PTEP
Prog. Theor. Exp. Phys. **2018**, 033E03 (19 pages)
DOI: 10.1093/ptep/pty017

Energy-dependent topological anti-de Sitter black holes in Gauss–Bonnet Born–Infeld gravity

S. H. Hendi[1,2,*], H. Behnamifard[1], and B. Bahrami-Asl[1]
[1] *Physics Department and Biruni Observatory, College of Sciences, Shiraz University, Shiraz 71454, Iran*
[2] *Research Institute for Astrophysics and Astronomy of Maragha (RIAAM), PO Box 55134-441, Maragha, Iran*
*E-mail: hendi@shirazu.ac.ir

Received June 28, 2017; Revised January 11, 2018; Accepted Feburary 1, 2018; Published March 27, 2018

Employing higher-curvature corrections to Einstein–Maxwell gravity has garnered a great deal of attention motivated by the high-energy regime in the quantum nature of black hole physics. In addition, one may employ gravity's rainbow to encode quantum gravity effects into black hole solutions. In this paper, we regard an energy-dependent static spacetime with various topologies and study its black hole solutions in the context of Gauss–Bonnet Born–Infeld (GB–BI) gravity. We study the thermodynamic properties and examine the first law of thermodynamics. Using a suitable local transformation, we endow the Ricci-flat black hole solutions with a global rotation and study the effects of rotation on thermodynamic quantities. We also investigate thermal stability in a canonical ensemble by calculating the heat capacity. We obtain the effects of various parameters on the horizon radius of stable black holes. Finally, we discuss a second-order phase transition in the extended phase space thermodynamics and investigate the critical behavior.

Subject Index A70, E03, E04, E05, E31

1. Introduction

James Clerk Maxwell is one of the physicists who played the most decisive part in developing electrodynamics. He collected and modified four equations, known as Maxwell's equations, which are the foundation of classical electromagnetism. Maxwell electrodynamics contains various problems that motivate one to consider nonlinear electrodynamics (NED). One of the main problematic aspects of Maxwell theory is having an infinite self-energy for a point-like charge. To remove such divergence, Born and Infeld introduced one of the interesting theories of NED, which is identified as Born–Infeld (BI) theory [1,2]. Recently, such nonlinear theories have acquired new impetus, since they can naturally arise in the low-energy limit of the open string theory [3–7].

From the gravitational point of view, Einstein gravity is a traditional theory of general relativity. Although such a theory is successful in various aspects and is consistent with some observational evidence, it has some problems in higher-curvature regimes. The natural extension of Einstein gravity is to consider higher-curvature gravity, which appears as the next-to-leading term in heterotic string effective action. Keeping quadratic curvature terms (and ignoring higher-curvature terms) in such stringy effective action [8–14] leads to the Gauss–Bonnet (GB) Lagrangian, which is a topological invariant in four dimensions. Various aspects of GB gravity and its thermodynamic properties have been investigated in several papers [15–33].

The electromagnetic and gravitational fields near charged black holes are very strong, and hence, both the nonlinear electromagnetic effects and higher-curvature gravitational terms should be taken into consideration. In other words, Maxwell theory and Einstein gravity can be, respectively, considered as approximations of NED theory and higher-curvature gravity in the weak field limit. Regarding the mentioned subjects, one is motivated to provide an analytically solvable theory that contains higher-curvature corrections of gauge–gravity theories. Among generalizations of the Einstein–Maxwell action to a higher-curvature gauge–gravity theory, the GB–BI gravity is of particular interest because it is a ghost-free theory on the gravitational side and a divergence-free field on the electromagnetic side. In addition, both GB and BI theories emerge in the effective low-energy action of string theory [3–14]. The influences of GB gravity and BI theory have been investigated in various physical phenomena [34–55]. Moreover, on the subject of black hole physics, various properties of GB–BI solutions have been studied (for a very incomplete list of references, see Refs. [56–58]).

On the other hand, regarding the high-energy regime of gravitational physics, it is important to find the UV generalization of general relativity. There are various attempts to obtain the UV completion of general relativity, which should reduce to general relativity in the IR limit. One approach for obtaining the UV completion of general relativity is based on the deformation of the usual energy–momentum dispersion relation in the UV limit, which is called gravity's rainbow [59]. Considering gravity's rainbow, one finds that the geometry of spacetime is made energy dependent through the introduction of rainbow functions [60,61].

On the other hand, one of the fundamental challenging subjects in cosmology is the big bang singularity. It is shown that nonlinear electrodynamics can remove both the black hole and big bang singularities [62–66]. Besides, one can construct a nonsingular universe by considering an effective energy-dependent spacetime [67–71]. In addition, an isotropic perfect fluid model is considered in the context of gravity's rainbow to obtain a nonsingular bouncing universe [72,73]. Moreover, the absence of singular black holes at the Large Hadron Collider (LHC) is explained with the formalism of gravity's rainbow [74]. In other words, gravity's rainbow is an effective approach for helping us to avoid the mentioned singularities.

It has been shown that for a certain choice of rainbow functions, there is a close relation between gravity's rainbow and Horava–Lifshitz gravity [75]. In addition, one may regard string theory as motivation for considering gravity's rainbow. The background fluxes in string theory produce a noncommutative deformation of the geometry [76,77], and such noncommutativity has also been used to motivate one of the most important classes of rainbow functions in gravity's rainbow [78,79]. Also, applying spontaneous breaking of the Lorentz symmetry, one should use a deformation of the usual energy–momentum relations, and such a modification can be used as a motivation for introducing gravity's rainbow [80].

It is worthwhile to mention that if the energy of the particle, E, was just a non-dynamical parameter, one could gauge it away by rescaling. However, it dynamically depends on the coordinates and it does break the original diffeomorphism's symmetry of the full metric. In fact, even the local symmetry in gravity's rainbow is not Lorentz symmetry, as it is based on the modified dispersion relation. An explicit form of coordinate dependence of the energy for a particular solution was discussed in Ref. [75]. Although it is hard to find explicit dependence of the energy on the coordinates for various solutions, it is important to know that E is a dynamical parameter that cannot be gauged away by rescaling.

After discovering the fact that the Hawking temperature and entropy are, respectively, proportional to a black hole's surface gravity and the area of its event horizon, investigation of black hole thermodynamics becomes a very interesting subject. In addition, regarding the cosmological constant as a dynamical pressure, one can study the critical behavior of a system in the extended phase space.

In this regard, one finds an analogy between a van der Waals liquid–gas system and the black hole as a thermodynamical system (for an incomplete list of references, see Refs. [81–91]).

The main goal of this paper is to find topological black hole solutions of the GB–BI gravity with an energy-dependent spacetime, although the system seems to be complex, as we have mentioned, considering that each item is based on a fundamental theory. It is worth mentioning that all higher-curvature gravity, higher-dimensional physics, higher-curvature terms in gauge theory, and the existence of an upper limit for the energy of a particle can be originated from heterotic string theory and the modified dispersion relation, which are two fundamental acceptable theories. In this paper, we consider the most important theory as well as the simplest ones. In other words, between all higher-curvature gravity (such as higher orders of Lovelock gravity and also $F(R)$ gravity theories), we select Gauss–Bonnet gravity, which is the simplest second-order derivative theory. On the other hand, regarding various models of nonlinear electrodynamics, we take into account the most interesting theory that has acceptable results. Regarding this theory, we could solve some problematic subjects from field theory to cosmological scales. Furthermore, regarding an energy-dependent line element is the simplest way to consider the energy of particles on the geometry of spacetime. Hence, in this paper, we try to combine some interesting subjects, while we take care to retain the simplicity. We also discuss the thermodynamic properties of such static topological black holes and generalize Ricci-flat anti-de Sitter (AdS) solutions to rotating black branes. We show how the thermodynamical properties and possible phase transitions of black holes can be affected by the higher-order curvature terms of gauge/gravity fields and the energy at which spacetime is probed. In other words, we study thermal stability and analyze the effects of GB, BI, rotation parameters, and rainbow functions on stability criteria.

2. Topological black hole solutions

In this section, we are going to obtain d-dimensional black hole solutions with various horizon topologies. In the context of gravity's rainbow, one may introduce a one-parameter family of an energy-dependent orthonormal frame field. As a result, a one-parameter family of the energy-dependent metric has been created, such as $g^{\mu\nu}(\varepsilon) = e^{\mu}_{a}(\varepsilon)e^{a\nu}(\varepsilon)$, where $\varepsilon = E/E_p$ is the energy ratio [59]. In this relation, E and E_p are, respectively, the energy of the test particle and the Planck energy. At the first step, we consider the following energy-dependent spherically symmetric line element:

$$d\tau^2 = -ds^2 = A(r,\varepsilon)dt^2 - B(r,\varepsilon)dr^2 - C(r,\varepsilon)d\Omega_k^2, \tag{1}$$

where $A(r,\varepsilon)$, $B(r,\varepsilon)$, and $C(r,\varepsilon)$ are functions of energy and radial coordinate. In addition, $d\Omega_k^2$ indicates the Euclidean metric of a $(d-2)$-dimensional hypersurface with constant curvature $(d-2)(d-3)k$ and volume V_{d-2}, where k is a constant. Hereafter, we take $k = \pm 1, 0$, without loss of generality. For more clarifications, we present the explicit form of $d\Omega_k^2$ as

$$d\Omega_k^2 = \begin{cases} d\theta_1^2 + \sum\limits_{i=2}^{d-2}\prod\limits_{j=1}^{i-1}\sin^2\theta_j d\theta_i^2 & k=1 \\ \sum\limits_{i=1}^{d-2} d\phi_i^2 & k=0\ . \\ d\theta_1^2 + \sinh^2\theta_1 (d\theta_2^2 + \sum\limits_{i=3}^{d-2}\prod\limits_{j=2}^{i-1}\sin^2\theta_j d\theta_i^2) & k=-1 \end{cases} \tag{2}$$

