

The Gauss observables of the gravity part are same as in 3+1 pure case, that is, T^0 and T^{1a}. Here we do not discuss the problem of whether these observables completely cover the gauge invariant phase space of the gravity part. Thus we investigate the matter and gravity mixed parts of the Gauss observables and simply describe the result here.

There are many Gauss observables which are SO(2,1) gauge invariant. For <u>local type</u> we have already known gauge invariant objects in the gauge field theory such as $\Psi^A(x)\Psi_A(x)$, $\Psi D_a\Psi=\frac{1}{2}\partial_a(\Psi^A\Psi_A)$, $\Psi\tilde{E}^aD_a\Psi$, ...for the matter configuration part and $\Pi^A\Psi_A$, $\Pi D_a\Psi$, $\Pi\tilde{E}^a\Psi$, $\Pi\tilde{E}^aD_a\Psi$, ...for the momentum part where we used the property that $(\tau_i)_{AB}\equiv(\tau_i)_A{}^C\epsilon_{CB}$ is symmetric in its indices. Based on the above local type, one can obtain the line type which is obtained putting $H(s,t)_A{}^B$ with $\alpha(s)\neq\alpha(t)$ between spinor fields as $\Psi(\alpha(s))H(s,t)\Psi(\alpha(t))$, $\Psi H(s,t)D_a\Psi$, $\Psi H(s,u)\widetilde{E}^a(u)H(u,t)\Psi$, $\Pi H(s,t)\Psi$, $\Pi H D_a\Psi$, $\Pi H\widetilde{E}^a H\Psi$ and so on, where we used the transformation property of $H(u,t)_A{}^B$;[#1] $\{G_N, H(u,t)_A{}^B\}=[H(u,t)N(t)-N(u)H(u,t)]_A{}^B$. Although these observables are naturally considered as the extension of the minimal coupling between the spin connection and spinor fields, we have to abandon those as a candidate of the physical observable since these are not manifestly invariant under the diffeomorphism.

There are more interesting and nontrivial gravity coupled Gauss observables, that is, the loop type which is obtained by connecting two end points $\alpha(s)=\alpha(t)$ in the above line type. At present, it is not certain for us whether physical observables exist in this loop type. So we are studing hard on this problem, with a hope that it may be solvable by passing from the connection representaion to the loop representation. In the 3+1 pure gravity (and also in our theory), $T^0[\alpha]$ and $T^{1a}[\alpha]$ are not physical observables, but they are very useful for finding the quantum states and the physical observables in the roop representation.[3),8)] On the other hand, they have transformation properties under the diffeomorphism very similar to the loop type Gauss observables of spinor fields. For example, $T^0[\alpha]$ and $\Psi(\alpha(s))H\Psi(\alpha(s+1))$ transform as follows,

$$\{D_{\vec{N}},T^0\} \equiv \{D_N, H(s,s+1)_A{}^A\} = N^b(\alpha(s))\partial_b[H(s,s+1)_A{}^A] - \oint du\dot{\alpha}^a(u)N^b(\alpha(u))[H(s,u)F_{ab}(\alpha(u))H(u,s+1)]_A{}^A, \qquad (11.a)$$

and

$$\{D_{\vec{N}},\Psi[\alpha(s)]H(s,s+1)\Psi[\alpha(s+1)]\} = N^b(\alpha(s))\partial_b[\Psi H(s,s+1)\Psi] - \oint du\dot{\alpha}^a(u)N^b(\alpha(u))[\Psi H(s,u)F_{ab}(\alpha(u))H(u,s+1)\Psi], \qquad (11.b)$$

where $\partial_b\equiv\partial/\partial\alpha^b(s)$. Therefore, we use these loop type Gauss observables as the gauge invariant phase space variables instead of Ψ_A and Π^A.

Finally we note that there exists a trivial but <u>physical</u> observable, that is, the <u>fermion number</u> $\int d^2x \Pi^A(x)\Psi_A(x)$, which commutes with all the constraints. Because we are treating the theory on the spatially compact Σ and the Hamiltonian is a linear combination of constraints, the physical observable must be a constant of motion. Therefore it is understandable that the global quantities of the system like the fermion number become physical.

5. The quantum constraint (ordering) and the quantum state

To carry out the Dirac constrained quantization, we have to decide the order of operators in the quantum constraints. We set the ordering of the quantum constraints such that the Gauss-law and diffeomorphism (vector) constraints generate respectively the gauge and coordinate (coordinate in the gauge $N^a\Lambda_a{}^i=0$) transformations. For the scalar constraint we decide it such that the algebra of quantum constraints be consistent. Here we report only the results for the quantum constraints and the worrying constraint algebra,

$$\hat{G}_N \equiv \int d^2x N^i \hat{G}_{Ni} \equiv -\int d^2x N^i [(D_a\hat{E}^a{}_i)+(\tau_i\Psi)_A\hat{\Pi}^A], \tag{12.a}$$

$$\hat{D}_{\vec{N}} \equiv \int d^2x N^a \hat{D}_a \equiv \int d^2x [(\mathcal{L}_N\Lambda_{ai})\hat{E}^{ai}+(\mathcal{L}_N\Psi_A)\hat{\Pi}^A], \tag{12.b}$$

$$\hat{V}_{\vec{N}} \equiv \int d^2x N^a \hat{V}_a \equiv \int d^2x N^a [F_{abi}\hat{E}^{bi}+(D_a\Psi)_A\hat{\Pi}^{\Gamma}], \tag{12.b'}$$

$$\hat{S}_{\underset{\sim}{N}} \equiv \int d^2x \underset{\sim}{N}\hat{S} \equiv \int d^2x \underset{\sim}{N}[-\tfrac{1}{2}\epsilon^{ijk}F_{abk}\hat{E}^a{}_i\hat{E}^b{}_j+2(\tau_i D_a\Psi)_A\hat{\Pi}^A\hat{E}^{ai}], \tag{12.c}$$

$$[\hat{S}_{\underset{\sim}{N}},\hat{S}_{\underset{\sim}{M}}] = i\hbar\hat{V}_{\vec{K}} = -i\hbar\int d^2x \hat{V}_a\hat{K}^a \tag{13}$$

where $\hat{E}^a{}_i(x)\equiv -i\hbar\delta/\delta\Lambda_a{}^i(x)$, $\hat{\Pi}^A(x)\equiv -i\hbar\delta/\delta\Psi_A(x)$ and $\hat{K}^a\equiv(\underset{\sim}{N}\partial_b\underset{\sim}{M}-\underset{\sim}{M}\partial_b\underset{\sim}{N})\hat{E}^{bi}\hat{E}^a{}_i$. Note the order of $\hat{V}_a$ and the structure functional $\hat{K}^a$ in the right hand side of the constraint algebra (13).

Now we discuss the quantum states (wavefunction) $\Phi(\Lambda,\Psi)$ that satisfy the WDW equation $\hat{S}\Phi(\Lambda,\Psi)=0$ and briefly report the results obtained up to the present. We have to seek the solutions from the gauge invariant configuration variables (configuration Gauss observables). Note that the 2+1 dimensional versions of the solutions obtained in the 3+1 pure gravity are also exact solutions in the 2+1 matter coupled gravity. If any functional, which purely consists of the connections $A_a{}^i$, is annihilated by the pure gravity part of the scalar constraint operator (12.c), then it becomes an exact solution of (12.c) since it is also annihilated by the remaining matter part.

As one of the nontrivial exact solutions, we can obtain the $T^{0f}[\alpha] \equiv \mathrm{Tr}H^f(s,s+1)$ where $H^f(s,s+1)$ is the smeared holonomy with the one-dimensional smearing density $f(\sigma)$ which approaches the delta function. See ref.2) for the detailed definition. Although this quantity only consists of $A_a{}^i$ and the gravitational local degrees of freedom are not exist in 2+1 dimensions, note that $A_a{}^i$ is determined by the matter fields through the constraints. There is a matter dependent but trivial solution as $\Psi^A(x)\Psi_A(x)$. This solution corresponds to the eigenfunction for the trivial physical observable $\int d^2x\Psi_A(x)\hat{\Pi}^A(x)$ described in sec.4.

As nontrivial matter dedendent solutions, we can consider the $\Psi^A(\alpha(s))H^f(s,s+1)_A{}^B\Psi_B(\alpha(s+1))$ but unfortunately this quantity does not satisfy the quantum scalar constraint equation as follows,

$$\begin{aligned}
&\hat{\underset{\sim}{S}}_N[\Psi^A(\alpha(s))H(s,s+1)_A{}^B\Psi_B(\alpha(s+1))]=\hat{\underset{\sim}{S}}_N(\text{matter part})[\Psi H\Psi] \\
&\quad =(3/2)J^{-1}(s)f(s)N(s)\dot{\alpha}^a(s)[(\partial_a\Psi^A)H(s,s+1)_A{}^B\Psi_B-\Psi^A H(s,s+1)_A{}^B(\partial_a\Psi_B) \\
&\qquad -\Psi^A\{A_a(\alpha(s))H(s,s+1)\}_A{}^B\Psi_B-\Psi^A\{H(s,s+1)A_a(\alpha(S+1))\}_A{}^B\Psi_B] \\
&\quad \neq 0, \qquad (14)
\end{aligned}$$

where J(s) is the Jacobian when one changes the coordinates $\alpha(S,\sigma)$ to the coordinates (s,σ) and $\partial_a \equiv \partial/\partial\alpha^a(S)$. Therefore, we must find the solutions from the more comlicated configuration Gauss observables presented in sec.4. Note that above solutions satisfy the WDW equation but are not physical states since both solutions are not invariant with respect to the diffeomorphism. We hope that one can solve this problem by passing from the connection representation to the loop representation such as in the 3+1 pure gravity case.[3]

6. Discussion

In this talk we have discussed the 2+1 dimensional quantum gravity coupled with spinor fields in the Palatini formalism. We have shown that the canonical structure of the theory, which is equivalent to the Einstein-Cartan theory in 2+1 dimensions, is similar to the case of the pure[1] and spinor coupled[7] gravity systems in 3+1 dimensions.

We have presented the Gauss observables which can be used as the gauge invariant phase space variables of the theory, instead of the fundamental phase space variables of the gravitational and spinor fields. We have also presented a trivial but physical observable which corresponds to the fermion number on the two dimensional compact space.

We have discussed the solutions of the quantum Gauss-law constraint and WDW equation. The trace of holonomy of the spin connection is also a nontrivial exact solution in our theory as has already been shown to be the case for the pure gravity in 2+1[6] and 3+1[2] dimensions. Up to the present, we have not found nontrivial solutions with respect to the spinor fields but presented a trivial solution that corresponds to the eigenfunction of the physical observable (fermion number).

Finally we comment on the immediate tasks. First we would like to find the nontrivial, fundamental solutions of the WDW equation from the loop type matter dependent configuration Gauss observables. Second we have to investigate those algebraic relations to see whether they become the solutions or not for the WDW equation. Then we want to investigate the quantum theory by passing from the connection representation to the loop representation. We note that the obtainable results in our theory are almost applicable to the spinor coupled system in 3+1 dimensions.

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Footnote

#1. We would like to thank Yoshiaki Tanii for useful comments on the explicit calculation for this transformation property.

Wormhole-induced Vertex Operators †

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1 Introduction

In the Euclidean path integral formulation of the quantum gravity, one of the most important issues is to include topology changing processes and to clarify physical effects of these processes. A wormhole is a Euclidean spacetime configuration which has two asymptotically flat regions connected by a narrow tube[2,3], and one of the simplest examples of topology changing processes. Recent development on the wormhole effects[4,5] shows that this topology-changing processes in quantum gravity may give dramatic effects on low energy physics.

Most of discussions on effects of wormholes, including a vanishing cosmological constant mechanism and the Big Fix, are based on the argument that the effects of microscopic wormholes are represented by insertions of bilocal operators at the low energy scale[5]. A form of the wormhole-induced bilocal operators is simply assumed in the previous works[4,5,6]. However, it is necessary for detailed discussions on the effects of wormholes to calculate explicitly what kinds of operators are induced in a given model[6].

We perform an explicit calculation of wormhole-induced bilocal operators, taking a massless scalar field coupled to the Einstein gravity. As explained below, it is sufficient for our present purpose to calculate the asymptotic behavior of a scalar field Green's function on the wormhole background. In the conformal coupling case[7] the Green's function has been analytically obtained by using a conformal transformation. However, for general massless scalar field, especially the minimally coupled case, the analytic solution is not

†This work is done with Yasuhiro Okada (Tohoku University) [1].

found. In such a case, we evaluate the Green's function numerically by reducing to the one-dimensional potential barrier problem in quantum mechanics, and show that the long distance effect of the wormhole can be represented by the bilocal operators.

2 Wormhole Solution and Background Operator

As a preliminary to our calculation, we introduce the axionic wormhole solution and give a general procedure to obtain the wormhole-induced operators. Our calculations are based on the Euclidean path integral formulation of quantum gravity and the validity of the semiclassical approximation is assumed. Thus, the path integral is supposed to be dominated by wormhole solutions which satisfy the classical Euclidean Einstein equation.

We use the axionic wormhole solution[3] which is found in a model of a $U(1)$ Nambu-Goldstone boson field coupled to gravity. The Euclidean action is given as

$$S_E = \int d^4x\sqrt{g}\left\{\frac{1}{16\pi G}R + \frac{1}{2}\nabla^\mu\theta\nabla_\mu\theta\right\} + iQ(\theta_F - \theta_I), \tag{1}$$

where Q is a $U(1)$ charge. The last term of (1) is derived from the treatment of global charge conservation law which is described in Ref. [8], and the boundary condition for the equations of motion is obtained from this term. When we put a spherically symmetric Ansatz on the metric and θ as

$$ds^2 = d\tau^2 + a^2(\tau)d\Omega^2_{(3)}, \qquad \theta = \theta(\tau), \tag{2}$$

where $d\Omega^2_{(3)}$ is a line element on a unit three sphere, the field equation of θ and its boundary conditions become

$$\begin{aligned} \frac{d}{d\tau}\left(a^3(\tau)\dot\theta(\tau)\right) &= 0, \\ 2\pi^2 a^3\dot\theta\Big|_{\tau=\tau_I\to-\infty} &= 2\pi^2 a^3\dot\theta\Big|_{\tau=\tau_F\to\infty} = -iQ, \end{aligned} \tag{3}$$

and the (00)-component of the Einstein equation is

$$\dot a^2(\tau) = 1 - \frac{a_0^4}{a^4}, \qquad a_0^4 = 16\pi G Q^2. \tag{4}$$

The solution of these equations is obtained analytically:

$$\frac{|\tau|}{a_0} = \frac{1}{\sqrt{2}}F\left(\cos^{-1}\frac{a_0}{a},\frac{1}{\sqrt{2}}\right) - \sqrt{2}E\left(\cos^{-1}\frac{a_0}{a},\frac{1}{\sqrt{2}}\right) + \frac{a}{a_0}\sqrt{1-\frac{a_0^4}{a^4}}, \tag{5}$$

$$\theta(\tau) = -\frac{iQ}{4\pi^2 a_0^2}\sin^{-1}\frac{a_0^2}{a^2(\tau)} \equiv \bar{\theta}(\tau), \tag{6}$$

where F and E are elliptic integrals of the first and second kinds, respectively, and a_0 is a radius of the wormhole neck. Asymptotic forms of $a(\tau)$ and $\bar{\theta}(\tau)$ for $\tau \to \pm\infty$ are written as

$$\frac{a(\tau)}{a_0} \sim \frac{|\tau|}{a_0} + K + O\left(\left|\frac{a_0}{\tau}\right|^3\right), \tag{7}$$

$$\bar{\theta}(\tau) \sim -\frac{iQ}{4\pi^2}\frac{1}{\tau^2} + O\left(\left|\frac{a_0}{\tau}\right|^6\right). \tag{8}$$

where $K \equiv \sqrt{2}E(\frac{\pi}{2},\frac{1}{\sqrt{2}}) - \frac{1}{\sqrt{2}}F(\frac{\pi}{2},\frac{1}{\sqrt{2}}) \approx 0.59907\cdots$ is a numerical constant.

To obtain wormhole-induced operators, we follow the general procedure described in Ref. [7]. First, we take a wormhole solution as a fixed background metric, and calculate an expectation value of local operators inserted in asymptotically flat regions:

$$\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\ \mathcal{O}(x_1')\cdots\mathcal{O}(x_m')\rangle_W, \tag{9}$$

where $\mathcal{O}$ is a local operator, $x_1,\cdots,x_n$ are coordinates of the points in one asymptotic region, $x_1',\cdots,x_m'$ are those in another asymptotic region, and $\langle\cdots\rangle_W$ means that the expectation value is calculated on the wormhole background. Next, we obtain a bilocal operator $C^{ij}\Phi_i(x_0)\Phi_j(x_0')$ which reproduces the expectation value (9) as a product (or a sum of products) of expectation values on two flat backgrounds, i.e. $C^{ij}\Phi_i(x_0)\Phi_j(x_0')$ satisfies

$$\begin{aligned}&\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\ \mathcal{O}(x_1')\cdots\mathcal{O}(x_m')\rangle_W \\ &\approx\ \mathrm{e}^{-S_W}C^{ij}\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\Phi_i(x_0)\rangle_F\langle\Phi_j(x_0')\mathcal{O}(x_1')\cdots\mathcal{O}(x_m')\rangle_F,\end{aligned} \tag{10}$$

where $\langle\cdots\rangle_F$ stands for an expectation value on the flat background, x_0 and x_0' are the coordinates of the wormhole ends in two flat spaces and S_W is the action of the wormhole solution. We then identify the bilocal operator $e^{-S_W} C^{ij} \Phi_i(x_0)\Phi_j(x_0')$ as the operator induced by the wormhole solution.

Since the background value $\bar{\theta}(\tau)$ is not vanishing, the expectation values of a product of θ's on the wormhole background is written as

$$\langle \theta(x_1)\cdots\theta(x_n)\theta(x_1')\cdots\theta(x_m')\rangle_W \approx e^{-S_W}\bar{\theta}(x_1)\cdots\bar{\theta}(x_n)\bar{\theta}(x_1')\cdots\bar{\theta}(x_m'), \qquad (11)$$

in the semiclassical approximation. This expectation value for any n and m is reproduced by the bilocal operator $e^{-S_W} e^{iQ\theta(x_0)} e^{-iQ\theta(x_0')}$ as described in Ref. [7]:

$$\begin{aligned}&\langle \theta(x_1)\cdots\theta(x_n)\theta(x_1')\cdots\theta(x_m')\rangle_W \\ \approx\ & e^{-S_W}\langle \theta(x_1)\cdots\theta(x_n)e^{iQ\theta(x_0)}\rangle_F \langle \theta(x_1')\cdots\theta(x_m')e^{-iQ\theta(x_0')}\rangle_F. \end{aligned} \qquad (12)$$

When other matter fields are included, whole induced operator is written in a form as

$$e^{-S_W} C^{ij} e^{iQ\theta(x_0)}\Phi_i(x_0) e^{-iQ\theta(x_0')}\Phi_j(x_0'). \qquad (13)$$

Here, the bilocal operator $C^{ij}\Phi_i(x_0)\Phi_j(x_0')$ is determined by expectation values which do not include the Nambu-Goldstone boson field θ. We calculate this operator for a massless scalar field.

3 Green's Function of a Minimally Coupled Scalar Field

Now, let us consider matter fields other than the Nambu-Goldstone boson field θ. Here, we consider a massless scalar field. The Euclidean action of a massless scalar φ is given as

$$S_{\text{matter}} = \frac{1}{2}\int d^4x\sqrt{g}\left(\nabla^\mu\varphi\nabla_\mu\varphi - \eta R\varphi^2\right), \qquad (14)$$

where $\eta = \frac{1}{6}$ corresponds to the conformal coupling and $\eta = 0$ to the minimal coupling. We will calculate the Green's function of φ on the wormhole background. To obtain the Green's function $G(x,x') = \langle\varphi(x)\varphi(x')\rangle_W$, we have to solve the equation

$$\left(\nabla^\mu\nabla_\mu + \eta R\right) G(x,x') = -\frac{1}{\sqrt{g}}\delta(x,x'). \qquad (15)$$

In the conformal coupling case, the calculation of the Green's function is reduced to that on the flat background by the conformal transformation[7]. However, if $\eta \neq \frac{1}{6}$ we cannot use this technique. Therefore we need a more general procedure to obtain the Green's function, which is described below.

If we obtain a complete set of eigenfunctions $\varphi_k^{(\sigma)}(x)$ which satisfies the eigenvalue equation

$$(\nabla^\mu \nabla_\mu + \eta R)\,\varphi_k^{(\sigma)}(x) = -k^2 \varphi_k^{(\sigma)}(x), \tag{16}$$

where k^2 is the eigenvalue and σ represents other quantum numbers collectively, and the completeness condition

$$\sum_\sigma \int_0^\infty \frac{dk}{2\pi} \varphi_k^{(\sigma)}(x) \varphi_k^{*(\sigma)}(x') = \delta(x, x'), \tag{17}$$

the Green's function is written in terms of $\varphi_k^{(\sigma)}(x)$ as

$$G(x, x') = \sum_\sigma \int_0^\infty \frac{dk}{2\pi} \frac{1}{k^2} \varphi_k^{(\sigma)}(x) \varphi_k^{*(\sigma)}(x'). \tag{18}$$

We solve the equation (16) first, and then calculate the Green's function using the formula (18).

