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Contents

List of Figures **vi**

Abstract **vii**

Dedication **viii**

Acknowledgments **ix**

I The Holographic Principle **1**

1 Introduction **2**

2 Entropy Bounds for Black Holes **6**
2.1 Black Holes Thermodynamics . . . 6
2.1.1 Area Theorem . . . 7
2.1.2 No-hair Theorem . . . 7
2.2 Bekenstein Entropy and the Generalized Second Law . . . 8
2.2.1 Hawking Radiation . . . 10
2.2.2 Bekenstein Bound: Geroch Process . . . 12
2.2.3 Spherical Entropy Bound: Susskind Process . . . 12
2.2.4 Relation to the Bekenstein Bound . . . 13

3 Degrees of Freedom and Entropy **15**
3.1 Degrees of Freedom . . . 15
3.1.1 Fundamental System . . . 16
3.2 Complexity According to Local Field Theory . . . 16
3.3 Complexity According to the Spherical Entropy Bound . . . 18
3.4 Why Local Field Theory Gives the Wrong Answer . . . 19

4 The Covariant Entropy Bound **20**
4.1 Light-Sheets . . . 20

4.1.1 The Raychaudhuri Equation 20
4.1.2 Orthogonal Null Hypersurfaces 24
4.1.3 Light-sheet Selection 26
4.1.4 Light-sheet Termination 28
4.2 Entropy on a Light-Sheet 29
4.3 Formulation of the Covariant Entropy Bound 30

5 Quantum Field Theory in Curved Spacetime 32
5.1 Scalar Field Quantization 32
5.2 Bogolubov Coefficients 35
5.3 Green Functions 36

II Publications 39

6 De Sitter Vacua, Renormalization and Locality 40
6.1 Introduction 40
6.2 Interacting Scalar Field Theory in an α Vacuum 44
6.2.1 Notation 44
6.2.2 Computation 46
6.3 The Wave Functional in the Antipodal Vacuum 53
6.4 Conclusion 56

7 Microscopic Quantum Mechanics of the $p = \rho$ Universe 58
7.1 Introduction 58
7.2 Local framework for a holographic theory of quantum gravity 61
7.2.1 The hilbert space of an observer 65
7.2.2 SUSY and the holoscreens: the degrees of freedom of quantum gravity 71
7.2.3 Rotation invariance 74
7.3 Quantum cosmology of a dense black hole fluid 75
7.3.1 The random operator ansatz 75
7.3.2 Homogeneity, isotropy and flatness 80
7.3.3 Time dependence - scaling laws 82
7.3.4 Time dependence: a consistency relation, and a failure 84
7.4 More general space-times 88
7.5 Discussion 92
7.6 Appendix 97
7.6.1 Intersection of causal diamonds 97
7.6.2 Holographic relations in a general FRW cosmology 102
7.6.3 Computation of c_e from geometry and constant in front of H_N . 108
7.7 Figures 111

8 Infrared Divergences in dS/CFT **120**
8.1 **Introduction** 120
8.2 Review of dS/CFT 123
8.3 Review of QFT in dS space 125
8.3.1 Coordinate Systems 126
8.3.2 Geodesic Distance 127
8.3.3 The Cut-off Prescription 128
8.3.4 Wave Function of the Universe 130
8.3.5 Representations of the dS^d Group 132
8.4 Scalar Green Functions 133
8.4.1 Maximally Symmetric Bitensors 133
8.4.2 Bulk Two-Point Function 135
8.4.3 Bulk to Boundary Propagators: dS/AdS 139
8.4.4 Boundary Two-point Function: dS/AdS 140
8.5 General Structure of the Computation 140
8.6 Models 144
8.7 1-loop Computation: Scalar Fields with Cubic Interaction 145
8.8 1-loop Computation: Scalar Fields with Derivative Coupling 148
8.9 1-loop Computation: Spinor Field with Derivative Coupling 150
8.10 Analysis Divergences 153
8.10.1 Three Massive Scalar Fields with Cubic Interaction in dS^d 153
8.10.2 Two Massive and One Massless field in dS^d 154
8.11 The Meaning of the Divergences 155
8.12 Generalization to a Model with Gravity 157
8.13 Appendix: Comparison with AdS 159
8.13.1 Three Scalar Fields AdS 159
8.14 Appendix: Spinor Green Functions 161
8.14.1 Spinor Parallel Propagator 161
8.14.2 Bulk Two-Point Function 162
8.14.3 Bulk to Boundary Propagators: dS/AdS 164

Bibliography **166**

List of Figures

4.1 Lightsheets . . . 21
4.2 Caustic . . . 25

7.1 Causal diamonds defining one observer . . . 112
7.2 Causal diamonds defining a pair of observers . . . 113
7.3 The maximal causal diamond . . . 114
7.4 Intersection of base spheres in 3D . . . 115
7.5 Intersection of base spheres in 2D . . . 116
7.6 Intersection of causal diamonds in 2D . . . 117
7.7 Detail of the the maximal causal diamond 1 . . . 118
7.8 Detail of the the maximal causal diamond 2 . . . 119

8.1 Penrose Diagram: Global coordinates . . . 129
8.2 Penrose Diagram: Flat coordinates . . . 129
8.3 Tree diagram . . . 141
8.4 1-loop diagram . . . 141

Abstract

Quantum Gravity and Cosmology

by

Lorenzo Mannelli

The main theme of this Thesis is the connection among Quantum Gravity and Cosmology. In the First Part (Chapters 1 to 5) I give an introduction to the Holographic Principle. The Second Part is a collection of my research work and it is articulated as follows. Chapter 7 is dedicated to analyze the renormalization properties of quantum field theories in de Sitter space. It is shown that only two of the maximally invariant vacuum states of free fields lead to consistent perturbation expansions. Chapter 8 first present a complete quantum mechanical description of a flat FRW universe with equation of state $p = \rho$. Then show a detailed correspondence with our heuristic picture of such a universe as a dense black hole fluid. In the end it is explained how features of the geometry are derived from purely quantum input. Chapter 9 study the problem of infrared renormalization of particle masses in de Sitter space. It is shown, in a toy model in which the graviton is replaced with a minimally coupled massless scalar field, that loop corrections to these masses are infrared (IR) divergent. It is argued that this implies anomalous dependence of masses on the cosmological constant, in a true theory of quantum gravity.

To My Family,

for their constant support and encouragement.

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Part I

The Holographic Principle

Chapter 1

Introduction

The main theme of this Thesis is the connection among Quantum Gravity and Cosmology and it mostly contain the research work that I have been doing in this field.

The quantization of gravity is probably the most outstanding unsolved problem in theoretical Physics. It has been studied for more than 70 years, nonetheless we haven't been able yet to find a complete formulation of a Quantum Theory of Gravity (QTG). The most promising candidate in this direction is String Theory.

Even if we don't know what the QTG is going to be there are strong evidence that must be "Holographic". The word Holography in this contest refer to a property of the fundamental degrees of freedom of the theory. In a QTG these are not extensive (growing with volume) as we would expect from a local Field Theory description. They rather scale like the area of a surface.

This surprising property has been first discovered studying Black Holes. In this case the degrees of freedom (dof) describing the black hole grow as the area A of the event horizon. More precisely the dof are counted by the entropy which satisfy the relation

$$S = \frac{A}{4}$$

in Planck units.

The generalization and study of this property in a QTG has initially been carried on by t' Hooft and Susskind. More recently a completely covariant formulation of the "Holographic Principle" has been given by Bousso.

String Theory being a QTG give us a description of the fundamental degrees of freedom of Nature. In the String Theory examples where has been explicitly possible to count the dof we have seen that they are Holographic in nature i.e. they scale as an area. The most remarkable among these examples is probably AdS/CFT.

In conclusion even if we still don't have a fundamental theory of nature we strongly believe that Holography is going to be one of its main feature.

The ideas of the Holographic Principle can be applied to the study of cosmology. This has been one of the dominant topics of my research in these years.

In the First part of this Thesis I give an introduction to the Holographic Principle. I mostly refer to the excellent review of Bousso [1]. The Second Part of the Thesis is a collection of the research work I published during the years of my doctorate. I will now briefly summarize the content of these articles.

In the first paper "De Sitter Vacua, Renormalization and Locality", written in collaboration with Tom Banks, we analyze the renormalization properties of quantum field theories in de Sitter space and show that only two of the maximally invariant vacuum states of free fields lead to consistent perturbation expansions. One is the Euclidean vacuum and the other can be viewed as an analytic continuation of Euclidean functional integrals on RP^d. The corresponding Lorentzian manifold is the future half of global de Sitter space with boundary conditions on fields at the origin of time. We argue that the perturbation series in this case has divergences at the origin which render the future evolution of the system indeterminate, without a better understanding of high energy physics.

In the second paper "Microscopic Quantum Mechanics of the $p = \rho$ Universe", written in collaboration with Tom Banks and Willy Fischler, we first present a complete quantum mechanical description of a flat FRW universe with equation of state $p = \rho$. We then find a detailed correspondence with our heuristic picture of such a universe as a dense black hole fluid. Finally we show how features of the geometry are derived from purely quantum input.

In the last paper "Infrared Divergences in dS/CFT", written in collaboration with Tom Banks and Willy Fischler, we study the problem of infrared renormalization of particle masses in de Sitter space. We use the conjectured dS/CFT correspondence introduced by Strominger and collaborators. dS/CFT gives a perturbatively gauge invariant definition of particle masses in de Sitter (dS) space. We show, in a toy model in which the graviton is replaced with a minimally coupled massless scalar field, that loop corrections to these masses are infrared (IR) divergent. We argue that this implies anomalous dependence of masses on the cosmological constant, in a true theory of quantum gravity. This is in accord with the hypothesis of Cosmological SUSY Breaking

(CSB).

Chapter 2

Entropy Bounds for Black Holes

In this section we discuss black hole entropy and some of the entropy bounds that have been derived from it.

The entropy bounds discussed in this section are independent of the specific characteristics and composition of matter systems. However, they apply only when gravity is weak.

2.1 Black Holes Thermodynamics

The notion of black hole entropy is motivated by two results in general relativity.

2.1.1 Area Theorem

The *area theorem* (Hawking [2]) states that: *during the evolution of a black hole the area of the event horizon never decreases with time:*

$$dA \geq 0 \tag{2.1}$$

Moreover, if two black holes merge, the horizon area of the new black hole will be greater than the total horizons area of the original black holes.

Consider for example an object falling into a Schwarzschild black hole this will increase the mass of the black hole, M. Hence the horizon area, $A = 16\pi M^2$ in $D = 4$, increases. On the other hand in any classical process the black hole cannot emit particles and so the mass and the area cannot decrease.

The theorem suggests an analogy between black hole area and thermodynamic entropy.

2.1.2 No-hair Theorem

The *no-hair theorem* affirm that [2], [3], [4], [5]: *A stationary black hole is characterized by only three quantities: mass, angular momentum, and charge.*

Consider a matter system, such as a star, that collapses to form a black hole. The black hole will eventually evolve into a final, stationary state. The no-hair theorem implies that this state is unique.

From the point of view of an outside observer, the formation of a black hole appears to violate the second law of thermodynamics. The collapsing system may have arbitrarily large entropy, but the final state has none at all. Different initial conditions will lead to indistinguishable results.

A similar problem arises when a matter system is dropped into an existing black hole.

2.2 Bekenstein Entropy and the Generalized Second Law

Thus, following the no-hair theorem, seems like when an object fall into a black hole his entropy disappear, on the other side the area of the black hole event horizon increase. This strongly suggest that we may associate the entropy of a black hole with its area.

Based on this reasoning, Bekenstein [6],[7],[8] suggested that a black hole actually has an entropy equal to its horizon area, $S_{\rm BH} = \eta A$, where η is a number of order unity. We will show in Sec. 2.2.1 that $\eta = \frac{1}{4}$

$$S_{\rm BH} = \frac{A}{4}. \tag{2.2}$$

[In full, $S_{\rm BH} = kAc^3/(4G\hbar)$.] The entropy of a black hole is given by a quarter of the area of its horizon in Planck units. In ordinary units, it is the horizon area divided by about $10^{-69}\mathrm{m}^2$.

Furthermore, Bekenstein [6],[7],[8] proposed that the second law of thermodynamics it is valid only for the *sum* of black hole entropy and matter entropy

$$dS_{\rm total} \geq 0. \tag{2.3}$$

where $S_{\text{total}} = S_{\text{matter}} + S_{\text{BH}}$

In other words for ordinary matter systems alone, the second law need not to be valid. But if the entropy of black holes, Eq. (2.2), is included in the balance, the total entropy will never decrease. This is referred to as the *generalized second law* or *GSL*. More precisely we can state the *generalized second law* as follows.

Consider a thermodynamic system $\mathcal{T}$, consisting of well-separated, non-interacting components. Label the components of the system made of ordinary matter as $\mathcal{C}_i$ and their entropy with $S(\mathcal{C}_i)$. On the other side we will label the black holes as $\mathcal{B}_j$ and their horizon areas as A_j.

The total entropy of $\mathcal{T}$ is given by

$$S_{\text{total}}^{\text{initial}} = S_{\text{matter}} + S_{\text{BH}}. \tag{2.4}$$

Here, $S_{\text{matter}} = \sum S(\mathcal{C}_i)$ is the total entropy of all ordinary matter. $S_{\text{BH}} = \sum \frac{A_j}{4}$ is the total entropy of all black holes present in $\mathcal{T}$.

Now let the components of $\mathcal{T}$ to interact until a new equilibrium is established. At the end of the evolution, the system $\mathcal{T}$ will consist of a new set of components $\hat{\mathcal{C}}_i$ and $\hat{\mathcal{B}}_j$, which total entropy is $S_{\text{total}}^{\text{final}}$. The *GSL* states that

$$S_{\text{total}}^{\text{final}} \geq S_{\text{total}}^{\text{initial}}. \tag{2.5}$$

What are the microstates counting the entropy of a black hole is still an open question, the answer will most likely require the knowledge of a fundamental theory of quantum gravity.

However, one result stands out because of its quantitative accuracy. Recent developments in string theory have led to description of limited classes of black holes in which the microstates can be identified and counted (Strominger and Vafa [9]).

The formula $S = A/4$ was precisely confirmed by this calculation.

2.2.1 Hawking Radiation

Purely from General Relativity we can infer the following relation, sometimes called "first law of black hole mechanics", among mass M surface gravity κ and area A of a black hole

$$dM = \frac{\kappa}{8\pi} dA. \tag{2.6}$$

for a definition of κ, see Wald [10]; e.g., a Schwarzschild black hole in $D = 4$ has $\kappa = (4M)^{-1}$.

If we interpretate κ as the temperature and A as the entropy of the black hole (as explained in the previous section) Eq. (2.6) reproduce the first law of thermodynamic for a black hole

$$dM = T dS_{\mathrm{BH}}. \tag{2.7}$$

A further confirmation that is correct to identify κ with the temperature of a black hole has been given by Hawking [11],[12], he showed with a semi-classical calculation that black holes do in fact radiate via a quantum process. A distant observer will detect a thermal spectrum of particles coming from the black hole, at a temperature

$$T = \frac{\kappa}{2\pi}. \tag{2.8}$$

For a Schwarzschild black hole in $D = 4$, this temperature is $\hbar c^3/(8\pi GkM)$, or about 10^{26} Kelvin divided by the mass of the black hole in grams. Note that such black holes have negative specific heat.

The discovery of Hawking radiation clarified and further strength the interpretation of the thermodynamic description of black holes.

In particular, Hawking's result affirmed that the entropy of black holes should be considered a genuine contribution to the total entropy content of the universe, as Bekenstein [6],[7],[8] had anticipated.

Via the first law of thermodynamics, Eq. (2.8), Hawking's calculation fixes the coefficient η in the Bekenstein entropy formula, Eq. (2.2), to be 1/4.

A radiating black hole loses mass, shrinks, and eventually disappears unless it is stabilized by charge or a steady influx of energy. Over a long time of order $M^{\frac{D-1}{D-3}}$, this process converts the black hole into a cloud of radiation.

In the following we will study two processes that will help to clarify the GSL and establish bounds on the entropy of matter systems.

First we will discuss the case in which a matter system is dropped into an existing black hole. Then we will turn to the process in which a black hole is formed by the collapse of ordinary matter. In both cases, ordinary entropy is converted into horizon entropy.

2.2.2 Bekenstein Bound: Geroch Process

Consider a weakly gravitating stable thermodynamic system of total energy E. Let R be the radius of the smallest sphere enclosing the system. To obtain an entropy bound the strategy is to move the system from infinity into a Schwarzschild black hole of radius b much larger than R. To optimize the tightness of the entropy bound we want to add as little energy as possible to the black hole. So we imagine to move the system slowly until is right outside the event horizon and then dropping it inside the black hole.

The mass added to the black hole is given by the energy E of the system, redshifted according to the position of the center of mass at the drop-off point. It can be easily seen that the entropy added to the black hole is

$$\delta S_{\rm BH} = \frac{dS_{\rm BH}}{dM}\,\delta M \leq 2\pi E R.$$

By the generalized second law, the entropy cannot decrease in the process: $\delta S_{\rm BH} - S_{\rm matter} \geq 0$. Hence,

$$S_{\rm matter} \leq 2\pi E R. \tag{2.9}$$

2.2.3 Spherical Entropy Bound: Susskind Process

Let us consider an isolated matter system of mass E and entropy $S_{\rm matter}$ located in a spacetime $\mathcal{M}$. Moreover let us assume that $\mathcal{M}$ is asymptotically flat. We define A to be the area of the circumscribing sphere, i.e., the smallest sphere that fits around the system. Note that A is well-defined only if the metric near the system is spherically symmetric or gravity is weak.

Let us further assume that the matter system is stable on a timescale much greater than $A^{1/2}$, so that the time-dependence of A will be negligible.

The mass of the system must be less than the mass M of a black hole of the same surface area. Otherwise, the system could not be gravitationally stable, and would collapse in a black hole. The system can be converted into a black hole of area A by collapsing a shell of mass $M - E$ onto the system. Let the shell of entropy S_{shell} be well separated from the black hole, The initial entropy is

$$S_{\text{total}}^{\text{initial}} = S_{\text{matter}} + S_{\text{shell}}.$$

The final state is a black hole of entropy

$$S_{\text{total}}^{\text{final}} = S_{\text{BH}} = \frac{A}{4}.$$

Following the generalized second law of thermodynamic the entropy cannot decrease in the process, thus we arrive at the spherical entropy bound

$$S_{\text{matter}} \leq \frac{A}{4}, \tag{2.10}$$

2.2.4 Relation to the Bekenstein Bound

The spherical entropy bound can be derived from the Bekenstein bound, if the latter is assumed to be valid for strongly gravitating system. The requirement that the system be gravitationally stable implies $2M \leq R$ in four dimensions. From Eq. (2.9), one thus obtains:

$$S \leq 2\pi MR \leq \pi R^2 = \frac{A}{4}. \tag{2.11}$$

We see that the spherical entropy bound is weaker than the Bekenstein bound, in situations where both can be applied.

However as we will see in the following the spherical entropy bound is more suited to a covariant generalization.

Chapter 3

Degrees of Freedom and Entropy

3.1 Degrees of Freedom

The Holographic principle is a statement about the number of degrees of freedom (dof) of a fundamental system. It affirm that the number of degrees of a system scale with the area.

In order to make this statement precise in these Sections we will define the concept of degrees of freedom and fundamental system. Moreover we will show why local field theories do not count correctly the number of degrees of freedom. In the following we will restrict us to a finite spherical region of volume V and boundary area A. Assume, for now, that gravity is weak so that these quantities are well defined, and that spacetime is asymptotically flat.

Let us define the *number of degrees of freedom* of a quantum-mechanical system, N, to be the logarithm of the dimension $\mathcal{N}$ of its Hilbert space $\mathcal{H}$:

$$N = \ln \mathcal{N} = \ln \dim(\mathcal{H}). \tag{3.1}$$

Note that a harmonic oscillator has $N = \infty$ with this definition. The number of degrees of freedom is equal (up to a factor of $\ln 2$) to the number of bits of information needed to characterize a state. For example, a system with 100 spins has $\mathcal{N} = 2^{100}$ states, $N = 100 \ln 2$ degrees of freedom, and can store 100 bits of information.

3.1.1 Fundamental System

By "fundamental system" we mean the description of a system in term of a fundamental theory of nature.

We assume here and in the following that this fundamental theory of nature will admit a description in term of Hilbert spaces and so it is appropriate to talk about degrees of freedom as we defined them in the previous section.

We yet don't know what the fundamental theory of nature will be (and if there will be one !) but that best candidate that we have at today i.e. String Theory (from the point of view of the author !) admit such a description in term of Hilbert spaces.

3.2 Complexity According to Local Field Theory

Let us assume here that the fundamental theory is quantum field theory in a fixed background. We will give a rough estimate of the number of degrees of freedom and then compare with the counting of degrees of freedom given by the entropy bound.

Quantum field theory consist of one or more oscillator at every point in space-time so apparently we have an infinite number of degrees of freedom. However in this reasoning we have disregarded the effects of gravity altogether.

If we introduce gravity in a crude way we may expect that we cannot resolve distances smaller than the Planck length, $l_{\mathrm{P}} = 1.6 \times 10^{-33}$cm. So we will discretize the space into a Plank grid and assume that there is an oscillator per Plank volume.

Moreover, the oscillator spectrum is discrete and bounded from below by finite volume effects. It is bounded from above because it must be cut off at the Planck energy, $M_{\mathrm{P}} = 1.3 \times 10^{19}$GeV. This is the largest amount of energy that can be localized to a Planck cube without producing a black hole. Thus, the total number of oscillators is V (in Planck units), and each has a finite number of states, n. Hence, the total number of independent quantum states in the specified region is

$$\mathcal{N} \sim n^V.$$

The number of degrees of freedom is given by

$$N \sim V \ln n \gtrsim V.$$

This result is in agreement with our prejudice that the degrees of freedom in the world are local in space, and that, therefore, grows with volume. It turns out, however, that this view conflicts with the entropy bound.

3.3 Complexity According to the Spherical Entropy Bound

Thermodynamic entropy has a statistical interpretation. Let S be the thermodynamic entropy of an isolated system at some specified value of macroscopic parameters such as energy and volume. Then $\mathcal{N} = e^S$ is the number of independent quantum states corresponding to these macroscopic parameters.

In Section 2.2.3 we derived the spherical entropy bound. It states that the entropy of the fundamental system is bounded by its area

$$S \leq \frac{A}{4},$$

In particular Black Holes saturate the bound

$$S_{\mathrm{BH}} = \frac{A}{4},$$

Therefore, the number of degrees of freedom in a region bounded by a sphere of area A is given by

$$N = \frac{A}{4};$$

on the other hand the number of states is

$$\mathcal{N} = e^{A/4}. \tag{3.2}$$

We assume that all physical systems are larger than the Planck scale. Hence, their volume will exceed their surface area, in Planck units. (For a proton, the volume is larger than the area by a factor of 10^{20}; for the earth, by 10^{41}).

The result obtained from the spherical entropy bound is thus in contradiction with the much larger number of degrees of freedom estimated from local field theory.

Which of the two conclusions should we believe?

3.4 Why Local Field Theory Gives the Wrong Answer

We will now show that the Quantum field theory computation overcounted the number of degrees of freedom, because we didn't appropriately take in account the effects of gravitation. We assume in the following $D = 4$ and neglect factors of order unity.

A spherical surface cannot contain more mass than a black hole of the same area. According to the Schwarzchild solution the mass of a black hole is given by its radius. Hence, the mass M contained within a sphere of radius R satisfy

$$M \lesssim R. \tag{3.3}$$

Previously, in Section 3.2 we imposed the same bound but only for a Planck cell i.e. a sphere of radius ($R = 1$).

For a larger region this cutoff would allow $M \sim R^3$, in contradiction with Eq. (3.3). In other worlds we didn't take into account that we would form a black hole well before exciting all the degrees of freedom allowed by quantum field theory.

This resolve the mismatch in the counting of degrees of freedom among field theory and the holographic bound.

Because of gravity not all the degrees of freedom allowed by field theory can be excited.

Chapter 4

The Covariant Entropy Bound

The aim of this and the following Sections is to give a covariant generalization of the spherical entropy bound (2.10) that we will call *the covariant entropy bound.* This generalization will find application in a variety of scenarios like for example cosmological universe and AdS/CFT.

To construct the *covariant entropy bound* we need to find a generalization of the concept of volume contained inside a $D-2$ spatial dimensions surface. It turns out that this is encoded in the notion of light-sheet introduced in the following.

4.1 Light-Sheets

4.1.1 The Raychaudhuri Equation

In this Section we discuss the dynamics of families of light rays in General Relativity this is necessary to give a precise mathematical definition of the concept of

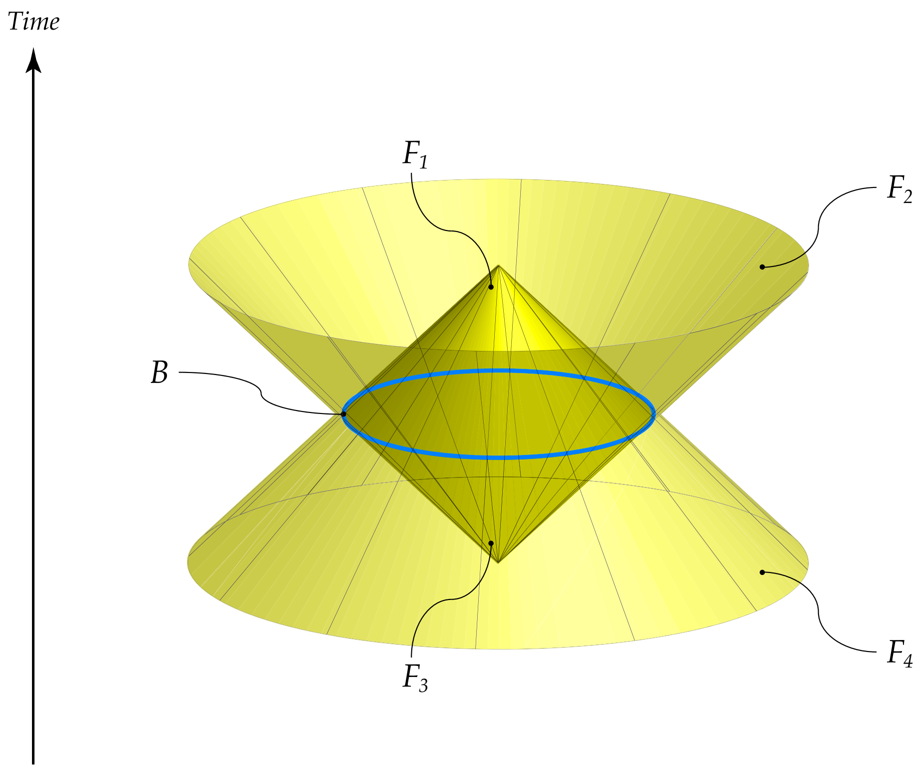

Figure 4.1: **Lightsheets.**
The four null hypersurfaces orthogonal to a spherical surface B. The two cones F_1, F_3 have negative expansion and hence correspond to light-sheets. The other two families of light rays, F_2 and F_4, generate the skirts. Their cross-sectional area is increasing, so they are not light-sheets.

light-sheet.

A family of light rays is locally characterized by its expansion, shear, and twist, which we define in the following.

Let B be a surface of $D-2$ spatial dimensions, parametrized by coordinates $x^\alpha = 1, ..., D-2$. Choose one of the four families of light rays $F_1, \ldots, F_4$ that originate from B into the past and future directions to either side of B Fig. 4.1. Each light ray satisfies the geodesics equation

$$\frac{dk^a}{d\lambda} + \Gamma^a_{bc} k^b k^c = 0$$

where λ is an affine parameter and the tangent vector k^a is defined by

$$k^a = \frac{dx^a}{d\lambda}$$

and satisfies the null condition $k^a k_a = 0$. The light rays generate a null hypersurface L parametrized by coordinates (x^α, λ). This means that in a neighborhood of B, each point on L is unambiguously defined by the light ray on which it lies (x^α) and the affine distance from B (λ). Let l^a be the null vector field on B that is orthogonal to B and satisfies $k^a l_a = 2$. (This means that l^a has the same time direction as k^a and is tangent to the orthogonal light rays constructed on the other side of B).

The induced $D-2$ dimensional metric on the surface B is given by

$$h_{ab} = g_{ab} + \frac{1}{2}(k_a l_b + k_b l_a)$$

In a similar way, it is possible to find an induced metric for all other spatial cross-sections of L. The *null extrinsic curvature* defined as

$$B_{ab} = h_a^c h_b^d \nabla_c k_d$$

contains information about the *expansion*, θ , *shear*, σ_{ab}, and *twist*, ω_{ab}, of the family of light rays, L

$$\begin{aligned}
\theta &= h^{ab} B_{ab} \\
\sigma_{ab} &= \frac{1}{2}\left(B_{ab} + B_{ba}\right) - \frac{1}{D-2}\theta h_{ab} \\
\omega_{ab} &= \frac{1}{2}\left(B_{ab} - B_{ba}\right)
\end{aligned}$$

Note that all of these quantities are functions of (x^α, λ).

In the next Sections we will need the following geometrical interpretation of the expansion parameter $\theta(\lambda)$. Define $\mathcal{A}\left(\bar{\lambda}\right)$ as the area of the submanifold $\lambda = \bar{\lambda}$ on the null hypersurface L Fig. 4.2, then we have

$$\theta(\lambda) = \frac{d\mathcal{A}}{d\lambda}$$

The Raychaudhuri equation describes the change of the expansion θ along the light rays:

$$\frac{d\theta}{d\lambda} = -\frac{1}{D-2}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - 8\pi T_{ab} k^a k^b \tag{4.1}$$

For a surface-orthogonal family of light rays, such as L, the twist ω_{ab} vanishes (Wald [10]). The final term, $-T_{ab}k^a k^b$, will be non-positive if the null energy condition is satisfied by matter, which we assume. Then the right hand side of the Raychaudhuri equation is manifestly non-positive. It follows that the expansion never increases.

We obtain the following differential inequality

$$\frac{d\theta}{d\lambda} \leq -\frac{1}{D-2}\theta^2$$

By solving it one arrives at the *focussing theorem*: If the expansion of a family of light rays takes the negative value θ_1 at any point λ_1, then θ will diverge to $-\infty$ at some affine parameter

$$\lambda_2 \leq \lambda_1 + \frac{D-2}{|\theta_1|}$$

The divergence of θ indicates that the cross-sectional area $\mathcal{A}\left(\bar{\lambda}\right)$ is locally vanishing, at such point infinitesimally neighboring light rays intersect and this is by definition a caustic point.

4.1.2 Orthogonal Null Hypersurfaces

Consider a $D-2$ spatial dimensions surface B it has precisely four orthogonal null directions Fig. 4.1. We will call them *future directed ingoing*, *future directed outgoing*, *past directed ingoing*, and *past directed outgoing*. Locally, these directions generate four null hypersurfaces $F_1, \ldots, F_4$ that border on B.

The F_i are generated by the past and the future directed light rays orthogonal to B, on either side of B.

If we consider a 2-sphere B embedded in Minkowski space Fig. 4.1, the two ingoing cones F_1 and F_3, and the two outgoing "skirts", F_2 and F_4, are easily seen to be null and orthogonal to B. However, the existence of four null hypersurfaces bordering on B is guaranteed in Lorentzian geometry independently of the shape and location of

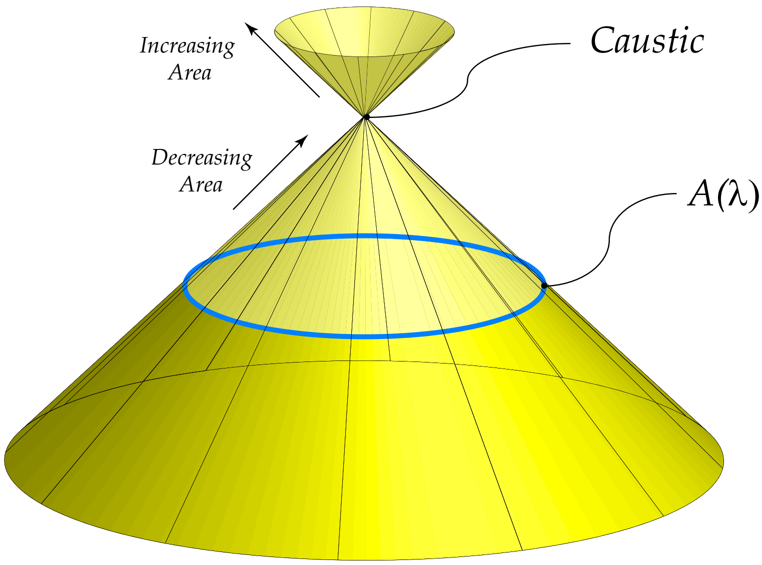

Figure 4.2: **Caustic.**
Ingoing rays perpendicular to a convex surface in a Euclidean geometry span decreasing area. This motivates the following local definition. "Inside" is the direction in which the cross-sectional area $A(\lambda)$ decreases . After light rays locally intersect, they begin to expand. Hence, light-sheets must be terminated at caustics.