The energy-dependent metric functions are considered to make a connection between these new tetrad fields and the usual frame fields $\tilde{e}_a^\mu$ in general relativity, $f(\varepsilon)e_0^\mu(\varepsilon) = \tilde{e}_0^\mu$ and $g(\varepsilon)e_i^\mu(\varepsilon) = \tilde{e}_i^\mu$, where i is the spatial index and $f(\varepsilon)$ and $g(\varepsilon)$ are rainbow functions. We also expect to recover the general relativity in the IR limit, as $\lim_{\varepsilon\to 0} f(\varepsilon) = \lim_{\varepsilon\to 0} g(\varepsilon) = 1$. Hereafter, we use the following separation of variables for simplicity and also consistency with the energy-dependent modified dispersion relation:

$$A(r,\varepsilon) = \frac{\psi(r)}{f(\varepsilon)^2}, \quad B(r,\varepsilon) = \frac{1}{\psi(r)g(\varepsilon)^2}, \quad C(r,\varepsilon) = \frac{r^2}{g(\varepsilon)^2}, \tag{3}$$

where in the IR limit the Schwarzschild-like metric is recovered. Although there are different models of the rainbow functions with various phenomenological motivations (see Ref. [60] for more details), we try to present all analytical relations with the closed form of rainbow functions. This general form may help us to follow the trace of temporal ($f(\varepsilon)$) and spatial ($g(\varepsilon)$) rainbow functions, separately. However, for plotting the diagrams and other numerical analysis, we have to choose a class of rainbow functions by adjusting the free parameters.

In order to discuss the geometric properties, we should calculate the scalar curvatures. Regarding the metric (1) with Eq. (3), it is easy to show that the Ricci and Kretschmann scalars are, respectively,

$$R = g(\varepsilon)^2 \left[-\psi''(r) - 2(d-2)\frac{\psi'(r)}{r} + (d-2)(d-3)\frac{k-\psi(r)}{r^2} \right], \tag{4}$$

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = g(\varepsilon)^4 \left[\psi''^2(r) + 2(d-2)\left(\frac{\psi'(r)}{r}\right)^2 + 2(d-2)(d-3)\left(\frac{k-\psi(r)}{r^2}\right)^2 \right], \tag{5}$$

where prime and double prime are the first and second derivatives with respect to r, respectively. Calculating the other curvature invariants (such as the Ricci square, Weyl square, and so on), we find that they are functions of ψ'', ψ'/r, and $(k-\psi)/r^2$ and therefore it is sufficient to study the Kretschmann scalar for investigating the spacetime curvature. Regarding Eqs. (4) and (5), one finds the considerable effect of the rainbow function on all terms of curvature scalars. This indicates that the curvature of spacetime can be related to the energy at which the spacetime is probed.

Since gravity's rainbow has considerable effects in the UV regime, one may be motivated to consider high-energy and quantum corrections of gravity and gauge fields. In this regard, we are interested in the GB gravity coupled to a nonlinear $U(1)$ gauge field.

In addition, we should note that the constants may depend on the scale at which a theory is probed [92]. Such dependence is based on the renormalization group flow. Also, according to the supergravity solutions, one may expect to consider energy-dependent constants. As a result, it is interesting to take into account energy-dependent constants. The action under study is

$$I_G = -\frac{1}{16\pi}\int_{\mathcal{M}} d^dx\sqrt{-g}\left[R - 2\Lambda(\varepsilon) + \alpha(\varepsilon)L_{\mathrm{GB}} + L(\mathcal{F})\right] \tag{6}$$

where $\Lambda(\varepsilon)$ and $\alpha(\varepsilon)$ are, respectively, the energy-dependent cosmological constant and GB parameter. In addition, L_{GB} and $L(\mathcal{F})$ refer to the Lagrangians of the GB and BI theories, which can be written as

$$L_{\mathrm{GB}} = R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta} - 4R_{\mu\nu}R^{\mu\nu} + R^2, \tag{7}$$

$$L(\mathcal{F}) = 4\beta(\varepsilon)^2\left(1 - \sqrt{1 + \frac{\mathcal{F}}{2\beta(\varepsilon)^2}}\right), \tag{8}$$

where $\beta(\varepsilon)$ is called the BI parameter, $\mathcal{F} = F_{\mu\nu}F^{\mu\nu}$ is the Maxwell invariant, in which $F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$ is the Faraday tensor, and A_μ is the gauge potential. Taking into account the action (6) and using the variational principle, we can obtain the following field equations:

$$G^{(0)}_{\mu\nu} + G^{(1)}_{\mu\nu} + G^{(2)}_{\mu\nu} = \frac{1}{2}L(\mathcal{F})g_{\mu\nu} - 2L_{\mathcal{F}}F_{\mu\lambda}F_{\nu}^{\ \lambda}, \tag{9}$$

$$\nabla_\mu \left(L_{\mathcal{F}}F^{\mu\nu}\right) = 0, \tag{10}$$

where $L_{\mathcal{F}} = \frac{dL(\mathcal{F})}{d\mathcal{F}}$, $G^{(0)}_{\mu\nu} = \Lambda(\varepsilon)g_{\mu\nu}$, $G^{(1)}_{\mu\nu}$ is the Einstein tensor, and $G^{(2)}_{\mu\nu}$ is the divergence-free symmetric tensor of the GB contribution:

$$G^{(2)}_{\mu\nu} = -\alpha(\varepsilon)\left(4R^{\rho\sigma}R_{\mu\rho\nu\sigma} - 2R_{\mu}^{\ \rho\sigma\lambda}R_{\nu\rho\sigma\lambda} - 2RR_{\mu\nu} + 4R_{\mu\lambda}R^{\lambda}_{\ \ \nu} + \frac{L_{\mathrm{GB}}}{2}g_{\mu\nu}\right). \tag{11}$$

Regarding the metric (1) with the field equations (9) and (10), we can obtain an energy-dependent metric function. At first, we start with a consistent gauge potential A_μ with the following form:

$$A_\mu = \frac{h(r)}{f(\varepsilon)}\delta^0_\mu. \tag{12}$$

The functional form of $h(r)$ can be obtained from Eq. (10):

$$h(r) = -\frac{q(\varepsilon)}{(d-3)r^{d-3}}\,\mathcal{H}, \tag{13}$$

where

$$\mathcal{H} = {}_2F_1\left(\left[\frac{1}{2}, \frac{d-3}{2d-4}\right], \left[\frac{3d-7}{2d-4}\right], -\eta\right), \tag{14}$$

$$\eta = \frac{g(\varepsilon)^2 q(\varepsilon)^2}{\beta(\varepsilon)^2 r^{2d-4}}, \tag{15}$$

in which $q(\varepsilon)$ is an integration constant proportional to the electric charge and ${}_2F_1\left([a,b],[c],x\right)$ is a hypergeometric function. Using series expansion for large r (or large $\beta(\varepsilon)$), we find that the gauge potential is well behaved, asymptotically:

$$h(r)|_{\mathrm{Large}\ r} = -\frac{q(\varepsilon)}{(d-3)r^{d-3}} + \frac{g(\varepsilon)^2 q(\varepsilon)^3}{2(3d-7)\beta(\varepsilon)^2 r^{3d-7}} + O\left(\frac{1}{\beta(\varepsilon)^4 r^{5d-11}}\right). \tag{16}$$

Equation (16) confirms that for large values of r (or $\beta(\varepsilon)$), the dominant (first) term of $h(r)$ is the same as the gauge potential in d-dimensional linear Maxwell theory and the other terms are the Maxwell corrections. It is also notable that the rainbow function affects the correction terms.

