QED as a Theory of Quantized Connection Forms

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1. Introduction

Classical gauge theories still play their role as conceptual building blocks of elementary particle theories. Using the differential geometric formulation, the geometric interpretation and topological dependence of these basic concepts become most apparent. The quantized versions of these theories essentially follow two different approaches: a.) that of Lagrangian quantum field theory based on perturbation theory and developped with the help of physically motivated model-Lagrangians, b.) that of axiomatic indefinite-metric operator theories or axiomatic C*-algebra theories based on physically motivated first principles (generalized Wightman axioms). In both domains Quantum Electrodynamics is the most elaborated and best established theory of fundamental interactions.

The QCC approach [1],[2], where QCC stands for Quantization of Connection and Curvature, takes neither Lagrangian densities nor Wightman axioms but the classical geometry as a starting-point. Its intention is to give - in a mathematical rigorous way - a complete chain of procedures leading from classical geometry (i.e. the 'bundle picture' of gauge theories) to the basic structures of indefinite-metric operator gauge field theories. This is achieved by using two intermediate formalisms: That of 'generalized geometry' and the *-algebra formalism of quantum field theory [4],[5]. The term 'generalized geometry' refers to the fact, that in the classical theory differential forms are replaced by deRham currents [6] and differential operators by generalized operators sending deRham currents (and differential forms) to deRham currents.

The motivation for introducing 'generalized fields' in the above sense is that the equations these fields satisfy should represent the behavior of the components of the N-point functions of a corresponding algebraic gauge quantum field theory. This is also the content of a quantization rule, which determines the link between the generalized geometry and the *-algebra formalism. Then a Gelfand-Naimark-Segal (GNS) construction [4b],[5a] is applied to derive an indefinite-metric operator theory.

We will first give a short introduction to the main ideas of the QCC-approach; details can be found in [1] and a more detailed introduction also in [2]. Then some results [3] obtained from the QCC-procedure when applied to a U(1)-gauge theory are summarized, answering the questions: a.) Is the derived structure of QED comparable to that of other approaches? b.) Does the QCC-procedure provide more information?.

2. The QCC-approach

For reasons of technical and formal simplicity the QCC-approach uses global formulations as long as possible; i.e. it works on the total space of a convenient principal bundle avoiding references to local sections, if possible. In this spirit the classical gauge theories are given in terms of a principal bundle P(M,G) over a space-time manifold M with metric g, where G denotes a suitable gauge group (in general a compact connected semi-simple Lie group) with Lie algebra $\mathcal{G}$. The classical fields are then represented by G-equivariant differential forms on the total space P with values in a vector space corresponding to a representation of the gauge group G (e.g. the Lie algebra $\mathcal{G}$ corresponding to the adjoint representation). To be more specific, a connection form Γ represents a gauge potential, its curvature Ω the field strength; external currents J are given by G-equivariant $\mathcal{G}$-valued one forms, etc. Recall that a connection form Γ together with the metric g on the base manifold M defines a unique metric $g^\wedge$ on P making P into a Kaluza-Klein space $(P,g^\wedge)$. Field equations are given in terms of canonical operators. We will focus on the field equations for the gauge fields, which are given by the Bianchi identity $\nabla\Omega = 0$, and by

$$(2.1)\qquad \not\nabla\,\Omega = J\ ,$$

where ∇, $\not\nabla$ denote the covariant derivative and the covariant coderivative, respectively. Equation (2.1) will be interpreted as the defining equation for J in this context.

Staying within the global approach, we have to implement gauge transformations in an active way; i.e. either by vertical bundle automorphisms or, infinitesimally, by G-equivariant $\mathcal{G}$-valued functions f on P. Then Γ, Ω and J transform according to

$$(2.2)\qquad \Gamma \longrightarrow \Gamma + \nabla f,\quad \Omega \longrightarrow \Omega + [\Omega,f],\quad J \longrightarrow J + [J,f].$$

The first step in the quantization procedure reformulates the classical theory in terms of deRham currents [6]. DeRham currents may be viewed as differential forms, the coefficents of which are distributions (test functions). To be able to formulate the theory also locally; i.e. by objects on the base manifold M, we are forced to define "smearing" of deRham currents with respect to the Kaluza-Klein metric $g^\wedge$ on P. So we understand p-deRham currents as linear functionals over the space $D^p(P,\mathcal{G})$ of $\mathcal{G}$-valued p-forms on P with compact support, making use of the Hodge star isomorphism $*: D^p(P,\mathcal{G}) \longrightarrow D^{n-p}(P,\mathcal{G})$, $n = \dim P$; i.e. formally we have

$$S(\alpha) = \int_P S \wedge *\alpha \quad , \qquad S \in (D^p(P,\mathcal{G}))' \; , \; \alpha \in D^p(P,\mathcal{G}) \; . \tag{2.3}$$

