FRW-metric and Friedmann Equations in a generalized cosmological model

P.C.Stavrinos

Department of Mathematics, University of Athens, 15784 Greece

E-mail: `pstavrin@math.uoa.gr`

A.P.Kouretsis

5 Festou St., 16674 Glifada, Athens, Greece

E-mail: `a.kouretsis@yahoo.com`

Abstract. A generalized model of space-time is given, taking into consideration the anisotropic structure of fields which are depended on the position and the direction (velocity).In this framework a generalized FRW-metric and Friedmann equations are studied. A long range vector field of cosmological origin is considered in relation with the physical geometry of space-time in which Cartan connection has a fundamental role.

1. Introduction

During the last few years considerable studies with respect to observable anisotropies of the universe have been done[1, 2].These are connected to the very early state of the universe and related to the estimations of WMAP of CMB, the anisotropic pressure or the incorporation of a primordial vector field(e.g. a magnetic field) to the metrical spatial stucture of the universe[5, 4, 16].In this case the form of scale factor will be influenced by the introductory field. A geometry which may connect the Riemann metric structure of the space-time to the physical vector fields is the class of Finsler-Randers type spaces. In these spaces an electromagnetic field, a magnetic field or a gauge vector field can be emerged out by a physical source of the universe and be incorporated to the geometry causing an anisotropic structure. The possible formation of singularities can be studied by a generalised Raychaudhuri equation initially presented in [8, 7]. The Cartan's torsion tensor[19, 21] characterises all the geometrical machinery of Finsler Geometry and appears to all the expressions of geometrical objects such as connection and curvuture. A Finsler geometry can be considered as a physical geometry on which matter dynamics takes place while the Riemann geometry is the gravitational geometry[20, 11, 13].

In the following we present some basic elements of Finsler geometry[13, 12]. In 1854 B.Riemann, before arriving at Riemannian metric was concerned with the concept of a more generalised metric

$$ds^2 = \mathcal{F}(x^1, x^2, ..., x^n, dx^1, ..., dx^n) \tag{1.1}$$

where n is the dimension of the space.A Finsler structure is endowed by a n-dimensional C^∞ manifold M^n, a function F(x,y) C^∞ on the tangent bundle $\tilde{T}M = TM/(0)$ $F(x,y): \tilde{T}M \to R$

that satisfies the conditions

$$(F1) \qquad \mathcal{F}(x,y) > 0 \ \forall \ y \neq 0 \tag{1.2}$$

$$(F2) \quad \mathcal{F}(x, py) = p\mathcal{F}(x,y) \ \text{for any } p > 0 \tag{1.3}$$

where y denotes the directions or velocities on the considered manifold with the previous coordinates. The metric tensor(Hessian)

$$f_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 \mathcal{F}^2}{\partial y^i \partial y^j}(x,y) \tag{1.4}$$

has rank $f_{ij} = n$ which is homogeneous of zero degree with respect to y due to the Euler's theorem. The lenght s of a curve $C : x^i(t), a \leq t \leq b$ on the manifold is

$$s = \int_a^b \mathcal{F}(x(t), y(t))dt. \tag{1.5}$$

The integral of the length is independent of the parameter if and only if the condition (F2) is valid. The condition of homogeniety enables us to define the line element

$$ds = \mathcal{F}(x, dx) \tag{1.6}$$

and the variation of the arclength $\delta \int ds = 0$ implies the Euler-Lagrange equations $\frac{d}{ds}\left(\frac{\partial \mathcal{F}}{\partial y}(x,y)\right) - \frac{\partial \mathcal{F}}{\partial x}(x,y) = 0$ which represent the geodesics of the Finsler space. The equation of geodesics then becomes analogous to the ones of the Riemann space

$$\frac{d^2 x^i}{ds^2} + \gamma^i_{jk}y^j y^k = 0 \tag{1.7}$$

where the Christoffel symbols are defined by the usual formula

$$\gamma^i_{jk}(x,y) = \frac{1}{2}g^{ir}(x,y)\left(g_{rj,k}(x,y) + g_{rk,j}(x,y) - g_{jk,r}(x,y)\right). \tag{1.8}$$

The notion of torsion tensor is crucial within the Finsler Geometry's framework. *A Finsler space is a Riemann space if and only if $C_{ijk} = 0$* where C_{ijk} is a torison tensor defined by E.Cartan as

$$C_{ijk} = \frac{1}{2}\frac{\partial f_{ij}}{\partial y^k} \tag{1.9}$$

therefore a Finsler space can be treated as a natural generalisation of a Riemann space.

