Riemann Surfaces, Conformal Fields and Strings

The role of Riemann surfaces in modern particle physics is discussed. Mathematically, quantum field theories can be defined on these manifolds if they are conformally invariant. Physically, Riemann surfaces provide a model for the world sheet swept out by a propagating relativistic string. Thus Riemann surfaces are the natural setting for conformal field theory, and both these concepts together provide a formulation of string theory.

Key Words: *string theory, conformal symmetry, world sheets, Riemann surfaces*

I. FIELD THEORY AND RIEMANN SURFACES

The space-time on which the particle physicist writes a Lagrangian field theory and deduces equations of motion, classical solutions and quantum scattering amplitudes is usually taken to be a Riemannian manifold.[1] The idea is central to general relativity and gravitation,[2] and has become more important in particle physics with increasing emphasis on unified theories of all forces including gravitation.

Riemannian geometry is usually incorporated as a series of prescriptions, starting with the introduction of a second-rank symmetric tensor field, the metric $g_{\mu\nu}(x)$, and combinations of its derivatives such as the Christoffel connection $\Gamma^{\lambda}_{\mu\nu}(x)$ and the Riemann curvature tensor $R^{\rho}_{\mu\nu\lambda}(x)$. These are then coupled invariantly to themselves and to other classical fields such as scalars $\phi(x)$ and

Comments Nucl. Part. Phys.
1988, Vol. 18, No. 3, pp. 133–155
Reprints available directly from the publisher
Photocopying permitted by license only

vector gauge fields $A^a_\mu(x)$. Some typical field-theory actions on Riemannian manifolds are:

(i) $(1/4\kappa^2) \int d^4x\sqrt{g}R$: the Einstein–Hilbert action (κ is the gravitational constant);
(ii) $(1/2) \int d^4x\sqrt{g}g^{\mu\nu}\, \partial_\mu\phi\partial_\nu\phi$: the scalar field coupled to gravity;
(iii) $-(1/4) \int d^4x\sqrt{g}g^{\mu\lambda}g^{\nu\rho}\, \mathrm{tr}(F_{\mu\nu}F_{\lambda\rho})$: the Yang–Mills field coupled to gravity. (1)

From the mathematician's point of view, a Riemannian manifold is a somewhat specialized structure. To define it, one starts with a topological space and introduces a differentiable structure (a set of coordinate charts such that coordinate changes across overlaps are given by differentiable functions). This gives rise to a differentiable manifold. At the next stage a Riemannian metric is introduced, and one gets a Riemannian manifold. For the particle physicist it has generally been this last attribute which has played a major role; the topology and differentiable structure have (until recently) played a relatively minor part.

An interesting specialization of the notion of a differentiable structure is that of a complex structure.[3] The coordinate charts are chosen to be complex, and coordinate changes across overlaps are given by complex *analytic* functions. In this way one gets a complex manifold. This can in turn be endowed with a metric. The various structures we have discussed can be listed in order of complexity (Fig. 1).

Riemann surfaces belong in the lower branch of this flow chart. They are, quite simply, one-complex-dimensional (connected) manifolds.[4] They need not be thought of as metric spaces at all. An important fact about them is that they are relatively simple

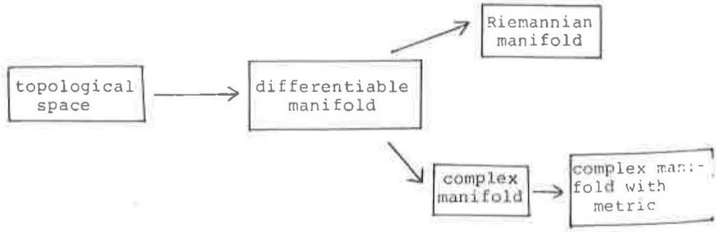

FIGURE 1

and rather well-understood objects. Their topologies and complex structures are completely classified. In sharp contrast, differentiable manifolds in four dimensions, and possible Riemannian metrics on them, are far from being classified.

From the physicist's point of view, Riemann surfaces are scarcely interesting as models of space-time! For one thing, they are two-dimensional as real manifolds. But their mathematical simplicity makes it very tempting to try and construct field-theory models on them.

II. STRING DYNAMICS AND RIEMANN SURFACES

Had physicists two decades ago reasoned things out in the manner discussed above, they would probably have come to a halt at this point. Fortunately for us, things worked out quite differently. Two-dimensional "space-times" entered particle physics in the description of the "world sheet" swept out by a propagating relativistic string[5] (Fig. 2).

It is a theorem[4] that a two-real-dimensional manifold can be described as a one-complex-dimensional manifold if it is orientable (which means that one can define a unique normal vector smoothly everywhere). For closed strings, this will be true of their world sheets if the strings themselves are assigned an orientation which is preserved in interactions. Thus closed oriented strings sweep out Riemann surfaces.

It does not (yet) follow that, in describing the dynamics of such a string, it will be sufficient to deal with Riemann surfaces, independent of a metric. In fact, as presently formulated, the basic

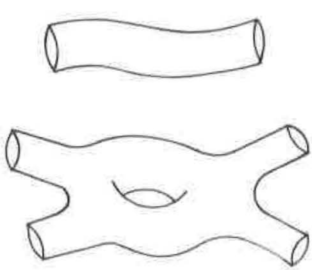

FIGURE 2

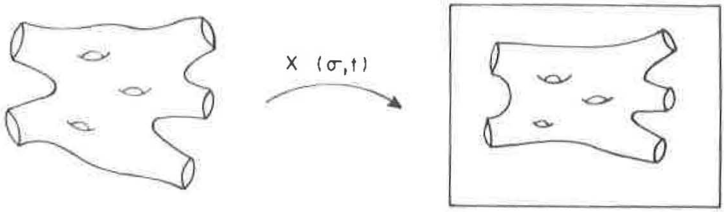

FIGURE 3

postulate of string theory does require a two-dimensional metric on the Riemann surface, as we will see, but the dependence on the metric ultimately cancels out in physical quantities. In a somewhat roundabout way, then, string theories are field theories on Riemann surfaces.