B. They are always uniquely generated by the four sets of surface-orthogonal light rays.

At least two of the four null hypersurfaces $F_1, \ldots, F_4$ will be selected as light-sheets, according to the condition of non-positive expansion discussed next.

4.1.3 Light-sheet Selection

In order to generalize the Entropy Bounds Section 2.2.3 we want to define the notion of "volume inside" the 2-dimensional surface B. We could choose a spacelike surface Σ passing through B and define the "volume inside" B in the usual way. However this definition is not covariant as it depends on the choice of the spatial slice Σ.

An appropriate covariant definition of "inside" of B is found considering the case in which B is a 2-sphere embedded in Minkowski space Fig. 4.1 In this example the family of null-hypersurfaces is made of two "cones" F_1 and F_3... and two "skirts" F_2 and F_4. We will define the "cones" F_1 and F_3 as the "inside" of B and we will call them *light-sheets*.

This definition is appropriate because is covariant and furthermore the two light-sheets terminate at some point (in this example the tips of the cones) and so represent bounded hypersurfaces.

We note that the fundamental property that characterize the light-sheets is that, as we move away from the boundary, the area $\mathcal{A}(\lambda)$, introduced in Section 4.1.1, decrease. Taking this property as the one that characterize light-sheets we will now extend the definition to general spacetime.

Let F_i be one of the four families $F_1, \ldots, F_4$ of null hypersurfaces originating from B, furthermore assume that F_i is parametrized with coordinates $x^\alpha = 1, ..., D-2$ and an affine parameter λ as explained in Section 4.1.1, we will say that F_i is a light-sheet if

$$\theta\left(\lambda\right) \leq 0 \;\; \text{for} \;\; \lambda = \lambda_0 \tag{4.2}$$

where $\lambda = \lambda_0$ on B and λ increase as we move away from B.

As explained previously if we define $\mathcal{A}\left(\bar{\lambda}\right)$ as the area of the submanifold $\lambda = \bar{\lambda}$ on the light-sheet Fig. 4.2, then we have the following geometrical interpretation for $\theta\left(\lambda\right)$

$$\theta\left(\lambda\right) = \frac{d\mathcal{A}}{d\lambda}$$

So a light-sheet is defined by the condition that as we move away from the boundary B of an infinitesimal quantity $d\lambda$ the area $A\left(\lambda\right)$ decrease.

By repeating this procedure for $i = 1, \ldots, 4$, one finds all light-sheets of B. Because the light rays generating opposite pairs of null directions (e.g. F_1 and F_4) are continuations of each other, it is clear that at least one member of each pair will be considered a light-sheet. If the light rays are locally neither expanding nor contracting, both members of a pair will be light-sheets. Hence, there will always be at least two light-sheet directions. In degenerate cases, there may be three or even four.

For the simple case of the spherical surface in Minkowski space, the condition (4.2) reproduces the intuitive answer. The area is decreasing in the F_1 and F_3 directions—the past and future directed light rays going to the center of the sphere.

4.1.4 Light-sheet Termination

In the example of a 2-sphere embedded in Minkowski space Fig. 4.1 the light-sheets ends at the tips of the cones F_1 and F_3. Strictly speaking, however, there was no particular reason to stop at the tip, where all light rays intersect. On the other hand, it would clearly be disastrous to continue the light-sheet beyond the tip. It would generate another cone which would grow indefinitely, containing unbounded entropy. One must enforce, by some condition, that the light-sheet is bounded. In all but the most special cases, the light rays generating a light-sheet will not intersect in a single point, so the condition must be more general.

A suitable condition is to demand that the expansion be non-positive *everywhere* on the light-sheet, and not only near B

$$\theta(\lambda) \leq 0, \tag{4.3}$$

for all values of the affine parameter on the light-sheet.

By construction, Section 4.1.3, the expansion is negative or zero on the boundary B. Raychaudhuri's equation Eq. (4.1) guarantees that the expansion can only decrease. The only way θ can become positive is if light rays intersect, for example at the tip of the light cone.

However, it is not necessary for all light rays to intersect in the same point. By Eq. (4.1), the expansion becomes positive at any *caustic*, that is, any place where a light ray crosses an infinitesimally neighboring light ray in the light-sheet Fig. 4.2.

Thus, Eq. (4.3) implies that light-sheets end at *caustics*. In general, each light ray in a light-sheet will have a different caustic point, and the resulting caustic surfaces can be very complicated. The case of a light cone is special in that all light rays share the same caustic point at the tip.

4.2 Entropy on a Light-Sheet

The geometric construction of light-sheets is well-defined. But how is "the entropy on a light-sheet", S_{matter}, determined? Let us start with an example where the definition of S_{matter} is evident. Suppose that B is a 2-sphere around an isolated, weakly gravitating thermodynamic system. Given certain macroscopic parameters, for example an energy or energy range, pressure, volume, etc., the entropy of the system can be computed either thermodynamically, or statistically as the logarithm of the number of quantum states associated with the configuration defined by the macroscopic parameters.

With the previous assumptions, the two light-sheets of B are to good approximation, a past and a future light cone. Let us consider the future directed light-sheet. The cone contains the matter system completely , in the same sense in which a $t = \text{const}$ surface contains the system completely . A light-sheet is just a different way of enclosing a matter system. (In fact, this is much closer to how the system is actually observed in practice.) Hence, the entropy on the light-sheet is simply given by the entropy of the matter system.

Let us now consider cosmological spacetimes, in this case the entropy is usually well approximated as a continuous fluid, as a consequence S_{matter} will be the integral of the entropy density over the light-sheet.

4.3 Formulation of the Covariant Entropy Bound

Now that we have introduced the appropriate mathematical formalism we are ready to state the covariant entropy bound.

Consider a D-dimensional Lorentzian spacetime M. Let $A(B)$ be the area of an arbitrary $D-2$ dimensional spatial surface B (which need not be closed). A $D-1$ dimensional hypersurface L is called a light-sheet of B if L is generated by light rays which begin at B, extend orthogonally away from B, and have non-positive expansion,

$$\theta \leq 0,$$

everywhere on L. Let S be the entropy on any light-sheet of B. Then

$$S \leq \frac{A(B)}{4}. \tag{4.4}$$

Let us restate the covariant entropy bound one more time, in a constructive form most suitable for applying and testing the bound.

1. Pick any $D-2$ dimensional spatial surface B, and determine its area $A(B)$. There will be four families of light rays projecting orthogonally away from B: $F_1 \ldots F_4$.

2. Usually additional information is available, such as the macroscopic spacetime metric everywhere or in a neighborhood of B. Then the expansion θ of the or-

thogonal light rays can be evaluated for each family. Of the four families, at least two will not expand ($\theta \leq 0$). Determine which.

3. Pick one of the non-expanding families, F_j. Follow each light ray no further than to a caustic, a place where it intersects with neighboring light rays. The light rays form a $D-1$ dimensional null hypersurface, a light-sheet $L(B)$.

4. Determine the entropy $S[L(B)]$ of matter on the light-sheet L, as described in Section 4.2.

5. The quantities $S[L(B)]$ and $A(B)$ can then be compared. The covariant entropy bound states that the entropy on the light-sheet will not exceed a quarter of the area: $S[L(B)] \leq \frac{A(B)}{4}$. This must hold for any surface B, and it applies to each non-expanding null direction, F_j, separately.

In particular, the bound is predictive and can be tested by observation, in the sense that the entropy and geometry of real matter systems can be determined (or, as in the case of large cosmological regions, at least estimated) from experimental measurements.

Chapter 5

Quantum Field Theory in Curved Spacetime

We briefly introduce some concepts of Quantum Field Theory in Curved Space that will be useful in the Second Part.

5.1 Scalar Field Quantization

For the sake of simplicity we will only study the quantization of a scalar field on a fixed background. Nevertheless this example is rich enough to contain all the features we are interested in.

The field quantization proceeds in strict analogy with the case of Minkowski space. We start with the Lagrangian density

$$\mathcal{L}(x) = \frac{1}{2}[-g(x)]^{\frac{1}{2}} \left\{ g^{\mu\nu}(x)\phi(x)_{,\mu}\phi(x)_{,\nu} - [m^2 + \xi R(x)]\phi^2(x) \right\} \tag{5.1}$$

where $\phi(x)$ is a scalar field of mass m and

$$g_{\mu\nu}, \quad \mu, \nu = 0, 1, \ldots, n-1$$

is the metric on a n-dimensional, globally hyperbolic, pseudo-Riemannian manifold and $g = \sqrt{\det(g_{\mu\nu})}$.

The coupling among the scalar field and the gravitational field is given by the term $\xi R(x)\phi^2(x)$ where ξ is a constant and $R(x)$ is the Ricci scalar.

The action is given as usual by

$$S = \int \mathcal{L}(x)\ d^n x$$

Considering the variation of the action respect to $\phi(x)$ we get the equation of motion

$$[\Box + m^2 + \xi R(x)]\ \phi(x) = 0 \tag{5.2}$$

where the Laplacian is given by

$$\Box\ \phi = g^{\mu\nu}\nabla_\mu\nabla_\nu\ \phi = (-g)^{\frac{1}{2}}\partial_\mu[(-g)^{\frac{1}{2}} g^{\mu\nu}\partial_\nu\phi]$$

The correct normalization for the states is obtained imposing the scalar product

$$(\phi_1, \phi_2) = -i \int_\Sigma \phi_1(x)\overleftrightarrow{\partial_\mu}\phi_2^*(x)[-g_\Sigma(x)]^{\frac{1}{2}}\ d\Sigma^\mu \tag{5.3}$$

$$\overleftrightarrow{\partial}$$

where $d\Sigma^\mu = n^\mu d\Sigma$, with n^μ a future-direct unit vector orthogonal to the spacelike hypersurface Σ and $d\Sigma$ is the volume element in Σ. The hypersurface Σ is assumed to be a Cauchy surface in the (globally hyperbolic) spacetime and one can show, using Gauss' theorem that the value of (ϕ_1, ϕ_2) is independent of Σ.

There will exists a complete set of functions $u_i(x)$ solutions of Eq. (5.2) which are orthonormal respect to the scalar product Eq. (5.3).

$$\begin{aligned}
(u_i, u_j) &= \delta_{ij} \\
(u_i^*, u_j^*) &= -\delta_{ij} \\
(u_i, u_j^*) &= 0
\end{aligned}$$

It is possible to decompose the field $\phi(x)$ respect to this base of functions

$$\phi(x) = \Sigma_i [a_i u_i(x) + a_i^\dagger u_i^*(x)]$$

The quantization of the fields is obtained in the usual way, imposing the commutation relation

$$\begin{aligned}
[a_i, a_j^\dagger] &= \delta_{ij} \\
[a_i^\dagger, a_j^\dagger] &= 0 \\
[a_i, a_j] &= 0
\end{aligned}$$

The vacuum state $|0\rangle$ has the property that is annihilated by all the $a_{\mathbf{k}}$ operators

$$a_j \,|0\rangle = 0, \quad \forall\; j$$

The *Fock space* is constructed acting on the vacuum with creation operators

$$|1_{j_1}, 1_{j_2}, \ldots, 1_{j_k}\rangle = a_{j_1}^\dagger a_{j_2}^\dagger \ldots a_{j_k}^\dagger \,|0\rangle$$

Differently from Minkowski spacetime there is not a unique prescription to choose the modes decomposition of the field $\phi(x)$. Different decomposition will generate in general different Fock spaces. In the following Section we analyze how these Fock spaces are related.

5.2 Bogolubov Coefficients

Let us consider a second set of orthonormal modes $\bar{u}_i(x)$. The field may be decomposed respect to this set also

$$\phi(x) = \Sigma_i[\bar{a}_i\bar{u}_i(x) + \bar{a}_i^\dagger \bar{u}_i^*(x)]$$

This decomposition of ϕ define a new vacuum state $|\bar{0}\rangle$

$$\bar{a}_j |\bar{0}\rangle = 0, \quad \forall\ j$$

and a new Fock space

$$|\bar{1}_{j_1}, \bar{1}_{j_2}, \ldots, \bar{1}_{j_k}\rangle = \bar{a}_{j_1}^\dagger \bar{a}_{j_2}^\dagger \ldots \bar{a}_{j_k}^\dagger |0\rangle$$

Given that both sets are complete, the new modes $\bar{u}_j$ can be expanded in terms of the old u_i

$$\bar{u}_j = \Sigma_i[\alpha_{ji}u_i + \beta_{ji}u_i^*]$$

and vice versa

$$u_i = \Sigma_j[\alpha_{ji}^*\bar{u}_j + \beta_{ji}\bar{u}_j^*]$$

These relations are called Bogolubov transformations. The matrices α_{ji}, β_{ji} are called Bogolubov coefficients and using the scalar product (5.3) are given by

$$\alpha_{ij} = (\bar{u}_i, u_j), \ \beta_{ij} = -(\bar{u}_i, u_j^*)$$

Moreover, it is possible to find the following expression for the annihilation operators

$$a_i = \Sigma_j(\alpha_{ji}\bar{a}_j + \beta_{ji}^*\bar{a}_j^\dagger) \tag{5.4}$$

and

$$\bar{a}_i = \Sigma_i(\alpha_{ji}^* a_i + \beta_{ji}^* a_i^\dagger)$$

It follows immediately from Eq. (5.4) that two Fock spaces based on two choices of modes u_i and $\bar{u}_j$ are different as long as $\beta_{ji} \neq 0$. For example $|\bar{0}\rangle$ will not be annihilated by a_i

$$a_i\,|\bar{0}\rangle = \Sigma_j\beta_{ji}^*\,|\bar{1}_j\rangle \neq 0$$

Furthermore, if we consider the expectation value of the number operator $N_i = a_i^\dagger a_i$ in the new vacuum $|\bar{0}\rangle$

$$\langle\bar{0}|\,N_i\,|\bar{0}\rangle = \Sigma_j\,|\beta_{ji}|^2$$

we find that the new vacuum contains $\Sigma_j\,|\beta_{ji}|^2$ particles associated with the old mode u_i.

5.3 Green Functions

We will now describe how to derive the several Green functions of the theory.

Let us first introduce the Wightman functions

$$\begin{aligned} G^{+}(x,y) &= \langle 0|\,\phi(x)\phi(y)\,|0\rangle \\ G^{-}(x,y) &= \langle 0|\,\phi(y)\phi(x)\,|0\rangle \end{aligned}$$

All the others Green functions can be derived from those. In particular we have

Pauli Jordan or Schwinger function

$$iG(x,y) = \langle 0|\,[\phi(x),\phi(y)]\,|0\rangle$$

Hadamard's elementary function

$$G^{(1)}(x,y) = \langle 0|\,\{\phi(x),\phi(y)\}\,|0\rangle$$

Feynman propagator

$$\begin{aligned} G_F(x,y) &= \langle 0|\,T\,(\phi(x)\phi(y))\,|0\rangle \\ &= \theta(t_x - t_y)G^{+}(x,y) + \theta(t_y - t_x)G^{-}(x,y) \end{aligned}$$

Retarded and advanced Green functions

$$\begin{aligned} G_R(x,y) &= -\theta(t_x - t_y)G(x,y) \\ G_A(x,y) &= \theta(t_y - t_x)G(x,y) \end{aligned}$$

$G,\ G^1,\ G^{\pm}$ all satisfy the homogeneous equation

$$\left[\Box_x + m^2 + \xi R(x)\right]\,\mathcal{G}(x,y) = 0$$

on the other hand $G_F,\ G_R,\ G_A$ satisfy the equations

$$[\Box_x + m^2 + \xi R(x)]\ G_F(x,y) = -(-g(x))^{\frac{1}{2}}\delta^n(x-y)$$

$$[\Box_x + m^2 + \xi R(x)]\ G_{R,A}(x,y) = (-g(x))^{\frac{1}{2}}\delta^n(x-y)$$

In Minkowski space all the Green functions can be obtained from the same integral in momentum space, choosing an appropriate path of integration in the complex plane, this correspond in turn to choose different boundary conditions for the differential equations. In curved space the situation is more complicated and the appropriate boundary conditions must be specified case by case.

Part II

Publications

Chapter 6

De Sitter Vacua, Renormalization and Locality

6.1 Introduction

In the recent outbreak of interest in de Sitter spacetimes, attention has been drawn again to the existence of a one (complex) parameter family of vacuum states (called the α-vacua) for free quantum fields in de Sitter spacetime [13]. Experts in the field have long harbored a vague suspicion that only the standard Euclidean vacuum was sensible, but until now there has been no conclusive argument to this effect. The purpose of this note is to present one.

The argument is, in essence, very simple. Propagators in quantum field theory are singular on the light cone. The propagators in the α-vacua are linear superpositions of a Euclidean[1] propagator evaluated between two points x, y, and the same propagator

[1] We use the short phrase Euclidean propagator to denote the propagator of a field in dS space, which

evaluated between x and the antipodal point to y, y^A. The Feynman diagrams of interacting quantum field theory contain products of propagators between the same two points. These are not distributions, and a subtraction procedure must be supplied to define them. The key point of standard renormalization theory is that the subtractions all take the form of local contributions to the effective action, and can thus be viewed as renormalizations of couplings in the theory. We will show by simple examples that in the α-vacua this is no longer true. The subtractions include non-local contributions to the effective action of the form *e.g.*

$$\delta S = \delta\lambda \int \phi(x)\phi(x^A) \tag{6.1}$$

where $\delta\lambda$ is a divergent constant. Thus, renormalized interacting field theory in a generic α vacuum is intrinsically non-local, and presumably has no sensible physical interpretation.

There are only two values of α for which this catastrophe is avoided. The first is $\Re(\alpha) = a = -\infty$ which gives the standard Euclidean vacuum and has no antipodal singularity . The second is $\alpha = 0$, which is the unique vacuum state invariant under the antipodal map. The Green's function in this vacuum appears to be the analytic continuation of a Euclidean functional integral on RP^{d}[2]. In this vacuum state, which we call the *antipodal vacuum* we must view the Lorentzian spacetime manifold as the orbifold of de Sitter space by the antipodal map. Every point is identified with its antipode, and the interaction 6.1 is local. From a physical point of view we have a

is obtained by analytic continuation of the Euclidean functional integral on a sphere.

[2]To our knowledge, E. Witten[14] was the first to point out the significance of this special value of alpha and its Euclidean interpretation.

manifold with a past spacelike singularity and an asymptotic de Sitter future. We call this spacetime the *antipodal universe*.

In discussions of inflationary cosmology, one often invokes a Quantum No-Hair Theorem for de Sitter space . According to this theorem, generic initial states of quantum fields in dS space, evolve into a state indistinguishable from the Euclidean vacuum after enough e-foldings. A crucial assumption in this theorem, is that the initial state approaches the Euclidean vacuum for very high angular momentum modes (in global coordinates - in planar coordinates we would say ordinary momentum modes). Modes of any finite comoving wave number are redshifted to a size larger than the horizon volume after a sufficient number of e-foldings, and are no longer observable by a local measurement. If the initial state is the Euclidean vacuum for sufficiently high momentum modes, then the local observer will eventually see a state indistinguishable from the Euclidean vacuum.

The state implied by the orbifold boundary conditions does not satisfy the conditions of this theorem. In global coordinates the Euclidean vacuum for a boson field is a Gaussian with time dependent covariance, for each angular momentum mode. The orbifold boundary conditions imply instead that the initial wave function of the even angular momentum states is a field eigenstate , while that of the odd modes is an eigenstate of the canonical momentum. These are non-normalizable states,for each angular momentum mode, and differ from the Euclidean vacuum for arbitrarily large angular momentum. They do not obey the de Sitter no hair theorem. Thus, the future evolution of the antipodal universe depends on the initial conditions.

We argue further that the initial conditions may be subject to infinite ultraviolet corrections in higher orders of perturbation theory. These are the standard UV divergences of fixed time Schrodinger picture states in quantum field theory. If this were true , we would have to claim that, without a nonperturbative understanding of the state near the orbifold singularity, we could not make reliable predictions in the antipodal universe.

These considerations cast doubt on the identification of the Lorentzian antipodal vacuum with the analytic continuation of a Euclidean functional integral on RP^d. The latter is renormalized by the standard counterterms for quantum field theory on smooth manifolds without boundary. It may be that the boundary conditions defined by the RP^d functional integral are a fixed point of the boundary renormalization group of the Lorentzian orbifold field theory, but we have not done enough computations to verify this conjecture.

All of these arguments are made in the context of quantum field theory in a fixed spacetime background. In quantum gravity, we have the additional problem that the antipodal initial state has infinite energy density, which leads us to expect a large back reaction. A much more extensive discussion of the back reaction problem in α-vacua will be presented in [15].

Our conclusion is that only the Euclidean vacuum state has a chance of describing sensible physical processes in de Sitter space. The rest of this note is devoted to calculations which explicate the argument made above.

We note that after we submitted this paper to arXiv.org, two related papers appeared which have some overlap with our work. The first, by Einhorn and Larsen[16], discusses aspects of higher loop graphs in α vacua, and also concludes that these are generally ill-defined. The second[17] discusses the Z_2 orbifold of dS space (and points out that it was first introduced long ago by Schrodinger). It is not clear to us that their definition of the quantum theory is the same as ours. They do not discuss divergences near the origin of time in this system.

6.2 Interacting Scalar Field Theory in an α Vacuum

In this section we will present a calculation of the two point function in a simple scalar field theory. We hope the reader will realize that our conclusions are quite general. In particular, we began this project by computing the two point function of the renormalized stress tensor in an α vacuum. This computation would enter into any perturbative theory of quantum gravity in de Sitter space. This calculation is more divergent than any we will actually present, but exhibits the same non-locality that we find in our simple example. We decided that the extra indices and the subtleties of covariance would only distract the reader from the main point.

6.2.1 Notation

In the following we will consider 4-dimensional de Sitter space dS^4. It may be realized as the manifold

$$- X_0^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 = l^2 \tag{6.2}$$

embedded in the 5-dimensional Minkowski space $M^{4,1}$. We will use lower case x to indicate 4-dimensional coordinates on dS^4 and upper case X to denote embedding coordinates. We will denote the antipodal points by $X^A \equiv -X$. Henceforth we will set $l = 1$.

We are considering an interacting scalar field theory in dS^4 with action

$$S = \frac{1}{2}\int \mathrm{d}^4x\,(-g(x))^{\frac{1}{2}}[(\nabla\phi)^2 - m^2\phi^2 - \frac{\lambda}{3!}\phi^3] \tag{6.3}$$

In dS^4 there is a one complex parameter, α, family of dS invariant vacua [13] that we will denote $|\alpha\rangle$. The associated de Sitter invariant family of two point Wightman functions is

$$\langle\alpha|\phi(x)\phi(y)|\alpha\rangle = W_\alpha(x,y) =$$

$$n^2\left(W_e(x,y) + e^{\alpha+\alpha^*}\,W_e(y,x) + e^{\alpha}\,W_e(x,y^A) + e^{\alpha^*}\,W_e(x^A,y)\right) \tag{6.4}$$

with

$$\alpha \in \mathcal{C}$$

$$\Re(\alpha) = a < 0 \tag{6.5}$$

$$n = n(\alpha) = \frac{1}{\sqrt{1 - e^{\alpha+\alpha^*}}} \tag{6.6}$$

Here we use the Euclidean two point Wightman function $W_e(x, y)$ defined in [18]. The Euclidean Wightman function and vacuum correspond to $a = -\infty$.

6.2.2 Computation

In this section we will compute a term in the *1-loop* effective action, in a general α-vacuum. The computation will lead to divergent non-local counterterms. Only the Euclidean vacuum produces a completely local counterterm action.

The 1-loop, two point contribution to the effective action in our simple field theory is

$$\Gamma(\phi) \sim \int \mathrm{d}^4x\, \mathrm{d}^4y\, (-g(x))^{\frac{1}{2}} (-g(y))^{\frac{1}{2}} \phi_{cl}(x) F_\alpha(x, y) F_\alpha(x, y) \phi_{cl}(y) \tag{6.7}$$

The Feynman propagator $F_\alpha(x, y)$ can be expressed in terms of the Wightman functions and the parameter α as

$$F_\alpha(x, y) = \Theta(x_0 - y_0)\, W_\alpha(x, y) + \Theta(y_0 - x_0)\, W_\alpha(y, x) \tag{6.8}$$

$$W_\alpha(x, y) = n^2 \left(W_e(x, y) + e^{\alpha+\alpha^*}\, W_e(y, x) + e^{\alpha}\, W_e(x, y^A) + e^{\alpha^*}\, W_e(x^A, y) \right) \tag{6.9}$$

with

$$n = n(\alpha) = \frac{1}{\sqrt{1 - e^{\alpha+\alpha^*}}} \tag{6.10}$$

The behavior of the two point Euclidean Wightman function near the light cone is for dS^4

$$W_e(x,y) \sim \frac{C}{(x_0 - y_0 - i\,\epsilon)^2 - (x_s - y_s)^2}, \tag{6.11}$$

where $x = (x_0, x_s)$, $y = (y_0, y_s)$ and C is a constant whose value is not relevant for the following considerations.

We will show now that in $F^2_\alpha(x,y)$ only the terms W_e^2, having a singular behavior near the light cone of the form

$$W_e^2 \sim \frac{C^2}{(x-y)^4} \tag{6.12}$$

$$W_e^2 \sim \frac{C^2}{(x^A-y)^4} \tag{6.13}$$

$$W_e^2 \sim \frac{C^2}{(x-y^A)^4} \tag{6.14}$$

$$W_e^2 \sim \frac{C^2}{(x-y^A)^2(x^A-y)^2} \tag{6.15}$$

contribute to the divergent part of the effective action. In these equations, we suppress the $i\epsilon$ prescription because it is not relevant at this point. Considering $W^2(x,y)$ as a distribution on the space of test function $\phi(x)$ we have

$$T_{W_e^2}[\phi] = \int \mathrm{d}^4x\, W_e^2(x,y)\phi(x)$$

$$= \int \mathrm{d}^4x \left(W_e^2(x,y) - \frac{C^2}{(x-y)^4} \right) \phi(x) + \int \mathrm{d}^4x \, \frac{C^2 \phi(x)}{(x-y)^4}$$

$$= \int \mathrm{d}^4x \left(W_e^2(x,y) - \frac{C^2}{(x-y)^4} \right) \phi(x)$$

$$+ \int \mathrm{d}^4x \, \frac{C^2 \left(\phi(x) - \phi(y)\right)}{(x-y)^4} + \phi(y) \int \mathrm{d}^4x \, \frac{C^2}{(x-y)^4}$$

$$= Regular + \int \mathrm{d}^4z \, \delta(z-y)\phi(z) \int \mathrm{d}^4x \, \frac{C^2}{(x-y)^4} \tag{6.16}$$

where the regular part does not contribute to the divergent part of the effective action. Similarly, in the terms which contain squares of Wightman functions evaluated between points and their antipodes, we have

$$T_{W_e^2}[\phi] = \int \mathrm{d}^4x \, W_e^2(x^A, y)\phi(x)$$

$$= Regular + \int \mathrm{d}^4z \, \delta(z-y^A)\phi(z) \int \mathrm{d}^4x \, \frac{C^2}{(x^A-y)^4} \tag{6.17}$$

and

$$T_{W_e^2}[\phi] = \int \mathrm{d}^4x \, W_e^2(x, y^A)\phi(x)$$

$$= Regular + \int \mathrm{d}^4z \, \delta(z-y^A)\phi(z) \int \mathrm{d}^4x \, \frac{C^2}{(x-y^A)^4} \tag{6.18}$$

Similarly

$$T_{W_e^2}[\phi] = \int \mathrm{d}^4x\, W_e(x^A, y) W_e(x, y^A)\phi(x)$$

$$= Regular + \int \mathrm{d}^4z\, \delta(z - y^A)\phi(z) \int \mathrm{d}^4x\, \frac{C^2}{(x^A - y)^2 (x - y^A)^2} \tag{6.19}$$

All the other terms in W_e^2 are regular and do not contribute to the divergent part of the effective action.