2. Anisotropy and Randers type Finslerian spaces

An alternative way of considering physical phenomena is to incorporate the dynamics to the active geometrical background following the Einstein's interpretation of gravity. Our investigation is based on the introduction of a Lagrangian metric[14, 8] facing universe's present anisotropy[1] as an embodied characteristic of the geometry of space-time. A similar investigation has been applied in case of electromagnetism [6, 17] together with some recent progress in gravity, cosmology and fluid dynamics[7, 14, 15]. We consider the geodesics of the

4-dimensional space-time[1] to be produced by a Lagrangian identified to the Randers-type metric function

$$F(x,y) = \sigma(x,y) + \phi(x)\hat{k}_\alpha y^\alpha \tag{2.1}$$

$$\sigma(x,y) = \sqrt{a_{\kappa\lambda}(x)y^\kappa y^\lambda} \tag{2.2}$$

where $a_{\kappa\lambda}(x)$ is the Robertson-Walker metric defined as

$$a_{\kappa\lambda}(x) = diag(1, -\frac{a^2}{1-kr^2}, -a^2 r^2, -a^2 r^2 \sin^2\theta). \tag{2.3}$$

where $k = 0, \pm 1$ for a *flat, closed* and *hyperbolic* geometry respectively. The spatial coordinates are comoving and the time coordinate represents the proper time measured by the comoving observer. The vector $y^\mu = \frac{dx^\mu}{ds}$ represents the tangent 4-velocity of a comoving observer along a preferred family of worldlines(fluid flow lines) in a locally anisotropic universe;the arclength parameter s stands for the proper time.We proceed considering the natural Lorentzian units i.e. $c = 1$. If we fix the direction $y = \dot{x}$ then $\sigma(x,\dot{x}) = 1$.The vector field

$$u_\alpha(x) = \hat{k}_\alpha \phi(x) \tag{2.4}$$

stands for a weak primordial electromagnetic field $|u_\alpha| \ll 1$ incorporated to the geometry of space-time as an intrinsic characteristic. The whole information about the anisotropy is coded to the scalar $\phi(x)$[14].We consider a linearised variation of anisotropy therefore the approximation $\phi(x) \approx \phi(0) + \partial_\mu \phi(0)x^\mu$ is valid for small x.

3. The osculating metric and the choice of the connection $A^\kappa_{\lambda\mu}(x)$

The metric of the Finsler space can be directly calculated from the metric function F. Since $f_{\mu\nu}(x,y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^\mu \partial y^\nu}(x,y)$ we derive[14]

$$f_{\mu\nu} = g_{\mu\nu} + \frac{1}{\sigma}(u_\mu y_\nu + u_\nu y_\mu) - \frac{\beta}{\sigma^3}y_\mu y_\nu + u_\mu u_\nu \tag{3.1}$$

where

$$g_{\mu\nu}(x,y) = \frac{F}{\sigma}(x,y)a_{\mu\nu}(x) \tag{3.2}$$

and

$$\beta(x,y) = \phi(x)\hat{k}_\alpha y^\alpha = u_\alpha(x)y^\alpha, \tag{3.3}$$

$\alpha_{\mu\nu}(x)$ represents the RW-metric.