In our wormhole background, the equation (16) is written as

$$\left\{ \frac{\partial^2}{\partial \tau^2} + \frac{3\dot{a}}{a}\frac{\partial}{\partial \tau} + \frac{\nabla_{(3)}^2}{a^2} + \frac{6\eta a_0^4}{a^6} \right\} \varphi_k^{(\sigma)}(x) = -k^2 \varphi_k^{(\sigma)}(x), \tag{19}$$

where $\nabla_{(3)}^2$ is a Laplacian on a unit three sphere. The angular part of the eigenfunction is separated by the spherical harmonics Y: $\nabla_{(3)}^2 Y_{lm_1m_2}(\Omega) = -l(l+2) Y_{lm_1m_2}(\Omega)$, $|m_2| \leq m_1 \leq l = 0, 1, 2, \cdots$, and we define the radial eigenfunction χ as

$$\varphi_{k,lm_1m_2}^{(\sigma)}(\tau, \Omega) = \frac{1}{a^{3/2}} \chi_{kl}^{(\sigma)}(\tau) Y_{lm_1m_2}(\Omega). \tag{20}$$

$\chi_{kl}^{(\sigma)}(\tau)$ satisfies the equation

$$\left\{ \frac{d^2}{d\tau^2} + k^2 - V_l(\tau) \right\} \chi_{kl}^{(\sigma)}(\tau) = 0,$$
$$V_l(\tau) = \frac{(l+1)^2 - \frac{1}{4}}{a^2(\tau)} + \frac{(\frac{9}{4} - 6\eta) a_0^4}{a^6(\tau)}, \tag{21}$$

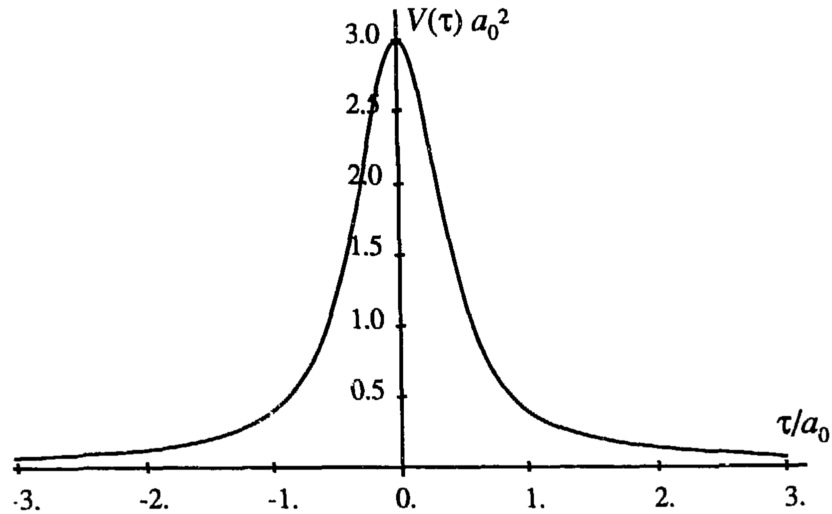

Figure 1: The potential barrier $V(\tau)$ for $l = 0$ in the minimal coupling case.

and the Green's function is written as

$$\begin{aligned} G(\tau,\Omega;\tau',\Omega') &= \sum_{\sigma}\sum_{l,m_1,m_2}\int_0^\infty \frac{dk}{2\pi}\frac{1}{k^2}a^{-3/2}(\tau)a^{-3/2}(\tau') \\ &\quad\times\chi_{kl}^{(\sigma)}(\tau)\chi_{kl}^{*(\sigma)}(\tau')Y_{lm_1m_2}(\Omega)Y^*_{lm_1m_2}(\Omega'). \end{aligned} \tag{22}$$

The equation (21) has the same structure as the Schrödinger equation in one dimension, with the potential $V_l(\tau)$ which is shown in Fig. 1. Consequently the problem is reduced to a one-dimensional potential barrier problem in quantum mechanics.

Since the relative angles between the angular coordinates in one asymptotic region and those in the other asymptotic region have no physical meaning, we have to average over the relative angles. In the present case we integrate over Ω' with Ω fixed in equation (22). The averaged Green's function $\bar{G}$ becomes

$$\begin{aligned} \bar{G}(\tau,\tau') &\equiv \int d^3\Omega' G(\tau,\Omega;\tau',\Omega') \\ &= \sum_{\sigma}\int_0^\infty \frac{dk}{2\pi}\frac{1}{k^2}a^{-3/2}(\tau)a^{-3/2}(\tau')\chi_{k0}^{(\sigma)}(\tau)\chi_{k0}^{*(\sigma)}(\tau'). \end{aligned} \tag{23}$$

Notice that the contributions from $l \neq 0$ terms all vanish under this averaging, and this is the reason why the induced operator becomes rotationary invariant. In the following discussion we solve the equation (21) for $l = 0$ only.

Equation (21) has two independent solutions $\chi_k^{(R)}(\tau)$ which has an asymptotic form

$$\chi_k^{(R)}(\tau) \sim \begin{cases} e^{ik\tau} + r(k)e^{-ik\tau} & \text{for } \tau \to -\infty, \\ t(k)e^{ik\tau} & \text{for } \tau \to \infty, \end{cases} \tag{24}$$

and $\chi_k^{(L)}(\tau) \equiv \chi_k^{(R)}(-\tau)$. We are interested in the asymptotic behavior of the Green's function $\bar{G}(\tau,\tau')$ for $\tau \to \infty$ and $\tau' \to -\infty$. It turns out that the asymptotic form (24) is not enough since we need $\chi_k^{(R,L)}(\tau)$ in a wider range of k for a fixed τ, and (24) is not applicable for $k|\tau| \lesssim 1$. The asymptotic form of $\chi_k^{(R)}(\tau)$ for $|\tau| \gg a_0$ which is a better approximation for any k is

$$\begin{aligned} \chi_k^{(R)}(\tau) \ \sim\ & \sqrt{\frac{\pi k(|\tau| + Ka_0)}{2}} \left\{ H_1^{(2)}\left(k(|\tau| + Ka_0)\right) + \tilde{r}(k) H_1^{(1)}\left(k(|\tau| + Ka_0)\right) \right\} \\ & \qquad \text{for } \tau \to -\infty, \\ & \sqrt{\frac{\pi k(|\tau| + Ka_0)}{2}} \tilde{t}(k) H_1^{(1)}\left(k(|\tau| + Ka_0)\right) \qquad \text{for } \tau \to \infty, \end{aligned} \tag{25}$$

where $H_1^{(1)}$ and $H_1^{(2)}$ are the Hankel function of the first and second kind, respectively. This asymptotic form is equivalent to (24) for $k|\tau| \gg 1$ and the relations between $r(k)$, $t(k)$ and $\tilde{r}(k)$, $\tilde{t}(k)$ are

$$\tilde{r}(k) = -ie^{-i2Kka_0} r(k), \qquad \tilde{t}(k) = -ie^{-i2Kka_0} t(k). \tag{26}$$

Substituting (25) into the formula (23), we obtain the asymptotic form of the Green's function $\bar{G}(\tau,\tau')$ for $\tau \to \infty$ and $\tau' \to -\infty$ as

$$\begin{aligned} \bar{G}(\tau,\tau') \ \sim\ & \frac{1}{4\pi^3 z^{3/2} z'^{3/2}} \int_0^\infty \frac{dk}{k} \frac{\pi\sqrt{zz'}}{2} \\ & \times \left\{ \tilde{t}(k) H_1^{(1)}(kz) H_1^{(1)}(kz') + \tilde{t}^*(k) H_1^{(2)}(kz) H_1^{(2)}(kz') \right. \\ & \quad \left. + \left[\tilde{t}(k)\tilde{r}^*(k) + \tilde{r}(k)\tilde{t}^*(k) \right] H_1^{(1)}(kz) H_1^{(2)}(kz') \right\}, \\ & \tau \to \infty, \quad \tau' \to -\infty, \end{aligned} \tag{27}$$

where $z = \tau + Ka_0$ and $z' = -\tau' + Ka_0$. The last term in (27) vanishes because of the symmetric property of the potential $V(-\tau) = V(\tau)$. Therefore, if $\tilde{t}(k)$ (or equivalently, $t(k)$) for $0 < k < \infty$ is given, we can calculate the Green's function $\bar{G}(\tau,\tau')$.

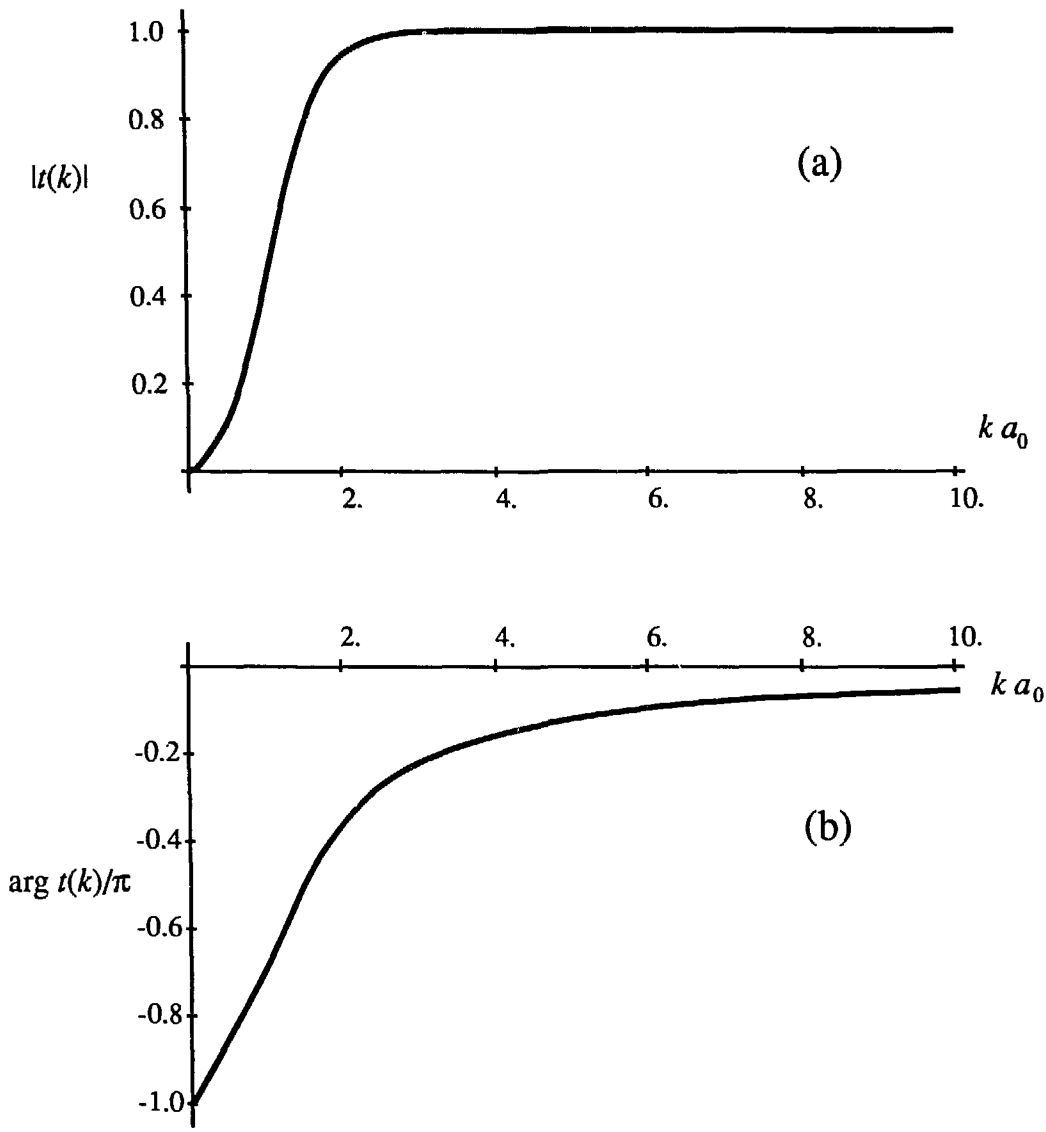

Figure 2: The numerical values of $t(k)$; (a) absolute value; (b) argument.

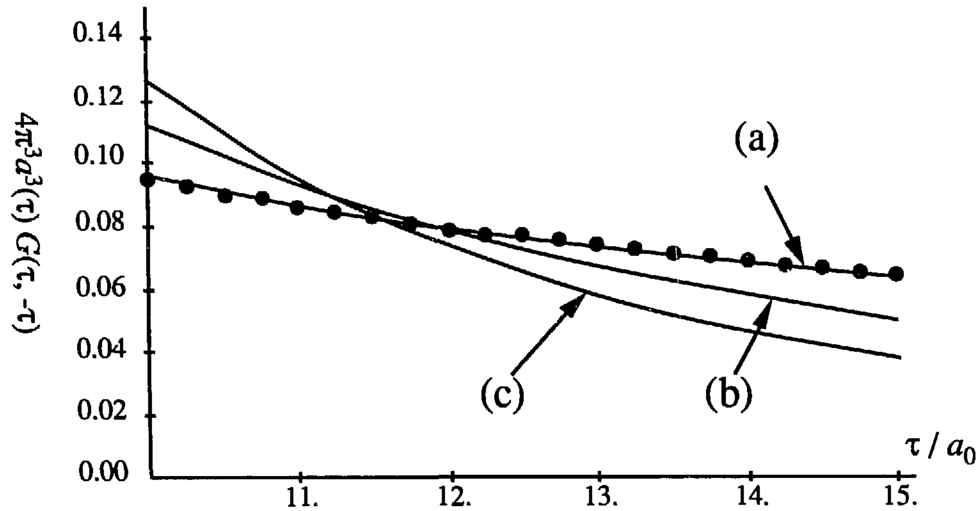

Figure 3: The numerical values of $4\pi^3 a^3(\tau)\bar{G}(\tau,-\tau)$ in the minimal coupling case and their curve fit by powers of τ; (a) τ^{-1}; (b) τ^{-2}; (c) τ^{-3}.

We evaluate $t(k)$ by a numerical method, and the results are given in Fig. 2. We see that $t(k)$ approaches to 1 for large k. For small k, $\tilde{t}(k)$ is proportional to k^2 with a pure imaginary coefficient, and this factor suppresses the contributions from $k \lesssim O(1/|\tau|)$ region in (27). This can be understood physically: a fluctuation mode with large wavelength cannot go through a small wormhole.

Using the numerical values of $t(k)$ (or $\tilde{t}(k)$), we calculate the Green's function $\bar{G}(\tau,\tau')$ for $\tau \to \infty$ and $\tau' \to -\infty$. Fig. 3 represents the numerical values of the Green's function $\bar{G}(\tau,-\tau)$ for large τ and their curve fits. We see from the result that the Green's function $\bar{G}(\tau,-\tau)$ is proportional to $a_0^2/|\tau|^4$, i.e.

$$G(\tau,\tau') \sim c_{\text{minimal}} \frac{a_0^2}{|\tau|^2|\tau'|^2} \qquad \text{for} \quad \begin{cases} \tau \to \infty \\ \tau' \to -\infty \end{cases}, \tag{28}$$

and the coefficient c_{minimal} is approximately 1/130.

This τ-dependence is reproduced by a bilocal operator $\sim \varphi(x_0)\varphi(x_0')$ where x_0 and x_0' are the coordinates of the wormhole ends in two flat spaces:

$$\bar{G}(\tau,\tau') \sim \langle\varphi(x)\varphi(x_0)\rangle_F \langle\varphi(x')\varphi(x_0')\rangle_F, \tag{29}$$

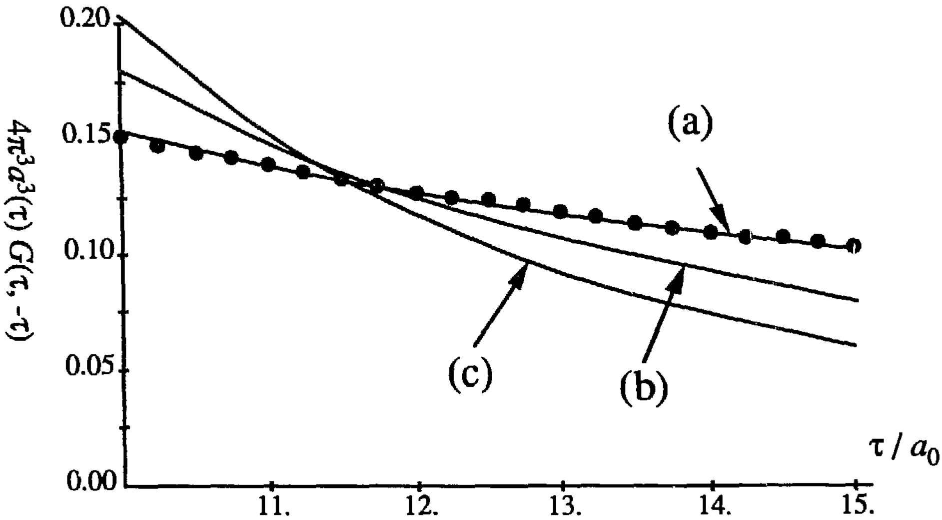

Figure 4: The numerical values of $4\pi^3 a^3(\tau)\bar{G}(\tau,-\tau)$ in the conformal coupling case and their curve fit by powers of τ; (a) τ^{-1}; (b) τ^{-2}; (c) τ^{-3}.

since the scalar propagator on the flat background is written as

$$\langle\varphi(x)\varphi(x_0)\rangle_F = -\frac{1}{4\pi^2|x-x_0|^2} \sim \frac{1}{\tau^2}, \tag{30}$$

for $\tau \to \infty$.

As a check of our calculation, we apply the method described above to the conformal coupling case. In this case, the Green's function is obtained analytically[7], and the asymptotic form is

$$\frac{1}{8\pi^2}\frac{a_0^2}{|\tau|^2|\tau'|^2}. \tag{31}$$

Our numerical calculation in this case is the same as before except that η is now set to 1/6 in the eigenvalue equation (21). The numerical values of the Green's function are shown in Fig. 4. We see that the asymptotic form of the Green's function is

$$\sim \frac{1}{81}\frac{a_0^2}{|\tau|^2|\tau'|^2}, \tag{32}$$

which is a good approximation of the analytic result (31).

4 Summary

We have evaluated a Green's function of a minimally coupled scalar field on a wormhole background. The Green's function is written in terms of the eigenfunctions and eigenvalues of the Laplacian operator with a spherical symmetry, thus the problem being reduced to a one-dimensional potential barrier problem in quantum mechanics, and is finally solved by a numerical calculation. We have shown that the asymptotic form of the Green's function connected by a wormhole has the τ-dependence $\sim 1/|\tau|^2|\tau'|^2$. This τ-dependence is interpreted as an appearance of a pair of local operators in the wormhole ends, $\sim \varphi(x_0)\varphi(x_0')$.

The procedure we used in this paper may be extended to calculate a Green's function of a field with higher spin, in which case the angular variables are separated with appropriate (spinor, vector, tensor, $\cdots$) harmonics and the eigenvalue equation is reduced to one-dimensional potential barrier problem just as the calculation of the scalar Green's function. Averaging over the relative angles between the two asymptotic regions is rather complicated, however.

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Lectures on the Matrix-Model Approach to 2D Gravity [1]

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Part 1 Introduction to the Matrix Model Approach to 2D Gravity

1. Historical background
2. Essentials of the double-scaling limit
3. Formal structure of the matrix model

Part 2 Toward a Canonical Formalism of 2D Gravity [2]

1. The phase space representation of the algebra of Psd operators
2. Action principle and its symmetry
3. Bilinear string equation
4. A derivation of the Schwinger-Dyson equation
5. Discussions

[1]Lecture given at the Institute for Nuclear Study, University of Tokyo, February, 1991.
[2]For the contents of part 2, please refer to preprint UT-Komaba 90-8.

Foreword

In part 1 of this lecture, I have given an elementary introduction to the recent developments of the matrix-model approach to the theories of strings and 2D gravity. In part 2, my own attempt toward a unified description of the structure of the continuum non-perturbative theory of 2D gravity using a simple action principle has been presented. Since a fuller account of the content of part 2 has already been given in my recent preprint (Preprint, UT-Komaba 90-8, "**Toward a Canonical Formalism of Non-Perturbative Two-Dimensional Gravity**"), I shall only give a brief summary of the part 1 of the lecture in this report. Section 1.3 presents a discussion on the action principle in the matrix model before taking the scaling limit, which has not been published elsewhere.

1 Introduction to the matrix model approach to 2D gravity

1.1 Historical background

Historically, the motivation for studying the large N limit of matrix field theories came from t'Hooft's observation [1] in 1974 that the Feynman diagrams in QCD are dominated by the planar graphs in the limit in which the number of colors increases indefinitely. The dominance of the planar diagrams conforms to, at least qualitatively, the properties of hadronic interactions, especially, with its string picture with confined quarks. In fact, it is formally possible to rewrite QCD entirely as a sort of string field theory[2]. However, unlike the case of σ models (or spin models in statistical mechanics), QCD is not solvable in this limit. This led Brezin et. al. [3] to study in detail the toy models in 'zero' and one dimensional spacetimes, i. e., defined on a point and a line, respectively. Among many interesting results derived by them, a crucial observation is that in general the matrix models exhibit critical behaviors which are analogous to the critical behaviors known in statistical models in the thermodynamic limit.