This comes about in the following way. Introduce a string coordinate $X^\mu(\sigma,t)$, where X^μ labels points in a Euclidean space-time and (σ,t) are local real coordinates on an oriented two-dimensional surface. A string configuration may be thought of as a map from the Riemann surface into physical space-time (Fig. 3).

Choosing a metric $g_{ab}(\sigma,t)$ on the two-dimensional surface, we postulate an action[6,7]

$$S = \frac{-1}{2\pi} \int d\sigma dt \, \sqrt{g} g^{ab} \, \partial_a X^\mu \partial_b X_\mu \tag{2}$$

where the integral is performed over the surface. The motivation for this classical action may be seen from the fact that the classical equation of motion for the metric (varied as an independent field) is

$$\partial_a X^\mu \partial_b X_\mu - \frac{1}{2} g_{ab} g^{cd} \, \partial_c X^\mu \partial_d X_\mu = 0 \tag{3}$$

This is solved by

$$g_{ab} = e^{\phi(\sigma,t)} \partial_a X^\mu \partial_b X_\mu$$

where $\phi(\sigma,t)$ is an arbitrary function. Substituting this into Eq. (2)

one gets the classical action

$$S = \frac{-1}{\pi} \int d\sigma dt \, (\det(\partial_a X^\mu \partial_b X_\mu))^{1/2} \qquad (4)$$

This is just the *area* of the world sheet swept out by the string in n-dimensional space.[8] Note that the ϕ-dependence has dropped out, as a consequence of Weyl invariance (invariance of Eq. (2) under local rescalings of the metric: $g_{ab}(\sigma,t) \rightarrow e^{\phi(\sigma,t)} g_{ab}(\sigma,t)$ for arbitrary $\phi(\sigma,t)$).

The connection between this action principle and Riemann surfaces as one-complex-dimensional manifolds comes about through three important theorems[4]:

(i) Every orientable two-real-dimensional manifold admits a complex structure, hence can be made into a Riemann surface.
(ii) Every Riemann surface admits a unique metric of constant curvature, denoted $\hat{g}_{ab}(\sigma,t)$.
(iii) All possible metrics on a given Riemann surface are conformally related: if $g_{ab}(\sigma,t)$ and $g'_{ab}(\sigma,t)$ are two metrics, then

$$g'_{ab}(\sigma,t) = e^{\phi(\sigma,t)} g_{ab}(\sigma,t) \qquad (5)$$

for some function $\phi(\sigma,t)$.

From (i) and (iii), every metric is conformally related to a unique one of constant curvature. Thus in the action (2), we can substitute any arbitrary g_{ab} by $e^{\phi} g_{ab}$ for a suitable $\phi(\sigma,t)$. The ϕ-dependence now drops out, and we are left with

$$S = \frac{-1}{2\pi} \int d\sigma dt \, \sqrt{\hat{g}} \hat{g}^{ab} \, \partial_a X^\mu \partial_b X_\mu \qquad (6)$$

But now theorem (ii) tells us that each $\hat{g}_{ab}$ is uniquely associated to a given complex structure, that is, to a given Riemann surface. Thus the string action is (classically) an action on a Riemann surface.

III. THE CLASSIFICATION OF COMPACT RIEMANN SURFACES

The question of how many inequivalent $\hat{g}_{ab}$'s there are reduces to that of classifying inequivalent Riemann surfaces. Let us make somewhat more precise what is meant by a complex structure.

The concepts involved in defining a manifold can be illustrated by a diagram (Fig. 4). μ_1 and μ_2 are two overlapping patches on the manifold. ϕ_1 and ϕ_2 associate subsets of Euclidean space, v_1 and v_2, to their respective patches. These are the coordinate maps. The "transition function" f describes how the shaded regions on the left and right are related, by virtue of being different coordinatizations of the same shaded region in M.

For a Riemann surface the coordinate patches v_1 and v_2 lie in the complex plane. If the complex coordinate on v_1 is z and on v_2, w, then the transition function is

$$f: z \rightarrow w = w(z) \tag{7}$$

The fact that w is an *analytic* function of z, on every overlap of coordinate patches, is what defines a complex structure. Another choice of patches, coordinates and analytic transition functions would define a different complex structure, *unless* the transition functions on overlaps of the old and new patches are also analytic. Thus a single complex structure is not just one collection of analytic charts, but all choices of such collections which agree analytically with each other.

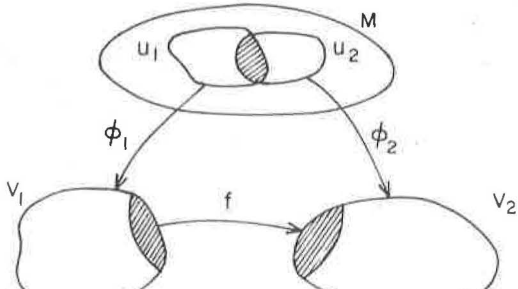

FIGURE 4

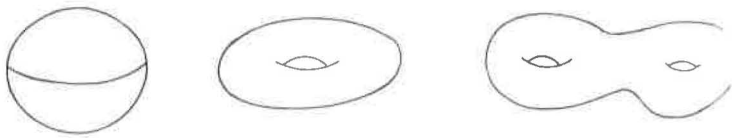

FIGURE 5

How many inequivalent Riemann surfaces are there? Let us concentrate on compact surfaces without boundary. Topologically, these are labelled by "genus," or number of "handles" (Fig. 5). Within each genus one has to ask how many inequivalent complex structures there are, in the sense described above. The result is:

(i) For $g = 0$, there is a unique complex structure.
(ii) For $g = 1$, there is a one-complex-parameter family of complex structures.
(iii) For $g \geq 2$, there is a $(3g - 3)$-complex-parameter family of complex structures.

