Journal of Physics: Conference Series **91** (2007) 012011 doi:10.1088/1742-6596/91/1/012011

The Poynting vector of a charged magnetic dipole: two limiting cases

J. Sod–Hoffs and V. S. Manko

Departamento de Física, Centro de Investigación y de Estudios Avanzados del I.P.N., A.P. 14-740, 07000 México D.F., Mexico

E-mail: `vsmanko@fis.cinvestav.mx, jordi@fis.cinvestav.mx`

Abstract. We consider the Poynting vector of two exact solutions describing a charged magnetized non–rotating mass in the following limiting cases: (i) $m^2 = q^2$, and (ii) $m = 0$. Whereas the former limit leads to a non–vanishing Poynting vector only for one of the solutions, the latter limit in both solutions results in non–zero expressions of the azimuthal component of the Poynting vector, thus providing evidence that Bonnor's frame–dragging effect takes place even in the case of a charged *massless* magnetic dipole.

1. Introduction

In a recent paper [1] it has been shown that in the case of stationary axially symmetric electrovac spacetimes the knowledge of the Ernst complex electromagnetic potential Φ [2] is sufficient to answer the question of whether or not the component S^φ of the Poynting vector $S^\alpha = (S^\rho, S^z, S^\varphi)$ is a zero quantity (two other components, S^ρ and S^z, are equal to zero identically). The formula obtained in [1] for S^φ reads

$$S^\varphi = \frac{\sqrt{f}\mathrm{e}^{-2\gamma}}{4\pi\rho}\mathrm{Im}(\bar{\Phi}_{,\rho}\Phi_{,z}), \tag{1}$$

where f and γ are the metric coefficients entering the Papapetrou line element [3]

$$\mathrm{d}s^2 = f^{-1}[e^{2\gamma}(\mathrm{d}\rho^2 + \mathrm{d}z^2) + \rho^2\mathrm{d}\varphi^2] - f(\mathrm{d}t - \omega\mathrm{d}\varphi)^2, \tag{2}$$

ρ and z are the Weyl–Papapetrou cylindrical coordinates, a bar over a symbol means complex conjugation, and a comma as subindex denotes partial differentiation. Note that the metric function ω does not enter the expression for S^φ.

In the ellipsoidal coordinates x and y defined by the relations

$$x = \frac{1}{2\kappa}(r_+ + r_-), \quad y = \frac{1}{2\kappa}(r_+ - r_-), \quad r_\pm = \sqrt{\rho^2 + (z \pm \kappa)^2}, \quad \kappa = \mathrm{const}, \tag{3}$$

formula (1) takes the form

$$S^\varphi = \frac{\sqrt{f}\mathrm{e}^{-2\gamma}}{4\pi\kappa^3(x^2 - y^2)}\mathrm{Im}(\bar{\Phi}_{,x}\Phi_{,y}), \tag{4}$$

while the line element (2) rewrites as

$$\mathrm{d}s^2 = \kappa^2 f^{-1}\left[\mathrm{e}^{2\gamma}(x^2 - y^2)\left(\frac{\mathrm{d}x^2}{x^2 - 1} + \frac{\mathrm{d}y^2}{1 - y^2}\right) + (x^2 - 1)(1 - y^2)\mathrm{d}\varphi^2\right] - f(\mathrm{d}t - \omega\mathrm{d}\varphi)^2. \tag{5}$$

Journal of Physics: Conference Series **91** (2007) 012011 doi:10.1088/1742-6596/91/1/012011

Formula (4) was applied in [1] to a particular metric [4] representing the field of a charged massive magnetic dipole. The resulting expression for S^φ, namely,

$$S^\varphi = \frac{qb\kappa^5(x^2-y^2)^3F(x,y)}{64\pi\sqrt{E}D^{5/2}(m^2-q^2)^6}, \tag{6}$$

contains the factor $(m^2-q^2)^6$ that suggests a possible singular behavior of S^φ when m tends to $\pm q$. This motivates us to analyze the limit $m^2=q^2$ in detail for clarifying the physical content of formula (6). In the next section we will show how such limit can be performed, yielding a specific electrostatic solution with vanishing Poynting vector. This, in turn, gives rise to a question about a possibility of performing the limit $m^2=q^2$ (which would result in a non–zero Poynting vector) making use of another known exact solution [5] for a charged massive magnetic dipole, and that question will be given a positive answer in section 3.

In the present article we also find it interesting to consider the limit $m=0$ (vanishing mass) in the solutions [4] and [5], in which case the latter solutions describe a charged *massless* magnetic dipole. We shall show that in this limit the component S^φ of the Poynting vector is a non–zero quantity for both solutions, thus underlying the purely electromagnetic nature of Bonnor's frame–dragging effect caused by a charged magnetic dipole [6].