Let us therefore consider qualitatively how such a critical behavior arises in the simplest case of the one-matrix model. The model is given by the following integral

$$Z = \int d^{N^2}M \exp -\beta U(M) \tag{1.1}$$

$$\propto \int \prod_{i=1}^{N} d\lambda_i \prod_{i<j}(\lambda_i - \lambda_j)^2 \exp -\beta \sum_{i=1}^{N} U(\lambda_i), \tag{1.2}$$

where $U(M) = \mathrm{Tr}(M^2 + g_4 M^4 + \cdots)$. The prefactor in front of the exponential comes from the Jacobian in the change of the integration variables from the components of the matrix M to its eigenvaues $\{\lambda_i\}$ Including this Jacobian factor in the action, we have a system with the following effective action,

$$S_{\text{eff}} = \sum_{i=1}^{N} U(\lambda_i) - \frac{1}{\beta}\sum_{i\neq j} \ln|\lambda_i - \lambda_j|. \tag{1.3}$$

In the limit of large β, this system can be approximated as a classical system of N particles, interacting with a repulsive 2-body potential $-\beta^{-1}\ln|\lambda_i - \lambda_j|$, in an external potential $U(\lambda)$. The two contributions are of equal order of magnitude provided that $N \sim \beta$. The average distance between adjacent particles is of the order β^{-1}, and hence the distribution of particles in the large N limit becomes continuous and of finite range. However, depending on the form of the potentials, there can occur one or more gaps in the particle distributions. As an example, suppose that the external potential $U(\lambda)$ has several wells. When $\beta^{-1}N$ is sufficiently small, there is a stable configuration in which all of the particles reside within one of the wells. But as β^{-1} increases, the range of the distribution increases and at some critical point $\beta \simeq N$, the particles suddenly spill over the ridge. Hence we expect a singular behavior of the distribution function at this point. The nature of the singularities does not depend on the detailed form of the potential except at the vicinity of the top of the ridge, and the singularity types may be classified into universal classes.

Let us now consider the meaning of the above qualitative property in terms of Feynman diagrams. To be specific, we consider the model with only a quartic term in the interaction potential. Then the partition function Z has the expansion,

$$\ln Z = \sum_{\text{connected Feynman diagrams}} \beta^{-P+V}(-g_4)^V N^L \tag{1.4}$$

where P,V,and L are the numbers of, respectively, propagators, vertices, and index loops and satisfy $2P = 4V, \chi$ (Eulernumber) $= V - P + L$. Here the Euler number is the one with respect to the closed surfaces consisting of the surface elements corresponding the index loops. Thus,

$$\ln Z = \sum_{\text{surfaces}} N^{\chi}(-\frac{N}{\beta}g_4)^V. \tag{1.5}$$

This expression ca. be regarded as a discretized version of the partition function of two-dimensional gravity

$$\sum_{\text{dualsurfaces}} \exp[-\Lambda \int \sqrt{g}d^2\xi + \frac{\ln N}{2\pi}\int R\sqrt{g}d^2\xi] \tag{1.6}$$

where the 'bare' cosmological constant Λ is equal to $-a^{-2}\ln(-\frac{N}{\beta}g_4)$ with a^2 being the area of the *dual* surface element corresponding to the M^4 vertices. This interpretation of the Feynman diagram expansion in terms of 2D gravity or random surface was first proposed by David and Kazakov [4]. Note that $g_4 < 0$ is required for this interpretation.

We have seen that the matrix model can be regarded as a regularization of 2D gravity. Furthermore, it is easy to see that, if the matrix field is assumed to live in a nontrivial target space, the model can be regarded as a regularization of string theories. Then we come to a natural question about the meaning of the critical behaviors. We know that a continuum limit of a regulalized field theory on a spacetime lattice can only be defined at the vicinity of critical points of second or higher orders. Is it possible to define the continuum limit of 2D gravity or continuum string theories using the above critical points of the matrix models? If this is the case, the matrix models could be a good candidate for defining string theories in a nonperturbative way. Non-perturbative formulation of string theories has been one of the major unsolved problems in unified theories of fundamental interactions including quantum gravity and also in QCD. The recent development is that the above possibility was shown to be indeed partially realized, at least, in the case of the linear matrix chains. In the following we will briefly summarize the basic reasoning [5] in the simplest case of the one-matrix model.

1.2 Essentials of the "double" scaling limit

1 *Behavior of Z near the critical point*

Let us consider the free energy of the one-matrix model.

$$F_N(\beta)(\equiv \ln Z) = \sum_{surfaces} N^{\chi}(-\frac{N}{\beta}g_4)^V, \tag{1.7}$$

$$= \sum_{\chi} N^{\chi} \sum_{V} n_{\chi,V}(-\frac{N}{\beta}g_4)^V \tag{1.8}$$

where $n_{\chi,V}$ is the number of nonequivalent surfaces with given χ and N. We expect that $n_{\chi,V}$ behaves for large V as

$$n_{\chi,V} \sim V^{\gamma(\chi)-3}\alpha^V, \ (\alpha > 1) \tag{1.9}$$

with $\gamma(\chi)$ and α being numerical constants. The exponent $\gamma(\chi)$ is called the string susceptibility exponent. Choosing the normalization of the coupling constant β such that $\alpha g_4 = -1$, we see that the contribution, $F_N^{\chi}(\beta)$, of the surfaces with a given fixed genus χ to the free energy behaves, in the limit $\frac{N}{\beta} \to 1-$, as

$$F_N^{\chi} = \sum_V V^{\gamma(\chi)-3}\frac{N}{\beta}^V \sim (1-\frac{N}{\beta})^{2-\gamma(\chi)} + \text{regular terms.} \tag{1.10}$$

This singularity is interpreted as the one expected from the qualitative discussion of the previous section.

2 *The double scaling limit*

Thus, we have, at the vicinity of the critical point,

$$F_N(\beta) = \sum_{\chi} N^{\chi} F_N^{\chi}(\beta), \tag{1.11}$$

$$\sim \sum_{\chi} N^{\chi}(1-\frac{N}{\beta})^{2-\gamma(\chi)} + \text{regular contribution} \tag{1.12}$$

This shows that if we can tune N and β such that

$$\lim_{N\to\infty,\beta\to N} N^{\chi}(1-\frac{N}{\beta})^{2-\gamma(\chi)} = \text{finite}, \tag{1.13}$$

we can extract a finite nontrivial result from the critical point of the model. In fact, the result from the continuum Liouville field theory [6] suggests $\gamma(\chi)-2=(\gamma(2)-2)\chi/2$ which indicates that in this limit the contributions come from all genus. Defining the finite renormalized cosmological constant x by

$$x = a^{-2}(\frac{\beta}{N}-1), \tag{1.14}$$

the genus $=0$ free energy is proportional to $x^{2-\gamma(2)}$. This is a desired continuum limit, since by choosing N to behave $a^{-(2-\gamma(2))}$, the number of the surfaces elements V increases as a^{-2} as $a\to 0$. The remaining regular terms can be shown to contribute only to zero area surfaces in the continuum limit.

3 *Differential equations for free energy*

One of the important results in the recent development is that we can write down a class of exact differential equations governing the free energy in the double scaling limit as a function of the renormalized cosmological constant x. Let us next briefly review its derivation.

The measure factor in the partition function of the matrix model is the square of the so-called Vandelmonde determinant,

$$\Delta(\lambda_1,\lambda_2,\dots\lambda_N) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_N \\ \vdots & \vdots & & \vdots \\ \lambda_1^{N-1} & \lambda_2^{N-1} & \dots & \lambda_N^{N-1} \end{vmatrix}, \tag{1.15}$$

which is rewritten, using an arbitrary set of polynomials $\{P_i(\lambda): P_i(\lambda)=\lambda^{i-1}+\dots\}$, as

$$\Delta(\lambda_1,\lambda_2,\dots\lambda_N) = \begin{vmatrix} P_1(\lambda_1) & P_1(\lambda_2) & \cdots & P_1(\lambda_N) \\ P_2(\lambda_1) & P_2(\lambda_2) & \cdots & P_2(\lambda_N) \\ \vdots & \vdots & & \vdots \\ P_N(\lambda_1) & P_N(\lambda_2) & \cdots & P_N(\lambda_N) \end{vmatrix}. \tag{1.16}$$

Thus the partition function of the matrix model is the norm of the Slater determinant of an N fermion system with the single-particle wave functions $P_i(\lambda)$ with respect to the measure $\exp -\beta U(\lambda)$. Choose the polynomials to be mutually orthogonal with respect

$$\int d\lambda e^{-\beta U}(\lambda) P_i(\lambda) P_j(\lambda) = h_i \delta_{ij}. \tag{1.17}$$

Then,

$$Z = N! \prod_{i=1}^{N} h_i. \tag{1.18}$$

We see that correspondingly to the semi-classical picture in the large N limit, the finite N partition function has a one-to- one correspondence with a ground state of an N-fermion system with the orthogonal polynomial wave functions. The basic equation determining the partition function can be formulated in terms of a 'matrix mechanics'.

Let $\{\hat{P}_i\}$ be the ortho*normal* set of polynomials corresponding to $\{P_i\}$, and define infinite dimensional matrices

$$Q_{ij} = \int e^{-\beta U} \hat{P}_i \lambda \hat{P}_j, \tag{1.19}$$

$$P_{ij} = \int e^{-\beta U} \hat{P}_i \frac{d}{d\lambda} \hat{P}_j. \tag{1.20}$$

By partial integration, it is easy to see that

$$P = \beta U'(Q)_+ \tag{1.21}$$

where the suffix $+$ indicates the strictly upper triangular part of the matrix. Substituting this relation to the canonical commutation relation $[P, Q] = 1$, we obtain the equation for determining the infinite dimensional matrix Q,

$$[\beta U'(Q)_+, Q] = 1. \tag{1.22}$$

Summing the diagonal elements up to $(n+1)$-th terms, this leads to

$$U'(Q)_{n,n+1} Q_{n+1,n} = \frac{n+1}{\beta}. \tag{1.23}$$

Let us first examine the possibility of the double scaling limit in the sphere limit. Using

$$Q_{i,j} = \sqrt{R_i}\delta_{i,j+1} + \sqrt{R_j}\delta_{i,j-1} \tag{1.24}$$

with $R_i \equiv h_i/h_{i-1}$ and (1.14), (1.23) is rewritten in the following form in this limit,

$$\oint \frac{dz}{2\pi i} z^{-1} U'(z + R/z) = 1 - a^2 z. \tag{1.25}$$

Then, by defining a scaling function $u(x)$ by $R_i - 1 = a^\sigma u(x)$ as $a \longrightarrow 0$, we see that a well defined double scaling limit exist if the polynomial potential U is tuned such that

$$\oint \frac{dz}{2\pi i} z^{-1} U'(z + R/z) = 1 - (1 - R)^k, \ (k = 1, 2, \ldots) \tag{1.26}$$

and the exponent σ is equal to $2/k$. Since the free energy $F_N = F(X) = -\sum_{i=1}^{N} i \log R_i \Rightarrow -\int^X dx(X - x)u(x)$ is finite if $\gamma_0(\equiv \chi(2)) = -\sigma$, the genus$= 0$ susceptibility exponent defined by (1.12) is given as

$$\gamma_0 = -\frac{1}{k}. \tag{1.27}$$

Next we shall briefly explain how we can obtain exact differential equations governing the two-point function ('specific heat') $u(x)$. In terms of the renormalized cosmological constant x, the matrix Q is written as a differential operator,

$$\begin{aligned} Q &= \sqrt{R(x - a^{-\gamma_0})} e^{-a^{-\gamma_0} d/dx} + \sqrt{R(x + a^{-2\gamma_0})} e^{a^{-\gamma_0} d/dx}, & (1.28) \\ &= 2 + a^{-2\gamma_0}(u + d^2/dx^2) + \text{higher orders}. & (1.29) \end{aligned}$$

The existence of the scaling limit guarantees that the matrix P also reduces to a differential operator of the form (const.$+a^{2\gamma_0}$(differentialoperator)$+$higher orders) such that $[P, Q] = 1$. Redefining the operator $Q \equiv d^2/dx^2 + u$, the most general form of such a differential operator can be shown to be

$$P = \sum_{k=1} c_k (Q^{(2k-1)/2})_+, \ (c_k = \text{constant}). \tag{1.30}$$

Here we used the notation of pseudo-differential calculus [7]. In particular, $[(Q^{3/2})_+, Q] = 1$ is equivalent with the Painlevé I equation $x = u^2 - u''/3$. The general equation which describes the continuum theories with arbitrary number of the coupling constants c_k, $[\sum c_k (Q^{(2k-1)/2})_+, Q] = 1$ [8] is called the 'string equation'. The critical theories which correspond to 2D gravity coupled with massless matter systems are represented by the condition that only one of the infinite number of the coupling constants c_k, apart from the cosmological constant x, is nonzero. Otherwise the continuum theory contains dimensionfull parameters other than the 2D cosmological constant.

Let us finally summarize the properties of the solutions. For the simplest nontrivial case $k = 2$ (critical case with $c_2 \neq 0$), it is known that a real solution for the partition function always exhibits an infinite number of zeros on the real axis x which accumulate to minus infinity. The perturbative expansion with respect to genus is not Borel summable, and there are in general nonperturbative contributions of the form $\exp -4\sqrt{2}x^{5/4}/5$, associated with parameters which are not determined by the boundary condition for large x (i. e., the sphere limit). It is remarkable that the nonperturbative effect is of the form $e^{-1/g}$ with respect to the loop expansion parameter g^2, instead of the familiar e^{-1/g^2} in local field theories. However, David [9] has argued that if one notes that the critical potential corresponding to $k = 2$ is unbounded from below, a more natural choice of the solutions is to take complex solutions which are inevitable when one deforms the integration contours such that the integral of the partition function is well defined. A similar structure can be seen much more explicitly in a simpler "1D gravity" which can be derived by taking the double scaling limit of $O(N)$ vector models [10]. Since these properties are not visible in the spherical limit (= the naive large N limit), the lesson we have to learn is the importance of the nonperturbative effects in 2D gravity or string theory.

1.3 Formal structure of the matrix model: action principle

In the second part of this lecture, I have presented a unified treatment of the differential equations and the Schwinger-Dyson equations on the partition function for the local observables of the continuum non-perturbative 2D gravity. There are two remarkable properties in the mathematical structure of the theory, namely, that (1) the flow property of the theory under the change of the coupling constants is described by the KP hierarchy, and that (2) the Schwinger-Dyson equations [11] exhibit a W_∞ algebraic structure. These properties are neatly summarized in an extremely simple action principle [12]. Actually, the action principle has a precursor already in the finite N matrix model which itself is a completely integrable system [13] and exhibits the W_∞ algebraic structure.

Consider the partition function

$$Z = \int \prod_{i=1}^{N} d\lambda_i \Delta(\lambda_1, \cdots, \lambda_N)^2 \exp -\beta \sum_{i=1}^{N} U(\lambda_i). \tag{1.31}$$

Introduce an arbitrary set, $\{\phi_i^{(0)} = \lambda^{i-1} + \cdots ; i = 1, 2, \cdots\}$, of polynomial basis. Then,

$$\Delta(\lambda_1, \cdots, \lambda_N) = \det \phi_i^{(0)}(\lambda_j). \tag{1.32}$$

Define the 'bare' coordinate and momentum matrices by

$$\lambda \phi_i^{(0)}(\lambda) = (Q_0 \phi)_i(\lambda), \tag{1.33}$$

$$-\frac{d}{d\lambda} \phi_i^{(0)} = (P_0 \phi)_i(\lambda). \tag{1.34}$$

A convenient basis for the one-matrix case is the one in which $Q_{0ij} = \delta_{i,j+1} + \delta_{i+1,j}$. The whole dynamical information is then contained in the transformation matrix K which sends the bare basis $\{\phi^{(0)}\}$ to the orthonormal basis $\{\phi\}$ satisfying

$$\int \phi_i(\lambda) \phi_j(\lambda) e^{-\beta U(\lambda)} = \delta_{ij}, \tag{1.35}$$

$$\phi_i = (K \phi^{(0)})_i. \tag{1.36}$$

By choosing the ordering of the basis polynomials appropriately, we can always assume that the *strictly upper triangular* part of K is zero, $K_+ = 0$. Clearly, the partition function for finite N is then given as

$$Z = (\det K)_N^{-2} = \exp -2\mathrm{Tr}_N \log K, \tag{1.37}$$

where $(\cdots)_N$ indicates N-dimensional operations. (Namely, the trace is taken for the first $N \times N$ small matrix of K. Note that K itself is an infinite dimensional matrix.) Corresponding to (1.33) and (1.34), it is convenient to define the 'dressed' coordinate and momentum by

$$\lambda\phi = Q\phi, \tag{1.38}$$

$$-\frac{d}{d\lambda}\phi = \tilde{P}\phi. \tag{1.39}$$

We have $[\tilde{P}, Q] = 1 = [P_0, Q_0]$, and $Q = KQ_0K^{-1}, \tilde{P} = KP_0K^{-1}$. Also, we have $\tilde{P} = \tilde{P}_-$ (i. e., strictly lower triangular), $Q = Q^t$. Note that Q is a Jacobi matrix for the above choice of the bare coordinate (namely, $Q_{ij} = 0, |i-j| > 2$) provided the symmetric nature of Q is imposed.

The action is then assumed to be

$$A = \mathrm{Tr}(PKQ_0K^{-1} + \log K), \tag{1.40}$$

where a new matrix which is antisymmetric, $P = -P^t$, is a lagrange multiplier imposing the symmetric nature of the dressed coordinate Q. Variational equations are

$$Q = Q^t, \tag{1.41}$$

$$[P, Q] = 1. \tag{1.42}$$

Let us consider the general solution to the variational equations. Since $[\tilde{P}_0, Q_0] = 1$, the most general form of P satisfies

$$K^{-1}PK = P_0 + f(Q_0) \tag{1.43}$$

for some function f, which leads to $P = \tilde{P}_- + f(Q)$. From this we obtain

$$P_+ = f(Q)_+ \tag{1.44}$$

and hence

$$P = f(Q)_+ - f(Q)_-. \tag{1.45}$$

Substituting this result to the commutation relation, we finally arrive at

$$[f(Q)_+ - f(Q)_-, Q] = 1 \tag{1.46}$$

which is equivalent with the matrix-string equation (1.22) for the general potential U with the identification $f(Q) = \beta \frac{dU(Q)}{dQ}/2$. We have shown that the general one-matrix model is the general solution of the action principle.

There are two kinds of symmetries in this action characterizing the solution space of the regularized random surfaces based on the one-matrix model.

(1) Toda flow:

$$\delta_r K = -(KQ_0^r K^{-1})_- K, \tag{1.47}$$

$$\delta_r P = \frac{1}{2}[(KQ_0^r K^{-1})_+ - (KQ_0^r K^{-1})_-, P]. \tag{1.48}$$

(2) Canonical 'gauge' symmetry:

$$\delta K = -GK, \tag{1.49}$$

$$\delta P = [G, P] \tag{1.50}$$

where G is an arbitrary antisymmetric matrix. The symmetry (1) shows that the solution space is parametrized by an completely integrable differential equations, Toda-lattice hierarchy. The symmetry (2) corresponds to the infinite dimensional orthogonal group changing the orthonormal basis, and contains as a special case, a 'conformal symmetry',

G = antisymmetric part of $Q^{n+1}P$ ($n \geq 1$), which leads to the Virasoro condition [11] on the partition function, since it induces the transformation $Q \to Q + \epsilon_n Q^{n+1}$. More generally, G contains a W_∞ algebra generated by the antisymmetric part of $Q^i P^j$ ($i \geq 0, j \geq 1$).

All of these symmetry properties have counterparts in the continuum theory. In the finite N case, however, the action principles for the multi- matrix models become increasingly complex as the number of the matrices increases and the form of the action varies. In contrast with this, a remarkable feature in the continuum action principle [12] is its universality. Namely, all the so-called (p, q) models can be embedded in one and the same form of the action principle, in which only the constraint varies. At present, it is not clear whether the action principle has deeper significance than as a convenient rewriting of the matrix model. I myself have a hope that the continuum action principle, in view of its simplicity and universality, might contain some clue towards further understanding of the meaning of the non-perturbative string theories. It should be emphasized that the action principle describes the structure of the theory space, and that the theory space in the critical string theories is essentially the space of all possible background spacetimes (=target space of matter systems in 2D gravity). It is also tempting to speculate a possibility of relating the commutation relation $[P, Q] = 1$ with the duality and minimal distance property (see, e. g., [14]) in perturbative string theories. I would now like to refer the reader to my recent preprint, mentioned in Foreword and future works for further discussion.

