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# How many futures on Finsler spacetime?

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**Abstract.** Some recent results by the author on the geometry and dynamics of Finsler spacetimes are reviewed. It is shown that in Finslerian generalizations of general relativity the number of predicted lightlike cones is two, one past and one future, as in general relativity. This result is non-trivial as it can fail, for instance, in spacetime dimension two. It is also shown that suitable versions of the reverse Cauchy-Schwarz and reverse triangle inequalities hold on Finsler spacetimes. Finally, a long standing problem of Finslerian gravity concerns the development of dynamical equations which imply a conservation law. We make some progress following a recent proposal by the author according to which physical Finsler spacetimes have affine sphere indicatrices of hyperbolic type.

## 1. Introduction

Although Finsler geometry is a quite venerable subject it is usually a good idea to remind the reader what it is all about. Shiing-Shen Chern tried to popularize this theory publishing an article entitled "Finsler geometry is just Riemannian geometry without the quadratic restriction". In fact Riemann suggested to consider spaces for which the arc-length of curves is given by $\int_\gamma \sqrt{2|\mathscr{L}(x,\dot{x})|}\mathrm{d}t$ where

$$\mathscr{L}(x,v)=\frac{1}{2}\,g_{\mu\nu}(x,v)v^\mu v^\nu, \tag{1}$$

namely where $g$ although positive definite depends on $v$ and hence does not define a quadratic form. Observe that the arc-length calculated with the above formula must be independent of the parametrization, and so it is natural to require $\mathscr{L}$ to be positive homogeneous of degree two

$$\forall s>0, \qquad \mathscr{L}(x,sv)=s^2\mathscr{L}(x,v). \tag{2}$$

In this way $g_{\mu\nu}$ can be defined as the Hessian with respect to the velocities of $\mathscr{L}$. In index free notation we shall also write $g_v=g_{\mu\nu}\mathrm{d}x^\mu\otimes\mathrm{d}x^\nu$, $g_v\colon E\mapsto T^*M\otimes_M T^*M$, $E=TM\backslash 0$, with a $v$ index to stress the dependence on the velocity. It can be easily shown, differentiating twice Eq. (2) with respect to $s$, that $g_v$ cannot be extended to the whole $TM$ while remaining continuous otherwise it would be independent of $v$ and we would be back to the Riemannian case. The Finsler Lagrangian $\mathscr{L}$ is said to be reversible if $\mathscr{L}(x,-v)=\mathscr{L}(x,v)$.

While it is true that Finsler geometry is obtained dropping the quadratic restriction on $\mathscr{L}$, this fact does not really explain why this theory is interesting and why we should study it. After all any theory could be made more general dropping some axiom. Instead, we want the theory

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to be simple and rich while being based on few axioms. The importance and naturalness of Finsler geometry is better understood with the following observation:

> *Riemannian geometry gives each tangent space of a manifold the structure of a finite dimensional Hilbert space; Finsler geometry gives each tangent space the structure of a Banach space whose unit balls are strongly convex.*

In this work we shall be interested in Lorentz-Finsler geometry which is obtained from the above axioms replacing the positive definite signature of $g$ with the Lorentzian signature $(-,+,\cdots,+)$. John Beem tried this option [1, 2] but could not establish the existence of two cones, namely the fact that the set of timelike vectors at $p \in M$

$$I_p := \{v \in T_pM\backslash 0 : \ \mathscr{L}(p,v) < 0\},$$

consists of two convex cones to be interpreted as the future and the past causal cones. Other authors [3] that considered the causal aspects of the theory simply stipulated the existence of a future timelike convex cone $I_p^+$ where $\mathscr{L} < 0$.

It seems that most authors working in Finsler geometry did not care very much of the signature of the metric, as it was simpler to leave aside, at least temporarily, the difficulties connected with the causal and geometrical interpretation of the theory. Also, they were essentially looking for tensorial generalizations of the Einstein's equations so they were not very much concerned with the underlying geometry [4–10]. Actually, understanding the geometry of the theory turned out to be very important since, as in Finsler geometry there are many connections, there were several tensorial candidates for a generalization of Einstein's equations and their physical interpretation was not entirely clear.