After eliminating the regular terms in $F_\alpha^2(x,y)$ and doing the replacements $\Theta(x_0 - y_0)\,\Theta(y_0 - x_0) \to 0$, $\Theta(x_0 - y_0)^2 \to \Theta(x_0 - y_0)$, and $\Theta(y_0 - x_0)^2 \to \Theta(y_0 - x_0)$, we get

$$F_\alpha(x,y)^2 = n^4\, \Theta(x_0 - y_0)\, W_e(x,y)^2 + e^{2\,\alpha + 2\,\alpha^*}\, n^4\, \Theta(y_0 - x_0)\, W_e(x,y)^2$$

$$+ e^{2\,\alpha^*}\, n^4\, \Theta(x_0 - y_0)\, W_e(x, y^A)^2 + 2\, e^{\alpha + \alpha^*}\, n^4\, \Theta(x_0 - y_0)\, W_e(x, y^A)\, W_e(x^A, y)$$

$$+ e^{2\,\alpha}\, n^4\, \Theta(x_0 - y_0)\, W_e(x^A, y)^2 + 2\, e^{\alpha + \alpha^*}\, n^4\, \Theta(x_0 - y_0)\, W_e(x,y)\, W_e(y,x)$$

$$+ 2\, e^{\alpha + \alpha^*}\, n^4\, \Theta(y_0 - x_0)\, W_e(x,y)\, W_e(y,x) + e^{2\,\alpha + 2\,\alpha^*}\, n^4\, \Theta(x_0 - y_0)\, W_e(y,x)^2$$

$$+n^4\,\Theta(y_0-x_0)\,W_e(y,x)^2+e^{2\,\alpha^*}\,n^4\,\Theta(y_0-x_0)\,W_e(y,x^A)^2$$

$$+\,2\,e^{\alpha+\alpha^*}\,n^4\,\Theta(y_0-x_0)\,W_e(y,x^A)\,W_e(y^A,x)+e^{2\,\alpha}\,n^4\,\Theta(y_0-x_0)\,W_e(y^A,x)^2 \quad (6.20)$$

Replacing the W_e terms with their singular behavior near the light cone, we find

$$F_\alpha(x,y)^2\sim$$

$$\delta(x-y)\left(\frac{C^2\,n^4\,\Theta(x_0-y_0)}{\Big((x_0-y_0-i\,\epsilon)^2-(x_s-y_s)^2\Big)^2}+\frac{C^2\,e^{2\,\alpha+2\,\alpha^*}\,n^4\,\Theta(x_0-y_0)}{\Big((y_0-x_0-i\,\epsilon)^2-(x_s-y_s)^2\Big)^2}\right.$$

$$+\frac{2\,C^2\,e^{\alpha+\alpha^*}\,n^4\,\Theta(x_0-y_0)}{\Big((x_0-y_0-i\,\epsilon)^2-(x_s-y_s)^2\Big)\,\Big((y_0-x_0-i\,\epsilon)^2-(x_s-y_s)^2\Big)}$$

$$+\frac{C^2\,e^{2\,\alpha+2\,\alpha^*}\,n^4\,\Theta(y_0-x_0)}{\Big((x_0-y_0-i\,\epsilon)^2-(x_s-y_s)^2\Big)^2}+\frac{C^2\,n^4\,\Theta(y_0-x_0)}{\Big((y_0-x_0-i\,\epsilon)^2-(x_s-y_s)^2\Big)^2}$$

$$\left.+\frac{2\,C^2\,e^{\alpha+\alpha^*}\,n^4\,\Theta(y_0-x_0)}{\Big((x_0-y_0-i\,\epsilon)^2-(x_s-y_s)^2\Big)\,\Big((y_0-x_0-i\,\epsilon)^2-(x_s-y_s)^2\Big)}\right)$$

$$+\delta(x-y^A)\left(\frac{C^2\,e^{2\,\alpha}\,n^4\,\Theta(x_0-y_0)}{\Big(\big(x_0^A-y_0-i\,\epsilon\big)^2-(x_s^A-y_s)^2\Big)^2}+\frac{C^2\,e^{2\,\alpha^*}\,n^4\,\Theta(x_0-y_0)}{\Big(\big(x_0-y_0^A-i\,\epsilon\big)^2-(x_s-y_s^A)^2\Big)^2}\right.$$

$$+\frac{2\,C^2\,e^{\alpha+\alpha^*}\,n^4\,\Theta(x_0-y_0)}{\left(\left(x_0^A-y_0-i\,\epsilon\right)^2-(x_s^A-y_s)^2\right)\,\left(\left(x_0-y_0^A-i\,\epsilon\right)^2-(x_s-y_s^A)^2\right)}$$

$$+\frac{C^2\,e^{2\,\alpha^*}\,n^4\,\Theta(y_0-x_0)}{\left(\left(y_0-x_0^A-i\,\epsilon\right)^2-(x_s^A-y_s)^2\right)^2}+\frac{C^2\,e^{2\,\alpha}\,n^4\,\Theta(y_0-x_0)}{\left(\left(y_0^A-x_0-i\,\epsilon\right)^2-(x_s-y_s^A)^2\right)^2}$$

$$\left.+\frac{2\,C^2\,e^{\alpha+\alpha^*}\,n^4\,\Theta(y_0-x_0)}{\left(\left(y_0-x_0^A-i\,\epsilon\right)^2-(x_s^A-y_s)^2\right)\,\left(\left(y_0^A-x_0-i\,\epsilon\right)^2-(x_s-y_s^A)^2\right)}\right) \tag{6.21}$$

The $\delta(x-y^A)$ term gives rise to a non local, divergent, contribution to the effective action . The coefficient of $\delta(x-y^A)$ is

$$\left(\frac{C^2\,e^{2\,\alpha}\,n^4\,\Theta(x_0-y_0)}{\left(\left(x_0^A-y_0-i\,\epsilon\right)^2-(x_s^A-y_s)^2\right)^2}+\frac{C^2\,e^{2\,\alpha^*}\,n^4\,\Theta(x_0-y_0)}{\left(\left(x_0-y_0^A-i\,\epsilon\right)^2-(x_s-y_s^A)^2\right)^2}\right.$$

$$+\frac{2\,C^2\,e^{\alpha+\alpha^*}\,n^4\,\Theta(x_0-y_0)}{\left(\left(x_0^A-y_0-i\,\epsilon\right)^2-(x_s^A-y_s)^2\right)\,\left(\left(x_0-y_0^A-i\,\epsilon\right)^2-(x_s-y_s^A)^2\right)}$$

$$+\frac{C^2\,e^{2\,\alpha^*}\,n^4\,\Theta(y_0-x_0)}{\left(\left(y_0-x_0^A-i\,\epsilon\right)^2-(x_s^A-y_s)^2\right)^2}+\frac{C^2\,e^{2\,\alpha}\,n^4\,\Theta(y_0-x_0)}{\left(\left(y_0^A-x_0-i\,\epsilon\right)^2-(x_s-y_s^A)^2\right)^2}$$

$$\left.+\frac{2\,C^2\,e^{\alpha+\alpha^*}\,n^4\,\Theta(y_0-x_0)}{\left(\left(y_0-x_0^A-i\,\epsilon\right)^2-(x_s^A-y_s)^2\right)\,\left(\left(y_0^A-x_0-i\,\epsilon\right)^2-(x_s-y_s^A)^2\right)}\right) \tag{6.22}$$

After the substitutions $(x_0^A,\, x_s^A) \to (-x_0,\, -x_s)$, $(y_0^A,\, y_s^A) \to (-y_0,\, -y_s)$, $\Theta(x_0 - y_0) + \Theta(y_0 - x_0) \to 1$,

we find that the non-local part of the divergent counterterm is

$$\left(\frac{C^2\, e^{2\,\alpha}\, n^4}{\Big((x_0 + y_0 + i\,\epsilon)^2 - (x_s + y_s)^2 \Big)^2} + \frac{C^2\, e^{2\,\alpha^*}\, n^4}{\Big((x_0 + y_0 - i\,\epsilon)^2 - (x_s + y_s)^2 \Big)^2} \right.$$

$$\left. + \frac{2\, C^2\, e^{\alpha + \alpha^*}\, n^4}{\Big((x_0 + y_0 + i\,\epsilon)^2 - (x_s + y_s)^2 \Big) \Big((x_0 + y_0 - i\,\epsilon)^2 - (x_s + y_s)^2 \Big)} \right) \tag{6.23}$$

The three terms in this expression are different because they have distinct $i\epsilon$ prescriptions and therefore diverse poles in the complex plane.

As a consequence to eliminate all the divergent, non local terms in the effective action we must set

$$e^{\alpha} = e^{a+ib} = 0 \tag{6.24}$$

$$e^{\alpha^*} = e^{a-ib} = 0 \tag{6.25}$$

$$\Rightarrow e^{a} = 0 \tag{6.26}$$

$$\Rightarrow a = -\infty \tag{6.27}$$

This corresponds to the choice of the Euclidean vacuum as previously stated. We should remark that the constant $n = \frac{1}{\sqrt{1-e^{\alpha+\alpha^*}}} = \frac{1}{\sqrt{1-e^{2a}}}$. can never be zero because the family of de Sitter invariant vacua is defined by $\alpha \in \mathcal{C}$, $\Re(\alpha) = a < 0$. There is however another way to obtain a system with local effective action. The nonlocalities are all products of fields at points and their antipodes. For $\alpha = 0$ we can interpret the Green's functions as living on an orbifold of dS space, the antipodal universe, in which a point is identified with its antipode. On this spacetime, all of our counterterms can be viewed as local operators.

Witten[14] has suggested that for this value of α the Green's functions can be viewed as analytic continuations of the Euclidean functional integral on the real projective space RP^4. Since RP^4 is a smooth manifold without boundary, this Euclidean functional integral should be renormalized by the same local counterterms that define the field theory on the sphere. We will discuss this interpretation in the next section.

6.3 The Wave Functional in the Antipodal Vacuum

We have seen that, with the exception of the Euclidean and Antipodal vacua, field theory in an α vacuum cannot be renormalized by local counterterms. We now want to investigate whether the Antipodal vacuum forms the basis for a sensible quantum field theory. Certainly, the Euclidean functional integral on RP^d is well defined. However, it is not immediately apparent that the Green's functions defined by this functional integral have a Hamiltonian interpretation. The conventional reflection positivity argument requires the reflected Euclidean points to be distinct from the points themselves.

Indeed, it would appear that the Lorentzian version of the Antipodal universe requires more renormalization than the corresponding Euclidean functional integral. RP^d is a smooth manifold without boundary and the Euclidean functional integral on this manifold will be renormalized by the same local subtractions that are required for the Euclidean functional integral on the sphere. However, the Lorentzian version of the theory describes the evolution of a quantum field theory starting from a fixed state at a sharp time. It has been known since the work of Symanzik[19] that in renormalizable quantum field theories, the wave functional at a sharp time requires additional renormalizations, above and beyond those which render the Green's functions finite. In modern parlance, the sharp time state introduces a boundary into the system and one must introduce counterterms for all relevant boundary operators at the fixed point of the bulk renormalization group.

Thus, it would seem that, if field theory is to be defined in the antipodal vacuum it requires additional definitions to determine the initial state. These remarks also seem to indicate that the connection between the Lorentzian theory and the Euclidean theory on RP^d must somehow be valid only in the absence of boundary renormalizations. We have remarked that the Euclidean antipodal Green's functions do not seem to require additional subtractions. It is possible that this means that the Lorentzian boundary conditions implied by continuation from RP^d are automatically fixed points of the boundary renormalization group.

Indeed, the above discussion of boundary renormalization is valid for boundary conditions of the form $\phi(t = 0, x) = \phi_0(x)$, which would define the Schrodinger wave functional. We then think of the orbifold boundary condition as a restriction on the allowed Schrodinger functionals. Perhaps , since the Lorentzian orbifold does not have a geometric boundary, all boundary counterterms will vanish in such a state[3]. We have not been able to determine the validity of such a conjecture. In particular, in general field theories there would seem to be marginal and relevant boundary operators which are not projected out by the orbifold condition. We do not understand why additional counterterms proportional to these relevant operators are not generated by the Lorentzian Feynman rules.

Even apart from these additional renormalization effects, the state defined by the antipodal boundary conditions is somewhat singular. Classically, the field is required to be invariant under simultaneous reflection in the global coordinate time and spatial sphere. If we expand the field into spherical harmonics, then (for free field theory in dS space) each mode ϕ_L is a time dependent harmonic oscillator, with a frequency that is even under reflection about the point of minimal size. Under reflection in the sphere, $\phi_L \to (-1)^L \phi_L$. Thus, invariance under the antipodal map is equivalent to the quantum mechanical statement that the initial state is annihilated by ϕ_L for odd L and by the conjugate momentum Π_L for even L. The quantum system is then studied as a collection of time dependent oscillators, with these initial conditions, on the interval $t \in [0, \infty]$. Note that the quantum state defined by this boundary condition differs from

[3] TB thanks M. Douglas for a discussion of this point.

the Euclidean state of the same system even for $L \to \infty$. The dS No-Hair Theorem is not applicable, and this state does not approach the Euclidean vacuum at large times.

Finally, we note that the deviation from the Euclidean vacuum for large L also implies that the matrix elements of the renormalized stress tensor between states of the form

$$a^{\dagger}_{L_1} \dots a^{\dagger}_{L_n} |A> \tag{6.28}$$

where $|A>$ is the antipodal vacuum and the operators are its associated creation operators, will blow up as $t \to 0$.

These additional divergences have little to do with renormalization. They are more analogous to the singularities at particular places in Lorentzian momentum space that one finds in the analytic continuation of renormalized Euclidean Green's functions in any field theory. That is, they represent real physical processes, rather than virtual contributions to the effective action.

The consequence of these remarks is that, although the antipodal vacuum does not suffer from the renormalization problems of the generic α vacuum, its physics is not under control at $t = 0$.

6.4 Conclusion

We have investigated the perturbative renormalizability of quantum field theories in rigid dS space, when the vacuum state of the free fields is chosen to be one of the non-Euclidean, dS invariant vacua. In general the renormalization program fails. Non-local counterterms, involving products of fields at both points and their antipodes, are necessary to render the interacting Green's functions finite. Even if it were possible to prove that this nonlocal renormalization program could be carried out to all orders (which is by no means obvious) , the resulting theory would probably not have a Hamiltonian interpretation. We consider this as evidence that quantum field theory in generic α-vacua does not make sense.

Apart from the Euclidean vacuum, the antipodal vacuum is the only one where the non-local renormalization problem can be avoided. This vacuum state can be interpreted in terms of field theory on an orbifold of dS space, in which the non-local operators are local. It is possible that the resulting theory is just the analytic continuation of the Euclidean functional integral on RP^d, though one would have to do a more thorough study of boundary renormalizations in the Lorentzian orbifold in order to prove this.

Independently of this renormalization problem, there are clearly divergent matrix elements of local operators like the stress tensor on the fixed plane $t = 0$ of the Lorentzian orbifold. If we tried to couple gravity to the system this would lead to large back reaction effects. At the very least, a straightforward perturbative approach to the system would fail. Back reaction effects and the failure of the semiclassical approximation in general α-vacua are discussed in more detail in [15].

Chapter 7

Microscopic Quantum Mechanics of the $p = \rho$ Universe

7.1 Introduction

A little over two years ago, two of us (TB and WF) introduced a new approach to cosmological initial conditions called holographic cosmology [20]. The basic principle on which it was based is the holographic entropy bound [21][22]. In a Big Bang cosmology, the bound implies a finite entropy for any causal diamond[1] whose future boundary is a finite timelike separation from the Big Bang. This entropy decreases to zero as we approach the initial singularity. We interpreted this entropy as the entropy of the maximally uncertain density matrix for measurements done inside the causal diamond,

[1] In fact, all of our previous work referred instead to the causal past of a point. Raphael Bousso has repeatedly emphasized the greater virtues of causal diamonds (where every point can be both seen and influenced by an observer) and we have realized that all of our actual formulae could be taken to refer to causal diamonds rather than causal pasts.

a conjecture with several attractive features.

Our approach led us both to a tentative set of rules for defining a general quantum space-time, and to a heuristic approach to the Big Bang singularity. In this paper we close the circle of these ideas. We find a solution of the consistency conditions we have formulated for quantum cosmology, which behaves qualitatively like the dense black hole fluid which was the basis for our heuristic description.

The mathematical formalism which we will present in this paper was alluded to in several of our previous publications [23]. It is motivated by the results of Belinskii, Khalatnikov and Lifshitz (BKL) and subsequent workers, which suggest that dynamics near a Big Bang singularity is chaotic [24]. This leads us to postulate that the time dependent Hamiltonian near the Big Bang, is, at each instant chosen independently from a certain random distribution of Hamiltonians. We will describe the distribution in more detail in section 3. For large causal diamonds, this hypothesis leads to a time independent spectral density for the time dependent Hamiltonian; that of a $1+1$ dimensional conformal field theory. Thus, the system is given a random kick at each time, but the spectral density of the time dependent Hamiltonian approaches a universal limit. The energy/entropy density relation $\sigma \sim \sqrt{\rho}$ of this system is precisely that of our heuristic black hole fluid, and is the relation following from thermodynamics and extensivity in any dimension, for a fluid with equation of state $p = \rho$.

Guided by this correspondence, we argue that the energy per unit length of the $1+1$ dimensional system should be taken as the space time Hamiltonian for an observer in a given causal diamond in the $p = \rho$ background. Using the transformation between entropy and cosmological time, we show that this observer, in most of the states of the $1+1$ dimensional system, sees an energy precisely equal to the mass of a horizon filling black hole.

We then show that the basic structure of our quantum formalism allows us to derive the d dimensional space time metric, which is a flat FRW universe with perfect fluid matter satisfying the equation of state $p = \rho$. The scaling symmetry of the $1+1$ CFT is reinterpreted as invariance of the dynamics under the conformal Killing vector of this cosmology. This symmetry was crucial to our derivation [20] of a scale invariant fluctuation spectrum for the cosmic microwave background.

We have structured this paper in the following manner: In the next section we present a general framework for the local quantum dynamics of gravitational systems. The formalism associates operator algebras with causal diamonds in a space time. The details of the mapping depend on the nature of the boundaries of space time. The fundamental quantum variables are associated with holographic screens for a causal diamond by the Cartan-Penrose [25] equation. Heuristically, we view them as "quantum pixels on the holographic screen of a causal diamond". They transform as spinors under local Lorentz transformations and inherit a natural Z_2 gauge invariance from the classical CP equation. We use this gauge symmetry to transform them into fermions, explaining the conventional connection between spin and statistics.

In Section 3 we apply this general formalism in cosmology. We argue that it introduces a natural arrow of time. The relation between this and the thermodynamic arrow of time must be derived at a later stage. We suggest that a random, time dependent dynamics is the proper description of physics near the Big Bang, and propose a particular class of random Hamiltonians for this purpose, with results outlined above.

In the conclusions we recall the outline of our heuristic description of holographic cosmology and its application to observational cosmology. We sketch a program for deriving the assumptions and parameters of the heuristic picture from the mathematical formalism presented in this paper. We also introduce a more general model which describes a "gas of causally disconnected, asymptotically de Sitter (dS) universes" embedded in a $p = \rho$ background. Such a model can implement the anthropic principle for the cosmological constant, without requiring other parameters of low energy physics to be anthropically selected.

7.2 Local framework for a holographic theory of quantum gravity

Thirty years of work on perturbative and non-perturbative formulations of string theory, have presented us with ample evidence for the holographic nature of this theory of quantum gravity. Every gauge invariant quantity in all versions of the theory, refers to an observable associated with the conformal boundary of a spatially infinite space-time.

There is a simple intuitive argument, which suggests why this should be the case. A theory of gravitation must describe the apparatus which might measure any given prediction of the theory, because all physical objects gravitate. In a quantum theory, this is problematic, because the mathematical predictions of quantum theory refer to limits of measurements made by an arbitrarily large measuring apparatus. In a theory of gravity, such a measuring apparatus would have large effects on the system being measured unless it were moved an infinite distance away. This suggests that the pattern we have observed in string theory is an inevitable consequence of the marriage of gravitation and quantum mechanics. *All gauge invariant observables in a quantum theory of gravity describe the response to measurements made by infinite machines on infinitely distant surfaces.* String theory in asymptotically flat, asymptotically AdS, and asymptotically linear dilaton space-times obeys this rule.

Stringy evidence and simple physical intuition thus both point to the impossibility of defining gauge invariant quantities for local systems. But the necessity of describing a real world, which is cosmological in nature, suggests that we need a more local description of physics. This can be reconciled with the arguments above only by recognizing that no local description will be gauge invariant.

Indeed, this is a lesson we have already learned from attempts to quantize gravity in the semi-classical approximation. In order to define a concept of time and a quantum mechanics with unitary time evolution in this framework, we must choose a classical background solution [26][2]. The background plays the role of the infinite measuring device that we need to define a gauge invariant notion of time. The resulting

[2]The examples of the relativistic particle and world sheet string theory (viewed as two dimensional gravity) show that one can quantize a generally covariant system beyond the semiclassical expansion only by second quantizing it. This evidence suggested the notion of Third Quantization, but there is no consistent formulation of a Third Quantized theory above two dimensions. Practitioners of loop quantum gravity have also encountered the unitarity problem of the Wheeler DeWitt equation. They tend to either put it off to future research, or try to live with non-unitary time evolution.

formalism is quantum field theory in curved space-time. Time evolutions defined by different classical solutions, or even by different coordinatizations of the same classical solution, do not commute with each other and cannot be easily reconciled. This leads to the notion of Black Hole Complementarity, which gives a conceptual (though not yet a mathematical) resolution of the black hole information paradox. Two of us (TB and WF) generalized this to Cosmological Complementarity for Asymptotically dS (AsdS) space-times, and E. Verlinde has suggested the name Observer Complementarity to describe general space-times with event horizons.

Quantum field theory in curved space-time leads to the familiar paradox of black hole decay, and fails decisively in the presence of space-time singularities. The evidence is that the same is true for weakly coupled string theory, which also relies on a classical space-time background. We need a better way.

For some time, the present authors have felt that the fundamental clue to a local formulation of quantum gravity could be found in Bousso's general formulation of the holographic principle [27]. A fundamental notion in Lorentzian geometry is the concept of causal diamond. This is the region of intersection of the causal past of a point P with the causal future of a point Q which is in the causal past of P. The covariant entropy bound implies that for any causal diamond, the entropy that can flow through its boundary is bounded by the area of the maximal area $d-2$ surface on the boundary. We have conjectured [28] that in the quantum theory of gravity, this entropy should be associated with the logarithm of the dimension of the Hilbert space necessary to describe all measurements done inside the causal diamond. In every Lorentzian space-time, the covariant entropy bound for a causal diamond, is finite for sufficiently small time-like separation between P and Q.

Of course, finiteness of the entropy of a density matrix does not by itself imply that the Hilbert space of the system is finite. But finite entropy density matrices in infinite systems, rely on special sets of operators (typically the Hamiltonian) whose spectrum defines a natural restriction of the Hilbert space. Our general discussion of quantum gravity suggests that a local description should contain no such special operators. That is, in general we expect the Hamiltonian of a local observer to be time dependent, and different observers will have different, generally non-commuting, time dependent Hamiltonians. The only natural density matrix, whose definition does not depend on a special operator, is the unit matrix.

The finite dimensional Hilbert space conjecture meshes with the arguments above, because a finite dimensional system cannot describe the infinite machines which make operational sense of the precise mathematical predictions of quantum theory. Thus we view a small causal diamond as defined in quantum theory by a (generally time dependent) Hamiltonian on a finite dimensional Hilbert space. Since such a system can never make arbitrarily precise measurements on itself, its Hamiltonian and other observables cannot be fixed. That is, given the *a priori* restriction on the precision of measurements, we will always be able to find many alternative mathematical descriptions, which agree up to the specified level of precision allowed by the size of the causal diamond. We view this statement as the quantum origin of the *Problem of Time* in semiclassical general relativity[3] and we view any given Hamiltonian description of a causal diamond as a gauge fixing. The aptness of this metaphor will become more apparent as we get deeper into the formalism.

We have not yet pointed out the most important aspect of our conjecture, namely that it provides a derivation of a notion of locality from the holographic principle itself. Indeed, what could it mean to assert the finiteness of the operator algebra associated with a causal diamond, if not the statement that it formed a tensor factor of the operator algebra of the entire space-time? The operators of the causal diamond $\mathcal{D}$ commute with all other operators necessary to describe the physics in any larger causal diamond $\mathcal{D}'$ containing $\mathcal{D}$.

The algebraic formulation of quantum field theory similarly assigns an operator algebra to each causal diamond. The field theory operator algebras are all infinite, and the detailed relation between algebraic and space-time structure will be different than what we propose here. However, the similarities of the two frameworks may eventually provide us with a better understanding of how field theory arises as a limit of a real theory of quantum gravity.

7.2.1 The hilbert space of an observer

The basic idea of our program is to use the holographic conjecture about the dimension of the Hilbert space of a causal diamond, to translate geometrical concepts into quantum mechanics. We urge the reader to think of the geometrical pictures as

[3] More generally, it is the quantum origin of general coordinate invariance.

"guides to the eye" at this stage, and to think of the quantum formalism as fundamental. At a later stage, one would hope to obtain a mathematical derivation of the rules of Einsteinian geometry from the quantum formalism. In this paper, we will provide one example of such a derivation, in a very special case.

We will use the word *observer* to denote a large, localized quantum system, which is capable of carrying out "almost classical" measurements on its environment. Any such observer will follow a timelike trajectory through space-time. We can describe this trajectory in terms of causal diamonds in the following manner. First consider space-times such that the observer's trajectory has infinite timelike extent in both past and future. Pick a point P on the trajectory and a segment of equal length to the past and future of P. Take the causal diamond defined by the endpoints of this segment. As we make the interval smaller, the FSB area of this diamond gets smaller. If we want to associate this area with the logarithm of the dimension of a Hilbert space, this process must stop at some smallest length. Let $\mathcal{K}$ be the dimension of this smallest Hilbert space. We will make a proposal for $\mathcal{K}$ in a moment.

Now we extend the interval around the point P, until the area of the causal diamond has increased by the logarithm of the dimension of $\mathcal{K}$[4]. By continuing this procedure, we describe the information that can be measured in experiments done by an observer in terms of a sequence of Hilbert spaces, $\mathcal{H}_N$ of dimension $(\dim\mathcal{K})^N$. This corresponds to a sequence of causal diamonds, as shown in Fig. 7.1. The entropy of

[4]One could imagine a formalism in which one changes the dimension of the Hilbert space by one at each step. It is harder to describe this in terms of an attractive operator algebra. Our motivation for tensoring in a fixed Hilbert space at each step is the concept of a holographical pixel, to be defined below.

the maximally uncertain density matrix for this system is $N \ln(\dim \mathcal{K})$. This is to be identified with one quarter of the area of the causal diamond in Planck units.

For Big Bang cosmologies, we can do something similar, but it is convenient to choose causal diamonds whose past tip lies on the Big Bang, and extend them only into the future. The smallest causal diamond for any observer, is that observer's view of the Big Bang hypersurface. Note that it will be completely finite. In our view, the Big Bang looks singular in general relativity, because one is thinking of the theory as a field theory and trying to describe all of the degrees of freedom of that theory in each horizon volume. The holographic principle suggests instead that near the Big Bang surface, small causal diamonds contain very few degrees of freedom, and have a completely non-singular quantum description.

Although the quantum mechanics of a causal diamond is always independent of that in other causal diamonds in the same space-time[5], one should not imagine that the initial state in a generic causal diamond is pure. Interactions to the past of the diamond could have entangled its degrees of freedom with those of other disjoint diamonds. Our fundamental cosmological hypothesis will be that the state in a causal diamond whose past tip is on the Big Bang, is pure. This corresponds to the familiar notion of *particle horizon.* All quantum correlations between the degrees of freedom of the system are to be generated by the dynamics, rather than put in as initial conditions.

We would like to emphasize that this hypothesis introduces the Arrow of Time

[5]That is, the Hilbert space of a causal diamond contains all the degrees of freedom necessary to describe measurements in that region. There *will* be mappings between the Hilbert spaces of different causal diamonds, and consistency relations among the different time evolution operators.

as a fundamental input to the definition of cosmology. That is, we could define both Big Bang and Big Crunch cosmologies (with, for simplicity, a past or future with the asymptotic causal structure of Minkowski space), in terms of semi-infinite sequences of Hilbert spaces. However, in the Big Bang case, the initial conditions would be subject to our purity constraint for causal diamonds whose tip lies on the singularity. By contrast, in the Big Crunch, the initial conditions would be described in terms of scattering data in the remote past. Even if we discussed finite causal diamonds whose future tip lay on the Big Crunch, it would not make sense to assume the final state in those causal diamonds was pure. It has been correlated with the states in each other causal diamond, by the evolution of the scattering data down to the singularity. *Thus we contend that the intrinsic formulation of a theory of quantum cosmology, forces us to introduce a time asymmetry, when there is a cosmological singularity*[6].

The causal diamond formalism automatically introduces an ultraviolet energy cutoff, because it discretizes the time step. Notice however that the cutoff is not uniform in time. In a region of space-time (and a given foliation) where the spatial curvature is negligible, the area of causal diamonds scales like the proper time to the $d-2$ power. So a fixed area cutoff, corresponds to a finer and finer slicing of proper time, as N increases. To get an intuitive feeling for this scaling note that it is the same as what one gets by applying the time energy uncertainty relation and saying that the time step is the inverse of the energy of the largest black hole that can fit in to the causal diamond

[6]If there is a reasonable description of a universe which undergoes a Big Bang followed by a Big Crunch, the time direction will be specified by the purity constraint. We would describe such a universe in terms of pure states in causal diamonds with their tip on the Big Bang. The range of N would be finite, and only the last causal diamond in the sequence would touch the Big Crunch.

at step N[7].

Note that, while we have introduced geometrical notions (area), our construction says nothing as yet about the actual geometry of space-time. One can introduce trajectories via sequences of causal diamonds with fixed area step, in any Lorentzian space-time. Certain global aspects of the space-time are encoded in the behavior of $\mathcal{H}_N$ for large N. In space-times with asymptotic causal structure like that of Minkowski space, the area of the causal diamond goes to infinity continuously as the time-like separation between its tips goes to infinity. In asymptotically AdS space-times, the area goes to infinity at finite time-like separation, when the causal diamond hits the time-like boundary of AdS. After that point the operator algebra becomes infinite and is equal to the algebra of conformal fields on the boundary, smeared with functions of compact support in boundary time. In asymptotically dS spaces, we expect the operator algebra to remain finite even in the limit of infinite proper time. We have already discussed the modification of the formalism necessary to the description of space-times with cosmological singularities. Thus, the boundary geometry of space-time affects the nature of the index set N (in AdS, the mapping between N, which counts area, and time, becomes singular at a finite time. After this point, the time becomes a continuous parameter while the area is infinite). In asymptotically dS space-time we can choose N to parametrize a discrete global time. Then N is allowed to go to infinity, but we stop adding degrees of freedom at a finite value of N). More generally, we expect the

[7] Here and henceforth, we will use a rough definition of a black hole as a localized concentration of energy and entropy, which maximizes the entropy for a given energy. We are aware that none of these concepts has an absolutely rigorous definition in general relativity.

geometry to emerge from an interplay between area and the time evolution operators in each Hilbert space $\mathcal{H}_N$.

In each Hilbert space, we postulate a sequence of unitary operators $U_N(k) \equiv e^{-iH_N(k)}$ for $1 \leq k \leq N$. In a Big Bang space-time $U_N(k)$ is supposed to represent the evolution of the system between the future tips of the k-th and $(k-1)$-th causal diamond.[8] Here we encounter the first of the fundamental consistency conditions of quantum gravity. The Hilbert space $\mathcal{H}_N$ contains a tensor factor isomorphic to $\mathcal{H}_K$ for $K < N$. Inside this factor the dynamical description of the later observer, must coincide with its own past history. That is

$$U_N(k) = U_K(k) \otimes V_{NK}(k), \tag{7.1}$$

for $k \leq K$. We should view the operator $V_{NK}(k)$ as describing the dynamics of degrees of freedom, which are, at time k, not observable by the observer under discussion. It acts only on the tensor complement of $\mathcal{H}_K$ in $\mathcal{H}_N$. It will become important when trying to make the dynamics consistent with the descriptions given by other observers.

We hope that this discussion of the Hilbert space of a single observer has been relatively easy to follow. By contrast, it is extraordinarily difficult to get one's head around the consistency conditions relating observers with different time-like trajectories. We attack this question by first introducing the $p = \rho$ cosmology, where there is a simple solution of all of the consistency conditions. Only at the end of our discussion of this cosmology will we return to the consistency conditions in a general space-time.

[8]From now on we will concentrate on the cosmological case. Much of the discussion has an obvious generalization to other boundary conditions.

First however, we introduce our parametrization of the operator algebras in terms of holographic pixels, and define the Hilbert space $\mathcal{K}$.

7.2.2 SUSY and the holoscreens: the degrees of freedom of quantum gravity

We now want to make an ansatz for the Hilbert space $\mathcal{K}$ which will connect our formalism to Riemannian geometry. If we associate the degrees of freedom with the holographic screen of a causal diamond, then the most fundamental thing that occurs when we increase the size of the diamond is that we "add a pixel" to the screen. The minimal new information must tell us about the size and orientation of that pixel, and about the null direction along which information from the bulk is projected onto the pixel.

There is a classical geometrical description of the orientation of a holographic screen in terms of *pure spinors* [25]. A pure spinor in d dimensions satisfies

$$\bar{\psi}\gamma^{\mu}\psi\gamma_{\mu}\psi = 0 \tag{7.2}$$

The defining equation is homogeneous and classically one views two pure spinors as identical if $\psi_1 = \lambda\psi_2$, where λ is real or complex depending on the reality of the spinor representation. In $3, 4, 6$ and 10 dimensions, a general spinor in the smallest irreducible spinor representation of the Lorentz group is automatically pure. The CP equation comes up repeatedly in superstring theory, particularly in the *super-embedding* approach [29].

The CP equation defines neither the position nor the size of the holographic pixel. Only the direction of the null vector and the orientation of its screen are fixed. This is in accord with the intuition that metrical notions, like area, are measured in Planck units, and should not appear until we quantize the theory.

To quantize the pixel variable ψ, we first note that it has half the components of a general Dirac/Majorana spinor (we impose Majorana conditions in those dimensions in which they exist). Denote the non-vanishing components as $\hat{S}_a$. They transform as the spinor representation of $SO(d-2)$, the transverse rotation group which leaves n^μ invariant. Note that in choosing to quantize only the physical components of the pure spinor, we are partially choosing the gauge for local Lorentz invariance, leaving over only an $SO(d-2)$ subgroup. Quantization of the pixel variable is dimension dependent. In the remainder of the paper, we will treat $p=\rho$ universes with arbitrary dimension, but in order to be specific, we will here discuss only the case $d=11$, where $\hat{S}_a$ has 16 real components. The finite Hilbert space $\mathcal{K}$ of the previous section will be identified with the Hilbert space of a single quantized pixel. The most general $SO(9)$ invariant quantization rule, which is representable in a Hilbert space with a finite number of states is

$$[\hat{S}_a, \hat{S}_b]_+ = 2\delta_{ab} \tag{7.3}$$

Note that this rule breaks the projective invariance of the classical CP equation, except for a Z_2 subgroup. We view this residual Z_2 as a gauge symmetry, which should be implemented in the quantum theory.

We now utilize these variables to construct the Hilbert spaces of the previous section. For a single observer we add a single copy of the $\hat{S}_a$ algebra at each time step. The new operators, $\hat{S}_a(N)$, commute with the operators, $\hat{S}_a(t); t < N$, describing the smaller causal diamond at the previous time step. The Hilbert space we tensor in is the irreducible representation of this Clifford algebra. It is easy to satisfy the consistency conditions for the evolution operators, by choosing $H_N(k)$, $N > k$, to be a sum of two terms. The first depends only on the $\hat{S}_a(t)$ for $t \leq k$, and the second only on those with $t > k$. The first term is chosen equal to $H_k(k)$.

Z_2 gauge invariance is guaranteed by choosing each Hamiltonian to contain only even polynomials in the pixel operators. We can then perform a Z_2 gauge transformation, to define new variables by

$$S_a(n) = (-1)^{F_n} \hat{S}_a(n), \tag{7.4}$$

where $(-1)^{F_n}$ is the product of all of the $\hat{S}_k$ for $k \neq n$. We then obtain the fermionic algebra

$$[S_a(m), S_b(n)]_+ = 2\delta_{ab}\delta_{mn} \tag{7.5}$$

Fermi statistics is thus seen to be a quantum remnant of the projective invariance of the CP equation, and the spin statistics connection is built in to our formalism[9]

[9]The cosmology we will describe in this paper has no particle excitations, so the relation between these fermionic commutation relations and the statistics of particles will not be evident. In [30] one of the authors will present a holographic description of 11 dimensional SUGRA in flat space-time which will exhibit the precise connection.