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(q,p) Critical Point from Two-Matrix Models [1]

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最近2次元重力の理論において非摂動的取扱を可能にする著しい進展があった[2]。即ちすべての種数の2次元面を考慮することができるようになった。この事は行列模型を用いることによって可能になった。行列模型自身は以前から考えられていたが連続理論の進展と併せてこの非摂動的取扱が可能になったのである。この2次元重力の非摂動的取扱からの結果はこれらの理論の背景に数学的背景があることを強く示唆している。特に1行列模型においてはその弦方程式と呼ばれる模型を決定する為の方程式の形を一般的にもとめることができたが、その形はＫｄＶ階層との関連を強く示唆する形となっている。一方2行列模型をふくめ多行列模型についてはこのように一般的に弦方程式を求めることはできなかった。そこでＤｏｕｇｌａｓは（ｑ－１）行列模型によっていわゆる（ｐ，ｑ）模型が記述され、その弦方程式はそれぞれｑ次、ｐ次の微分演算子Ｑ，Ｐを用いて

$$[P,Q]=1 \tag{1}$$

と与えられるという予想を提案した[3]。この予想は数学的背景を持っているのでこの予想に基づいて理論の数学的背景が議論された。しかし乍ら具体的に多行列模型を解くことは難しく３行列模型から（４、５）模型などが求められただけであった。筆者は山口昌宏との共同研究[4]においてこの困難を解決し、２行列模型に６次までの項をいれることにより（４、５）模型などを求められることを示した。本講演ではこの結果を拡張し２行列模型で一般の（ｐ，ｑ）模型を得るための処方箋を示す。またこの一般的な枠組みの観点から先の研究を再考察する。

行列模型の解析には直交多項式を用いる。この直交多項式は１行列模型

の場合には、

$$\lambda P_n = P_{n+1} + R_n P_{n-1} \tag{2}$$

と展開される。このこきR_nを用いて分配関数が、

$$Z_N = N! h_0^N R_1^{N-1} \cdots R_{N-2}^2 R_{N-1} \tag{3}$$

と表される。即ちR_nに対する方程式、所謂弦方程式を求めることが目標である。このときNを無限大にすることにより弦方程式を導くことができる。このとき臨界点に系を近づける近づけ方とNを無限大に持っていく割合を調節することによりすべての種数の2次元面からの寄与を含んだ弦方程式を得ることができる。この極限のとりかたはdouble scaling limitと呼ばれる。一方多行列模型の場合には直交多項式の展開が一般に複数の係数による

$$xP_m(x) = P_{m+1}(x) + R_m^{(\alpha)} P_{m-1}(x) + S_m^{(\alpha)} P_{m-3}(x) + \cdots \tag{4}$$

の形になる。このため弦方程式が連立方程式になるため臨界点を求めることが難しくなる。そこでよりＤｏｕｇｌａｓの予想に添った形で模型を解析することを考える。通常多行列模型でも複数のポテンシャルは同じ形に導入するが、本講演では２行列模型に二つのことなるポテンシャルを導入する。ここでは具体的に（３、５）模型を求める。この時２つのポテンシャルに対応して直交多項式も二つ必要であり、その展開は

$$\begin{aligned} xP_m(x) &= P_{m+1}(x) + R_m^{(\alpha)} P_{m-1}(x) + S_m^{(\alpha)} P_{m-3}(x) \\ &\quad + T_m^{(\alpha)} P_{m-5}(x) + U_m^{(\alpha)} P_{m-7}(x) \end{aligned} \tag{5}$$

$$\begin{aligned} yQ_m(y) &= Q_{m+1}(y) + R_m^{(\beta)} Q_{m-1}(y) + S_m^{(\beta)} Q_{m-3}(y) \\ &\quad + T_m^{(\beta)} Q_{m-5}(y) + U_m^{(\beta)} Q_{m-7}(y) \end{aligned} \tag{6}$$

となる。

$$\int dxdy e^{-w(x,y)} \tilde{Q}_j(y) \begin{pmatrix} x \\ y \end{pmatrix} \tilde{P}_i(x) \equiv \begin{pmatrix} \bar{Q}_{ij} \\ \bar{P}_{ji} \end{pmatrix}. \tag{7}$$

を定義することにより、これらは離散レベルで

$$[\bar{P}, \bar{Q}] = 1 \tag{8}$$

を満たし、連続極限で弦方程式を与えることが期待される。ここでは臨界点を具体的に求める事により、$\bar{P}$、$\bar{Q}$の連続極限が弦方程式を与えることを示す。

連続極限では

$$\begin{aligned} \bar{Q}_{i,j} \underset{\tilde{a}\to 0}{\longrightarrow} \quad & \tilde{Q}(z) \\ = \quad & const. + \tilde{a}(\bigcirc d + \ldots) \\ & +\tilde{a}^2(\bigcirc d^2 + \ldots) \\ & +\tilde{a}^3(\bigcirc d^3 + \ldots); \end{aligned} \tag{9}$$

となる。これが3次の微分演算子になるためにはd及びd^2の前の係数が消えることが必要である。この必要条件はパラメーターに対して線形の方程式を与えるので簡単に解くことができる。同様の必要条件を$\bar{P}$に対しても課すことにより理論のパラメータを完全に決定することができる。するとこの要求が十分であることが具体計算によって示せる。本講演では（３、５）模型の場合について具体計算を示し、じっさいにＰおよびＱ演算子がＤｏｕｇｌａｓの予想で与えられるものと一致することを示す。このようにして従来臨界点を決定することが困難であった高次の２行列模型に対して、任意の（ｐ、ｑ）模型の臨界点を得るための処方が示された。

一方この観点から先の山口氏との共同研究は２つのポテンシャルが一致した特別の場合を考察していたことが明らかになった。この場合には上の定義による$\bar{P}$と$\bar{Q}$が一致してしまう。しかしながら$\bar{P}$と$\bar{Q}$に要求される微分の次数は異なるため、ｐ次およびｑ次の部分を抜き出してくれば良い。但しこの方法の限界はポテンシャルの形にたいしてどのような模型がえられるのか前もって知ることが出来ない点にある。しかしながら対称性の観点から言えばいわゆるユニタリ系列に対してはこの対称的な模型を考えるのが自然であり、ユニタリ模型にかぎれば一般的な処方箋が与えられる。

このように、ここでは２行列模型の臨界点を調べるための一般的な処方箋を提唱し、いくつかの興味深い例について具体的に臨界点を調べた。ここ

で必要条件のみから臨界点を決定できたことから行列模型による量子重力の定式化は非常に高い対称性を持っていることが推測される。これらの数学的背景を探り出すことなどが今後の課題である。

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Weyl invariant formulation of 2D Quantum Gravity

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Introduction

We investigate two-dimensional Einstein gravity coupled to N massless scalar fields, which is described by the following action;

$$S_0 = \int d^2x\sqrt{-g}[R + \frac{1}{2}g^{\mu\nu}G_{AB}\partial_\mu\phi^A \cdot \partial_\nu\phi^B], \tag{1}$$

where G_{AB} is a metric of ϕ's space. Here, we shall consider the case when the bare cosmological constant is exactly zero. The Hilbert-Einstein term becomes a total derivative in two-dimensional space-time, hence, gravity itself has no kinetic term. As will be shown in the later, this is not the problem when the Weyl invariance is taken into account. This action also appears in string theories of Brink-Di Veccia-Howe-Polyakov type action. However, we shall consider the different situation from string theories, where the space-time has no boundary and has an infinite volume.

The action (1) is invariant under Weyl transformation

$$\delta g_{\mu\nu} = \rho(x)g_{\mu\nu}, \quad \delta\phi^A = 0, \quad \delta x^\mu = 0$$

in addition to the general covariance because of the absence of the cosmological term and mass term. In two-dimensional space-time, Weyl invariance is realized without introducing the Weyl's gauge field in Einstein gravity. In the following, we shall take care of the Weyl invariance, *i.e.*, typical feature of this system.

Now, we are going to quantize the system in the operator formalism. First, we would like to take care so that three components of the metric are treated as dynamical variables. This is important in order to realize the Weyl transformation which we should treat. If one chooses only the Weyl invariant variables as dynamical variables, Weyl invariance becomes trivial and there is no Weyl transformation. In this connection, we do not consider conformal or light-cone type gauge because, one cannot regard the metric as tensor under the Lorentz transformation in these condition. So, we would like to adopt the harmonic ($\partial_\mu \tilde{g}^{\mu\nu} = 0$) and the zero scalar curvature gauge ($R = 0$) as gauge condition. There are many possible gauge choice so that dynamical freedom of the metric is three. However, we would like to choose them. Because, these gauge conditions are independent each other and there are no first-order derivatives written by only the metric, which is invariant under the general coordinate transformation. In these gauge choice, there are a global Weyl invariance and a general linear (GL(2)) invariance, at least, in the operator level. Secondly, the quantization procedure is based on BRS invariances corresponding to both general covariance and Weyl invariance under the subsidiary conditions of Kugo-Ojima's type; *i.e.*, we shall regard the Weyl invariance as an ordinary local gauge invariance and perform the gauge fixing and Faddeev-Popov procedure for both invariances. Hence, we introduce two kinds of BRS transformation corresponding to general covariance and Weyl invariance, and the generators are Q_b^{GCT} and Q_b^{Weyl} [1].

$$
\begin{aligned}
&\delta^* x^\mu = 0,\ \delta^* C^\mu = -C^\lambda \partial_\lambda C^\mu,\ \delta^* \bar{C}_\mu = iB_\mu - C^\lambda \partial_\lambda \bar{C}_\mu,\ \delta^* B_\mu = -C^\lambda \partial_\lambda B_\mu, \\
&\delta^* g_{\mu\nu} = -\partial_\mu C^\lambda \cdot g_{\lambda\nu} - \partial_\nu C^\lambda \cdot g_{\mu\lambda} - \partial_\lambda (C^\lambda g_{\mu\nu}),\ \delta^* \sqrt{-g} = -\partial_\lambda (C^\lambda \sqrt{-g}), \\
&\delta^* \phi^A = -C^\lambda \partial_\lambda \phi^A,\ \delta^* B = -C^\lambda \partial_\lambda B,\ \delta^* C = -C^\lambda \partial_\lambda C,\ \delta^* \bar{C} = -C^\lambda \partial_\lambda \bar{C},
\end{aligned}
\tag{2}
$$

$$
\begin{aligned}
&\delta^W g_{\mu\nu} = C(x) g_{\mu\nu},\ \delta^W C(x) = 0,\ \delta^W \bar{C}(x) = iB(x),\ \delta^W B(x) = 0, \\
&\delta^W \phi^A = \delta^W x^\mu = \delta^W B_\mu(x) = \delta^W C^\nu = \delta^W \bar{C}_\lambda = 0,
\end{aligned}
\tag{3}
$$

It is easy to check that both BRS transformations are nilpotent and independent of each other by definition;

$$(\delta^*)^2 = (\delta^W)^2 = 0, \quad \delta^*\delta^W + \delta^W\delta^* = 0. \tag{4}$$

(3) is similar to the BRS transformation of Abelian gauge theories. This is a reflection of the abelian nature of the Weyl transformation.

Now, we consider the BRS invariant action for the two-dimensional quantum gravity;

$$\begin{aligned} S = \int d^2x[\frac{1}{2}\tilde{g}^{\mu\nu}G_{AB}\partial_\mu\phi^A \cdot \partial_\nu\phi^B + \partial_\mu\tilde{g}^{\mu\nu} \cdot B_\nu - i\tilde{g}^{\mu\nu}\partial_\mu\bar{C}_\lambda \cdot \partial_\nu C^\lambda \\ + \sqrt{-g}RB - i\tilde{g}^{\mu\nu}\partial_\mu\bar{C} \cdot \partial_\nu C] \end{aligned} \tag{5}$$

Recently, Abe and Nakanishi [2] proposed a unitary theory of two-dimensional quantum gravity different from ours. They start from the action which is obtained by eliminating the Faddeev-Popov ghosts concerning Weyl invariance from (5). Instead, their formulation is based on two kinds of BRS invariances; one is ordinary BRS invariance corresponding to general covariance, and another is a vector type which also exists in our theory as residual symmetries. Although the starting actions are different, their work gave us a good hint, *i.e.*, the physical state condition is not enough in order to find the physical subspace.

Canonical quantization and extension of BRS algebra

We can show that the canonical commutation relations are consistently imposed because the gauge fixing procedure for the Weyl invariance. For example, the momenta which are canonically conjugate to g_{11} and B are given by

$$\begin{aligned} \Pi^{11} &= \frac{\sqrt{-g}}{2g}(g^{1\mu}\partial_1 g_{0\mu} + 2\partial_0)B + \frac{\sqrt{-g}}{2g}[B_0 - g^{01}(g_{00}B_1 - g_{01}B_0)], \\ \Pi_B &\equiv \frac{\delta S}{\delta \dot{B}} = \frac{\sqrt{-g}}{g}(\partial_1 g_{01} - \partial_0 g_{11}), \end{aligned} \tag{6}$$

respectively. It is to be noted that Π^{11} contains $\dot{B}$ and that Π_B contains $\dot{g}_{11}$. These results means that Nakanishi-Lautrup field B is no longer an auxiliary field. Therefore, the physical degrees of the freedom except for the scalar fields becomes -2, not -3. This is to be expected. Negative degree of freedom means that gravity itself has no meaning. If the other fields exist together with the gravitational field, and if the negative freedom of the metric can *eat* the other freedom, one obtains a consistent theory. This is one of the basic problems in two-dimensional quantum gravity. We would like to study this problem in the next section.

There are more symmetries than BRS and other familiar symmetries in the present theory. In our case, Einstein equation becomes $T_{\mu\nu} = 0$ Using the Einstein equation, we can show that all fields except for the metric satisfy d'Alembert equation of the scalar type in our formulation [1] [3]. In the following, X,Y,Z,..., denote all fields which appear in this theory except for the metric. If X, Y, Z, ... satisfy d'Alembert equation, it is easy to check that the currents

$$\begin{aligned} \mathbf{P}^{\mu}(X) &= \tilde{g}^{\mu\lambda}\partial_{\lambda}X, \\ \mathbf{M}^{\mu}(X,Y) &= \sqrt{\epsilon(X,Y)}(X\mathbf{P}^{\mu}(Y) - \epsilon(X,Y)Y\mathbf{P}^{\mu}(X)) \end{aligned} \tag{7}$$

satisfy the continuity equations $\partial_{\mu}\mathbf{P}^{\mu}(X) = 0$ and $\partial_{\mu}\mathbf{M}^{\mu}(X,Y) = 0$ in the harmonic gauge. Here, $\epsilon(X,Y)$ is the sign factor that, when both X and Y satisfy Fermi statistics, takes -1. Hence, the charges $P(X) = \int dx^1\mathbf{P}^0(X)$ and $M(X,Y) = \int dx^1\mathbf{M}^0(X,Y)$ are conserved. As the action (5) has the corresponding symmetries, we can also derive the above result from Noether's theorem by using equation of motion $T_{\mu\nu} = 0$. As X, Y denote $x^{\mu}, \phi^A, B_{\mu}, C^{\mu}, \bar{C}_{\mu}, B, C$, and $\bar{C}$ in our case, the number of the charges amount to $\frac{1}{2}N(N+23)+72$. It is to be remembered that there are the conserved charges $M(\phi^A, C^{\mu})$ which generate the ' super-rotation ' in the ϕ^A, C^{μ}, $\bar{C}_{\nu}$'s space.

Physical subspace and additional subsidiary condition

In this section, we shall study the structure of the physical subspace. In the following, we assume the existence of the asymptotic fields for the elementary fields and assume the asymptotic completeness, and we consider the flat background $\langle 0| g_{\mu\nu} |0\rangle = \eta_{\mu\nu}$. Asymptotic fields are defined by

$$g_{\mu\nu} \underset{x^0 \to \pm\infty}{\longrightarrow} \eta_{\mu\nu} + \kappa\varphi_{\mu\nu}, \quad \cdots ,$$

and are governed by the quadratic part of the Lagrangian;

$$\begin{aligned}\mathcal{L}^{(2)} =& \frac{1}{2}\eta^{\mu\nu}G_{AB}\partial_\mu\phi^A \cdot \partial_\nu\phi^A - \frac{1}{2}\eta^{\mu\nu}\partial_\mu\varphi \cdot \partial_\nu B - i\eta^{\mu\nu}\partial_\mu\bar{C} \cdot \partial_\nu C \\ & - \partial_\mu\bar{\varphi}^{\mu\nu} \cdot \bar{B}_\nu - i\eta^{\mu\nu}\partial_\mu\bar{C}_\lambda \cdot \partial_\nu C^\lambda,\end{aligned} \tag{8}$$

where we define $\varphi \equiv \eta^{\mu\nu}h_{\mu\nu}$, $\bar{\varphi}_{\mu\nu} \equiv \varphi_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\varphi$ and $\bar{B}_\mu \equiv B_\mu - \partial_\mu B$. Equations of motion are the same type $\eta^{\mu\nu}\partial_\mu\partial_\nu X = 0$ for all the fields. Hence, the momentum representation is possible for the all fields, though we encounter a serious infrared problem. It is to be remarked that $\bar{B}_\mu$ and $\bar{\varphi}_{\mu\nu}$ satisfy the first order equations;

$$\partial^\mu\bar{\varphi}_{\mu\nu} = 0, \qquad \partial_\mu\bar{B}_\nu + \partial_\nu\bar{B}_\mu - \eta_{\mu\nu}\partial^\lambda\bar{B}_\lambda = 0. \tag{9}$$

For the convenience, we introduce the new variables as follows;

$$\begin{aligned}&\varphi^\pm = \frac{1}{\sqrt{2}}(\bar{\varphi}_{00} \pm \bar{\varphi}_{01}), \quad \bar{B}_\pm = \frac{1}{\sqrt{2}}(\bar{B}_0 \pm \bar{B}_1), \quad \partial_\pm B = \frac{1}{\sqrt{2}}(\partial_0 B \pm \partial_1 B), \\ &C^\pm = \frac{1}{\sqrt{2}}(C^0 \pm C^1), \quad \bar{C}_\pm = \frac{1}{\sqrt{2}}(\bar{C}_0 \pm \bar{C}_1).\end{aligned} \tag{10}$$

We rewrite BRS charges and ghost number charges as

$$\begin{aligned}Q_b^{GCT} &= M(B_\mu, C^\mu) \equiv Q_b^{(+)} + Q_b^{(-)}, \\ Q_b^{(\pm)} &= i\int dp(c^{\pm\dagger}(p)b_\pm(p) - b_\pm^\dagger(p)c^\pm(p)), \\ Q_b^{Weyl} &= M(B, C) = i\int dp(c^\dagger(p)b(p) - b^\dagger(p)c(p)), \\ Q_c^{(\pm)} &= M(C^\pm, \bar{C}_\pm) = \int dp(\bar{c}_\pm^\dagger(p)c^\pm(p) + c^{\pm\dagger}(p)\bar{c}_\pm(p)), \\ Q_c^{Weyl} &= M(C, \bar{C}) = \int dp(\bar{c}^\dagger(p)c(p) + c^\dagger(p)\bar{c}_(p)).\end{aligned}$$

In order to obtain a unitary theory, we define the physical subspace by

$$Q_b^{(+)}\,|\text{phys}\rangle = 0 \quad Q_b^{(-)}\,|\text{phys}\rangle = 0 \quad \text{and} \quad Q_b^{Weyl}\,|\text{phys}\rangle = 0. \tag{11}$$

For later convenience, we shall consider, in the following, the representations of $Q_b^{(\pm)}$ instead of Q_b^{GCT}. So, we consider the representation of the three independent BRS algebra;

$$\begin{aligned}
(Q_b^{(+)})^2 = 0, &\quad [Q_b^{(+)}, Q_c^{(+)}] = -iQ_b^{(+)}, \\
(Q_b^{(-)})^2 = 0, &\quad [Q_b^{(-)}, Q_c^{(-)}] = -iQ_b^{(-)}, \\
(Q_b^{Weyl})^2 = 0, &\quad [Q_b^{Weyl}, Q_c^{Weyl}] = -iQ_b^{Weyl}, \quad \text{others} = 0.
\end{aligned}$$

By the way, remember the equations (9). Solving the first order differential equations (9), we obtain

$$\varphi^{+} = 0, \quad \bar{b}_{-} = 0, \quad \partial_{-}b = 0, \quad b_{-} = 0, \quad Q_b^{(-)} = 0, \tag{12}$$

for $p > 0$. Here, p is a space momentum. In the similar fashion, for $p < 0$, the relations in a opposite sign are satisfied. Therefore, only one component of the $\bar{\varphi}_{\mu\nu}$ and one component of the $\bar{b}_\mu$ are independent variables. The constraints (12) bring us a difficult problem in finding the quartet representation for $Q_b^{(\pm)}$ and Q_b^{Weyl}. Let us consider the case of $p > 0$. It is no problem in order to find the two independent quartets for the $Q_b^{(\pm)}$ and Q_b^{Weyl}. But, for the remaining one, the problem arises. Because, the algebra $[Q_b^{(-)}, Q_c^{(-)}] = -iQ_b^{(-)}$ becomes trivial from (12), and we show that the ghost states

$$|-\rangle \equiv \prod_{m,n} c^{-\dagger}(p_m)\bar{c}_{-}^{\dagger}(p_n)\,|0\rangle$$

, which can not rewritten as BRS exact form, satisfy the subsidiary condition (11). Therefore, the physical states $|-\rangle$ are infinitely degenerate in terms of the ghost number $Q_c^{(-)}$. This is the problem, which has relation to the negative physical freedom (-2).

