

2 Scalar 1-loop Feynman integrals in arbitrary space-time dimension d – an update

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2.1 Introduction

The study and use of analyticity of scattering amplitudes was founded by R. Eden, P. Landshoff, D. Olive and J. Polkinghorn in their famous book “The Analytic S-Matrix” in 1966 [1]. Indeed, already in 1969 J. Schwinger quotes: “One of the most remarkable discoveries in elementary particle physics has been that of the complex plane,” “... the theory of functions of complex variables plays the role not of a mathematical tool, but of a fundamental description of nature inseparable from physics. ...” [2].

It took many years to make the use of analyticity and unitarity, together with renormalizability and gauge invariance of quantum field theory a practical tool for the calculation of cross sections at real colliders. When the analysis of LEP 1 data, around 1989, was prepared it became evident that the S-matrix language helps to efficiently sort the various perturbative contributions of the Standard Model.

The scattering amplitude for the reaction $e^+e^- \rightarrow (Z, \gamma) \rightarrow f\bar{f}$ at LEP energies depends on two variables s and $\cos\theta$, and the integrated cross section may be described by an analytical function of s with a simple pole, describing mass and width of the Z resonance:

$$A = \frac{R}{s - M_Z^2 + iM_Z\Gamma_Z} + \sum_{i=0}^{\infty} a_i (s - M_Z^2 + iM_Z\Gamma_Z)^i. \quad (2.1)$$

Here, position $s_0 = M_Z^2 - iM_Z\Gamma_Z$ and residue R of the pole as well as the background expansion are of interest. The analytic form of (2.1) has to be respected when deriving a Z amplitude at multiloop accuracy; see [3] and the references therein.

Shortly after the work by Eden et al., physical amplitudes were proposed to be considered also as complex functions of space-time dimension d (dimensional regularization), in C. Bollini and J. Giambiagi, “Dimensional Renormalization: The Number of Dimensions as a Regularizing Parameter” (1972) [4] and G. ’t Hooft and M. Veltman, “Regularization and renormalization of gauge fields” (1972) [5].

In perturbative calculations with dimensional regularization, Feynman integrals I are complex functions of the space-time dimension $d = 4 - 2\epsilon$. In fact, they are meromorphic functions of d and may be expanded in Laurent series around poles at *e.g.* $d_s = 4 + 2N_0$, $N_0 \geq 0$. Be J_n an n -point one-loop Feynman integral as shown in figure C.1:

$$J_n \equiv J_n(d; \{p_i p_j\}, \{m_i^2\}) = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{D_1^{\nu_1} D_2^{\nu_2} \dots D_n^{\nu_n}} \quad (2.2)$$

with

$$D_i = \frac{1}{(k + q_i)^2 - m_i^2 + i\epsilon} \quad (2.3)$$

and

$$\nu = \sum_{i=1}^n \nu_i, \quad \sum_{e=1}^n p_e = 0. \quad (2.4)$$

The Feynman integrals are analytical functions of d everywhere with exclusion of isolated singular points d_s , where they behave not worse than

$$\frac{A_s}{(d - d_s)^{N_s}}. \quad (2.5)$$

In physics applications, we need the Feynman integrals at a potentially singular point, $d = 4$, so that the general behaviour of them at non-singular points is not in the original focus. Nevertheless, the question arises:

Can we determine the general d -dependence of a Feynman integral?

For one-loop integrals, the question has been answered recently, in K.H. Phan, T. Riemann, “Scalar 1-loop Feynman integrals as meromorphic functions in space-time dimension d ” (2018) [6].

At the begin of systematic cross-section calculations in d dimensions, there are two seminal papers on 1-loop Feynman integrals in dimensional regularization: Passarino, Veltman (Feb. 1978), “One Loop Corrections for e^+e^- Annihilation into $\mu^+\mu^-$ in the Weinberg Model” [7] and ’t Hooft, Veltman (Nov. 1978), “Scalar oneloop integrals” [8]. Later, many improvements and generalizations were introduced in various respects.

We see several reasons to study the d -dependence of one-loop Feynman integrals and will discuss them shortly in the next subsection.

2.2 Interests in the d -dependence of one-loop Feynman integrals

2.2.1 Interest from mathematical physics

There is a general interest to know the Feynman integrals as meromorphic functions of space-time dimension d , and the easiest case is that of one loop. Early attempts, for the massless case, trace back to E. Boos and A. Davydychev (1986), “A Method of the Evaluation of the Vertex Type Feynman Integrals” [9]. The general one-loop integrals were tackled systematically by O. Tarasov et al. since the nineteen nineties; see e.g. [10–13] and references therein. In J. Fleischer, F. Jegerlehner, O. Tarasov (2003), “A new hypergeometric representation of one loop scalar integrals in d dimensions” [14, 15], the class of generalized hypergeometric functions for massive one-loop Feynman integrals with unit indices was determined and studied with a novel approach based on dimensional difference equations:

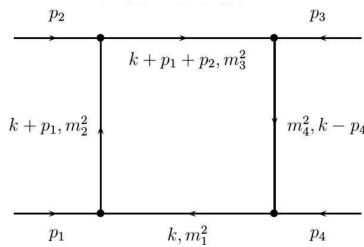


Fig. C.1: *One-loop Feynman integral.*

- ${}_2F_1$ Gauss hypergeometric functions are needed for self-energies;
- F_1 Appell functions are needed for vertices;
- F_S Lauricella-Saran functions are needed for boxes.

Finally, the correct, general massive one-loop one- to four-point functions with unit indices at arbitrary kinematics were determined by K.H. Phan and T.R. (2018) in [6], where also the numerics of the generalized hypergeometric functions was worked out.

2.2.2 Interest from tensor reductions of n -point functions in higher space-time dimensions

For many-particle calculations, there appear at certain kinematical configurations p_i inverse Gram determinants $1/G(p_i)$ from tensor reductions. These terms $1/G(p_i)$ may diverge, because Gram determinants can exactly vanish: $G(p_i) \equiv 0$. One may perform tensor reductions so that no inverse Gram determinants appear. But then one has to calculate scalar 1-loop integrals in higher dimensions, $D = 4 + 2n - 2\epsilon, n > 0$. See [16, 17]. In fact, one introduces new scalar integrals [16]. Let us take as an example a rank-5 tensor of an n -point function:

$$\begin{aligned}
 I_n^{\mu\nu\lambda\rho\sigma} &= \int \frac{d^d k}{i\pi^{d/2}} \frac{k^\mu k^\nu k^\lambda k^\rho k^\sigma}{\prod_{j=1}^n c_j} \\
 &= - \sum_{i,j,k,l,m=1}^n q_i^\mu q_j^\nu q_k^\lambda q_l^\rho q_m^\sigma n_{ijklm} I_{n,ijklm}^{[d+]^5} \\
 &\quad + \frac{1}{2} \sum_{i,j,k=1}^n g^{[\mu\nu} q_i^\lambda q_j^\rho q_k^\sigma] n_{ijk} I_{n,ijk}^{[d+]^4} - \frac{1}{4} \sum_{i=1}^n g^{[\mu\nu} g^{\lambda\rho} q_i^\sigma] I_{n,i}^{[d+]^3}. \tag{2.6}
 \end{aligned}$$

The integrals $I_{n,ab\dots}^{[d+]^l}$ are special cases of $I_{n,ab\dots}^{[d+]^l, s}$, defined in $[d+]^l = 4 - 2\epsilon + 2l$ dimensions, by shrinking line s and raising the powers of propagators (indices) a, b, \dots

At this step, the tensor integral is represented by scalar integrals with higher space-time dimensions and higher propagator powers. The publicly available Feynman integral libraries deliver, though, ordinary scalar integrals in $d = 4 - 2\epsilon$ dimensions and with unit propagator powers. With the usual integration-by-parts reduction technique [18, 19] one may shift indices, i.e. reduce propagator powers to unity:

$$\nu_j \mathbf{j}^+ I_5 = \frac{1}{\binom{0}{0}_5} \sum_{k=1}^5 \binom{0j}{0k}_5 \left[d - \sum_{i=1}^5 \nu_i (\mathbf{k}^- \mathbf{i}^+ + 1) \right] I_5. \tag{2.7}$$

The operators $\mathbf{i}^\pm, \mathbf{j}^\pm, \mathbf{k}^\pm$ act by shifting the indices ν_i, ν_j, ν_k by ± 1 .