Later, when we speak of maps between Hilbert spaces corresponding to spatially separated, but overlapping causal diamonds, $\mathcal{H}(\mathcal{D}_1)$ and $\mathcal{H}(\mathcal{D}_2)$ we will view these maps as implemented by isomomorphisms between subalgebras of the pixel operators on each Hilbert space. Note that these need not be linear mappings between the generators. We can find non-linear functions of the pixel operators, which satisfy the same Clifford algebra. The homomorphism might be a linear map between the fundamental pixel operators of one Hilbert space, and such "composite" pixel operators in another.

7.2.3 Rotation invariance

A model of a homogeneous isotropic universe, should be invariant under spatial rotations. In our $11D$ example, the 16 real S_a operators transform as a spinor of $SO(9)$ but not of $SO(10)$. There is an analogy, which we believe will be helpful in understanding rotation invariance [30], between the $S_a(n)$ operators and sections of the spinor bundle over the 9-sphere. Any such section is given locally, by a map $S_a(\mathbf{\Omega})$, from the sphere to the spinor representation of the $SO(9)$ which preserves a point $\mathbf{\Omega}$. We should think of the $S_a(n)$ as finite dimensional analogs of sections of the spinor bundle over the sphere.

The seminal idea of non-commutative geometry [31] is to replace the commutative C^* algebra of continuous complex valued functions on a manifold, with a general non-commutative C^* algebra. In particular, if we choose finite dimensional matrix algebras we obtain *fuzzy spaces*. Particular infinite sequences of matrix algebras lead to fuzzy approximations to Riemannian manifolds.

In non-commutative geometry, the concept of vector bundle is replaced by the (equivalent in the commutative case) notion of a *projective module*. A projective module R over an associative algebra $\mathcal{A}$ is a representation of $\mathcal{A}$ with the property that there exists another representation $\bar{R}$ such that $R \oplus \bar{R} = \mathcal{A}^p$, where the power means pth tensor product of the regular representation of $\mathcal{A}$ on itself by left multiplication. This is the analog of the existence of an anti-bundle $\bar{V}$ for each vector bundle V over a commutative manifold, such that $V \oplus \bar{V}$ is trivial.

Our $S_a(n)$ variables should belong to an operator valued projective module for a finite dimensional associative algebra on which $SO(10)$ acts. Finite dimensional representations of the Clifford-Dirac algebra γ^M of SO(10) are examples of such fuzzy 9 spheres. The smallest one is given by the irreducible representation of the Clifford-Dirac algebra and has real dimension 32. In formulas below, we will use this doubling of the indices of $S_a(n)$ to ensure $SO(d-1)$ rotation invariance.

We will not pursue these rotational properties further in this paper, but note merely that they may be helpful in resolving a puzzle we will encounter later.

7.3 Quantum cosmology of a dense black hole fluid

7.3.1 The random operator ansatz

We now want to present a complete solution of the general constraints on quantum cosmology. We will argue that this solution corresponds to a flat FRW universe with equation of state $p = \rho$. This is the system which we have studied heuristically in previous publications under the name of "a dense black hole fluid". The mathematical analysis of this section will, we believe, amply justify that colorful terminology. We emphasize that we are presenting this solution of the constraints before making a general statement of what the constraints are. We hope that this order of presentation will help readers to understand the general construction.

A fundamental clue to our mathematical formalism is the result of BKL [24] that the dynamics of general relativity near a space-like singularity is chaotic. This suggests that the quantum theory should be described by a random Hamiltonian. The causal diamond formalism and its description in terms of fermionic holopixels suggests a particular ensemble of random Hamiltonians.

Let us begin by considering the quadratic term in the Hamiltonian $H_N(N)$. It has the form

$$H_N^{(2)}(N) = i\frac{1}{N}S^a(n)h^{mn}S^a(m) \equiv \frac{1}{N}H_{FT}, \tag{7.6}$$

where h^{mn} is a real anti-symmetric $N \times N$ matrix. We have imposed $SO(d-1)$ invariance by using the invariant scalar product on the component indices of S[10]. Our ansatz will be to choose h to be a gaussian random matrix with the standard probability distribution $P(h) = e^{N\mathrm{tr}h^2}$. For large N the distribution is described by a master field, with spectral density given by the Wigner semi-circle law, $\rho_h(x) = \sqrt{1-x^2}$. The distribution is flat near the origin and has a cutoff of order one for its eigenvalues. It then follows that the large N thermodynamics of $H_{FT} \equiv NH_N^{(2)}(N)$ is that of a free $1+1$ dimensional

[10]Here we are assuming that the appropriate fuzzy spinor bundle is just the direct sum of copies of the minimal one in which we double the indices of S_a to extend it to an $SO(d-1)$ representation. This doubling should be understood in the above formula. It may be that this missing factor of $d-2$ we encounter below is an indication that this is the wrong choice.

fermionic field theory [32]. The entropy is of order N, the eigenvalue spacing is of order $\frac{1}{N}$. Thus H_{FT} should be viewed as a $1+1$ dimensional free fermion system with UV cutoff of order 1, living on an interval of length of order N. The $1+1$ dimensional entropy and energy densities are related by $\sigma_{1+1} \propto \sqrt{\rho_{1+1}}$. We will identify these as the space-time entropy and energy densities of our cosmology. This equation of state would be appropriate for an FRW universe with equation of state $p = \rho$. Before pursuing this relationship, let us extend our ansatz for the basic Hamiltonian.

The thermodynamics of this system is dominated by the IR physics of $1+1$ CFT. This will be unchanged by a wide class of perturbations of H_{FT}. Indeed, the only relevant perturbations of this system are the fermion mass and the marginally relevant four fermi operators. Our random matrix ansatz has automatically set the fermion mass to zero. The marginally relevant perturbations are marginally irrelevant if their sign is appropriately chosen. Thus we can add to H_{FT} an arbitrary even function of the pixel operators of degree ≥ 4, *whose coefficients in the eigenvalue basis of h_N are smooth functions of the eigenvalue in the large N limit*, as long as the sign of the quartic terms is chosen correctly. We see that a very wide class of random Hamiltonians for our system, will have identical large N thermodynamics. Thus, our full ansatz for the cosmological time evolution is that for each N we make an independent choice of random Hamiltonian, $H_N(N)$, from the distribution defined in the last two paragraphs.

The operators $H_N(k)$ with $k < N$ are partially fixed by the requirement that $H_N(k) = H_k(k) \otimes 1 + 1 \otimes O_N(k)$, where $O_N(k)$ depends only on the variables $S_a(t)$ with $N \geq t > k$. The universe experienced by the observer in this causal patch is unaffected by the choice of these operators. One might however have thought that they were constrained by the spatial overlap conditions. For our choice of overlap conditions in the $p = \rho$ universe, this turns out to be untrue. The $O_N(k)$ are completely unconstrained. We suspect that this might not be the case for more general space-times. We will see below, that although our ansatz reproduces the scaling laws of the $p = \rho$ universe, it fails to reproduce certain more refined features of the geometry. This leads us to surmise that the ansatz needs to be modified. The necessary modification is likely to require us to specify $O_N(k)$.

A full definition of a quantum space-time must include the descriptions of other observers. A coordinate system can be thought of as a way of covering space-time by the trajectories of observers. We will choose time-like observers and will choose a time slicing such that at a given time, along any trajectory defining our coordinate system, the area of the maximally past extended causal diamond is the same. We call this equal area slicing of a Big Bang space-time. At (say) the initial time the ends of the trajectories form a lattice. We specify the topology of this spatial slice, including its dimension by choosing a particular topological lattice. For simplicity of exposition, we will choose the $d-1$ dimensional hypercubic lattice. At large N this choice will not matter and our ansatz would work for any lattice with the same continuum topology.

Each trajectory is specified by a sequence of Hilbert spaces and unitary operators as above. Two neighboring trajectories would correspond to two overlapping sequences of causal diamonds, as shown in Fig. 7.2 . *A priori* one could imagine making independent choices of Hamiltonian at each point on the spatial lattice. We will argue that this is inconsistent with the random operator hypothesis, and that in fact the sequence of Hamiltonians defining a given observer will be identical at all spatial points. Only the initial state can differ from point to point. Indeed, the causal diamonds of two trajectories will generally have an overlap Fig. 7.2 . The overlap will not be a causal diamond, but will contain some maximal area causal diamond. It is reasonable to postulate that the information which could be accessed in the overlap can be encoded in a Hilbert space which is (isomorphic to) a tensor factor in each of the individual causal diamond Hilbert spaces. Furthermore, if we look at the actions of the time evolution operators of the individual diamonds, on this common factor space, they must agree. Since there are many such overlaps, this is a very strong constraint on the dynamics.

In the $p = \rho$ cosmology, our ansatz for spatial overlap Hilbert spaces is simple and general. If we consider two Hilbert spaces $\mathcal{H}_N(\mathbf{x})$ and $\mathcal{H}_N(\mathbf{y})$ which are s steps away from each other on the lattice, we choose the overlap to be $\mathcal{H}_{N-s}(\mathbf{x}) = \mathcal{H}_{\mathbf{N-s}}(\mathbf{y})$. In finer detail, we identify the individual $S_a(t, \mathbf{x})$ operators, with their counterparts in the Hilbert space at $\mathbf{y}$. If we now require that the Hamiltonian evolutions of each sequence of causal diamonds are identical, then all of our consistency conditions are satisfied, in the following sense. For each geometrical overlap between causal diamonds, we have defined a Hilbert space and a sequence of time evolution operators, which purports to describe the physics in the overlap region of space-time. The overlap Hilbert space is a tensor factor in each of the individual observer's Hilbert space. Furthermore, the dynamics in this tensor factor is consistent with that defined by either of the individual observers.

It seems likely, but we have not been able to prove, that there is no other solution of the overlap conditions which would be compatible with each observer having a random sequence of Hamiltonians.

7.3.2 Homogeneity, isotropy and flatness

Our construction is homogeneous on the spatial lattice. We have built isotropy into our construction in a formal way, by insisting on $SO(10)$ invariance. The overlap rules give us further indications that our system is isotropic. We will have occasion to refer both to the Euclidean distances and angles on our hypercubic lattice, and the actual Riemannian distance in the space-time metric we claim to be constructing. The reader should be careful to keep these two ideas completely separate. We have defined a space-time lattice with lattice points labeled $(N, \mathbf{x})$. Define the base of the causal past of the point $(N, \mathbf{x})$ to be the set of all points on the lattice, whose Hilbert space at time N has an overlap with $\mathcal{H}_N(\mathbf{x})$. According to our overlap rules, the boundary of this set is given by the endpoints of walks on the lattice, starting at $\mathbf{x}$ and increasing the Euclidean distance on the lattice at each step. The base of the causal past thus forms a hypercube oriented at forty five degrees to the coordinate axes. Each step along the walk reduces the area of overlap by one unit, and so should be thought of as increasing the Riemannian distance by some (N dependent) unit. Thus, the boundary of the base of the causal past consists of points which are the same Riemannian distance away from $\mathbf{x}$.

Think of a *carpenter's ruler* which follows a walk along the lattice to the boundary of the base of the causal past. The map between the coordinate (lattice) space and the real geometry, is given by "straightening out the carpenter's ruler". The tilted hypercube is mapped into a sphere.

We have thus derived homogeneity and isotropy of our cosmology from our definition of the overlap rules. Given the non-compact topology of the lattice, the spatial curvature is non-positive. There are three different arguments that it is zero. The first is simply that our model saturates the entropy bound. At any given late time, the excited states of our system are generic states of the Hilbert space, because they are obtained by the action of a sequence of random Hamiltonians. We know that even the maximally stiff equation of state $p = \rho$ cannot saturate the entropy bound in a universe of negative curvature.

The second argument for flatness also shows us that our spin connection is Riemannian. The overlap conditions have forced us to identify the S_a operators in Hilbert spaces at different points. Thus, the parallel transporter is the identity in $SO(10)$ and the curvature of the spin connection vanishes.

Finally, note that for large N the spectrum of our system has a scaling symmetry because it is that of a $1+1$ CFT. If it is to be identified with an FRW universe, that universe should have a conformal isometry corresponding to the symmetry[11]. Such an isometry exists for any FRW universe with flat spatial sections and a single component equation of state. Curved spatial sections introduce a scale and such geometries do not have a conformal isometry.

[11] We also learn that the "matter" in this universe must be invariant under this conformal isometry.

The last argument can be stated in another way. We have defined a sequence of physical spheres, *the causal boundaries at time* N on our $d-1$ dimensional coordinate lattice. If the spatial geometry were curved, we would expect to see a scale, the radius of curvature, at which the behavior of the geometry changed. As we take N to infinity we will sweep through this scale. However, the dynamics does not have such a scale in it. It becomes scale invariant for large N.

To summarize, we have shown that the random Hamiltonian ansatz, which obeys our consistency conditions for a quantum cosmology, gives a spatial geometry which is homogeneous, isotropic and flat. It also obeys two laws which suggest that it is in fact the quantum realization of $p = \rho$ cosmology. The entropy bounds are saturated for all time, and the energy entropy relation of an extensive $p = \rho$ fluid is valid at all times. In the next subsection we will provide further evidence that this is the right interpretation of our system.

7.3.3 Time dependence - scaling laws

In order to discuss the time dependence of our geometry, we have to identify the conventional cosmological time parameter in terms of the parameters of our quantum system. In any flat FRW cosmology, the area of causal diamonds at cosmological time t, scales as t^{d-2}. Thus, we should write $N \sim t^{d-2}$. The logarithm of the N dependent time evolution operator is $-i\Delta N H_N$, where ΔN is N independent. Writing

$$\Delta N \sim t^{d-3}\Delta t \tag{7.7}$$

we see that the cosmological time dependent Hamiltonian is

$$H(t) \sim N^{\frac{(d-3)}{(d-2)}} H_N \tag{7.8}$$

$H_N(N)$ is the Hamiltonian as viewed by an observer in a given causal diamond. To the extent that one can really talk about such an observer in the heuristic picture of a dense black hole fluid one views it as hovering about the maximal black hole at a distance of order its Schwarzchild radius. The energy of the system is just the energy of the black hole for such an observer. In our quantum mechanical model, for most states of that system, the energy per unit length is of order 1 (*i.e.* N independent). Thus

$$H(t) \sim N^{\frac{(d-3)}{(d-2)}} \tag{7.9}$$

This implies that the local cosmological observer sees an energy which scales like the mass of the maximal black hole, exactly as required by our heuristic picture. Note that this calculation works in any dimension.

We can get further confirmation by noting that we have outlined an order of magnitude calculation of the physical size of the particle horizon in the previous subsection. It is N lattice steps in coordinate space, while the UV cutoff scales like $N^{-\frac{(d-3)}{(d-2)}}$. Thus, the physical size of the particle horizon scales like $N^{-\frac{1}{d-2}}$. Since the spatial geometry is flat, this implies a horizon volume

$$V_H \sim N^{-\frac{d-1}{d-2}}$$

The cosmological energy density is obtained by dividing 7.9 by this volume. Thus,

$$\rho \sim N^{-\frac{2}{d-2}} \sim \frac{1}{t^2} \tag{7.10}$$

where at the last stage we have again used the relation between entropy and cosmological time. Similarly, the total entropy is N so the entropy density is

$$\sigma \sim N^{-\frac{1}{d-2}} \sim \frac{1}{t} \tag{7.11}$$

Thus, we have obtained both the $\sigma \sim \sqrt{\rho}$ equation of state of the $p = \rho$ universe, as well as the $\frac{1}{t^2}$ dependence of energy density, usually derived from the Friedmann equation, from a purely quantum mechanical calculation.

7.3.4 Time dependence: a consistency relation, and a failure

Another interesting geometrical quantity is the area of the overlap causal diamond, as a function of N and of the geodesic separation between the trajectories. In the Appendix 7.6 we calculate this area for a general flat FRW space-time. Not surprisingly, it scales like Δ^{d-2} where Δ is the geodesic separation. On the other hand, in our quantum definition of overlap, the entropy in the overlap is $(N-k)L_S$, where k is the minimal number of lattice steps separating the tips of the two causal diamonds, and $L_S = \ln(\dim \mathcal{K})$. The overlap entropy is linear in k. We have argued that for fixed N, the number of steps is linear in the geodesic distance Δ.

This is not necessarily a contradiction. The quantum calculation is only supposed to agree with the geometrical picture in the limit that N is large, and for causal diamonds which have large area. The area of the overlap diamond decreases to zero as $k \to N$. Thus, it might be reasonable to require agreement with geometry only for $\frac{k}{N} \ll 1$. In this limit, both expressions are linear in k and we can compare how they scale with N.

Consider two diamonds in a flat FRW space-time, whose future tips lie at conformal time η_0. Let these two diamonds be separated by co-moving coordinate distance Δx. Then, according to our calculations in the Appendix 7.6, the area of the maximal causal diamond which fits in their intersection is, to leading order in Δx,

$$A_{int}^{Geo} = A\left(1 - \frac{d-2}{\eta_0}\Delta x\right) \tag{7.12}$$

To fit with the quantum mechanical picture, where the entropy associated with this intersection is $(N-k)L_s$ for two diamonds separated by k lattice steps , we must choose $A = 4NL_s$, and $\frac{N}{\eta_0}(d-2)\Delta x = 1$, for the co-moving separation corresponding to a single step on our coordinate lattice. The geodesic distance at time $\frac{\eta_0}{2}$ (the time of maximal area on the causal diamonds) represented by a step is thus

$$\Delta d = a\left(\frac{\eta_0}{2}\right)\frac{\eta_0}{N(d-2)} \tag{7.13}$$

There is now a consistency condition. We can compute the area of the causal diamond at time η_0 in two ways. On the one hand, in order to causally separate two causal diamonds, we must, according to our overlap rules, move N steps on the lattice. This indicates that the radius of the maximal sphere on the causal diamond is $\frac{N}{2}$ lattice steps. This corresponds to an area

$$A = \Omega_{d-2} \left(\frac{N}{2} \Delta d \right)^{d-2} \tag{7.14}$$

Ω_{d-2} is the area of a unit $d-2$ sphere. This area (in Planck units, and we have set $G_N = 1$) must be $4NL_s$. This gives us a second equation for Δd

$$\Delta d = \frac{2}{N} \left(\frac{4NL_s}{\Omega_{d-2}} \right)^{\frac{1}{d-2}} \tag{7.15}$$

Note that this has an attractive scaling property $\Delta d \sim N^{-\frac{d-3}{d-2}}$. We have suggested that the proper time cutoff scales like the inverse of the energy of the maximal black hole, which fits in a causal diamond. Here we find a spatial distance cutoff of the same order of magnitude.

To compare the two expressions for Δd we use the Friedmann equation for $p = \rho$ geometry to write

$$a \left(\frac{\eta_0}{2} \right) = a_0 \left(\frac{(d-2) a_0 \eta_0}{2(d-1)} \right)^{\frac{1}{d-2}} \tag{7.16}$$

We also express η_0 in terms of the area, and thence the entropy

$$\eta_0 a_0 = 2^{\frac{d+1}{d-1}} \left(\frac{NL_s(d-1)}{(d-2)\Omega_{d-2}} \right)^{\frac{1}{d-1}} \tag{7.17}$$

Plugging these expressions into 7.13 we obtain

$$\Delta d = \frac{1}{d-2} \frac{2}{N} \left(\frac{4NL_s}{\Omega_{d-2}} \right)^{\frac{1}{d-2}} \tag{7.18}$$

Thus, the two expressions for the geodesic distance scale the same, but differ by a factor $d-2$. We have not been able to explain this discrepancy. It is clearly related to the fact that the relation of overlap area to geodesic distance in geometry is $A \sim \Delta^{d-2}$. We suspect the discrepancy indicates the need for a slight modification of our overlap rules, and is connected to another disturbing feature of these calculations. One might have expected the numerical factors in the matching of geometry to quantum mechanics would depend on the dimension of the pixel Hilbert space $\mathcal{K}$, which in turn depends on the space-time dimension. Further, one might have expected the overlap rules to have a directional dependence on the lattice which should break the local $SO(d-1)$ invariance of the individual fermionic Hilbert spaces, leaving only a global $SO(d-1)$. Neither of these expectations is realized in our current rules, and we expect that when the rules are modified to take this into account, the discrepant factor of $d-2$ will disappear.

We emphasize that the calculation of the area of overlaps *does* have consistent scaling behavior with N. This is an independent check that our quantum system satisfies the scaling laws of $p = \rho$ geometry. In order to achieve this we had to insist on comparing geometric and quantum predictions only at leading order in the area. For a more normal space-time background this would probably not be sufficient to reproduce what we know of the physics. The $p = \rho$ fluid appears to be a system in which the laws of geometry are satisfied only in a very coarse grained sense.

We have tried to find other detailed numerical comparisons between our quantum formalism and space-time physics. Unfortunately they all seem to lead simply to a definition of constants in the quantum formalism. We record these calculations in the

Appendix 7.6.

7.4 More general space-times

The general kinematic framework for discussing holographic space-times is very similar to what we outlined above. We will distinguish two different kinds of temporal asymptotics: Scattering universes and Big Bang universes. Big Crunch space-times pose additional problems, which we will ignore in this paper.

A Scattering universe has past and future asymptotics which are describable in terms of QFT in curved space-time. That is to say, in both the past and the future there is a complete set of scattering states, which may be viewed as localized excitations propagating on a classical geometry. The Penrose diagram of a true scattering universe will be like that of Minkowski space, or the universal cover of AdS. In the semi-classical approximation, dS space is a scattering universe, but if one accepts the conjecture that the quantum theory has a finite number of states, this is no longer precisely correct. Nonetheless, we will include dS space under the rubric of scattering universes. The reason for this is our belief [28] that as the c.c. goes to zero, the theory of dS space will contain a unitary operator which converges to the scattering matrix of an asymptotically flat space-time. The definition of this operator will contain ambiguities which go to zero exponentially with the c.c., as long as the scattering energies are kept fixed as Λ goes to zero. We will reserve the phrase *true scattering universes* to describe space-times with a Penrose diagram similar to that of Minkowski space. This does not imply that the geometry is asymptotically flat. Non-accelerating FRW universes are also true scattering universes. Big Bang space-times can asymptote either to a future scattering universe or to dS space.

In a scattering universe, one describes the quantum theory by picking a point on a time-like trajectory, and considering the causal diamonds defined by successively larger intervals around that point, as in Fig. 7.1. For each causal diamond we have a sequence of unitary transformations $U_N(k)$ which describe time evolution in each of the sub-diamonds contained in it. These must satisfy the causality requirement

$$U_N(k) = U_k(k) \otimes W_N(k),$$

where $W_N(k)$ acts only on the tensor complement of $\mathcal{H}_k$ in $\mathcal{H}_N$. As $N \to \infty$, in a true scattering universe, we will have

$$U_N(N) \to U_+(N) S U_-(N),$$

where $U_\pm(N)$ describe free asymptotic propagation and S is the scattering matrix. In an asymptotically (past and future) dS universe there should be, in the limit of small cosmological constant, a similar construction [28], [33]. However, in this case we cannot take the large entropy limit. After some time, the dimension of the Hilbert space stops increasing. Nonetheless, in the limit of small cosmological constant, we expect an approximate S-matrix to exist. It would describe a single observer's experience of excitations coming in through its past cosmological horizon and passing out through its future cosmological horizon. However, most of the states in the system cannot be viewed in this way. From the point of view of any given observer, they are instead quantum fluctuations bound to the cosmological horizon. The interaction between the horizon states and the “scattering states” introduces a thermal uncertainty in the scattering matrix. This uncertainty cannot be removed by local measurements, because the locus of the horizon states is an extreme environment from the point of view of a given observer. It cannot perform observations near the horizon without a large expense of energy, which distorts the measurement [34].

Thus, in the *AsdS* case, the S-matrix is only approximately defined. Paban, and two of the present authors [33] have argued that the S-matrix for energies[12] that are kept fixed as the c.c. goes to zero, should have a well defined but non-summable small Λ asymptotic expansion, with errors of order (in four dimensions) $e^{-(\frac{M_P 4}{\Lambda})^{3/2}}$.

In both a true scattering universe, and an AsdS universe the description of a single observer suffices from an operational point of view. However, the constraints on the quantum mechanics of a single observer are not very strong. As in the $p = \rho$ universe, we introduce other observers as a lattice of sequences of Hilbert spaces $\mathcal{H}_N(\mathbf{x})$. The lattice has the topology of R^{d-1} [13]. For each pair of points on the lattice, we introduce, at each N, a tensor factor $\mathcal{O}_N(\mathbf{x}, \mathbf{y})$ of both $\mathcal{H}_N(\mathbf{x})$ and $\mathcal{H}_N(\mathbf{y})$. For nearest neighbor points, the dimension of $\mathcal{O}_N(\mathbf{x}, \mathbf{y})$, ($\equiv D(N, \mathbf{x}, \mathbf{y})$) is $(\dim \mathcal{K})^{N-1}$. For fixed

[12]In this sentence, energy refers to the eigenvalue of an operator which approaches a timelike component of the momentum in the Poincare algebra, as the c.c. goes to zero. This is not the same as the Hamiltonian of the static observer, though the commutator between these generators is expected to be small in the subspace with fixed Poincare energy.

[13]Compact or partially compact spatial topologies present new difficulties, with which we are not yet prepared to deal.

N, $D(N, \mathbf{x}, \mathbf{y})$ should be a monotonically decreasing function of the lattice distance between $\mathbf{x}$ and $\mathbf{y}$. The specification of this function is part of the definition of the quantum space-time.

Most importantly, the time evolution operators in each sequence of Hilbert spaces $\mathcal{H}_N(\mathbf{x})$ are constrained by the requirement that they be compatible on all overlaps. This is such a complicated system of constraints, that one might have despaired of finding a solution to it, if it were not for the example of the $p = \rho$ universe discussed in the previous section. We have yet to find a clue, which would help us to construct an example of a universe that supports localized excitations.

For true scattering universes, the initial state is pure only as $N \to \infty$. The Hilbert spaces of different observers must all coincide in this limit. The S-matrix is expected to be unique and mathematically well defined. The most interesting question for such space-times is how one can express the constraints of compatibility of the descriptions of different observers as equations for the S-matrix. We conjecture that these equations will be generalizations of the usual criteria of crossing symmetry and analyticity, and that, together with unitarity, and a specification of the boundary geometry, they will completely determine the S-matrix.

For Big Bang cosmologies, the construction is similar except that there is an initial time slice, and all causal diamonds begin on that slice[14].

[14]Remember that we are working in a gauge-fixed formalism. This condition is part of the gauge fixing.

7.5 Discussion

The phenomenological discussion of holographic cosmology presented in [20] begins from a system close to the $p = \rho$ cosmology, but requires inhomogeneous defects as input. We have treated these defects heuristically as a network of spheres joined together in a "tinker toy". This was motivated by the observation that the Israel junction condition applied to a single sphere of radiation or matter dominated cosmology embedded in a $p = \rho$ background, requires the sphere to shrink in FRW coordinates. The tinker toy is supposed to be the maximal entropy configuration[15] for which this collapse does not occur. To maximize the entropy we minimize the initial volume of the normal region. The initial ratio of volumes is called ϵ and is assumed small. We then argued that the volume of normal region, in equal area slicing, grows relative to that of the $p = \rho$ region. Eventually, the physical volume of the initial coordinate sphere is dominated by the normal region. The $p = \rho$ regions are large black holes embedded in the normal region. From this point on, the evolution can be treated by conventional field theory methods, and we argued that it is plausible, if the low energy degrees of freedom include an appropriate inflaton field, for the universe to undergo a brief period of inflation. Depending on the value of ϵ (and another parameter which we cannot calculate), the fluctuations of the microwave background can be generated either in the $p = \rho$ phase, or during inflation. The two possibilities are incompatible with each other and the experimental signatures of them are, in principle, distinguishable.

[15] which fits inside a given initial coordinate sphere. We will return to what determines the initial size of this sphere.

In order to put this cosmology on a mathematical basis, we have to find a holographic description of a normal radiation dominated universe. Next we must understand how the consistency conditions which we have discussed in this paper, can be used to define an infinite hyperplanar boundary between a normal phase and the dense black hole fluid. This would be the quantum analog of the Israel junction condition. At this stage of development one might hope to get a crude estimate of ϵ. More detailed questions, such as whether the fluctuations generated during the $p = \rho$ era have Gaussian statistics, will probably require us to understand the more complicated boundary of the tinker toy.

These problems seem hard, but before the present work we had despaired of ever finding a solution to the consistency conditions for holographic cosmology.

We want to end this paper with a metaphysical speculation. The Israel junction condition applied to the large sphere inside of which the tinker toy fits, would seem to require that that region collapse in coordinate volume. One way to avoid this catastrophe would be to imagine that both the initial black hole fluid, and the tinker toy had infinite extent in space.

There is a more attractive way out of this problem. If we try to embed a (future) asymptotically de Sitter space into the $p = \rho$ fluid, we can satisfy the Israel condition by matching the cosmological horizon to a sphere of fixed physical size in the $p = \rho$ background. Now we imagine an infinite $p = \rho$ background, littered with tinker toys of various sizes, with the proviso that low energy physics inside each tinker toy universe is compatible with eventual evolution to a stationary state of fixed positive cosmological constant. From a global point of view, we would have a collection of finite, asymptotically dS universes, embedded in an infinite, flat $p = \rho$ background.

We can also understand the stability of this sort of cosmology from an entropic point of view. We have advocated the $p = \rho$ cosmology as the most entropic initial condition for the universe. In fact, in the more general cosmology consisting of an infinite $p = \rho$ background, filled with a collection of dS bubbles, any causal diamond which includes complete dS bubbles, has the same number of states "excited" as the pure $p = \rho$ fluid. It is only when we look at causal diamonds inside a dS bubble that we find observers which observe less than the maximal amount of entropy. We have argued that the most generic way for such low entropy regions to arise is for the interior of the dS bubble to begin as a tinker toy embedded in a $p = \rho$ background. This then goes through a stage where the localized entropy increases and is eventually followed by an AsdS stage where the localized entropy is very small because everything has been swept out of the observer's horizon.

Our notion of a generic state in an AsdS universe should be compared with that of [35]. These authors organize the states according to the eigenvalues of the static Hamiltonian. They then require that cosmological evolution be viewed as a typical thermal fluctuation with certain constraints[16]. Among these constraints is the anthropic principle. They then argue that a typical cosmology consistent with these constraints will not look like the world we observe. From our point of view, the choice of initial conditions made by these authors is not the maximally entropic one for a local observer.

[16]The explicit model is a scalar field with an inflationary maximum and a dS minimum with small c.c. . The typical cosmological fluctuation is one which puts the scalar at the inflationary maximum.

They impose global constraints on the states (thermality with respect to the static Hamiltonian of the asymptotic future, and homogeneity over the inflationary horizon size) at arbitrarily early times. On the contrary, in most early horizon volumes we allow an absolutely random state to be acted on by a random sequence of Hamiltonians. Certain horizon volumes, which contain parts of the tinker toy, are somewhat more structured. In a previous paper we have argued that these initial conditions have much more entropy than inflationary ones. In our model, inflation only becomes possible in large normal regions in which the black hole fluid has become dilute.

The $p = \rho$ universe with a distribution of AsdS bubbles is a model which naturally provides us with an ensemble of universes with varying cosmological constant. If we wish, we can apply the anthropic mode of reasoning to this model. If the physics of a stable dS universe approaches a limit as Λ goes to zero, with the parameters which determine the primordial density fluctuations and the dark matter density at the beginning of the matter dominated era, both becoming independent of Λ in the limit, then Weinberg's anthropic argument for the value of the c.c. would more or less explain the value that we see. At the very least, it explains most of the "fine tuning" that we find so disturbing.

We are of two minds as to the virtues of such a model. Much of our previous work on the asymptotic dS universe simply postulates the cosmological constant as an input, whose value will never have an explanation. The model under discussion views that input as being determined by a very weak form of the anthropic principle. We gain some degree of understanding[17], but at the expense of introducing a large set of degrees of freedom which will never be observed. Occam would surely complain!