A generalized connection form $T \in (D^1(P,\mathcal{G}))'$ on P can be shown to decompose into a given classical connection form Γ and a basic (i.e. horizontal and G-equivariant) 1-deRham current $E \in (D^1(P,\mathcal{G}))'$:

$$T = \Gamma + E. \tag{2.4}$$

The field equation for E corresponding to the classical equation (2.1) e.g. takes the form [2]

$$\tilde{\nabla} \nabla E + \tfrac{1}{2}[E,E] + *^{-1}[E,*E] + *^{-1}[E,*\Omega] = J \quad , \tag{2.5}$$

where the external current $J \in (D^1{}_H(P,\mathcal{G}))'$ now is a basic 1-deRham current. This equation corresponds to a "quantized" potential in a classical background field. Observe, that the classical background field (i.e. the classical connection form Γ) is here introduced via the metric $\hat{g}$ on P: Fixing the metric $\hat{g}$ means fixing the classical connection, and it is this correspondence which, in the light of the QCC-approach, forces one to discuss the introduction of gauge fixing terms, ghost fields, etc. by determining the background field physically (i.e. in general locally).

Classical gauge transformations are implemented in the generalized formalism in a straightforward way. A generalized connection form T and its components Γ and E transform under these so-called 'c-transformations' according to

$$T \longrightarrow T + \tilde{\nabla} f, \quad \Gamma \longrightarrow \Gamma + \nabla f, \quad E \longrightarrow E + [E,f], \tag{2.6}$$

where $\tilde{\nabla}$ denotes the generalized covariant derivative, and f in general will be a $\mathcal{G}$-valued distribution on P. Of course after having fixed the metric $\hat{g}$ (i.e. the connection form Γ) on P it does not make sense to gauge transform Γ anymore. We therefore replace the c-transformations by *q-transformations*, which leave the classical connection Γ invariant. For T, Γ and E the q-transformations are given by

$$T \longrightarrow T + \tilde{\nabla} f, \quad \Gamma \longrightarrow \Gamma, \quad E \longrightarrow E + \tilde{\nabla} f. \tag{2.7}$$

We observe that the field E now transforms like a gauge potential although it is a basic 1-deRham current, and we shall refer to it as the *generalized potential*.

The space $D^1{}_H(P,\mathcal{G})$ of basic 1-test forms contains a subspace $D^1{}_{inv}(P,\mathcal{G})$ restricted to which two gauge equivalent generalized potentials E, E' coincide. The subspace $D^1{}_{inv}(P,\mathcal{G})$ is given by the kernel of the generalized covariant coderivative $\tilde{\bar{\nabla}}$; i.e. by test forms $\alpha \in D^1{}_H(P,\mathcal{G})$ satisfying [2]

$$\bar{\nabla}\alpha \;-\; *^{-1}[E,*\alpha] = 0 \; . \tag{2.8}$$

It is exactly this subspace which in the QCC-approach generates the physical states of the constructed indefinite metric operator theory. Of course the mathematical difficulties to determine this kernel in the general case should be mentioned here.

We now construct a gauge quantum field theory in terms of the *-algebra formulation [4],[5]. But instead of starting with an abstract *-algebra, we will work with concrete algebras B^p, called *test form algebras*, which are defined as topological direct sums of completed r-fold tensor products of suitable chosen test form spaces $D^p(P,G)$; i.e. $B^p = \mathbb{C} + \Sigma_r \otimes^r D^p(P,G)$. Field functionals $C \in (B^p)'$ are then defined with respect to a 'canonical' quantization rule, which states e.g. for a potential functional $C^E \in (B^1)'$ that C^E is a real, normed, linear functional (i.e. not a necessarily positive state) over the test form algebra B^1, the r-point functions of which behave like the generalized potential $E \in (D^1(P,G))'$ in each variable separately.