In order to avoid the infinite degeneracy in terms of the ghost number Q_c^-, we need a new quantum number which can be regarded as BRS charge. Fortunately, we have already obtained wider symmetries than the symmetries from which we start. Among those charges, we find the conserved charge $Q_b^{\mu A} \equiv M(C^\mu, \phi^A)$. If there is a scalar field which creates negative norm states and we call the field as ϕ^1, we can always obtain the following two fields;

$$A_\pm \equiv \frac{1}{\sqrt{2}}(\phi^1 \pm \phi^2), [A_+(x), A_-(y)] = -1 \cdot D(x-y), \qquad [a_+(p), a_-^\dagger(q)] = -1 \cdot \delta(p-q). \tag{13}$$

Let us consider the case where diag $G_{AB} = (-1, 1, \ldots, 1)$, which is compatible to the above case. Then, Q_b^{+-} and Q_b^{--} are nilpotent and hermitian charges like a BRS charge. Next, we would like to find the quartet representation for the third BRS transformation. We take the quartet for $Q_b^v \equiv Q_b^{\mp -}$ as

$$\chi_v^\dagger(p) = a^{+\dagger}(p), \quad \gamma_v^\dagger(p) = c_\mp^\dagger(p), \quad \beta_v^\dagger(p) = a^{-\dagger}(p), \quad \bar{\gamma}_v^\dagger(p) = \bar{c}_\mp^\dagger(p), \tag{14}$$

It is easy to prove that Q_b^v anti-commutes with $Q_b^{(\pm)}$ and Q_b^{Weyl}. Hence, we find the physical subspace as

$$\mathcal{V}_{phys} = \mathcal{V}_{\phi^3 \ldots \phi^N} \oplus \mathcal{V}_0^\pm \oplus \mathcal{V}_0^{Weyl} \oplus \mathcal{V}_0^v,$$

which is defined by

$$Q_b^\pm \,|\text{phys}\rangle = 0, \quad Q_b^{Weyl}\,|\text{phys}\rangle = 0 \quad \text{and} \quad Q_b^v\,|\text{phys}\rangle = 0, \tag{15}$$

where $\mathcal{V}_0^i$ are the subspaces in which only BRS quartets live and which have the zero-norm; $\mathcal{V}_{\phi^3 \ldots \phi^N}$ is positive-definite physical state in which only $\{\phi^3, \ldots, \phi^N\}$ live. If $\mathcal{V}_{\phi^3 \ldots \phi^N}$ has positive-definite norm, total physical subspace is positive semi-definite due to the Kugo-Ojima's quartet mechanism. If not, we can, at least, confirm that the contribution from the quartets is exactly zero*

At first, scalar fields are not constrained. In quantum theory, the unitarity require the relevant signatures in the ϕ's space. Scalar field which has a negative-signatured kinetic term is not familiar in ordinary particle physics. In this situation, the scalar fields becomes a harmonic map which isometrically embeds a curved two-dimensional manifold into N-dimensional Minkowski *space-time*.

Conformal invariance in quantum gravity

In this note, we mainly study the structure of the physical subspace of two-dimensional quantum gravity. We would like to mention other results.

Though local Weyl invariance is broken by the gauge fixing procedure, the invariance under its linearized version still remains;

$$\delta g_{\mu\nu} = (a_\lambda x^\lambda + b) g_{\mu\nu}, \tag{16}$$

where a_μ and b are arbitrary infinitesimal quantities. The generators of the above symmetries are $W = P(B) + M(\bar{C}, C)$ and $W^\mu = M(x^\mu, B)$. As is well known, in the harmonic gauge, the general linear invariance exists. The GL(2) transformation and corresponding generators are given by

$$\delta x^\mu = a^\mu{}_\nu x^\nu, \quad G^\mu{}_\nu = M(x^m u, B_\nu) \tag{17}$$

The translation, the GL(2) and the linearized Weyl symmetries are maximally extended global symmetries in terms of the space-time coordinates. We shall give

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* As we are treating the massless scalar field in two-dimensional space-time, the positive and negative frequency part of the field is ill-defined. If we introduce a small fictitious mass for all fields, we may expect that the quartets have zero-norm as long as the all commutation relations do not change. In this prescription, the space $\mathcal{V}_{\phi^3 \ldots \phi^N}$ is not well-defined because, in this Hilbert-space, the norm cancellation does not occur and infrared divergence emerge as small fictitious mass goes to zero.

their algebras;

$$
\begin{aligned}
&[P_\mu, P_\nu] = 0,\ [P_\mu, G^\nu{}_\lambda] = -i\delta^\nu{}_\mu P_\lambda,\ [G^\mu{}_\nu, G^\rho{}_\lambda] = i(\delta^\rho{}_\nu G^\mu{}_\lambda - \delta^\mu{}_\lambda G^\rho{}_\nu),\\
&[W, W] = [W, W^\mu] = [W^\nu, W^\lambda] = 0, \qquad\qquad (18)\\
&[P_\mu, W] = [G^\nu{}_\lambda, W] = 0,\ [P_\mu, W^\nu] = i\delta^\nu{}_\mu W,\ [G^\mu{}_\nu, W^\lambda] = 2i\delta^\lambda{}_\nu W^\mu.
\end{aligned}
$$

Now, we are going to investigate the conformal invariance. In general curved space, conformal transformation is defined by some combination of the coordinate transformation whose parameter satisfies the conformal Killing equation and Weyl transformation. In ordinary field theories including conformal field theories, one discuss the conformal invariance using conformal Killing equation on some background metric which is external field. In quantum gravity, the metric is no longer external field. So, there are no meaning to solve Killing equation under given metric. In quantum theory, we must take care of the state vectors in order to discuss the symmetries of the quantum system. For examples, we study the vacuum which have the expectation value $\langle 0|\, g_{\mu\nu}\, |0\rangle = \eta_{\mu\nu}$. The symmetries (18) exist in the operator level. We find the spontaneous symmetry breaking

$$IGL(2) \oplus WEYL \to ISO(1,1) \oplus Dilatation$$

occurs; individual symmetries are broken but the special combination $D \equiv G^\mu{}_\mu + 2W$ can be unbroken, this symmetry is just the dilatation. It is to be important that the symmetries under any kind of Weyl transformation are broken under the non-zero background.

Discussions

We obtain " unitary " theory by introducing additional subsidiary condition. The lack of subsidiary condition is a common feature in two-dimensional gravity theory. In Abe and Nakanishi's formulation, gauge fixing term can not be expressed by BRS coboundary. If one can consider the classical (which means non BRS-coboundary term) system in theirs, it becomes scalar-tensor type which have only the general covariance. From the the standpoint the scalar-tensor type, one must introduce additional subsidiary condition by using vector charge which Abe and Nakanishi call vector type Weyl-BRS. This situation is very similar to ours. In string theory, the vector type condition is need. In bosonic string, Kato and Ogawa presented the no-ghost theorem with a subsidiary condition $c_0 |phys\rangle = 0$. Without introducing the above condition, Hamiltonian is doubly degenerate in terms of ghost zero mode. The FP ghost is a world sheet vector. So, this may be vector type. In the present stage, we do not know the relation among them. For this point, further investigation is need.

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RENORMALIZATION GROUP FLOW AND VIRASORO CONSTRAINTS IN TWO-DIMENSIONAL QUANTUM GRAVITY

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ABSTRACT

The idea of Wilson's renormalization group is applied to the two-dimensional Liouville theory coupled to matter fields. The Virasoro structures including those of Liouville field are explicitly derived at the fixed point of the renormalization group flow. The Virasoro operators are transformed into another set of Virasoro operators acting in the target space and it is argued that the latter could be interpreted as those discovered recently in matrix models.

1. Introduction

The two-dimensional gravity has been intensively studied since the advent of the work of Polyakov and his collaborators [1]. We have also witnessed in the last year a remarkable progress in the matrix model [2,3], which is a discretized model for the two-dimensional gravity and offers a poweful method of non-perturbative analysis in the double scaling limit [3].

More recently, the authors of Refs. [4] and [5] have proposed a new way of the matrix model analysis. Their approach is based on the Schwinger-Dyson equation [2,6] satisfied by correlation functions of puncture operators and they derived Virasoro constraints on the partition function. It was argued that these

Virasoro structures would be playing the central role in the matrix model with the help of soliton equations.

The algebraic constraints à la Virasoro, however, raised a new problem, namely, the problem of physical interpretation of the Virasoro algebra. There have been several works [7-10] uncovering various features of the Virasoro structures , but it will be apparently more helpful if we could gain more physical insight into the nature of the analysis of Refs. [4,5]. The present work is motivated by these recent developments in the matrix model and is an outcome of modest attempts to understand the physical meaning of the Virasoro algebra.

In the present work we will investigate Liouville field theories [11-14] coupled to matter fields, and apply to this system the idea of the renormalization group of the Wilson type [15,16]. The renormalization group of the Wilson type has recently been studied extensively in the context of string theories[17-19]. We will modify the work of Hughes, Liu and Polchinski [18] by including the Liouville field, and will show that Virasoro generators show up quite naturally from the stability conditions of the renormalization group flow.

Our emphasis is put on the fact that the method to be developed in the Liouville field theory bears the closest analogy with the matrix model. In fact we convert the Virasoro operators associated with the two-dimensional world sheet to corresponding ones defined in the target space, and suggest that the latter could be identified as those discovered in matrix models.

2. Preliminaries

Let us begin with the basics of the Liouville field theories following the line of David [20] and of Distler and Kawai [21]. (See also Ref. [22].) We consider the two-dimensional system consisting of matter fields $X^\mu(x)$ $(\mu = 1, \cdots, D)$ and the Liouville field $\phi(x)$ together with the ghost $c^m(x)$ and anti-ghost $b_{mn}(x)$. The generating functional is given by the path integral

$$Z[\Psi] = \int \mathcal{D}X\mathcal{D}\phi\mathcal{D}b\mathcal{D}c \exp\{-S[\Psi]\}, \tag{2.1}$$

where the action $S[\Psi]$ consists of the free part and the interacting one $S_{int}[\Psi]$,

$$S[\Psi] = S_X + S_g + S_\phi + S_{int}[\Psi], \tag{2.2}$$

$$S_X = \frac{1}{16\pi\alpha'} \int d^2x \sqrt{\hat{g}}\hat{g}^{mn}\partial_m X^\mu \partial_n X^\mu, \tag{2.3}$$

$$S_g = \frac{1}{4\pi\alpha'} \int d^2x \sqrt{\hat{g}} \Big(2b_{mn} \nabla^m c^n + \nabla^m b_{mn} c^n \Big), \tag{2.4}$$

$$S_\phi = \frac{1}{8\pi} \int d^2x \sqrt{\hat{g}} \Big(\hat{g}^{mn} \partial_m \phi \partial_n \phi - Q\hat{R}\phi + \mu e^{\alpha\phi} \Big). \tag{2.5}$$

The Liouville field $\phi(x)$ has been extracted from the two-dimensional metric $g_{mn}(x)$ by the formula

$$g_{mn}(x) = e^{\alpha\phi(x)} \hat{g}_{mn}(x). \tag{2.6}$$

It is apparent that, since the original theory depends only on $g_{mn}(x)$, there should be an invariance under the simultaneous change

$$\hat{g}_{mn}(x) \to e^{\sigma(x)} \hat{g}_{mn}(x), \qquad \phi(x) \to \phi(x) - \frac{1}{\alpha}\sigma(x). \tag{2.7}$$

To determine the coefficients Q and α, we follow the argument of Refs. [20] and [21]. In fact, the condition for the conformal anomaly to vanish gives us

$$Q = \sqrt{\frac{25 - D}{3}}. \tag{2.8}$$

The other coefficient α is determined by requiring that $e^{\alpha\phi(x)}$ be a conformal tensor of weight (1,1), i.e.,

$$\alpha = -\frac{1}{2\sqrt{3}}(\sqrt{25 - D} - \sqrt{1 - D}). \tag{2.9}$$

With these choices of the coefficients, the path integral measures in Eq. (2.1) are the usual ones referring to the metric $\hat{g}_{mn}(x)$. The renormalized cosmological constant μ may be set equal to zero by adjusting the unrenormalized one. The action in Eq. (2.1) has been supplemented by the interaction part $S_{int}[\Psi]$, which consists of an infinite number of background fields. More explicitly, $S_{int}[\Psi]$ is expressed as an infinte series of vertex operators with gravitational dressing

$$\begin{aligned} S_{int}[\Psi] \sim & \int d^2x \sqrt{\hat{g}} \int d^Dk \int d\beta \;\; T(k,\beta) e^{ik\cdot X} e^{\beta\phi} \\ & + \int d^2x \sqrt{\hat{g}} \int d^Dk \int d\beta \;\; h_{\mu\nu}(k,\beta) \hat{g}^{mn} \partial_m X^\mu \partial_n X^\nu e^{ik\cdot X} e^{\beta\phi} \\ & + \cdots\cdots\cdots. \end{aligned} \tag{2.10}$$

We have denoted the "tachyon" and "graviton" fields in the target space by

$T(k,\beta)$ and $h_{\mu\nu}(k,\beta)$ and ellipses in Eq. (2.10) stand for all the other excitation modes. The background fields, $T(k,\beta)$, $h_{\mu\nu}(k,\beta)$ and so forth are denoted collectively by Ψ in Eq. (2.10), which is an analogue of the string field in the second-quantized string theory. We will later describe in more detail the techniques of incorporating the infinite towers of excited states into $S_{int}[\Psi]$ together with the precise definition of the string field Ψ.

3. Renormalization Group Flow

We are now in a position to apply the idea of Wilson's renormalization group. We will look for an equation describing the stable points of the renormalization group flow in the Liouville theory along the line of Hughes et al [18]. The stability condition of the flow turns out to be a constraint on the interaction action $S_{int}[\Psi]$.

To explain the procedures in general terms let us consider a transformation of the path integral variables

$$X^{\mu}(x) \rightarrow X^{\mu}(x) + \Delta X^{\mu}(x), \tag{3.1a}$$

$$b_{mn}(x) \rightarrow b_{mn}(x) + \Delta b_{mn}(x), \tag{3.1b}$$

$$c^{m}(x) \rightarrow c^{m}(x) + \Delta c^{m}(x), \tag{3.1c}$$

$$\phi(x) \rightarrow \phi(x) + \Delta\phi(x). \tag{3.1d}$$

The partition function (2.1) will be unchanged provided that our dynamical system has the symmetry under the transformation (3.1). More explicitly we have the following equation

$$\int d^2x\sqrt{\hat{g}}\Big\{ -\Delta X^{\mu}(x)\frac{\delta S}{\delta X^{\mu}(x)} - \Delta b_{mn}(x)\frac{\delta S}{\delta b_{mn}(x)} - \Delta c^{m}(x)\frac{\delta S}{\delta c^{m}(x)} - \Delta\phi(x)\frac{\delta S}{\delta\phi(x)}$$
$$+\frac{\delta\Delta X^{\mu}(x)}{\delta X^{\mu}(x)} - \frac{\delta\Delta b_{mn}(x)}{\delta b_{mn}(x)} - \frac{\delta\Delta c^{m}(x)}{\delta c^{m}(x)} + \frac{\delta\Delta\phi(x)}{\delta\phi(x)}\Big\} = 0. \tag{3.2}$$

The terms in the second line of Eq. (3.2) come from the Jacobian due to the change of the path integral variables.

To qualify Eq. (3.2) as an equation describing the stable point of the renormalization group flow, the change of variables (3.1) should be due to the two-dimensional conformal transformation

$$x^m \to x^m + v^m(x), \tag{3.3}$$

where we demand that the infinitesimal parameter function $v^m(x)$ satisfy locally $\nabla_m v_n(x) + \nabla_n v_m(x) = g_{mn}(x)\nabla \cdot v(x)$

Hughes, Liu and Pochinski [18] argued that the transformation of X^μ due to the local scale transformations is accompanied by a generalized anomalous dimension. In our case the transformation (3.1) should be

$$\begin{aligned}\Delta X^\mu(x) = {} & -v^m(x)\partial_m X^\mu(x) - \frac{1}{2}\int d^2y\sqrt{\hat{g}}\ \ \delta_v N(x,y)\frac{\delta S}{\delta X^\mu(y)} \\ & + \int d^2y\sqrt{\hat{g}}\ \ (\delta_v N \cdot N^{-1})(x,y)X^\mu(y),\end{aligned} \tag{3.4a}$$

$$\begin{aligned}\Delta b_{mn}(x) = {} & -v^l(x)\nabla_l b_{mn}(x) - \nabla_m v^l(x) b_{ln}(x) - \nabla_n v^l(x) b_{lm}(x) \\ & - \frac{1}{4}\int d^2y\sqrt{\hat{g}}\ \ \delta_v G_m(x,y)\frac{\delta S}{\delta c^n(y)} \\ & + \frac{1}{2}\int d^2y\sqrt{\hat{g}}\ \ \delta_v G_m \cdot (G^{-1})^l(x,y) b_{ln}(y),\end{aligned} \tag{3.4b}$$

$$\begin{aligned}\Delta c^m(x) = {} & -v^n(x)\nabla_n c^m(x) + c^n(x)\nabla_n v^m(x) \\ & - \frac{1}{4}\int d^2y\sqrt{\hat{g}}\ \ \delta_v G_n(x,y)\frac{\delta S}{\delta b_{mn}(y)} \\ & + \frac{1}{2}\int d^2y\sqrt{\hat{g}}\ \ \delta_v G_n \cdot (G^{-1})^n(x,y) c^m(y),\end{aligned} \tag{3.4c}$$

$$\begin{aligned}\Delta\phi(x) = {} & -v^m(x)\partial_m\phi(x) + \frac{1}{2}Q\nabla \cdot v(x) \\ & - \frac{1}{2}\int d^2y\sqrt{\hat{g}}\ \ \delta_v N_\phi(x,y)\frac{\delta S}{\delta\phi(y)} \\ & + \int d^2y\sqrt{\hat{g}}\ \ (\delta_v N_\phi \cdot N_\phi^{-1})(x,y)\phi(y).\end{aligned}$$

(3.4d)

The symmetrization with respect to the two indices of Δb_{mn} is always assumed in (3.4b). We have introduced in Eq. (3.4) the two-point functions

$$\langle X^\mu(x) X^\nu(y)\rangle = \delta^{\mu\nu} N(x,y), \tag{3.5a}$$

$$\langle b_{mn}(x) c^n(y)\rangle = G_m(x,y), \tag{3.5b}$$

$$\langle \phi(x)\phi(y)\rangle = N_\phi(x,y). \tag{3.5c}$$

The variations of the two-point functions under (3.3) are denoted by $\delta_v N(x,y)$, $\delta_v G_m(x,y)$, and $\delta_v N_\phi(x,y)$, respectively. The generalized anomalous dimensions give rise to contributions containing these variations in Eq. (3.4). The term proportional to Q in Eq. (3.4d) was introduced so that the variation of the scalar curvature term in Eq. (2.5) is compensated.

Putting Eq. (3.4) into Eq. (3.2) altogether, we arrive at the equation constraining the interaction action $S_{int}[\Psi]$

$$\mathcal{G}_X[S_{int}] + \mathcal{G}_g[S_{int}] + \mathcal{G}_\phi[S_{int}] = 0, \tag{3.6}$$

where we have defined the renormalization group operations $\mathcal{G}$ as follows

$$\begin{aligned}\mathcal{G}_X[S_{int}] = &\int d^2x\sqrt{\hat{g}} v^m(x)\partial_m X^\mu(x)\frac{\delta S_{int}[\Psi]}{\delta X^\mu(x)} \\ &-\frac{1}{2}\int d^2x\sqrt{\hat{g}}\int d^2y\sqrt{\hat{g}}\ \ \delta_v N(x,y)\left\{\frac{\delta^2 S_{int}[\Psi]}{\delta X^\mu(x)\delta X^\mu(y)} - \frac{\delta S_{int}[\Psi]}{\delta X^\mu(x)}\frac{\delta S_{int}[\Psi]}{\delta X^\mu(y)}\right\},\end{aligned} \tag{3.7a}$$

$$\begin{aligned}\mathcal{G}_g[S_{int}] = &\int d^2x\sqrt{\hat{g}}\Bigg[\left\{v^l(x)\nabla_l b_{mn}(x) + \nabla_m v^l(x) b_{ln}(x) + \nabla_n v^l(x) b_{lm}(x)\right\}\frac{\delta S_{int}[\Psi]}{\delta b_{mn}(x)} \\ &+\left\{v^m(x)\nabla_m c^n(x) - \nabla_m v^n(x) c^m(x)\right\}\frac{\delta S_{int}[\Psi]}{\delta c^n(x)}\Bigg] \\ &+\frac{1}{2}\int d^2x\sqrt{\hat{g}}\int d^2y\sqrt{\hat{g}}\ \ \delta_v G_m(x,y)\left\{\frac{\delta^2 S_{int}[\Psi]}{\delta b_{mn}(x)\delta c^n(y)} - \frac{\delta S_{int}[\Psi]}{\delta b_{mn}(x)}\frac{\delta S_{int}[\Psi]}{\delta c^n(y)}\right\},\end{aligned} \tag{3.7b}$$

$$\begin{aligned}\mathcal{G}_\phi[S_{int}] = &\int d^2x\sqrt{\hat{g}}\left\{v^m(x)\partial_m\phi(x) - \frac{1}{2}Q\nabla\cdot v(x)\right\}\frac{\delta S_{int}[\Psi]}{\delta\phi(x)} \\ &-\frac{1}{2}\int d^2x\sqrt{\hat{g}}\int d^2y\sqrt{\hat{g}}\ \ \delta_v N_\phi(x,y)\left\{\frac{\delta S_{int}[\Psi]}{\delta\phi(x)\delta\phi(y)} - \frac{\delta S_{int}[\Psi]}{\delta\phi(x)}\frac{\delta S_{int}[\Psi]}{\delta\phi(y)}\right\}.\end{aligned} \tag{3.7c}$$

4. Interaction $S_{int}[\Psi]$

Before launching into the analysis of the stability condition (3.6), we have to digress a little bit to construct $S_{int}[\Psi]$ in the most general settings. The methods we are going to make use of are those proposed by Ichinose and Sakita [23] to construct string emission vertex operators and generalization thereof [24,25]. It is not difficult to include the Liouville field into their framework. Hereafter we will work in the conformally flat metric for $\hat{g}_{mn}$

$$\hat{g}_{z\bar{z}} = \hat{g}_{\bar{z}z} = \frac{1}{2}, \qquad \hat{g}_{zz} = \hat{g}_{\bar{z}\bar{z}} = 0 \tag{4.1}$$

and the interaction is given by an integral of the string emission vertex operator over $(z, \bar{z})$

$$S_{int}[\Psi] = \int d^2 z V[z, \bar{z}; \Psi]. \tag{4.2}$$

The string emission vertex in Ref. [24] is generalized here by including the Liouville field

$$V[z, \bar{z}; \Psi] = \int DpD\xi D\eta Dt \;\; E(p, \xi, \eta, t; X, b, c, \phi)\Psi(p, \xi, \eta, t). \tag{4.3}$$

The integration measures are understood as

$$Dp = \prod_{n=-\infty}^{\infty} dp_n, \quad D\xi = \prod_{n=0}^{\infty} d\xi_n d\bar{\xi}_n, \quad D\eta = \prod_{n=0}^{\infty} d\eta_n d\bar{\eta}_n, \quad Dt = \prod_{n=-\infty}^{\infty} dt_n, \tag{4.4}$$

where ξ and η are both grassmann numbers.