On the positive side, one should compare this use of the anthropic principle with others which have been contemplated in the literature. First of all, in this model we imagine that all of the physics in a given tinker-toy universe is completely determined by the value of a single parameter, the cosmological constant. Thus, our model is required to calculate most physical quantities successfully, from first principles. Only one parameter is determined anthropically, and it is one for which the anthropic range is quite narrow if everything else is fixed at its measured value. Secondly, the anthropic argument we use is quite broad, and would apply to any form of life whose existence depends on structures as complicated as galaxies. This fixes the c.c. to be no larger than a factor of 100 times its observed value. Even the more refined arguments of Vilenkin [36] , which reduce this factor to something of order one, do not depend on crucial details of nuclear physics or organic chemistry, as long as we view the c.c. as the only parameter which varies among the different universes in our ensemble.

To summarize, we have described a well defined quantum mechanical model, which obeys a plausible set of axioms for quantum cosmology. At large scales it obeys scaling laws which are the same as those obeyed by a flat FRW universe with equation of state $p = \rho$. The detailed dynamics of the model realizes many of the properties of such a system that two of the authors have proposed based on the intuitive idea of a dense black hole fluid. The constants in the geometrical equations can mostly be fit by choices

[17] avoiding the introduction, by hand, of a huge integer, the number of physical states, into our model of the world

of constants in the quantum mechanics, but we have found one constant which seems to be determined unambiguously. Unfortunately it misses the geometric prediction by a factor of $d-2$.

7.6 Appendix

7.6.1 Intersection of causal diamonds

In this sub-appendix we will determine the causal diamond $\mathcal{D}_M$ with maximal FSB area, which is contained in the intersection of two causal diamonds $\mathcal{D}_1$ and $\mathcal{D}_2$ both starting at time η_1 and ending at time η_2. We will solve the problem first in the simple case of Minkowski spacetime and then in a general conformally flat spacetime.

So let's first consider Minkowski spacetime with dimension $d=4$

$$ds^2 = d\eta^2 - d\mathbf{x}^2$$

where we use the following notation $\mathbf{x} = (x, y, z)$ for the spatial coordinates.

It will be clear in the following that identical considerations apply to spacetimes of general dimension.

Given the two causal diamonds $\mathcal{D}_1$ and $\mathcal{D}_2$, both starting at time η_1 and ending at time η_2, we will indicate with $\mathcal{D}_M$ the maximal causal diamond belonging to the intersection of $\mathcal{D}_1$ and $\mathcal{D}_2$ Fig. 7.3 and Fig. 7.6. Let's indicate with Σ the spatial surface to which both the base sphere $S_{\mathcal{D}_1}$ and $S_{\mathcal{D}_2}$ of $\mathcal{D}_1$ and $\mathcal{D}_2$ belong. Let $\widetilde{S}$ be the maximal sphere that fits into the intersection of $S_{\mathcal{D}_1}$ and $S_{\mathcal{D}_2}$. $\widetilde{S}, S_{\mathcal{D}_1}$ and $S_{\mathcal{D}_2}$ are represented in Fig. 7.4 and Fig. 7.5. Furthermore let $(\eta_i, \mathbf{x}_i)$ $(\eta_f, \mathbf{x}_f)$ be the points of $\mathcal{D}_1 \cap \mathcal{D}_2$ with the minimum and maximum values of η Fig. 7.6.

It is obvious that the maximal causal diamond $\mathcal{D}_M$, belonging to the intersection of two causal diamonds $\mathcal{D}_1$ and $\mathcal{D}_2$, must start at $(\eta_i, \mathbf{x}_i)$ end at $(\eta_f, \mathbf{x}_f)$ and have as base sphere $S_{\mathcal{D}_M} = \widetilde{S}$. In the η, x -plane Fig. 7.6 we will indicate with Δx the separation among the tips of $\mathcal{D}_1$ and $\mathcal{D}_2$ at time η_1. The maximal causal diamond $\mathcal{D}_M$ will start at conformal time η_i and end at conformal time η_f.

Denote by $r_{\mathcal{D}_M}$ the radius of the base sphere on $\mathcal{D}_M$ and with $h = \eta_f - \eta_i$ the height of the causal diamond $\mathcal{D}_M$ Fig. 7.6 Fig. 7.7. Defining $h = 2a$ we see from the pictures Fig. 7.6 Fig. 7.7 we have $r_{\mathcal{D}_M} = a$. Furthermore we can see inspecting Fig. 7.6 that Δx is given by

$$\eta_i - \eta_1 = \frac{\Delta x}{2}$$

and so

$$\begin{aligned} r_{\mathcal{D}_M} &= a = \frac{1}{2}(\eta_2 - \eta_1) - (\eta_i - \eta_1) \\ &= \frac{1}{2}(\eta_2 - \eta_1) - \frac{\Delta x}{2} \end{aligned}$$

The quantities that we have determined, i.e. the radius of the base sphere $r_{\mathcal{D}_M}$, the height h and the initial and final times η_i, η_f, are all the parameters that describe the geometry of $\mathcal{D}_M$.

We will now turn to the general problem of determining the maximal causal diamond $\mathcal{D}_M$ in an FRW cosmology

$$ds^2 = a^2(\eta)\left(d\eta^2 - d\mathbf{x}^2\right)$$

Since the space is conformally flat all the previous considerations continue to apply and the maximal causal diamond is still $\mathcal{D}_M$ Fig. 7.3, Fig. 7.4 and Fig. 7.5. Moreover the parameters that determine completely the geometry of $\mathcal{D}_M$ are, as before, the radius of the base sphere $r_{\mathcal{D}_M} = \frac{1}{2}(\eta_2 - \eta_1) - \frac{\Delta x}{2}$, the height h and the initial and final times $\eta_i,\ \eta_f$.

Next we determine the sphere of maximal area (maximal sphere) on the causal diamond $\mathcal{D}_M$ in an FRW cosmology

$$ds^2 = a^2(\eta)\left(d\eta^2 - d\mathbf{x}^2\right)$$

The area of a *generic* 2-sphere S of radius r, given by the intersection of $\mathcal{D}_M$ and the spatial section at time η Fig. 7.6 Fig. 7.8, is

$$A(\eta) = 4\pi r^2 a^2(\eta) \tag{7.19}$$

As mentioned before we want to determine the maximal area sphere S_M.

Assume that the spacetime contracts monotonically as we move toward the past: ($a(\eta)$ decreases monotonically as η goes to zero). Then the maximal sphere is always in the upper half of the causal diamond Fig. 7.8 and its radius is

$$r = \eta_f - \eta \tag{7.20}$$

To determine the maximal sphere we have to maximize the area $A(\eta)$ in the interval $(\eta_f, \overline{\eta})$, where we defined $\overline{\eta} = \frac{1}{2}(\eta_f - \eta_i)$.

The Friedmann's equations in conformal coordinate are

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi\rho a^2}{3} - k \tag{7.21}$$

where $k = 0$ for the conformally flat metric that we are considering. We will assume as usual for an FRW cosmology that the matter content of the universe is a perfect fluid with stress tensor

$$T_a^b = diag(-\rho, p, p, p)$$

Assume that the pressure p and energy density ρ are related by the equation of state

$$p = w\rho$$

With this ansatz for the matter content the Friedmann's Equations (7.21) can be solved and we find the conformal factor

$$a(\eta) = a_0 \left(\frac{\eta}{q}\right)^q, \quad q = \frac{2}{1+3w} \tag{7.22}$$

The extremum of area $A(\eta)$ is given by

$$\frac{dA(\eta)}{d\eta} = 0$$

using the Eq. (7.22) for the conformal factor and the expression (7.20) for the radius we find

$$\widetilde{\eta} = \frac{q\eta_f}{1+q}$$

furthermore we have

$$\frac{d^2A(\widetilde{\eta})}{d\eta^2} = -\frac{2(1+q)}{q}\left(\frac{\eta_f}{1+q}\right)^{2q} < 0, \quad \forall\ q,\ \eta_f$$

showing that $\widetilde{\eta}$ is actually a maximum.

It is clear from Fig. 7.8, that if

$$\widetilde{\eta} > \overline{\eta} = \frac{1}{2}(\eta_f - \eta_i) \tag{7.23}$$

then the point where we have the maximal sphere is at $\eta_M = \widetilde{\eta}$, otherwise the maximal sphere it is at $\eta_M = \overline{\eta}$.

The condition given by Eq. (7.23) is equivalent to

$$\begin{aligned} \frac{q\eta_f}{1+q} &> \frac{1}{2}(\eta_f - \eta_i) \\ w &< \frac{1}{3}\frac{\eta_f + 3\eta_i}{\eta_f - \eta_i} \end{aligned} \tag{7.24}$$

where the last quantity is clearly always greater than zero.

The previous condition (7.23) is always verified for dust $w = 0$ and for space-time with a positive cosmological constant $w = -1$, implying that in these case the maximal sphere is at $\eta_M = \widetilde{\eta}$.

For a radiation dominated universe we have $w = \frac{1}{3}$ and the condition (7.24) becomes

$$\begin{aligned} \frac{1}{3} &< \frac{1}{3}\frac{\eta_f + 3\eta_i}{\eta_f - \eta_i} \\ &\Rightarrow \\ \eta_i &> 0 \end{aligned} \tag{7.25}$$

this is always true and so even in this case we have $\eta_M = \widetilde{\eta}$.

The interesting case for the bulk of this paper is $w = 1$. In this case if $\eta_f \gg 1$ (large enough causal diamonds) the condition (7.24)

$$w < \frac{1}{3}\frac{\eta_f + 3\eta_i}{\eta_f - \eta_i} \tag{7.26}$$

is never verified. As a consequence in this limiting case we always have $\eta_M = \overline{\eta}$, or in other words the maximal sphere coincides with the base sphere of $\mathcal{D}_M$. This gives the area formula we used in the text.

7.6.2 Holographic relations in a general FRW cosmology

In this sub-appendix, we want to show how the relation between area and conformal time for a general FRW universe, filled with a combination of perfect fluids, can be used to extract the equation of state. This indicates that in a more general holographic cosmology, we can expect the formula for the Hamiltonian as a function of the area to determine the background metric.

The metric for an FRW universe is

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right)$$

To analyze this problem it's more useful to work with conformal time η and comoving coordinate χ

$$d\eta = \frac{dt}{a(t)}, \quad d\chi = \frac{dr}{\sqrt{1-kr^2}}$$

$$ds^2 = a^2(\eta)\left(-d\eta^2 + d\chi^2 + f^2(\chi)d\Omega^2\right)$$

Where as usual $k = -1, 0, 1$ and $f(\chi) = \sinh\chi, \chi, \sin\chi$ correspond to open, flat and closed universes, respectively.

We want to analyze a flat universe $f(\chi) = \chi$. Consider the FRW universe with a big bang singularity, given any point p in the space-time consider the backward light cone, it initially expands and then starts contracting when we approach the singularity.

Let B be the *apparent horizon* ,i.e. the spatial surface with the maximum area on the light cone. According to the covariant entropy bound, the total number of degrees of freedom is bounded by the area of B

$$N \leq \frac{A(B)}{2}$$

The apparent horizon is found geometrically as the sphere at which at least one pair of lightsheets has zero expansion. The radius of the apparent horizon $\chi_{AH}(\eta)$, as a function of time, is given by the equation

$$\frac{\dot{a}}{a}(\eta) = \pm\frac{f'}{f} = \pm\frac{1}{\chi}$$

The proper area of the apparent horizon is given by

$$A_{AH}(\eta) = 4\pi a^2(\eta) f^2[\chi_{AH}(\eta)]$$

In the case of a flat universe $f(\chi) = \chi$

$$A_{AH}(\eta) = \frac{4\pi a^2(\eta)}{\frac{\dot{a}^2}{a^2}}$$

Using the Friedmann's equations (in conformal time)

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi\rho a^2}{3} - k$$

with $k = 0$, we have

$$A_{AH}(\eta) = \frac{3}{2\rho(\eta)}$$

All these results are valid for cosmologies with a generic ρ. Thus, the time dependence of the area of the apparent horizon determines the time dependence of the energy density and vice versa.

We will first write everything as a function of cosmological scale factor, so that the previous equation reads

$$A_{AH}(a) = \frac{3}{2\rho(a)}$$

We want to determine $\rho(a)$ for a fluid with many components. The equation of energy conservation for one fluid is

$$d\left(a^3(\rho + p)\right) = a^3 dp$$

Assuming an equation of state

$$p = w\rho$$

this can be rewritten

$$\frac{d\rho}{da} + \alpha\frac{\rho}{a} = 0$$

with

$$\alpha = 3(1 + w)$$

In general for many fluids we will have

$$\sum_i \frac{d\rho_i}{da} + \alpha_i\frac{\rho_i}{a} = 0$$

with

$$\alpha_i = 3(1 + w_i)$$

To keep things simple we will consider the case of two fluids, but the results will be valid in the general case.

A general solution is given by

$$\rho = \rho_1 + \rho_2$$

with

$$
\frac{d\rho_1}{da} + \alpha_1 \frac{\rho_1}{a} = f(a)
$$
$$
\frac{d\rho_2}{da} + \alpha_2 \frac{\rho_2}{a} = -f(a)
$$

so

$$
\rho_1 = C_1 a^{-\alpha} + a^{-\alpha} \int_{a_1}^{a} d\bar{a}\ \bar{a}^{\alpha} f(\bar{a})
$$
$$
\rho_2 = C_2 a^{-\alpha} - a^{-\alpha} \int_{a_1}^{a} d\bar{a}\ \bar{a}^{\alpha} f(\bar{a})
$$

with $a_1 = a(\eta = 1)$ and C_1 and C_2 integration constants.

In this context the form of the function $f(a)$ is not determined and so we will consider $f(a)$ to be arbitrary. The function $f(a)$ describes how the two fluids exchange energy and is determined by the dynamics of the system.

The area A_{AH} will not depend on w_i $\forall a$ iff

$$
\begin{aligned}
\frac{\partial A_{AH}}{\partial w_i} &= \frac{-3}{2\rho^2}\frac{\partial \rho}{\partial w_i} = 0,\ \forall a \\
&\Longleftrightarrow \\
\frac{\partial \rho}{\partial w_i} &= 0,\ \forall a
\end{aligned}
$$

where we assumed $\rho \neq \infty$.

It turns out that there are no values of $C_i,\ a_i,\ f(a),\ \alpha$ for which

$$
\frac{\partial \rho}{\partial w_i} = 0,\ \forall\ a
$$

In fact considering for example $\frac{\partial \rho}{\partial w_1}$ we have

$$\begin{aligned}
\frac{\partial \rho}{\partial w_1} &= \frac{\partial \rho_1}{\partial w_1} \\
&= -3a^{-\alpha}\left(C_1 \log(a) + \log(a)\int_{a_1}^{a} d\bar{a}\ \bar{a}^{\alpha} f(\bar{a}) \right. \\
&\quad \left. - \left(\int_{a_1}^{a} d\bar{a}\ \bar{a}^{\alpha} \log(\bar{a}) f(\bar{a})\right)\right)
\end{aligned}$$

a necessary condition for this to be zero $\forall\ a$ is that the derivative respect to a is zero $\forall\ a$, where we assumed that $a \neq 0$. We have

$$\frac{\partial}{\partial a}\left(\frac{1}{-3a^{-\alpha}}\frac{\partial \rho_1}{\partial w_1}\right) = \frac{1}{a}\left(C_1 + \int_{a_1}^{a} d\bar{a}\ \bar{a}^{\alpha} f(\bar{a})\right)$$

Assuming $a \neq \infty$ this can be zero $\forall\ a$ iff $C_1 = f(a) = 0$ but in this case we would have $\rho = 0,\ \forall\ a$.

As far as the dependence of A_{AH} on the energy densities at some initial time $\tilde{\rho}_i = \rho_i(\tilde{a})$, the area A_{AH} will not depend on $\tilde{\rho}_i$ $\forall\ a$ iff

$$\begin{aligned}
\frac{\partial A_{AH}}{\partial \tilde{\rho}_i} &= \frac{-3}{2\rho^2}\frac{\partial \rho}{\partial \tilde{\rho}_i} = 0,\ \forall a \\
&\Longleftrightarrow \\
\frac{\partial \rho}{\partial \tilde{\rho}_i} &= 0,\ \forall a
\end{aligned}$$

where we assumed $\rho \neq \infty$. It turns out that even in this case there are no values of $C_i,\ a_i,\ f(a),\ \alpha$ for which

$$\frac{\partial \rho}{\partial \tilde{\rho}_i} = 0,\ \forall a$$

In fact

$$\begin{aligned}
\tilde{\rho}_i &= \rho_i(\tilde{a}) = C_i \tilde{a}^{-\alpha} + \tilde{a}^{-\alpha} \int_{a_1}^{\tilde{a}} d\bar{a}\ \bar{a}^{\alpha} f(\bar{a}) \\
&\Longrightarrow \\
C_i &= \left(\frac{\tilde{\rho}_i}{\tilde{a}^{-\alpha}} - \int_{a_1}^{\tilde{a}} d\bar{a}\ \bar{a}^{\alpha} f(\bar{a}) \right)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \rho}{\partial \tilde{\rho}_i} &= \frac{\partial \rho_i}{\partial \tilde{\rho}_i} = \frac{\partial C_i}{\partial \tilde{\rho}_i} a^{-\alpha} \\
&= \frac{a^{-\alpha}}{\tilde{a}^{-\alpha}} \neq 0,\ \forall a
\end{aligned}$$

always assuming that $a \neq 0$. Thus, we can always extract the parameters w_i from the scale factor dependence of the energy density, and consequently, from the scale factor dependence of the area of the apparent horizon.

We now return to the problem of studying the dependence of ρ as a function of η on the parameter w_i, $\tilde{\rho}_i$, which we will now denote generically as β_i. We have

$$\rho = \rho\left(a(\eta, \beta_i), \beta_i\right)$$

and so

$$\frac{\partial \rho}{\partial \beta_i} = \frac{\partial \rho}{\partial \beta_i} + \frac{\partial \rho}{\partial a} \frac{\partial a}{\partial \beta_i}$$

The problem is slightly more complicated but can still be solved exactly, in fact the dependence of a on β_i can be found by solving the Friedmann equations by quadrature

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi \rho\left(a(\eta), \beta_i\right) a^2}{3} - k$$

We conclude that the component equations of state of an arbitrary multi-component fluid, can be extracted from the dependence of the horizon area on conformal time. In this derivation we have used the Friedmann equation. In the quantum approach to cosmology, which we have discussed at length in this paper, we believe that the replacement for the Friedmann equation is the equation determining the N dependence of the Hamiltonians $H_N(k, \mathbf{x})$. There are strong constraints on these Hamiltonians, coming from the overlap conditions. We have found one solution of these equations and argued that it corresponds to a $p = \rho$ FRW universe. We conjecture that other solutions will also represent Big Bang cosmologies.

7.6.3 Computation of c_e from geometry and constant in front of H_N

The Einstein equations in d space-time dimensions are

$$G_{\mu\nu} = 2\Omega_{d-1} G_N T_{\mu\nu}$$

where Ω_{d-1} is the surface of a sphere in $l = d-1$ spatial dimensions. In 4 dimension we recover the usual result

$$G_{\mu\nu} = 8\pi G_N \; T_{\mu\nu}$$

Through a standard computation we recover the Friedmann's equation in d dimensions

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\left(\Omega_{d-1} G_N\right)}{d(1-d)}\rho$$

For a $p = \rho$ cosmology the expression for the energy density as a function of the entropy density is

$$\rho = {c_e}^2 \sigma^2$$

substituting in the Friedmann's equation we have

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\left(\Omega_{d-1} G_N\right)}{d(1-d)} \rho = \frac{c_d^2 c_e^2 \sigma_0^2}{a^{2(d-1)}}$$

the solution of the previous equation is

$$a(\eta) = a_0 \eta^{\frac{1}{d-2}} \left(a_0 \frac{d-2}{d-1}\right)^{\frac{1}{d-2}} \tag{7.27}$$

with

$$a_0 = \left(c_d c_e \sigma_0 (d-1)\right)^{\frac{1}{d-1}} \tag{7.28}$$

$$c_d = \sqrt{\frac{8\left(\Omega_{d-1} G_N\right)}{d(d-1)}}$$

in the following we will set

$$G_N = 1$$

For a $p = \rho$ cosmology the value of the constant c_e can be obtained saturating the entropy bound.

We have for a causal diamond of maximal FSB area A

$$A = a_0^{d-1} \left(\frac{d-2}{d-1}\right) \Omega_{d-1} \left(\frac{\eta}{2}\right)^{d-1} = 4 N l_s = 4 \sigma_0 \Omega_{d-1} \eta^{d-1} \left(\frac{1}{d-1}\right)$$

substituting the equations (7.27) and (7.28) we find

$$c_e = 2^{d+1} \frac{1}{c_d} \frac{1}{(d-1)(d-2)} = 2^{d+1} \frac{1}{\sqrt{\frac{8(\Omega_{d-1})}{d(d-1)}}} \frac{1}{(d-1)(d-2)}$$

This expression for c_e can be used to fix the constant in front of the Hamiltonian H_N.

There do not seem to be any further consequences of requiring that our quantum cosmology obey the equations of classical $p = \rho$ cosmology, not just as scaling relations, but including the constants. This only serves to define Newton's constant, and the constant in front of our quantum hamiltonian. The one classical relation from which these constants scale out is the relation between overlap areas. Here we have a chance for a numerical triumph, but our current definitions miss by a factor of $d - 2$.

7.7 Figures

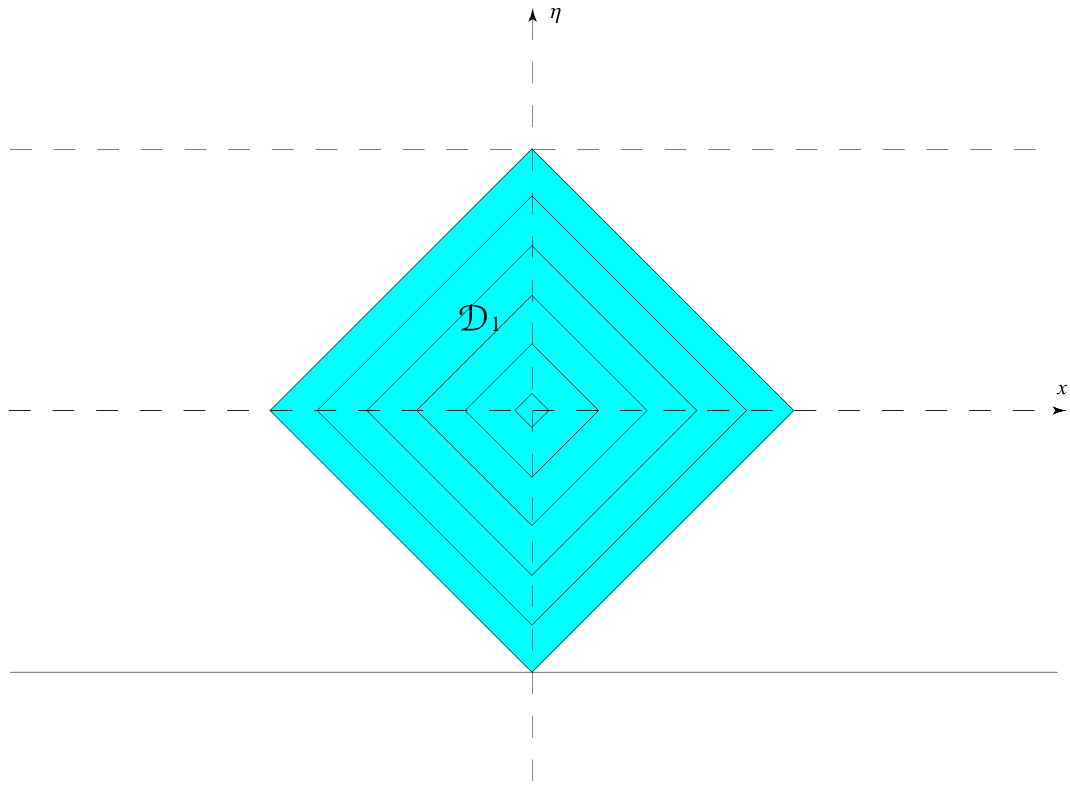

Figure 7.1:
Nested causal diamonds defining an observer in a time symmetric space-time

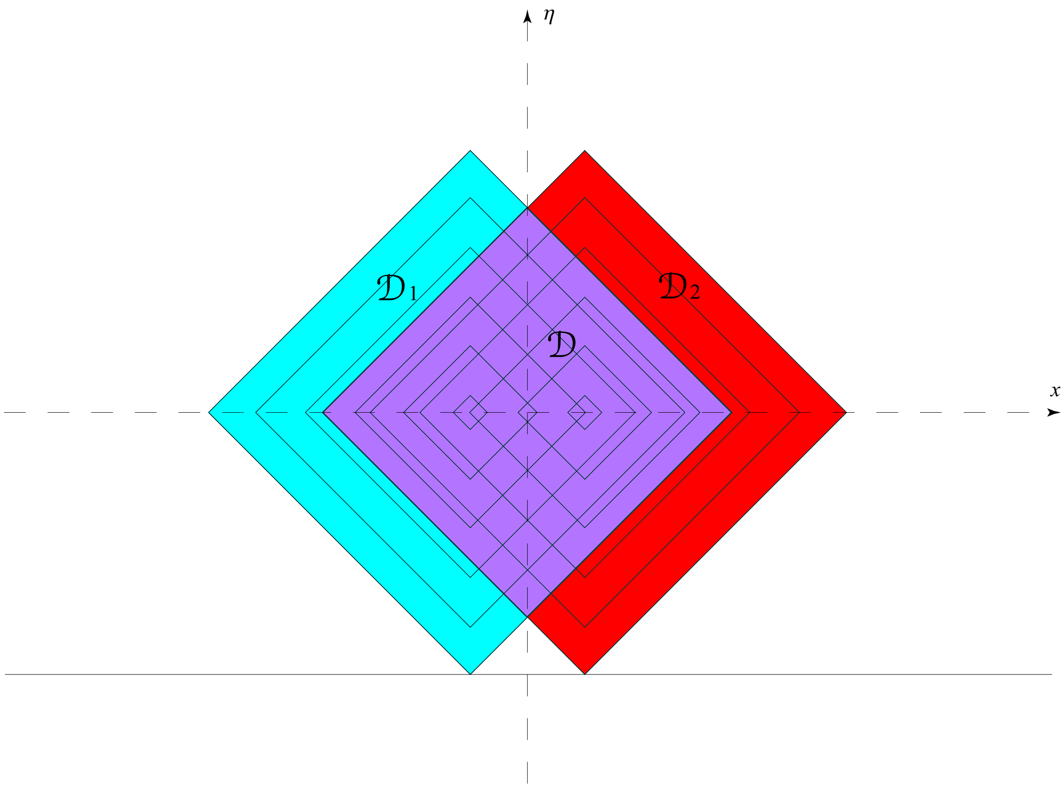

Figure 7.2:
Nested causal diamonds defining a nearest neighbors pair of observers in a time symmetric space-time

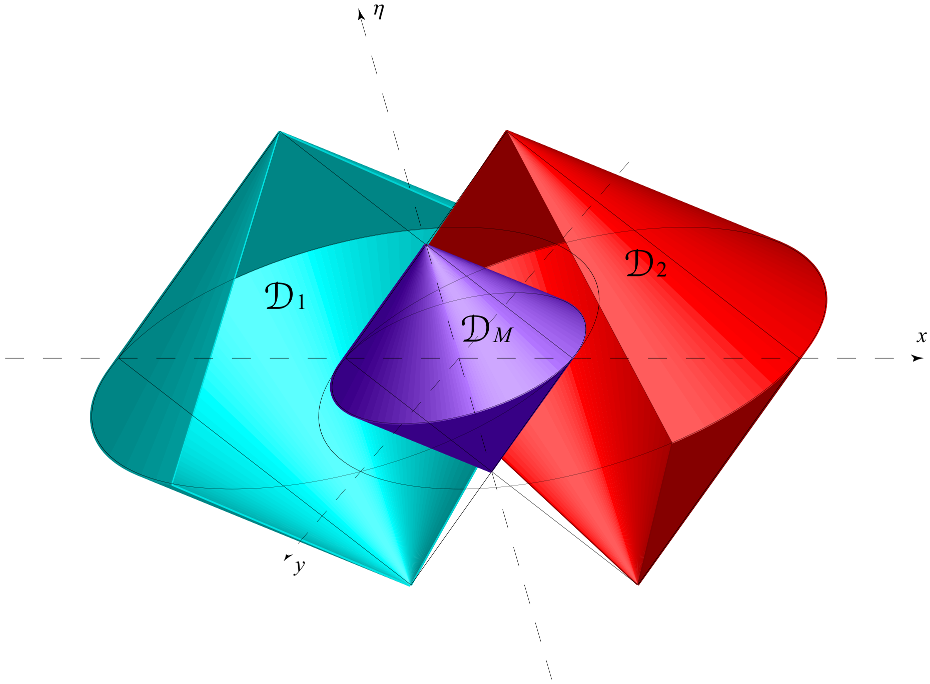

Figure 7.3:
The two causal diamonds $\mathcal{D}_1$ and $\mathcal{D}_2$ (z spatial coordinate suppressed) and the maximal causal diamond $\mathcal{D}_M$ that fits in the intersection $\mathcal{D}_1 \cap \mathcal{D}_2$. The picture is valid for Minkowski spacetime and more generaly conformally flat spacetimes.

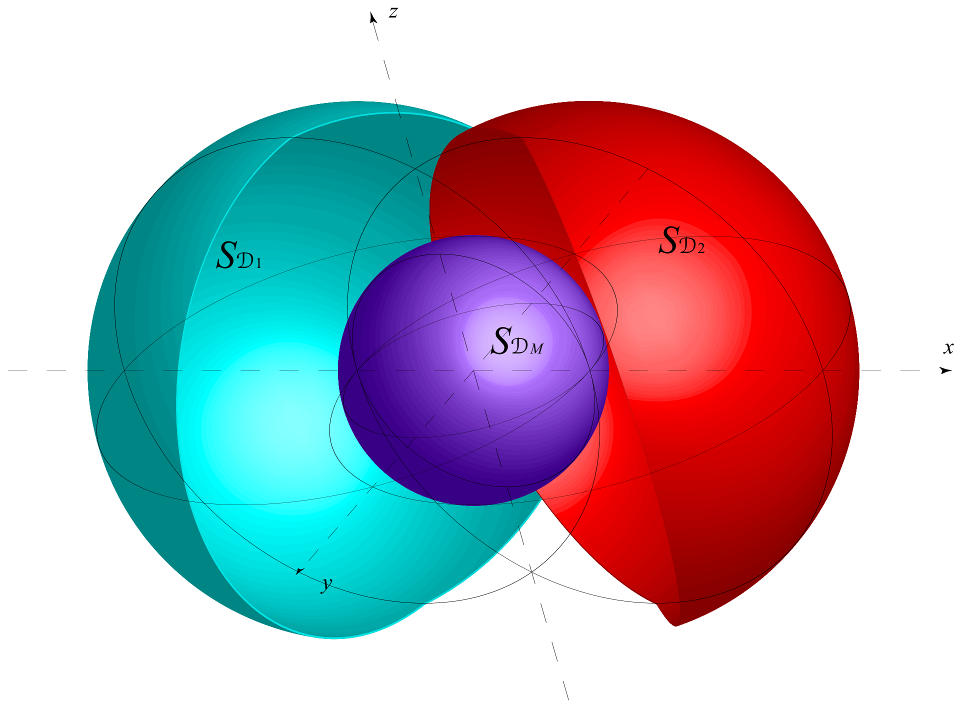

Figure 7.4: The base spheres $S_{\mathcal{D}_1}$ and $S_{\mathcal{D}_2}$of the two causal diamonds $\mathcal{D}_1$ and $\mathcal{D}_2$ (Fig. 7.3) $\widetilde{S}$ is the maximal sphere belonging to the intersection of $S_{\mathcal{D}_1}$ and $S_{\mathcal{D}_2}$. $\widetilde{S}$ coincide with $S_{\mathcal{D}_M}$ the base sphere of $\mathcal{D}_M$.