2. Limits of the Poynting vector of MSM solution

We remind that the Ernst potentials and the corresponding metric functions f and γ of MSM solution for a charged massive magnetic dipole are defined by the expressions [4]

$$\begin{aligned}
\mathcal{E} &= \frac{A-B}{A+B}, \quad \Phi = \frac{C}{A+B}, \quad f=\frac{E}{D}, \quad \mathrm{e}^{2\gamma} = \frac{E}{16\kappa^8(x^2-y^2)^4},\\
A &= 2[(\kappa^2x^2-\delta y^2)^2-d^2]-2\mathrm{i}\kappa qbxy(1-y^2),\\
B &= m[2\kappa^3x(x^2-1)+(1-y^2)(2\kappa\delta x-\mathrm{i}qby)],\\
C &= 2\kappa^2(x^2-1)(\kappa qx+\mathrm{i}by)+(1-y^2)[2\kappa q\delta x-\mathrm{i}by(q^2-2\delta)],\\
D &= 4[(\kappa^2x^2-\delta y^2)^2+\kappa^3mx(x^2-1)+\kappa m\delta x(1-y^2)-d^2]^2\\
&\quad +q^2b^2y^2(2\kappa x+m)^2(1-y^2)^2,\\
E &= 4[\kappa^2(x^2-1)+\delta(1-y^2)]^4-4\kappa^2q^2b^2y^4(x^2-1)(1-y^2),\\
\kappa &= \sqrt{d+\delta}, \quad d=\frac{1}{4}(m^2-q^2), \quad \delta=\frac{b^2}{m^2-q^2},
\end{aligned} \tag{7}$$

where κ is the parameter involved in formulae (3), and the real parameters m, q and b are the mass, charge and magnetic dipole moment of the source, respectively. The non–zero component S^φ of the Poynting vector of the MSM solution is given by formula (6) in which $F(x,y)$ is some function of x and y.

The limit $m^2=q^2$ can most easily be worked out if one first considers the axis data of the solution (7) on the upper part of the symmetry axis, namely,

$$\mathcal{E}(\rho=0,z) = \frac{z(z-m)+d-\delta}{z(z+m)+d-\delta}, \quad \Phi(\rho=0,z) = \frac{qz+\mathrm{i}b}{z(z+m)+d-\delta}. \tag{8}$$

In the limit $m^2=q^2$ the parameter δ, as it follows from its definition, becomes an infinitely large quantity. However, after the redefinition of the magnetic dipole parameter b according to

$$b=\tilde{b}(m^2-q^2)^{1/2}, \tag{9}$$

Journal of Physics: Conference Series **91** (2007) 012011 doi:10.1088/1742-6596/91/1/012011

the limit $m^2 = q^2$ does not show any pathology and leads to the axis data

$$\mathcal{E}(\rho = 0, z) = \frac{z(z-q) - \tilde{b}^2}{z(z+q) - \tilde{b}^2}, \quad \Phi(\rho = 0, z) = \frac{qz}{z(z+q) - \tilde{b}^2}, \tag{10}$$

which define a specific electrostatic solution because $\mathcal{E}(\rho = 0, z)$ and $\Phi(\rho = 0, z)$ in (10) are real functions. Hence, we immediately infer that S^φ is equal to zero in this case, like in any other electrostatic case in general. This can be also verified directly by substituting formulae (7), in which the limit $m^2 = q^2$ must be performed after the redefinition (9) of the parameter b, into (4). It should be emphasized that vanishing of the magnetic field in the limit $m^2 = q^2$ must be considered only as a specific feature of MSM solution which is explained by certain restrictions on the parameters introduced during the process of construction of that solution, and this property is not necessarily shared by other solutions for a charged massive magnetic dipole (see next section).

Another limit which is likely to be considered for the Poynting vector of MSM solution corresponds to vanishing mass of the source, i.e., $m = 0$. This limit resulting in a non–zero expression for S^φ would mean that Bonnor's frame–dragging effect is originated exclusively by the electromagnetic part of the solution, and not by its "gravitational" component involving mass and angular momentum.

By setting $m = 0$ in the axis data (8), we get

$$\mathcal{E}(\rho = 0, z) = 1, \quad \Phi(\rho = 0, z) = \frac{qz + \mathrm{i}b}{z^2 - (q^2/4) + (b^2/q^2)}, \tag{11}$$

so that both the electric and magnetic fields are present in the axis expression of the potential Φ in (11).