Inserting metric (1) into the gravitational field equations with the obtained gauge potential, one finds the following nonzero components of Eq. (9):

$$e_1 = \left[2\alpha(\varepsilon)(d-4)\frac{g(\varepsilon)^2}{r^2}\left[\psi(r)-k\right] - \frac{1}{d-3}\right]\frac{\psi'(r)}{r} - \frac{\left[\psi(r)-k\right]}{r^2}$$
$$+ (d-4)(d-5)\alpha(\varepsilon)g(\varepsilon)^2\left[\frac{\psi(r)-k}{r^2}\right]^2 + \frac{4\beta(\varepsilon)^2(1-\sqrt{1+\eta}) - 2\Lambda(\varepsilon)}{(d-2)(d-3)g(\varepsilon)^2} = 0, \tag{17}$$

$$e_2 = \left[\frac{1}{(d-3)(d-4)} - 2\alpha(\varepsilon)g(\varepsilon)^2\frac{[\psi(r)-k]}{r^2}\right]\psi''(r) - 2\alpha(\varepsilon)g(\varepsilon)^2\left(\frac{\psi'(r)}{r}\right)^2$$
$$- 2\alpha(\varepsilon)g(\varepsilon)^2\left[2(d-5)\frac{[\psi(r)-k]}{r^2} - \frac{1}{\alpha(\varepsilon)(d-4)g(\varepsilon)^2}\right]\frac{\psi'(r)}{r} + \frac{[\psi(r)-k]}{r^2}$$
$$- (d-5)(d-6)\alpha(\varepsilon)g(\varepsilon)^2\frac{[\psi(r)-k]^2}{r^{d-3}} + \frac{2\Lambda(\varepsilon) - 4\beta(\varepsilon)^2\left(1-\frac{1}{\sqrt{1+\eta}}\right)}{(d-3)(d-4)g(\varepsilon)^2} = 0. \quad (18)$$

It is a matter of calculation to show that the following metric function satisfies all field equations simultaneously:

$$\psi(r) = k + \frac{r^2}{2\alpha(\varepsilon)(d-4)(d-3)g(\varepsilon)^2}\left(1-\sqrt{\digamma(r)}\right) \quad (19)$$

with

$$\digamma(r) = 1 + \frac{8(d-3)(d-4)\alpha(\varepsilon)}{(d-1)(d-2)}\left(\Lambda(\varepsilon) + \frac{(d-1)(d-2)m(\varepsilon)}{2r^{d-1}} - \Theta(r)\right), \quad (20)$$

$$\Theta(r) = 2\beta(\varepsilon)^2\left[\left(1-\sqrt{1+\eta}\right) + \frac{d-2}{d-3}\eta\mathcal{H}\right], \quad (21)$$

where $m(\varepsilon)$ is an integration constant. For large values of nonlinearity parameter ($\beta(\varepsilon) \to \infty$), this solution reduces to a GB–Maxwell gravity's rainbow black hole [60]. In addition, the obtained solutions reduce to GB–BI black holes for the IR limit $f(\varepsilon) = g(\varepsilon) = 1$.

Using series expansion of the metric function for large distances, we find that the dominant term is proportional to the cosmological constant. Therefore, the obtained solutions are asymptotically (A)dS with an effective cosmological constant (see Ref. [58] for more details). Depending on the free parameters, one finds that the function $\digamma(r)$ may be positive, zero, or negative. In order to obtain real (physical) solutions, we can use one of the following two methods. First, we can limit ourselves to the set of parameters that leads to non-negative $\digamma(r)$ for $0 \leq r < \infty$. Second, we can define r_R as the largest root of $\digamma(r = r_R) = 0$, in which $\digamma(r)$ is positive for $r > r_R$. One can use a suitable coordinate transformation ($r \longrightarrow r'$) to obtain real solutions for $0 \leq r' < \infty$ (see the last reference in Ref. [58] for more details). Hereafter, we follow the first method.

Now, we focus on obtaining black hole solutions. Inserting Eq. (19) in Eqs. (4) and (5), we find that both the Kretschmann and Ricci scalars diverge at $r = 0$ and are finite for $r \neq 0$. Since there is at least one real positive root for the metric function, the mentioned singularity can be covered with an event horizon, and therefore the solutions can be interpreted as black holes. In order to discuss the type of horizon, we can follow Refs. [93,94]. It was shown that there is a critical value for the nonlinearity parameter, $\beta_c(\varepsilon)$, in which, for $\beta(\varepsilon) < \beta_c(\varepsilon)$, the horizon geometry of nonlinear charged solutions behaves like Schwarzschild solutions and the singularity is spacelike. For $\beta(\varepsilon) > \beta_c(\varepsilon)$, the horizon geometry is the same as a Reissner–Nordström black hole and the singularity is timelike. The mentioned singularity is null for $\beta(\varepsilon) = \beta_c(\varepsilon)$ (see Fig. 1 for more details).

3. Thermodynamics

In this section, we discuss the thermodynamical behavior of the black hole solutions. First we calculate the Hawking temperature. To do so, we can use the surface gravity interpretation or follow

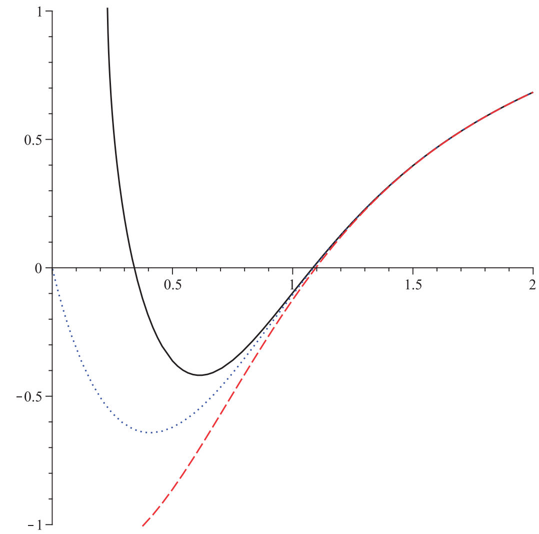

Fig. 1. $\psi(r)$ versus r for $k = 1$, $d = 5$, $g(\varepsilon) = f(\varepsilon) = 0.95$, $\Lambda(\varepsilon) = -0.1$, $\alpha(\varepsilon) = 0.1$, $q(\varepsilon) = 1$, and $\beta(\varepsilon) > \beta_c(\varepsilon)$ (continuous line: timelike singularity), $\beta(\varepsilon) = \beta_c(\varepsilon) \approx 1.032$ (dotted line: null singularity), and $\beta(\varepsilon) < \beta_c(\varepsilon)$ (dashed line: spacelike singularity).

the absence of a conical singularity at the horizon in the Euclidean sector of the black hole solutions. Both methods lead to the following temperature:

$$T = \frac{g(\varepsilon)}{4\pi f(\varepsilon)} \left.\frac{d\psi(r)}{dr}\right|_{r=r_+}. \tag{22}$$

After some manipulations, we obtain

$$T = \frac{r_+\left[\frac{4\beta(\varepsilon)^2\left[1-\sqrt{1+\eta_+}\right]-2\Lambda(\varepsilon)}{(d-2)} + \frac{(d-3)kg(\varepsilon)^2\left(1+\frac{k\alpha(\varepsilon)g(\varepsilon)^2}{r_+^2}(d-5)(d-4)\right)}{r_+^2}\right]}{4\pi f(\varepsilon)g(\varepsilon)\left[1+\frac{2k(d-3)(d-4)\alpha(\varepsilon)g(\varepsilon)^2}{r_+^2}\right]}, \tag{23}$$

$$\eta_+ = \eta|_{r=r_+},$$

which shows that rainbow functions, the nonlinearity, and the GB parameters affect the black hole temperature.

Now, we are going to calculate the entropy. Regarding Einstein gravity, it has been shown that the entropy of black holes satisfies the so-called area law, which states that the black hole entropy is equal to one-quarter of the horizon area [95–99]. However, we could not use the area law for higher-curvature gravity [100,101]. It is known that the entropy of asymptotically flat solutions ($\Lambda(\varepsilon) = 0$) of GB gravity can be obtained from the Wald formula [102–106]:

$$S = \frac{1}{4}\int d^{d-2}x\sqrt{\tilde{g}}\left(1+2\alpha(\varepsilon)\widetilde{R}\right) = \frac{r_+^{d-2}}{4g(\varepsilon)^{d-2}}\left(1+\frac{2k\,(d-2)\,(d-3)\alpha(\varepsilon)g(\varepsilon)^2}{r_+^2}\right), \tag{24}$$

where the integration is done on the $(d-2)$-dimensional spacelike hypersurface with the induced metric $\tilde{g}_{\mu\nu}$. We should note that $\tilde{g}$ is the determinant of $\tilde{g}_{\mu\nu}$ and $\tilde{R}$ is the Ricci scalar for the induced metric. Following the Gibbs–Duhem relation, one can show that the entropy of asymptotically (A)dS black holes in GB gravity is the same as Eq. (24). It is notable that, although the (nonlinear) electromagnetic source changes the locations of the inner and outer horizons, it does not change the functional form of the entropy formula and also the area law (see Ref. [107] for more details).

Calculating the flux of the electric field through a given closed hypersurface at infinity, one can obtain the electric charge per unit volume V_{d-2} of the black hole, yielding

$$Q = \frac{q(\varepsilon)f(\varepsilon)}{4\pi g(\varepsilon)^{d-3}}, \tag{25}$$

which shows that, although the total charge does not depend on the nonlinearity of the electromagnetic field, it depends on the energy functions. This behavior is expected, since, for large values of the radial coordinate, the electric field of BI theory reduces to a linear Maxwell field, but the metric components depend on the energy functions. Regarding ∂_t as the Killing vector (χ^μ) of static spacetime, we can obtain the electric potential of the event horizon with respect to the potential reference:

$$\Phi = A_\mu \chi^\mu \left|_{r\to\infty} - A_\mu \chi^\mu\right|_{r=r_+} = \frac{q(\varepsilon)\mathcal{H}_+}{(d-3)r_+^{d-3}}, \tag{26}$$

where $\mathcal{H}_+ = \mathcal{H}(r = r_+)$.