Under the weak field assumption[2] we can approximate the Finsler metric $f_{\mu\nu}(x,y) \approx g_{\mu\nu}(x,y)$ by considering of a small perturbation of the last three terms in (3.1). The metric is considered to have signature $(+, -, -, -)$ for any (x,y). The square of the length of an arbitrary contravariant vector X^μ is to be defined $|X|^2 = f_{\mu\nu}(x,y)X^\mu X^\nu \approx g_{\mu\nu}(x,y)X^\mu X^\nu$. We insert the metric $g_{\mu\nu}(x,y)$ into (3.2) to calculate the connection components of the metric

$$r^\kappa_{\lambda\mu}(x,y) = \left[\frac{1}{2}g^{\kappa\rho}\left(g_{\rho\lambda,\kappa} + g_{\rho\kappa,\lambda} - g_{\lambda\mu,\rho}\right)\right](x,y) \tag{3.4}$$

---

[1] The greek indices belong to $\{0,1,2,3\}$ and the latin ones to $\{1,2,3\}$

[2] In General Relativity a weak vector field in space (e.g. primordial magnetic field) can be treated as first order perturbation upon the Riemann metric tensor

In many cases we consider a convenient Finsler metric to approximate the gravitational theories[19, 21]. This metric is connected with a Riemannian one, $r_{\mu\nu}(x)$ refered as *osculating Riemannian metric*[19]

$$r_{\mu\nu}(x) = f_{\mu\nu}(x, y(x)) \approx g_{\mu\nu}(x, y(x)) \tag{3.5}$$

the equation of geodesics is given by the usual formula

$$\frac{d^2 x^\mu}{ds^2} + r^\mu_{\rho\sigma}(x) y^\rho y^\sigma = 0 \tag{3.6}$$

where $r^\kappa_{\lambda\mu}(x)$ are the Christoffel components coming from the metric $r_{\mu\nu}(x)$. The equations (3.6) are identified to ones of Finsler space (1.7) due to the homogeneity properties of the Cartan's torsion($C_{\alpha\beta\gamma} y^\alpha = C_{\alpha\beta\gamma} y^\beta = C_{\alpha\beta\gamma} y^\gamma = 0$).

The Cartan's torsion tensor can be easily deduced from (2.1) and (1.9)and the full expression is[14]

$$C_{\mu\nu\lambda} = \frac{1}{2} \left\{ \frac{1}{\sigma} \mathcal{S}_{(\mu\nu\lambda)}(a_{\mu\nu} u_\lambda) - \frac{1}{\sigma^3} \mathcal{S}_{(\mu\nu\lambda)}(y_\mu y_\nu u_\lambda) - \frac{\beta}{\sigma^3} S_{(\mu\nu\lambda)}(a_{\mu\nu} y_\lambda) \right\} \tag{3.7}$$

where $\mathcal{S}_{\mu\nu\lambda}$ denotes the sum over the cyclic permutation of the indices. Every single term of (3.7) is proportional to the components of the field u_α thus $C_{\mu\nu\lambda} \approx 0$ under the condition $|u_\alpha| \ll 1$ and then we can drop all the torsion dependent terms at the calculation of the $r^\kappa_{\lambda\mu}(x)$. A direct calculation in virtue of (3.4) leads to the following expressions for the nonzero $A^\kappa_{\lambda\mu}$ osculating affine connection coefficients

$$
\begin{aligned}
A^0_{11} &= \frac{a\dot{a}}{1-kr^2} + \frac{a^2}{1-kr^2} z_t, \\
A^0_{22} &= a\dot{a}r^2 + a^2 r^2 z_t, \\
A^0_{33} &= a\dot{a}r^2 \sin^2\theta + a^2 r^2 \sin^2\theta z_t,
\end{aligned}
$$

$$
\begin{aligned}
A^1_{01} &= A^2_{02} = A^3_{03} = \frac{\dot{a}}{a} + z_t, \\
A^1_{22} &= -r(1-kr^2)(1-rz_t), \\
A^1_{11} &= \frac{kr}{1-kr^2} + z_t, \\
A^1_{33} &= -r(1-kr^2)\sin^2\theta(1+rz_r),
\end{aligned}
$$

$$
\begin{aligned}
A^2_{12} &= A^3_{13} = \frac{1}{r} + z_r, \\
A^2_{33} &= -\sin\theta\cos\theta + \sin^2\theta z_\theta, \\
A^3_{23} &= \cot\theta + z_\theta
\end{aligned} \tag{3.9}
$$

where we have taken into account that $F/\sigma = 1 + u_\alpha y^\alpha/\sigma \approx 1$. The tensorial quantities $z_t\ z_\theta, z_r$ are defined by