The investigations by Beem confirmed that causality in Finsler space is non-trivial. Although each connected component of the the timelike region $I_p$ had to be a convex cone [1, 2, 11, 12], he found that $I_p$ could consist of more than two convex cones. For instance, he gave the example of a 2-dimensional spacetime $\mathbb{R}^2$ endowed with a Finsler Lagrangian

$$\mathscr{L}(x,v) = \frac{(v^1)^3 v^2 - v^1 (v^2)^3}{(v^1)^2 + (v^2)^2},$$

where $I_p$ is independent of $x \in M$ and has four convex cone components. Actually, Beem believed to have found similar examples for higher-dimensional spacetimes [2], but his examples can be shown to be incorrect since the metric has vanishing determinant at some point of the slit tangent bundle.

In this work I shall outline further progress by the author which has shown that, at least for reversible Lagrangians, Beem's pathological examples can be found only in two dimensional spacetimes. In the physical 4-dimensional spacetime, the Lorentzianity condition on $g_v$ implies that $I_p$ consists precisely of two convex cones. Thus, Finsler causality predicts that there is just one future as desired. This is really a good news since the mathematical theory of Finsler connections, sprays, exponential maps and convex neighborhoods has been developed using the slit tangent bundle $E = TM\backslash 0$. We do not have to restrict the domain of $\mathscr{L}$ to a convex cone thus we can use the standard theory of Finsler connections in the study of Lorentz-Finsler geometry.

In our recent works we have also solved other problems. In fact we proved for the first time the reverse Cauchy-Schwarz and reverse triangle inequalities for Lorentz-Finsler geometry. These results are fundamental in many geometrical arguments. They have been used to establish that lightlike geodesics locally maximize the Lorentzian length [13], a fact which implies by the usual interpolation argument [14, Prop. 2.8] that any causal geodesic which is not an achronal lightlike geodesic can be deformed to a timelike curve keeping endpoints fixed. Let us denote with $J_p^\alpha$

a connected component of the set of causal vectors, namely the closed set of the slit tangent bundle given by

$$J_p = \{v \in T_pM\backslash 0 : \ \mathscr{L}(p,v) \le 0\}.$$

Then the inequalities read as follows ($V = T_pM$).

**Theorem 1.1** (Finslerian reverse Cauchy-Schwarz inequality)**.** *Let $v_1, v_2 \in J_p^\alpha$ then*

$$-g_{v_1}(v_1,v_2) \ge \sqrt{-g_{v_1}(v_1,v_1)}\,\sqrt{-g_{v_2}(v_2,v_2)},$$

*with equality if and only if $v_1$ and $v_2$ are proportional. In particular, if $v_1, v_2 \in J_p^\alpha$ then $g_{v_1}(v_1,v_2) \le 0$ with equality if and only if $v_1$ and $v_2$ are proportional and lightlike.*

**Theorem 1.2** (Finslerian reverse triangle inequality)**.** *Let $v_1, v_2 \in J_p^\alpha$ then defined $v = v_1 + v_2$, we have $v \in J_p^\alpha$ and*

$$\sqrt{-g_v(v,v)} \ge \sqrt{-g_{v_1}(v_1,v_1)} + \sqrt{-g_{v_2}(v_2,v_2)}.$$

*with equality if and only if $v_1$ and $v_2$ are proportional. In particular, if $v_1$ is timelike and $v_2$ is causal then $v$ is timelike.*

We also showed that the Legendre map is a diffeomorphism.

**Theorem 1.3** (The Legendre map is a diffeomorphism for spacetime dimension $n+1 \ge 3$)**.** *Suppose that $\dim V \ge 3$, then the map $\ell\colon V\backslash 0 \to V^*\backslash 0$ defined by*

$$v \mapsto g_v(v,\cdot) = \partial \mathscr{L}/\partial v$$

*is a bijection and hence a diffeomorphism. Its extension to the whole $V$ with $\ell(0) := 0$ is a Lipeomorphism (locally Lipschitz homeomorphism with locally Lipschitz inverse). Moreover, without any condition on dimension, it is always true that $\ell$ establishes a bijection between $J_p^\alpha$ and its polar cone*

$$J_p^{\alpha *} = \{q \in T_M^*\backslash 0 : \ q(v) \le 0, \ \textit{for every } v \in J_p^\alpha\}.$$

Since this results plays a role in the proof that $I_p$ has two components we sketch the proof.