Relations between different fields defined by classical linear operations σ (e.g. d, δ, ∇, $\not\nabla$, or gauge transformations) are now converted to canonical *-algebra homomorphisms Θ_σ relating subalgebras of the corresponding test form algebras. To give an

Example: The relation between generalized fields $S_1 \in (D^{p-1}(P,G))'$ and $S_2 \in (D^p(P,G))'$ is given by $dS_1 = S_2$; i.e. $dS_1(\alpha) = S_1(\delta\alpha) = S_2(\alpha)$, with $\alpha \in D^p(P,G)$. Then the corresponding functionals $C_1 \in (B^{p-1})'$ and $C_2 \in (B^p)'$ should satisfy the equation $\Theta_d C_1 = C_2$; i.e. $\Theta_d C_1(k) = C_1(\Theta_\delta k) = C_2(k)$, $k \in B^p$, where $\Theta_\delta : B^p \longrightarrow B^{p-1}{}_\delta$ denotes the induced *-algebra homomorphism onto its image $B^{p-1}{}_\delta$ in B^{p-1}.

Of special interest is the subalgebra $B^1{}_{inv}$ of B^1 constructed from the test form space $D^1{}_{inv}(P,G)$. Potential functionals are defined to be *gauge equivalent* iff they are identical when restricted to $B^1{}_{inv}$. Motivated by the fact, that all physical informations are gauge independent, we postulate: Potential functionals C^E are positive when restricted to $B^1{}_{inv}$.

Having established the general structur of a gauge quantum field theory in its *-algebra formulation, where of course physical properties like Poincaré invariance, the spectral condition, etc. may still have to be implemented, we derive an indefinite metric operator theory from it by applying the Gelfand-Naimark-Segal (GNS) construction [4b],[5a]: every functional $C \in (B^p)'$ generates a strong cyclic representation D^C of the corresponding test form algebra B^p in a topological vector space H^C with cyclic vector Φ^C and non-degenerate Hermitian form $\langle .,. \rangle^C$.

Relations between generalized fields, first encoded into *-algebra homomorphisms, now find their analogue in isometric intertwining maps between subspaces of the corresponding representation spaces. For instance starting with the situation of the example given above we obtain the following diagram:

$$\begin{array}{ccc} B^p & \overset{\Theta_\delta}{--->} & B^{p-1}{}_\delta \subset B^{p-1} \\ \Pi^{c2} \downarrow & & \downarrow \Pi^{c1} \\ H^{c2} & \underset{W_\delta}{<---} & H^{c1}{}_\delta \subset H^{c1} \end{array}$$

where Π^{c1} and Π^{c2} are canonical projections and W_δ is a surjective isometric intertwining map sending the cyclic vector Φ^{c1} in H^{c1} onto the cyclic vector Φ^{c2} in H^{c2}. The map W_δ is an isomorphism iff $\langle .,. \rangle^{c2}$ is positive-definite on $H^{c2}{}_\delta$.

If $\{H^E, D^E, \Phi^E, \langle .,. \rangle^E\}$ denotes the operator theory constructed from a potential functional $C^E \in (B^1)'$, then the physical subspace $H^E{}_{phys}$ in H^E is induced by the invariance algebra $B^1{}_{inv}$:

$$H^E{}_{phys} := \{\phi \in H^E \ / \ \phi = D^E(k)\Phi^E, \ k \in B^1{}_{inv}\} \ .$$

The hermitian form $\langle .,. \rangle^E$ in general is positive semi-definite on $H^E{}_{phys}$. It is positive definite on the subspace $H^E{}_+ := H^E{}_{phys}/H^E{}_0$ in $H^E{}_{phys}$, where $H^E{}_0$ denotes the space of zero norm states.

Instead of going into the intricacies of the general formalism (see [2]) we concentrate on its results [3] for an U(1)-gauge theory.

3. U(1) gauge quantum field theories

Abelian gauge theories have some very special features with respect to the QCC-approach. We first observe that the classical background field decouples completely from the "first quantized" fields. Furthermore, not only the field strength and the external currents but also the quantized potentials E are invariant under classical gauge transformations; but the q-transformations (2.7) replacing them look the same; i.e. $E \;{--->}\; E + df$. Since E is also horizontal, we may formulate the theory as well on the space-time manifold M.

In the following we will as usual denote the (generalized) potential by $A \in (D^1(M))'$, the field strength by $F \in (D^2(M))'$ and the external current by $J \in (D^1(M))'$. For these deRham currents on M the basic equations reduce to

$$(3.1) \qquad F = dA \quad , \quad \delta F = J \ ,$$

from which the Bianchi identity $dF=0$ and the conservation of the current $\delta J=0$ follows trivially. F and J are invariant under q-transformations and A transforms like $A \;{--->}\; A + df$, $f \in (D^0(M))'$.