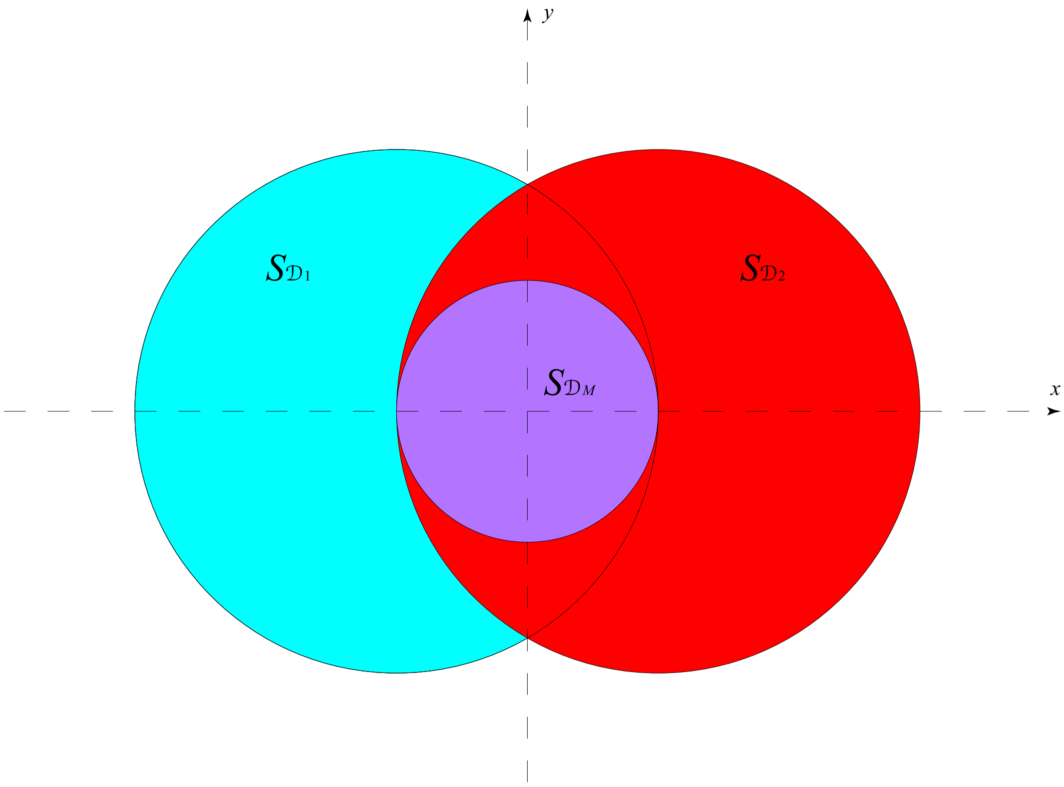

Figure 7.5:
The base spheres $S_{\mathcal{D}_1}$ and $S_{\mathcal{D}_2}$ of the two causal diamonds $\mathcal{D}_1$ and $\mathcal{D}_2$ (Fig. 7.3), coordinate z suppressed.

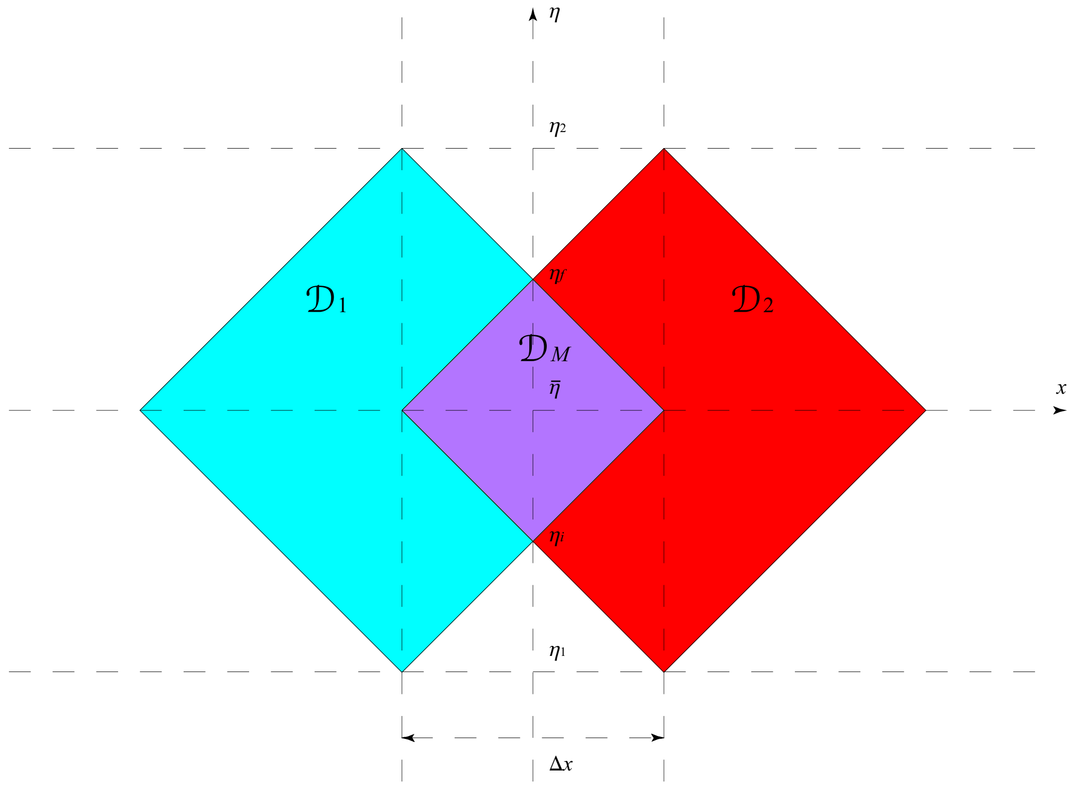

Figure 7.6:
The two causal diamonds $\mathcal{D}_1$ and $\mathcal{D}_2$ (z, y spatial coordinates suppressed) and the maximal causal diamond $\mathcal{D}_M$ that fits in the intersection $\mathcal{D}_1 \cap \mathcal{D}_2$. η_i, η_f are the points of $\mathcal{D}_1 \cap \mathcal{D}_2$with the minimum and maximum values of η. Δx is the separation among the tips of $\mathcal{D}_1$ and $\mathcal{D}_2$ at time η_1.

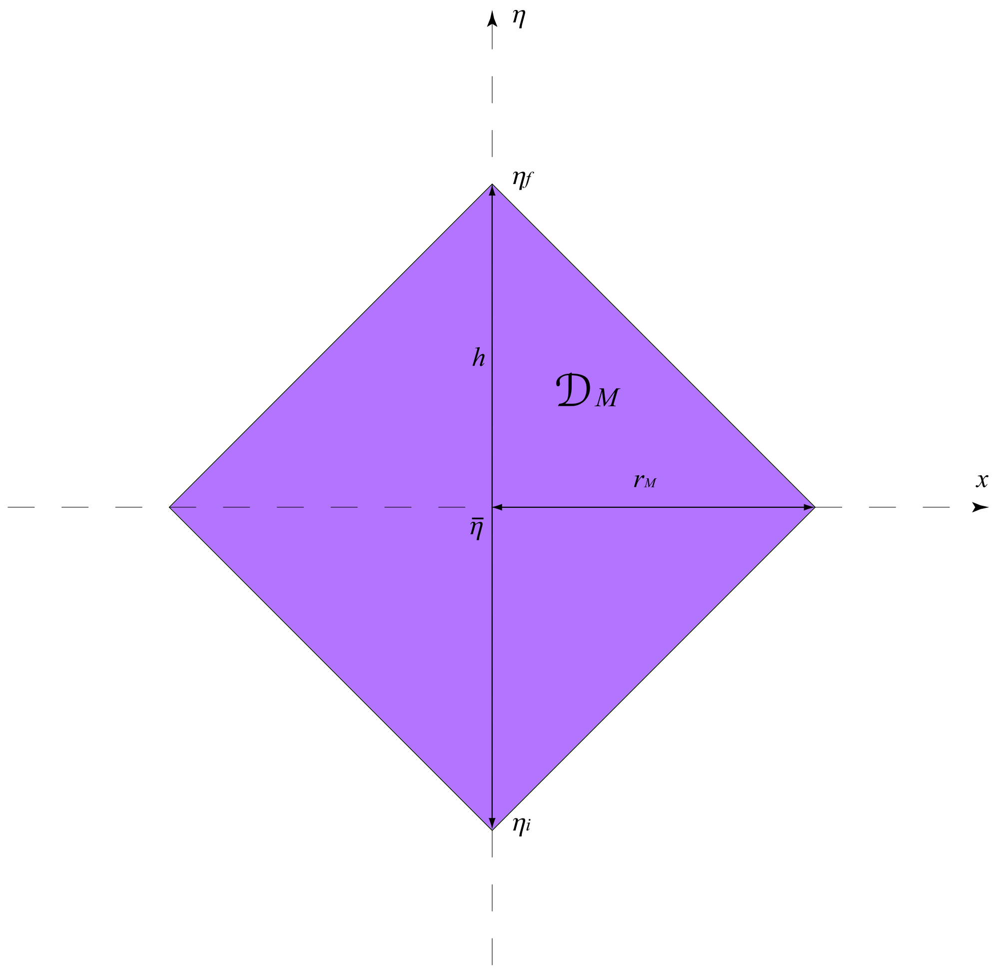

Figure 7.7:
Detail of the the maximal causal diamond $\mathcal{D}_M$. $h = \eta_f - \eta_i$ is the height of $\mathcal{D}_M$ and $r_{\mathcal{D}_M}$ is the radius of the base sphere on $\mathcal{D}_M$.

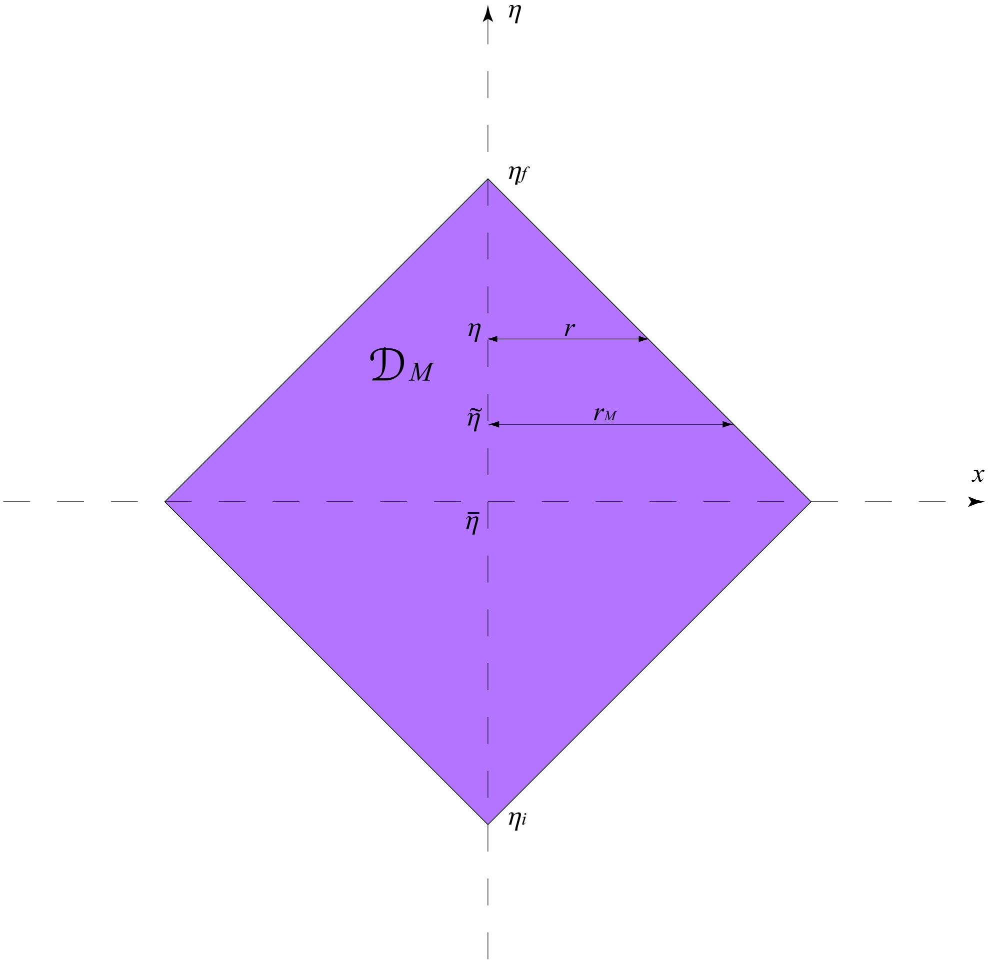

Figure 7.8:
Detail of the themaximal causal diamond $\mathcal{D}_M$. r is the radius of a generical sphere on $\mathcal{D}_M$, r_M is the radius of the sphere of maximal area on $\mathcal{D}_M$ i.e. the maximal sphere.

Chapter 8

Infrared Divergences in dS/CFT

8.1 Introduction

The hypothesis of Cosmological Supersymmetry Breaking (CSB) is based on the idea [37] [38] [39] that quantum theories of stable, asymptotically de Sitter (AsdS) space-times exist and have a finite number of physical states. The (positive) cosmological constant, Λ,is an input parameter, which controls the number of states. The limit of vanishing Λ is a super-Poincare invariant theory, but SUSY is broken for finite Λ: the operator which converges to the Poincare Hamiltonian P_0, does not commute with the SUSY charges.

Classical SUGRA supports such a picture, but suggests a relation between the gravitino mass and the c.c.: $m_{3/2} \sim \Lambda^{1/2}/M_P$. CSB is the proposal that the exponent 1/2 in this relation is replaced by 1/4 in the quantum theory. Given the interpretation of Λ as a parameter controlling the number of states, this is a critical exponent, and it is plausible that it has fluctuation corrections. Indeed, low energy effective field theory cannot calculate the real relation between the gravitino mass and the c.c., since the c.c. is a relevant parameter and one must introduce a counterterm for it. The exponent above is just the "natural" relation of classical SUGRA, without fine tuning of the constant in the superpotential. If we accept such fine tuning, we can get any relation we want between $m_{3/2}$ and Λ in effective field theory.

However, the necessity of canceling an infinite c.c. appears to be a short distance problem in effective field theory, and as such, does not seem to depend on the value of the c.c. . As a consequence, there has been considerable skepticism about CSB.

In [40], one of the authors presented an argument for the exponent 1/4, based on crude approximations to the dynamics of the cosmological horizon in the static observer gauge for dS space. It was clear that from the static observer's point of view, the enhanced exponent is an IR effect. However, since the argument relied on conjectures about the horizon dynamics, it has not convinced anyone. Skeptical observers want to understand where effective field theory reasoning breaks down. The work of [41] provided an important clue. In the static gauge most of the states in a quantum theory of dS space live on the horizon of the static observer. Local field theory can describe only a negligible fraction of the entropy. On the contrary, it was argued in [41] that in global coordinates, the entire Hilbert space may be well described by field theory. The contradiction between a finite number of states and the field theoretic description can be viewed as an IR cutoff, which restricts the global time coordinate to an interval of order $|t| \leq \frac{R}{6} \ln{(RM_P)}$ around the time symmetric point. The field theory also has a UV cutoff at a scale $M_c \sim \left(\frac{M_P}{R}\right)^{1/2}$. This description is inappropriate for states containing black holes whose size scales like R, but there is a basis of field theoretic states in global coordinates, which may span the Hilbert space.

A simple way to restate this conclusion is to invoke the fact that the global description of dS space in field theory does not seem to break down until we contemplate introducing black holes on early time initial data slices, whose entropy exceeds that of the space-time. The combined UV and IR cutoffs prevent us from introducing such objects, and describes a cutoff field theory with a finite number of states. The field theory description of many of these states breaks down near the time-symmetric point, but near the upper and lower limits of t, it is a good approximation to their properties.

We have thus set up a framework in which IR divergences in a field theoretic treatment of dS space can be thought of as introducing non-classical dependence on the c.c. . It has often been argued that perturbative quantum gravity expanded around dS space is fraught with IR divergences. These claims have been less than convincing, because no-one had identified gauge invariant observables with which to check the physical meaning of the logarithmically growing graviton propagator. This problem is solved by dS/CFT [42][43][44]. In particular, the mass of a field in dS space is given a gauge invariant meaning: it is related to the dimension of a conformal field on the boundary.

The plan of this paper is as follows: in the next section we review dS/CFT, in the Wheeler-DeWitt formalism proposed by Maldacena. This allows perturbative calculations to be performed in a straightforward manner, apparently troubled only by conventional UV divergences. In section 3 we perform one loop calculations of boundary dimensions in a variety of non-gravitational field theories. We find that when the theory contains a massless, minimally coupled scalar field with soft couplings, the dimensions are infected with IR logarithms. In the conclusions, we discuss the difficulties attendant on an extension of these calculations to perturbative quantum gravity.

8.2 Review of dS/CFT

In his talk at Strings 2001 in Mumbai [42], Witten proposed a sort of scattering theory for de Sitter space. The fundamental object was the path integral with fixed boundary conditions on $\mathcal{I}_{\pm}$. It was implicitly assumed that, as in asymptotically flat and Anti-deSitter spaces, a field theoretic approximation became exact near the boundaries of space-time. This assumption is open to criticism. It is likely that generic boundary conditions on fields on $\mathcal{I}_{-}$ will lead to Big Crunch space-times, rather than space-times which are future asymptotically dS. However, this criticism does not apply to perturbation theory, where the boundary conditions are infinitesimal perturbations of those corresponding to the dS vacuum. Witten's prescription provides a perturbative definition of amplitudes in dS quantum gravity, which are invariant under diffeomorphisms that approach the identity near $\mathcal{I}_{\pm}$.

Somewhat later, Strominger proposed [43] that suitably defined boundary amplitudes should be the correlation functions of a Euclidean conformal field theory (CFT). An apparent difference with Witten's proposal is the role of conformally covariant, rather than invariant amplitudes in dS/CFT. However, Maldacena [45] has emphasized that the operator dimensions, OPE coefficients and the like, of dS/CFT, are gauge invariant observables in the sense of Witten.

The boundary correlation functions defined by Strominger should certainly be conformally invariant, but it is not clear that they should obey the axioms of field theory. Analogous arguments would lead us to believe that the holographic dual of linear dilaton backgrounds [46] was a Lorentz invariant field theory. The calculations of Peet and Polchinski [47] show that it is not. In the dS/CFT case, the form of the two point function follows from conformal invariance alone, and does not give us enough of a clue to the nature of the holographic dual. As believers in the proposition that quantum dS space has only a finite number of states, the present authors are inclined to disbelieve that a CFT will be the exact description of the quantum theory.

For our present purposes, all of these issues of principle are somewhat beside the point. We want a definition of correlation functions on $\mathcal{I}_\pm$ which is perturbatively well defined and gauge invariant. Furthermore, we will be interested only in two point functions, and will not have to address the question of whether higher order correlators obey the axioms of CFT. We have found that the dS/CFT prescription advocated by Maldacena [44] is the most appropriate for our purposes. Maldacena observes that the Euclidean path integral on a space with the topology of a hemisphere defines a "wave function of the universe" which is a functional of fields on the boundary of the hemisphere. In leading semiclassical approximation, the geometry is the section of the round sphere metric

$$ds^2 = d\theta^2 + sin^2(\theta)d\Omega^2$$

with $0 \leq \theta \leq \theta_0$. Maldacena defines boundary correlators as the expansion coefficients of the logarithm of the wave function of the universe for fixed θ_0. The analytic extrapolation $\theta_0 \to \frac{\pi}{2} + it$, $t \to \infty$ defines correlation functions on $\mathcal{I}_+$. If the limiting correlation functions exist, they should be covariant under the conformal group of the sphere. In particular, if we work in planar coordinates on the upper triangle of the dS Penrose diagram

$$ds^2 = \frac{1}{\eta^2}\left(-d\eta^2 + d\mathbf{x}^2\right)$$

($\mathcal{I}_+$is at $\eta = 0$) then the boundary two point function should have the form $|\mathbf{x}|^{-\Delta}$. For a free scalar field of mass m^2 this is indeed true, and the relation between mass and dimension is given by

$$\Delta_\pm = a = \frac{1}{2}\left(d - 1 \pm \sqrt{(d-1)^2 - 4m^2R^2}\right)$$

This is an analytic continuation (in the c.c.) of analogous formulas in AdS/CFT. Indeed, Maldacena's proposal for the correlation functions is the direct analog of the calculation of Euclidean correlation functions in AdS/CFT.

The purpose of the present paper is to compute one loop corrections to $\Delta_\pm$ in simple field theory models. We will see that when the theory has a massless, minimally coupled scalar with soft couplings, these corrections are IR divergent.

8.3 Review of QFT in dS space

In this section we will introduce the principal formulae of QFT in d-dimensional de Sitter (dS^d) space, and fix our notation .

For a more complete discussion we refer to the excellent review paper [48].

8.3.1 Coordinate Systems

d-dimensional de Sitter dS^d can be realized as the manifold, embedded in $d+1$ dimensional Minkowski $\mathcal{M}^{d,1}$ space, defined by the equation

$$-X_0^2 + X_1^2 + \cdots X_d^2 = R^2 \tag{8.1}$$

where R is the de Sitter radius.

The de Sitter metric is the standard metric induced by immersion in $\mathcal{M}^{d,1}$ with the usual flat metric. The isometry group of dS^d is $O(d,1)$ in fact this leave invariant both the hyperboloid defined by the equation (8.1) and the flat metric of $\mathcal{M}^{d,1}$.

For the most part, we will use planar coordinates

$$\begin{aligned} X^0 &= \sinh t - \frac{1}{2}x_i x_i e^{-t} \\ X^i &= x^i e^{-t} \\ X^d &= \cosh t - \frac{1}{2}x_i x_i e^{-t} \end{aligned} \tag{8.2}$$

with $i = 1, \ldots, d$ the metric take the form

$$ds^2 = -dt^2 + e^{-2t} dx_i dx_i$$

In these coordinates the spatial sections have flat *Euclidean* metric.

It is useful to introduce conformal coordinates too, defined by the transformation

$$\eta = e^t$$

The metric is conformally flat and takes the form

$$ds^2 = \frac{1}{\eta^2}\left(-d\eta^2 + dx_i dx_i\right)$$

with $i = 1, \dots, d$. In the following, unless otherwise stated, we will consider the *Euclidean* section of dS^d defined by the analytical continuation

$$\eta \to i x_0 \tag{8.3}$$

after the transformation (8.3) the metric become

$$ds^2 = -\frac{1}{x_0^2}\left(dx_0^2 + dx_i dx_i\right) \tag{8.4}$$

in these coordinates the boundary of dS^d Σ is given by the submanifold $x_0 = \epsilon$ where $\epsilon \to 0$.

8.3.2 Geodesic Distance

The geodesic distance between two points x and x' is

$$\mu(x, x') = \int_0^1 \left[g_{ab}\dot{x}^a(\lambda)\dot{x}^b(\lambda)\right]^{\frac{1}{2}} d\lambda, \quad x^a(0) = x, \quad x^a(1) = x'$$

In the following we will often use the new variable

$$z = \cos^2\left(\frac{\mu}{2R}\right)$$

It is possible to show that

$$\cos\left(\frac{\mu(x, x')}{R}\right) = \frac{\eta_{ab}X^a(x)X^b(x')}{R^2}$$

with $X^a(x),\ X^b(x') \in \mathcal{M}^{d,1}$ embedding coordinates and η_{ab} =diag$(-1, 1, \dots, 1)$.

Consequently we have

$$\begin{aligned} z &= \cos^2\left(\frac{\mu}{2R}\right) \\ &= \frac{1}{2}\left(1+\cos(\frac{\mu}{R})\right) \\ &= \frac{1}{2}\left(1+\frac{\eta_{ab}X^a(x)X^b(x')}{R^2}\right) \end{aligned}$$

In the *Euclidean* conformally flat coordinates (8.4) we have

$$z = -\frac{(x_0-y_0)^2+(\bar{x}-\bar{y})^2}{x_0y_0} = -2+\frac{x_0^2+{y_0}^2+(\bar{x}-\bar{y})^2}{x_0y_0}$$

8.3.3 The Cut-off Prescription

Maldacena's prescription defines the boundary correlators by analytic continuation in global time. We have proposed that these formulae should be cut off at a fixed global time T. IR divergences will appear as divergent behavior at large T. It is most convenient to do calculations in conformal coordinates. Thus we have to understand the effect of a global time cut-off in conformal coordinates.

The relation between the two coordinate systems is most simply understood by writing the embedding coordinates in terms of conformal coordinates. The slices of fixed embedding time and global time coincide:

$$X^0 = \frac{R}{2}\left(\frac{x^0}{R}-\frac{R}{x^0}\right)-\frac{\mathbf{x}^2}{2x^0}$$

At $X^0 = T$, see Fig. 8.1 and Fig. 8.2. This relation implies a maximal value of $|\mathbf{x}|$ for fixed x^0, as well as a maximal value of x^0 (which runs between $-\infty$ and 0 in the

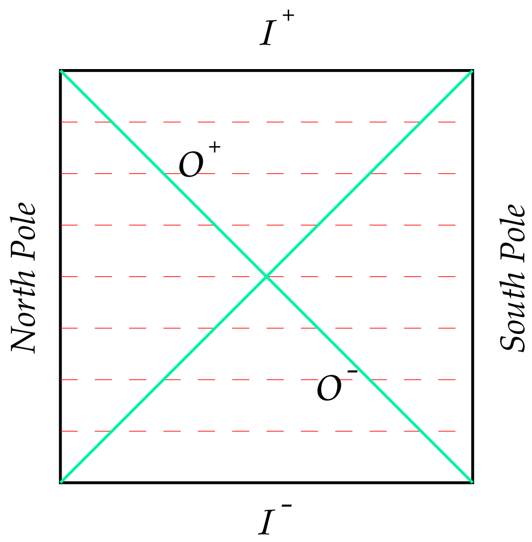

Figure 8.1: **Global coordinates.** *Foliation of dS with compact spatial sections (spheres).*

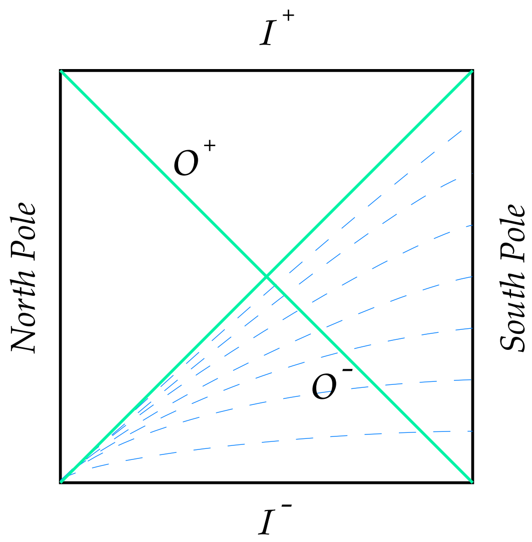

Figure 8.2: **Flat coordinates.** *Foliation of dS with flat spatial sections.*

conformal coordinate patch). The relation is

$$x_{\max}^2 = -2x^0 \left(T - x^0 + \frac{R^2}{x^0} \right)$$

The maximal value of x^0 is the point at which $x_{max} = 0$.

$$x^0_{max} \approx -\frac{R^2}{T} \quad T \gg R$$

The maximal geodesic distance between two points on a give x^0 slice is $\frac{x_{max}}{x^0_{max}}$. The slice on which this distance is maximal is given by $x^0_* = -\frac{2R^2}{T}$. The geodesic distance on this slice is $o(T)$, while the maximum coordinate distance is $o(R)$. IR divergences will come predominantly from slices near this maximal slice.

Dirichlet boundary conditions on the $X^0 = T$ surface become *spatial* Dirichlet boundary conditions on the spatial slices of conformal coordinates. On most of the slice of maximal geodesic size, the Dirichlet propagator will coincide with the usual Euclidean propagator defined by analytic continuation from the entire sphere. Thus, the boundary conditions will not affect the IR divergences.

8.3.4 Wave Function of the Universe

We are looking for a gauge invariant definition of the IR renormalization of the particle mass. The Wave Function of the Universe (WFU) will provide us with such a definition.

The WFU $\Psi[h_{ij}, \phi_0]$ was first introduced by Hartle and Hawking in [49]. If $I[g, \phi]$ is the Euclidean action for gravity and a set of fields indicated by ϕ, the *Euclidean* WFU is defined as the path integral

$$\Psi[h_{ij}, \phi_0] = \int_C [dg][d\phi] e^{-I[g,\phi]} \tag{8.5}$$

over a class C of space-times with a compact space-like boundary Σ on which the induced metric is h_{ij} and over the field configurations ϕ with boundary value ϕ_0. The boundary Σ has only one connected component.

In the case $\Lambda > 0$ we imagine a semiclassical expansion of the integral over Riemannian spaces with the topology of a hemisphere, expanded around the metric on the portion of the round sphere below polar angle θ_0. We then analytically continue to the future half of Lorentzian dS space. This prescription corresponds to the choice of Euclidean vacuum in de Sitter space.

Given the WFU we can define the "boundary two-point function" in the limit where the boundary is taken to $\mathcal{I}^+$

$$\frac{\delta\Psi[h_{ij},\phi_0]}{\delta\phi_0(\bar{x})\delta\phi_0(\bar{y})}$$

Once we expand around dS^d we find

$$\frac{\delta\Psi[h_{ij},\phi_0]}{\delta\phi_0(\bar{x})\delta\phi_0(\bar{y})} = C_+\frac{1}{(\bar{x}-\bar{y})^{2\Delta_+}} + C_-\frac{1}{(\bar{x}-\bar{y})^{2\Delta_-}} \tag{8.6}$$

, where $C_\pm$ are constants This form is dictated by conformal invariance. If λ and m are the coupling and the mass of the field ϕ, in the classical Lagrangian, then Δ will be a function of λ and m and will provide a gauge invariant definition of the renormalized mass.

The Eq. (8.6) is the analogue of the boundary correlators defined in the AdS/CFT correspondence

$$Z[\phi_0] = \left\langle e^{\int d^4x\phi_0(x)O(x)}\right\rangle_{CFT} \quad , \ \phi(x_0=\epsilon)\sim\phi_0$$

$$\langle 0|O(\bar{x})O(\bar{y})|0\rangle = \frac{\delta Z}{\delta\phi_0(\bar{x})\delta\phi_0(\bar{y})} = \tilde{C}\frac{1}{(\bar{x}-\bar{y})^{2\tilde{\Delta}}}$$

There are however, important differences between the two cases. They stem from the fact that the Euclidean section of dS space is a spherical cap and has a conventional Dirichlet problem, different from the singular Dirichlet boundary conditions on the boundary of Euclidean AdS space. There are no large volume divergences in the Euclidean calculation. They appear only after extrapolation to infinite Lorentzian time. As a consequence, the divergent behavior comes as a combination of both powers $\Delta_\pm$. For fields corresponding to the principal series of dS representation theory, the real parts of $\Delta_\pm$ are equal.

The prescription to extract boundary two-point function in dS^d given by (8.6) was first pointed out by Maldacena in [44] and it is, as explained in this paper, different from the prescription used by Strominger and collaborators in [43], [50].

8.3.5 Representations of the dS^d Group

The scalar representation of the de Sitter group $SO(1,d)$ are classified according to the mass m in the following series, see [51], [52]: the principal series

$$m^2 \geqslant \left(\frac{d-1}{2R_{dS}}\right)$$

the complementary series

$$0 < m^2 < \left(\frac{d-1}{2R_{dS}}\right)$$

and the discrete series, whose only case of physical interest is $m^2 = 0$.

Under a Wigner-Inönü contraction to the Poincare group, only the representations of the principal series contract to representation of the Poincare Group.

Lowe and Güijosa [53] and Lowe [54] use the principal series to construct the dS/CFT correspondence. They stress the fact that when one replaces the dS isometry group with a q-deformed version, the unitary principal representation deform to a finite dimensional unitary representation of the quantum group[1].

The massive scalar particles in our formulae will always correspond to the principle series representations, so that the boundary dimensions all have the same real

[1]The idea that a q-deformed version of the dS group might have finite dimensional unitary representations, resolving the contradiction between dS invariance and a finite number of states, was pointed out to one of the authors (TB) by A. Rajaraman in the fall of 1999. There seemed to be a problem with this idea, because the dS group has no highest weight unitary representations, but Lowe and Güijosa made the crucial observation that the cyclic representations of the quantum group (which are not highest weight) converged to the principal unitary series.

part. We will also use a massless, minimally coupled scalar, which is our toy model of the graviton.

8.4 Scalar Green Functions

In the next few subsections we will derive the scalar Green Functions relevant for our computations and their asymptotic behavior. As explained in the section on the cut-off procedure, we will not impose Dirichlet boundary conditions on the bulk propagators. The IR divergences, which are our principal concern, are not affected by the boundary conditions on the bulk propagator. For a more detailed discussion of dS Green functions, see for example [55], [56].