To obtain a concise analytical expression defining the Poynting vector, we can restrict our consideration exclusively to the equatorial plane which in the ellipsoidal coordinates is given by $y = 0$. Then, taking the limit $m = 0$ in (7) and calculating the corresponding S^φ using formula (4), we arrive at the following elegant final expression:

$$S^\varphi = \frac{64q^7 b(q^4 + 4b^2)^3 x^6}{\pi[(q^4 + 4b^2)x^2 - q^4][(q^4 + 4b^2)x^2 - 4\kappa m q^2 x + q^4]^5}. \tag{12}$$

This clearly demonstrates the electromagnetic character of Bonnor's frame–dragging effect.

3. The case of Manko's solution

We now turn to another known exact solution for a charged massive magnetic dipole whose Ernst potentials and metric functions f and γ are defined by the expressions [5]

$$\mathcal{E} = \frac{A - B}{A + B}, \quad \Phi = \frac{C}{A + B}, \quad f = \frac{A\bar{A} - B\bar{B} + C\bar{C}}{(A + B)(\bar{A} + \bar{B})}, \quad \mathrm{e}^{-2\gamma} = \frac{16\kappa_+^4 \kappa_-^4 R_+ R_- r_+ r_-}{A\bar{A} - B\bar{B} + C\bar{C}},$$

$$\begin{aligned}
A = {} & \kappa_+^2[(m^2 - q^2 - b)(R_+ r_- + R_- r_+) + \mathrm{i}q\kappa_-(R_+ r_- - R_- r_+)] + \kappa_-^2[(m^2 - q^2 + b) \\
& \times (R_+ r_+ + R_- r_-) - \mathrm{i}q\kappa_+(R_+ r_+ - R_- r_-)] - 4b^2(R_+ R_- + r_+ r_-), \\
B = {} & m\kappa_+\kappa_-\{\kappa_+\kappa_-(R_+ + R_- + r_+ + r_-) - (m^2 - q^2)(R_+ + R_- - r_+ - r_-) \\
& + \mathrm{i}q[(\kappa_+ - \kappa_-)(R_+ - R_-) - (\kappa_+ + \kappa_-)(r_+ - r_-)]\}, \\
C = {} & \kappa_+\kappa_-\{q\kappa_+\kappa_-(R_+ + R_- + r_+ + r_-) - q(m^2 - q^2)(R_+ + R_- - r_+ - r_-) \\
& + \mathrm{i}[\kappa_+(q^2 + b)(R_+ - R_- - r_+ + r_-) - \kappa_-(q^2 - b)(R_+ - R_- + r_+ - r_-)]\},
\end{aligned} \tag{13}$$

with

$$R_\pm = \sqrt{\rho^2 + [z \pm \tfrac{1}{2}(\kappa_+ + \kappa_-)]^2},$$
$$r_\pm = \sqrt{\rho^2 + [z \pm \tfrac{1}{2}(\kappa_+ - \kappa_-)]^2},$$
$$\kappa_\pm = \sqrt{m^2 - q^2 \pm 2b}. \tag{14}$$

The interpretation of the parameters m, q and b in the above formulae is the same as in MSM solution, i.e., the mass, charge and magnetic dipole moment, respectively.

On the upper part of the symmetry axis the potentials $\mathcal{E}$ and Φ from (14) assume the form

$$\mathcal{E}(\rho=0,z) = \frac{z-m}{z+m}, \quad \Phi(\rho=0,z) = \frac{qz + \mathrm{i}b}{z(z+m)}, \tag{15}$$

and one can see that, unlike in the axis data (8), the constants m and q in (15) admit arbitrary relations between them without annihilating the magnetic field.

We are especially interested in the limit $m^2 = q^2$ which led to a zero Poynting vector in the case of MSM solution. The axis data (15) hint that now such limit is feasible and might lead to a non–zero value of S^φ. To verify this, let us assume for definiteness that $q > 0$ and $b > 0$; also, like in the previous section, we shall be calculating S^φ in the equatorial plane $z = 0$ in order to obtain a concise analytic expression for the azimuthal component of the Poynting vector. Then, carrying out the limit $m = q$ in the formulae (14) and calculating the corresponding S^φ with the aid of formula (1), we arrive, after some tedious but straightforward calculations, to the desired result

$$S^\varphi = -\frac{16q[b(R_+ + r_+ + 2q) + \mathrm{i}q^2(r_+ - R_+)]}{\pi(R_+ + r_+)^2(R_+ + r_+ + 2q)^5}, \tag{16}$$