$$z_\mu(x) = \left(\frac{F}{2\sigma}\right)_{,\mu}(x, y(x)) \tag{3.10}$$

and related to the *variation of anisotropy*. Since the vector field y^α has been picked up to satisfy the condition $y^\alpha_{;\mu} = 0$ this yields the comoving nature of $y^\alpha = (1, 0, 0, 0)$

$$z_\mu(x) = u_{0,\mu}(x)/2. \tag{3.11}$$

The right choice of the connection components gives directly the curvature tensor[3].We insert (3.9) into

$$L^\kappa_{\lambda\mu\nu} = A^\kappa_{\lambda\nu,\mu} - A^\kappa_{\lambda\mu,\nu} + A^\rho_{\lambda\nu} A^\kappa_{\rho\mu} - A^\rho_{\lambda\mu} A^\kappa_{\rho\nu} \tag{3.12}$$

___
[3] The curvature which is associated to the commutation relations of the δ-derivatives

$$L_{\mu\nu} = L^{\alpha}_{\mu\alpha\nu} \tag{3.13}$$

$$L = g^{\mu\nu} L_{\mu\nu}. \tag{3.14}$$

The calculation of the Ricci tensor can be drastically simplified in virtue of $z_{\mu,\nu} = 0$ and $z_{\mu}^{2} \approx 0$. This approximation is valid since $\phi(x)$ is linearly expressed while z_{μ} can be cosidered very small at the first stages of a highly accelerated expanding universe[24]. We arrive then at the following nonzero components

$$
\begin{array}{ll}
L_{00} = & -3\left(\frac{\ddot{a}}{a} + \frac{\dot{a}}{a}z_t\right) \\
L_{11} = & (a\ddot{a} + 2\dot{a}^2 + 2k + 4a\dot{a}z_t)/(1 - kr^2) \\
L_{22} = & \left(a\ddot{a} + 2\dot{a}^2 + 2k + 4a\dot{a}z_t\right)r^2 - kr^3 z_r - \cot\theta z_{\theta} \\
L_{33} = & \left(a\ddot{a} + 2\dot{a}^2 + 2k + 4a\dot{a}z_t\right)r^2\sin^2\theta + 2\sin\theta\cos\theta z_{\theta}
\end{array} \tag{3.15}
$$

The geodesic deviation equation in the case of a perfect fluid along the neighbouring world lines can be generalised within the Finslerian framework (ξ^{μ} is the deviation vector)[9, 12]

$$\frac{\delta^2 \xi^{\mu}}{\delta s^2} + L^{\mu}_{\nu\rho\sigma} y^{\nu} y^{\rho} \xi^{\sigma} \tag{3.16}$$

where the operator $\frac{\delta}{\delta s}$ denotes the Finslerian $\delta-$connection along the geodesics.

The formula that characterises isotropic points in a Finslerian space-time of constant curvature K reduced to the familiar form[19, 22, 14]

$$L_{\kappa\lambda\mu\nu} = K(g_{\kappa\mu}g_{\lambda\nu} - g_{\kappa\nu}g_{\lambda\mu}) \tag{3.17}$$

4. The Einstein's field equations for an anisotropic universe

The energy-momentum tensor of a Finslerian perfect fluid for a comoving observer is defined to be[21, 7]

$$T_{\mu\nu}(x, y(x)) = (\mu + P)y_{\mu}(x)y_{\nu}(x) - Pg_{\mu\nu}(x, y(x)) \tag{4.1}$$

where $P \equiv P(x), \mu \equiv \mu(x)$ is the pressure and the energy density of the cosmic fluid respectively. The vector $y^{\alpha} = \frac{dx^{\alpha}}{d\tau}$ is a 4-velocity of the fluid since $y = (1, 0, 0, 0)$ with respect to comoving coordinates. Thus $T_{\mu\nu}$ becomes[4] [23]

$$
\begin{array}{lll}
T_{00} = & \mu & \tag{4.2}\\
T_{ij} = & -P\frac{F}{\sigma}a_{ij} & \tag{4.3}\\
T = & T^{\mu}_{\mu} & \\
= & \frac{\sigma}{F}\mu - 3P & \tag{4.4}
\end{array}
$$

where $F/\sigma \approx 1$ at the weak field limit.

