Mass quantization in quantum and susy cosmological models with matter content

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Abstract. We present the study of the quantum closed Friedmann-Robertson-Walker (FRW) cosmological model with a matter content given by a perfect fluid with barotropic state equation $p = \gamma \rho$. The Wheeler-DeWitt equation is viewed as the Schrödinger equation for the linear harmonic oscillator with energy E. Such type of Universe has quantized masses of the order of the Planck mass and harmonic oscillator wave functions. Then, we consider the $n = 2$ supersymmetric superfield approach for the same model and obtain a normalizable wave function (at zero energy) of the universe. Besides, the mass parameter spectrum is found in the Schrödinger picture, being similar to those obtained by other methods, using a black hole system.

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1. Introduction

The quantum solution of the FRW cosmological model has been calculated in many works [1, 2, 3, 4], but not related to mass quantization.

The main purpose of this work is to consider a time independent Schrödinger equation and its SUSY generalization to obtain a mass spectrum for the closed FRW model, in which dust matter is filling the universe, as well as the wave function of the FRW cosmological model in both formalisms. It was made following the canonical quantization procedure.

The rest of the paper is organized as follow. In sec. II, using the canonical formalism, we construct the correspondig Hamiltonian for the FRW cosmological model. In sec. III the time independent Schrödinger equation is obtained, promoting the classical Hamiltonian to operators, and applying it to the wave function ψ, $\hat{H}\psi = 0$. Here we introduce the quantization rules for the energy, which depends on an integer number n. These quantization rules were obtained using the creation-annihilation representation. In sec IV, the mass spectrum is calculated. In Sec. V, we present the SUSY version, obtaining the corresponding susyquantum solution in

Sec. VI with aid of the quantum constraints and the matrix representation. Also, we give the supersymmetric quantum conditions for the energy in the Schrödinger picture. Sec. VII is devoted to conclusions.

2. The canonical Hamiltonian

Starting with the FRW model we consider the classical lagrangian for a pure gravitating system and the corresponding term of matter content, perfect fluid with barotropic state equation $p = \gamma\rho$, and cosmological term [4]

$$L = -\frac{c^2 R}{2NG}\left(\frac{dR}{dt}\right)^2 + N\frac{\kappa c^4}{2G}R + N\frac{c^4\Lambda}{6G}R^3 - NM_\gamma c^2 R^{-3\gamma}. \tag{1}$$

In particular, we will consider the dust case $\gamma = 0$, with $\kappa = 1$ and $\Lambda = 0$, The action for this system has the form

$$S = \int\left[-\frac{c^2}{2GN}R\dot{R}^2 + \frac{c^4}{2G}NR - NE_s\right]dt, \tag{2}$$

with $E_s = Mc^2$, where M corresponds to the mass parameter of the closed Universe and dust scenario.

Note that if we take the lapse function as

$$N(t) = \tilde{N}(t)R(t)\frac{c^2}{M_{pl}G}, \tag{3}$$

we have an invariant action, obtaining the following canonical Hamiltonian using the usual scheme

$$H = \tilde{N}\left[-\frac{P_R^2}{2M_{pl}} - \frac{M_{pl}}{2}\omega_0^2\left(R - \frac{MG}{c^2}\right)^2 + \frac{M}{2M_{pl}}Mc^2\right], \tag{4}$$

with the fundamental frequency of the system $\omega_0 = \frac{c^3}{M_{pl}G}$. The lapse function $\tilde{N}(t)$ is a Lagrange multiplier, which enforces the first class constraint $H = 0$.