8.4.1 Maximally Symmetric Bitensors

The relevant geometric objects in maximally symmetric spaces, like dS, are the geodesic distance $\mu(x,x')$ between two points x and x', the unit tangent vectors $n_\sigma(x,x')$ and $n_{\sigma'}(x,x')$ to the geodesic at x and at x', the vector parallel propagator $g^{\mu}{}_{\nu'}(x,x')$ and the spinor parallel propagator $\Lambda^{\alpha}{}_{\beta'}(x,x')$.

The geodesic distance is by definition the distance along the geodesic $x^a(\lambda)$ connecting x and x'

$$\mu(x,x') = \int_0^1 \left[g_{ab}\dot{x}^a(\lambda)\dot{x}^b(\lambda)\right]^{\frac{1}{2}} d\lambda, \quad x^a(0) = x, \quad x^a(1) = x'$$

The vectors $n_\sigma, n_{\sigma'}$ are defined by

$$n_\sigma = \nabla_\sigma \mu(x,x') \qquad \text{and} \qquad n_{\sigma'} = \nabla_{\sigma'}\mu(x,x')$$

where ∇_σ is the covariant derivative. We note that

$$n_\sigma = -g_\sigma{}^{\rho'} n_{\rho'}$$

The vector and spinor parallel propagators are defined by

$$\begin{aligned} V^\mu(x) &= g^\mu{}_{\nu'}(x,x')V^{\nu'}(x') && (8.7) \\ \psi^\alpha(x) &= \Lambda^\alpha{}_{\beta'}(x,x')\psi^{\beta'}(x') && (8.8) \end{aligned}$$

for every parallel-transported vector $V^\mu(x)$ and spinor $\psi^\alpha(x)$, respectively.

Tensors that depend on two points x and x' on the manifold are called *bitensor.* We will say that a bitensor is *maximally symmetric* if is invariant under any isometry of the manifold. It can be proved that any maximally symmetric bitensor can be expressed as a sum of products of $g^\mu{}_{\nu'}$, $g_{\mu\nu}$, $g_{\mu'\nu'}$, μ, n_σ and $n_{\sigma'}$. Furthermore the coefficients of these terms are functions only of the geodesic distance $\mu(x,x')$.

The covariant derivatives of the above bitensors are given by

$$\begin{aligned} \nabla_\mu n_\nu &= A\,(g_{\mu\nu} - n_\mu n_\nu) \\ \nabla_{\mu'} n_\nu &= C\,(g_{\mu'\nu} + n_{\mu'} n_\nu) \\ \nabla_\mu g_{\nu\rho'} &= -(A+C)\,(g_{\mu\nu} n_{\rho'} + g_{\mu\rho'} n_\nu) \\ \nabla_\mu \Lambda^\alpha{}_{\beta'} &= \frac{1}{2}(A+C)\,[\,(\Gamma_\mu \Gamma^\nu n_\nu - n_\mu)\,\Lambda]^\alpha{}_{\beta'} \\ \nabla_{\mu'} \Lambda^\alpha{}_{\beta'} &= -\frac{1}{2}(A+C)\,[\,(\Gamma_{\mu'} \Gamma^{\nu'} n_{\nu'} - n_{\mu'})\,\Lambda]^\alpha{}_{\beta'} \end{aligned} \qquad (8.9)$$

where A and C are the following functions of the geodesic distance:

$$\begin{aligned} &\text{for } \mathbf{R}^{\mathrm{d}}: \qquad A(\mu) = \frac{1}{\mu} \qquad C(\mu) = -\frac{1}{\mu} \\ &\text{for dS and AdS}: \qquad A(\mu) = \frac{1}{R}\cot\frac{\mu}{R} \qquad C(\mu) = -\frac{1}{R\sin\left(\frac{\mu}{R}\right)} \end{aligned} \tag{8.10}$$

The radius R is real for dS^d and it is $R = i\tilde{R}$ with $\tilde{R}$ real for AdS^d. The covariant gamma matrices satisfy the usual relation $\{\Gamma^\mu, \Gamma^\nu\} = 2g^{\mu\nu}$.

8.4.2 Bulk Two-Point Function

In this subsection we will evaluate the scalar two-point function

$$G(x, x') = \langle\psi|\phi(x)\phi(x')|\psi\rangle$$

We will assume that the state $|\psi\rangle$ is maximally symmetric, this implies that for spacelike separated points $G(x, x')$ depends only on the geodesic distance $\mu(x, x')$. For timelike separation the symmetric and Feynman functions also depend only on μ but the commutator function depend on the time ordering too. Doing an analytical continuation from spacelike separation $\mu^2 > 0$ to timelike separation $\mu^2 < 0$, it is possible to obtain all these two-point functions.

We now derive a differential equation for $G(x, x')$. Applying the Laplacian operator to $G(x, x')$ we have

$$\begin{aligned} \Box G(\mu) &= \nabla^\nu\nabla_\nu G(\mu) = \nabla^\nu(G'(\mu)n_\nu) \\ &= G''(\mu)n^\nu n_\nu + G'(\mu)\nabla^\nu n_\nu \\ &= G''(\mu) + (d-1)A(\mu)G'(\mu) \end{aligned}$$

where we have used the formulae (8.10) and the notation $G' = \frac{dG}{d\mu}$.

Using the equation of motion $(\Box - m^2)\phi = 0$ we find

$$G''(\mu) + (d-1)A(\mu)G'(\mu) - m^2 G = 0 \tag{8.11}$$

as long as $x \neq x'$.

Defining the change of variable

$$z = \cos^2\left(\frac{\mu}{2R}\right)$$

the Eq. (8.11) for G becomes

$$z(1-z)\frac{d^2G}{dz^2} + [c - (a+b+1)z]\frac{dG}{dz} - abG = 0 \tag{8.12}$$

where we defined

$$a = \Delta_+ = \frac{1}{2}\left(d - 1 + \sqrt{(d-1)^2 - 4m^2R^2}\right) \tag{8.13}$$

$$b = \Delta_- = \frac{1}{2}\left(d - 1 - \sqrt{(d-1)^2 - 4m^2R^2}\right) \tag{8.14}$$

$$c = \frac{1}{2}d \tag{8.15}$$

8.4.2.1 De Sitter Space: Massive Scalar

De Sitter space corresponds to choosing R real in the Eq. (8.10). There are two linearly independent solution $G(z)$ to Eq. (8.12). Any of the solutions of Eq. (8.12) is associated with a particular vacuum $|\psi\rangle$.

The Two-point function

$$G_E(x, x') = \langle E|\phi(x)\phi(x')|E\rangle$$

associated with the Euclidean vacuum $|E\rangle$ Introduced in Section 8.3.4 and defined as analytical continuation from the sphere is given by

$$G_E(x, x') = qF(a, b; c; z) \tag{8.16}$$

where $F(a, b; c; z)$ is the hypergeometric function.

The two-point function in the Euclidean vacuum turns out to have the following properties:

1. has only one singular point at $\mu(x, x') = 0$ and therefore regular at $\mu(x, x') = \pi R$

2. Has the same strength $\mu \to 0$ singularity as in flat space.

The constant q in Eq. (8.16) is determined by the condition that as $\mu \to 0$ $G_E(x, x')$ has to approach the flat two point function

$$G_{flat}(\mu) \sim \frac{\Gamma\left(\frac{d}{2}\right)}{2(d-2)\pi^{\frac{d}{2}}} \mu^{-d+2}, \quad \mu \to 0$$

we find

$$q = \frac{\Gamma(a)\Gamma(b)}{\Gamma\left(\frac{d}{2}\right) 2^d \pi^{\frac{d}{2}}} R^{-d+2}$$

For the computation it will be useful to derive the asymptotic expansion of $G(z)$ for $z \to -\infty$ that correspond to $x_0 \to 0$ or $y_0 \to 0$.

The geodesic distance in conformally flat coordinate was given in Section 8.3.2 and it is

$$z = -\frac{(x_0 - y_0)^2 + (\bar{x} - \bar{y})^2}{x_0 y_0}$$

we have

$$\lim_{\substack{x_0\to 0\\ y_0\to 0}} z \sim -\frac{(\bar{x}-\bar{y})^2}{x_0 y_0}$$

so that the asymptotic expansion of $G(z)$ for $z \to -\infty$ is

$$\lim_{z\to\infty} G(z) \sim C_+ \frac{1}{z^{\Delta_+}} + C_- \frac{1}{z^{\Delta_-}} = C_+ \left(\frac{-x_0 y_0}{(\bar{x}-\bar{y})^2}\right)^{\Delta_+} + C_- \left(\frac{-x_0 y_0}{(\bar{x}-\bar{y})^2}\right)^{\Delta_-} \tag{8.17}$$

with

$$C_+ = q\,\frac{\Gamma(\frac{d}{2})\Gamma(\Delta_- - \Delta_+)}{\Gamma(\Delta_-)\Gamma(\frac{d}{2}-\Delta_+)}, \quad C_- = q\,\frac{\Gamma(\frac{d}{2})\Gamma(\Delta_+ - \Delta_-)}{\Gamma(\Delta_+)\Gamma(\frac{d}{2}-\Delta_-)}$$

8.4.2.2 De Sitter Space: Massless Scalar

The two-point function for a massless minimally coupled scalar field in de Sitter space was studied in [57], [56]. They find the following expression for the two-point function

$$\begin{aligned} G_0(z) &= \frac{R^2}{192\pi^2 m^2} + \frac{R}{48\pi^2}\left(\ln(1-z) + \frac{1}{1-z}\right) \\ &= C_0\left(\ln(1-z) + \frac{1}{1-z}\right) + \tilde{C} \end{aligned} \tag{8.18}$$

We will not need the actual values of the constants C_0 and $\tilde{C}$ in our computation.

The asymptotic expansion for $z \to -\infty$ of the massless two-point function (8.18) is

$$G_0(z) \sim C_0\left(\ln\frac{(\bar{x}-\bar{y})^2}{x_0 y_0} + \frac{x_0 y_0}{(\bar{x}-\bar{y})^2}\right) \tag{8.19}$$

8.4.3 Bulk to Boundary Propagators: dS/AdS

The Bulk to Boundary propagator for AdS^d were derived by Witten in [58]. They obey the equations

$$(\Box_x - m^2)\tilde{K}(x, \bar{y}) = 0$$

$$\tilde{K}(\bar{x}, x_0; \bar{y}) \to (x_0)^{((d-1)-\Delta)}\delta^d(\bar{x}-\bar{y}), \text{ for } x_0 \to 0$$

and their explicit form in the Poincare coordinates in AdS^d is

$$\tilde{K}(\bar{x}, x_0; \bar{y}) = \frac{\Gamma(\Delta)}{\pi^{\frac{d-1}{2}}\Gamma\left(\Delta - \frac{d-1}{2}\right)}\left(\frac{x_0}{x_0^2 + (\bar{x}-\bar{y})^2}\right)^{\Delta}$$

with

$$\Delta = \Delta_+ = a = \frac{1}{2}\left(d - 1 + \sqrt{(d-1)^2 + 4m^2\tilde{R}^2}\right)$$

If we consider the conformally flat coordinates (8.4) in dS^d the equations defining the Bulk to Boundary propagator become

$$\left(\Box_x + m^2\right)K(x, \bar{y}) = 0$$

We impose Dirichlet boundary conditions, $K \to \delta(x-\bar{y})$ as x approaches the boundary of a spherical cap. The cap is then continued to a hemisphere, and analytically continued to $\theta = \frac{\pi}{2} + it$. In our conformal coordinates for the Lorentzian section, $t \to \infty$, corresponds to $x^0 \to 0$. In this limit

$$K(\bar{x}, x_0; \bar{y}) \to C_+(x_0)^{((d-1)-\Delta_+)}\delta^d(\bar{x}-\bar{y}) + C_-(x_0)^{((d-1)-\Delta_-)}\delta^d(\bar{x}-\bar{y}), \text{ for } x_0 \to 0$$

with

$$\Delta_{\pm} = a = \frac{1}{2}\left(d - 1 \pm \sqrt{(d-1)^2 - 4m^2R^2}\right)$$

8.4.4 Boundary Two-point Function: dS/AdS

The boundary two point function for AdS^d in the Poincare patch as given for example in [46] is

$$\langle 0|O(\bar{x})O(\bar{y})|0\rangle = \frac{\delta Z}{\delta\phi(\bar{x})\delta\phi(\bar{y})} = C\frac{1}{(\bar{x}-\bar{y})^{2\Delta}}$$

with

$$\Delta = \Delta_{+} = a = \frac{1}{2}\left(d - 1 + \sqrt{(d-1)^2 + 4m^2\tilde{R}^2}\right)$$

For dS^d in the conformally flat coordinates (8.4) we have

$$\frac{\delta\Psi_0[h_{ij},\phi_0]}{\delta\phi_0(\bar{x})\delta\phi_0(\bar{y})} = C_{+}\frac{1}{(\bar{x}-\bar{y})^{2\Delta_{+}}} + C_{-}\frac{1}{(\bar{x}-\bar{y})^{2\Delta_{-}}}$$

with

$$\Delta_{\pm} = a = \frac{1}{2}\left(d - 1 \pm \sqrt{(d-1)^2 - 4m^2R^2}\right)$$

8.5 General Structure of the Computation

In this section we want to give a general description of the calculation we will perform for three specific models.

As we have already discussed in Section 8.3.4 we are interesting in computing at *1-loop* the Wave Function of the Universe (WFU)

$$\Psi[h_{ij},\phi_0] = \int_C [dg][d\phi]e^{-I[g,\phi]}$$

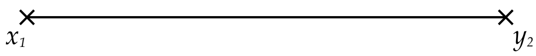

Figure 8.3: **Tree diagram.** *This diagram represent the tree-level contribution to the Wave Function of the Universe (WFU). The points x_1 and x_2 are on the boundary.*

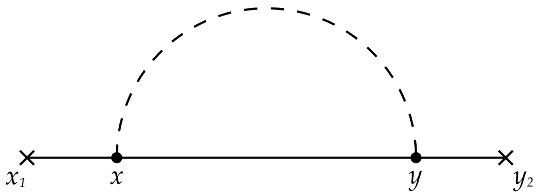

Figure 8.4: **1-loop diagram.** *1-Loop contribution to the WFU. The points x_1 and x_2 are on the boundary while x and y are bulk points.*

for the models described in Section 8.6. The *tree-level* and *1-loop* diagrams are represented respectively in Fig. 8.3 and Fig. 8.4.

Given the WFU we want to find the "boundary two-point function"

$$\frac{\delta\Psi[h_{ij},\phi_0]}{\delta\phi_0(\bar{x})\delta\phi_0(\bar{y})} \tag{8.20}$$

this will provide us with a gauge invariant definition of the renormalized mass.

We consider a general action of the form

$$S = \int d^dx\sqrt{g}\left(\phi_A\triangle\phi_A + \phi_B\triangle\phi_B + \phi_C\triangle\phi_C\right) + \lambda\sqrt{g}\phi_A\phi_B\phi_C$$

where

$$S_0 = \int d^dx\sqrt{g}\phi_\alpha\triangle\phi_\alpha, \quad \alpha = A, B, C$$

is the quadratic part of the action i.e.

$$S_0 = \int d^dx\sqrt{g}\frac{1}{2}\left[(\partial\phi_A)^2 + m_A^2\phi_A^2\right]$$

for a scalar field and

$$S_0 = S_M + S_{\partial M} = \int_M d^dx\sqrt{g}\bar{\psi}\left(\not{D} - m\right)\psi + \int_{\partial M} d^dx\ \sqrt{h}\bar{\psi}\psi$$

for a spinor field.

In the WFU we are integrating over fields with the following boundary conditions

$$\phi_\alpha|_\Sigma = \phi_{\alpha 0}, \quad \alpha = A, B, C$$

where by the symbol $\phi_\alpha|_\Sigma$ we mean the field evaluated on the boundary of the Euclidean spherical cap. To impose the boundary condition we decompose the field in the following way

$$\phi_\alpha = \phi_{\alpha 1} + \phi_{\alpha 2}$$

with

$$\phi_{\alpha 1}|_\Sigma = \phi_{\alpha 0}, \quad \phi_{\alpha 2}|_\Sigma = 0$$

The field $\phi_{\alpha 1}$ is the solution of the free wave equation with Dirichlet boundary conditions, and can be written in terms of the appropriate Bulk to Boundary propagator

$$\phi_{\alpha 1} = K_\alpha \circ \phi_{\alpha 0} = \int_\Sigma d\bar{y}\ K_\alpha(\bar{x}, x_0; \bar{y})\phi_{\alpha 0}(\bar{y}), \quad \bar{y} \in \Sigma, \quad \alpha = A, B, C$$

described in the Sections 8.4.3 and 8.14.3.

To compute the *1-loop* correction to the "boundary two-point function" (8.20) we have to evaluate the terms in $\Psi[h_{ij}, \phi_0]$ that are quadratic both in ϕ_0 and in the coupling constant λ. Expanding the path integral we have

$$\begin{aligned}
\Psi &= \int [d\phi_A][d\phi_B]\ [d\phi_C] e^{-S_0[\phi_A,\phi_B,\phi_C] - \int d^dx\ \sqrt{g(x)}\lambda\phi_A\phi_B\phi_C} \\
&= \int [d\phi_A][d\phi_B]\ [d\phi_C] e^{-S_0[\phi_A,\phi_B,\phi_C]} \Big[1 - \lambda \int d^dx\ \sqrt{g(x)}\phi_A(x)\phi_B(x)\phi_C(x) \\
&\quad + \frac{\lambda^2}{2} \int d^dx \int d^dy\ \sqrt{g(x)}\sqrt{g(y)}\phi_A(x)\phi_B(x)\phi_C(x)\phi_A(y)\phi_B(y)\phi_C(y) + O(\lambda^3)\Big]
\end{aligned}$$

where we indicated with $S_0[\phi_A, \phi_B, \phi_C]$ the quadratic part of the action.

The terms quadratic in $\phi_{\alpha 0}$ $\alpha = A, B, C$ come from the expansion of the term

$$\phi_A(x)\phi_B(x)\phi_C(x)\phi_A(y)\phi_B(y)\phi_C(y)$$

we have

$$\begin{aligned} & \phi_A(x)\phi_B(x)\phi_C(x)\phi_A(y)\phi_B(y)\phi_C(y) \\ = \; & \phi_{A1}(x)\phi_{A1}(y)\left[\phi_{B2}(x)\phi_{B2}(y)\phi_{C2}(x)\phi_{C2}(y)\right] \\ & + \cdots \end{aligned}$$

We will compute only the correction to the two-point function of the field ϕ_A. The part of the path integral relevant to this calculation is

$$\begin{aligned} \Psi^A_{\textit{1-loop}}[\phi_{A0}] \;=\; & \int [d\phi_B]\;[d\phi_C] e^{-\int d^dx\sqrt{g}\phi_B\triangle\phi_B - \int d^dx\sqrt{g}\phi_C\triangle\phi_C} \\ & \times\frac{\lambda^2}{2}\int d^dx\int d^dy\;\sqrt{g(x)}\sqrt{g(y)}\phi_{A1}(x)\phi_{A1}(y)\left[\phi_{B2}(x)\phi_{B2}(y)\phi_{C2}(x)\phi_{C2}(y)\right] \end{aligned}$$

The only parts of the fields that fluctuate in the path integral are $\phi_{\alpha 2}$, in fact $\phi_{\alpha 1}$ is fixed by the boundary conditions. For this reason the measure of integration is given by

$$[d\phi_B]\;[d\phi_C] = [d\phi_{B2}]\;[d\phi_{C2}]$$

Standard manipulation give us the following expression for the path integral

$$
\begin{aligned}
\Psi^A_{\textit{1-loop}}[\phi_{A0}] &= \int [d\phi_{B2}]\ [d\phi_{C2}] e^{-\int d^dx\sqrt{g}\phi_B\triangle\phi_B - \int d^dx\sqrt{g}\phi_C\triangle\phi_C} \\
&\times \frac{\lambda^2}{2}\int d^dx\int d^dy\ \sqrt{g(x)}\sqrt{g(y)}\phi_{A1}(x)\phi_{A1}(y)[\phi_{B2}(x)\phi_{B2}(y)\phi_{C2}(x)\phi_{C2}(y)] \\
&= \frac{\lambda^2}{2}\int d^dx\int d^dy\ \sqrt{g(x)}\sqrt{g(y)}\phi_{A1}(x)\phi_{A1}(y)\langle E|\phi_{B2}(x)\phi_{B2}(y)\phi_{C2}(x)\phi_{C2}(y)|E\rangle \\
&= \frac{\lambda^2}{2}\int d^dx\int d^dy\ \sqrt{g(x)}\sqrt{g(y)}\phi_{A1}(x)\phi_{A1}(y)\langle E|\phi_{B}(x)\phi_{B}(y)\phi_{C}(x)\phi_{C}(y)|E\rangle \\
&= \frac{\lambda^2}{2}\int d^dx\int d^dy\ \sqrt{g(x)}\sqrt{g(y)}\phi_{A1}(x)\phi_{A1}(y)G_B(x,y)G_C(x,y) \\
&= \frac{\lambda^2}{2}\int d^dx\int d^dy\ \sqrt{g(x)}\sqrt{g(y)}\int_\Sigma d\bar{x}_1\ K_A(x;\bar{x}_1)\phi_{A0}(\bar{x}_1) \\
&\times \int_\Sigma d\bar{x}_2\ K_A(y;\bar{x}_2)\phi_{A0}(\bar{x}_2)G_B(x,y)G_C(x,y)
\end{aligned}
$$

The boundary two-point function at *1-loop* is given by

$$
\begin{aligned}
\frac{\delta\Psi^A}{\delta\phi_{A0}(\bar{x}_1)\delta\phi_{A0}(\bar{x}_2)} &= C\frac{1}{(\bar{x}_1-\bar{x}_2)^{2\Delta}} \\
&+ \frac{\lambda^2}{2}\int d^dx\int d^dy\ \sqrt{g(x)}\sqrt{g(y)}\ K_A(x;\bar{x}_1)G_B(x,y)G_C(x,y)K_A(y;\bar{x}_2)
\end{aligned}
$$

We have similar expressions for the boundary two-point functions of the others fields $\phi_B,\ \phi_C$.

8.6 Models

We have computed the *1-loop* boundary two point function for the following models:

Scalar Fields with Cubic Interaction

$$S = \int d^dx\sqrt{g}\frac{1}{2}\left[(\partial\phi)^2 + m^2\phi^2 + (\partial\phi_1)^2 + m^2\phi_1^2 + (\partial\phi_0)^2\right] + \sqrt{g}\lambda\phi\phi_0\phi_1$$

where the field ϕ_0 is massless.

Scalar Fields with Derivative Couplings

$$S=\int d^dx\sqrt{g}\frac{1}{2}\left[(\partial\phi_A)^2+m^2\phi_A^2+(\partial\phi_B)^2+m^2\phi_B^2+(\partial\phi)^2\right]+\sqrt{g}\lambda\phi g^{\mu\nu}\partial_\mu\phi_A\partial_\nu\phi_B$$

where the field ϕ is massless.

Spinor Field with Derivative Coupling

$$S_0+S_I=\int d^dx\frac{1}{2}\sqrt{g}(\partial\phi)^2+\int_M d^dx\sqrt{g}\bar{\psi}(\not{D}-m)\psi+\int_{\partial M} d^dx\ \sqrt{h}\bar{\psi}\psi+\int_M d^dx\ \sqrt{g}\lambda\partial_a\phi\bar{\psi}\Gamma^a\psi$$

where the field ϕ is massless. The surface term for the fermions is explained in [59],[60],[61].

We have chosen these models in order to see whether the fact that the massless boson is derivatively coupled effects the IR divergence, and to study the effect of fermion chirality. In the conclusions we will discuss the issues that these results raise for the analogous calculations in quantum supergravity.

8.7 1-loop Computation: Scalar Fields with Cubic Interaction

In this section we will compute the *1-loop* boundary two point function for the massive field ϕ interacting with a massive scalar field ϕ_1 and a massless scalar field ϕ_0. The lagrangian is

$$\mathcal{L}=\sqrt{g}\frac{1}{2}\left[(\partial\phi)^2+m^2\phi^2+(\partial\phi_1)^2+m^2\phi_1^2+(\partial\phi_0)^2\right]+\sqrt{g}\lambda\phi\phi_0\phi_1$$

The asymptotic expansions of both the bulk and bulk to boundary propagators, at large Lorentzian time and space-like separation, contain terms with both powers $(x_0)^{\Delta_\pm}$. For the principal series, these powers differ in the sign of their imaginary part. We have found that the most divergent terms as $x_0 \to 0$ come from products of terms from individual propagators that all have the same power of x_0. We call these the *pure* terms. Mixed terms have rapidly oscillating phases, which lead to more convergent integrals. We will find that in this model the pure terms look like the tree level results, but with a divergent correction to the mass. The mixed terms are sub-leading, and do not have the same form as the tree level result. We will explicitly show only our results for the pure terms.

As explained in Section 8.5 the *1-loop* correction to the boundary two-point function

$$\frac{\delta\Psi_{\textit{1-loop}}}{\delta\phi_0(\bar{x}_1)\delta\phi_0(\bar{x}_2)} = G_{\textit{1-loop}}(\bar{x}_1, \bar{x}_2)$$

is given by

$$\begin{aligned} G_{\textit{1-loop}}(\bar{x}_1, \bar{x}_2) &= \frac{\lambda^2}{2} \int d^d x \int d^d y \, \sqrt{g(x)}\sqrt{g(y)} \, K(x;\bar{x}_1) G_1(x,y) G_0(x,y) K(y;\bar{x}_2) \\ &= \frac{\lambda^2}{2} \int d^{d-1}\bar{x} \int d^{d-1}\bar{y} \int dx_0 \int dy_0 \, \frac{1}{x_0^d}\frac{1}{y_0^d} \, K(x;\bar{x}_1) G_1(x,y) G_0(x,y) K(y;\bar{x}_2) \end{aligned}$$

In principle, the bulk propagators in these equations should satisfy (vanishing) Dirichlet boundary conditions at a fixed global time, T. We have seen that in conformal coordinates this corresponds to an x^0 dependent Dirichlet boundary condition on a sphere in $\mathbf{x}$ space, as well as an upper cut-off $x^0_{max} \sim -R^2/T$. The IR divergences will come from the regions of maximal spatial geodesic size, and, because of the Dirichlet boundary conditions, from regions where the two integrated bulk points are far from the spatial boundary sphere. Thus considering only the leading IR divergent part of the answer, we can use the usual Euclidean vacuum Green's function (without Dirichlet boundary conditions) and approximate it by its asymptotic form at large geodesic distance:

$$\begin{aligned}
G^{IR}_{\text{1-loop}}(\bar{x}_1,\bar{x}_2) &\sim \frac{\lambda^2}{2}\int d^{d-1}\bar{x}\int d^{d-1}\bar{y}\int dx_0\int dy_0\,\frac{1}{x_0^d}\frac{1}{y_0^d}\,(x_0y_0)^{((d-1)-\Delta_\pm)} \\
&\quad\times\delta^{d-1}(\bar{x}-\bar{x}_1)G_1(x,y)G_0(x,y)\delta^{d-1}(\bar{y}-\bar{x}_2) \\
&= \frac{\lambda^2}{2}\int dx_0\int dy_0\,\frac{1}{x_0^d}\frac{1}{y_0^d}\,(x_0y_0)^{((d-1)-\Delta_\pm)}G_0(\bar{x}_1,x_0;\bar{x}_2,y_0)G_1(\bar{x}_1,x_0;\bar{x}_2,y_0) \\
&\sim \frac{\lambda^2}{2}\int_\alpha^\epsilon dx_0\int_\beta^\epsilon dy_0\,\frac{1}{x_0}\frac{1}{y_0}\,C_0C_-\ln\left(\frac{(\bar{x}-\bar{y})^2}{x_0y_0}\right)\left(\frac{1}{(\bar{x}-\bar{y})^2}\right)^{\Delta_\pm}
\end{aligned}$$

Here we used the fact that bulk to boundary propagators satisfy

$$K(\bar{x},x_0;\bar{y})\to C_+(x_0)^{((d-1)-\Delta_+)}\delta^d(\bar{x}-\bar{y})+C_-(x_0)^{((d-1)-\Delta_-)}\delta^d(\bar{x}-\bar{y}),\ \text{for } x_0\to 0$$

explained in Section 8.4.3 and the asymptotic expansion (8.17), (8.19) for the bulk two-point functions[2] .

Integrating in x_0 and y_0 and keeping the leading part in $\epsilon\to 0$ we find

$$\begin{aligned}
G^{IR}_{\text{1-loop}}(\bar{x}_1,\bar{x}_2) &\sim \frac{\lambda^2}{2}\frac{1}{(\bar{x}_1-\bar{x}_2)^{2\Delta_\pm}}\times\left(\ln\left(\frac{(\bar{x}_1-\bar{x}_2)^2}{\epsilon}\right)\right)^3 \\
&\quad+\text{Subleading terms in }\epsilon
\end{aligned}$$

[2]In tree level calculations involving two bulk to boundary propagators, only one of them can be replaced by a δ function, since the other ends up evaluated at separated points. The powers of x^0 that would set it equal to zero are part of the renormalization factor that defines the limiting boundary two point function. In our calculation, both bulk to boundary propagators are legitimately replaced by δ functions.

8.8 1-loop Computation: Scalar Fields with Derivative Coupling

In this section we will compute the *1-loop* boundary two points function for the massive scalar field ϕ derivatively coupled to a massless scalar field ϕ_A and a massive scalar field ϕ_B. The action is

$$S = \int d^d x \sqrt{g} \frac{1}{2} \left[(\partial\phi)^2 + m^2\phi^2 + (\partial\phi_B)^2 + m^2\phi_B^2 + (\partial\phi_A)^2 \right] + \sqrt{g}\lambda\phi g^{\mu\nu}\partial_\mu\phi_A\partial_\nu\phi_B$$

Following the general lines of the computation done in Section 8.5 we find for the *1-loop* WFU

$$\begin{aligned}
\Psi_{\textit{1-loop}} &= \int [d\phi_{B2}]\ [d\phi_{C2}] e^{-\int d^d x \sqrt{g}\frac{1}{2}\left[(\partial\phi)^2 + m^2\phi^2 + (\partial\phi_B)^2 + m^2\phi_B^2 + (\partial\phi_A)^2\right]} \\
&\quad \times \frac{\lambda^2}{2} \int d^d x \int d^d y\ \sqrt{g(x)}\sqrt{g(y)} \\
&\quad \times (\phi(x) g^{\mu\nu}(x)\partial_\mu\phi_A(x)\partial_\nu\phi_B(x))(\phi(y) g^{\rho\lambda}(y)\partial_\rho\phi_A(y)\partial_\lambda\phi_B(y)) \\
&= \frac{\lambda^2}{2} \int d^d x \int d^d y\ \sqrt{g(x)}\sqrt{g(y)}\phi_1(x)\phi_1(y) \\
&\quad \times g^{\mu\nu}(x) g^{\rho\lambda}(y)\partial^x_\mu\partial^y_\rho G_A(x,y)\partial^x_\nu\partial^y_\lambda G_B(x,y) \\
&= \frac{\lambda^2}{2} \int d^d x \int d^d y\ \sqrt{g(x)}\sqrt{g(y)} \int_\Sigma d\bar{x}_1\ K_A(x;\bar{x}_1)\phi_0(\bar{x}_1) \int_\Sigma d\bar{x}_2\ K_A(y;\bar{x}_2)\phi_0(\bar{x}_2) \\
&\quad \times g^{\mu\nu}(x) g^{\rho\lambda}(y)\partial^x_\mu\partial^y_\rho G_A(x,y)\partial^x_\nu\partial^y_\lambda G_B(x,y)
\end{aligned}$$

The *1-loop* two point function is

$$\frac{\delta\Psi_{\textit{1-loop}}}{\delta\phi_0(\bar{x}_1)\delta\phi_0(\bar{x}_2)} = G_{\textit{1-loop}}(\bar{x}_1, \bar{x}_2)$$

Considering only the leading IR divergent part we have

$$
\begin{aligned}
G^{IR}_{\textit{1-loop}}(\bar{x}_1,\bar{x}_2) &\sim \frac{\lambda^2}{2}\int dx_0 \int dy_0\ \frac{1}{x_0^d}\frac{1}{y_0^d}\ (x_0y_0)^{((d-1)-\Delta_\pm)} \\
&\quad \times g^{\mu\nu}(x)g^{\rho\lambda}(y)\partial^x_\mu\partial^y_\rho C_0C_- \ln\left(\frac{(\bar{x}_1-\bar{x}_2)^2}{x_0y_0}\right)\partial^x_\nu\partial^y_\lambda\left(\frac{x_0y_0}{(\bar{x}_1-\bar{x}_2)^2}\right)^{\Delta_\pm} \\
&= \frac{\lambda^2}{2}\int dx_0 \int dy_0\ \frac{1}{x_0^d}\frac{1}{y_0^d}\ x_0^2y_0^2(x_0y_0)^{((d-1)-\Delta_\pm)}\partial^x_\mu\partial^y_\rho C_0C_- \\
&\quad \times \ln\left(\frac{(\bar{x}_1-\bar{x}_2)^2}{x_0y_0}\right)\partial^x_\mu\partial^y_\rho\left(\frac{x_0y_0}{(\bar{x}_1-\bar{x}_2)^2}\right)^{\Delta_\pm} \\
&= \frac{\lambda^2}{2}\int dx_0 \int dy_0\ \frac{1}{x_0^d}\frac{1}{y_0^d}\ x_0^2y_0^2(x_0y_0)^{((d-1)-\Delta_\pm)}\partial^x_i\partial^y_j C_0C_- \\
&\quad \times \ln\left((\bar{x}_1-\bar{x}_2)^2\right)\partial^x_i\partial^y_j\left(\frac{x_0y_0}{(\bar{x}_1-\bar{x}_2)^2}\right)^{\Delta_\pm} \\
&= \frac{\lambda^2}{2}\int dx_0 \int dy_0\ x_0^1y_0^1(-4\Delta(3+2\Delta))C_0C_-\left(\frac{1}{(\bar{x}_1-\bar{x}_2)^2}\right)^{2+\Delta_\pm}
\end{aligned}
$$

where we used the bulk to boundary propagators property explained in Section 8.4.3 and the asymptotic expansion (8.17), (8.19) for the bulk two-point functions. Furthermore we used the fact that

$$
\partial^x_0\partial^y_0\left(\ln\frac{(\bar{x}-\bar{y})^2}{x_0y_0}\right)=0,\quad \partial^x_0\partial^y_j\left(\ln\frac{(\bar{x}-\bar{y})^2}{x_0y_0}\right)=0
$$

with $i,j=1,\ldots,d$.