The parameters labelling inequivalent complex structures are called "Teichmuller parameters" or "moduli."

For the torus ($g = 1$), one can give a simple intuitive description of the moduli. Consider the complex plane, and a lattice of points on it (Fig. 6). A lattice on the complex plane is simply a discrete set of points which are integral linear combinations of two basic

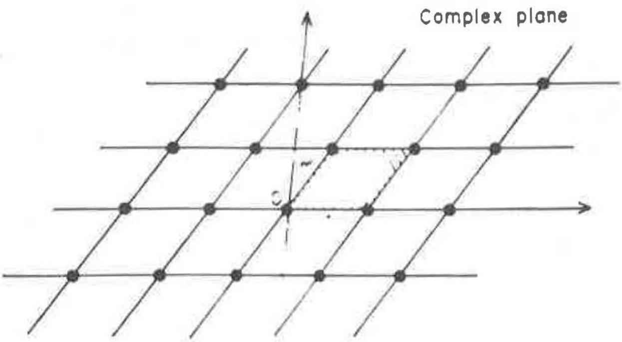

FIGURE 6

vectors, which we have chosen to be the unit vector along the real axis, and an arbitrary independent vector labelled by a complex number τ. Now if we identify all points in the complex plane which differ by a lattice vector $m + n\tau$ (for arbitrary integers m, n), then we are left with the shaded parallelogram shown in the figure, which is equivalent to all other such parallelograms under the identification. Moreover, the opposite edges of the shaded region are identified, and this is therefore a representation of the torus. The complex number τ labels the torus: for each different number, we get a different lattice and hence a different torus. However, it may happen that certain different values of τ generate the *same* torus: for example, τ and $\tau + 1$. The transformation $\tau \to \tau + 1$ is called a modular transformation. Physically meaningful quantities which are defined on the surface labelled by τ must be invariant under modular transformations, since these are discrete transformations of the given torus onto itself.

IV. THE PATH INTEGRAL OF STRING THEORY

In modern language, the basic postulate of string theory[7] is that scattering amplitudes of various string states are given by the insertion of certain fields (called "vertex operators,"[10] one for each string state) into the path integral whose action is Eq. (2). The path integral is evaluated, in principle, by summing over metrics on compact Riemann surfaces of a given genus, and the result is summed up over all genus:

$$A(k_1, \ldots, k_N) = \sum_{\text{genus } g} c^g \int [dg_{ab}][dX^\mu] \times e^{-S[X,g]} V(X,k_1) \ldots V(X,k_N) \quad (8)$$

Here $V[X,k_i]$ is the vertex operator for the emission of some string state of momentum k_i, and c is a coupling constant. What we have written looks similar to a loop expansion in ordinary quantum field theory, with the genus counting the number of loops.

As formulated, there appears to be a formidable functional integral over all possible metrics in a given genus. But because of

Eq. (6), only inequivalent Riemann surfaces give different actions, and this forms a *finite*-parameter family in each genus, labelled by the moduli. This is true as long as the classical Weyl invariance continues to hold in the quantum theory. Polyakov showed that there is in general an anomaly in the Weyl symmetry, but this vanishes in 26 space-time dimensions for the action Eq. (2). In this critical dimension the amplitude in Eq. (8) is evaluated purely by summing over all inequivalent Riemann surfaces, regardless of a metric.[11] Thus the integrand to be performed is a finite-dimensional one, over modular parameters. For consistency, the integrand must be modular invariant.

It is a somewhat puzzling feature of string theory that a two-dimensional metric has to be carried along until the end, at which point it does not affect physical results. Possibly a new formulation based directly on Riemann surfaces (or generalizations of this concept) will simplify the situation.

V. CONFORMAL FIELD THEORY: THE BASIC CONCEPTS

Let us rewrite the action, Eq. (6), in terms of local isothermal complex coordinates z, $\bar{z}$ on the Riemann surface. These coordinates are defined by the requirement that in a given patch the line element be given by:

$$ds^2 = e^{\phi(z,\bar{z})} dz d\bar{z} \tag{9}$$

for some function $\phi(z,\bar{z})$. That such coordinates can always be chosen is intuitively evident from the fact that two of the three components of the metric g_{ab} can be fixed at will by the two available coordinate choices, while the third is the conformal factor $\phi(z,\bar{z})$. In these coordinates, the action is:

$$S = -\frac{1}{\pi} \int dz d\bar{z}\ \partial_z X^\mu \partial_{\bar{z}} X_\mu \tag{10}$$

In this form the entire information about the complex structure is contained in the way the coordinates patch up over the Riemann surface. On the overlap of two isothermal coordinate patches, the transition functions are, as usual, analytic:

$$Z \rightarrow w(z) \tag{11}$$

Clearly this is a symmetry of the two-dimensional classical field theory Eq. (10). This is called *conformal invariance*, which is really just that property of a field theory which tells us that it may be defined on a Riemann surface.

