TENSOR PRODUCTS FOR AFFINE KAC-MOODY ALGEBRAS

R.C. King and T.A. Welsh
Mathematics Department, University of Southampton,
Southampton, SO9 5NH England

Following Kac [1], let $\mathcal{G} = \mathcal{G}(A)$ be the affine Kac-Moody algebra of rank ℓ associated with a symmetrisable generalised Cartan matrix A having matrix elements $A_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle = 2(\langle \alpha_i, \alpha_j \rangle)/(\langle \alpha_j, \alpha_j \rangle)$ for $i, j \in I$ where $I = \{0, 1, \ldots, \ell\}$. Each simple root α_i, with $i \in I$ lies in $\mathcal{H}^*$, the dual of the Cartan subalgebra $\mathcal{H}$ of $\mathcal{G}$. The corresponding co-roots are defined by $\alpha_i^\vee = 2\alpha_i/\langle \alpha_i, \alpha_i \rangle$ for $i \in I$.

It is convenient to introduce vectors δ and ω_i, with $i \in I$, which together span $\mathcal{H}^*$. In terms of the integer marks c_i and co-marks $c_i^\vee$ for $i \in I$ we have $\delta = \sum_{i=0}^{\ell} c_i \alpha_i = \sum_{i=0}^{\ell} c_i^\vee \alpha_i^\vee$. The marks are chosen [1] so that $\langle \delta, \alpha_j^\vee \rangle = 0$ for $j \in I$. In addition $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$ for $i, j \in I$, $\langle \omega_i, \omega_0 \rangle = 0$ for $i \in I$, and $\langle \delta, \omega_0 \rangle = c_0^\vee$. Every weight $\lambda \in \mathcal{H}^*$ may then be expressed in the form $\lambda = \sum_{i=0}^{\ell} \lambda_i \omega_i - k\delta = (\lambda_0, \lambda_1, \ldots \lambda_\ell; k)$ where the Dynkin components of λ are given by $\lambda_i = \langle \lambda, \alpha_i^\vee \rangle$ for $i \in I$. The level and depth of the weight λ are defined by $L(\lambda) = \langle \lambda, \delta \rangle = \sum_{i=0}^{\ell} c_i^\vee \lambda_i$ and $D(\lambda) = -(c_0^\vee)^{-1} \langle \lambda, \omega_0 \rangle = k$. A weight $\lambda \in \mathcal{H}^*$ is said to belong to the set of integral weights, P, if $\langle \lambda, \alpha_i^\vee \rangle \in \mathbf{Z}$ for $i \in I$. Such an integral weight λ is dominant and in P_+ if $\langle \lambda, \alpha_i^\vee \rangle \geq 0$ for $i \in I$, and strongly dominant and in P_{++} if $\langle \lambda, \alpha_i^\vee \rangle > 0$ for $i \in I$.

The Weyl group, W, of $\mathcal{G}$ is generated by the Weyl reflections whose action on $\mathcal{H}^*$ are defined by $r_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i$ for $i \in I$. The orbit of λ under the Weyl group action is the set $W_\lambda = \{w\lambda \mid w \in W\}$. For each λ in P there exists a unique dominant weight λ^+ in P_+ such that $\lambda^+ = w_\lambda \lambda$ for some $w_\lambda \in W$. If λ^+ is in P_{++} then all the elements $w\lambda$ with $w \in W$ are distinct, and in particular w_λ is unique.

The null depth of λ is then defined to be $d(\lambda) = D(\lambda) - D(\lambda^+)$. By exploiting the Weyl group invariance, which implies $\langle \lambda^+, \lambda^+ \rangle = \langle w_\lambda \lambda, w_\lambda \lambda \rangle = \langle \lambda, \lambda \rangle$, it is possible to show that $d(\lambda) = (1/2L(\lambda)) \sum_{i,j=1}^{\ell} (G_{ij}\ (\lambda_i \lambda_j - \lambda_i^+ \lambda_j^+))$, where $G = S^{-1}$ with $S_{ij} = c_i (c_i^\vee)^{-1} A_{ij}$ for $i, j \in I_+ = \{1, 2, \ldots, \ell\}$.

Each irreducible highest weight integrable module V^λ of $\mathcal{G}$ is labelled by a dominant integral weight λ. Such a module has a weight space decomposition $V^\lambda = \oplus_{\kappa \in \mathcal{H}^*} V_\kappa^\lambda$ and the character of this module is formally given by $\operatorname{ch} V^\lambda = \sum_{\kappa \in P} m_\kappa^\lambda e^\kappa$, where the weight multiplicity m_κ^λ is the dimension of V_κ^λ. Kac [1] has established the character formula:

$$\operatorname{ch} V^\lambda = \sum_{w \in W} \epsilon(w) e^{w(\lambda+\rho)} \Big/ \sum_{w \in W} \epsilon(w) e^{w\rho}, \tag{1}$$

where $\rho \in \mathcal{H}^*$ is defined by $\rho = \sum_{i=0}^{\ell} \omega_i$ so that $\langle \rho, \alpha_i^\vee \rangle = 1$ for $i \in I$.