Our notations in Eq. (4.3) are

$$\begin{aligned} &E(p, \xi, \eta, t; X, b, c, \phi) \\ &= \exp\{i \sum_{n=-\infty}^{\infty} p_{n\mu} K_n X^\mu(z, \bar{z})\} \exp\{ \sum_{n=-\infty}^{\infty} t_n K_n \phi(z, \bar{z})\} \\ &\times \exp \sum_{n=0}^{\infty} \{\xi_n K_n c(z) + \eta_n K_n b(z)\} \exp \sum_{n=0}^{\infty} \{\bar{\xi}_n K_{-n} \bar{c}(\bar{z}) + \bar{\eta}_n K_{-n} \bar{b}(\bar{z})\}, \end{aligned} \tag{4.5}$$

$$K_n = \frac{1}{n!}\partial_z^n, \qquad K_{-n} = \frac{1}{n!}\bar{\partial}_{\bar{z}}^n, \qquad (n \geq 0). \tag{4.6}$$

Perhaps a few examples for the string wave functions $\Psi(p, \xi, \eta, t)$ will be useful for better understanding of the interaction action (4.2). The low-lying states such as tachyon and graviton are given, respectively, by

$$\Psi_T(p,\xi,\eta,t) = T(p_0)\frac{\partial}{\partial\xi_0}\frac{\partial}{\partial\bar{\xi}_0}\Psi_0(p,\xi,\eta,t), \tag{4.7a}$$

$$\Psi_G(p,\xi,\eta,t) = h_{\mu\nu}(p_0)\frac{\partial}{\partial p_{1\mu}}\frac{\partial}{\partial p_{-1\nu}}\frac{\partial}{\partial\xi_0}\frac{\partial}{\partial\bar{\xi}_0}\Psi_0(p,\xi,\eta,t). \tag{4.7b}$$

Here $\Psi_0(p,\xi,\eta,t)$ is defined by

$$\Psi_0(p,\xi,\eta,t) = \prod_{n=1}^{\infty}\delta(p_n)\delta(p_{-n})\delta(t_n)\delta(t_{-n})\prod_{n=1}^{\infty}\delta(\xi_n)\delta(\bar{\xi}_n)\delta(\eta_n)\delta(\bar{\eta}_n) \tag{4.8}$$

which corresponds to the vacuum and the derivatives in Eq. (4,7) give rise to excitation of various oscillation modes. It is in fact easy to confirm that Eq. (4.2) with the string function $\Psi = \Psi_T + \Psi_G + \cdots\cdots$ produces Eq. (2.10) (with suitable ghost insertion).

The advantage of using the interaction action (4.2) together with (4.3) lies in the fact that the dependence of the path integral variables $(X, b, \bar{b}, c, \bar{c}, \phi)$ is factored out of the string wave function Ψ. This makes it possible to perform the functional derivatives in Eq. (3.7) without touching on the string states. The fixed point condition will in turn provide us with constraints upon Ψ.

5. Virasoro Structures

Let us now have a closer look at the fixed point condition (3.6). We would like to derive the Virasoro structure from Eq. (3.6) and will confine ourselves to the linear terms in Eq. (3.7) with respect to $S_{int}[\Psi]$.

It is rather straightforward to put Eq. (4.2) into (3.7), thereby taking the functional derivatives. In fact Eq. (3.7a) is put in the form

$$\begin{aligned}\mathcal{G}_X[S_{int}] = \int d^2z\sqrt{\hat{g}}\int DpD\xi D\eta Dt \;\; E(p,\xi,\eta,t;X,b,c,\phi)\Psi(p,\xi,\eta,t) \\ \times\left\{i\sum_{n=0}^{\infty}p_{n\mu}K_n[v(z)\partial_z X^\mu(z,\bar{z})]\right. \\ \left. +\frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}p_n\cdot p_m K_n K'_m\delta_v N(z,z')\,|_{z'=z}\right\} + C.C.\end{aligned} \tag{5.1}$$

Here $C.C.$ stands for the anti-holomorphic part due to the variation $\bar{v}(\bar{z})$.

The first term in the brackets above is a classical part. It can be rewritten after some combinatrics as

$$i\sum_{n=0}^{\infty} p_n \cdot K_n[v(z)\partial_z X(z,\bar{z})] = i\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(m+1)p_{m+n}\cdot K_n v(z) K_{m+1} X(z,\bar{z}) \quad (5.2)$$

and we can replace $iK_{m+1}X^{\mu}(z,\bar{z})$ in Eq. (5.2) by the derivative $\partial/\partial p^{\mu}_{m+1}$ acting on $E(p,\xi,\eta,t;X,b,c,\phi)$. The second term in (5.1), on the other hand, is of quantum origin and depends on the regularization of the two-point function. Here we employ the conventional regularization, the so-called conformal ordering method [26], which implies that the variation is given by the formula

$$K_m K'_n \delta_v N(z,z')\,|_{z'=z} = 2\alpha' K_{m+n+1} v(z). \quad (m,\ n \geq 0) \qquad (5.3)$$

By putting the formula (5.3) into Eq. (5.1) and after some manipulations, we end up with a simple expression

$$\mathcal{G}_X[S_{int}] = \int d^2 z\sqrt{\hat{g}}\int DpD\xi D\eta DtE(p,\xi,\eta,t;X,b,c,\phi) \times \sum_{n=0}^{\infty}\left\{K_n v(z) L^{(X)}_{n-1} + K_{-n}\bar{v}(\bar{z})\bar{L}^{(X)}_{n-1}\right\}\Psi(p,\xi,\eta,t), \qquad (5.4)$$

where we have introduced the Virasoro generators

$$L^{(X)}_n = -\sum_{m=1}^{\infty} m\frac{\partial}{\partial p_m}\cdot p_{m+n} + \alpha'\sum_{m=0}^{n} p_m \cdot p_{n-m}, \qquad (n \geq 0) \qquad (5.5a)$$

$$L_{-1} = -\sum_{m=1}^{\infty} m\frac{\partial}{\partial p_m}\cdot p_{m-1}. \qquad (5.5b)$$

The p-derivatives in Eq. (5.5) has been made to act on $\Psi(p,\xi,\eta,t)$ in Eq. (5.4) by partial integration. The definition of $\bar{L}^{(X)}_n$ is obtained by replacing p_n by p_{-n} in the above expression.

In a similar way, we can shuffle the renormalization group operations, $\mathcal{G}_g[S_{int}]$ and $\mathcal{G}_\phi[S_{int}]$ in Eq. (3.6). The regularization rule for the two-point functions for ghost and Liouville fields are:

$$K_m K'_n \delta_v G_z(z,z')\,|_{z'=z} = -2(n+2m+3)K_{m+n+2}v(z), \quad (m,\ n \geq 0) \qquad (5.6a)$$

$$K_m K'_n \delta_v N_\phi(z,z')\,|_{z'=z} = K_{m+n+1}v(z). \qquad (5.6b)$$

The final formula for the stability condition (3.6) turns out to be

$$\begin{aligned} 0 &= \mathcal{G}_X[S_{int}] + \mathcal{G}_g[S_{int}] + \mathcal{G}_\phi[S_{int}] \\ &= \int d^2z\sqrt{\hat{g}} \int DpD\xi D\eta Dt \;\; E(p,\xi,\eta,t;X,b,c,\phi) \\ &\times \sum_{n=0}^{\infty}\Big\{K_n v(z)L_{n-1} + K_{-n}\bar{v}(\bar{z})\bar{L}_{n-1}\Big\}\Psi(p,\xi,\eta,t), \end{aligned} \qquad (5.7)$$

where

$$L_n = L_n^{(X)} + L_n^{(g)} + L_n^{(\phi)}, \quad \bar{L}_n = \bar{L}_n^{(X)} + \bar{L}_n^{(g)} + \bar{L}_n^{(\phi)}. \qquad (5.8)$$

The Virasoro generators of ghost and Liouville sectors are defined by

$$\begin{aligned} L_n^{(g)} &= \sum_{m=0}^{\infty}\Big\{(-n-1+m)\frac{\partial}{\partial\xi_m}\xi_{m+n} + (2n+2+m)\frac{\partial}{\partial\eta_m}\eta_{m+n}\Big\} \\ &+ \sum_{m=1}^{n}(m+n+1)\xi_{n-m}\eta_{m-1}, \quad (n \geq 0) \end{aligned} \qquad (5.9a)$$

$$L_{-1}^{(g)} = \sum_{m=0}^{\infty}(m+1)\Big\{\frac{\partial}{\partial\xi_{m+1}}\xi_m + \frac{\partial}{\partial\eta_{m+1}}\eta_m\Big\}, \qquad (5.9b)$$

$$L_n^{(\phi)} = -\sum_{m=1}^{\infty} m\frac{\partial}{\partial t_m}t_{m+n} - \frac{1}{2}Q(n+1)t_n - \frac{1}{2}\sum_{m=0}^{n} t_m t_{n-m}, \quad (n \geq 0) \qquad (5.10a)$$

$$L_{-1}^{(\phi)} = -\sum_{m=1}^{\infty} m\frac{\partial}{\partial t_m}t_{m-1}. \qquad (5.10b)$$

The generators $\bar{L}_n^{(g)}$ and $\bar{L}_n^{(\phi)}$ are defined in a similar way. Note that Eqs. (5.5),(5.9) and (5.10) constitute the Virasoro algebras with the central charges

D, -26 and $1+3Q^2 = 26-D$, respectively and thus the total central charge vanishes as is expected. The correspondence to the usual oscillator mode notations for (5.5) and (5.9) is established by the following identification

$$p_m \leftrightarrow \frac{1}{\sqrt{2\alpha'}}\alpha_m, \quad \frac{\partial}{\partial p_m} \leftrightarrow -\frac{\sqrt{2\alpha'}}{m}\alpha_{-m}, \quad (m \geq 1) \tag{5.11a}$$

$$\xi_m \leftrightarrow b_{m-1}, \quad \frac{\partial}{\partial \xi_m} \leftrightarrow c_{1-m}, \quad (m \geq 0) \tag{5.11b}$$

$$\eta_m \leftrightarrow c_{2+m}, \quad \frac{\partial}{\partial \eta_m} \leftrightarrow b_{-2-m}. \quad (m \geq 0) \tag{5.11c}$$

6. Liouville Field Theories versus Matrix Models

It is very interesting to compare the above considerations based on the Wilson's renormalization group with the matrix model analysis. The partition function of the matrix model is defined by

$$Z(g) = \int dM \ \exp\{-\beta U(M)\}, \tag{6.1}$$

where M is an $N \times N$ hermitian matrix and the potential is given by

$$U(M) = \sum_{l=0}^{\infty} g_l tr(M^l). \tag{6.2}$$

We have introduced in (6.2) an infinite number of coupling constants $\{g_l\}$, which are matrix model counterparts of the string wave functions $\Psi = (T, h_{\mu\nu}, \cdots)$ in Liouville theories. The integration measure in (6.1) is defined as $dM = \prod_i dM_{ii} \prod_{i<j} d(\Re M_{ij}) d(\Im M_{ij})$.

As was discussed in Ref. [27], the derivation of the Virasoro structures is facilitated by considering the change of integration variables

$$M \to M + \epsilon M^{n+1}. \tag{6.3}$$

The variations of the potential and the integration measure under (6.3) become

$$U(M) \to U(M) + \epsilon \sum_{l=0}^{\infty} l g_l tr(M^{l+n}), \tag{6.4}$$

$$dM \to dM + \epsilon dM \sum_{l=0}^{n} tr(M^l) tr(M^{n-l}). \tag{6.5}$$

The invariance of the partition function $Z(g)$ under the change of integration variables amounts to the condition

$$\begin{aligned} 0 = \delta Z(g) &= \epsilon \Big\{ -\frac{1}{\beta} \sum_{l=0}^{\infty} l g_l \frac{\partial}{\partial g_{l+n}} + \frac{1}{\beta^2} \sum_{l=0}^{n} \frac{\partial}{\partial g_l} \frac{\partial}{\partial g_{n-l}} \Big\} Z(g) \\ &\equiv \frac{\epsilon}{\beta} \hat{\mathcal{L}}_n Z(g). \end{aligned} \tag{6.6}$$

It is easy to confirm that the differential operators $\{\hat{\mathcal{L}}_n\}$ satisfy the Virasoro algebra.

Now it is almost self-evident that there exists a strong resemblance between the matrix model and Liouville theory in deriving each Virasoro structure. Although we do not know the precise relation between the matrix M and the dynamical variables $(X^\mu, b_{mn}, c^m, \phi)$, the change of integration variables (6.3) is reminiscent of the scale transformation (3.1) and (3.4). The physical meaning of the Virasoro operators is, however, rather subtle. The operators $\{\hat{\mathcal{L}}_n\}$ is defined in the space of the coupling constants $\{g_l\}$, while those in Liouville theory (5.8) are associated with oscillation modes on the two-dimensional world surface. To strengthen the relation between matrix and Liouville approaches, all we need is to find in the latter the analogue of $\{\hat{\mathcal{L}}_n\}$.

If we could confine ourselves to the linear terms in $S_{int}[\Psi]$, it would be extremely tempting to introduce the following set $\{\mathcal{L}_n\}$

$$\mathcal{L}_n Z[\Psi] \equiv \int \mathcal{D}X\mathcal{D}\phi\mathcal{D}b\mathcal{D}c \exp\{-S_X - S_g - S_\phi\} S_{int}[L_n \Psi]. \tag{6.7}$$

The operators $\{\mathcal{L}_n\}$, defined rather indirectly in (6.7), are supposed to act in the space of $\Psi = (T, h_{\mu\nu} \cdots)$. One can easily see that the algebra of $\{L_n\}$ is transcribed into another Virasoro algebra

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n}. \tag{6.8}$$

We can not resist identifying the operators $\{\mathcal{L}_n\}$ as the analogue of the matrix model Virasoro generators $\{\hat{\mathcal{L}}_n\}$.

At the present stage we are not able to express $\{\mathcal{L}_n\}$ in an explicit and closed form, nor are we quite sure if $\{\mathcal{L}_n\}$ are well-defined by (6.7) without any ambiguities. Further investigations on these matters are apparently necessary for a deeper understanding of the algebraic structures.

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2D Topological Gravity and String Theory*

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近年 Random Surface の量子論が急速に進展し、又多くの関心を集めている。この問題は（特にsubcriticalな次元での）弦理論、又二次元量子重力と不可分なものである。Random Surface の量子論へのアプローチとしては、本研究会での講演でもあったように、およそ以下の三通りの考え方があると思われる。

1) Liouville mode の量子化

2) Large N Matrix Models

3) Topological Field Theories

特に最近2)と3)の関係として、Multi-Matrix model と Topological (minimal) matter と Topological gravity がcoupleした系（以下このような系をTopological String [2]と呼ぶ）との同等性が指摘され、Topological Field Theory によるアプローチもますます重要視されつつある。いずれにしても上記のアプローチの包括的理解は重要な問題の一つである。

本研究会では2D Topological Gravity 及び Topological String のBRST Formulationに対するコメントを行った。

2D Topological Gravity のactionの導出[3]においては、先ず2D metric に対するtopologicalな変換

$$\text{T)} \quad \delta g_{\mu\nu} = \psi_{\mu\nu}$$

そして同時にdiffeomorphism

$$\text{D)} \quad \delta g_{\mu\nu} = \nabla_\mu c_\nu + \nabla_\nu c_\mu$$

を考える。ここで $\psi_{\mu\nu}$ 及び c^μ はghostである。適当なゲージ固定の後、BRST不変なactionとして

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$$S^{Gravity} = \int d^2z \, (b_{zz} \partial_{\bar{z}} c^z + b_{\bar{z}\bar{z}} \partial_z c^{\bar{z}}) + (\beta_{zz} \partial_{\bar{z}} \gamma^z + \beta_{\bar{z}\bar{z}} \partial_z \gamma^{\bar{z}})$$

又BRST Charge として

$$Q^{Gravity} = \oint \{ b_{zz} \gamma^z + \frac{1}{2} c^z (c^z \partial_z b_{zz} + 2 \partial_z c^z b_{zz}) + c^z (\gamma^z \partial_z \beta_{zz} + 2 \partial_z \gamma^z \beta_{zz}) \} + c.c.$$

を得る。ここに γ^μ は変換T)及びD)が独立でないために必要になったghost for ghost で そのghost numberは２である。

一方最も簡単なTopologocalなMatterとして、作用

$$S = \int d^2\sigma \;\; \epsilon^{\mu\nu} J_{ab}(x) \partial_\mu x^a \partial_\nu x^b$$

を考える。ここで a,b は space-time target manifold M の足で、今Mは２次元であるとする。$J_{ab}(x)$は M上のcomplex structure である。（この作用はナイーブには2D Nambu-Goto action

$$S = - \int d^2\sigma \sqrt{\det (\partial_\mu x^a \partial_\nu x_a)}$$

の書換と見えるが、平方根を開く際surfaceのparity(±)の違いがある。その為にこの作用はworld-sheetの面積ではなくworld-sheetからtarget manifold M への wrapping number ($\times 4\pi$) になっている。） この作用も topological な変換

T) $\delta x^a = \xi^a$

の下で不変である。 又一方world-sheetのdiffeomorphism

D) $\delta x^a = c^\mu \partial_\mu x^a$

の下でも明らかに不変である。

ここで取り上げたい問題は topologicalな変換T) のみならず diffeomorphism D) も同時に考慮して量子的作用を構成するとどうなるかという事である。2D Topological Gravity の場合の様に T) と D) とが独立でない (reducible)為にghost for ghost γ^μ の導入が必要になる。 Matter X^a 及び ghost場のBRST変換は

$$\delta X^a = \xi^a + c^\mu \partial_\mu X^a$$

$$\delta \xi^a = c^\mu \partial_\mu \xi^a - \gamma^\mu \partial_\mu X^a$$

$$\delta c^\mu = c^\nu \partial_\nu c^\mu + \gamma^\mu$$

$$\delta \gamma^\mu = c^\nu \partial_\nu \gamma^\mu - \gamma^\nu \partial_\nu c^\mu$$

となる。

我々は二つの異なるgauge条件を課し、BRST不変な作用を導く事を考察した。一つ目のgauge条件は(以下world-sheet 及び target manifold いずれについてもcomplex coordinateをとって、)

$$\partial_{\bar z} X^+ = 0$$

$$\partial_z X^- = 0$$

$$D_{\bar z} \xi^+ = \partial_{\bar z} \xi^+ + \partial_{\bar z} X^+ \Gamma^+_{++}(X) \xi^+ = 0$$

$$D_z \xi^- = \partial_z \xi^- + \partial_z X^- \Gamma^-_{--}(X) \xi^- = 0$$

である。antighost η, ρ , Nakanisi-Lautrap field B,E を導入してgauge固定すれば 作用は

$$
\begin{aligned}
S^{I} = {} & G_{+-}\left[B_{\bar z}{}^{-}\partial_{\bar z}X^{+} + B_{\bar z}{}^{+}\partial_{z}X^{+} + E_{z}{}^{-}D_{\bar z}\xi^{+} + E_{\bar z}{}^{+}D_{z}\xi^{-}\right] \\
& + G_{+-}(\eta_{z}{}^{-}\partial_{z}X^{+} + \rho_{z}{}^{-}D_{z}\xi^{+})\partial_{\bar z}C^{z} \\
& + G_{+-}(\eta_{\bar z}{}^{+}\partial_{\bar z}X^{-} + \rho_{\bar z}{}^{+}D_{\bar z}\xi^{-})\partial_{z}C^{\bar z} \\
& + G_{+-}(\rho_{z}{}^{-}\partial_{z}X^{+})\partial_{\bar z}\gamma^{z} + G_{+-}(\rho_{\bar z}{}^{+}\partial_{\bar z}X^{-})\partial_{z}\gamma^{\bar z}.
\end{aligned}
$$

となる。ここで新しい場を以下のように導入する。

$$b_{zz} = G_{+-}(x)(\eta_{z}{}^{-}\partial_{z}X^{+} + \rho_{z}{}^{-}D_{z}\xi^{+})$$

$$b_{\bar z\bar z} = G_{+-}(x)(\eta_{\bar z}{}^{+}\partial_{\bar z}X^{-} + \rho_{\bar z}{}^{+}D_{\bar z}\xi^{-})$$

$$\beta_{zz} = G_{+-}(x)\,\rho_{z}{}^{-}\partial_{z}X^{+}$$

$$\beta_{\bar z\bar z} = G_{+-}(x)\,\rho_{\bar z}{}^{+}\partial_{\bar z}X^{-}$$

ここで変数変換にともなうJacobianは1となる事に注意。 こうして作用Sは2D Topological Gravity の作用に一致する。 又BRST変換及びBRST chargeも 2D Topological Gravity のそれに一致する事が容易に示される。

又一方 次のgauge条件

$$\partial_{\bar z}X^{+} = \frac{1}{2}B_{\bar z}{}^{+}$$

$$\partial_{z}X^{-} = \frac{1}{2}B_{z}{}^{-}$$

$$\partial_{\bar z}C^{z} = 0$$

$$\partial_{z}C^{\bar z} = 0$$

を考えてみる。ghost ξ^{a} と b の再定義をすれば、今度は

$$S^{II} = \int d^2z \left\{ \partial_{\bar z} X^+ \partial_z X^- + (\eta_z{}^- \partial_{\bar z} \xi^+ + \eta_{\bar z}{}^+ \partial_z \xi^-) + (b_{zz} \partial_{\bar z} c^z + b_{\bar z \bar z} \partial_z c^{\bar z}) + (\beta_{zz} \partial_{\bar z} \gamma^z + \beta_{\bar z \bar z} \partial_z \gamma^{\bar z}) \right\}$$

という作用になる事が解る。この場合のBRST charge も

$$Q = Q^{Matter} + Q^{Gravity}$$

で与えられ、系として Topological matter が Topological Gravity とcoupleした系、いわゆる Topological String になっている事が解る。

以上では便宜上 Target manifold の次元を2としたが、実は任意の次元でのTopological sigma model の導出の際にdiffeomorphismを考慮してやればやはり同様の結果を得る事ができる。この様に異なるgauge固定により異なるphysical space（local な dynamical な自由度はもちろんgaugeに依らずこの場合には存在しないわけであるが、いわゆるmoduliと呼ばれるglobalな自由度はgaugeの取り方に依存する。）を持つ理論を得たわけであるが、今後このgauge条件の幾何学的意味そしてobservablesについて考察する必要があるとおもわれる。

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Superconformal Topological Field Theory*

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abstract

We obtain conformal invariant topological field theories with $N = 2$ supersymmetry by twisting Sevrin, Troost and Van Proeyen's $SU(2) \times SU(2) \times U(1)$ -extended $N = 4$ superconformal field theories. We expect that the number of physical states is finite although the original $N = 4$ theories have continuous spectra. It is shown that the number of physical states is actually finite when the central charge $c < 6$ in the corresponding $N = 4$ theories. The physical states inherit the structure of chiral ring in $N = 2$ superconformal *minimal* series which is obtained by the reduction from $N = 4$ theories. We also show that the algebra contains topological $N = 1$ superconformal algebra as subalgebra. Therefore a closed set of finite number of physical states in the topological $N = 1$ superconformal algebra can be also obtained.

