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Classification of Periodic Alternating-Solenoid Lattices

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Abstract

The classification of different types of periodic alternating-solenoid lattices based on the symmetry of the main harmonic content was first considered by G. Penn. In this note we extend his analysis and look in more detail at the characteristics of possible solutions. A classification scheme closely tied to earlier work is proposed. Regions of parameter space corresponding to various classes of solution are mapped out in detail.

1 Introduction

Most of the ionization cooling systems designed so far have used a periodic focusing lattice. Any cooling lattice needs reversals in the field direction to counteract the build-up of canonical angular momentum. For this reason these lattices consist of solenoids with alternating polarity.

Historically the first system considered had a simple sinusoidal dependence of the longitudinal field. At the design momentum the beta function had minima as a function of distance at the zero crossing points of the magnetic field. This was designated a FOFO lattice [1]. Later, it was found that the performance of these lattices could be improved significantly by the addition of harmonic terms to the on-axis fields. The first lattice of this type had the minima in the beta function at the location of the maxima of the magnetic field. This design was unfortunately designated the "alternating solenoid" lattice [2], even though this term more correctly applies to the whole set of lattices considered here. Additional lattice types with significant harmonic content were given the generic name "super-FOFO"[3]. These lattices have minima in the beta function at the zero crossings of the magnetic field. Super-FOFO designs in which alternate lattice cells had the opposite polarity in the magnetic field were called SFOFO, whereas designs in which the field reversal takes place inside a single cell were called RFOFO.

The first systematic study of the effects of harmonic content on the properties of super-FOFO lattices was made by G. Penn [3,4]. He had the clever idea to consider two types of field profile with even

$$B_Z(z) = B_1 \sin\left(\frac{2\pi z}{L}\right) + B_2 \sin\left(\frac{4\pi z}{L}\right) \quad \textbf{(1)}$$

and odd

$$B_Z(z) = B_1 \sin\left(\frac{\pi z}{L}\right) + B_3 \sin\left(\frac{3\pi z}{L}\right) \quad \textbf{(2)}$$

harmonics, where L is the length of a lattice cell. He showed that the properties of the lattices could be related to the scaling parameter

$$\chi = \frac{B_o L}{p_Z} \quad \textbf{(3)}$$

where B_o is the peak value of the magnetic field on-axis. Using these parameters he was able to produce the phase diagrams of lattice behavior shown in Fig. 1. In this note we will continue this type of study and examine the parameter space in greater detail, clearly identifying the regions of parameter space that correspond to various types of lattices.