where we have taken into account that $\kappa_+ = \sqrt{2b}$, $\kappa_- = \mathrm{i}\sqrt{2b}$ in the limit $m = q$ (and under assumption of the positiveness of b); besides, in the equatorial plane in our case the equalities $R_- = R_+$ and $r_- = r_+$ take place. We point out that by further setting $q = 0$ or $b = 0$ in (16), one gets $S^\varphi = 0$, the fact apparent in the limit $q = 0$, whereas in the limit $b = 0$ the annulling of S^φ is due to vanishing of the factor $(r_+ - R_+)$ in the numerator of (16) (note that $\mathrm{i}(r_+ - R_+)$ is a real quantity). Therefore. we have shown that in the limit $m^2 = q^2$, frame–dragging by a charged magnetic dipole can also take place, provided this limit does not lead to vanishing of the electric or magnetic field.

Another limit, $m = 0$, can be worked out too using formulae (14). In this case representing a charged *massless* magnetic dipole the axis data (15) reduces to

$$\mathcal{E}(\rho=0,z) = 1, \quad \Phi(\rho=0,z) = \frac{qz + \mathrm{i}b}{z^2}, \tag{17}$$

showing that both the electric and magnetic fields survive this limit which, in turn, suggests the non–zero character of the corresponding component S^φ of the Poynting vector. To verify this, we calculate the principal electromagnetic part, $\mathrm{Im}(\bar\Phi_{,\rho}\Phi_{,z})$, of S^φ in (1), the result being

$$\mathrm{Im}(\bar\Phi_{,\rho}\Phi_{,z}) = \frac{qb(4b^2 - q^4)\rho}{4(\rho^4 - q^2\rho^2 + b^2)[(q^4 - 2b^2)R_+ r_+ - 2b^2\rho^2 + q^2 b^2]^4}$$
$$\times[\kappa_+\kappa_-(R_+ + r_+)(q^4\rho^2 - 4b^2\rho^2 - q^2b^2) + (R_+ - r_+)(q^6\rho^2 + q^4b^2 + 4b^4)]^2. \tag{18}$$

Like in the 'massless' case of MSM solution considered in the previous section, we have arrived at the non–zero expression for the Poynting vector, having this time the solution (14) as a starting point. It should be noted that the case $q^4 = 4b^2$ in (18) does not cause the Poynting vector to vanish since, as can be seen from the definition of $\kappa_\pm$, $R_\pm$, $r_\pm$ in (14), the special choice $m = 0$, $2b = \pm q^2$ of the parameters makes κ_+ or κ_- equal to zero. Hence we arrive at a particular case which can be considered as a sort of an extreme one because $R_+ = r_+$, $R_- = r_-$ for $2b = -q^2$, and $R_+ = r_-$, $R_- = r_+$ for $2b = q^2$ independently of the value of z, so that the corresponding limit needs a more subtle tackling. If, for instance, $2b = -q^2$, i.e., $\kappa_- = 0$, then formula (18) rewrites, after appropriately performing in it the limit $b = -q^2/2$, as

$$\mathrm{Im}(\bar{\Phi}_{,\rho}\Phi_{,z}) = -256q^5\rho/R_+^6, \tag{19}$$

once again getting a non–zero value for the Poynting vector.

4. Conclusion

In this article, which can be considered as a useful complement of the paper [1], we have considered two limits in the formulae for the azimuthal component of the Poynting vector employing the known exact solutions for a charged massive magnetic dipole. The analysis carried out in sections 2 and 3 permits us to conclude that the 'mass equal to charge' limit results in a non–vanishing Poynting vector only for one of the solutions, and the zero value of S^φ in the case of MSM solution is explained by a specific choice of the parameters in the axis data (8) leading to vanishing of the magnetic field in the limit $m^2 = q^2$. On the other hand, the *massless* limit of a charged magnetic dipole yields a non–zero expression of S^φ for both solutions, thus clearly demonstrating the electromagnetic nature of Bonnor's frame–dragging effect.

Acknowledgments

We are thankful to Máximo López–López for drawing our attention to the possible singular behavior of formula (6) in the limit $m \to \pm q$. This work was supported by Project 45946–F from CONACyT of Mexico.

References

[1] Manko V S, Rodchenko E D, Sadovnikov B I and Sod–Hoffs J 2006 *Class Quantum Grav.* **23** 5389
[2] Ernst F J 1968 *Phys. Rev.* **168** 1415
[3] Papapetrou A 1953 *Ann. Physik* **12** 309.
[4] Manko V S, Sanabria–Gómez J D and Manko O V 2000 *Phys. Rev.* D **62** 044048
[5] Manko V S 1993 *Phys. Lett.* A **181** 349
[6] Bonnor W B 1991 *Phys. Lett.* A **158** 23