Conformal invariance is a property of a large class of field-theoretic models in two dimensions.[12,13] Indeed, it was realized many years ago[14] that a two-dimensional field theory with scale invariance actually possesses conformal invariance. This latter invariance, being parametrized by arbitrary analytic functions, forms an infinite-dimensional group of symmetries. This enables one to solve many such models completely by using very general properties of their symmetry algebra.[12]

We have thus found an answer to two basic questions: what kind of field theories can be written on Riemann surfaces, and what physical use they might be to the particle physicist. To the first, the answer is conformal field theory, while to the second the answer lies in string theory. (It is beyond the scope of this article to discuss in detail why string theories are believed to be useful in particle physics.[5])

Now although the specific conformal field theory which describes the closed bosonic string propagating in a flat space-time is described by the simple action Eq. (10), one can take the point of view that the space of all possible conformal field theories has applicability to string physics. The logic is that one is not ultimately interested only in the propagation of a single string in a flat background space-time. Specifically, one may be interested in a string propagating in arbitrary curved background space-times.[15,16] This brings in a large class of conformal field theories, which are specific types of non-linear σ-models with vanishing β-function.[16,17] But besides these examples, which have an obvious physical motivation, one can argue that every conformal field theory is a candidate "vacuum" configuration for a many-string theory, subject to the condition that Weyl invariance be maintained. This proposal, if implemented in a dynamical framework, could even supplant conventional string field theory. Thus, to study string theory, we need to understand all conformal field theories on all Riemann surfaces.

VI. CONFORMAL FIELD THEORY: LOCAL PROPERTIES

We may list some simple classical actions which describe conformal field theories:

(i) The free scalar field theory:

$$S = -\frac{1}{\pi}\int dz d\bar{z}\ \partial_z X \partial_{\bar{z}} X \tag{12a}$$

$X(z,\bar{z})$ is a scalar field.

(ii) The (bosonic) non-linear σ-model:

$$S = -\frac{1}{\pi}\int dz d\bar{z}\ G_{\mu\nu}(X)\ \partial_z X^\mu \partial_{\bar{z}} X^\nu \tag{12b}$$

X^μ, $\mu = 1 \ldots d$, are scalar fields, and $G_{\mu\nu}(X)$ is an arbitrary non-singular matrix function of the X^μ, which has the geometrical interpretation of a metric on the space of fields.

(iii) The free Majorana spinor field theory:

$$S = \frac{1}{2\pi}\int dz d\bar{z}\ (\psi\partial_{\bar{z}}\psi - \bar{\psi}\partial_z\bar{\psi}) \tag{12c}$$

ψ is a complex one-component Euclidean spinor. In Minkowski space it would represent a real, or Majorana, spinor field.

(iv) The free Dirac spinor field theory:

$$S = \frac{1}{\pi}\int dz d\bar{z}\ (b\partial_{\bar{z}}c - \bar{c}\partial_z\bar{b}) \tag{12d}$$

In Minkowski space b, c would represent two independent real spinors, hence can be thought of as components of a Dirac spinor field.

The local fields of these actions transform as tensors under analytic coordinate changes $z \to w(z)$. A generic field with this property is called a primary conformal field[12]:

$$A(z,\bar{z}) \to A'(w,\bar{w}) = \left(\frac{\partial z}{\partial w}\right)^{\Delta_A} \left(\frac{\partial \bar{z}}{\partial \bar{w}}\right)^{\bar{\Delta}_A} A(z,\bar{z}) \qquad (13)$$

The pair of real numbers $(\Delta_A, \bar{\Delta}_A)$ is called the *conformal dimension* of the field A.

In the actions listed above, the field $\partial_z X$ has dimension (1,0) while ψ has dimension (1/2,0). For Dirac field, invariance of the Lagrangian only requires that the pair of independent but canonically conjugate fields b, c have dimensions $(J,0)$ and $(1 - J,0)$, respectively, where J is usually chosen to be an integer or half-integer.

Classically, a consequence of conformal symmetry is that the energy-momentum tensor $T_{\mu\nu}$ is traceless. It is also conserved as a consequence of translation invariance:

$$T^{\mu}_{\mu} = 0, \quad D^{\mu} T_{\mu\nu} = 0 \qquad (14)$$

In isothermal coordinates, these equations reduce to:

$$T_{z\bar{z}} = 0, \quad D_z T_{\bar{z}z} + D_{\bar{z}} T_{zz} = 0$$

It follows that the non-vanishing components are (anti-)analytic[12]:

$$\partial_{\bar{z}} T_{zz} = 0 = \partial_z T_{\bar{z}\bar{z}} \qquad (15)$$

We may list the analytic energy-momentum tensors for the simple conformal field theories listed in Eq. (12):

$$\begin{aligned}
&\text{(i) } T_{zz} = \frac{1}{2} \partial_z X \partial_z X \\
&\text{(ii) } T_{zz} = \frac{1}{2} G_{\mu\nu}(X)\, \partial_z X^{\mu} \partial_z X^{\nu} \\
&\text{(iii) } T_{zz} = -\frac{1}{2} \psi\, \partial_z\, \psi \\
&\text{(iv) } T_{zz} = -(J b \partial_z c + (1 - J) c \partial_z b)
\end{aligned} \qquad (16)$$

It is easy to check that they are all holomorphic as a consequence of the corresponding equations of motion.