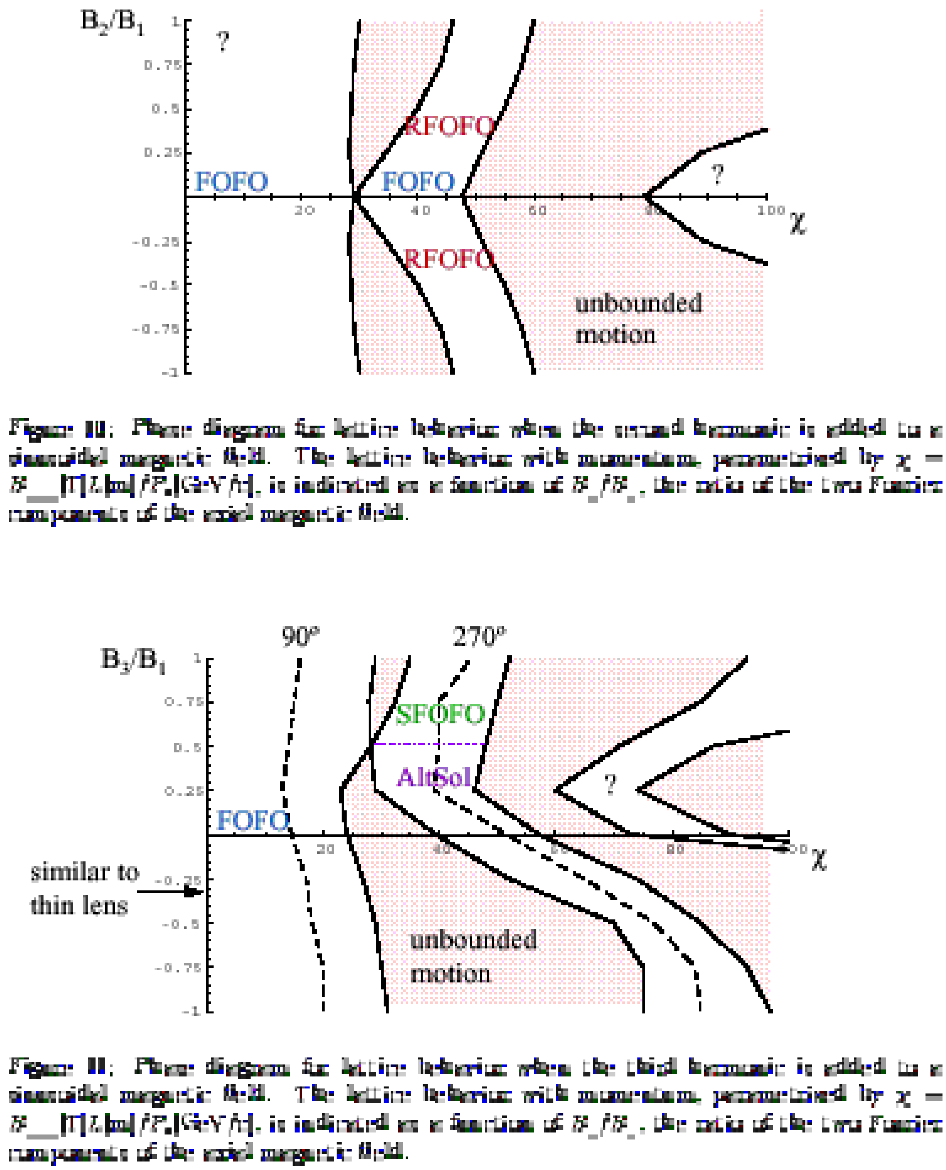

Figure 1. Penn's phase diagram of lattice behavior from reference 4.

2 General classification of alternating solenoid lattices

It has been found useful to classify cooling lattices depending on the relationship between (a) the on-axis profile of the longitudinal component of the solenoidal field, and (b) the corresponding profile of the beta function. However, it is important to keep in mind that the dependence of the beta function on the magnetic field profile is momentum dependent.

The most general form of the magnetic field on-axis is

$$B_Z(z) = \sum_{n=0}^{\infty} \left(a_n \cos nkz + b_n \sin nkz\right) \tag{4}$$

where

$$k = \frac{2\pi}{\lambda} \tag{5}$$

and λ is the period of the magnetic field. In general this can be a very complicated function of z. However, the cosine terms only differ from the sine terms by a shift of a quarter period along z. If we adopt the definition that the location z = 0 corresponds to a zero crossing point of the magnetic field, we can ignore the cosine terms in Eq. 4. The focusing function in a solenoidal lattice is related to the square of the magnetic field through

$$K(z) = \left(\frac{eB(z)}{2p_Z}\right)^2 \tag{6}$$

The symmetry of the focusing depends on the cross terms in the expansion of the square of the magnetic field. If just the fundamental and odd harmonics are present, K(z) has the period $\lambda/2$. We refer to this as an S-lattice. If the fundamental and any even harmonics are present, K(z) has the period λ and we refer to this as an R-lattice. We call the period of K(z) the lattice cell length. These definitions are consistent with prior usage.

In addition to the harmonic content, the location of the minimum of the beta function in the lattice cell is another important property of a solution. We adopt the term FOFO to mean the minimum in the beta function occurs at the beginning of the cell. Recall that the zero-crossing of the magnetic field also occurs there. We use the term ASOL, instead of “alternating solenoid”, to mean the minimum in the beta function occurs at the midpoint of the lattice cell. Again these definitions are consistent with prior usage.

This classification scheme is summarized in Table 1.

Table 1 Classification of alternating solenoid lattice types

Type	B harmonic content	β_{MIN} at
FOFO	none	zero-crossing in B
ASOL	none	midpoint of cell
SFOFO	odd	zero-crossing in B
SASOL	odd	midpoint of cell
RFOFO	even	alternate zero-crossing in B
RASOL	even	midpoint of cell

There are many solutions that do not fit in the classification of Table 1. One could easily invent a classification based on, for example, the number of minima and their symmetry in the cell. However, up until now no one has shown any advantage in using these kinds of lattices for cooling, so we just lump them all together in a general “mixed” symmetry category.

Another lattice name that appeared in the literature was called a DFOFO. [5]. We believe that further proliferation of lattice prefix letters should be discouraged. The DFOFO lattice corresponds to what we are calling an RFOFO.