Recently the great progress has been made in the study of two dimensional gravity by using matrix models.[2,3,4] The recursion relations for the correlation functions have structures similar to those of topological field theories.[5] We expect[6] that two dimensional gravity can be described by two dimensional topological gravity[7] coupled with some topological matter.[8,9] The topological conformal matter is obtained by twisting $N = 2$ superconformal field theories ($N = 2$ SCFT).[10] Especially topological *minimal* series corresponding to *minimal* conformal series coupled with gravity is obtained by twisting the *minimal* series of $N = 2$ SCFT.[8,9] When the energy momentum (EM) tensor is twisted by the $U(1)$ current, the conformal dimensions of all the charged operators are modified. If

⋆ This report is mainly based on Ref.1.
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we consider a supersymmetric extension of topological gravity,[11,12] we expect that there exists a topological conformal matter with supersymmetry.

Topological conformal matter with $N = 1$ supersymmetry has been obtained by twisting $N = 3$ SCFT.[13] $N = 3$ SCFT contains $SO(3)$ currents. When twisted by the current of $U(1) \subset SO(3)$, one of the three supercurrents, which is neutral with respect to the $U(1)$ current,* remains conformal dimension 3/2. When we construct topological superstrings, topological $N = 2$ SCFT ($N = 2$ TSCFT) will be required. Recently proposed matrix models having space-time supersymmetry[14,15] might be described by topological supergravity coupled with topological $N = 2$ superconformal matter. An example of $N = 2$ SCFT has been obtained[16] by twisting Schoutens' $SO(4)$-extended $N = 4$ SCFT ($SO(4)$ $N = 4$ SCFT).[17] Notice that this $SO(4)$ $N = 4$ algebra contains the $N = 3$ algebra as algebra.‡

In this report, we construct $N = 2$ TSCFT by twisting Sevrin, Troost and Van Proeyen's $SU(2) \times SU(2) \times U(1)$-extended $N = 4$ SCFT (new $N = 4$ SCFT).[18] The point is that new $N = 4$ SCFT contains two sets of $SU(2)$ currents.♮ If we twist the EM tensor by a current of $U(1) \subset SU(2)_{\text{diagonal}} \subset SU(2) \times SU(2)$, there still remains $N = 2$ supersymmetry since two of the four supercurrents can be neutral with respect to the $U(1)$ current.◇ We expect that the number of physical

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* Three supercurrents are vector representation of $SO(3)$ algebra and they have the $U(1)$ charges ± 1 and 0.

‡ The algebra of conventional $N = 4$ SCFT does *not* contain the $N = 3$ algebra as subalgebra. The $N = 4$ SCFT contains $SU(2)$ currents and four supercurrents are spinor representation of the $SU(2)$ algebra. If we twist the EM tensor by any current of $U(1) \subset SU(2)$, there remains no supersymmetry since there is no supercurrent which is neutral with respect to the $U(1)$ current.

♮ $SO(4)$ $N = 4$ SCFT is given by a special case of new $N = 4$ SCFT and the algebra of new $N = 4$ SCFT does not contain that of $N = 3$ SCFT as subalgebra except this case. When the two sets of $SU(2)$ currents in new $N = 4$ SCFT have the common level, $SU(2) \times SU(2)$ current algebra become $SO(4)$ ($\sim SU(2) \times SU(2)$) current algebra and new $N = 4$ SCFT is reduced to $SO(4)$ $N = 4$ SCFT.

◇ Four supercurrents are spinor representations in both of two $SU(2)$ algebra. Since $2 \times 2 = 1 + 3$, these supercurrents are trivial (1) and vector (3) representations in $SU(2)_{\text{diagonal}}$ algebra. The supercurrent of the trivial representation is $U(1)$ neutral and those of the vector representation has the $U(1)$ charges 0, ± 1.

states in the $N=2$ TSCFT is finite although the original $N=4$ SCFT have continuous spectra. In case the central charge $c<6$ in the corresponding $N=4$ SCFT, we show that the number of physical states is actually finite. The physical states inherit the structure of chiral ring[19] in the *minimal* series of $N=2$ SCFT which was obtained from the reduction of $N=4$ SCFT.[20] Hence the present $N=2$ TSCFT may be regarded as natural analogue of the minimal $N=2$ SCFT. Furthermore we also show that the algebra contains $N=1$ TSCFT as subalgebra. Therefore we can also obtain a closed set of finite number of physical states in the $N=1$ SCFT.

New $N=4$ SCFT[18] algebra are generated by energy momentum tensor $L(z)$, two complex supercurrents $G^{\pm\pm}(z)$, two sets of $SU(2)$ currents $\{A^3(z), A^{\pm}(z)\}$, $\{B^3(z),\ B^{\pm}(z)\}$, two complex fermions $Q^{\pm\pm}(z)$ and a $U(1)$ current $U(z)$. The algebra is summarized in Ref. 1 . The central charge c is determined by the levels of two $SU(2)$ currents $k_A=\frac{c}{6\xi_+}$ and $k_B=\frac{c}{6\xi_-}$: $c=\frac{6k_Ak_B}{k_A+k_B}$. $N=2$ TSCFT is obtained by twisting this new $N=4$ SCFT. We define new energy momentum tensor $\tilde{L}(z)$, new supercurrents $G^{\pm}(z)$, a $U(1)$ current $J(z)$, another $U(1)$ current $\hat{J}(z)$, complex free fermions $\Psi^{\pm}(z)$, a BRS current $Q(z)$, an anti-ghost $b(z)$, a ghost number current $J^{\text{ghost}}(z)$, two anti-superghosts $\beta^{\pm}(z)$ and their conjugates $\gamma^{\pm}(z)$, a $U(1)$-ghost and anti-$U(1)$-ghost $\theta(z)$, $\chi(z)$ as follows,

$$
\begin{aligned}
&\tilde{L}_n = L_n + \frac{1}{2}n(A^3_n + B^3_n) + \frac{i}{2}(\xi_- - \xi_-)nU_n - \frac{c}{24}\delta_{n,0}\ ,\\
&G^{\pm}_n = G^{\pm\mp}_n \pm 2i\xi_{\mp}nQ^{\pm\mp}_n\ ,\\
&J_n = A^3_n - B^3_n + iU_n\ ,\ \hat{J}_n = (\xi_- - \xi_+)(A^3_n + B^3_n) + iU_n\ ,\ \Psi^{\pm}_n = Q^{\pm\mp}_n\ ,\\
&b_n = G^{++}_n + i(\xi_- - \xi_+)nQ^{++}_n\ ,\quad Q_n = G^{--}_n + i(\xi_- - \xi_+)nQ^{--}_n\ ,\\
&J^{\text{ghost}}_n = -A^3_n - B^3_n + \frac{c}{24\xi_+\xi_-}\delta_{n,0}\ ,\\
&\gamma^+_n = B^-_n\ ,\ \gamma^-_n = A^-_n\ ,\ \beta^+_n = A^+_n\ ,\ \beta^-_n = B^+_n\ ,\ \theta_n = Q^{--}_n\ ,\ \chi_n = Q^{++}_n\ .
\end{aligned}
\tag{1}
$$

$Q(z)$, $b(z)$, $\gamma^{\pm}(z)$, $\beta^{\pm}(z)$, $\theta(z)$ and $\chi(z)$ are primary fields with conformal dimensions 1, 2, $\frac{1}{2}$, $\frac{3}{2}$, 0, 1 and ghost number 1, -1, 1, -1, 1, and -1, respectively. We summarize the algebra of $N=2$ SCFT in Ref. 1. We find that the algebra has an isomorphism called spectral flow,

$$\begin{aligned}
&\tilde{L}_n^{(\eta,\zeta)} = \tilde{L}_n - \eta J_n - \zeta J_n^{\text{ghost}} + \frac{c}{24\xi_-}(\zeta^2-\zeta)\delta_{n,0} \ , \\
&G_{n\pm\eta}^{\pm\,(\eta,\zeta)} = G_n^{\pm} \pm i\zeta\Psi_n^{\pm} \ , \ b_{n+\zeta}^{(\eta,\zeta)} = b_n - 2i\eta\chi_n \ , \ Q_{n-\zeta}^{(\eta,\zeta)} = Q_n \ , \\
&J_n^{(\eta,\zeta)} = J_n + \frac{c(\xi_- - \xi_+)}{12\xi_+\xi_-}\zeta\delta_{n,0} \ , \ \hat{J}_n^{(\eta,\zeta)} = \hat{J}_n + \frac{1}{3}c\eta\delta_{n,0} - \frac{c(\xi_- - \xi_+)}{12\xi_+\xi_-}\zeta\delta_{n,0} \ , \\
&J_n^{\text{ghost}\,(\eta,\zeta)} = J_n^{\text{ghost}} - \frac{c(\xi_- - \xi_+)}{12\xi_+\xi_-}\eta\delta_{n,0} - \frac{c}{12\xi_+\xi_-}\zeta\delta_{n,0} \ , \\
&\Psi_{n\pm\eta}^{\pm\,(\eta,\zeta)} = \Psi_n^{\pm} \ , \ \theta_{n-\zeta}^{(\eta,\zeta)} = \theta_n \ , \ \chi_{n+\zeta}^{(\eta,\zeta)} = \chi_n \ .
\end{aligned} \tag{2}$$

Here η and ζ are parameters of flow.

We now discuss the representation of $N=2$ TSCFT in Ref. 1 . In the following, we assume that the indices n of b_n and Q_n is an integer. In the Neveu-Schwarz (Ramond) sector, the index n of supercurrents $G_n^{\pm}$ is a half-integer (an integer). We define 'physical' state $|\Phi>$ as follows:

$$Q_0|\Phi>=0 \ . \tag{3}$$

In addition we choose the following 'gauge' condition for physical states,[8]*

$$b_0|\Phi>=0 \ . \tag{4}$$

Equations (3) and (4) tell that the conformal dimension h of any physical state $|\Phi>$ vanishes since $\{b_0, Q_0\} = 4L_0$ (Ref. 1). If the physical state $|\Phi>$ has a positive norm, we have an inequality with respect to its conformal dimension h and its $U(1)$ charge q : $h \geq 2|q|$ in the Neveu-Schwarz sector.[19] Since the conformal dimension h of any physical state vanishes: $h=0$, its $U(1)$ charge q

* If the difference of two physical states $|\Phi>$ and $|\Phi>'$ is BRS-exact, *i.e.* $|\Phi>=|\Phi>'$ $+Q_0|\Lambda>$ for a state $|\Lambda>$, we should identify $|\Phi>$ with $|\Phi>'$. This ambiguity is fixed by this gauge condition.

also vanishes: $q = 0$. By using the spectral flow (2), we find that the $U(1)$ charge of the physical state vanishes even in the Ramond sector,

$$J_0|\Phi>= 0 \ . \tag{5}$$

Since $U(1)$ charge J_0 is given by $J_0 = A_0^3 - B_0^3 + iU_0$ (1), we have $U_0 = i(A_0^3 - B_0^3)$ on the physical state. Therefore the eigenvalue of U_0 is quantized although U_0 has a continuous spectrum in the original $N = 4$ superconformal theory.

Since anti-ghost b is transformed inhomogeneously under the spectral flow (2), the gauge condition (4) changes in general. To preserve the gauge condition, we may impose

$$\chi_0|\Phi>= 0 \ . \tag{6}$$

This condition will be necessary when we construct a topological superstring theory. We note that the condition (6) can be imposed consistently since $\{Q_0, \chi_0\} = -2iJ_0$ and J_0 vanishes on the physical state.

We expect that the number of physical states in $N = 2$ TSCFT should be finite although the original $N = 4$ theories have continuous spectra. In the following, we consider a series of $N = 2$ TSCFT corresponding to a special series of new $N = 4$ SCFT, where $k_B = 1$ and $k_A = 1, 2, \cdots$ and hence the central charge $c < 6$.† We show that the number of physical states in this series of $N = 2$ TSCFT is actually finite. This series is expected to correspond to the *minimal* series of $N = 2$ SCFT when coupled with topological supergravity.

The series of $N = 4$ SCFT is realized by super $SU(2) \times U(1)$ current algebra *i.e.* $SU(2)$ currents $\{J^3(z), J^{\pm}(z)\}$, complex fermions $\tilde{Q}^{\pm\pm}(z)$ and $U(1)$ current

† It has been known that there is a relation between this series of $N = 4$ SCFT and the *minimal* series of $N = 2$ SCFT.[20]

(free boson) $J^0(z)$,[18] whose operator product expansions are standard,

$$\begin{aligned}
&J^+(z)J^-(w) = \frac{k}{(z-w)^2} + \frac{2J^3(w)}{z-w} + O(1) \ , \\
&J^3(z)J^3(w) = \frac{\frac{k}{2}}{(z-w)^2} + O(1) \ , \ J^3(z)J^\pm(w) = \pm\frac{J^\pm(w)}{z-w} + O(1) \ , \\
&\tilde{Q}^{++}(z)\tilde{Q}^{--}(w) = \frac{1}{z-w} + O(1) \ , \ \tilde{Q}^{+-}(z)\tilde{Q}^{-+}(w) = \frac{1}{z-w} + O(1) \ , \\
&J^0(z)J^0(w) = \frac{\frac{1}{2}}{(z-w)^2} + O(1) \ .
\end{aligned} \tag{7}$$

Then the operators in new $N=4$ SCFT are realized by, ($k_A = k+1$, $k_B = 1$)

$$\begin{aligned}
G^{\pm\pm}(z) &= \frac{1}{\sqrt{k+2}}[\tilde{Q}^{\pm\pm}(z)\{\mp i(\tilde{Q}^{-+}(z)\tilde{Q}^{+-}(z) - \tilde{Q}^{+-}(z)\tilde{Q}^{-+}(z)) \\
&\quad \pm 2iJ^3(z) + 2\sqrt{k+2}J^0(z)\} \pm 2i\tilde{Q}^{\mp\pm}(z)J^\pm(z)] \ , \\
G^{\pm\mp}(z) &= \frac{1}{\sqrt{k+2}}[\tilde{Q}^{\pm\mp}(z)\{\pm i(\tilde{Q}^{++}(z)\tilde{Q}^{--}(z) - \tilde{Q}^{--}(z)\tilde{Q}^{++}(z)) \\
&\quad \pm 2iJ^3(z) + 2\sqrt{k+2}J^0(z)\} \mp 2i\tilde{Q}^{\mp\mp}(z)J^\pm(z)] \ , \\
A^3(z) &= \frac{1}{4}(\tilde{Q}^{++}(z)\tilde{Q}^{--}(z) - \tilde{Q}^{--}(z)\tilde{Q}^{++}(z)) \\
&\quad - \frac{1}{4}(\tilde{Q}^{-+}(z)\tilde{Q}^{+-}(z) - \tilde{Q}^{+-}(z)\tilde{Q}^{-+}(z)) + J^3(z) \ , \\
B^3(z) &= \frac{1}{4}(\tilde{Q}^{++}(z)\tilde{Q}^{--}(z) - \tilde{Q}^{--}(z)\tilde{Q}^{++}(z)) \\
&\quad + \frac{1}{4}(\tilde{Q}^{-+}(z)\tilde{Q}^{+-}(z) - \tilde{Q}^{+-}(z)\tilde{Q}^{-+}(z)) \ , \\
A^\pm(z) &= \pm\tilde{Q}^{\pm\pm}(z)\tilde{Q}^{\pm\mp}(z) + J^\pm(z) \ , \ B^\pm(z) = \pm\tilde{Q}^{\pm\pm}(z)\tilde{Q}^{\mp\pm}(z) \ , \\
Q^{\pm\pm}(z) &= \sqrt{k+2}\tilde{Q}^{\pm\pm}(z) \ , \ U(z) = \sqrt{k+2}J^0(z) \ .
\end{aligned} \tag{8}$$

Notice that this theory has a finite number of primary states if we neglect the contribution from the $U(1)$ currents $U(z) = \sqrt{k+2}J^0(z)$. Since the $U(1)$ charge is restricted after twisting by Eq.(5), there exist only a finite number of physical states in the corresponding $N=2$ TSCFT.

We now show that the physical states can be regarded as a chiral and primary states[19] of the *minimal* series in $N=2$ SCFT. At first we note that $G^{--}(z)$ takes the form: $G^{--}(z) = [-2i\tilde{Q}^{--}(z)J(z) + 2i\tilde{Q}^{+-}(z)J^{-}(z)]/\sqrt{k+2}$. Here $J(z)$ is a $U(1)$ current in Eq.(1). Since a physical state $|\Phi>$: $Q_0|\Phi>= G_0^{--}|\Phi>= 0$ (3) has a vanishing $U(1)$ charge: $J_0|\Phi>= 0$ (5), we obtain

$$(\tilde{Q}^{+-}J^{-})_0|\Phi>= 0\ . \tag{9}$$

Here $(\tilde{Q}^{+-}J^{-})_0 \equiv \frac{1}{2\pi i}\oint_0 dz\, z^{\frac{1}{2}}\tilde{Q}^{+-}(z)J^{-}(z)$. Furthermore since $G^{++}(z)$ has a structure: $G^{++}(z) = [-2i\tilde{Q}^{++}(z)\{\cdots\} + 2i\tilde{Q}^{-+}(z)\ J^{+}(z)]/\sqrt{k+2}$, by using the gauge condition (4) : $b_0|\Phi>= G_0^{++}|\Phi>= 0$ and the condition to preserve the gauge condition : $\chi_0|\Phi> \propto \tilde{Q}_0^{++}|\Phi>= 0$ (6), we find

$$(\tilde{Q}^{-+}J^{+})_0|\Phi>= 0\ . \tag{10}$$

Here $(\tilde{Q}^{-+}J^{+})_0 \equiv \frac{1}{2\pi i}\oint_0 dz\, z^{\frac{1}{2}}\tilde{Q}^{-+}(z)J^{+}(z)$. If we bosonize $\tilde{Q}^{\pm\mp}(z) =: e^{\pm i\phi(z)}:$ and $J^{\mp}(z) = \Psi^{\pm}(z) : e^{\pm i\frac{\tilde{\phi}(z)}{\sqrt{k}}}:$, we obtain $\hat{G}^{\pm}(z) \equiv \tilde{Q}^{\pm\mp}(z)J^{\mp}(z) = \Psi^{\pm}(z) : e^{\pm i\sqrt{1+\frac{1}{k}}\hat{\phi}(z)}:$. Here $\Psi^{\pm}(z)$ is an $SU(2)$ parafermion field and $\hat{\phi}(z) = \sqrt{\frac{k}{k+1}}\phi(z) + \frac{1}{\sqrt{k+1}}\tilde{\phi}(z)$. Therefore, we can regard $\hat{G}^{\pm}(z)$ as supercurrents in the *minimal* series of $N = 2$ SCFT. The "supercurrents" $\hat{G}^{\pm}(z)$ are nothing but the supercurrents which appeared in Ref. 20 when $c < 6$ $N = 4$ SCFT was reduced to the *minimal* series of $N = 2$ SCFT. Equations (9) and (10) tell that, in the sector* where the indices n of G_n^{++}, G_n^{--} and also $\hat{G}_n^{\pm}$ are half-integers, the state $|\bar{\Phi}>$ corresponding to $|\Phi>$ by the spectral flow† satisfies: $\hat{G}^{+}_{-\frac{1}{2}}|\bar{\Phi}>= \hat{G}^{-}_{\frac{1}{2}}|\bar{\Phi}>= 0$, *i.e.* $|\bar{\Phi}>$ is a chiral and primary state with respect to the "supercurrents" $\hat{G}^{\pm}(z)$. Therefore the physical states in $N = 2$ TSCFT inherit the structure of chiral ring in the *minimal* series of $N = 2$ SCFT.