In the abelian case the subspace $D^1{}_{inv}(M)$ of $D^1(M)$ is given by the kernel of the coderivative δ; i.e. $D^1{}_{inv}(M) = \ker \delta$. Choosing Minkowski space as space-time manifold M, we have $\ker \delta = \operatorname{im} \delta$; i.e. $D^1{}_{inv}(M) = D^1{}_\delta(M)$.

In the following let B^p denote the p-test form algebra on M; i.e. $B^p = \mathbb{C} + \Sigma_r \otimes^r D^p(M)$. The potential, field strength and current functionals are given by states $C^A \in (B^1)'$, $C^F \in (B^2)'$, and $C^J \in (B^1)'$, respectively. According to (3.1) they satisfy

$$(3.2) \qquad \Theta_d C^A = C^F \ , \qquad \Theta_\delta C^F = C^J \ .$$

Furthermore the Bianchi identity $\Theta_d C^F = 0$ states that C^F vanishes on the subalgebra B^2_δ of B^2, and the current conservation $\Theta_\delta C^J = 0$ states, that C^J vanishes on the subalgebra B^1_d of B^1. On Minkowski space M the subalgebra B^1_{inv} of B^1, constructed from the test form space $D^1_{inv}(M)$, can be identified with the subalgebra B^1_δ of B^1. With respect to the quantization rule we postulate: C^F and C^J are positive states, C^A is positive when restricted to the subalgebra B^1_{inv}. Such a triple (C^A, C^F, C^J) of functionals, satisfying (3.2) is called a *field configuration.*

Two potential functionals C^A, $C^{A'}$ are defined to be *gauge equivalent* iff $C^A(k) = C^{A'}(k)$, $k \in B^1_{inv}$. Induced by the fact that im $\delta \subset$ ker δ two gauge equivalent functionals C^A, $C^{A'}$ satisfy $\Theta_d C^A = \Theta_d C^{A'} = C^F$. Furthermore we have the

Lemma: If C^A, $C^{A'}$ are gauge equivalent, then there exists a real, normed functional $C^T \in (B^1)'$, vanishing on B^1_{inv}, such that

$$(3.3) \qquad C^{A'} = C^A *_s C^T \ .$$

Here $*_s$ denotes the s-product in the sense of [5b].

The GNS-construction assigns to a field configuration (C^A, C^F, C^J) three different operator theories $\{H^A, D^A, \Phi^A, \langle.,.\rangle^A\}$, $\{H^F, D^F, \Phi^F, \langle.,.\rangle^F\}$ and $\{H^J, D^J, \Phi^J, \langle.,.\rangle^J\}$. Again the Hermitian form $\langle.,.\rangle^A$ is positive semi-definite on the physical subspace H^A_{phys} of H^A defined by $H^A_{phys} := \{\phi \in H^A \ / \ \phi = D^A(k)\Phi^A, \ k \in B^1_{inv}\}$, and it is positive definite on H^A_+. The Hermitian forms $\langle.,.\rangle^F$ and $\langle.,.\rangle^J$ are positive definite on H^F and H^J, respectively. The relations between the three representation spaces are summarized in the two diagrams

$$\Theta_\delta C^F = C^J : \qquad\qquad\qquad \Theta_d C^A = C^F :$$

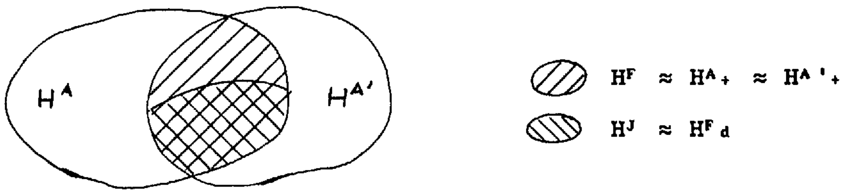
$$\begin{array}{ccccccc}
B^1 & \xrightarrow{\ \Theta_d\ } & B^2_d & \subset\!\longrightarrow & B^2 & \xrightarrow{\ \Theta_\delta\ } & B^1_\delta \\
\Pi^C \downarrow & & \downarrow \Pi^C & & \Pi^C \downarrow & & \downarrow \Pi^C \\
H^J & \xleftarrow[\ W_d\ ]{} & H^F_d & \subset\!\longrightarrow & H^F & \xleftarrow{} & H^A_{phys} \\
 & & & & \nwarrow^{W_\delta}_{\simeq} & & \downarrow \\
 & & & & & & H^A_+
\end{array}$$

The isometric intertwining map W_d is an isomorphism, the map W_δ is not an isomorphism, but it induces an isometric isomorphism between H^F and H^A_+.