3 Sinusoidal lattices

For the following we will adopt a slightly different definition of the scaling variable

$$\chi = \frac{eB_o\lambda}{p_Z} \tag{7}$$

where λ is the period of the magnetic field. The units used are {T, m, GeV/c, e=0.3}. In order to determine the stability and beta function for a given field configuration, we will use the symplectic matrix method described in a previous note [6]. We have seen that this method agrees well with tracking simulations and other analytic calculations. As a further check on the validity of the method, we show in Table 2 the computed stop bands for a pure sinusoidal field with a magnitude of 2.8 T and a period of 5.5 m. The momentum is related to the dimensionless parameter q in the Mathieu theory by [7]

$$q = \left(\frac{eB_o\lambda}{8\pi p_Z}\right)^2 = \left(\frac{\chi}{8\pi}\right)^2 \tag{8}$$

Table 2 Stop bands for a pure sinusoidal field

Δp_{SYMP} [MeV/c]	Δq_{SYMP}	Δq_{MATH}	$\Delta\chi$
321 - 195	0.328 - 0.889	0.329 - 0.890	14.4 - 23.7
135 - 106	1.85 - 3.01	1.86 - 3.04	34.3 - 43.8
85.6 - 72.6	4.61 - 6.41	4.63 - 6.42	54.1 - 63.7
62.6 - 55.4	8.62 - 11.0	8.63 - 11.0	73.8 - 83.4

We see that the results of the symplectic matrix calculation agree very well with stop bands determined analytically from the roots of the Mathieu equation.

We show in Fig. 2 an example of a solution from the first pass band, where p > 321 MeV/c. This is a FOFO solution.

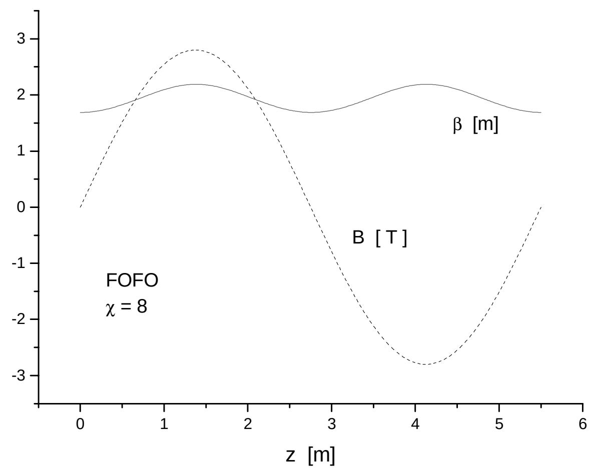

Figure 2. Magnetic field and beta function for a FOFO solution.

In the next pass band, where 136 < p < 194 MeV/c, the solution is an ASOL lattice.

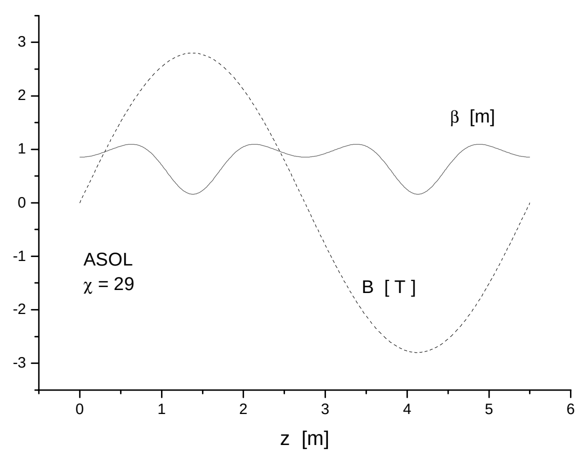

Figure 3. Magnetic field and beta function for a ASOL solution.

Finally in the following pass band, where 86 < p < 105 MeV/c, the solution has mixed symmetry

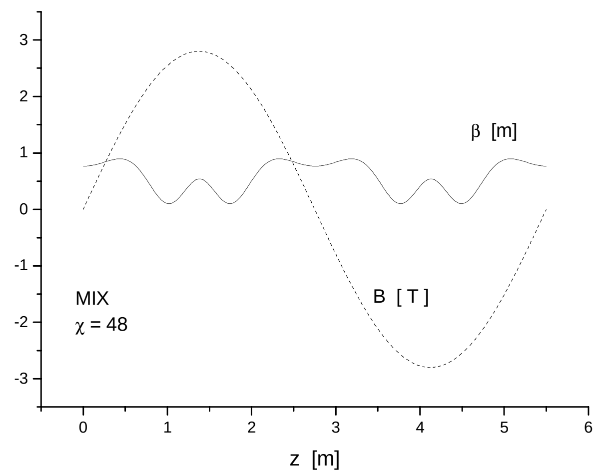

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Figure 4. Magnetic field and beta function for a MIX solution.

4 S lattices: odd harmonics

The magnetic fields in S-type lattices are made up from a fundamental sine wave plus odd harmonics. As the simplest example for this type of field we consider fields of the form

$$B_Z(z) = b_1 \sin(kz) + b_3 \sin(3kz)$$

Fig. 5 shows the region of parameter space with stable solutions as a function of the scaling variable χ and of the amount of harmonic content b_3 / b_1.