Considering the behavior of the metric at large r, one can obtain the ADM (Arnowitt–Deser–Misner) mass of a black hole for asymptotically flat solutions [108]. In addition, we use the counterterm method to calculate the finite mass of (A)dS solutions. Both calculations (ADM and counterterm methods) lead to the following unique relation for the mass per unit volume V_{d-2}:

$$M = \frac{(d-2)m(\varepsilon)}{16\pi f(\varepsilon)g(\varepsilon)^{d-1}}. \tag{27}$$

Since all conserved and thermodynamic quantities have been obtained, we can examine the validity of the first law of thermodynamics. Considering the expressions for the entropy, the electric charge, and the mass given in Eqs. (24), (25), and (27), we can obtain the total mass M as a function of the extensive quantities Q and S, such as the Smarr-type formula

$$M(S,Q) = \frac{f(\varepsilon)^2 r_+^{d-1}}{16(d-1)}\left(\frac{4(d-2)\Omega\beta(\varepsilon)^2\mathcal{A}}{(d-3)} + 4\beta(\varepsilon)^2(1-\sqrt{1+\Omega}) - 2\Lambda(\varepsilon) + \Sigma\right), \tag{28}$$

where $r_+ = r_+(S)$ and

$$\Sigma = \frac{(d-1)(d-2)kg(\varepsilon)^2}{r_+^2}\left[1 + \frac{(d-3)(d-4)k\alpha(\varepsilon)g(\varepsilon)^2}{r_+^2}\right],$$

$$\mathcal{A} = {}_2F_1\left(\left[\frac{1}{2}, \frac{d-3}{2d-4}\right], \left[\frac{3d-7}{2d-4}\right], -\Omega\right),$$

$$\Omega = \frac{16\pi^2 g(\varepsilon)^{2d-4}Q^2}{\beta(\varepsilon)^2 f(\varepsilon)^2 r_+^{2d-4}}.$$

Now, we regard Q and S as extensive parameters and introduce their conjugate intensive parameters, which are, respectively, the temperature and the electric potential, as

$$T = \left(\frac{\partial M}{\partial S}\right)_Q = \left(\frac{\partial M}{\partial r_+}\right)_Q \bigg/ \left(\frac{\partial S}{\partial r_+}\right)_Q, \tag{29}$$

$$\Phi = \left(\frac{\partial M}{\partial Q}\right)_S = \left(\frac{\partial M}{\partial q}\right)_S \bigg/ \left(\frac{\partial Q}{\partial q}\right)_S. \tag{30}$$

It is straightforward to show that Eqs. (29) and (30) are equal to Eqs. (23) and (26), respectively, and therefore one can conclude that these quantities satisfy the first law of thermodynamics:

$$dM = TdS + \Phi dQ. \tag{31}$$

Now, we are going to check thermal stability. The thermodynamic stability of black holes can be investigated in various ensembles. Referring to the thermal stability criterion in the canonical ensemble, stable black holes have positive heat capacity ($C_Q = T_+/\left(\partial^2 M/\partial S^2\right)_Q$). As an additional note, we should mention that, since size plays an essential role in gravitational thermodynamics, the negativity of the heat capacity does not, necessarily, mean that a system is unstable in the canonical ensemble. There are some finite-size black hole systems with negative heat capacity but stable state [109]. Since the black hole systems here have infinite size, negative C_Q indicates an unstable state.

The canonical ensemble instability criterion is sufficiently strong to veto some gravity toy models. The thermal stability of a typical thermodynamic system should be considered as the result of small variations of the thermodynamic coordinates. In this regard, the energy $\mathcal{M}(S, Q)$ should be a convex function of its extensive variable. The electric charge is a fixed parameter in the canonical ensemble and the positivity of heat capacity is sufficient to ensure (local) thermal stability. After some calculations, we find

$$C_Q = \frac{(d-2)r_+^d\left[1+\frac{2(d-3)(d-4)k\alpha(\varepsilon)g(\varepsilon)^2}{r_+^2}\right]^2\left[2\Lambda(\varepsilon)-4\beta(\varepsilon)^2(1-\sqrt{1+\eta_+})-A_0\right]}{4(d-3)g(\varepsilon)^d\left[2(d-4)k\alpha(\varepsilon)A_1+(d-2)kA_2-\frac{4q(\varepsilon)^2(1+A_3)}{r_+^{2d-6}\sqrt{1+\eta_+}}\right]}, \tag{32}$$

where

$$A_0 = \frac{(d-2)(d-3)kg(\varepsilon)^2}{r_+^2}\left[1+\frac{(d-4)(d-5)\alpha(\varepsilon)kg(\varepsilon)^2}{r_+^2}\right],$$

$$A_1 = \frac{(d-2)(d-3)kg(\varepsilon)^2}{r_+^2}\left(\frac{(d-4)(d-5)k\alpha(\varepsilon)g(\varepsilon)^2}{r_+^2}-1\right)-12\left(\beta(\varepsilon)^2-\frac{\Lambda(\varepsilon)}{2}\right),$$

$$A_2 = 1+\frac{3(d-4)(d-5)k\alpha(\varepsilon)g(\varepsilon)^2}{r_+^2}-\frac{4r_+^2}{(d-2)(d-3)kg(\varepsilon)^2}\left(\beta(\varepsilon)^2-\frac{\Lambda(\varepsilon)}{2}\right),$$

$$A_3 = \frac{2(d-4)(d-5)k\alpha(\varepsilon)g(\varepsilon)^2}{r_+^2}-\frac{\beta(\varepsilon)^2 r_+^{2d-4}}{q(\varepsilon)^2}\left(\frac{1}{(d-3)g(\varepsilon)^2}+\frac{6(d-4)k\alpha(\varepsilon)}{r_+^2}\right).$$

It is a nontrivial task to analytically find the positivity of the heat capacity. Instead, we use numerical analysis with various plots to discuss the sign and divergence points of the heat capacity. Regarding Figs. 2–4 with selected parameters, one finds that two different cases may take place for the heat capacity.

In the first case, C_Q is positive definite for $r_+ > r_0$, in which r_0 is the real positive root of temperature. Also, variation of free parameters in this case leads to changing r_0. In this regard, we study the effects of the GB coefficient ($\alpha(\varepsilon)$), BI parameter ($\beta(\varepsilon)$), and rainbow function ($g(\varepsilon)$). The left-hand panels of Figs. 2–4 and numerical analysis show that, although variation of $\alpha(\varepsilon)$ does not have a significant effect on r_0, r_0 is an increasing function of $\beta(\varepsilon)$ (decreasing function of $g(\varepsilon)$).

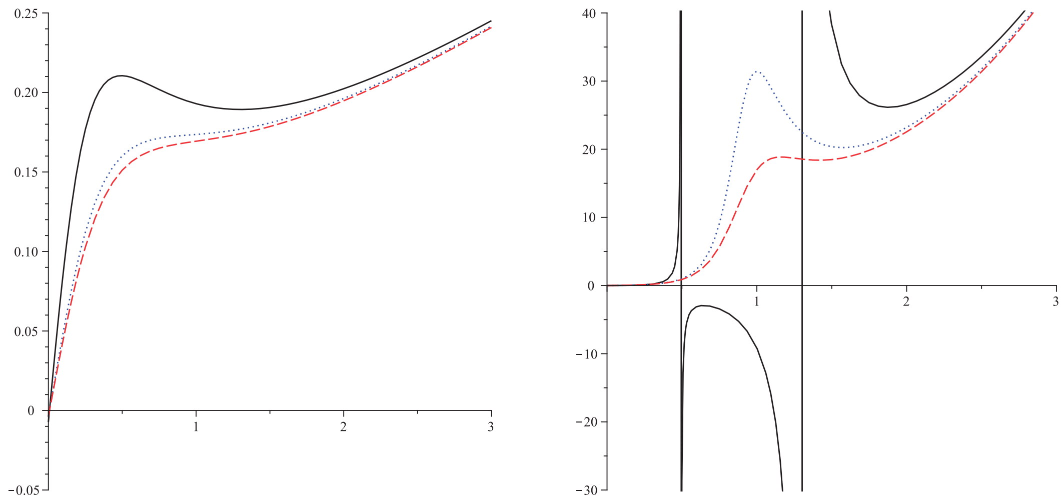

Fig. 2. T (left) and C_Q (right) diagrams versus r_+ for $d = 5$, $g(\varepsilon) = f(\varepsilon) = 0.9$, $\Lambda(\varepsilon) = -1$, $\beta(\varepsilon) = 0.1$, $q(\varepsilon) = 0.1$, and $\alpha(\varepsilon) = 0.05$ (continuous line), $\alpha(\varepsilon) = 0.09$ (dotted line), and $\alpha(\varepsilon) = 0.10$ (dashed line).

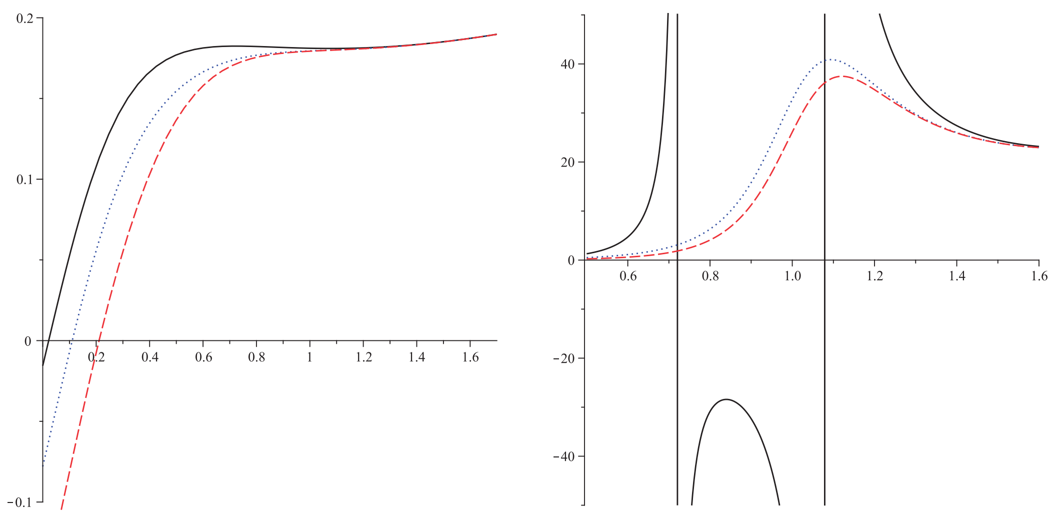

Fig. 3. T (left) and C_Q (right) diagrams versus r_+ for $d = 5$, $g(\varepsilon) = f(\varepsilon) = 0.9$, $\Lambda(\varepsilon) = -1$, $\alpha(\varepsilon) = 0.07$, $q(\varepsilon) = 0.3$, and $\beta(\varepsilon) = 0.1$ (continuous line), $\beta(\varepsilon) = 0.5$ (dotted line), and $\beta(\varepsilon) = 1.0$ (dashed line).