The symmetry algebra of a conformal field theory may be deduced from the behavior of the product of energy-momentum tensors at short distances. T_{zz} generates conformal transformations on fields by:

$$\delta_\epsilon A(w,\overline{w}) = \left[\oint \epsilon(z) T_{zz} dz,\ A(w,\overline{w})\right] \tag{17}$$

We have chosen some disk on the Riemann surface and a coordinate z which vanishes at the origin of the disk. Quantization is carried out by associating time with the radial variable on the disk, so that the conserved charge associated with T_{zz} is its contour integral around a closed contour encircling the origin. By inserting an arbitrary analytic function $\epsilon(z)$ on the disk into the integral, we pick out combinations of the modes of T_{zz}. By simple manipulations, the above expression can be rewritten

$$\delta_\epsilon A(w,\overline{w}) = \oint_{w} \epsilon(z) T_{zz} A(w,\overline{w}) dz \tag{18}$$

where now the contour encircles w. Thus the conformal transformation of A comes only from singular terms in its operator-product expansion (OPE) with T_{zz}. If A is a primary field, the infinitesimal form of Eq. (13), under the transformation $z \rightarrow w = z + \epsilon(z)$, implies:

$$\delta_\epsilon A(z,\overline{z}) = \Delta_A \epsilon'(z) A(\overline{z},\overline{z}) + \epsilon(z)\, \partial A(z,\overline{z}) \tag{19}$$

Comparing Eqs. (18) and (19), we find the operator-product expansion

$$T_{zz}(z) A(w,\overline{w}) = \frac{\Delta_A A(w,\overline{w})}{(z-w)^2} + \frac{\partial_w A(w,\overline{w})}{z-w} + \text{non-singular terms} \tag{20}$$

This equation is to be interpreted in terms of the behavior of any correlation function (in the path integral sense) of products of arbitrary local fields with $T_{zz}(z)$ and $A(w,\overline{w})$.

T_{zz} is not itself a primary field. On dimensional grounds its OPE with itself can contain an extra term besides the ones describing a primary field[12]:

$$T_{zz}(z)T_{ww}(w) = \frac{c}{2}\frac{1}{(z-w)^4} + \frac{2T_{ww}}{(z-w)^2} + \frac{\partial_w T_{ww}}{(z-w)} + \text{non-singular terms} \tag{21}$$

The c-number coefficient c is called the "central charge" or "anomaly" in the OPE. This is, in general, nonzero. One can check, using the short-distance behavior of two-point functions on the disk:

$$\begin{aligned} \langle\psi(z)\ \psi(w)\rangle &\sim \frac{1}{z-w} \\ \langle b(z)\ c(w)\rangle &\sim \frac{1}{z-w} \end{aligned} \tag{22}$$

that $c = 1/2$ for the free Majorana field, and $c = -2(6J^2 - 6J + 1)$ for the Dirac field. For free scalar fields, $c = 1$.

It is useful to define an operator formalism for conformal fields on Riemann surfaces. In such a formalism, the expectation values in the path-integral sense discussed above are converted into the matrix elements between suitable states of "time-ordered" products of operators. In order to implement this, one makes use of the fact that the infinite cylinder can be conformally mapped onto the complex plane (with the origin deleted). In the former picture, the cylinder represents the time evolution of a quantum state, where the state at $-\infty$ and $+\infty$ is the vacuum of the theory. On the plane, the in-vacuum is at the origin, the out-vacuum is the state at $|z| = \infty$, and the constant time slices on the cylinder have gone into concentric circles on the plane. This is known as radial quantization.

For a general Riemann surface, one can perform radial quantization in the following way: take the semi-infinite cylinder, and map it conformally onto the unit disk. Now the in-vacuum is associated with the origin of the disk, as before, but the state at the other end of the cylinder, which is not a vacuum state (since the other end is not at $+\infty$) is mapped onto the state at the boundary of the disk. This unit disk is taken to be a coordinate patch D of a Riemann surface Σ (Fig. 7).

Now, operators can be constructed on the disk D using the mapping onto the semi-infinite cylinder. Their expectation values on the disk are taken with the vacuum state $\langle o|$ on the left and some other state $|g\rangle$ on the right, and this state $|g\rangle$ is determined by the Riemann surface.

The local coordinate on the disk, z, is chosen to vanish at the origin. Operators on the disk can be decomposed into modes in terms of these coordinates. Thus, for the energy-momentum tensor we have:

$$T_{zz} = \sum_{n=-\infty}^{+\infty} \frac{L_n}{z^{n+2}} \tag{23}$$

It is easy to show, from Eq. (21), that the operators L_n satisfy the algebra:

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12} n(n^2 - 1)\delta_{n+m,o} \tag{24}$$

This is the infinite-dimensional symmetry algebra of conformal field theory. It is called the Virasoro algebra.[18,12]

A great deal of work has been done on representations of the Virasoro algebra.[12,13,19] The central charge c is a local property, independent of the Riemann surface, and its allowed values in unitary representations have been classified. The mathematical structure of these representations is very profound and allows one to derive powerful constraints on the correlation functions of conformal fields. One can obtain differential equations for correlation

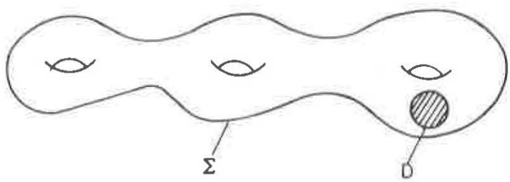

FIGURE 7

functions of a large class of theories; these can be solved explicitly on the sphere, but the extension to higher genus is not that simple. For free theories one can use Wick's theorem to derive all n-point functions from the two-point function, and much more is known explicitly in this case. Infinite-dimensional algebras are known which contain the Virasoro algebra as a subalgebra. These correspond to conformal field theories with additional symmetries besides conformal invariance. Example are the superconformal algebras[20,13] and the current-conformal algebras.[21] Conformal theories based on these symmetry algebras are the subject of active research, both from the point of view of mathematical physics and for their possible applications to critical phenomena[22] and to string theory.

VII. STRING AMPLITUDES

As indicated previously, scattering amplitudes in string theory are given by insertion of certain fields called vertex operators in the string path integral. These operators are conformal fields of dimension (0,0): this follows when one requires that Weyl invariance of the string theory be maintained at the quantum level. The amplitudes are thus given by correlation functions of vertex operators in conformal field theories. More precisely, the expression (Eq. (8)) does not require us to divide out by the vacuum path integral ("partition function") in computing string amplitudes, while this division is implicit in the definition of correlation function, which is a statistical average. Thus a string amplitude is the product of a correlation function of vertex operators with the partition function of the theory.