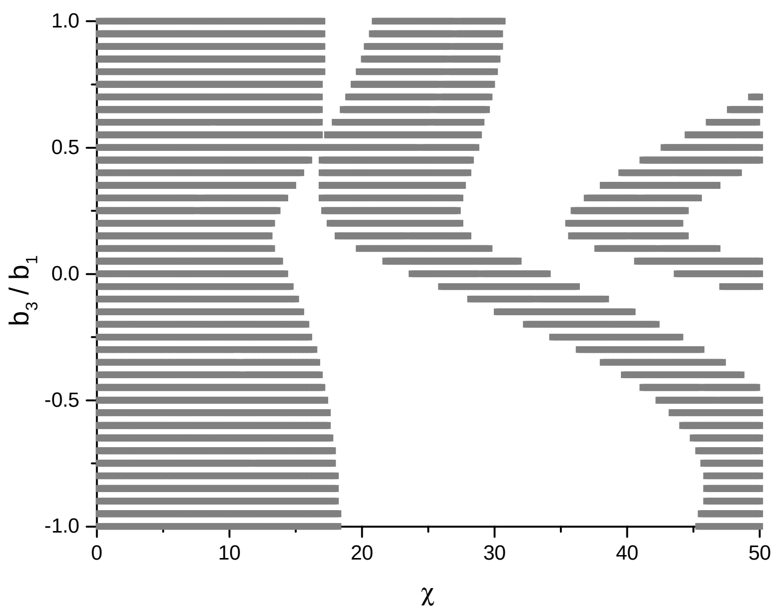


Figure 5. Stability space diagram for S-type lattices. Dark bars show regions with stable solutions.

Three bands of stable solutions can be seen in Fig. 5.

The contour of minimum beta function for this space is shown in Fig. 6.

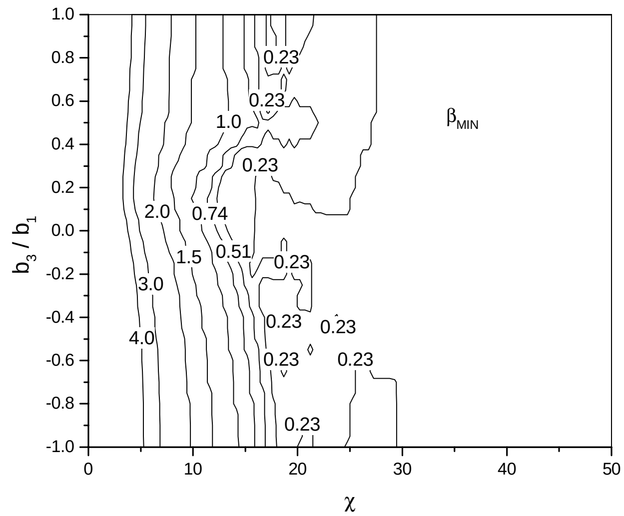


Figure 6. Contour plot of the minimum value of the beta function.

We see the general trend of the minimum value of the beta function decreasing to the right. This is simply the consequence of large values of the parameter χ corresponding to small momentum, where focusing is easier.

The SFOFO solutions correspond to the subset of Fig. 5 that have a minimum in the beta function at the beginning of the cell, where the magnetic field has a zero crossing. These solutions are shown in Fig. 7.

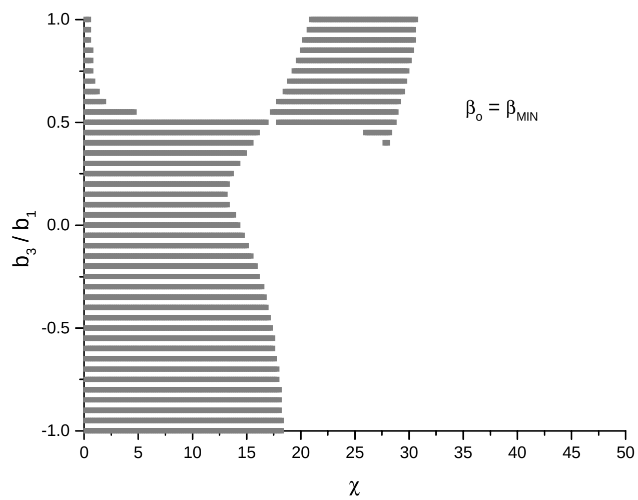


Figure 7. Stability diagram for SFOFO solutions.

Note the boundary line for $b_3 / b_1 \sim 0.5$ and the singular point with $\chi \sim 17.5$. A typical SFOFO solution taken from the space of Fig. 7 is shown in Fig. 8.