In the second case, there are two divergence points (r_{+1} and r_{+2} with $r_{+1} < r_{+2}$) for the heat capacity. In other words, the heat capacity is positive for $r_0 < r_+ < r_{+1}$ and $r_+ > r_{+2}$ and the black hole is stable in these ranges. However, there is an unstable state for $r_{+1} < r_+ < r_{+2}$. Considering phase space conception, one may consider a phase transition between the two stable states mentioned. It is worthwhile to mention that variation of free parameters may change the locations of r_{+1} and r_{+2}. In addition, the right-hand panels of Figs. 2–4 indicate that the unstable range of the event horizon is an increasing function of $g(\varepsilon)$ and a decreasing function of $\alpha(\varepsilon)$ and $\beta(\varepsilon)$. In other words, one can find a limit value for the GB coefficient ($\alpha_0(\varepsilon)$), BI parameter ($\beta_0(\varepsilon)$), and rainbow function ($g_0(\varepsilon)$), in which for $\alpha(\varepsilon) > \alpha_0(\varepsilon)$ ($\beta(\varepsilon) > \beta_0(\varepsilon)$ or $g(\varepsilon) < g_0(\varepsilon)$), there is no unstable phase and the heat capacity is positive definite in the range of $r_+ > r_0$.

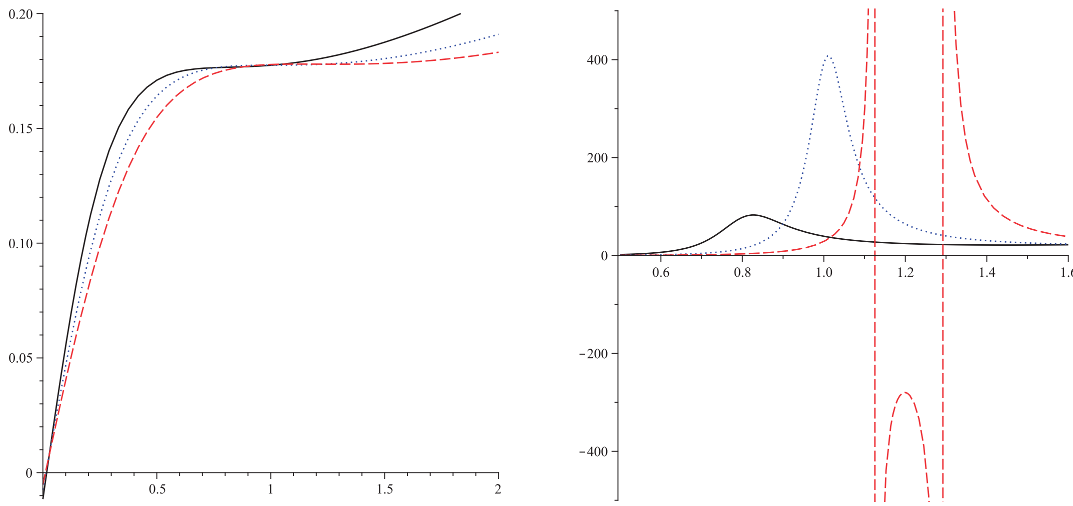

Fig. 4. T (left) and C_Q (right) diagrams versus r_+ for $d = 5$, $\Lambda(\varepsilon) = -1$, $\alpha(\varepsilon) = 0.08$, $\beta(\varepsilon) = 0.1$, $q(\varepsilon) = 0.2$, and $f(\varepsilon) = g(\varepsilon) = 0.8$ (continuous line), $f(\varepsilon) = g(\varepsilon) = 1.0$ (dotted line), and $f(\varepsilon) = g(\varepsilon) = 1.2$ (dashed line).

4. Asymptotically AdS rotating black branes

In this section, we are going to add a global rotation into the zero-curvature horizon ($k = 0$). The maximum number of independent rotation parameters is equal to $[\frac{d-1}{2}]$ ($[x]$ is the integer part of x), which is due to the fact that the rotation group in d dimensions is $SO(d-1)$. Now, we perform the following local boost transformation in the $t - \phi_i$ planes for adding the angular momentum:

$$\frac{t}{f(\varepsilon)} \mapsto \Xi(\varepsilon)\frac{t}{f(\varepsilon)} - a_i(\varepsilon)\frac{\phi_i}{g(\varepsilon)}, \qquad \frac{\phi_i}{g(\varepsilon)} \mapsto \Xi(\varepsilon)\frac{\phi_i}{g(\varepsilon)} - \frac{a_i(\varepsilon)}{l(\varepsilon)^2}\frac{t}{f(\varepsilon)}, \tag{33}$$

where $l(\varepsilon)$ is an energy-dependent scale parameter with length dimension, $a_i(\varepsilon)$ are rotation parameters in which $i = 1...[\frac{d-1}{2}]$, and $\Xi(\varepsilon) = \sqrt{1 + \sum_{i=1}^{p} \frac{a_i(\varepsilon)^2}{l(\varepsilon)^2}}$. Applying such a boost into the metric (1) with $k = 0$, one can obtain the following d-dimensional Ricci-flat solutions with $p \leq [\frac{d-1}{2}]$ rotation parameters:

$$\begin{aligned} ds^2 = &-\psi(r)\left(\frac{\Xi(\varepsilon)}{f(\varepsilon)}dt - \frac{1}{g(\varepsilon)}\sum_{i=1}^{p} a_i(\varepsilon)d\phi_i\right)^2 + \frac{r^2}{l^4}\sum_{i=1}^{p}\left(\frac{a_i(\varepsilon)}{f(\varepsilon)}dt - \frac{\Xi(\varepsilon)l(\varepsilon)^2}{g(\varepsilon)}d\phi_i\right)^2 \\ &+ \frac{dr^2}{g(\varepsilon)^2\psi(r)} - \frac{r^2}{g(\varepsilon)^2 l(\varepsilon)^2}\sum_{i<j}^{p}[a_i(\varepsilon)d\phi_j - a_j(\varepsilon)d\phi_i]^2 + \frac{r^2}{g(\varepsilon)^2}\sum_{i=p+1}^{d-2} d\phi_i^2. \end{aligned} \tag{34}$$

Using Eq. (10), we can obtain the following consistent gauge potential:

$$A_\mu = h(r)\left(\frac{\Xi(\varepsilon)}{f(\varepsilon)}\delta_\mu^0 - \frac{a_i(\varepsilon)}{g(\varepsilon)}\delta_\mu^i\right) \quad \text{(no sum on } i\text{)}, \tag{35}$$

where $h(r)$ is the same as Eq. (13). In addition, straightforward calculations show that all components of the gravitational field equation (9) can be satisfied by the metric function of Eq. (19). It is worthwhile to mention that the local transformation (33) generates a new metric, because it is not a proper coordinate transformation on the entire manifold [110]. In other words, the static and rotating metrics can be locally mapped into each other but not globally, and therefore they are distinct.

Following the same method, one can find that there is a curvature singularity at $r = 0$, which can be covered with an event horizon. Therefore, one may interpret such a singularity as a black brane. Using the fact that $\chi = \partial_t + \sum_i \Omega_i \partial_{\phi_i}$ is the null generator of the horizon for the mentioned rotating black branes, we can obtain

$$T_{\text{rotating}} = \left.\frac{T_{\text{static}}}{\Xi(\varepsilon)}\right|_{k=0}, \tag{36}$$

$$\Phi_{\text{rotating}} = \frac{\Phi_{\text{static}}}{\Xi(\varepsilon)}, \tag{37}$$

$$Q_{\text{rotating}} = \Xi(\varepsilon) Q_{\text{static}}, \tag{38}$$

$$S_{\text{rotating}} = \Xi(\varepsilon) S_{\text{static}}|_{k=0}, \tag{39}$$

$$M_{\text{rotating}} = \left(\frac{(d-1)\Xi(\varepsilon)^2 - 1}{d-2}\right) M_{\text{static}}, \tag{40}$$

and for rotating quantities, one can, respectively, obtain the following functional forms of angular velocities and angular momenta:

$$\Omega_i = \frac{a_i(\varepsilon)}{\Xi(\varepsilon) l(\varepsilon)^2}, \tag{41}$$

$$J_i = \frac{d-1}{16\pi} \Xi(\varepsilon) m(\varepsilon) a_i(\varepsilon). \tag{42}$$