When evaluated on the 2-sphere (genus 0), these functions determine tree-level string amplitudes, while the corresponding quantities calculated on genus-g surfaces represent the g-loop corrections to these amplitudes. A complete understanding of partition functions and correlation functions for conformal field theories on Riemann surfaces will provide crucial insight into the properties of string theory.

Let us return to the simplest string action, Eq. (10). The free scalar field $X^\mu(z,\bar{z})$ appears to be a primary field of dimension (0,0), but, as is well known, its correlation functions have infrared singularities, so that it is not a well-defined quantum field at all. Nevertheless one can write down a regularized expression for its two-point function, from which those of other conformal fields can be deduced by differentiation and Wick's theorem.

On the infinite complex plane, one finds, by inverting the kinetic term in Eq. (10):

$$\langle X^\mu(z,\bar{z}) X^\nu(w,\bar{w})\rangle_{\text{plane}} = -\delta^{\mu\nu} \log|z - w|^2 \tag{25}$$

An infrared cutoff is implicit in the logarithm. From this one gets

$$\begin{aligned}\langle \partial_z X^\mu(z)\, \partial_w X^\nu(w)\rangle_{\text{plane}} &= -\delta^{\mu\nu} \frac{1}{(z-w)^2} \\ \langle \partial_z X^\mu(z)\, \partial_{\bar{w}} X^\nu(\bar{w})\rangle_{\text{plane}} &= -\delta^{\mu\nu}\delta^2(z-w)\end{aligned} \tag{26}$$

Here $\partial_z X^\mu$ and $\partial_{\bar{z}} X^\mu$ are genuine conformal fields of dimension (1,0) and (0,1), respectively. Another interesting conformal field can be defined using exponentials of the modes of the free scalar field. We make a mode expansion of $X^\mu(z,\bar{z})$ on the plane:

$$\begin{aligned}X^\mu(z,\bar{z}) = X_0^\mu + ip_0^\mu \ln|z| &+ \frac{i}{\sqrt{2}} \sum_{n\neq 0} \frac{\alpha_n^\mu z^n}{n} \\ &+ \frac{i}{\sqrt{2}} \sum_{n\neq 0} \frac{\bar{\alpha}_n^\mu \bar{z}^n}{n}\end{aligned} \tag{27}$$

where X_0^μ and p_0^μ are the center-of-mass position and momentum of the string, and α_n^μ, $\bar{\alpha}_n^\mu$ are oscillators describing the excitation modes. In operator language, quantization is performed via the canonical commutators:

$$\begin{aligned}[X_0^\mu, p_0^\nu] = i\delta^{\mu\nu} \quad & [\alpha_n^\mu, \bar{\alpha}_m^\nu] = 0 \\ [\alpha_n^\mu, \alpha_m^\nu] = n\delta_{n+m,0}\delta^{\mu\nu} &= [\bar{\alpha}_n^\mu, \bar{\alpha}_m^\nu]\end{aligned} \tag{28}$$

Now define the "vertex operator"[10]

$$V(a,z,\bar{z}) : e^{ia_\mu X^\mu(z,\bar{z})} : = V_o V_L V_R$$

$$\begin{aligned} V_o &\equiv \exp(-a_\mu p_o^\mu \ln|z|)\exp(ia_\mu X_o^\mu) \\ V_L &\equiv \exp\left(\frac{a_\mu}{\sqrt{2}} \sum_{n=1}^{\infty} \frac{\alpha_{-n}^\mu z^{-n}}{n}\right) \exp\left(\frac{-a_\mu}{\sqrt{2}} \sum_{n=1}^{\infty} \frac{\alpha_n^\mu z^n}{n}\right) \\ V_R &\equiv \exp\left(\frac{a_\mu}{\sqrt{2}} \sum_{n=1}^{\infty} \frac{\bar{\alpha}_{-n}^\mu \bar{z}^{-n}}{n}\right) \exp\left(\frac{-a_\mu}{\sqrt{2}} \sum_{n=1}^{\infty} \frac{\bar{\alpha}_n^\mu \bar{z}^n}{n}\right) \end{aligned} \tag{29}$$

$a_\mu = (a_1 \ldots a_{26})$ is a set of arbitrary real or complex numbers.

We have merely written down a precise prescription for normal-ordering an object as complicated as the exponential of a scalar field. One can now compute, on the plane, the correlation function of two vertex operators:

$$\begin{aligned} &\langle :e^{ia_\mu X^\mu(z,\bar{z})} : : e^{ib_\mu X^\mu(w,\bar{w})} : \rangle_{\text{plane}} \\ &= \delta(a + b)\,|z - w|^{a.b} \\ &= \delta(a + b)\,|z - w|^{-a^2} \end{aligned} \tag{30}$$

Although this can be checked directly using the OPE of T_{zz} with the vertex operator, the form of the two-point function immediately implies that $V(a,z,\bar{z})$ is a conformal field of dimension $(a^2/2, a^2/2)$.

In string theory, amplitudes are computed by associating generalized vertex operators to each particle state of the string. The operator defined in Eq. (29) corresponds to emission of a scalar particle, the tachyon, which corresponds to the ground state of the closed bosonic string. The numerical vector a_μ is associated to the space-time momentum k_μ of the tachyon. As mentioned earlier, scattering amplitudes for particle states of the string are obtained by multiplying correlation functions of the corresponding vertex operators by the partition function.