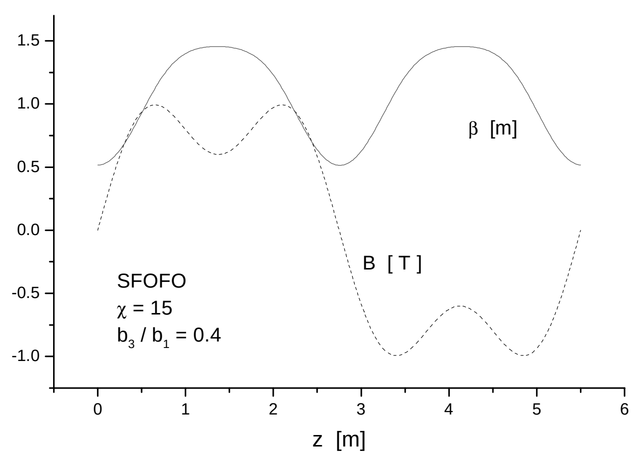


Figure 8. Magnetic field and beta function for typical SFOFO solution.

The SASOL solutions correspond to the subset of Fig. 5 that have the minimum in the beta function at the midpoint of the cell, where the magnetic field has a non-zero value. These solutions are shown in Fig. 9.

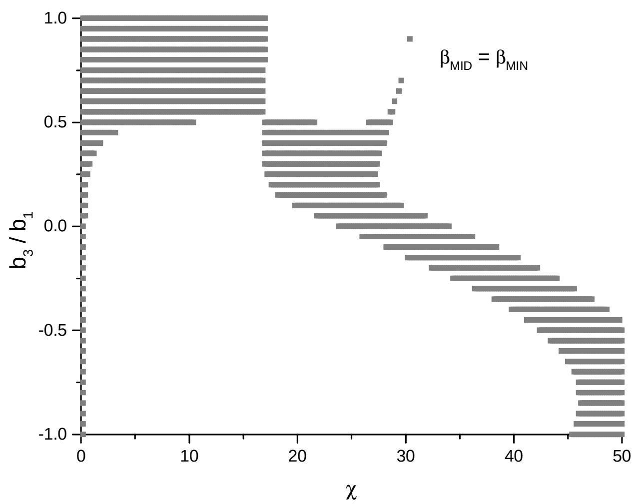


Figure 9. Stability diagram for SASOL solutions.

A typical solution of this type is shown in Fig. 10.

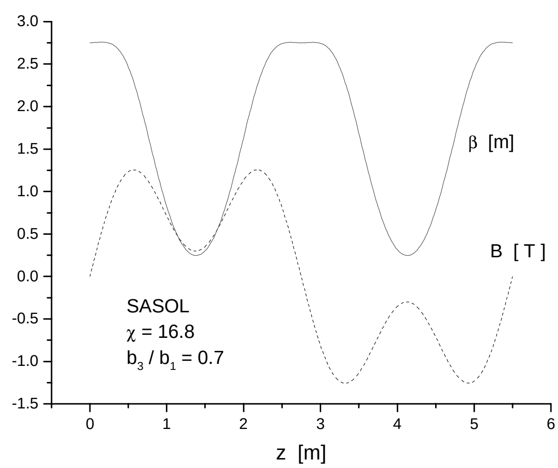


Figure 10. Magnetic field and beta function for a typical SASOL solution.

The stable solutions in Fig. 5 that don't fall in Figs. 7 or 9 are mixed solutions, where the minimum of the beta function does not occur at the beginning or the midpoint of the cell. Fig. 11 shows a typical mixed solution.

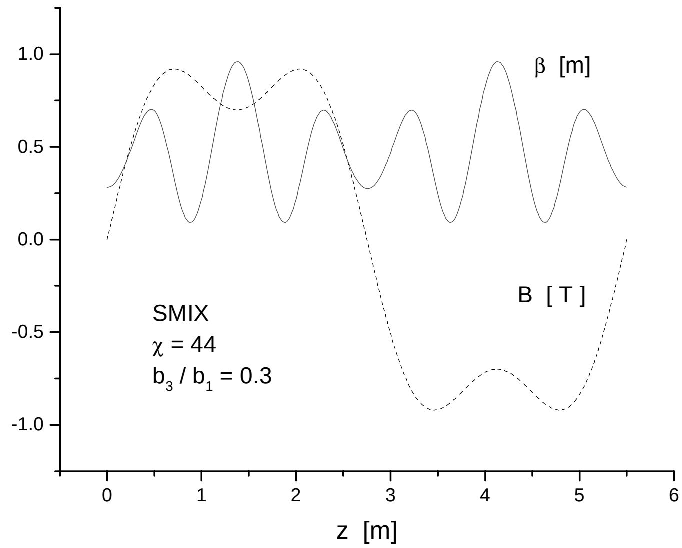


Figure 11. Magnetic field and beta function for a typical SMIX solution.