We transform Eq. (4) by defining $\xi = R - \frac{MG}{c^2}$, thus its momentum conjugate becomes $P_\xi = P_R$ and the constraint at the classical level reads as follows

$$H_{can} = \tilde{N}H = \tilde{N}\left[-\frac{P_\xi^2}{2M_{pl}} - \frac{M_{pl}}{2}\omega_0^2\xi^2 + \frac{M}{2M_{pl}}Mc^2\right] = 0. \tag{5}$$

3. Harmonic oscillator equation and quantization rules

Making the usual realization of the operator $\frac{\hat{P}_\xi^2}{2M_{pl}} = -\frac{\hbar^2}{2M_{pl}}\frac{d^2}{d\xi^2}$ and applying it to the wavefunction ψ, we get the following linear harmonic oscillator equation

$$\left[-\frac{\hbar^2}{2M_{pl}}\frac{d^2}{d\xi^2} + \frac{M_{pl}\omega_0^2}{2}\xi^2\right]\psi = \frac{M}{M_{pl}}\frac{E_s}{2}\psi. \tag{6}$$

In this point we make the transformation

$$E_s = \frac{c^4}{2G}R_{sup} \tag{7}$$

considering the form of E_s given in (2) we obtain that $R_{sup} = \frac{2MG}{c^2}$, being the radius for the closed universe. Making the transformation $\frac{\xi}{\ell_{pl}} = x$, one can rewrite (6) as

$$\frac{1}{2}\left[x^2 - \frac{d^2}{dx^2}\right]\psi = \frac{1}{4}\frac{R_{sup}E_s}{\ell_{pl}E_{pl}}\psi. \tag{8}$$

Using the creation-annihilation representation, with the usual algebra between them, $[a, a^\dagger] = 1$, we can rewrite Eq. (8) as

$$a^\dagger a\psi = \frac{1}{2}\left[x^2 - \frac{d^2}{dx^2}\right]\psi - \frac{1}{2}\psi = \left(-\frac{1}{2} + \frac{1}{4}\frac{R_{sup}E_s}{\ell_{pl}E_{pl}}\right)\psi = n\psi, \qquad n = 0, 1, 2, \cdots. \tag{9}$$

In this way, we have the following useful relations

$$R_{sup}E_s = 4(n + \tfrac{1}{2})\ell_{pl}E_{pl}$$
$$= 4(n + \tfrac{1}{2})\hbar c, \tag{10}$$
$$E_s^2 = 2(n + \tfrac{1}{2})E_{pl}^2, \tag{11}$$
$$\frac{E_s}{2} = (n + \tfrac{1}{2})\hbar\omega_0. \tag{12}$$

One can see that when n is big, we find

$$\frac{R_{sup}}{\ell_{pl}} = 2\sqrt{2n+1}, \tag{13}$$

such that, when $n \to \infty$, R_{sup} coincide with the maximum expansion of the scale factor R.

Let us write the equation (8) in the following form

$$\frac{d^2\psi}{dx^2} + (\alpha_n^2 - x^2)\psi = 0, \qquad \alpha_n = \frac{E_s}{E_{pl}} \tag{14}$$

where α_n is parameter associated with the energy of the nth eigenstate, the quantum solution is similar to the harmonic oscillator case

$$\psi_n(x) = \left(\frac{1}{\sqrt{\pi}n!2^n}\right)^{\frac{1}{2}} H_n(x)\, e^{-\frac{1}{2}x^2}, \tag{15}$$

with $H_n(x)$ the Hermite polynomials.

4. The discrete mass spectrum

Now, it is clear that the system, even in its lowest energy state $n = 0$, has a finite, minimal energy. Eq. (10) implies the following quantization mass rule

$$M_n = \sqrt{2n+1}\, M_{pl}. \tag{16}$$

We introduce the condition on the M_n parameter when $n \to \infty$. This parameter may be the classical mass parameter M_{sup}, for the closed universe, filled with dust matter, in the maximum expansion.

These results are similar to those obtained by other methods in the black hole scenario, [5, 6, 7, 8, 9].

The difference in mass between any two consecutive eigenvalues is given by

$$\Delta M_{n+1} \equiv M_{n+1} - M_n = \left[\sqrt{1 + \frac{2}{2n+1}} - 1\right]M_n \qquad n\,finite \tag{17}$$
$$= 0 \qquad n \to \infty,$$

the result when $n \to \infty$ is in agreement with the correspondence principle.