*Idea of proof.* Let $V = T_pM$ and let us introduce the equivalence relation on $V\backslash 0$ "$v_1 \sim v_2$ if there is $s > 0$ such that $v_1 = sv_2$", namely let us regard $V\backslash 0$ as a radial bundle over a base $Q$ diffeomorphic to $S^n$. Analogously, let us introduce the quotient $Q^*$ of $V^*\backslash 0$ with respect to the radial directions. The map $\ell$ satisfies $\ell(sv) = s\ell(v)$ for every $s > 0$, thus it passes to the quotient to a map $\tilde{\ell}\colon Q \to Q^*$. Since $Q$ and $Q^*$ are closed manifolds with the same dimension, $\tilde{\ell}$ is actually a covering. Since $Q^* \sim S^n$ is simply connected (here we use $n \ge 2$) and $Q \sim S^n$ is connected this covering is actually a homeomorphism. From here the proof is straightforward. □

**Lemma 1.4.** *Let $\dim V \ge 3$ and let $\mathscr{L}$ be reversible. Given any two distinct components $J^\alpha$ and $J^\beta$ we have $J^\alpha \cap (-J^\beta) \ne \emptyset$.*

*Idea of proof.* The proof is by contradiction. Suppose that $J^\alpha \cap (-J^\beta) = \emptyset$ then there is a hyperplane $W \subset V$ passing through the origin which separates $J^\alpha$ and $J^{-\beta} = -J^\beta$ and denoting with $u^\alpha$ and $u^\beta$ the points of tangency of the translations of $W$ with the indicatrix $\{v : 2\mathscr{L}(p,v) = -1\}$ which stays inside the cones

$$W = \ker g_{u^\alpha}(u^\alpha,\cdot) = \ker g_{u^\beta}(u^\beta,\cdot)$$

and from here it is easy to find $s > 0$ such that

$$g_{u^\alpha}(u^\alpha,\cdot) = g_{su^\beta}(su^\beta,\cdot)$$

contradicting the injectivity of the Legendre transform. □

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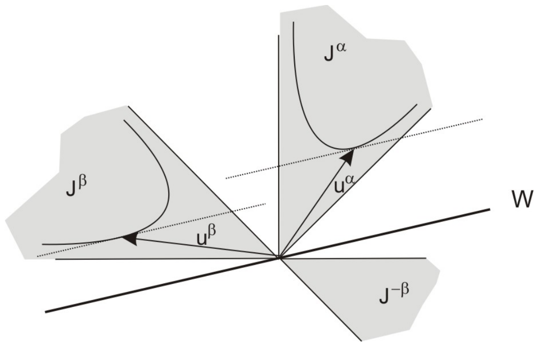


**Figure 1.** The idea of Lemma 1.4.

We arrive at the final step of the proof

**Theorem 1.5.** *Let $dim V \geq 3$ and suppose that $\mathscr{L}$ is reversible, then $I_p$ has two components.*

*Proof.* Let $I_p^\alpha$ be a component of $I_p$, and let $I_p^{-\alpha} = -I_p^\alpha$ be its opposite. Suppose that there is another component $I_p^\beta$ with $\beta \neq \alpha, -\alpha$, then $I^\beta$ cannot intersect neither $I_p^\alpha$ nor $I_p^{-\alpha}$ since the components of $I_p$ are disjoint, however it must intersect both of them. The contradiction proves that there are just two components. □