5 R lattices: even harmonics

The magnetic fields in R-type lattices are made up from a fundamental sine wave plus even harmonics. As the simplest example for this type of field we consider fields of the form

$$B_Z(z) = b_1 \sin(kz) + b_2 \sin(2kz)$$

Fig. 12 shows the region of parameter space with stable solutions as a function of the scaling variable χ and of the amount of harmonic content b_2 / b_1.

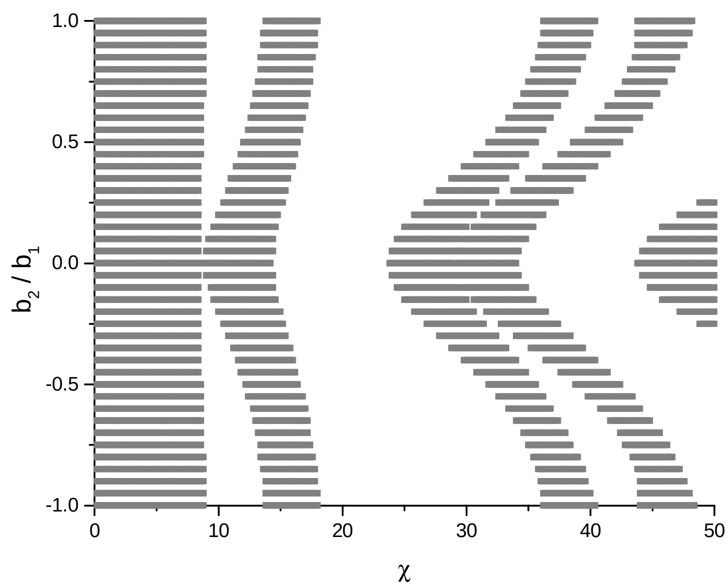


Figure 12. Stability space diagram for R-type lattices. Dark bars show regions with stable solutions.

We can see five bands of stable solutions in this parameter space. On the abscissa, where the magnetic field is a pure sinusoid, there are three bands corresponding to the three bands in Fig. 5.

The contour for the minimum of the beta function is shown in Fig. 13.

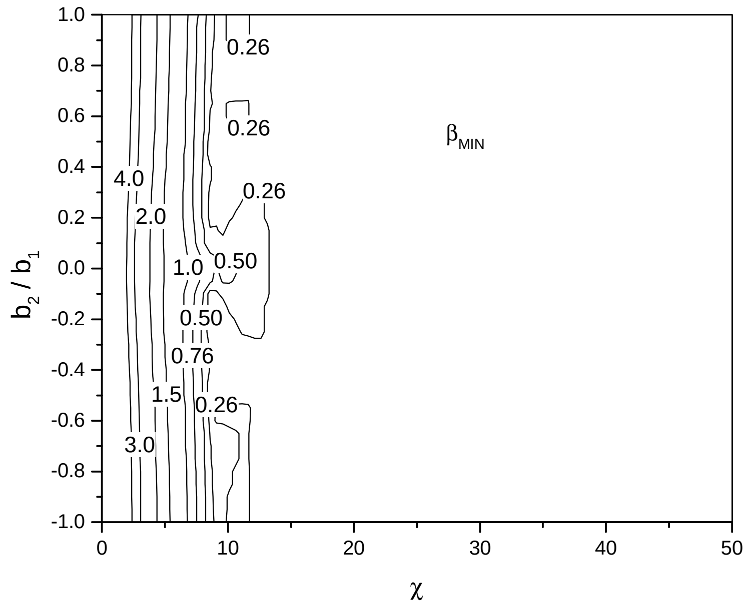


Figure 13. Contour plot for the minimum value of the beta function.

This also shows the general trend of solutions with smaller minimum beta at larger values of χ.

The RFOFO solutions correspond to the subset of Fig. 12 that have a minimum in the beta function at the beginning of the cell, where the magnetic field has a zero crossing. These solutions are shown in Fig. 14.

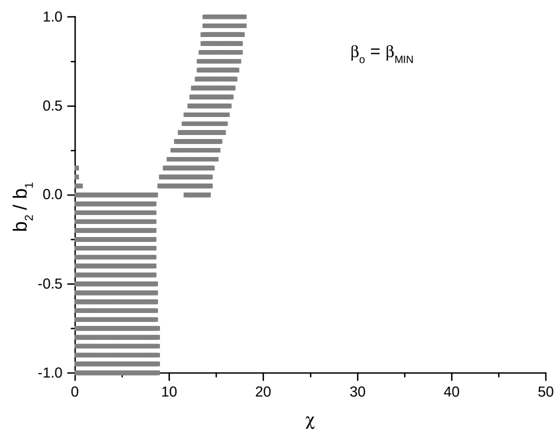


Figure 14. Stability diagram for RFOFO solutions.