After this step, one has yet to deal with scalar functions in $d = 4 - 2\epsilon + 2l$ dimensions. This may be further reduced by applying dimensional reduction formulas invented by O. Tarasov (1996) [10, 13]. Shift of dimension and index:

$$\nu_j (\mathbf{j}^+ I_5^{[d+]}) = \frac{1}{\binom{0}{0}_5} \left[- \binom{j}{0}_5 + \sum_{k=1}^5 \binom{j}{k}_5 \mathbf{k}^- \right] I_5, \tag{2.8}$$

and also shift of only dimension:

$$(d - \sum_{i=1}^5 \nu_i + 1) I_5^{[d+]} = \frac{1}{\binom{0}{0}_5} \left[\binom{0}{0}_5 - \sum_{k=1}^5 \binom{0}{k}_5 \mathbf{k}^- \right] I_5. \tag{2.9}$$

The procedure is elegant, but it introduces in both cases inverse powers of potentially vanishing Gram determinants. As a consequence, one has finally to treat in sophisticated ways their numerical implications.

At this stage one might try an alternative: Perform the reductions of tensor functions to scalar ones with unit indices, but allowing for the use of higher space-time dimensions. This avoids the vanishing inverse Gram problem, but introduces the need of a library of scalar Feynman integrals in higher dimensions. This idea makes it attractive to derive an algorithm allowing the systematic calculation of scalar one- to n -point functions in arbitrary dimensions, and to implement a numerical solution for that.

To be a bit more definite, we quote here some unpublished formulae from [17, 20]. The following reduction of a 5-point tensor in terms of tensor coefficients E_{ijklm}^s , with line s skipped from the 5-point integral, may be used as a starting point:

$$I_5^{\mu\nu\lambda\rho\sigma} = \sum_{s=1}^5 \left[\sum_{i,j,k,l,m=1}^5 q_i^\mu q_j^\nu q_k^\lambda q_l^\rho q_m^\sigma E_{ijklm}^s + \sum_{i,j,k=1}^5 g^{[\mu\nu} q_i^\lambda q_j^\rho q_k^{\sigma]} E_{00ijk}^s + \sum_{i=1}^5 g^{[\mu\nu} g^{\lambda\rho} q_i^{\sigma]} E_{0000i}^s \right]. \quad (2.10)$$

The tensor coefficients E_{ijklm}^s are expressed in terms of integrals $I_{4,i\dots}^{[d+],s}$, e.g.:

$$E_{ijklm}^s = -\frac{1}{\binom{0}{0}_5} \left\{ \left[\binom{0l}{sm}_5 n_{ijk} I_{4,ijk}^{[d+],s} + (i \leftrightarrow l) + (j \leftrightarrow l) + (k \leftrightarrow l) \right] + \binom{0s}{0m}_5 n_{ijkl} I_{4,ijkl}^{[d+],s} \right\}. \quad (2.11)$$

No factors $1/G_5 = \binom{0}{0}_5$ appear. Now, in a next step, one may avoid the appearance of inverse sub-Gram determinants $\binom{0}{0}_4$. Further, the complete dependence on the indices i of the tensor coefficients can be shifted into the integral's pre-factors with signed minors. One can say that the indices *decouple* from the integrals. As an example, we reproduce the 4-point part of $I_{4,ijkl}^{[d+],s}$:

$$\begin{aligned} n_{ijkl} I_{4,ijkl}^{[d+],s} &= \frac{\binom{0}{i} \binom{0}{j} \binom{0}{k} \binom{0}{l}}{\binom{0}{0} \binom{0}{0} \binom{0}{0} \binom{0}{0}} d(d+1)(d+2)(d+3) I_4^{[d+],4} \\ &+ \frac{\binom{0i}{0j} \binom{0}{k} \binom{0}{l} + \binom{0i}{0k} \binom{0}{j} \binom{0}{l} + \binom{0j}{0k} \binom{0}{i} \binom{0}{l} + \binom{0i}{0l} \binom{0}{j} \binom{0}{k} + \binom{0j}{0l} \binom{0}{i} \binom{0}{k} + \binom{0k}{0l} \binom{0}{i} \binom{0}{j}}{\binom{0}{0}^3} \\ &\times d(d+1) I_4^{[d+],3} \\ &+ \frac{\binom{0i}{0l} \binom{0j}{0k} + \binom{0j}{0l} \binom{0i}{0k} + \binom{0k}{0l} \binom{0i}{0j}}{\binom{0}{0}^2} I_4^{[d+],2} + \dots \end{aligned} \quad (2.12)$$

In (2.12), one has to understand the 4-point integrals to carry the corresponding index s of (2.10) and the signed minors are $\binom{0}{k} \rightarrow \binom{0s}{ks}_5$ etc. We arrived at:

✓ No scalar 5-point integrals in higher dimensions.

✓ No inverse Gram determinants $\binom{0}{0}_5$.

✓ The 4-point integrals are without indices.

† Scalar 4-point integrals in higher dimensions appear: $I_4^{[d+],s}$ etc.

† Inverse 4-point Gram determinants $\binom{0}{0}_5 \equiv \binom{0}{0}_4$ appear.

2.2.3 Interest from multi-loop calculations

Higher-order loop calculations need higher-order contributions from ϵ -expansions of one-loop terms, typically stemming from the expansions

$$\frac{1}{d-4} = -\frac{1}{2\epsilon} \quad (2.13)$$

and

$$\Gamma(\epsilon) = \frac{a_1}{\epsilon} + a_0 + a_1\epsilon + \dots \quad (2.14)$$

A seminal paper on the ϵ -terms of one-loop functions is U. Nierste, D. Müller, M. Böhm, (1992), “Two loop relevant parts of D-dimensional massive scalar one loop integrals” [21]. A general analytical solution of the problem of determining the general ϵ -expansion of Feynman integrals is unsolved so far, even for the one-loop case. Though, see the series of papers by G. Passarino et al. [22–25]. The determination of one-loop Feynman integrals as meromorphic functions of d might be a useful preparatory step for determining the pole expansion in d around e.g. $d = 4$.

2.2.4 Interest from Mellin-Barnes representations

A powerful approach to arbitrary Feynman integrals is based on Mellin-Barnes representations [26, 27]. One-loop integrals with variable, in general non-integer indices are needed in the context of the *loop-by-loop Mellin-Barnes approach* to multi-loop integrals. Details may be found in the literature on the Mathematica package AMBRE [28–35], and in references therein.

A crucial technical problem of the Mellin-Barnes representations arises from the rising number of dimensions of these representations with a rising number of physical scales. We will detail this in subsection 2.3.1. So there is an unresolved need of low-dimensional one-loop MB-integrals, with arbitrary indices.

2.3 Mellin-Barnes representations for one-loop Feynman integrals

There are two numerical MB-approaches advocated.

2.3.1 AMBRE

There are several ways to take advantage of Mellin-Barnes representations for the calculation of Feynman integrals. One approach is the replacement of massive propagators by Mellin-Barnes integrals over massless propagators, invented by N. Usyukina (1975) [36]. Another approach transforms the Feynman parameter representation with Mellin-Barnes representations into multiple complex path integrals, invented in 1999 by V. Smirnov for planar diagrams [26] and B. Tausk for non-planar diagrams [27]. This approach implies “automatically” a general solution of the infra-red problem and has been worked out in the AMBRE approach [28, 32, 34, 35, 37].

The general definitions for a multi-loop Feynman integral are:

$$J_n^L \equiv J_n^L(d; \{p_i p_j\}, \{m_i^2\}) = \int \prod_{j=1}^L \frac{d^d k_j}{i\pi^{d/2}} \frac{1}{D_1^{\nu_1} D_2^{\nu_2} \dots D_n^{\nu_n}} \quad (2.15)$$

with

$$D_i = \left(\sum_{l=1}^L a_{il} k_l + \sum_{e=1}^E b_{ie} p_e \right)^2 - m_i^2 + i\delta, \quad a_{il}, b_{ie} \in \{-1, 0, 1\}, \quad (2.16)$$

where m_i are the masses, p_e the external momenta, k_l the loop momenta, $i\delta$ the Feynman prescription and finally ν_i the complex variables.