## 2. A proposal for a Finslerian gravitational dynamics

The variational principle

$$\delta \int \mathscr{L}(x, \dot{x}) \mathrm{d}t = 0,$$

gives the geodesic equation

$$\ddot{x}^\mu + 2G^\mu(x, \dot{x}) = 0,$$

where the spray is determined by

$$2G^\alpha(x, v) = g^{\alpha\eta}\Big(\frac{\partial^2 \mathscr{L}}{\partial x^\nu \partial v^\eta}\, v^\nu - \frac{\partial \mathscr{L}}{\partial x^\eta}\Big). \tag{3}$$

The geodesic equation determines the motion of free falling particles in the theory. As we mentioned a lot of work was devoted to the study of the Finslerian generalization of Einstein's equations. The main difficulty is that of finding a divergence free stress-energy tensor, so as to establish a suitable conservation law. This result has been obtained under additional conditions, e.g. for spaces of constant curvature [6] or for spaces of scalar curvature [9].

Let us write $G^\alpha_\beta$ for $\partial G^\alpha / \partial v^\beta$ and $G^\alpha_{\beta\gamma}$ for $\partial G^\alpha_\beta / \partial v^\gamma$. Let us define

$$\frac{\delta}{\delta x^\mu} := \frac{\partial}{\partial x^\mu} - G^\alpha_\mu(x, v) \frac{\partial}{\partial v^\alpha},$$

and

$$\Gamma^\alpha_{\mu\nu}(x, v) = \frac{1}{2}\, g^{\alpha\eta}\Big(\frac{\delta}{\delta x^\nu}\, g_{\eta\mu} + \frac{\delta}{\delta x^\mu}\, g_{\eta\nu} - \frac{\delta}{\delta x^\eta}\, g_{\nu\mu}\Big),$$

so that the spray (3) can be written $2G^\alpha = G^\alpha_{\beta\gamma} v^\beta v^\gamma = \Gamma^\alpha_{\beta\gamma} v^\beta v^\gamma$. The coefficients $G^\alpha_{\beta\gamma}$ and $\Gamma^\alpha_{\beta\gamma}$ are examples of horizontal connection coefficients of a Finsler connection: the Berwald connection in the former case and the Chern-Rund connection in the latter case.

We assume that the mean Cartan torsion vanishes

$$I_\gamma := \frac{1}{2} g^{\alpha\beta} \frac{\partial}{\partial v^\gamma}\, g_{\alpha\beta} = \frac{\partial}{\partial v^\gamma} \left( \ln \sqrt{|\det g|} \right), \tag{4}$$

namely that for every $x \in M$, the indicatrix $\{v :\ 2\mathscr{L}(x,v) = -1\}$ is an affine sphere of hyperbolic type.

*Remark* 2.1. These spaces are really important, indeed using some standard results in affine differential geometry [15] it is not difficult to show that they are equivalent to a pair given by a distribution of convex sharp cones (the future light cones) and a volume form (the spacetime measure).

Moreover, we assume that the space is Landberg, namely the Berwald horizontal derivative is compatible with the metric ($\nabla^{HB} g = 0$)

$$\frac{\delta}{\delta x^\mu}\, g_{\alpha\beta} - G^\nu_{\alpha\mu} g_{\nu\beta} - G^\nu_{\beta\mu} g_{\alpha\nu} = 0. \tag{5}$$

This condition is equivalent to $G^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma}$, that is, the Berwald and Chern-Rund connections coincide. In particular we can simply write $\nabla^H$ for the horizontal derivative. The HH-curvature of the Berwald-Chern-Rund connection is

$$R^\alpha_{\beta\gamma\delta} = \frac{\delta}{\delta x^\gamma}\, G^\alpha_{\beta\delta} - \frac{\delta}{\delta x^\delta}\, G^\alpha_{\beta\gamma} + G^\alpha_{\mu\gamma} G^\mu_{\beta\delta} - G^\alpha_{\mu\delta} G^\mu_{\beta\gamma}.$$

Whenever needed we shall lower the first index to the left. Our suggestion [16] for the Einstein tensor is

$$E_{\alpha\beta} := \frac{1}{2}(R_\alpha{}^\mu{}_{\beta\mu} + R_\beta{}^\mu{}_{\alpha\mu} - g_{\alpha\beta} R^{\mu\nu}{}_{\mu\nu}), \tag{6}$$