The boundary line between RFOFO and RASOL solutions has $b_2 = 0$ here and the singular point occurs at $\chi \sim 9.5$. A typical RFOFO solution taken from the space of Fig. 14 is shown in Fig. 15.

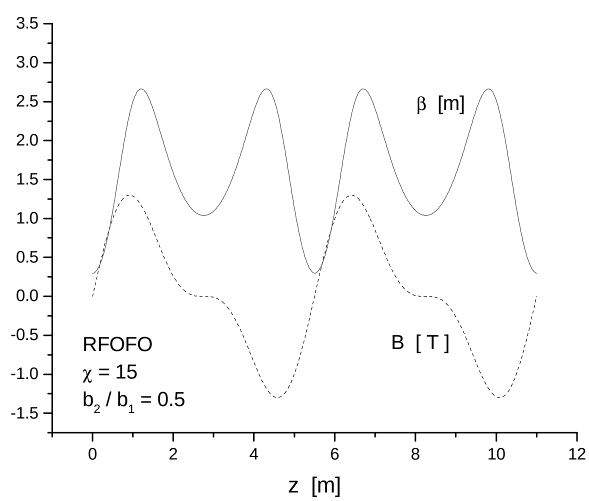


Figure 15. Magnetic field and beta function for typical RFOFO solution.

The RASOL solutions correspond to the subset of Fig. 12 that have a minimum in the beta function at the middle of the cell, where the magnetic field has an additional zero crossing. These solutions are shown in Fig. 16.

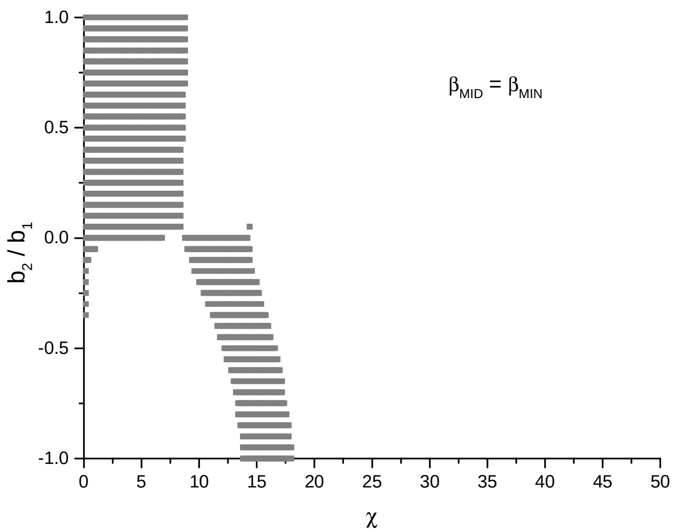


Figure 16. Stability diagram for RASOL solutions.

A typical example of an RASOL lattice is shown on Fig. 17.

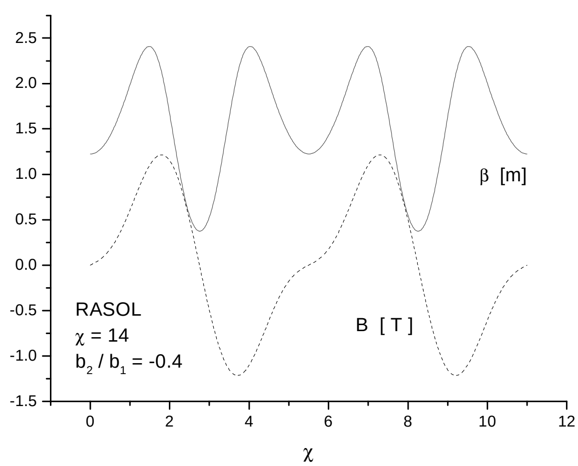


Figure 17. Magnetic field and beta function for typical RASOL solution.

The other solutions from Fig. 12 that do not belong to Figs. 14 or 16 have mixed symmetry. A typical solution is shown in Fig. 18.