Doing the integrals and keeping the leading parts in $\epsilon\to 0$ we find

$$
\begin{aligned}
G^{IR}_{\textit{1-loop}}(\bar{x}_1,\bar{x}_2) &\sim (\epsilon)^4\left(\frac{1}{(\bar{x}_1-\bar{x}_2)^2}\right)^{2+\Delta_-} \\
&\quad +\text{Subleading terms in } \epsilon
\end{aligned}
\tag{8.21}
$$

8.9 1-loop Computation: Spinor Field with Derivative Coupling

In this last section we will evaluate the *1-loop* boundary two-point function for a spinor field ψ derivatively coupled to a massless scalar field ϕ. The action in the tangent frame is

$$S_0+S_I = \int_M d^dx \frac{1}{2}\sqrt{g}(\partial\phi)^2 + \int_M d^dx \sqrt{g}\bar{\psi}(\not{D}-m)\psi + \int_{\partial M} d^dx\, \sqrt{h}\bar{\psi}\psi + \int_M d^dx\, \lambda\sqrt{g}\partial_a\phi\bar{\psi}\Gamma^a\psi$$

The surface term for the fermions is explained in [59],[60],[61].

More specifically we are using the metric

$$ds^2 = -\frac{1}{x_0^2}(dx^0dx^0 + d\bar{x}\cdot d\bar{x}) = -\frac{1}{x_0^2}(dx^0dx^0 + dx_idx_i)$$

and the vielbein e^a_μ, $a = 0,\dots,d-1$ such that $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$. The explicit form of the vielbein and is inverse is

$$\begin{aligned} e^a_\mu &= \frac{1}{x_0}\delta^a_\mu \\ e^\mu_a &= x_0\delta^\mu_a \end{aligned}$$

the spin connection has the form

$$\omega_i^{0j} = \omega_i^{j0} = \frac{1}{x_0}\delta_i^j$$

and all other component vanishing. The Dirac operator is given by

$$\not{D} = e^\mu_a(\partial_\mu + \frac{1}{2}\omega_\mu^{bc}\Sigma_{bc}) = x_0\Gamma^0\partial_0 + x_0\bar{\Gamma}\cdot\nabla - \frac{d-1}{2}\Gamma^0$$

where $\Gamma^a = (\Gamma^0, \Gamma^i) = (\Gamma^0, \bar{\Gamma})$ satisfy $\{\Gamma^a, \Gamma^b\} = 2\eta^{ab}$ and $\partial_\mu = (\partial_0, \partial_i) = (\partial_0, \nabla)$.

The explicit form of the interacting term is

$$\mathcal{L}_I = \lambda\sqrt{g}\partial_a\phi\bar{\psi}\Gamma^a\psi = \lambda\sqrt{g}e^\mu_a\partial_\mu\phi\bar{\psi}\Gamma^a\psi = \lambda\sqrt{g}x_0\delta^\mu_a\partial_\mu\phi\bar{\psi}\Gamma^a\psi$$

Again following the same reasoning of Section 8.5 we find for the *1-loop* WFU

$$\begin{aligned}
\Psi_{\textit{1-loop}} &= \int [d\psi]\ [d\bar{\psi}] e^{-\left(\int_M d^dx \frac{1}{2}\sqrt{g}(\partial\phi)^2 + \int_M d^dx \sqrt{g}\bar{\psi}(\not{D} - m)\psi + \int_{\partial M} d^dx\ \sqrt{h}\bar{\psi}\psi + \int_M d^dx\ \lambda\sqrt{g}\partial_a\phi\bar{\psi}\Gamma^a\psi\right)} \\
&= \int [d\psi]\ [d\bar{\psi}] e^{-S_0} \int d^dx \int d^dy\ \sqrt{g(x)}\sqrt{g(y)} \\
&\quad \times \frac{\lambda^2}{2} \left(\partial_a\phi(x)\bar{\psi}(x)\Gamma^a\psi(x)\right) \Big(\partial_b\phi(y)\bar{\psi}(y)\Gamma^b\psi(y)\Big) \\
&= \frac{\lambda^2}{2} \int d^dx \int d^dy\ \sqrt{g(x)}\sqrt{g(y)}\bar{\psi}_1(x)\langle E|\partial_a\phi(x)\Gamma^a\psi(x)\partial_b\phi(y)\bar{\psi}(y)\Gamma^b|E\rangle\psi_1(y) \\
&= \frac{\lambda^2}{2} \int d^dx \int d^dy\ \sqrt{g(x)}\sqrt{g(y)}\bar{\psi}_1(x)\Gamma^a S(x,y)\Gamma^b\partial^x_a\partial^y_b G_0(x,y)\psi_1(y) \\
&= \frac{\lambda^2}{2} \int d^dx \int d^dy\ \sqrt{g(x)}\sqrt{g(y)} \int d^{d-1}\bar{x}\ \bar{\psi}_0(\bar{x})K(y,\bar{x})\Gamma^a S(x,y)\Gamma^b\partial^x_a\partial^y_b G_0(x,y) \\
&\quad \times \int d^{d-1}\bar{x}\ K(x,\bar{x})\psi_0(\bar{x})
\end{aligned}$$

taking the limit $x_0 \to 0,\ y_0 \to 0$ we find the leading IR part of $\Psi_{\textit{1-loop}}$

$$\begin{aligned}
\Psi^{IR}_{\textit{1-loop}} &\sim \frac{\lambda^2}{2}\int d^dx \int d^dy\ \frac{1}{x_0^d}\frac{1}{y_0^d}\bar{\psi}_{0+}(\bar{x}_1)\Gamma^a S(x,y)\Gamma^b \partial^x_a \partial^y_b G_0(x,y)\psi_{0-}(\bar{x}_2) \\
&\sim \frac{\lambda^2}{2}\int d^dx \int d^dy\ \frac{1}{x_0^d}\frac{1}{y_0^d} \\
&\quad \times \bar{\psi}_{0+}(\bar{x}_1)\Gamma^a C_- C_0 \left(\frac{x_0 y_0}{(\bar{x}-\bar{y})^2}\right)^{\Delta_-} \frac{\bar{\Gamma}\cdot(\bar{x}-\bar{y})}{|\bar{x}-\bar{y}|}\Gamma^b \partial^x_a \partial^y_b \ln\left(\frac{(\bar{x}-\bar{y})^2}{x_0 y_0}\right)\psi_{0-}(\bar{x}_2) \\
&= \frac{\lambda^2}{2}\int d^dx \int d^dy\ \frac{1}{x_0^d}\frac{1}{y_0^d}(x_0 y_0)^{\Delta_1+1} \\
&\quad \times \bar{\psi}_{0+}(\bar{x}_1)\Gamma^a C_- C_0 \left(\frac{x_0 y_0}{(\bar{x}-\bar{y})^2}\right)^{\Delta_-} \frac{\bar{\Gamma}\cdot(\bar{x}-\bar{y})}{|\bar{x}-\bar{y}|}\Gamma^b \delta^\mu_a \partial^x_\mu \delta^\nu_b \partial^y_\nu \ln\left(\frac{(\bar{x}-\bar{y})^2}{x_0 y_0}\right)\psi_{0-}(\bar{x}_2) \\
&= \frac{\lambda^2}{2}\int d^dx \int d^dy\ \frac{1}{x_0^d}\frac{1}{y_0^d}(x_0 y_0)^{\Delta_-+1} \\
&\quad \times \bar{\psi}_{0+}(\bar{x}_1)\Gamma^a C_- C_0 \left(\frac{x_0 y_0}{(\bar{x}-\bar{y})^2}\right)^{\Delta_-} \frac{\bar{\Gamma}\cdot(\bar{x}-\bar{y})}{|\bar{x}-\bar{y}|}\Gamma^b \delta^i_a \partial^x_i \delta^j_b \partial^y_j \ln\left((\bar{x}-\bar{y})^2\right)\psi_{0-}(\bar{x}_2) \\
&= \frac{\lambda^2}{2}C_- C_0 \int d^dx \int d^dy\ \frac{1}{x_0^d}\frac{1}{y_0^d}(x_0 y_0)^{\Delta_-+1} \\
&\quad \times \bar{\psi}_{0+}(\bar{x}_1)\Gamma^i \left(\frac{1}{(\bar{x}-\bar{y})^2}\right)^{\Delta_-} \frac{\Gamma^k(\bar{x}-\bar{y})_k}{|\bar{x}-\bar{y}|}\Gamma^j \partial^x_i \partial^y_j \ln\left((\bar{x}-\bar{y})^2\right)\psi_{0-}(\bar{x}_2)
\end{aligned}$$

where we used the bulk to boundary propagators property

$$\lim_{x_0\to 0}(x_0)^{-\frac{d}{2}+m}\psi(x) = -c\psi_{0-}(\bar{x})$$

$$\lim_{x_0\to 0}(x_0)^{-\frac{d}{2}+m}\bar{\psi}(x) = c\bar{\psi}_{0+}(\bar{x})$$

explained in Appendix 8.14.3 and the asymptotic expansion (8.33), (8.19) for the bulk two-point functions. As in the previous section we noticed that

$$\partial^x_0 \partial^y_0 \ln\left(\frac{(\bar{x}-\bar{y})^2}{x_0 y_0}\right) = 0, \quad \partial^x_0 \partial^y_j \ln\left(\frac{(\bar{x}-\bar{y})^2}{x_0 y_0}\right) = 0, \quad i,j = 1,\ldots,d$$

The boundary two-point function at 1-loop is

$$\begin{aligned}
G_{\textit{1-loop}}(\bar{x}_1,\bar{x}_2) &= \frac{\delta\Psi^{IR}_{\textit{1-loop}}}{\delta\bar{\psi}_{0+}(\bar{x}_1)\delta\psi_{0-}(\bar{x}_2)} \\
&= \frac{\lambda^2}{2}C_0C_-\int_\alpha^\epsilon d^0x\int_\beta^\epsilon d^0y\ (x_0y_0)^{(\Delta_-+1-d)} \\
&\quad\times\Gamma^i\left(\frac{1}{(\bar{x}_1-\bar{x}_2)^2}\right)^{\Delta_-}\frac{\Gamma^k(\bar{x}_1-\bar{x}_2)_k}{|\bar{x}_1-\bar{x}_2|}\Gamma^j\partial_i^x\partial_j^y\ln\left((\bar{x}_1-\bar{x}_2)^2\right)
\end{aligned}$$

doing the integrals and keeping the leading terms in $\epsilon\to 0$ we find

$$\begin{aligned}
G_{\textit{1-loop}}(\bar{x}_1,\bar{x}_2) &\sim \lambda^2\epsilon^{2(\Delta_--d+2)} \\
&\quad\times\Gamma^i\Gamma^k\Gamma^j\left(\frac{1}{(\bar{x}_1-\bar{x}_2)^2}\right)^{\Delta_-}\frac{(\bar{x}_1-\bar{x}_2)_k}{|\bar{x}_1-\bar{x}_2|}\partial_i^x\partial_j^y\ln\left((\bar{x}_1-\bar{x}_2)^2\right) \\
&\quad+\text{Subleading terms in }\epsilon
\end{aligned} \tag{8.22}$$

8.10 Analysis Divergences

8.10.1 Three Massive Scalar Fields with Cubic Interaction in dS^d

8.10.1.1 Leading Terms

We didn't perform explicitly the computation in this case but it is easy to see that the leading IR divergent term (which is not in fact divergent in this case) in the boundary two-point function has the following form up to a constant

$$\begin{aligned}
G^{IR}_{\textit{1-loop}}(\bar{x}_1,\bar{x}_2) &\sim \frac{\lambda^2}{2}\epsilon^{2\Delta_2}\frac{1}{(\bar{x}_1-\bar{x}_2)^{2\Delta_2}}\frac{1}{(\bar{x}_1-\bar{x}_2)^{2\Delta_1}} \\
&\quad+\text{Subleading terms in }\epsilon
\end{aligned}$$

where $\Delta_2,\ \Delta_1$ correspond respectively to the fields ϕ_1 and ϕ_2. In this expression we have kept only pure terms. Other terms are no more divergent than these.

The leading IR term in $G^{IR}_{\textit{1-loop}}(\bar{x}_1, \bar{x}_2)$ is proportional to

$$\epsilon^{2\Delta_2}$$

We have

$$\Delta^i_\pm = \frac{1}{2}\left(d - 1 \pm \sqrt{(d-1)^2 - 4m_i^2 R_{dS}^2}\right) = \frac{1}{2}(d-1)\left(1 \pm \sqrt{(1-\alpha_i)}\right)$$

with

$$\alpha_i = \left(\frac{2m_i R_{dS}}{d-1}\right)^2$$

So we immediately see that $G^{IR}_{\textit{1-loop}}(\bar{x}_1, \bar{x}_2)$ is IR convergent for every α_i i.e. both for the complementary and principal series, see Section 8.3.5.

This computation shows that in the case of massive fields there is no IR divergence in the boundary two point function. This is in accord with naive expectations.

8.10.2 Two Massive and One Massless field in dS^d

The leading IR term in $G_{\textit{1-loop}}(\bar{x}_1, \bar{x}_2)$ is proportional to

$$(\log \epsilon)^3$$

So in this case $G_{\textit{1-loop}}(\bar{x}_1, \bar{x}_2)$ is IR divergent.

The analysis of divergences in the remaining case (8.21), (8.22) is very similar and we will not repeat it. We want only to remark that these cases are not IR divergent, due to the presence of derivative couplings, as can be seen inspecting the power dependence of the ϵ cutoff.

8.11 The Meaning of the Divergences

To understand the meaning of the divergences we have found, we compare our expressions to those obtained by perturbing the free massive theory by a term $\frac{1}{2}\delta m^2\phi^2$. That computation gives

$$\delta m^2 \int dx_0 \, \frac{1}{x_0^d} \int d^{d-1}\bar{x} \; K(x_0, \bar{x}; \bar{x}_b) K(x_0, \bar{x}; \bar{y}_b)$$

where K is the massive bulk to boundary propagator. The IR divergent contribution to this integral comes from $x_0 \sim 0$, where we can substitute one of the propagators by $K(x_0, \bar{x}; \bar{x}_b) \sim (x_0)^{d-1-\Delta}\delta(\bar{x} - \bar{x}_b)$. The result is

$$\delta m^2 \int dx_0 \, \frac{1}{x_0} \, |\bar{x}_b - \bar{y}_b|^{-2\Delta}$$

It is important to note that this expression for the perturbed two point function could be derived explicitly from the expression of the two point function as an integral over the boundary. One simply uses Green's theorem and a perturbative analysis of the Klein-Gordon equation. The same statement would *not* be true in AdS/CFT. In that context, the Euclidean boundary conditions depend on δm^2, and so the straightforward perturbative analysis of the path integral misses a term coming from the perturbation of the boundary conditions. It turns out that the missing term is sub-leading if the boundary operator is irrelevant, but is the dominant term if it is marginal or relevant.

By contrast, in the one loop computation with massless fields and non-derivative coupling, we obtained the IR divergent part

$$\int dx_0 \int dy_0 \, \frac{1}{x_0^d} \frac{1}{y_0^d} (x_0 y_0)^{d-1-\Delta} \left(\frac{x_0 y_0}{|\bar{x}_b - \bar{y}_b|^2} \right)^{\Delta} (\ln x_0 + \ln y_0)$$

The first term after the integration measure comes from the two bulk to boundary propagators, which we have approximated by their small x_0 limits. This enabled us to do the two spatial integrals using the δ functions. The first term in square brackets is the asymptotic form of the massive bulk propagator, while the second is that of the massless propagator. We note that if we had instead exchanged a massive field from the principle series in the loop, or if the massless scalar had derivative couplings, this last factor would have been a positive power of x_0 and all the integrals in the loop diagram would have been convergent. This means that for a purely massive theory the IR region of coordinate space does not contribute to the mass renormalization at all[3]. The value of the mass renormalization following from exchange of a minimal massless scalar, with soft couplings is thus

$$\delta m^2 \propto \int dx_0 \, \frac{1}{x_0} \ln x_0 \sim \ln^2 T \sim \ln^2 \Lambda$$

The last equality reflects our prejudice that the IR cutoff should be determined in terms of the c.c., by the requirement of finite entropy.

We note that minimally coupled scalars would generally arise as Nambu-Goldstone bosons and would be derivatively coupled. Our calculation shows that one would not expect IR mass divergences in models with NG bosons. However, we believe

[3]We would get contributions from the region where the two bulk points in the diagram were close together, corresponding to the usual UV mass renormalization.

that there are indications that gravity has IR divergence problems comparable to those of minimally coupled massless bosons with soft couplings. Thus, the divergence we have uncovered reflects our best guess at the behavior of perturbative quantum gravity in dS space.

8.12 Generalization to a Model with Gravity

The simplest generalization of the calculations we have done is to a model of gravity interacting with a massive scalar in a dS background. The Lagrangian is

$$\mathcal{L} = \sqrt{|g|}\left[M_P^2 R - \left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right)\right]$$

As always in perturbative quantum gravity calculations must be done in a fixed gauge. We first studied this problem in the gauge for fluctuations around the dS metric defined by

$$h_{\mu\nu} = \frac{1}{d} g_{\mu\nu} h + H_{\mu\nu}$$

$$g^{\mu\nu} H_{\mu\nu} = 0 = D^\mu H_{\mu\nu}$$

$g_{\mu\nu}$ is the background dS metric, and D^μ its Christoffel connection. In this gauge, the Lagrangian for h is that of a scalar field with tachyonic mass, while the components of $H_{\mu\nu}$ satisfy a massive Klein-Gordon equation. One might think that the IR divergences at one loop arise only from the exchange of h[4]. If this were the case, the calculation would be a simple generalization of our non-derivative trilinear scalar interaction, with the massless field replaced by a tachyon.

[4] In this gauge, ghosts couple only to gravitons and so there are no ghost contributions to the one loop boundary two point function of the massive scalar.

The result of this computation is disastrous and confusing. The IR divergence is power law rather than logarithmic (relative to the tree level calculation). Furthermore the power of $|\bar{x}_b - \bar{y}_b|$ differs from the tree level power, so we cannot interpret the effect as a mass renormalization. If this result were valid one would be led to the conclusion that the dS/CFT correlation functions simply did not exist, even in perturbation theory, and the divergence could not be explained as a divergent mass renormalization.

We gained insight by viewing the transverse gauge as the $\alpha \to 0$ limit of the one parameter family of gauge fixing Lagrangians

$$\delta\mathcal{L} = \frac{1}{2\alpha}\left(D^\mu H_{\mu\nu} + 2b\alpha\partial_\nu h\right)^2$$

The coefficient b is chosen to cancel the mixing between $H_{\mu\nu}$ and h in the classical Lichnerowicz Lagrangian for fluctuations around dS space. In this class of gauges, it is easy to see that the tachyonic mass, as well as the overall normalization of the h propagator, is α dependent. The same is therefore true of the power of T and of $|\bar{x}_b - \bar{y}_b|$ in the the IR divergent part of the h exchange graph.

Thus, either this contribution is canceled by $H_{\mu\nu}$ exchange, or the answer is not gauge invariant. Formal arguments using graphical Ward identities seem to suggest that the boundary two point function is indeed α independent. Thus, we expect the power law IR divergences to cancel at this order. This suggests the possibility that logarithmic divergences, which come from the behavior of the transverse, traceless part of the graviton propagator, may not cancel. Gravitational theories would then exhibit the same sort of IR divergences as our toy model. Of course, we really need to do a careful computation in order to verify gauge invariance of the results. We plan to return to this in a future publication. See [61] and references therein.

8.13 Appendix: Comparison with AdS

In this appendix we record comparisons of our computation of three massive scalars, with an analogous computation of AdS space. The purpose of this is to verify that there is no analogue of the divergences we have found, even when one of the scalars is massless. The essential reason for this difference is that the bulk AdS propagator is constructed only from normalizable modes. By contrast, in dS space the Euclidean propagator contains both solutions of the homogeneous wave equation at large proper distance.

8.13.1 Three Scalar Fields AdS

For comparison we will describe the case of three massive scalar fields with cubic interaction in AdS.

As before it is easy to see that in AdS the part of $G^{IR}_{\textit{1-loop}}(\bar{x}_1, \bar{x}_2)$ that is dependent on ϵ is proportional to

$$\epsilon^{2\Delta_+}$$

In AdS we consider only one type of modes

$$\Delta = \Delta_+ = \frac{1}{2}\left(d - 1 + \sqrt{(d-1)^2 + 4m_i^2 R_{AdS}^2}\right) = \frac{1}{2}(d-1)\left(1 + \sqrt{(1+\alpha_i)}\right)$$

with

$$\alpha_i = \left(\frac{2m_i R_{AdS}}{d-1}\right)^2$$

so

$$\Delta_+ > 0, \ \forall\ \alpha_i$$

and $G_{\textit{1-loop}}(\bar{x}_1, \bar{x}_2)$ is IR convergent for every α_i even when m_i is zero.

8.13.1.1 Anti de Sitter: Scalar Propagator

The two-point function for a scalar field of mass m in AdS^d has been derived for example in [55]. They find

$$\begin{aligned} G(z) &= rz^{-a}F(a, a-c+1; a-b+1; z^{-1}) \\ r &= \frac{\Gamma(a)\Gamma(a-c+1)}{\Gamma(a-b+1)\pi^{\frac{d}{2}}2^d}R^{2-d} \end{aligned} \tag{8.23}$$

with a, b, c given respectively by (8.13), (8.14), (8.15) and where for AdS^d we have $R = i\tilde{R}$, $\tilde{R} \in \mathbb{R}$.

The asymptotic expansion $z \to \infty$ of (8.23) is

$$F(a, a-c+1; a-b+1; z^{-1}) \to 1$$

$$\lim_{z\to\infty} G(z) \sim rz^{-\Delta}$$

with

$$\Delta = \Delta_+ = a = \frac{1}{2}\left(d - 1 + \sqrt{(d-1)^2 + 4m^2\tilde{R}^2}\right)$$

8.14 Appendix: Spinor Green Functions

Here we record the spinor Green Functions needed for the computations and their asymptotic behavior. For a more exhaustive discussion see for example [55], [62], [63].

8.14.1 Spinor Parallel Propagator

In this section we will derive a differential equation for the spinor parallel propagator $\Lambda(x', x)^{\alpha'}_{\beta}$ (8.8) whose action on a spinor is

$$\psi'(x')^{\alpha'} = \Lambda(x', x)^{\alpha'}_{\beta}\psi(x)^{\beta}$$

this equation for $\Lambda(x', x)^{\alpha'}_{\beta}$ will be a fundamental ingredient in the derivation of the spinor Green function $S(x, x')$.

$\Lambda(x', x)$ satisfy the following properties

$$n^{\mu}\nabla_{\mu}\Lambda(x, x') = 0 \tag{8.24a}$$

$$\Lambda(x', x) = [\Lambda(x, x')]^{-1} \tag{8.24b}$$

$$\Gamma^{\nu'}(x') = \Lambda(x', x)\Gamma^{\mu}(x)\Lambda(x, x')g^{\nu'}_{\mu}(x', x) \tag{8.24c}$$

(8.24a) follows from the definition of parallel transport of a spinor along a curve, (8.24b) derive from the fact that the $\Lambda(x', x)$ form a group and (8.24c) indicate how to parallel transport the gamma matrices.

Manipulating the previous equations we obtain

$$\nabla_\mu \Lambda(x,x') = \frac{1}{2}(A+C)\left(\Gamma_\mu \Gamma^\nu n_\nu - n_\mu\right)\Lambda(x,x') \tag{8.25}$$

and

$$\nabla_{\mu'} \Lambda(x,x') = -\frac{1}{2}(A+C)\Lambda(x,x')\left(\Gamma_{\mu'} \Gamma^{\nu'} n_{\nu'} - n_{\mu'}\right)$$

8.14.2 Bulk Two-Point Function

The spinor Green $S(x,x')$ function is defined by the equation

$$\left[(\not{D} - m)S(x,x')\right]^{\alpha}_{\ \beta'} = \frac{\delta(x-x')}{\sqrt{g(x)}}\delta^{\alpha}_{\beta'} \tag{8.26}$$

The most general form for $S(x,x')$ is

$$S(x,x') = \left[\alpha(\mu) + \beta(\mu) n_\nu \Gamma^\nu\right]\Lambda(x,x') \tag{8.27}$$

with $\alpha(\mu)$, $\beta(\mu)$ functions only of the geodesic distance.

Substituting (8.27) into (8.26) and using (8.25) we obtain two differential equations for $\alpha(\mu)$ and $\beta(\mu)$

$$\beta' + \frac{1}{2}(d-1)(A-C)\beta - m\alpha = \frac{\delta(x-x')}{\sqrt{g(x)}} \tag{8.28}$$

$$\alpha' + \frac{1}{2}(d-1)(A+C)\alpha - m\beta = 0, \tag{8.29}$$

Combining (8.28) and (8.29) we find the following differential equation for $\alpha(\mu)$

$$\alpha'' + (d-1)A\alpha' - \frac{1}{2}(d-1)C(A+C)\alpha - \left[\frac{(d-1)^2}{4R^2} + m^2\right]\alpha = m\frac{\delta(x-x')}{\sqrt{g(x)}} \tag{8.30}$$

8.14.2.1 De Sitter Space: Massive Spinor

To derive $S(x,x')$ in dS^d space we perform the change of variables

$$\begin{aligned} z &= \cos^2\frac{\mu}{2R} \\ \alpha(z) &= \sqrt{z}\gamma(z) \end{aligned}$$

the Eq. (8.30) become

$$H(a,b;c;z)\gamma(z) = 0 \tag{8.31a}$$

$$H(a,b;c;z) = z(1-z)\frac{d^2}{dz^2} + [c-(a+b+1)z]\frac{d}{dz} - ab$$

with

$$a = \frac{d}{2} - i|m|R, \quad b = \frac{d}{2} + i|m|R, \quad c = \frac{d}{2} + 1$$

As explained in Section 8.4.2.1, the solution corresponding to the *Euclidean vacuum* is the one that is singular only at $z=1$ i.e.

$$\gamma(z) = \lambda\,\mathrm{F}(a,b;c;z) = \lambda\,\mathrm{F}(d/2 - i|m|R, d/2 + i|m|R; d/2+1; z)$$

$$\alpha(z) = \lambda\sqrt{z}\,\mathrm{F}(d/2 - i|m|R, d/2 + i|m|R; d/2+1; z)$$

The constant λ is derived by the requirement that (8.27) has the same behavior of the flat spinor Green function for $R \to \infty$. We have

$$\lambda = -m\frac{\Gamma(d/2 - i|m|R)\Gamma(d/2 + i|m|R)}{\Gamma(d/2+1)\pi^{d/2}2^d}R^{2-d}$$

Finally $\beta(z)$ is determined by the Eq. (8.29)

$$\beta(z) = -\frac{1}{m}\left[\frac{1}{R}\sqrt{z(1-z)}\frac{d}{dz} + \frac{d-1}{2R}\sqrt{\frac{1-z}{z}}\right]\alpha(z) \tag{8.32}$$
$$= -\frac{\lambda}{mR}\sqrt{1-z}\Big[z\,\mathrm{F}(d/2+1-i|m|R, d/2+1+i|m|R; d/2+2; z)$$
$$+\frac{d}{2}\,\mathrm{F}(d/2-i|m|R, d/2+i|m|R; d/2+1; z)\Big]$$

The asymptotic $z \to -\infty$ expansion for the spinor two-point function is found to be

$$\lim_{\substack{x_0\to 0\\ y_0\to 0}} S(x,y) = \left(\left(C_+\frac{-x_0y_0}{(\overline{x}-\overline{y})^2}\right)^{\Delta_+} + C_-\left(\frac{-x_0y_0}{(\overline{x}-\overline{y})^2}\right)^{\Delta_-}\right)\frac{\bar{\Gamma}\cdot(\overline{x}-\overline{y})}{|\overline{x}-\overline{y}|} \tag{8.33}$$

with

$$\Delta_+ = \frac{d-1}{2} + im$$
$$\Delta_- = \frac{d-1}{2} - im$$

8.14.3 Bulk to Boundary Propagators: dS/AdS

The complete expression for the spinor Bulk to Boundary propagators:

$$\psi_1(x) = \int d^{d-1}\bar{x}\ K(x,\bar{x})\psi_0(\bar{x}) \tag{8.34}$$
$$\bar{\psi}_1(x) = \int d^{d-1}\bar{x}\ \bar{\psi}_0(\bar{x})K(x,\bar{x})\psi_0(\bar{x}) \tag{8.35}$$

has been given for example in [61].

For our purposes we will need only the asymptotic expansion $x_0 \to 0,\ y_0 \to 0$ for the propagators (8.34), (8.35), we have

$$\lim_{x_0\to 0}(x_0)^{-\frac{d}{2}+m}\left(-\frac{1}{c}\right)\psi(x) = \psi_{0-}(\bar{x}) - \frac{1}{c}\int d^{d-1}\bar{y}\ |\bar{x}-\bar{y}|^{-d-1+2m}(\bar{x}-\bar{y})\cdot\bar{\Gamma}\psi_{0+}(\bar{y}) \tag{8.36}$$

$$\lim_{x_0\to 0}(x_0)^{-\frac{d}{2}+m}\left(\frac{1}{c}\right)\bar{\psi}(x)=\bar{\psi}_{0+}(\bar{x})+\frac{1}{c}\int d^{d-1}\bar{y}\,\bar{\psi}_{0-}(\bar{y})(\bar{x}-\bar{y})\cdot\bar{\Gamma}\,|\bar{x}-\bar{y}|^{-d-1+2m} \quad (8.37)$$

where the constant is $c=\pi^{d/2}\Gamma(m+\frac{1}{2})/\Gamma(m+\frac{d+1}{2})$. And we have used the following decomposition for the fields

$$\begin{aligned}\psi_0(\bar{x}) &= \psi_{0+}(\bar{x})+\psi_{0-}(\bar{x})\\ \bar{\psi}_0(\bar{x}) &= \bar{\psi}_{0+}(\bar{x})+\bar{\psi}_{0-}(\bar{x})\end{aligned}$$

with

$$\begin{aligned}\Gamma^0\psi_{\pm}(\bar{x}) &= \pm\psi_{\pm}(\bar{x})\\ \bar{\psi}_{\pm}(\bar{x})\Gamma^0 &= \pm\bar{\psi}_{\pm}(\bar{x})\end{aligned}$$

For the right-hand side of (8.36), (8.37) to be integrable, with respect to the measure $d^{d-1}\bar{y}$ on the boundary Σ we have to impose the conditions

$$\begin{aligned}\psi_{+}(\bar{y}) &= 0\\ \bar{\psi}_{-}(\bar{y}) &= 0\end{aligned}$$

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