The correlators calculated above have been defined on the infinite complex plane. Actually this can be made into a Riemann surface. The complex plane can be stereographically projected

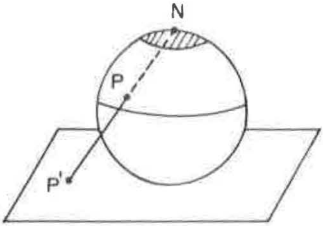

FIGURE 8

onto a 2-sphere with one point deleted (Fig. 8). Straight lines through the north pole identify pairs of points P and P' on the 2-sphere and the complex plane in a one-to-one fashion. So the complex plane is the punctured sphere. Adding one point to the complex plane makes it compact (we can think of this as the "point at infinity"). Thus the correlators we have computed in Eqs. (25), (26), and (30) are those for free scalar field theory on the sphere.

For Riemann surfaces of higher genus, mode expansions of fields may be performed on some region which is topologically a disk, as in Fig. 7. Correlation functions are calculated on the disk, with boundary conditions on the circle enclosing the disk determined by doing a path integral on the rest of the Riemann surface.[23,24] Alternatively (and more or less equivalently) one can try to determine the correlation function from its known singular behavior on the disk (which is a local property independent of the Riemann surface), from conformal Ward identities and from general properties of functions on Riemann surfaces.[25]

An important class of conformal fields, the *holomorphic* fields, are those whose equation of motion is an equation of analyticity. Examples are the free Majorana field ψ; the free Dirac fields b, c; the derivative of the scalar field $\partial_z X^\mu$ and the energy-momentum tensor T_{zz}. For such fields we have the equations of motion:

$$\begin{aligned} \partial_{\bar z}\psi &= 0 \\ \partial_{\bar z} b = \partial_{\bar z} c &= 0 \\ \partial_{\bar z}(\partial_z \phi) &= 0 \\ \partial_{\bar z} T_{zz} &= 0 \end{aligned} \tag{31}$$

The first three equations follow from the free Lagrangians Eqs. (10), (12c), and (12d) while the last one is a general property of conformal field theory, as we have seen.

A consequence of the analyticity of these fields is that their correlation functions are analytic (holomorphic) except at points where the arguments of fields coincide, where there is a singularity. Thus we are interested in meromorphic functions on Riemann surfaces. This puts a strong constraint on allowed functions, since the theory of analytic and meromorphic functions on Riemann surfaces is well-studied and has a very tightly constrained structure.[4,25]

The most convenient functions in terms of which correlators on Riemann surfaces can be expressed are the well-known θ-functions.[26,27] In genus 1 (the torus) we have the important θ-functions,

$$\theta\begin{pmatrix} a \\ b \end{pmatrix}(z|\tau) \equiv \sum_{n=-\infty}^{+\infty} e^{i\pi(n+a)^2\tau + 2\pi i(n+a)(z+b)}$$

where z is a coordinate on the torus and τ is the Teichmuller parameter which labels the complex structure (Fig. 6). The constants a and b are chosen to have the values 0 or 1/2. For the bosonic field, a particularly important construction is the "prime form" [25–27]

$$E(z,w) = \frac{\theta\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}(z - w|\tau)}{\theta'\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}(0|\tau)} \tag{32}$$

This has the property that it goes like $(z - w)$ as z approaches w, and is non-singular, nonzero and analytic everywhere else. It is the generalization appropriate to a torus of the function $(z - w)$ on the plane which generally appears in correlators, and can be easily generalized to higher genus Riemann surfaces. The only problem is that it is a multi-valued function, while correlators of free bosonic fields are single-valued. This can be remedied by multiplying or adding certain other functions. For example, the generalization of Eq. (25) to the torus is:

$$\langle X^{\mu}(z,\bar{z})X^{\nu}(w,\bar{w})\rangle_{\text{torus}}$$

$$= \delta^{\mu\nu}\left(\log|E(z,w)|^2 - 2\pi\,\frac{\mathrm{Im}(z-w)^2}{\mathrm{Im}\tau}\right) \tag{33}$$

In the coincident limit, the second term vanishes and the first one reduces to Eq. (25).

Similarly the two-point function of the tachyon vertex operator on the torus is:

$$\langle :e^{ia_{\mu}X^{\mu}(z,\bar{z})}:e^{ib_{\mu}X^{\mu}(z,\bar{z})}:\rangle_{\text{torus}}$$

$$= \delta(a+b)\left\{|E(z,w)|\exp\left(-\pi\,\frac{\mathrm{Im}(z-w)^2}{\mathrm{Im}\tau}\right)\right\}^{a.b} \tag{34}$$

which again has the same singular behavior as on the plane.

An aesthetically satisfying feature of string theory is that physical scattering amplitudes (which, in the correct string theory, should be related to experimentally measured cross sections) are given in terms of powers and derivatives of the θ-functions, which have very beautiful mathematical properties.

VIII. CONCLUSION

Whether or not string theories have immediate success in particle physics, the concept of conformal quantum field theory on Riemann surfaces is mathematically profound, and will presumably play a major role in the general framework of quantum field theory. It is already of importance in the study of critical phenomena in statistical systems. Quite conceivably, this subject will reappear in different and rather unexpected ways in physics. It might also happen that with mathematical advances in the study of differentiable and complex structures in *four* dimensions, the intuition gained from studying conformal field theory on Riemann surfaces will prove useful in understanding the right kind of field theory to describe quantum gravity.

Acknowledgments

Most of what I know about this subject has been learned over the last year in collaboration with Samir Mathur, and it is a pleasure to thank him. I am grateful to Probir Roy and S. M. Roy for reading the manuscript and for helpful criticism.