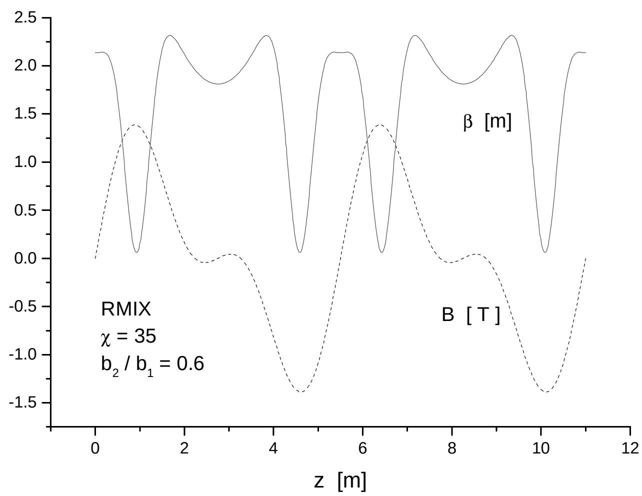


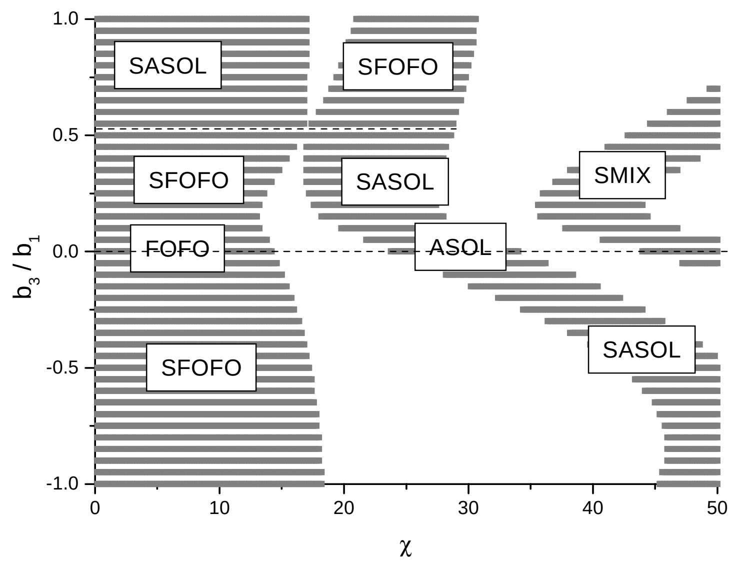
Figure 18. Magnetic field and beta function for typical RMIX solution.

6 Conclusions

A plot of the stable S and R parameter spaces with regions corresponding to various symmetry solutions is shown in Fig. 19.

It is interesting to ask where the Study-II 2.75 m lattice solution lies on this parameter space. At 200 MeV/c this lattice has $\chi \sim 23$ and $b_3 / b_1 \sim 0.42$ [6]. Looking at the upper diagram in Fig. 19, this point unfortunately lies in the SASOL region, just below the boundary line, while we know that the Study II lattice is a SFOFO solution. However, there is not an exact correspondence with this plot because the Study II lattice contains significant amounts of other odd harmonics. Apparently these other harmonics are sufficient to move the operating point across the boundary line.

In the lower diagram the horizontal axis with $b_2 = 0$ is similar to a line of "unstable equilibrium". Solutions along this line have cell lengths equal to $\lambda / 2$. As soon as any amount of b_2 harmonic is added to the fundamental field, the cell length immediately changes to λ.



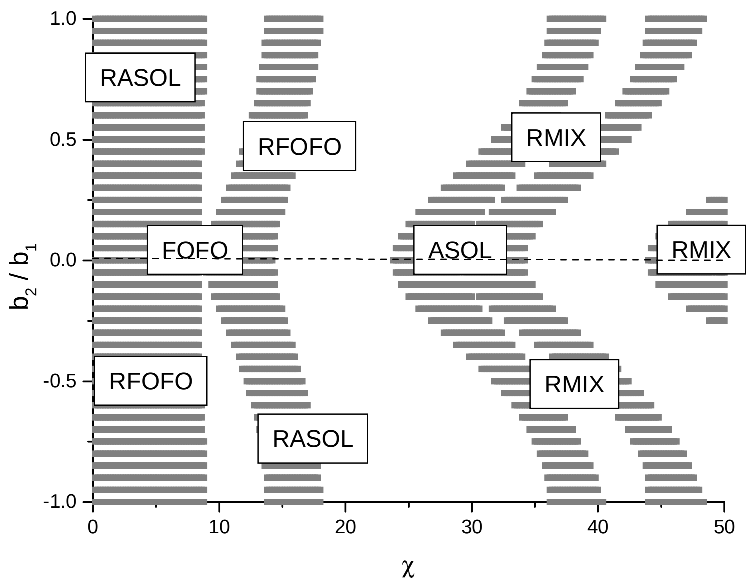


Figure 19. Location and identity of stable lattice solutions.

Notes and references

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