With the following Feynman trick we get a really neat parametric representation:

$$\frac{(-1)^\nu}{\prod_{j=1}^n (-D_j^{\nu_j})} = \frac{(-1)^\nu \Gamma(\nu) \left(\prod_{j=1}^n \int_{\{x_j \geq 0\}} \frac{dx_j x_j^{\nu_j-1}}{\Gamma(\nu_j)} \right) \delta(1 - \sum_{j=1}^n x_j)}{(-k_l^\mu M_{ll'} k_{l'\mu} + 2k_l^\mu Q_{ll'} + J - i\delta)^\nu}, \quad \nu = \sum_{j=1}^n \nu_j, \quad (2.17)$$

where

$$M_{ll'} = \sum_{j=1}^n a_{jl} a_{j l'} x_j \quad (2.18)$$

is an $L \times L$ symmetric matrix,

$$Q_l^\nu = - \sum_{j=1}^n x_j a_{jl} \sum_{e=1}^E b_{je} p_e^\nu \quad (2.19)$$

is a vector with L components and

$$J = - \sum_{j=1}^n x_j \left(\sum_{e=1}^E b_{je} p_e^\mu \sum_{e'=1}^E p_{e'}^\nu b_{je'} g_{\mu\nu} - m_j^2 \right), \quad (2.20)$$

where x_j are the Feynman parameters introduced with the Feynman trick. The metric tensor is $g_{\mu\nu} = \text{diag}(1, -1, \dots, -1)$.

The Feynman integral can now be written in the Feynman parameter integral representation:

$$J_n^L = (-1)^\nu \Gamma(\nu - LD/2) \left(\prod_{j=1}^n \int_{\{x_j \geq 0\}} \frac{dx_j x_j^{\nu_j-1}}{\Gamma(\nu_j)} \right) \delta(1 - \sum_{j=1}^n x_j) \frac{U(x)^{\nu-(L+1)D/2}}{F(x)^{\nu-LD/2}}, \quad (2.21)$$

where

$$U(x) = \det M, \quad (2.22)$$

$$F(x) = U(x) (Q_l^\mu M_{ll'}^{-1} Q_{l'\mu} + J - i\delta). \quad (2.23)$$

From these definitions it follows that the functions $F(x)$ and $U(x)$ are homogeneous in the Feynman parameters x_i . The function $U(x)$ is of degree L and the function $F(x)$ is of degree $L + 1$. The functions $U(x)$ and $F(x)$ are also known as Symanzik polynomials.

At one loop level the definition of the Feynman integral simplifies drastically and gives many insights straight away which we will bring to light in this work:

$$J_n \equiv J_n(d; \{p_i p_j\}, \{m_i^2\}) = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{D_1^{\nu_1} D_2^{\nu_2} \dots D_n^{\nu_n}} \quad (2.24)$$

with propagators depending only on one loop momenta:

$$D_i = \frac{1}{(k + q_i)^2 - m_i^2 + i\epsilon}, \quad (2.25)$$

with

$$q_i = \sum_{e=1}^i p_e. \quad (2.26)$$

We assume here for brevity

$$\nu_i = 1, \quad \sum_{e=1}^n p_e = 0. \quad (2.27)$$

If we take the argument of the Dirac delta function to be $1 - \sum_{j=1}^n x_j$ the Feynman parameter representation for one-loop Feynman integrals simplifies to:

$$J_n = (-1)^n \Gamma(n - d/2) \int_0^1 \prod_{i=1}^n dx_i \delta\left(1 - \sum_{j=1}^n x_j\right) \frac{1}{F_n(x)^{n-d/2}}. \quad (2.28)$$

Here, the F -function is the second Symanzik polynomial, which is just of second degree in the Feynman parameters:

$$F_n(x) = \frac{1}{2} \sum_{i,j} x_i Y_{ij} x_j - i\epsilon. \quad (2.29)$$

The Y_{ij} are elements of the Cayley matrix $Y = (Y_{ij})$,

$$Y_{ij} = Y_{ji} = m_i^2 + m_j^2 - (q_i - q_j)^2. \quad (2.30)$$

Gram and Cayley determinants were introduced by Melrose (1965) [38]; see also [13]. The $(n-1) \times (n-1)$ dimensional Gram determinant $G_n \equiv G_{12\dots n}$ is

$$G_n = - \begin{vmatrix} (q_1 - q_n)^2 & (q_1 - q_n)(q_2 - q_n) & \dots & (q_1 - q_n)(q_{n-1} - q_n) \\ (q_1 - q_n)(q_2 - q_n) & (q_2 - q_n)^2 & \dots & (q_2 - q_n)(q_{n-1} - q_n) \\ \vdots & \vdots & \ddots & \vdots \\ (q_1 - q_n)(q_{n-1} - q_n) & (q_2 - q_n)(q_{n-1} - q_n) & \dots & (q_{n-1} - q_n)^2 \end{vmatrix}. \quad (2.31)$$

The $2^n G_n$ equals notationally the G_{n-1} of [13]. Evidently, the Gram determinant G_n is independent of the propagator masses.

The Cayley determinant $\Delta_n = \lambda_{12\dots n}$ is composed of the Y_{ij} introduced in (2.30):

$$\text{Cayley determinant : } \Delta_n = \lambda_n \equiv \lambda_{12\dots n} = \begin{vmatrix} Y_{11} & Y_{12} & \dots & Y_{1n} \\ Y_{12} & Y_{22} & \dots & Y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1n} & Y_{2n} & \dots & Y_{nn} \end{vmatrix}. \quad (2.32)$$

We also define the modified Cayley determinant

$$\text{modified Cayley determinant : } ()_n = \begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & Y_{11} & Y_{12} & \dots & Y_{1n} \\ 1 & Y_{12} & Y_{22} & \dots & Y_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & Y_{1n} & Y_{2n} & \dots & Y_{nn} \end{vmatrix}. \quad (2.33)$$

The determinants Δ_n , $(\)_n$ and G_n are evidently independent of a common shifting of the momenta q_i .

One may use Mellin-Barnes integrals [39],

$$\frac{1}{(1+z)^\lambda} = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma(\lambda+s)}{\Gamma(\lambda)} z^s = {}_2F_1 \left[\begin{matrix} \lambda, b; \\ b; \end{matrix} -z \right], \quad (2.34)$$

in order to split the sum $F_n(x)$ in (2.29) into a product, getting nested MB-integrals to be calculated. For some mathematics behind the derivation, see the corollary at p. 289 in [40]). Eqn. (2.34) is valid if $|\text{Arg}(z)| < \pi$. The integration contour has to be chosen such that the poles of $\Gamma(-s)$ and $\Gamma(\lambda+s)$ are well-separated. The right hand side of (2.34) is identified as Gauss' hypergeometric function.

There are $N_n = \frac{1}{2}n(n+1)$ different Y_{ij} for n -point functions, leading to $N_n = [\frac{1}{2}n(n+1)-1]$ dimensional Mellin-Barnes integrals when splitting the sum in (2.29) into a product:

- $N_3 = 5$ MB-dimensions for the most general massive vertices;
- $N_4 = 9$ MB-dimensions for the most general massive box integrals;
- $N_5 = 14$ MB-dimensions for the most general massive pentagon integrals.

The introduction of N_n -dimensional MB-integrals allows to perform the x -integrations. The MB-integrations have to be performed afterwards, and this raises some mathematical problems with rising integral dimensions. This is, for Mellin-Barnes integrals numerical applications, one of the most important limiting factors.

For further details of this approach, we refer to the quoted literature on AMBRE and MBnumerics.

2.3.2 *MBOneLoop*

A completely different approach was initiated in [6, 41]. The idea is based on rewriting the F -function in (2.28) by exploring the factor $\delta(1 - \sum x_i)$ which makes the n -fold x -integration to be an integral over an $(n-1)$ -simplex.