In the following, we construct physical states which satisfies Eqs. (9) and (10) in the Ramond sector where the indices n of all the operators $\mathcal{O}_n$ are integer.

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* Here we do not call this sector the Neveu-Schwarz sector because $G^{++}(z)$ and $G^{--}(z)$ are not supercurrents but anti-ghost and BRS current after the twisting.

† This spectral flow corresponds to $\eta = 0$ and $\zeta = -\frac{1}{2}$ in Equation (2).

The physical states in the Neveu-Schwarz sector are straightforwardly given by using spectral flow in Eq. (2). We assume that the physical states $|\Phi>$ satisfies: $\mathcal{O}_n|\Phi>=0$, $(n>0)$ for all the operators $\mathcal{O}_n$. Then the conditions (9) and (10) reduce to:

$$\tilde{Q}_0^{\pm\mp} J_0^{\mp} |\Phi>=0 \ . \tag{11}$$

The state $|\Phi>$ is given by a direct product of the representations in the algebra of the zero modes $\{J_0^3, J_0^{\pm}\}$, $\tilde{Q}_0^{\pm\pm}$ and $U_0 = \sqrt{k+2}J_0^0$. We define:

$$\begin{aligned} &|l\ m>:\ \{\frac{1}{2}(J_0^+ J_0^- + J_0^- J_0^+) + J_0^3 J_0^3\}|l\ m>=l(l+1)|l\ m> \ , \\ &\qquad J_0^3|l\ m>=l|l\ m> \ , \ (l=0,\frac{1}{2},1,\cdots,\frac{k}{2} \ , \ m=-l,-l+1,\cdots,l) \\ &|-\mp>:\ \tilde{Q}_0^{-\mp}|-\mp>=0 \ , \ \ |+\mp>\equiv \tilde{Q}_0^{+\mp}|-\mp> \ , \\ &|u>_U\ :\ U_0|u>_U=u|u>_U \ . \end{aligned} \tag{12}$$

Then the solutions of Eqs. (9), (10) and also (5), (6) are given by,

$$\begin{aligned} &|l>_+ = |l\ l>\otimes|++>\otimes|+->\otimes|i(\frac{1}{2}+l)>_U \ , \\ &|l>_- = |l\ -l>\otimes|++>\otimes|-+>\otimes|i(-\frac{1}{2}+l)>_U \ . \end{aligned} \tag{13}$$

The physical state $|l>_+$ ($|l>_-$) has a ghost number $-l-\frac{k}{4}$ ($l-\frac{k}{4}$). By using spectral flow in Eq.(2) and setting $\eta=\frac{1}{2}$ and $\zeta=0$, the physical state $|l>'_+$ corresponding to $|l>_+$ has a ghost number $-l-\frac{k}{2}$ in the Neveu-Schwarz sector. By setting $\eta=\frac{1}{2}$ and $\zeta=0$, J_0^+ and $\tilde{Q}_0^{+-}$ become annihilation operators and the state $|l>'_+$ remains to be primary.* Similarly by setting $\eta=-\frac{1}{2}$ and $\zeta=0$, the physical primary state corresponding to $|l>_-$ has a ghost number l in the Neveu-Schwarz sector.

* If we set *e.g.* $\eta=-\frac{1}{2}$ $\zeta=0$, the state $|l>'_+$ satisfies $\tilde{Q}^{+-}_{-\frac{1}{2}}|l>'_+=0$ due to Eq.(12), which tells that $|l>'_+$ is not a primary state : $\tilde{Q}^{-+}_{\frac{1}{2}}|l>'_+\neq 0$ since $\{\tilde{Q}^{+-}_{-\frac{1}{2}}, \tilde{Q}^{-+}_{\frac{1}{2}}\}=1$ (7).

Finally we remark that the algebra of $N = 2$ TSCFT in Ref. 1 contains that of $N = 1$ TSCFT as subalgebra. If we define: $G_n = \frac{G_n^+ + G_n^-}{2}$, $\Psi_n = \frac{\Psi_n^+ + \Psi_n^-}{2i}$, $\beta_n = \beta_n^+ + \beta_n^-$, $\gamma_n = \gamma_n^+ + \gamma_n^-$, these operators $\{G_n, \Psi_n, \beta_n, \gamma_n\}$ and $\{Q_n, b_n, J_n^{ghost}\}$ give a closed algebra. Replacing $\frac{c}{4c_+ + c_-} = \frac{3}{2}(k_A + k_B)$ by the central charge in $N = 3$ SCFT $\hat{c} = \frac{3}{2}k$ (k is a positive integer), we find that the algebra of $N = 1$ TSCFT obtained from $N = 2$ TSCFT becomes identical with the $N = 1$ TSCFT obtained by twisting $N = 3$ SCFT[13] except $\hat{c} = \frac{3}{2}$ case.‡ Therefore we can obtain a closed set of finite number of physical states in $N = 1$ TSCFT from that in $N = 2$ TSCFT.♮

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‡ That's not so trivial because the algebra of $N = 3$ SCFT is not subalgebra of the algebra of new $N = 4$ SCFT except $k_A = k_B$ case.

♮ The number of physical states in the topological $N = 1$ superconformal algebra obtained by twisting $N = 3$ superconformal algebra should be also finite since the physical states will be given by so-called massless representation.[21]

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A GAUGE THEORY OF SPIN 1/2 FIELD AND ITS GRAVITATIONAL INTERACTION

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ABSTRACT

A gauge theory of spin 1/2 (Dirac) field in 4 dimensional spacetime is presented. The gravitational interaction and the algebra for grading are also investigated.

Since the success of the local field theory for electrweak, strong and gravitational forces, the local gauge symmetry is considered to be the most fundamentalconcept in particle physics.

For these viewpoints, local supersymmetry(SUSY)[1] is expected to play an important role for the unification of all matters and forces.

In superstring theory(SST), local SUSY is realized as the world-sheet supersymmetry(WSSUSY),which plays an important role for the advocated success of the theory as a promissing model for unifying matters and forces including gravitation[2]. In WSSUSY, the world space time is 10 dimensional flat space specified by fermionic vector-spinor coordinates $\psi_\mu(\tau,\sigma)$ in addition to bosonic vector coordinates $X_\mu(\tau,\sigma)$. Due to the mathematical difficulties, however , it is not clear at all whether SST is really the theory of everything.

While within the framework of 4 dimensional local field theory, despite the brilliant success mentioned above, we have not yet succeeded in constructing a promissing model of the unification. We may still drop some fundamental geometrical gauge symmetry to include gravitation in 4 dimensional local field theory.

In this letter, following WSSUSY of SST we extend the ordinary superspace (x_μ , $\xi(x)$) of SUSY to a generalized superspace(x_μ , $\xi_\mu(x)$) and look for a new local gauge symmetry in 4 dimensional local field theory.

As a simple generalization of the fermionic part of ordinary supergravity(SUGRA), it is natural to take the following fermionic gauge fields,

$$(\psi_{\mu\nu}(x) \ , \chi_{\mu\nu}(x) \ ) \tag{1}$$

and consider the following gauge transformations

$$\delta\psi_{\mu\nu}(x) = \partial_\mu \varepsilon_\nu(x) + \partial_\nu \varepsilon_\mu(x) \equiv \partial_{(\mu} \varepsilon_{\nu)}(x) \tag{2}$$

$$\delta\chi_{\mu\nu}(x) = \partial_\mu \varepsilon_\nu(x) - \partial_\nu \varepsilon_\mu(x) \equiv \partial_{[\mu} \varepsilon_{\nu]}(x) \tag{3}$$

where $\psi_{\mu\nu}(x)$ and $\chi_{\mu\nu}(x)$ are symmetric and antisymmetric tensor-spinor field respectively and $\varepsilon_\mu(x)$ is unconstrained vector-spinor gauge parameter.

Our starting point is the following Lagrangian in 4 dimensional flat space[3],

$$\begin{aligned}\mathcal{L}_\psi = {} & i\bar{\psi}_{\mu\nu}\partial\!\!\!/\psi^{\mu\nu} - (i\bar{\psi}_{\mu\nu}\gamma^\mu\partial_\rho\psi^{\rho\nu} + h.c.) + (i\bar{\psi}_{\mu\nu}\gamma^\mu\partial^\nu\psi_\rho{}^\rho + h.c.) \\ & - i\bar{\psi}_\rho{}^\rho\partial\!\!\!/\psi_\sigma{}^\sigma - \varepsilon^{\alpha\beta\gamma\delta}\bar{\psi}_{\alpha\rho}\gamma_5\gamma_\beta\partial_\gamma\psi_\delta{}^\rho\end{aligned} \tag{4}$$

$$\begin{aligned}\mathcal{L}_\chi = {} & -i\bar{\chi}_{\mu\nu}\partial\!\!\!/\chi^{\mu\nu} - (i\bar{\chi}_{\mu\nu}\gamma^\rho\partial^\mu\chi^\nu{}_\rho + h.c.) - (\varepsilon^{\alpha\beta\gamma\delta}\bar{\chi}_{\alpha\beta}\gamma_5\gamma^\rho\partial_\gamma\chi_{\rho\delta} + h.c.) \\ & - \varepsilon^{\alpha\beta\gamma\delta}\bar{\chi}_{\alpha\rho}\gamma_5\gamma_\beta\partial_\gamma\chi_\delta{}^\rho\end{aligned} \tag{5}$$

$$\begin{aligned}\mathcal{L}_{\psi\text{-}\chi} = {} & -(i\bar{\psi}_{\mu\nu}\gamma^\mu\partial_\sigma\chi^{\nu\sigma} + h.c.) + (i\bar{\psi}_{\mu\nu}\gamma_\sigma\partial^\nu\chi^{\mu\sigma} + h.c.) + (i\bar{\psi}_\rho{}^\rho\gamma_\mu\partial_\nu\chi^{\mu\nu} + h.c.) \\ & - (\varepsilon_\nu{}^{\rho\sigma\lambda}\bar{\psi}_{\mu\nu}\gamma_5\gamma_\lambda\partial_\sigma\chi_{\mu\rho} + h.c.) + (\varepsilon^{\sigma\mu\rho\kappa}\bar{\psi}_{\sigma\lambda}\gamma_5\gamma_\lambda\partial_\mu\chi_{\rho\kappa} + h.c.)\end{aligned} \tag{6}$$

The total Lagrangian

$$\mathcal{L} = \mathcal{L}_\psi + \mathcal{L}_\chi + \mathcal{L}_{\psi\text{-}\chi} \tag{7}$$

is parity conserving and invariant under the gauge transformations (2) and (3).

Now we show briefly that the total Lagrangian (7) has the following remarkable properties (i) and (ii).

(i) The contribution of spin 3/2 components of $\psi_{\mu\nu}(x)$ and $\chi_{\mu\nu}(x)$ to the total Lagrangian is the familliar (flat space) form of the Rarita-Schwinger field in SUGRA.

(ii) When the gravitational interaction is introduced in the minimal way by using full covariant derivatives, it is shown that Euler equations for $\psi_{\mu\nu}(x)$ and $\chi_{\mu\nu}(x)$ are invariant up to the nontrivial first order of $\psi_{\mu\nu}(x)$ and $\chi_{\mu\nu}(x)$, i.e. the variations of Euler equations produce only Einstein tensor terms up to the first order of $\psi_{\mu\nu}(x)$ and $\chi_{\mu\nu}(x)$. This situation is the same as in SUGRA, i.e. the variations of Einstein action can cancel so far the variations of (7).

(i) By replacing

$$\psi_{\mu\nu}(x) \longrightarrow \gamma_{(\mu}\psi_{\nu)}(x) \qquad (8)$$

$$\chi_{\mu\nu}(x) \longrightarrow \gamma_{[\nu}\psi_{\mu]}(x) \qquad (9)$$

$$\varepsilon_{\mu}(x) \longrightarrow \gamma_{\mu}\varepsilon(x) \qquad (10)$$

we can show straightforwardly

$$\mathcal{L} \longrightarrow \varepsilon^{\alpha\beta\gamma\delta}\bar{\psi}_{\alpha}\gamma_{\beta}\gamma_{5}\partial_{\gamma}\psi_{\delta} \qquad (11)$$

$$\delta\psi_{\mu\nu}(x),\ \delta\chi_{\mu\nu}(x) \longrightarrow \delta\psi_{\mu} = \partial_{\mu}\varepsilon(x) \qquad (12)$$

This shows that the pure spin 3/2 part of the action is just that of SUGRA. Alternatively this shows the possibility that by gauge conditions spin 3/2 components can be canceled each other (i.e. gauged away), which is argued later. It is interesting to note that by replacing $\psi_{\mu\nu}(x) \rightarrow \gamma_{(\mu}\psi_{\nu)}$ [or 0] and $\chi_{\mu\nu} \rightarrow 0$ [or $\gamma_{[\nu}\psi_{\mu]}$] instead of (8) and (9), $\mathcal{L}$ does not produce apparently SUGRA Lagrangian(11). This situation may indicate that the gauge condition will become coupled equations of $\psi_{\mu\nu}(x)$ and $\chi_{\mu\nu}(x)$.

(ii) The variations of the Lagrangian with respect to $\psi_{\mu\nu}(x)$ and $\chi_{\mu\nu}(x)$ give the following Euler equations respectively.

$$
\begin{aligned}
I_{(\mu\nu)} \equiv \Big(& i\not\partial\psi_{\mu\nu} - i\gamma_\mu\partial_\rho\psi^{\mu\rho} + i\gamma_\mu\partial_\nu\psi_\rho{}^\rho - \eta_{\mu\nu}\not\partial\psi_\lambda{}^\lambda - \varepsilon_\mu{}^{\beta\gamma\delta}\gamma_5\gamma_\beta\partial_\gamma\psi_{\delta\nu} - i\gamma_\mu\partial_\sigma\chi_\mu{}^\sigma + i\gamma_\sigma\partial_\nu\chi_\mu{}^\sigma \\
& + i\eta_{\mu\nu}\chi_\rho\partial_\sigma\chi^{\rho\sigma} - \varepsilon_\nu{}^{\rho\sigma\lambda}\gamma_5\gamma_\lambda\partial_\sigma\chi_{\mu\rho} + \varepsilon_\mu{}^{\sigma\rho\kappa}\gamma_5\gamma_\nu\partial_\sigma\chi_{\rho\kappa}\Big) + (\mu\leftrightarrow\nu) = 0
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
X_{[\mu\nu]} \equiv \Big(& -i\not\partial\chi_{\mu\nu} - i\gamma^\rho\partial_\mu\chi_{\nu\rho} - \varepsilon_{\mu\nu}{}^{\gamma\delta}\gamma_5\gamma^\rho\partial_\gamma\chi_{\rho\delta} - \varepsilon_\mu{}^{\beta\gamma\delta}\gamma_5\gamma_\beta\partial_\gamma\chi_{\delta\nu} - i\gamma^\rho\partial_\nu\psi_{\mu\rho} + i\gamma_\nu\partial^\rho\psi_{\mu\rho} \\
& + i\gamma_\mu\partial_\nu\psi_\rho{}^\rho + \varepsilon_{\rho\nu}{}^{\sigma\lambda}\gamma_5\gamma_\lambda\partial_\sigma\psi_\mu{}^\rho - \varepsilon^{\sigma\rho}{}_{\mu\nu}\gamma_5\gamma^\lambda\partial_\rho\psi_{\sigma\lambda}\Big) - (\mu\leftrightarrow\nu) = 0,
\end{aligned}
\tag{14}
$$

which are invariant under (2) and (3). Now we introduce the gravitational interaction in the minimal way and check the gauge invariance under the covariantized (13) and (14). This produces

$$\delta^{cov.} I^{cov.}_{(\mu\nu)} = \left\{ -G_{\mu\nu}\, i\not\varepsilon + G_{\mu\rho}\, i(\gamma^\rho\varepsilon_\nu - \gamma_\nu\varepsilon^\rho) \right\} \tag{15}$$

$$\delta^{cov} X_{[\mu\nu]} = \left[-\frac{1}{2}\left\{ G_{\mu\rho}\, i(\gamma^\rho\varepsilon_\nu - \gamma_\nu\varepsilon^\rho) + G^\sigma_\rho\, \varepsilon_{\mu\nu\sigma\kappa}\gamma_5(\gamma^\rho\varepsilon^\kappa + \gamma^\kappa\varepsilon^\rho)\right] - \left[\mu\leftrightarrow\nu\right], \tag{16}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ and the torsions are neglected. Remarkablely, these situations are the same as in SUGRA[4], i.e. the action consisting of Einstein action and the generally covariantized action (7) is invariant up to the first order of $\psi_{\mu\nu}(x)$ and $\chi_{\mu\nu}(x)$ under generalized transformations for $\delta\psi_{\mu\nu}(x)$, $\delta\chi_{\mu\nu}(x)$ and $\delta g_{\mu\nu}(x)$ determined uniquely by (15) and (16). We consider these cancelations very positive indications for all order invariance of the action. To prove the all order invariance, we must see the cancelations higher order terms (no derivatives) of

$\psi_{\mu\nu}(x)$ and $\chi_{\mu\nu}(x)$, which come from the torsions. The results obtained above may give a new insight to the difficulties of the gauge interaction of the higher rank(≥ 2) tensor gauge fields.

Next we discuss the physical degrees of freedom of our action, which shows the conjecture in ref(3) is incorrect. By using the properties of Dirac matrices, we can rewrite Lagrangian (4),(5) and (6) as follows

$$\mathcal{L}_{\psi} = \bar{\psi}_{\mu\nu}\left(-\frac{i}{2}\gamma^{\mu}\gamma^{\beta}\gamma^{\delta}\partial_{\beta}\psi_{\delta}{}^{\nu} + i\gamma^{\mu}\partial^{\nu}\psi_{\rho}{}^{\rho} - \frac{i}{2}\eta^{\mu\nu}\not{\partial}\psi_{\rho}{}^{\rho}\right) + (\text{h. c.}) \tag{17}$$

$$\mathcal{L}_{\chi} = \bar{\chi}_{\mu\nu}\left(-i\not{\partial}\chi^{\mu\nu} - \frac{i}{2}\gamma^{\mu}\gamma^{\beta}\gamma^{\delta}\partial_{\beta}\chi_{\delta}{}^{\nu} - 2i\gamma^{\rho}\partial^{\mu}\chi^{\nu}{}_{\rho} - \varepsilon^{\mu\nu\gamma\delta}\gamma_{5}\gamma^{\rho}\partial_{\gamma}\chi_{\rho\delta}\right) + (\text{h. c.}) \tag{18}$$

$$\mathcal{L}_{\psi-\chi} = \bar{\psi}_{\mu\nu}\left(-i\partial^{\nu}\gamma^{\rho}\gamma^{\sigma}\partial_{\sigma}\chi^{\mu}{}_{\rho} + i\eta^{\mu\nu}\gamma_{\rho}\partial_{\sigma}\chi^{\rho\sigma} + \varepsilon^{\mu\lambda\rho\kappa}\gamma_{5}\gamma^{\nu}\partial^{\lambda}\chi_{\rho\kappa}\right) + (\text{h. c.}) \tag{19}$$

Surprisingly, only the trce of $\psi_{\mu\nu}(x)$ appears in the action, which means 24 degrees of freedom of $\psi_{\mu\nu}(x)$ are absent from the action. Instead, $\chi_{\mu\nu}(x)$ (24 degrees of freedom) appears in the action. The total number of degrees of freedom is 40, which equals to that of spin 5/2 gauge field(symmetric), where the highest helicity state $\pm 5/2$ is replaced by $\pm 1/2$. Applying the general arguments for spin 5/2 of ref.(5) and (6), we show that the physical components of our system is helicity $\pm$ 1/2 state. An appropriate gauge condition for this is the following one which has antisymmetric tensor indices. The familiar vctor type as discussed in ref.(5) and (6) do not work in this case, for the tower structure of helicity states is different.

$$\mathcal{F}_{[\mu\nu]} \equiv \chi_{\mu\nu} - \frac{1}{2}\left(\gamma_{\mu}\gamma^{\rho}\psi_{\rho\nu} - \gamma_{\nu}\gamma^{\rho}\psi_{\rho\mu}\right) + \frac{2}{3}[\gamma_{\mu}, \gamma_{\nu}]\psi_{\rho}{}^{\rho} = 0 \tag{20}$$

By using this constraints, Lagrangian becomes as follows,

$$\mathcal{L}_\psi + \mathcal{L}_\chi + \mathcal{L}_{\psi-\chi} \longrightarrow i\bar{\psi}\not{\partial}\psi \ , \quad \psi \equiv \psi_\rho{}^\rho . \tag{21}$$

Surprisingly this is the familiar Lagrangian of massless (Dirac) spin 1/2 field, however a gauge particle. The mass term is forbidden by the gauge invariance (2) and (3).

(22) possesses chiral symmetry.

As for the grading of the action, we must prove the all order gauge invariance in the curved space. The algebraic consideration presented in ref.(7) may be useful.

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