We observe that in general for two gauge equivalent functionals C^A, $C^{A'}$ neither their state spaces H^A, $H^{A'}$ nor their physical subspaces $H^A{}_{phys}$, $H^{A'}{}_{phys}$ are isomorphic, but their positive state spaces $H^A{}_+$ and $H^{A'}{}_+$ are. Since in general q-gauge transformations are not given by *-algebra homomorphisms, they can not be represented by operator equations on the spaces H^A or $H^{A'}$. But using the lemma above; i.e.(3.3), there is an operator representation on the space $\mathcal{H} := H^A \times H^T$ given by $D^{A'} := D^A *^s D^T$, as described in [4b], with $*^s$ denoting the s-product.

The basic structure of QED as deduced from the QCC-approach may be resumed in the following diagrams:

$$\begin{array}{lllll} H^A \hookleftarrow H^A{}_+ & \overset{\simeq}{\longleftarrow} & & & \\ & & H^F \hookleftarrow H^F{}_d & \overset{\simeq}{\longleftarrow} & H^J \\ H^{A'} \hookleftarrow H^{A'}{}_+ & \overset{\simeq}{\longleftarrow} & & & \end{array}$$



To be able to compare the derived structure of QED with that of other approaches we shall choose a specific gauge: the Gupta-Bleuler gauge. For that we fix M to be Minkowski space-time. On the algebraic level of the derived field theory Poincaré transformations are implemented by *-algebra homomorphisms $\Theta_g : B^p \longrightarrow B^p$, $g \in P$, which are induced by the canonical action of the Poincaré group P on differential forms on M. We define a functional $C \in (B^p)'$ to be *Poincaré invariant* iff $\Theta_g C = C$, $g \in P$.

For physical reasons we will expect a potential functional $C^A \in (B^1)'$ to be Poincaré invariant only when restricted to the subalgebra $B^1{}_{inv}$. If a potential functional C^A is Poincaré invariant on $B^1{}_{inv}$ then the isometric intertwining maps W_g induced by the *-algebra homomorphisms Θ_g leave invariant the physical subspace $H^A{}_{phys}$; i.e. $W_g\, H^A{}_{phys} \subset H^A{}_{phys}$, $g \in P$.

The subalgebra $B^1{}_L \subset B^1$ generated by the span of the test form spaces $D^1{}_d(M)$ and $D^1{}_\delta(M)$ is called the *Lorentz subalgebra*. It has the property $B^1{}_{inv} \subset B^1{}_L \subset B^1$. A potential functional $C^A \in (B^1)'$ is called a *Gupta-Bleuler gauge* iff

i) $\Theta_\delta C^A = 0$,
ii) C^A is positive on $B^1{}_L$,
iii) $\langle .,. \rangle^A$ is positive definite on the subspace $H^A_\Delta \subset H^A$,

where $\Delta = d\delta + \delta d$ denotes the Laplace-deRham operator. A Gupta-Bleuler gauge C^A is consistent with the Poincaré invariance of C^A on $B^1{}_{inv}$. Furthermore for a Gupta-Bleuler gauge C^A the equation $\Delta A = J$ holds as an operator equation on $H^A_\Delta \subset H^A$.

It then can be shown [3] that an operator theory of the free field as given explicitely in Strocchi, Wightman [7c] is fully compatible with the corresponding operator theory derived by the QCC-approach.

4. Final remarks

First recall that the QCC-approach does work for every metric on the base (space-time) manifold M; i.e. not only for flat metrics. Furthermore it is highly sensitive for the topology of the space-time manifold M. The structure of QED given above may serve as an example.

a.) If M is topological non-trivial than ker $\delta \neq$ im δ. The subspace $D^1_{inv}(M)$ = ker $\delta \subset D^1(M)$, relevant for the definition of physical states in the operator formalism, and the subspace $D^1_s(M)$ = im $\delta \subset D^1(M)$ relevant for the relation between potential and field strength satisfy $D^1_{inv}(M) \supset D^1_s(M)$. This induces a segmentation of the physical subspace H^A_{phys} into different 'topological charged' sectors, classified by the cohomology of M.

b.) If M is a compact Riemannian manifold, then the techniques of Hodge theory, Fredholm alternative, etc. can be applied so that the substructure of a state space H^A can be given in more explicit terms.

c.) In the more physical case of a pseudo-Riemannian manifold M one may introduce a space-time splitting structure on M. If M then is isomorphic e.g. to R x S^3 one can apply the techniques b.) to the space-like parts of the theory.

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