The δ -function allows for the elimination of x_n , just one of the x_i , which creates linear terms in the remaining x_i -variables in the F -function:

$$F_n(x) = x^T G_n x + 2H_n^T x + K_n. \quad (2.35)$$

The $F_n(x)$ may be re-cast back into a bilinear form by shifts $x \rightarrow (x-y)$,

$$F_n(x) = (x-y)^T G_n (x-y) + r_n - i\varepsilon = \Lambda_n(x) + r_n - i\varepsilon = \Lambda_n(x) + R_n. \quad (2.36)$$

As a result, there is a separation of F into a homogeneous part $\Lambda_n(x)$,

$$\Lambda_n(x) = (x-y)^T G_n (x-y), \quad (2.37)$$

and an inhomogeneity R_n ,

$$R_n = r_n - i\varepsilon = K_n - H_n^T G_n^{-1} H_n - i\varepsilon = -\frac{\lambda_n}{g_n} - i\varepsilon = -\frac{\begin{pmatrix} 0 \\ 0 \end{pmatrix}_n}{(\)_n}. \quad (2.38)$$

It is only this inhomogeneity $R_n = r_n - i\varepsilon$ who carries the $i\varepsilon$ -prescription. The $(n - 1)$ components y_i of the shift vector y appearing here in $F_n(x)$ are:

$$y_i = -\left(G_n^{-1}K_n\right)_i, \quad i \neq n. \quad (2.39)$$

The following relations are also valid:

$$y_i = \frac{\partial r_n}{\partial m_i^2} = -\frac{1}{g_n} \frac{\partial \lambda_n}{\partial m_i^2} = -\frac{\partial_i \lambda_n}{g_n} = \frac{2}{g_n} \begin{pmatrix} 0 \\ i \end{pmatrix}_n, \quad i = 1 \dots n. \quad (2.40)$$

One further notation has been introduced in (2.40), namely that of *co-factors of the modified Cayley matrix*, also called signed minors in e.g. [38, 42]:

$$\begin{pmatrix} j_1 & j_2 & \dots & j_m \\ k_1 & k_2 & \dots & k_m \end{pmatrix}_n. \quad (2.41)$$

The signed minors are determinants, labeled by those rows j_1, j_2, \dots, j_m and columns k_1, k_2, \dots, k_m which have been discarded from the definition of the modified Cayley determinant $(\)_n$, with a sign convention:

$$\begin{aligned} \text{sign} \begin{pmatrix} j_1 & j_2 & \dots & j_m \\ k_1 & k_2 & \dots & k_m \end{pmatrix}_n &= (-1)^{j_1+j_2+\dots+j_m+k_1+k_2+\dots+k_m} \times \mathbf{Signature}[j_1, j_2, \dots, j_m] \\ &\times \mathbf{Signature}[k_1, k_2, \dots, k_m]. \end{aligned} \quad (2.42)$$

Here, **Signature** (defined like the Wolfram Mathematica command) gives the sign of permutations needed to place the indices in increasing order. The Cayley determinant is a signed minor of the modified Cayley determinant,

$$\Delta_n = \lambda_n = \begin{pmatrix} 0 \\ 0 \end{pmatrix}_n. \quad (2.43)$$

For later use, we introduce also

$$y_n = 1 - \sum_{i=1}^{n-1} y_i \equiv \frac{\partial r_n}{\partial m_n^2}. \quad (2.44)$$

The auxiliary condition $\sum_i^n y_i = 1$ is fulfilled. Further, the notations for the F -function are finally independent of the choice of the variable which was eliminated by the use of the δ -function in the integrand of (2.28). And the inhomogeneity R_n is the only variable carrying the causal $i\varepsilon$ -prescription, while e.g. $\Lambda(x)$ and the y_i are by definition real quantities. The R_n may be expressed by the ratio of the Cayley determinant (2.32) and the Gram determinant (2.31),

$$R_n = r_{1\dots n} - i\varepsilon = -\frac{\lambda_{1\dots n}}{g_{1\dots n}} - i\varepsilon. \quad (2.45)$$

One may use the Mellin-Barnes relation (2.34) in order to decompose the integrand of J_n given in (2.28) as follows:

$$J_n \sim \int dx \frac{1}{[F(x)]^{n-\frac{d}{2}}} \equiv \int dx \frac{1}{[\Lambda_n(x) + R_n]^{n-\frac{d}{2}}} \equiv \int dx \frac{R_n^{-(n-\frac{d}{2})}}{\left[1 + \frac{\Lambda_n(x)}{R_n}\right]^{n-\frac{d}{2}}}$$

$$= \int dx \frac{R_n^{-(n-\frac{d}{2})}}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma(n - \frac{d}{2} + s)}{\Gamma(n - \frac{d}{2})} \left[\frac{\Lambda_n(x)}{R_n} \right]^s, \quad (2.46)$$

for $|\text{Arg}(\Lambda_n/R_n)| < \pi$. The condition always applies. Further, the integration path in the complex s -plane separates the poles of $\Gamma(-s)$ and $\Gamma(n - \frac{d}{2} + s)$.

As a result of (2.46), the Feynman parameter integral of J_n becomes homogeneous:

$$\begin{aligned} \kappa_n &= \int dx \left[\frac{\Lambda_n(x)}{R_n} \right]^s \\ &= \prod_{j=1}^{n-1} \int_0^{1-\sum_{i=j+1}^{n-1} x_i} dx_j \left[\frac{\Lambda_n(x)}{R_n} \right]^s \equiv \int dS_{n-1} \left[\frac{\Lambda_n(x)}{R_n} \right]^s. \end{aligned} \quad (2.47)$$

In order to reformulate this integral, one may introduce the differential operator \hat{P}_n [43,44],

$$\frac{\hat{P}_n}{s} \left[\frac{\Lambda_n(x)}{R_n} \right]^s \equiv \sum_{i=1}^{n-1} \frac{1}{2s} (x_i - y_i) \frac{\partial}{\partial x_i} \left[\frac{\Lambda_n(x)}{R_n} \right]^s = \left[\frac{\Lambda_n(x)}{R_n} \right]^s, \quad (2.48)$$

into (2.47):

$$K_n = \frac{1}{s} \int dS_{n-1} \hat{P}_n \left[\frac{\Lambda_n(x)}{R_n} \right]^s = \frac{1}{2s} \sum_{i=1}^{n-1} \prod_{k=1}^{n-1} \int_0^{u_k} dx'_k (x_i - y_i) \frac{\partial}{\partial x_i} \left[\frac{\Lambda_n(x)}{R_n} \right]^s. \quad (2.49)$$

After performing now one of the x -integrations – by partial integration, eliminating this way the corresponding differential, and applying a Barnes relation [45] (item 14.53 at page 290 of [46]), one arrives at a recursion relation in the number of internal lines n :

$$\begin{aligned} J_n(d, \{q_i, m_i^2\}) &= \frac{-1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma(\frac{d-n+1}{2} + s) \Gamma(s+1)}{2\Gamma(\frac{d-n+1}{2})} \left(\frac{1}{R_n} \right)^s \\ &\quad \times \sum_{k=1}^n \left(\frac{1}{R_n} \frac{\partial r_n}{\partial m_k^2} \right) \mathbf{k}^- J_n(d+2s; \{q_i, m_i^2\}). \end{aligned} \quad (2.50)$$

The operator \mathbf{k}^- , introduced in (2.7), will reduce an n -point Feynman integral J_n to a sum of $(n-1)$ -point integrals J_{n-1} by shrinking propagators D_k from the original n -point integral. The starting term is the 1-point function, or tadpole,

$$J_1(d; m^2) = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{k^2 - m^2 + i\varepsilon} = -\frac{\Gamma(1-d/2)}{R_1^{1-d/2}}, \quad (2.51)$$

$$R_1 = m^2 - i\varepsilon. \quad (2.52)$$

The cases $G_n = 0$ and $\lambda_n = r_n = 0$ are discussed in subsection 2.5.