5. The classical supersymmetric lagrangian

We will proceed with the superfield formulation of the action (2). For this purpose we need to generalize the local time transformations $t \rightarrow (t, \eta, \bar{\eta})$ in the following way

$$
\begin{aligned}
\delta t &= a(t) + \frac{i}{2}(\eta \bar{\beta}'(t) + \bar{\eta} \bar{\beta}'), \\
\delta \eta &= \frac{1}{2}\bar{\beta}'(t) + \frac{1}{2}(\dot{a}(t) + ib(t))\eta + \frac{i}{2}\dot{\bar{\beta}}'\eta\bar{\eta}, \\
\delta \bar{\eta} &= \frac{1}{2}\beta'(t) + \frac{1}{2}(\dot{a}(t) - ib(t))\bar{\eta} - \frac{i}{2}\dot{\beta}'\eta\bar{\eta},
\end{aligned}
\tag{18}
$$

where η is a complex odd parameter (η odd "time" coordinates), $\beta(t)$ is the Grassmann complex parameter of the local "small" $n = 2$ supersymmetry (SUSY) transformation, and $b(t)$ is the parameter of local $U(1)$ rotations of the complex η. Then, the superfield generalization of action (2), which is invariant under the local $n = 2$ supersymmetry transformation (18) has the form

$$
S = \int \Big(-\frac{c^2}{2G} I\!N^{-1} I\!R D_{\bar{\eta}} I\!R D_{\eta} I\!R + \frac{c^3\sqrt{\kappa}}{2G} I\!R^2 - Mc I\!R \Big) d\eta d\bar{\eta} dt,
\tag{19}
$$

where

$$
D_{\eta} = \frac{\partial}{\partial \eta} + i\bar{\eta}\frac{\partial}{\partial t}, \qquad D_{\bar{\eta}} = -\frac{\partial}{\partial \bar{\eta}} - i\eta\frac{\partial}{\partial t}
\tag{20}
$$

are the supercovariant derivatives of the global "small" supersymmetry of the generalized parameter corresponding to t.

The Taylor series expansion for the superfields $I\!N(t, \eta, \bar{\eta})$ and $I\!R(t, \eta, \bar{\eta})$ are the following

$$
\begin{aligned}
I\!N(t, \eta, \bar{\eta}) &= N(t) + i\eta\bar{\psi}'(t) + i\bar{\eta}\psi'(t) + V'(t)\eta\bar{\eta}, \\
I\!R(t, \eta, \bar{\eta}) &= R(t) + i\eta\bar{\lambda}'(t) + i\bar{\eta}\lambda'(t) + B'(t)\eta\bar{\eta}.
\end{aligned}
\tag{21}
\tag{22}
$$

The components of the superfield $I\!N(t, \eta, \bar{\eta})$ are gauge fields of the one-dimensional $n = 2$ extended supergravity. $N(t)$ is the einbein, $\psi(t)$ and $\bar{\psi}(t)$ are the complex gravitino fields, and $V(t)$ is the $U(1)$ gauge field. The component $B(t)$ in (22) is an auxiliary degree of freedom (non-dynamical variable), and $\lambda, \bar{\lambda}$ are the fermion partners of the scale factor $R(t)$.

We get the Lagrangian only in terms of dynamical fields,

$$
\begin{aligned}
L &= -\frac{c^2}{2G}\frac{R(DR)^2}{N} + \frac{i}{2}(\bar{\lambda}D\lambda - D\bar{\lambda}\lambda) + N\Big[-\sqrt{\kappa}Mc^2 + \frac{\kappa c^4}{2G}R + \frac{1}{2}\frac{GM^2}{R} \\
&\quad + \frac{\sqrt{\kappa}c}{2}\frac{\bar{\lambda}\lambda}{R} + \frac{GM}{2c}\frac{\bar{\lambda}\lambda}{R^2}\Big] + \frac{c^2\sqrt{\kappa}R^{1/2}}{2G^{1/2}}(\bar{\psi}\lambda - \psi\bar{\lambda}) - \frac{MG^{1/2}}{2R^{1/2}}(\bar{\psi}\lambda - \psi\bar{\lambda}),
\end{aligned}
\tag{23}
$$