The substitution of (4.1) to the field equations

$$L_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right) \tag{4.5}$$

---

[4] $T_{\mu\nu} = diag(\mu, -\frac{F}{\sigma}a_{ij}P)$ in matrix form

implies the following O.D.E

$$\frac{\ddot{a}}{a} + 2\frac{\dot{a}}{a}z_t = - \quad \frac{4\pi G}{3}(\mu + 3P) \tag{4.6}$$

$$\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{k}{a^2} + 4\frac{\dot{a}}{a}z_t = \quad 4\pi G(\mu - P) \tag{4.7}$$

In virtue of (4.6),(4.7) we obtain the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{\dot{a}}{a}z_t = \frac{8\pi G}{3}\mu - \frac{k}{a^2} \tag{4.8}$$

which is similar to the one derived from the Robertson-Walker metric in the Riemannian framework, apart from the extra term $\frac{\dot{a}}{a}z_t$. We associate this extra term to the present Universe's anisotropy.

The quantity z_t is easily calculated from (3.10)

$$z_t = \frac{1}{2}\hat{k}_0\phi(x)_{,0}. \tag{4.9}$$

The physical quantity z_t depends on the scalar $\phi(x)$ which is the only quantity of the Lagrangian that gives us insight about the evolution of anisotropy. The parameter z_t is measured by the Hubble's units as (4.8) implies. It is significant that z_t depends on the geometrical properties of the Finslerian space-time manifold. Indeed, the component C_{000} can be directly calculated from (3.7) as

$$C_{000} = \frac{u_0}{2} \tag{4.10}$$

(as we previously considered $|u_\alpha| << 1$) after differentiating with respect to proper time we lead to *the direct dependence of z_t on the Cartan torsion component C_{000}*

$$z_t = C_{000,0} \tag{4.11}$$

hence the variation of anisotropy is closely related to the variation of the Cartan torsion tensor as an intrinsic object of the Finslerian space-time.

5. The cosmological anisotropic parameters

We list the main anisotropic parameters constructed within the Finslerian framework[21, 8, 7, 14]

The anisotropic scale factor is defined along each world line. $S(s)$ is the length scale introduced in [7, 2]. $S(s) = \tilde{a}(v(s))$ where $v(s)$ is the tangent vector field along the world lines.

The anisotropic Hubble parameter $\tilde{H}$ is given by the relation($\dot{S} = \frac{\partial \tilde{a}}{\partial y^\mu}\dot{v}^\mu(s)$)

$$\tilde{H} = \frac{\dot{S}}{S} = \frac{1}{3}\tilde{\Theta} \tag{5.1}$$

The term $\tilde{\Theta}$ is the expansion in the Finslerian space-time expressed by

$$\tilde{\Theta} = \nabla_\mu y^\mu - C^\lambda_{\mu\lambda}\dot{y}^\mu \tag{5.2}$$

Where the symbol ∇ means the Riemannian covariant derivative associated with the osculating Riemannian metric tensor $r_{\mu\nu}(x)$. The anisotropic Hubble parameter can be computed from (4.8)

$$\tilde{H}^2 = H^2 + Hz_t \tag{5.3}$$

thus we have to attribute Hubble's units to z_t.

The density parameter can be defined with respect to anisotropic Hubble parameter $\tilde{H}$

$$\tilde{\Omega}_\mu = \frac{8\pi G}{3\tilde{H}^2}\mu = \frac{\mu}{\tilde{\mu}_{crit}} \tag{5.4}$$

where

$$\tilde{\mu}_{crit} = \frac{3\tilde{H}^2}{8\pi G}. \tag{5.5}$$

The deceleration parameter is defined in terms of the anisotropic scale factor $S(s)$