Eq. (2.50) is the master integral for one-loop n -point functions in space-time dimension d , representing them by n integrals over $(n-1)$ -point functions with a shifted dimension $d+2s$. The recursion was first published in [41]. It implies a series of Mellin-Barnes representations for arbitrary massive one-loop n -point integrals with Mellin-Barnes integral dimensions $n-1$. This linear rise of the MB-dimension is highly advantageous compared to the number of MB-integral dimensions in the AMBRE approach (rising as n^2 with the number n of scales).

Based on (2.50), one has now several opportunities to proceed:

- (i) Evaluate the MB-integral in a direct numerical way.
- (ii) Derive ε -expansions for the Feynman integrals.
- (iii) Apply the Cauchy theorem for deriving sums and determine analytical expressions in terms of known special functions.

The first approach is based on AMBRE/MBOneLoop, the middle one is not yet finished, and the last approach was applied in [41] for massive vertex integrals and in [6] also for massive box integrals.

Few comments are at hand:

- Any 4-point integral e.g. is in the recursion a 3-fold Mellin-Barnes integral. While, with AMBRE, one gets for e.g. box integrals up to 9-fold MB-integrals.
- Euklidean and Minkoswkian integrals converge equally good. See J. Usovitsch’s talk at LL2018 [47, 48].
- There appear no numerical problems due to vanishing Gram determinants. See for few details table C.6 and [49].

2.4 The basic scalar one-loop functions

2.4.1 Massive two-point functions

From the recursion relation (2.50), taken at $n = 2$ and using the expression (2.51) with $d \rightarrow d + 2s$ for the one-point functions under the integral, one gets the following Mellin-Barnes representation:

$$\begin{aligned}
 J_2(d; q_1, m_1^2, q_2, m_2^2) &= \frac{e^{\varepsilon\gamma_E}}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma\left(\frac{d-1}{2} + s\right) \Gamma(s+1)}{2 \Gamma\left(\frac{d-1}{2}\right)} R_2^s \\
 &\times \left[\frac{1}{r_2} \frac{\partial r_2}{\partial m_2^2} \frac{\Gamma\left(1 - \frac{d+2s}{2}\right)}{(m_1^2)^{1 - \frac{d+2s}{2}}} + (m_1^2 \leftrightarrow m_2^2) \right]. \quad (2.53)
 \end{aligned}$$

One may close the integration contour of the MB-integral in (2.53) to the right, apply the Cauchy theorem and collect the residua originating from two series of zeros of arguments of Γ -functions at $s = m$ and $s = m - d/2 - 1$ for $m \in \mathbb{N}$. The first series stems from the MB-integration kernel, the other one from the dimensionally shifted 1-point functions. And then one may sum up analytically in terms of Gauss’ hypergeometric functions.

The 2-point function, with $R_2 \equiv R_{12}$, becomes:

$$\begin{aligned}
 J_2(d; Q^2, m_1^2, q_2, m_2^2) &= -\frac{\Gamma\left(2 - \frac{d}{2}\right) \Gamma\left(\frac{d}{2} - 1\right)}{(d-2) \Gamma\left(\frac{d}{2}\right)} \frac{\partial_2 R_2}{R_2} \\
 &\left[(m_1^2)^{\frac{d}{2}-1} {}_2F_1\left[1, \frac{d}{2} - \frac{1}{2}; \frac{m_1^2}{R_2}\right] + \frac{R_2^{\frac{d}{2}-1}}{\sqrt{1 - \frac{m_1^2}{R_2}}} \sqrt{\pi} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2} - \frac{1}{2}\right)} \right] \\
 &+ (m_1^2 \leftrightarrow m_2^2). \quad (2.54)
 \end{aligned}$$

The representation (2.54) is valid for $\left|\frac{m_1^2}{r_{12}}\right| < 1$, $\left|\frac{m_2^2}{r_{12}}\right| < 1$ and $\mathcal{R}e\left(\frac{d-2}{2}\right) > 0$. The result is in agreement with Eqn. (53) of Tarasov et al. (2003) [15].

The iterative determination of higher-point functions proceeds analogously. Closing the integration contours to the right or to the left will cover different kinematical regions in the invariants R_n .

2.4.2 Massive three-point functions

The Mellin-Barnes integral for the massive vertex is a sum of three terms [50]:

$$J_3 = J_{123} + J_{231} + J_{312}, \quad (2.55)$$

using the representation for e.g. J_{123}

$$\begin{aligned} J_{123}(d, \{q_i, m_i^2\}) &= -\frac{e^{\epsilon\gamma_E}}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma(\frac{d-2+2s}{2}) \Gamma(s+1)}{2 \Gamma(\frac{d-2}{2})} R_3^{-s} \\ &\times \frac{1}{r_3} \frac{\partial r_3}{\partial m_3^2} J_2(d+2s; q_1, m_1^2, q_2, m_2^2). \end{aligned} \quad (2.56)$$

After applying the Cauchy theorem and summing up, one gets an analytical representation. The integrated massive vertex has been published in [41]. We quote here the representation given in [6]:

$$\begin{aligned} J_{123} &= \Gamma\left(2 - \frac{d}{2}\right) \frac{\partial_3 r_3}{r_3} \frac{\partial_2 r_2}{r_2} \frac{r_2}{2\sqrt{1 - m_1^2/r_2}} \\ &\left[-R_2^{d/2-2} \frac{\sqrt{\pi} \Gamma(\frac{d}{2} - 1)}{2 \Gamma(\frac{d}{2} - \frac{1}{2})} {}_2F_1\left[\frac{d-2}{2}, 1; \frac{R_2}{R_3}\right] + R_3^{d/2-2} {}_2F_1\left[\frac{1}{2}, 1; \frac{R_2}{R_3}\right] \right] \\ &+ \Gamma\left(2 - \frac{d}{2}\right) \frac{\partial_3 r_3}{r_3} \frac{\partial_2 r_2}{r_2} \frac{m_1^2}{4\sqrt{1 - m_1^2/r_2}} \\ &\left[+ \frac{2(m_1^2)^{d/2-2}}{d-2} F_1\left(\frac{d-2}{2}; 1, \frac{1}{2}; \frac{d}{2}; \frac{m_1^2}{R_3}, \frac{m_1^2}{R_2}\right) - R_3^{d/2-2} F_1\left(1; 1, \frac{1}{2}; 2; \frac{m_1^2}{R_3}, \frac{m_1^2}{R_2}\right) \right] \\ &+ (m_1^2 \leftrightarrow m_2^2), \end{aligned}$$

with the short notations

$$R_3 = R_{123}, R_2 = R_{12}, \quad (2.57)$$

etc. For $d \rightarrow 4$, the bracket expressions vanish so that their product with the prefactor $\Gamma(2 - d/2)$ stays finite in this limit, as it must come out for a massive vertex function. For some numerics see tables C.1 and C.2 and C.3 and C.4.

2.4.3 Massive four-point functions

Finally we reproduce the box integral, as a three-dimensional Mellin-Barnes representation:

$$J_4(d; \{p_i^2\}, s, t, \{m_i^2\}) = \left(\frac{-1}{4\pi i}\right)^4 \frac{1}{\Gamma(\frac{d-3}{2})} \sum_{k_1, k_2, k_3, k_4=1}^4 D_{k_1 k_2 k_3 k_4} \left(\frac{1}{r_4} \frac{\partial r_4}{\partial m_{k_4}^2}\right)$$

Table C.1: Numerics for a vertex, $d = 4 - 2\epsilon$. Input quantities suggest that, according to Eqn. (73) in [15], one has to set $b_3 = 0$. Although b_3 of [15] deviates from our vanishing value, it has to be set to zero, $b_3 \rightarrow 0$. The results of both calculations for J_3 agree for this case.