This tensor is symmetric and does indeed satisfy the conservation law (for a proof see [16])

$$\nabla^H_\beta (E^\beta{}_\alpha v^\alpha) = 0, \tag{7}$$

namely, the energy-momentum is conserved for all observers. Indeed, using a divergence theorem due to Rund [17] for any section $s\colon M \to TM\backslash 0$, and vector field $Z^\gamma(x,v)$

$$\nabla^{s^*g} \cdot s^* Z = s^*(\nabla^{HC} \cdot Z) + s^*(I_\beta Z^\gamma + \frac{\partial Z^\gamma}{\partial v^\beta}) D_\gamma s^\beta, \tag{8}$$

where $\nabla^{s^*g}$ is the Levi-Civita connection of the pullback metric $s^*g(x) := g(x, s(x))$, $\nabla^{HC}$ is the horizontal Cartan (or Chern-Rund) Finslerian covariant derivative and $D$ is the non-linear covariant derivative $D_\gamma s^\beta = \frac{\partial s^\beta}{\partial x^\gamma} + G^\beta_\gamma(x, s(x))$. Thus

$$\nabla^{s^*g}_\gamma (E^\gamma_\alpha(x, s(x)) s^\alpha) = s^*(\frac{\partial}{\partial v^\beta}(E^\gamma_\alpha(x,v) v^\alpha)) D_\gamma s^\beta, \tag{9}$$

which is the analog of the general relativistic almost conservation equation $(T^{\alpha\beta} u_\beta)_{;\alpha} = T^{\alpha\beta} u_{\beta;\alpha}$.

*Remark* 2.2. Concerning the problem of constructing dynamical equations for Finsler gravity which imply a conservation law, Ishikawa wrote [8]

> There have been several attempts to construct the Einstein tensor in a Finsler space itself. [. . . ] However, it seems difficult or seems to be impossible to construct the Einstein tensor according to this line of approach (by using the Finslerian Bianchi's identity).

We have solved the problem though we had to impose some additional conditions, namely Eqs. (4)-(5). We wish to remark a key idea which played an important role in our arguments. Contrary to previous approaches we did not try to find a divergence free stress-energy *tensor* but rather a divergence free energy-momentum *current* dependent on the observer. In fact, in our opinion what really matters is the map $v \mapsto \xi$ where $\xi^\alpha(v) = -E^\alpha_\beta(v)v^\beta$. One can even try to define the notion of *dominant energy condition* using this map (assume $\Lambda = 0$ for simplicity). There are two inequivalent possibilities: (a) at every $p \in M$ the map $v^\alpha \mapsto -E_{\alpha\beta}(v)v^\beta$ sends the future causal cone $J^+_p$ to its polar cone $(J^+_p)^*$, that is: for every $v, w$ f.d.-causal, $-E_{\alpha\beta}(v)v^\beta w^\alpha \le 0$; or (b) the map $v^\alpha \mapsto -E^\alpha_\beta(v)v^\beta$ sends $J^+_p$ into $J^+_p$.

Let us end this work comparing our equations with previous proposals. Let $Ric(v) = R^\alpha_{\ \beta\alpha\gamma}v^\beta v^\gamma$. It can be observed that upon contraction with $v^\alpha v^\beta$ our vacuum equation $E_{\alpha\beta} = 0$ gives

$$Ric(v) = 0. \tag{10}$$

Since $Ric(v)$ is positive homogeneous of degree two, Eq. (10) can also be rewritten

$$\frac{1}{2}\frac{\partial^2 Ric(v)}{\partial v^\alpha \partial v^\beta} = 0 \tag{11}$$

where the expression on the left-hand side is the Akbar-Zadeh Ricci tensor [18]. All authors seem to agree on the validity of this equation in the vacuum case since it is implied by almost every choice of dynamical equations that has been proposed so far, starting from the first proposal by Horvath [4]. Not all authors obtained this equation from a tensorial generalization of Einstein's. Rutz [19], for instance, argued for its validity using an analogy based on the the Jacobi deviation equation. Its consequences have also been explored by Li and Chang [20].

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