In order to examine the validity of the first law of thermodynamics, we regard S, Q, and J_i as a complete set of extensive quantities for the mass $M(S, Q, J_i)$ and define intensive quantities conjugate to them. The conjugate quantities are T, Φ, and Ω_i:

$$T = \left(\frac{\partial \mathcal{M}}{\partial S}\right)_{Q,J_i}, \tag{43}$$

$$\Phi = \left(\frac{\partial \mathcal{M}}{\partial Q}\right)_{S,J_i}, \tag{44}$$

$$\Omega_i = \left(\frac{\partial \mathcal{M}}{\partial J_i}\right)_{S,Q}. \tag{45}$$

Considering the fact that the metric function vanishes at the event horizon with Eqs. (43), (44), and (45), we find that the intensive quantities calculated by Eqs. (43), (44), and (45) are, respectively, consistent with Eqs. (36), (37), and (41), and therefore one can confirm that the relevant thermodynamic quantities satisfy the first law of thermodynamics as

$$dM = TdS + \Phi dQ + \sum_i \Omega_i dJ_i. \tag{46}$$

Since we investigated the thermodynamic stability of AdS Ricci-flat static black holes in the previous section, here we are going to check the effect of rotation on thermal stability. For rotating solutions, the mass M is a function of the entropy S, the angular momentum J, and the electric charge Q. As we mentioned before, the positivity of the heat capacity $C_{Q,J} = T \Big/ \left(\partial^2 M / \partial S^2\right)_{Q,J}$ is sufficient to ensure the thermodynamic stability. Numerical calculations show that the effects of variation of rainbow functions, BI parameter, and GB coefficient are the same as those in the static case. In addition, as shown in Ref. [58], $C_{Q,J}$ is an increasing regular function of Ξ. Thus we ignore more explanations and, in the next section, we focus on the more interesting case $k = 1$ for investigating critical behavior.

5. Critical behavior

Here, we use the the extended phase space thermodynamics to investigate critical behavior. To do so, we take into account the AdS black holes with spherical horizons (it has been shown that there is no van der Waals-like phase transition for $k = 0$ and $k = -1$). In this regard, we treat the cosmological constant as a thermodynamic pressure. In fact, we do not work in a fixed AdS background, and therefore the cosmological constant is not a constant anymore, but a variable. If we treat the negative cosmological constant proportional to the thermodynamic pressure, its conjugate quantity will be the thermodynamic volume. Using the scaling argument, it has already been shown that the Smarr relation is consistent with the first law of thermodynamics by assuming the cosmological constant as a thermodynamic variable. Here, we regard the following relation between the cosmological constant and the pressure [81–91]:

$$P = -\frac{\Lambda(\varepsilon)}{8\pi}. \tag{47}$$

As a notable comment, it is worthwhile to mention that it was shown that the modified Smarr relation that can be calculated by the scaling argument is completely in agreement with the modified first law of thermodynamics in the extended phase space. In this regard, in addition to $\Lambda(\varepsilon)$, the nonlinearity parameter ($\beta(\varepsilon)$) and GB parameter ($\alpha(\varepsilon)$) are two other thermodynamical variables [111,112]. It is notable that in the presence of rainbow functions, the modified first law of thermodynamics can be written as

$$dM = TdS + \Phi dQ + VdP + \mathcal{A}d\alpha + \mathcal{B}d\beta, \tag{48}$$

where the conjugate quantities can be obtained as

$$T = \left(\frac{\partial M}{\partial S}\right)_{Q,P,\alpha,\beta},$$
$$\Phi = \left(\frac{\partial M}{\partial Q}\right)_{S,P,\alpha,\beta},$$
$$V = \left(\frac{\partial M}{\partial P}\right)_{S,Q,\alpha,\beta},$$
$$\mathcal{A} = \left(\frac{\partial M}{\partial \alpha}\right)_{S,Q,P,\beta},$$
$$\mathcal{B} = \left(\frac{\partial M}{\partial \beta}\right)_{S,Q,P,\alpha}.$$

Moreover, we can obtain the generalized Smarr relation for our asymptotically AdS solutions in the extended phase space:

$$(d-3)M = (d-2)TS + (d-3)Q\Phi - 2PV - 2\left(\frac{d-2}{d-4}\right)\mathcal{A}\alpha - \mathcal{B}\beta, \tag{49}$$

Table 1. Critical values for $q(\varepsilon) = 1, f(\varepsilon) = g(\varepsilon) = 1$, $\beta(\varepsilon) = 0.1$, and $d = 5$.

$\alpha(\varepsilon)$	r_c	T_c	P_c	$\frac{P_c r_c}{T_c}$
0.0100	0.4606	0.3456	0.1868	0.2490
0.0500	0.8822	0.1807	0.0505	0.2468
0.1000	1.2072	0.1328	0.0270	0.2458
0.5000	2.6086	0.0636	0.0062	0.2561
1.0000	3.5479	0.0456	0.0032	0.2528
10.0000	10.9575	0.0145	0.0003	0.2500

which is completely in agreement with the mentioned modified first law of thermodynamics.

Inserting Eq. (47) in the temperature equation (23), one can find the equation of state $P = P(T, r_+)$

$$P = \frac{4\beta(\varepsilon)^2}{16\pi}\left(\sqrt{1+\eta_+} - 1\right) + \left(1 + \frac{2(d-3)(d-4)\alpha(\varepsilon)g(\varepsilon)^2}{r_+^2}\right)\frac{(d-2)f(\varepsilon)g(\varepsilon)T}{4r_+} - \frac{(d-2)(d-3)g(\varepsilon)^2}{16\pi r_+^2}\left(1 + \frac{(d-4)(d-5)\alpha(\varepsilon)g(\varepsilon)^2}{r_+^2}\right). \tag{50}$$

Since the critical point is an inflection point on the critical isothermal P–r_+ diagrams, we use the following relations to obtain the proper equations for critical quantities:

$$\left(\frac{\partial P}{\partial r_+}\right)_T = 0, \tag{51}$$

$$\left(\frac{\partial^2 P}{\partial r_+^2}\right)_T = 0. \tag{52}$$

Using Eqs. (51) and (52), one can find the critical quantities. Numerical calculations confirm that by choosing suitable parameters, one can find critical values for the thermodynamic quantities. In order to analyze the effects of various parameters, we present different tables (see Tables 1–6). Accordingly, we can find that the GB and nonlinearity parameters, rainbow functions, and dimensionality affect the critical quantities. Based on Tables 1 and 2, one finds that increasing the GB parameter leads to an increase in the critical volume and a decrease in the critical temperature and pressure. In addition, for sufficiently large $\alpha(\varepsilon)$, one can obtain the universality ratio $1/4$ independent of other parameters. It is notable that if we increase $\alpha(\varepsilon)$ more, we cannot obtain real positive values for the critical quantities. In other words, there is an $\alpha_0(\varepsilon)$ in which for $\alpha(\varepsilon) > \alpha_0(\varepsilon)$ there is no phase transition and critical behavior (see Fig. 2 for more details). We should note that the value of $\alpha_0(\varepsilon)$ depends on other parameters. The same behavior takes place for the nonlinearity parameter. This means that the critical volume is (critical temperature and pressure are) an increasing function (decreasing functions) of the BI parameter, and also that there is a $\beta_0(\varepsilon)$ in which for $\beta(\varepsilon) > \beta_0(\varepsilon)$ the system is thermally stable and there is no phase transition (see Fig. 3 for more details). For the rainbow functions, we obtain different behavior. We find that by increasing these functions, all critical quantities increase, except critical temperature. In addition, there is a lower bound for the rainbow functions, in which if we regard the values of the rainbow functions less than such a bound, we cannot find any phase transition (see Fig. 4 for more details).

Table 2. Critical values for $q(\varepsilon) = 1, f(\varepsilon) = g(\varepsilon) = 1, \beta(\varepsilon) = 0.1$, and $d = 6$.

$\alpha(\varepsilon)$	r_c	T_c	P_c	$\frac{P_c r_c}{T_c}$
0.010 00	0.6790	0.3720	0.1869	0.3411
0.050 00	1.1013	0.1938	0.0547	0.3106
0.100 00	1.4543	0.1410	0.0295	0.3048
0.500 00	2.0513	0.0647	0.0064	0.2041
1.000 00	3.2858	0.0459	0.0033	0.2367
10.0000	10.9384	0.0145	0.0003	0.2496

Table 3. Critical values for $q(\varepsilon) = 1, f(\varepsilon) = g(\varepsilon) = 1, \alpha(\varepsilon) = 10^{-2}$, and $d = 5$.

$\beta(\varepsilon)$	r_c	T_c	P_c	$\frac{P_c r_c}{T_c}$
0.0100	0.3566	0.4464	0.3130	0.2500
0.0500	0.4000	0.3979	0.2485	0.2498
0.1000	0.4606	0.3456	0.1868	0.2490
0.5000	1.3362	0.1696	0.0365	0.2877
1.0000	1.4922	0.1633	0.0335	0.3059
10.0000	1.5265	0.1618	0.0328	0.3093

Table 4. Critical values for $q(\varepsilon) = 1, f(\varepsilon) = g(\varepsilon) = 1, \alpha(\varepsilon) = 10^{-2}$, and $d = 6$.