where $DR = \dot{R} - \frac{iG^{1/2}}{2cR^{1/2}}(\psi\bar{\lambda} + \bar{\psi}\lambda)$, $D\lambda = \dot{\lambda} - \frac{i}{2}V\lambda$ and its complex conjugate $D\bar{\lambda} = \dot{\bar{\lambda}} + \frac{i}{2}V\bar{\lambda}$.

The classical canonical Hamiltonian is calculated in the usual way for systems with constraints,

$$
H_c = NH + \frac{i}{2}\bar{\psi}S - \frac{i}{2}\psi\bar{S} + \frac{1}{2}VF,
\tag{24}
$$

where H is the Hamiltonian of the system, S and $\bar{S}$ are the supercharges and F is the $U(1)$ rotation generator. The form of the canonical Hamiltonian (24) explains the fact that $N, \psi, \bar{\psi}$ and V are Lagrangian multipliers enforcing the first-class constraints $H = 0, S = 0, \bar{S} = 0$ and $F = 0$, expressing the invariance under the conformal $n = 2$ supersymmetric transformations. We obtain the first-class constraints

$$H = -\frac{G}{2c^2R}\pi_R^2 - \frac{\kappa c^4 R}{2G} - \frac{M^2 G}{2R} + c^2 M\sqrt{\kappa} - \frac{c\sqrt{\kappa}}{2R}\bar{\lambda}\lambda - \frac{MG}{2cR^2}\bar{\lambda}\lambda, \tag{25}$$

$$S = \Big(\frac{iG^{1/2}}{cR^{1/2}}\pi_R - \frac{c^2\sqrt{\kappa}R^{1/2}}{G^{1/2}} + \frac{MG^{1/2}}{R^{1/2}}\Big)\lambda, \tag{26}$$

$$\bar{S} = \Big(\frac{iG^{1/2}}{cR^{1/2}}\pi_R + \frac{c^2\sqrt{\kappa}R^{1/2}}{G^{1/2}} - \frac{MG^{1/2}}{R^{1/2}}\Big)\bar{\lambda}, \tag{27}$$

$$F = -\lambda\bar{\lambda}, \tag{28}$$

where $\pi_R = -\frac{c^2 R}{GN}\dot{R} + \frac{icR^{1/2}}{2NG^{1/2}}(\bar{\psi}\lambda + \psi\bar{\lambda})$ is the canonical momentum associated to R.

The canonical Dirac brackets are defined as

$$\{R, \pi_R\} = 1, \qquad \{\lambda, \bar{\lambda}\} = i, \tag{29}$$

respect to these brackets, the super-algebra for the quantum generators $H, S, \bar{S}$ and F becomes

$$\{S, \bar{S}\} = -2iH, \qquad \{S, H\} = \{\bar{S}, H\} = 0, \qquad \{F, S\} = iS, \qquad \{F, \bar{S}\} = i\bar{S}. \tag{30}$$

We can choose the following matrix representation for the fermionic parameters λ, $\bar{\lambda}$ and ξ as

$$\lambda = \sqrt{\hbar}\sigma_-, \qquad \bar{\lambda} = -\sqrt{\hbar}\sigma_+, \qquad \xi = \sigma_3, \tag{31}$$

with $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm \sigma_2)$, where σ_1, σ_2, σ_3 are the Pauli matrices.