$[p_i^2], [m_i^2]$	[+100, +200, +300], [10, 20, 30]	
G_{123}	-160000	
λ_{123}	-8860000	
m_i^2/r_{123}	-0.180587, -0.361174, -0.541761	
m_i^2/r_{12}	-0.97561, -1.95122, -2.92683	
m_i^2/r_{23}	-0.39801, -0.79602, -1.19403	
m_i^2/r_{31}	-0.180723, -0.361446, -0.542169	
$\sum J$ -terms [15]	(0.019223879 - 0.007987267 i)	
$\sum b_3$ -terms (TR)	0	
J_3 (TR)	(0.019223879 - 0.007987267 i)	
b_3 -term [15]	(-0.089171509 + 0.069788641 i)	(0.022214414)/eps
$b_3 + \sum J$ -terms	(-0.012307377 - 0.009301346 i)	
J_3 (OT)	$\sum J$ -terms, b_3 -term $\rightarrow 0$, OK	
MB suite		
(-1) \times fiesta3 [51]	- (0.012307 + 0.009301 i)	+ (8 * 10 ⁻⁶ + 0.00001 i) \pm (1 + i)10 ⁻⁴
LoopTools [52]	0.0192238790286 - 0.00798726725497 i	

$$\begin{aligned}
 & \left(\frac{1}{r_{k_3 k_2 k_1}} \frac{\partial r_{k_3 k_2 k_1}}{\partial m_{k_3}^2} \right) \left(\frac{1}{r_{k_2 k_1}} \frac{\partial r_{k_2 k_1}}{\partial m_{k_2}^2} \right) (m_{k_1}^2)^{d/2-1} \quad (2.58) \\
 & \int_{-i\infty}^{+i\infty} dz_4 \int_{-i\infty}^{+i\infty} dz_3 \int_{-i\infty}^{+i\infty} dz_2 \left(\frac{m_{k_1}^2}{R_4} \right)^{z_4} \left(\frac{m_{k_1}^2}{R_{k_3 k_2 k_1}} \right)^{z_3} \left(\frac{m_{k_1}^2}{R_{k_2 k_1}} \right)^{z_2} \\
 & \Gamma(-z_4) \Gamma(z_4 + 1) \frac{\Gamma(z_4 + \frac{d-3}{2})}{\Gamma(z_4 + \frac{d-2}{2})} \Gamma(-z_3) \Gamma(z_3 + 1) \frac{\Gamma(z_3 + z_4 + \frac{d-2}{2})}{\Gamma(z_3 + z_4 + \frac{d-1}{2})} \\
 & \Gamma(z_2 + z_3 + z_4 + \frac{d-1}{2}) \Gamma(-z_2 - z_3 - z_4 - \frac{d+2}{2}) \Gamma(-z_2) \Gamma(z_2 + 1).
 \end{aligned}$$

The representation (2.58) can be treated by the Mathematica packages MB and MBnumerics of the MBSuite, replacing AMBRE by a derivative of MBnumerics: MBOneLoop [47,54]. For numerical examples, see table C.5.

After applying the Cauchy theorem and summing the residues, we get [6,55] :

$$J_4 = J_{1234} + J_{2341} + J_{3412} + J_{4123}, \quad (2.59)$$

with $R_4 = R_{1234}$, $R_3 = R_{123}$, $R_2 = R_{12}$ etc.:

$$\begin{aligned}
 J_{1234} = & \Gamma \left(2 - \frac{d}{2} \right) \frac{\partial_4 r_4}{r_4} \left\{ \right. \\
 & \left. \left[\frac{b_{123}}{2} \left(- R_3^{d/2-2} {}_2F_1 \left[\frac{d-3}{2}, 1; \frac{R_2}{R_3} \right] + R_4^{d/2-2} \sqrt{\pi} \frac{\Gamma(\frac{d}{2}-1)}{\Gamma(\frac{d}{2}-\frac{3}{2})} {}_2F_1(d \rightarrow 4) \right) \right] \right\}
 \end{aligned}$$

Table C.2: Numerics for a vertex, $d = 4 - 2\epsilon$. Input quantities suggest that, according to eq. (73) in [15], one has to set $b_3 = 0$. Further, we have set in the numerics for eq. (75) of [15] that the root of the Gram determinant is $\sqrt{-g_{123} + i \epsilon}$, what looks counter-intuitive for a “momentum”-like function. Both results agree if we do not set Tarasov’s $b_3 \rightarrow 0$. Table courtesy [53].

$[p_i^2], [m_i^2]$	$[-100, +200, -300], [10, 20, 30]$	
G_{123}	480000	
λ_3	-19300000	
m_i^2/r_3	0.248705, 0.497409, 0.746114	
m_i^2/r_{12}	0.248447, 0.496894, 0.745342	
m_i^2/r_{23}	-0.39801, -0.79602, -1.19403	
m_i^2/r_{31}	0.104895, 0.20979, 0.314685	
$\sum J$ -terms	$(-0.012307377 - 0.056679689 i)$	$(0.012825498 i)/\text{eps}$
$\sum b_3$ -terms	$(0.047378343 i)$	$(-0.012825498 i)/\text{eps}$
$J_3(\text{TR})$	$(-0.012307377 - 0.009301346 i)$	
b_3 -term	$(0.047378343 i)$	$(-0.012825498 i)/\text{eps}$
$b_3 + \sum J$ -terms	$(-0.012307377 - 0.009301346 i)$	
$J_3(\text{OT})$	$\sum J$ -terms, b_3 -term $\rightarrow 0$, gets wrong!	
MB suite		
$(-1)*\text{fiesta3}$	$-(0.012307 + 0.009301 i)$	$(8 * 10^{-6} + 0.00001 i) \pm (1 + i)10^{-4}$
LoopTools/FF, ϵ^0	$-0.01230737736778 - 0.009301346170 i$	

$$\begin{aligned}
 & + \left[+ \frac{R_2^{d/2-2}}{d-3} F_1 \left(\frac{d-3}{2}; 1, \frac{1}{2}; \frac{d-1}{2}; \frac{R_2}{R_4}, \frac{R_2}{R_3} \right) - R_4^{d/2-2} F_1(d \rightarrow 4) \right] \\
 & + \frac{m_1^2 \Gamma\left(\frac{d}{2} - 1\right)}{8 \Gamma\left(\frac{d}{2} - \frac{3}{2}\right)} \frac{\partial_3 r_3}{r_3} \frac{\partial_2 r_2}{r_2} \frac{r_3}{r_3 - m_1^2} \frac{r_2}{r_2 - m_1^2} \\
 & \left[- (m_1^2)^{d/2-2} \frac{\Gamma\left(\frac{d}{2} - 3/2\right)}{\Gamma\left(\frac{d}{2}\right)} F_S(d/2 - 3/2, 1, 1, 1, 1, d/2, d/2, d/2, d/2, \frac{m_1^2}{R_4}, \frac{m_1^2}{m_1^2 - R_3}, \frac{m_1^2}{m_1^2 - R_2}) \right. \\
 & \left. + R_4^{d/2-2} \sqrt{\pi} F_S(d \rightarrow 4) \right] + (m_1^2 \leftrightarrow m_2^2) \Big\}. \tag{2.60}
 \end{aligned}$$

For $d \rightarrow 4$, all three contributions in square brackets approach zero, so that the massive J_4 gets finite in this limit, as it should do. Table C.5 contains numerical examples.

2.5 The cases of vanishing Cayley determinant $\lambda_n = 0$ and of vanishing Gram determinant $G_n = 0$

We refer here to two important special cases, where the general derivations cannot be applied.

In the case of vanishing Cayley determinant, $\lambda_n = 0$, we cannot introduce the inhomogeneity $R_n = -\lambda_n/G_n$ into the Symanzik polynomial F_n . Let us assume that it is $G_n \neq 0$, so that $r_n = 0$. A useful alternative representation to (2.50) is known from the literature, see e.g.