$\beta(\varepsilon)$	r_c	T_c	P_c	$\frac{P_c r_c}{T_c}$
0.0100	0.2449	0.4332	0.2210	0.1250
0.0500	0.5769	0.4055	0.2317	0.3296
0.1000	0.6790	0.3720	0.1869	0.3411
0.5000	1.2425	0.2764	0.0892	0.4010
1.0000	1.3299	0.2708	0.0850	0.4170
10.0000	1.3515	0.2693	0.0838	0.4205

Table 5. Critical values for $q(\varepsilon) = 1$, $\beta(\varepsilon) = 0.1$, $\alpha(\varepsilon) = 0.1$, and $d = 5$.

$f(\varepsilon) = g(\varepsilon)$	r_c	T_c	P_c	$\frac{P_c r_c}{T_c}$
0.9000	1.1092	0.1444	0.0259	0.1986
0.9500	1.1578	0.1384	0.0265	0.2216
1.0000	1.2072	0.1328	0.0270	0.2458
1.0500	1.2573	0.1276	0.0275	0.2714
1.1000	1.3079	0.1227	0.0280	0.2981

Table 6. Critical values for $q(\varepsilon) = 1$, $\beta(\varepsilon) = 0.1$, $\alpha(\varepsilon) = 0.1$, and $d = 6$.

$f(\varepsilon) = g(\varepsilon)$	r_c	T_c	P_c	$\frac{P_c r_c}{T_c}$
0.9000	0.6452	0.1480	0.0147	0.0641
0.9500	1.4042	0.1477	0.0291	0.2767
1.0000	1.4543	0.1410	0.0295	0.3048
1.0500	1.5052	0.1348	0.0299	0.3343
1.1000	1.5566	0.1291	0.0303	0.3651

6. Closing remarks

We have studied black hole solutions of GB–BI gravity's rainbow in arbitrary dimensions ($d \geq 5$). All GB, BI, and gravity's rainbow make sense in the high-energy regime and are motivated by string theory corrections.

First, we presented a brief discussion regarding geometrical properties and found that the obtained solutions are asymptotically AdS with an effective cosmological constant. In addition, we found that, depending on the values of the parameters, the black hole singularity may be timelike, spacelike, or null. In other words, such a singularity may behave like Reissner–Nordström solutions, Schwarzschild black holes, or completely different from them.

Then, we focused on the thermodynamical behavior of the solutions. We calculated the conserved and thermodynamic quantities and found that they may be affected by the existence of rainbow functions and/or the GB parameter. We have shown that, although the BI parameter changes the type of singularity, it does not change the conserved quantities directly. Such behavior is expected, since its effect disappears for large distances ($r \to \infty$). We have also seen that, although rainbow functions modify the conserved and thermodynamic quantities, the first law of thermodynamics is valid in the energy-dependent spacetime. Then, we calculated the heat capacity and discussed thermal stability in the canonical ensemble. We found that, depending on the values of the free parameters, large black holes are thermally stable.

Next, we generalized horizon-flat solutions to the case of rotation and presented exact solutions. Since we used a local transformation between the static and rotating cases, we can confirm that rotating and static solutions are distinct. We calculated all related quantities and found that the rotation parameters affected all the conserved and thermodynamic quantities. We also examined the first law of thermodynamics and proved that its generalization (in the presence of angular momentum) is already valid.

Finally, we used extended phase space thermodynamics and regarded the cosmological constant as a thermodynamical pressure. We calculated critical quantities and found that, depending on the values of the free parameters, black hole solutions enjoy a phase transition.

We also studied the modified first law of thermodynamics in the extended phase space with an energy-dependent spacetime. We found that, in order to have a consistent Smarr relation and first law of thermodynamics in a differential form, one has to consider both $\alpha(\varepsilon)$ and $\beta(\varepsilon)$ as thermodynamical variables.

As a final comment, we should note that, in this paper, we have regarded energy-dependent constants without their functional forms. It would be interesting to take into account the explicit functional form of the energy-dependent constants and investigate their effects.

Acknowledgements

We gratefully thank the anonymous referee for enlightening comments and suggestions that helped in improving the quality of the paper. We also thank S. Panahiyan for reading the manuscript and for useful comments. We also thank Shiraz University Research Council. This work was supported financially by the Research Institute for Astronomy and Astrophysics of Maragha, Iran.

Appendix A.

In this paper, we have tried to present all analytical relations with general closed forms of rainbow functions. These general closed forms help us to follow the trace of temporal ($f(\varepsilon)$) and spatial ($g(\varepsilon)$) rainbow functions, separately. However, in order to plot diagrams and undertake other numerical

Table A1. Numerical calculations for rainbow functions.

λ	ε	$f(\varepsilon) = g(\varepsilon)$
−0.5555	0.2000	0.9000
−0.2631	0.2000	0.9500
0.0000	0.2000	1.0000
0.2381	0.2000	1.0500
0.4546	0.2000	1.1000

analysis, we have to choose a class of rainbow functions. Between all possible choices of rainbow functions, there are three known classes with various phenomenological motivations. The first model comes from the hard spectra of gamma-ray-burst motivation with the following explicit forms [114]:

$$f(\varepsilon) = \frac{\exp(\beta\varepsilon) - 1}{\beta\varepsilon}, \qquad g(\varepsilon) = 1. \tag{A.1}$$

Taking the constancy of the velocity of light into account, one can find the following relations for the rainbow functions as the second model [113]:

$$f(\varepsilon) = g(\varepsilon) = \frac{1}{1 - \lambda\varepsilon}. \tag{A.2}$$

The third model is motivated from loop quantum gravity and noncommutative geometry in which the rainbow functions are given as [78,79]

$$f(\varepsilon) = 1, \qquad g(\varepsilon) = \sqrt{1 - \eta\varepsilon^n}, \tag{A.3}$$

where β, λ, η, and n are arbitrary constants that can be fixed according to experimental evidence near the Planck energy. For numerical calculations, we are interested in the second model (A.2). In Table A1, we choose suitable values of ε and λ for obtaining the related rainbow functions that are used in Tables 5 and 6.

References

[1] M. Born, Proc. R. Soc. Lond. A **143**, 410 (1934).
[2] M. Born and L. Infeld, Proc. R. Soc. Lond. A **144**, 425 (1934).
[3] E. S. Fradkin and A. A. Tseytlin, Phys. Lett. B **163**, 123 (1985).
[4] R. R. Matsaev, M. A. Rahmanov, and A. A. Tseytlin, Phys. Lett. B **193**, 207 (1987).
[5] E. Bergshoeff, E. Sezgin, C. N. Pope, and P. K. Townsend, Phys. Lett. B **188**, 70 (1987).
[6] C. G. Callan, C. Lovelace, C. R. Nappi, and S. A. Yost, Nucl. Phys. B **308**, 221 (1988).
[7] O. D. Andreev and A. A. Tseytlin, Nucl. Phys. B **311**, 205 (1988).
[8] B. Zwiebach, Phys. Lett. B **156**, 315 (1985).
[9] D. G. Boulware and S. Deser, Phys. Rev. Lett. **55**, 2656 (1985).
[10] D. G. Boulware and S. Deser, Phys. Lett. B **175**, 409 (1986).
[11] R. R. Metsaev and A. A. Tseytlin, Phys. Lett. B **191**, 354 (1987).
[12] R. R. Metsaev and A. A. Tseytlin, Nucl. Phys. B **293**, 385 (1987).
[13] C. G. Callan, D. Friedan, E. J. Martinec and M. J. Perry, Nucl. Phys. B **262**, 593 (1985).
[14] A. Sen, Phys. Rev. Lett. **55**, 33 (1985).
[15] J. E. Kim, B. Kyae, and H. M. Lee, Phys. Rev. D **62**, 045013 (2000).
[16] Y. M. Cho and I. P. Neupane, Phys. Rev. D **66**, 024044 (2002).
[17] C. Charmousis and J.-F. Dufaux, Classical Quantum Gravity **19**, 4671 (2002).