$$\tilde{q} = -S\ddot{S}/\dot{S}^2. \tag{5.6}$$

Therefore the Friedmann equation can also be rewritten in the form

$$\tilde{\Omega} - 1 = k/(\tilde{H}^2 a^2) \tag{5.7}$$

6. Discussion

The study of a FRW-model in the framework of a generalised metric space with a weak vector field into the osculating Riemannian structure of the universe provides us with the extended Friedmann equation (4.8).The contribution of the variation of anisotropy is expressed by the z_t-constant produced by the Finslerian character of the geometry of spacetime.In classical relativity the entire evolution of a homogeneous isotropic universe is contained in the scale factor $a(t)$(e.g. evolution of matter,radiation and vacuum densities and every other physical quantity depending on the proper time t).In case we take into account the directional dependence of the scale factor we violate the assumption of isotropy. This consideration leads us to modify $a(t)$ to the form $\tilde{a}(\upsilon(s))$ to be compatible within the framework of locally anisotropic spacetime. The vector $\upsilon(s)$ represents the unit tangent vector in flow lines(wordlines of the cosmological fluid). The whole picture can be naturally interpreted in terms of the locally anisotropic metric structures of the universe.The solution of the FRW equations can be applied to the estimation of the variation of anisotropy and calculate with better precision the temperature of the CMB radiation depicted at the present picture of WMAP[25].

7. Acknowledments

 [1] Bennet et.al Astrophys.Jour. **583** 1 (2003)
 [2] G.F.Ellis,C.Hellaby,D.R.Matravers Astrophyphys.Jour. **364** 400 (1990)
 [3] C.G.Tsagas *Geometrical Aspects of Cosmic Magnetic Field* arXiv:gr-qc/0112077 (2001)
 [4] D.R.Matravers,C.G.Tsagas *Cosmic Magnetics, curvvuture and the expansion dynamics* arXiv:gr-qc/0003181 (2000)
 [5] C.G.Tsagas,R.Maartens Phys.Rev.D **61** 083519 (2000)
 [6] P.C.Stavrinos, D.Nikolopoulos, N.Prezas Sectilor Stiintifice Sir.IV Tomul XX (Academia Rumana Buchurest 9 (2000))
 [7] P.C.Stavrinos J.Phys.Conf. **8** 49 (2006)
 [8] P.C.Stavrinos Int.Journ.Theor.Phys. **44(2)** 245 (2005)
 [9] P.C.Stavrinos, G.S.Asanov Rep.Math.Phys **30** 63 (1991)
[10] P.C.Stavrinos,S.Ikeda Int.Jour.Theor.Phys. **43(2)** 437 (2004)
[11] S.Vacaru Annals of Physics **290** 83 (2001)
[12] P.C.Stavrinos *Remarks on the Applications of Finsler Geometry to Space-time* Conf.Proceed. MSRI Berkeley, USA June 2002 pp.30 http://www.msri.org/publications

[13] S.Vacaru,P.C.Stavrinos,E.Gaburov and D.Gonta *Clifford and Riemann-Finsler Structures in Geometric Mechanics and Gravity* Geometry Balcan Press (2005) arXiv:math-ph/0608024

[14] P.C.Stavrinos, F.I.Diakogiannis Gravitation and Cosmology **10(4)** 1 (2005)

[15] P.G.Lampropoulos, P.C.Stavrinos *A locally anisotropic geometrical model of space-time based on CMBR* arXiv:gr-qc/0509020 (2006)

[16] G.F.R.Ellis, R.Maartens *The emergent Universe:Inflationary Cosmology with no singularity* arXiv:gr-qc/0211082

[17] R.G.Beil Int.Journ.Th.Phys **26** 2 (1987)

[18] R.Miron, M.Anastasei, *Vector Bundles, Lagrange Spaces, Applications to Relativity*, Kluwer A.P. (1997)

[19] H.Rund *The Differential Geometry of Finsler Spaces* Springer, Berlin (1959)

[20] J.D.Bekenstein Phys.Rev.D **48(8)** 3641 (1993)

[21] G.S.Asanov *Finsler Geometry, Relativity and Gauge Theories* Reidel Pub.Com (1985)

[22] H.Yasuda, H.Shimada Rep.Math.Phys **11** 347 (1977)

[23] H.C.Ohanian, R.Ruffini *Gravitation and space-time* Norton (1994)

[24] P.J.E.Peebles *Principles of Physical Cosmology* Princeston Series in Physics (1993)

[25] P.C.Stavrinos, A.P.Kouretsis, M.Stathakopoulos *Friedmann Robertson-Walker model in generalised metric space with weak anisotropy* under preparation (2006)