Table C.3: Numerics for a vertex in space-time dimension $d = 4 - 2\epsilon$. Causal $\epsilon = 10^{-20}$. Agreement with [15]. Table courtesy [53].

p_i^2	-100, -200, -300	
m_i^2	10, 20, 30	
G_{123}	-160000	
λ_{123}	15260000	
m_i^2/r_{123}	0.104849, 0.209699, 0.314548	
m_i^2/r_{12}	0.248447, 0.496894, 0.745342	
m_i^2/r_{23}	0.133111, 0.266223, 0.399334	
m_i^2/r_{31}	0.104895, 0.20979, 0.314685	
$\sum J$ -terms	(0.0933877 - 0 i)	-(0.0222144 - 0 i)/eps
$\sum b$ -terms	-0.101249	+0.0222144/eps
$J_3(\text{TR})$	(-0.00786155 - 0 i)	
b_3	(-0.101249 + 0 i)	(0.0222144 + 0 i)/eps
$b_3 + J$ -terms	(-0.007861546 + 0 i)	
$J_3(\text{OT})$	$b_3 + J$ -terms \rightarrow OK	
MB suite	-0.007862014, 5.002549159 * 10 ⁻⁶ , 0	
(-1)*fiesta3	-0.007862	6 * 10 ⁻⁶ + 6 * 10 ⁻⁶ i \pm (1 + i)10 ⁻¹⁰
LoopTools/FF, ϵ^0	-0.00786154613229082290	

Eqn. (3) in [15]:

$$J_n(d) = \frac{1}{d - n - 1} \sum_{k=1}^n \frac{\partial_k \lambda_n}{G_n} \mathbf{k}^- J_n(d - 2). \quad (2.61)$$

Another special case is a vanishing Gram determinant, $G_n = 0$. Here, again one may use Eqn. (3) of [15] and the result is (for $\lambda_n \neq 0$):

$$J_n(d) = - \sum_{k=1}^n \frac{\partial_k \lambda_n}{2\lambda_n} \mathbf{k}^- J_n(d). \quad (2.62)$$

The representation was, for the special case of the vertex function, also given in Eqn. (46) of [60].

For the vertex function, a general study of the special cases has been carried through in [61].

2.6 Example: A massive 4-point function with vanishing Gram determinant

As a very interesting, non-trivial example we had re-studied the numerics of a massive 4-point function with a small or vanishing Gram determinant [47, 50, 54, 62]. The original example has been taken from Appendix C of [17].

The sample outcome is shown in Table C.6. The new iterative Mellin-Barnes representations deliver very precise numerical results for e.g. box functions, including cases of small or vanishing Gram determinants. The software used is MBOneLoop [63].

The notational correspondences are e.g.:

$$J_4(12 - 2\epsilon, 1, 5, 1, 1) \rightarrow I_{4,2222}^{[d+]} = D_{1111}.$$

Table C.4: Numerics for a vertex in space-time dimension $d = 4 - 2\epsilon$. Causal $\epsilon = 10^{-20}$. Input quantities suggest that, according to eq. (73) in [15], one has to set $b_3 = 0$. Agreement due to setting $b_3 = 0$ there. Table courtesy [53].

p_i^2	+100, -200, +300	
m_i^2	10, 20, 30	
G_{123}	480000	
λ_{123}	4900000	
m_i^2/r_{123}	-0.979592, -1.95918, -2.93878	
m_i^2/r_{12}	-0.97561, -1.95122, -2.92683	
m_i^2/r_{23}	0.133111, 0.266223, 0.399334	
m_i^2/r_{31}	-0.180723, -0.361446, -0.542169	
$\sum J$ -terms	(0.006243624 - 0.018272524 i)	
$\sum b_3$ -terms	0	
$J_3(\text{TR})$	(0.006243624 - 0.018272524 i)	
b_3 -term	(0.040292491 + 0.029796253 i)	(-0.012825498 i)/eps
$b_3 + \sum J$ -terms	(-0.012307377 - 0.009301346 i)	(4 * 10 ⁻¹⁸ - 6 * 10 ⁻¹⁸ i)/eps
$J_3(\text{OT})$	$\sum J$ -terms, b_3 -term $\rightarrow 0$, OK	
MB suite		
(-1)*fiesta3	-(-0.006322 + 0.014701 i)	+ (0.000012 + 0.000014 i) \pm (1 + i)10 ⁻²
LoopTools/FF, ϵ^0	0.00624362477277 - 0.0182725240487 i	

2.7 Calculation of Gauss hypergeometric function ${}_2F_1$, Appell function F_1 , and Saran function F_S at arbitrary kinematics

There is little known about the precise numerical calculation of generalized hypergeometric functions at arbitrary arguments. Numerical calculations of specific Gauss hypergeometric functions ${}_2F_1$, Appell functions F_1 (Eqn. (1) of [64]), and Lauricella-Saran functions F_S (Eqn. (2.9) of [65]) are needed for the scalar one-loop Feynman integrals:

$${}_2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k! (c)_k} x^k, \quad (2.63)$$

$$F_1(a; b, b'; c; y, z) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} (b)_m (b')_n}{m! n! (c)_{m+n}} y^m z^n, \quad (2.64)$$

$$F_S(a_1, a_2, a_2; b_1, b_2, b_3; c, c, c; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_m (a_2)_{n+p} (b_1)_m (b_2)_n (b_3)_p}{m! n! p! (c)_{m+n+p}} x^m y^n z^p. \quad (2.65)$$

The $(a)_k$ is the Pochhammer symbol. The specific cases needed here are discussed in the appendices of [6]. Here, we repeat only few definitions.

One approach to the numerics of ${}_2F_1$, F_1 , and F_S may be based on Mellin-Barnes representations. For the Gauss function ${}_2F_1$ and the Appell function F_1 , Mellin-Barnes representations are known since a while. See Eqn. (1.6.1.6) in [66],

$${}_2F_1(a, b; c; z) = \frac{1}{2\pi i} \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_{-i\infty}^{+i\infty} ds (-z)^s \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}, \quad (2.66)$$

Table C.5: Comparison of the box integral J_4 defined in (2.60) with the LoopTools function $D0(p_1^2, p_2^2, p_3^2, p_4^2, (p_1 + p_2)^2, (p_2 + p_3)^2, m_1^2, m_2^2, m_3^2, m_4^2)$ [56, 57] at $m_2^2 = m_3^2 = m_4^2 = 0$. Further numerical references are the packages K.H.P_D0 (PHK, unpublished) and MBOneLoop [58, 59]. External invariants: $(p_1^2 = \pm 1, p_2^2 = \pm 5, p_3^2 = \pm 2, p_4^2 = \pm 7, s = \pm 20, t = \pm 1)$.

Table from Phan, Riemann, PLB 2019 [6], licence: <https://creativecommons.org/licenses/by/4.0/>.

$(p_1^2, p_2^2, p_3^2, p_4^2, s, t)$	4-point integral
$(-, -, -, -, -, -)$	$d = 4, m_1^2 = 100$
J_4	0.00917867
LoopTools	0.0091786707
MBOneLoop	0.0091786707
$(+, +, +, +, +, +)$	$d = 4, m_1^2 = 100$
J_4	$-0.0115927 - 0.00040603 i$
LoopTools	$-0.0115917 - 0.00040602 i$
MBOneLoop	$-0.0115917369 - 0.0004060243 i$
$(-, -, -, -, -, -)$	$d = 5, m_1^2 = 100$
J_4	0.00926895
K.H.P_D0	0.00926888
MBOneLoop	0.0092689488
$(+, +, +, +, +, +)$	$d = 5, m_1^2 = 100$
J_4	$-0.00272889 + 0.0126488 i$
K.H.P_D0	(-)
MBOneLoop	$-0.0027284242 + 0.0126488134 i$
$(-, -, -, -, -, -)$	$d = 5, m_1^2 = 100 - 10 i$
J_4	$0.00920065 + 0.000782308 i$
K.H.P_D0	$0.0092006 + 0.000782301 i$
MBOneLoop	$0.0092006481 + 0.0007823090 i$
$(+, +, +, +, +, +)$	$d = 5, m_1^2 = 100 - 10 i$
J_4	$-0.00398725 + 0.012067 i$
K.H.P_D0	$-0.00398723 + 0.012069 i$
MBOneLoop	$-0.0039867702 + 0.0120670388 i$

and Eqn. (10) in [64], which is a two-dimensional MB-integral:

$$F_1(a; b, b'; c; x, y) = \frac{1}{2\pi i} \frac{\Gamma(c)}{\Gamma(a)\Gamma(b')} \int_{-i\infty}^{+i\infty} dt (-y)^t {}_2F_1(a+t, b; c+t, x) \frac{\Gamma(a+t)\Gamma(b'+t)\Gamma(-t)}{\Gamma(c+t)}. \quad (2.67)$$