[18] G. Kofinas, R. Maartens, and E. Papantonopoulos, J. High Energy Phys. **10**, 066 (2003).
[19] R.-G. Cai and Q. Guo, Phys. Rev. D **69**, 104025 (2004).
[20] A. Barrau, J. Grain, and S. Alexeyev, Phys. Lett. B **584**, 114 (2004).
[21] K. I. Maeda and T. Torii, Phys. Rev. D **69**, 024002 (2004).
[22] C. de Rham and A. J. Tolley, J. Cosmol. Astropart. Phys. **07**, 004 (2006).
[23] G. Dotti, J. Oliva, and R. Troncoso, Phys. Rev. D **76**, 064038 (2007).
[24] R. A. Brown, Gen. Relat. Gravit. **39**, 477 (2007).
[25] H. Maeda, V. Sahni, and Y. Shtanov, Phys. Rev. D **76**, 104028 (2007).
[26] Charmousis C. Higher Order Gravity Theories and Their Black Hole Solutions. in *Physics of Black Holes. Lecture Notes in Physics*, ed. E. Papantonopoulos (Springer, Berlin, Heidelberg 2009), Vol. 769.
[27] S. H. Hendi and B. Eslam Panah, Phys. Lett. B **684**, 77 (2010).
[28] M. Bouhmadi-López, Y.-W. Liu, K. Izumi, and P. Chen, Phys. Rev. D **89**, 063501 (2014).
[29] S. H. Hendi, S. Panahiyan, and E. Mahmoudi, Eur. Phys. J. C **74**, 3079 (2014).
[30] Y. Yamashita and T. Tanaka, J. Cosmol. Astropart. Phys. **06**, 004 (2014).
[31] A. Maselli, P. Pani, L. Gualtieri, and V. Ferrari, arXiv:1507.00680 [gr-qc] [Search INSPIRE].
[32] W.-K. Ahn, B. Gwak, B.-H. Lee, and W. Lee, Eur. Phys. J. C **75**, 372 (2015).
[33] A. Mandal and R. Biswas, arXiv:1602.03000 [physics.gen-ph].
[34] Q. Pan and B. Wang, Phys. Lett. B **693**, 159 (2010).
[35] R.-G. Cai, Z.-Y. Nie, and H.-Q. Zhang, Phys. Rev. D **82**, 066007 (2010).
[36] J. Jing, L. Wang, Q. Pan, and S. Chen, Phys. Rev. D **83**, 066010 (2011).
[37] Y.-P. Hu, P. Sun, and J.-H. Zhang, Phys. Rev. D **83**, 126003 (2011).
[38] Y.-P. Hu, H.-F. Li, and Z.-Y. Nie, J. High Energy Phys. **01**, 123 (2011).
[39] A. Barrau, J. Grain, and S. Alexeyev, Phys. Lett. B **584**, 114 (2004).
[40] N. Okada and S. Okada, Phys. Rev. D **79**, 103528 (2009).
[41] B. M. Leith and I. P. Neupane, J. Cosmol. Astropart. Phys. **05**, 019 (2007).
[42] A. K. Sanyal, Phys. Lett. B **645**, 1 (2007).
[43] X.-H. Ge and S.-J. Sin, J. High Energy Phys. **05**, 051 (2009).
[44] R.-G. Cai, Z.-Y. Nie, N. Ohta, and Y.-W. Sun, Phys. Rev. D **79**, 066004 (2009).
[45] J. Jing and S. Chen, Phys. Lett. B **686**, 68 (2010).
[46] S. Gangopadhyay and D. Roychowdhury, J. High Energy Phys. **05**, 156 (2012).
[47] S. L. Cui and Z. Xue, Phys. Rev. D **88**, 107501 (2013).
[48] H. Carley and M. K.-H. Kiessling, Phys. Rev. Lett. **96**, 030402 (2006).
[49] J. Franklin and T. Garon, Phys. Lett. A **375**, 1391 (2011).
[50] B. Vaseghi, G. Rezaei, S. H. Hendi, and M. Tabatabaei, Quantum Matter **2**, 194 (2013).
[51] S. Habib Mazharimousavi, I. Sakalli, and M. Halilsoy, Phys. Lett. B **672**, 177 (2009).
[52] H. S. Tan, J. High Energy Phys. **04**, 131 (2009).
[53] F. Fiorini and R. Ferraro, Int. J. Mod. Phys. A **24**, 1686 (2009).
[54] Z. Haghani, H. R. Sepangi, and S. Shahidi, Phys. Rev. D **83**, 064014 (2011).
[55] P. P. Avelino, Phys. Rev. D **85**, 104053 (2012).
[56] D. L. Wiltshire, Phys. Lett. B **169**, 36 (1986).
[57] D. L. Wiltshire, Phys. Rev. D **38**, 2445 (1988).
[58] M. H. Dehghani and S. H. Hendi, Int. J. Mod. Phys. D **16**, 1829 (2007).
[59] J. Magueijo and L. Smolin, Classical Quantum Gravity **21**, 1725 (2004).
[60] S. H. Hendi and M. Faizal, Phys. Rev. D **92**, 044027 (2015).
[61] S. H. Hendi, S. Panahiyan, B. Eslam Panah, M. Faizal, and M. Momennia, Phys. Rev. D **94**, 024028 (2016).
[62] E. Ayón-Beato, A. Garcia, Phys. Lett. B **464**, 25 (1999).
[63] V. A. De Lorenci, R. Klippert, M. Novello, and J. M. Salim, Phys. Rev. D **65**, 063501 (2002).
[64] C. Corda and H. J. M. Cuesta, Astropart. Phys. **34**, 587 (2011).
[65] C. Corda and H. J. M. Cuesta, Mod. Phys. Lett. A **25**, 2423 (2010).
[66] Y.-F. Cai and E. N. Saridakis, J. Cosmol. **17**, 7238 (2011).
[67] A. Awad, A. F. Ali, and B. Majumder, J. Cosmol. Astropart. Phys. **10**, 052 (2013).
[68] Y. Ling, J. Cosmol. Astropart. Phys. **08**, 017 (2007).
[69] G. Santos, G. Gubitosi, and G. Amelino-Camelia, J. Cosmol. Astropart. Phys. **08**, 005 (2015).
[70] S. H. Hendi, M. Momennia, B. Eslam Panah, and M. Faizal, Astrophys. J. **827**, 153 (2016).

[71] S. H. Hendi, M. Momennia, B. Eslam Panah, and S. Panahiyan, Phys. Dark Universe **16**, 26 (2017).
[72] B. Majumder, Int. J. Mod. Phys. D **22**, 1342021 (2013).
[73] M. Khodadi, K. Nozari, and H. R. Sepangi, Gen. Relat. Gravit. **48**, 166 (2016).
[74] A. F. Ali, M. Faizal, and M. M. Khalil, Phys. Lett. B **743**, 295 (2015).
[75] R. Garattini and E. N. Saridakis, Eur. Phys. J. C **75**, 343 (2015).
[76] N. Seiberg and E. Witten, J. High Energy Phys. **09**, 032 (1999).
[77] Y.-K. E. Cheung and M. Krogh, Nucl. Phys. B **528**, 185 (1998).
[78] G. Amelino-Camelia, Living Rev. Relat. **5**, 16 (2013).
[79] U. Jacob, F. Mercati, G. Amelino-Camelia, and T. Piran, Phys. Rev. D **82**, 084021 (2010).
[80] V. A. Kostelecký and S. Samuel, Phys. Rev. D **40**, 1886 (1989).
[81] D. Kastor, S. Ray, and J. Traschen, Classical Quantum Gravity **26**, 195011 (2009).
[82] M. Cvetič, G. W. Gibbons, D. Kubizňák, and C. N. Pope, Phys. Rev. D **84**, 024037 (2011).
[83] B. P. Dolan, Classical Quantum Gravity **28**, 125020 (2011).
[84] B. P. Dolan, Classical Quantum Gravity **28**, 235017 (2011).
[85] D. Kubizňák and R. B. Mann, J. High Energy Phys. **07**, 033 (2012).
[86] S. Gunasekaran, D. Kubizňák, and R. B. Mann, J. High Energy Phys. **11**, 110 (2012).
[87] R. Banerjee and D. Roychowdhury, Phys. Rev. D **85**, 104043 (2012).
[88] A. Lala and D. Roychowdhury, Phys. Rev. D **86**, 084027 (2012).
[89] R. Banerjee and D. Roychowdhury, Phys. Rev. D **85**, 044040 (2012).
[90] S. H. Hendi and M. H. Vahidinia, Phys. Rev. D **88**, 084045 (2013).
[91] D.-C. Zou, S.-J. Zhang, and B. Wang, Phys. Rev. D **89**, 044002 (2014).
[92] O. J. Rosten, Phys. Rep. **511**, 177 (2012).
[93] S. H. Hendi, J. High Energy Phys. **03**, 065 (2012).
[94] S. H. Hendi, Ann. Phys. (N.Y.) **333**, 282 (2013).
[95] J. D. Bekenstein, Lett. Nuovo Cimento **4**, 737 (1972).
[96] J. D. Bekenstein, Phys. Rev. D **7**, 2333 (1973).
[97] S. W. Hawking and C. J. Hunter, Phys. Rev. D **59**, 044025 (1999).
[98] S. W. Hawking, C. J. Hunter, and D. N. Page, Phys. Rev. D **59**, 044033 (1999).
[99] R. B. Mann, Phys. Rev. D **61**, 084013 (2000).
[100] M. Lu and M. B. Wise, Phys. Rev. D **47**, R3095(R) (1993).
[101] M. Visser, Phys. Rev. D **48**, 583 (1993).
[102] T. Jacobson and R. C. Myers, Phys. Rev. Lett. **70**, 3684 (1993).
[103] R. M. Wald, Phys. Rev. D **48**, R3427(R) (1993).
[104] M. Visser, Phys. Rev. D **48**, 5697 (1993).
[105] T. Jacobson, G. Kang, and R. C. Myers, Phys. Rev. D **49**, 6587 (1994).
[106] V. Iyer and R. M. Wald, Phys. Rev. D **50**, 846 (1994).
[107] J. P. Gauntlett, R. C. Myers, and P. K. Townsend, Classical Quantum Gravity **16**, 1 (1999).
[108] L. Brewin, Gen. Relat. Gravit. **39**, 521 (2007).
[109] J. W. York, Jr., Phys. Rev. D **33**, 2092 (1986).
[110] J. Stachel, Phys. Rev. D **26**, 1281 (1982).
[111] S. H. Hendi and A. Dehghani, Phys. Rev. D **91**, 064045 (2015).
[112] S. H. Hendi, S. Panahiyan, and M. Momennia, Int. J. Mod. Phys. D **25**, 1650063 (2016).
[113] J. Magueijo and L. Smolin, Phys. Rev. Lett. **88**, 190403 (2002).
[114] G. Amelino-Camelia, J. Ellis, N. E. Mavromatos, D. V. Nanopoulos, and S. Sarkar, Nature **393**, 763 (1998).