SUNIL MUKHI
Tata Institute of Fundamental Research,
Homi Bhabha Road,
Bombay-400005,
India

References

1. B. Riemann, *Gesammelte mathematische werke* (Dover, New York, 1953).
2. A. Einstein, *The Meaning of Relativity* (Princeton University Press, Princeton, 1946).
3. S. S. Chern, *Complex Manifolds without Potential Theory* (Springer-Verlag, 1979).
4. H. M. Farkas and I. Kra, *Riemann Surfaces* (Springer-Verlag, 1980).
5. M. Jacob (Ed.), *Dual Theory* (North-Holland, Amsterdam, 1974); M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory* (Cambridge University Press, Cambridge, 1987).
6. L. Brink, P. di Vecchia and P. Howe, Phys. Lett. **65B**, 471 (1976); S. Deser and B. Zumino, Phys. Lett. **65B**, 369 (1976).
7. A. M. Polyakov, Phys. Lett. **103B**, 207, 211 (1981).
8. Y. Nambu, Notes prepared for the Copenhagen High Energy Symposium 1970, reprinted in *Strings, Lattice Gauge Theory and High Energy Phenomenology*, eds. V. Singh and S. Wadia (World Scientific, Singapore, 1987).
9. L. Bers, Bull. Amer. Math. Soc. **5**, 131 (1981).
10. J. Lepowsky *et al.* (Eds.), *Vertex Operators in Mathematics and Physics* (Springer-Verlag, 1984).
11. O. Alvarez, Nucl. Phys. B **216**, 125 (1983); D. Friedan, in *Recent Advances in Field Theory and Statistical Mechanics*, eds. J. B. Zuber and R. Stora (North-Holland, Amsterdam, 1984); O. Alvarez, in *Workshop on Unified String Theories*, eds. M. Green and D. Gross (World Scientific, Singapore, 1986); E. D'Hoker and D. H. Phong, Nucl. Phys. B **269**, 205 (1986).
12. A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, Nucl. Phys. B **241**, 333 (1984).
13. D. Friedan, Z. Qiu and S. Shenker, Phys. Lett. **151B**, 37 (1985); D. Friedan, E. Martinec and S. Shenker, Nucl. Phys. B **271**, 93 (1986).
14. A. M. Polyakov, JETP Letters **12**, 538 (1970).
15. C. Lovelace, Phys. Lett. **135B**, 75 (1984); P. Candelas, G. Horowitz, A. Strominger and E. Witten, Nucl. Phys. **B258**, 46 (1985); D. Nemeschansky and S. Yankielowicz, Phys. Rev. Lett. **54**, 620 (1985); S. Jain, R. Shankar and S. Wadia, Phys. Rev. D **32**, 2713 (1985).
16. A. Sen, Phys. Rev. Lett. **55**, 1846 (1985); C. Callan, D. Friedan, E. Martinec and M. Perry, Nucl. Phys. B **262**, 593 (1985).
17. D. Friedan, Phys. Rev. Lett. **45**, 1057 (1980); L. Alvarez-Gaume, D. Freedman

and S. Mukhi, Ann. Phys. **134**, 85 (1981); E. Witten, Comm. Math. Phys. **92**, 455 (1984); L. Alvarez-Gaume and D. Freedman, Comm. Math. Phys. **80**, 443 (1980); S. Mukhi, Phys. Lett. **162B**, 345 (1985); S. P. de Alwis, Phys. Lett. **164B**, 67 (1985); M. Grisaru, A. Van de Ven and D. Zanon, Phys. Lett. **173B**, 423 (1986).
18. M. A. Virasoro, Phys. Rev. D **1**, 2933 (1969).
19. D. Friedan, Z. Qiu and S. Shenker, Phys. Rev. Lett. **52**, 1575 (1984); B. L. Feigin and D. B. Fuks, Funct. Analys. Appl. **16**, 114 (1982); V. L. Dotsenko and V. Fateev, Nucl. Phys. B **240**, 312 (1984).
20. A. Neveu and J. H. Schwarz, Nucl. Phys. B **31**, 86 (1971); P. Ramond, Phys. Rev. D **3**, 2415 (1971).
21. E. Witten, in Ref. 17; V. Knizhnik and A. B. Zamolodchikov, Nucl. Phys. B **247**, 83 (1984); P. Goddard and D. Olive, Int. J. Mod. Phys. **A1**, 303 (1986).
22. J. Cardy, "Conformal Invariance," in *Phase Transitions and Critical Phenomena*, Vol. 11, eds. C. Domb and M. Green (Academic Press, London, 1987).
23. N. Ishibashi, Y. Matsuo and H. Ooguri, Mod. Phys. Lett. **2**, 119 (1987); C. Vafa, Phys. Lett. **190B**, 47 (1987); L. Alvarez-Gaume, C. Gomez and C. Reina, Phys. Lett. **190B**, 55 (1987); E. Witten, Princeton preprint PUPT-1057 (May 1987).
24. S. Mukhi and S. Panda, Tata Institute preprint TIFR/TH/87-51 (Nov. 1987).
25. S. Hamidi and C. Vafa, Nucl. Phys. B **279**, 465; (1987); M. A. Namazie, K. S. Narain and M. H. Sarmadi, Phys. Lett. **177B**, 329 (1986); H. Sonoda, Phys. Lett. **178B**, 390 (1986); J. Atick and A. Sen, Nucl. Phys. B **286**, 189 (1987); T. Eguchi and H. Ooguri, Nucl. Phys. B **282**, 308 (1987); Phys. Lett. **187B**, 127 (1987); E. Verlinde and H. Verlinde, Nucl. Phys. B **288**, 357 (1987); M. J. Dugan and H. Sonoda, Nucl. Phys. B **289**, 227 (1987).
26. D. Mumford, *Tata Lectures on Theta* (Birkhauser, Basel, 1984).
27. J. Fay, *Theta Functions on Riemann Surfaces* (Springer-Verlag, Berlin, 1973).