6. Superquantum solutions

In the quantum theory the first-class constraints $H = 0$, $S = 0$, $\bar{S} = 0$ and $F = 0$, associated with the invariant action under the $n = 2$ local conformal supersymmetry become conditions on the wave function $\Psi(R)$. Furthermore, any physical state must satisfy the quantum constraints

$$H\,\Psi(R) = 0, \qquad S\,\Psi(R) = 0, \qquad \bar{S}\,\Psi(R) = 0, \qquad F\,\Psi(R) = 0, \tag{32}$$

where the first equation in (32) is the Wheeler-DeWitt equation for the minisuperspace model. The eigenstates have two components in the matrix representation (31)

$$\Psi = \left(\begin{array}{c} \Psi_1 \\ \Psi_2 \end{array}\right).$$

However, the supersymmetric physical states are obtained applying the supercharges operators $S\Psi = 0$, $\bar{S}\,\Psi = 0$. With the conformal algebra given by (30), these are rewritten in the following form

$$(\lambda\bar{S} - \bar{\lambda}S)\,\Psi = 0. \tag{33}$$

We use the matrix representation for λ and $\bar{\lambda}$ to solve (33), obtaining the differential equation for the $\Psi_1(R)$ and $\Psi_2(R)$ components

$$\left(-\frac{\hbar G^{1/2}}{c}R^{-1/2}\frac{\partial}{\partial R} - \frac{\sqrt{\kappa}c^2}{G^{1/2}}R^{1/2} + G^{1/2}MR^{-1/2}\right)\Psi_1 = 0, \tag{34}$$

$$\left(-\frac{\hbar G^{1/2}}{c}R^{-1/2}\frac{\partial}{\partial R} + \frac{\sqrt{\kappa}c^2}{G^{1/2}}R^{1/2} - G^{1/2}MR^{1/2}\right)\Psi_2 = 0 \tag{35}$$

whose solutions are

$$\Psi_1 \quad = C\, Exp\left[-\frac{\sqrt{\kappa}c^3}{2G\hbar}R^2 + \frac{Mc}{\hbar}R\right], \tag{36}$$

$$\Psi_2 \quad = \tilde{C}\, Exp\left[\frac{\sqrt{\kappa}c^3}{2G\hbar}R^2 - \frac{Mc}{\hbar}R\right]. \tag{37}$$

At this point, from (36) we can see that Ψ_1 has the right behaviour when $R \to R_{sup}$, where R_{sup} is the maximum radius for the closed universe, whereas Ψ_2 does not behave properly. Thus, there exist a normalizable component Ψ_1 for H, where this eigenstate corresponds to the state with eigenvalue $E = 0$.

Using (36), we obtain the wave function for closed universe ($\kappa = 1$), with mass M_0,

$$\Psi_1 = C\, Exp\left[-2n\frac{R}{R_{sup}}\left(\frac{R}{R_{sup}} - 1\right)\right]. \tag{38}$$

On the other hand, following the reference [10], we can view in the Schrödinger picture, that the Wheeler-DeWitt equation (25) is transformed into the bosonic and fermionic oscillators system. Thus, we have the supersymmetric quantum conditions for the energy parameter:

$$E_s^2 \quad = \quad nE_{pl}^2, \qquad n = 0, 1, 2, .. \tag{39}$$

$$E_s \quad = \quad \sqrt{n}E_{pl}, \tag{40}$$

$$E_s\, R_{sup} \quad = \quad 2n\hbar c. \tag{41}$$

Using Eq. (40), we have the quantization rule for the mass parameter of the universe in the dust model,

$$M_n = \sqrt{n}M_{pl}. \tag{42}$$

The difference in mass between any two consecutive eigenvalues is given by

$$\Delta M_{n+1} \equiv M_{n+1} - M_n \quad = \quad \left[\sqrt{1 + \frac{1}{n}} - 1\right]M_n \qquad n\, finite \tag{43}$$

$$= \quad 0 \qquad n \to \infty,$$

the result when $n \to \infty$ is in agreement with the correspondence principle, being the classical mass when the scale factor $R = R_{sup}$.

7. Conclusions

The main result of this work is that the universes of this type have a quantized mass of the order of the Planck mass $M_{pl} = 2.18 \times 10^{-8}Kg$. The mass spectrum does have corrections in the standard quantum cosmology, but in the susy approach does not, see Eqs. (16,42). In both approaches, the corresponding solutions were found.

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