For the Lauricella-Saran function F_S , the following, new, three-dimensional MB-integral was given in [6]:

$$F_S(a_1, a_2, a_2; b_1, b_2, b_3; c, c, c; x, y, z) = \frac{1}{2\pi i} \frac{\Gamma(c)}{\Gamma(a_1)\Gamma(b_1)} \int_{-i\infty}^{+i\infty} dt (-x)^t \frac{\Gamma(a_1+t)\Gamma(b_1+t)\Gamma(-t)}{\Gamma(c+t)} \times F_1(a_2; b_2, b_3; c+t; y, z). \quad (2.68)$$

Table C.6: *The Feynman integral $J_4(12 - 2\epsilon, 1, 5, 1, 1)$ compared to numbers from [17]. The $I_{4,2222}^{[d+]}^4$ is the scalar integral where propagator 2 has index $\nu_2 = 1 + (1 + 1 + 1 + 1) = 5$, the others have index 1. The integral corresponds to D_{1111} in notations of LoopTools [56]. For $x = 0$, the Gram determinant vanishes. We see an agreement of about 10 to 11 relevant digits. The deviations of the two calculations seem to stem from a limited accuracy of the Pade approximations used in [17]. Table courtesy [54, 62].*

x	value for $4! \times J_4(12 - 2\epsilon, 1, 5, 1, 1)$
0	$(2.05969289730 + 1.55594910118i)10^{-10}$ [J. Fleischer, T. Riemann, 2010]
0	$(2.05969289730 + 1.55594910118 i)10^{-10}$ MBOneLoop+Kira+MBnumerics
10^{-8}	$(2.05969289342 + 1.55594909187 i)10^{-10}$ [J. Fleischer, T. Riemann, 2010]
10^{-8}	$(2.05969289363 + 1.55594909187 i)10^{-10}$ MBOneLoop+Kira+MBnumerics
10^{-4}	$(2.05965609497 + 1.55585605343 i)10^{-10}$ [J. Fleischer, T. Riemann, 2010]
10^{-4}	$(2.05965609489 + 1.55585605343 i)10^{-10}$ MBOneLoop+Kira+MBnumerics

The numerics of the Gauss hypergeometric function is generally known in all detail.

For the Appell function F_1 , the numerical mean value integration of the one-dimensional integral representation of [67] may be advocated, being quoted in Eqn. (9) of [64]:

$$F_1(a; b, b'; c; x, y) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 du \frac{u^{a-1}(1-u)^{c-a-1}}{(1-xu)^b(1-yu)^{b'}}. \quad (2.69)$$

We need three specific cases, taken at $d \geq 4$. For vertices e.g.,

$$F_1^\nu(d) \equiv F_1\left(\frac{d-2}{2}; 1, \frac{1}{2}; \frac{d}{2}; x_c, y_c\right) = \frac{1}{2}(d-2) \int_0^1 \frac{du u^{\frac{d}{2}-2}}{(1-x_c u)\sqrt{1-y_c u}}. \quad (2.70)$$

Integrability is violated at $u = 0$ if not $\Re(d) > 2$. The stability of numerics is well controlled as exemplified in Table C.7.

For the calculation of the 4-point Feynman integrals, one needs additionally the Lauricella-Saran function F_S [65]. Saran defines F_S as a three-fold sum (2.65), see Eqn. (2.9) in [65]. He derives a 3-fold integral representation in Eqn. (2.15) and a 2-fold integral in Eqn. (2.16). We recommend to use the following representation, derived at p. 304 of [65]:

$$\begin{aligned} & F_S(a_1, a_2, a_2; b_1, b_2, b_3; c, c, c, x, y, z) \\ &= \frac{\Gamma(c)}{\Gamma(a_1)\Gamma(c-a_1)} \int_0^1 dt \frac{t^{c-a_1-1}(1-t)^{a_1-1}}{(1-x+tx)^{b_1}} F_1(a_2; b_2, b_3; c-a_1; ty, tz). \end{aligned} \quad (2.71)$$

For the box integrals one needs the specific case

$$\begin{aligned} F_S^b(d) &= F_S\left(\frac{d-3}{2}, 1, 1, 1, 1, \frac{1}{2}, \frac{d}{2}, \frac{d}{2}, \frac{d}{2}, x_c, y_c, z_c\right) \\ &= \frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d-3}{2})\Gamma(\frac{3}{2})} \int_0^1 dt \frac{\sqrt{t}(1-t)^{\frac{d-5}{2}}}{(1-x_c+xt)^{b_1}} F_1\left(1; 1, \frac{1}{2}; \frac{3}{2}; y_c t, z_c t\right). \end{aligned} \quad (2.72)$$

Table C.7: The Appell function F_1 of the massive vertex integrals as defined in (2.70). As a proof of principle, only the constant term of the expansion in $d = 4 - 2\varepsilon$ is shown, $F_1(1; 1, \frac{1}{2}; 2; x, y)$. Upper values from general numerics of appendices of [6], lower values from setting $d = 4$ and use of analytical formulae. Table courtesy from [6] under licence <http://creativecommons.org/licenses/by/4.0/>.

$x - i\varepsilon_x$	$y - i\varepsilon_y$	$F_1(1; 1, \frac{1}{2}; 2; x, y)$	
$+11.1 - 10^{-12} i$	$+12.1 - 10^{-12} i$	-0.1750442480735 $-0.17504424807351877884498289912$	$-0.0542281294732 i$ $-0.054228129473304027882097641167 i$
$+11.1 - 10^{-12} i$	$+12.1 + 10^{-12} i$	$+1.7108545293244$ $+1.71085452932433557134838204175$	$+0.0542281294732 i$ $+0.05422812947148217381589270924 i$
$+11.1 + 10^{-12} i$	$+12.1 - 10^{-12} i$	$+1.7108545304114$ $+1.71085452932433557134838204175$	$-0.0542281294732 i$ $-0.05422812947148217381589270924 i$
$+11.1 + 10^{-12} i$	$+12.1 + 10^{-12} i$	-0.1750442480735 $-0.17504424807351877884498289912$	$+0.0542281294733 i$ $+0.054228129473304027882097641167 i$
$+12.1 - 10^{-15} i$	$+11.1 - 10^{-15} i$	-0.1700827166484	$-0.0518684846037 i$
$+12.1 - 10^{-10} i$	$+11.1 - 10^{-15} i$	$-0.17008271664800058101165749279$	$-0.05186848460465674976556525621 i$
$+12.1 - 10^{-15} i$	$+11.1 + 10^{-15} i$	-0.1700827166484 $-0.17008271664844025647268817399$	$-1.7544202909955 i$ $-1.75442029099557688735842562038 i$
$+12.1 + 10^{-15} i$	$+11.1 - 10^{-15} i$	-0.1700827166484 $-0.17008271664844025647268817399$	$+1.7544202909955 i$ $+1.75442029099557688735842562038 i$
$+12.1 + 10^{-15} i$	$+11.1 + 10^{-15} i$	-0.1700827166484	$+0.0518684846037 i$
$+12.1 - 10^{-10} i$	$+11.1 - 10^{-15} i$	$-0.17008271664800058101165749279$	$+0.05186848460465674976556525621 i$
$+11.1 - 10^{-15} i$	-12.1	-0.0533705146518 $-0.05337051465189944473349401152$	$-0.1957692111557 i$ $-0.195769211155733985388920833693 i$
$+11.1 + 10^{-15} i$	-12.1	-0.0533705146518 $-0.05337051465189944473349401152$	$+0.1957692111557 i$ $+0.195769211155733985388920833693 i$
-11.1	$+12.1 - 10^{-12} i$	$+0.1060864084662$ $+0.10608640847651064287133527599$	$-0.1447440700082i$ $-0.144744070021333407167349619088 i$
-11.1	$+12.1 + 10^{-12} i$	$+0.1060864084662$ $+0.10608640847651064287133527599$	$+0.1447440700082i$ $+0.144744070021333407167349619088 i$
-12.1	-11.1	$+0.122456767687224028$ $+0.12245676768722402506513395161$	

Eqn. (2.72) is valid if $\Re(d) > 3$.

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