The tensor product $V^\mu \otimes V^\nu$ of two irreducible integrable highest weight modules of $\mathcal{G}$ is fully reducible into a direct sum of such modules. If the multiplicity of occurrence of modules V^λ in this tensor product is denoted by $g_{\mu\nu}^\lambda$ then $\operatorname{ch} V^\mu \operatorname{ch} V^\nu =$

$\sum_{\lambda\in P_+} g^\lambda_{\mu\nu}\,\mathrm{ch}\,V^\lambda$. Two problems which immediately present themselves are the explicit evaluation of the weight multiplicities m^λ_κ and the explicit evaluation of the tensor product multiplicities $g^\lambda_{\mu\nu}$. In what follows it is demonstrated not only that these problems may be solved algorithmically but also that they are intimately connected. Illustrations are confined for simplicity to the case $\mathcal{G} = A_1^{(1)}$.

The formal definition of $\mathrm{ch}\,V^\mu$ and the character formula (1) for $\mathrm{ch}\,V^\nu$ and $\mathrm{ch}\,V^\lambda$ imply

$$\sum_{\kappa\in P} m^\mu_\kappa e^\kappa \sum_{v\in W} \epsilon(v) e^{v(\nu+\rho)} = \sum_{\lambda\in P_+} g^\lambda_{\mu\nu} \sum_{w\in W} \epsilon(w) e^{w(\lambda+\rho)}. \tag{2}$$

For any $\lambda \in P_+$ it follows that $\lambda+\rho \in P_{++}$. Moreover $w(\lambda+\rho)\in P_{++}$ if and only if w is the identity element of W. Setting $\kappa = v\sigma$, using the fact that $m^\mu_{v\sigma} = m^\mu_\sigma$ and picking out those terms on both sides of (2) involving e^η with $\eta \in P_{++}$ leads to the identity

$$\sum_{\substack{\sigma\in P\\ (\sigma+\nu+\rho)^+\in P_{++}}} \epsilon(w_{\sigma+\nu+\rho}) m^\mu_\sigma e^{(\sigma+\nu+\rho)^+} = \sum_{\lambda\in P_+} g^\lambda_{\mu\nu} e^{\lambda+\rho}. \tag{3}$$

This identity provides a geometric procedure for determining the tensor product multiplicities $g^\lambda_{\mu\nu}$ from the weight multiplicities m^μ_σ of just one of the constituent irreducible modules in the product. Its use is a straightforward generalisation of a very well known technique [2,3,4] developed in the context of finite dimensional Lie algebras. The identity (3) actually provides an explicit formula [3] for tensor product multiplicities:

$$g^\lambda_{\mu\nu} = \sum_{\substack{\sigma\in P\\ (\sigma+\nu+\rho)^+=\lambda+\rho}} \epsilon(w_{\sigma+\nu+\rho}) m^\mu_\sigma = \sum_{w\in W} \epsilon(w) m^\mu_{\lambda+\rho-w(\nu+\rho)}, \tag{4}$$

where the sum over σ has been replaced in the second expression by a sum over $w \in W$ since the set of elements $\sigma+\nu+\rho$ such that $(\sigma+\nu+\rho)^+ = \lambda+\rho$ is precisely $W(\lambda+\rho)$. Use has also been made the fact that $m^\mu_\sigma = m^\mu_{w\sigma}$.

This formula not only allows the explicit calculation of tensor product multiplicities from a knowledge of weight multiplicities but also the converse. Indeed setting $\nu = 0$, so that $V^\nu = V^0$ is the trivial one-dimensional module and $g^\lambda_{\mu\nu} = g^\lambda_{\mu 0} = \delta^\lambda_\mu$, and taking $\lambda \neq \mu$ in (4) gives Racah's familiar recurrence relation [3] for weight multiplicities: $\sum_{w\in W} \epsilon(w) m^\mu_{\lambda+\rho-w\rho} = 0$. However, (4) can be used as it stands as a tool for determining weight multiplicities from tensor product multiplicities.

By way of illustration, in the case of $g = A_1^{(1)}$ we have $A = \begin{pmatrix} 2 & -2\\ -2 & 2\end{pmatrix}$, so that $S = (2)$, $G = (\frac{1}{2})$ and $d(\mu) = \frac{1}{4L(\mu)}(\mu_1^2 - \mu_1^{+2})$. For the simplest non-trivial module, $V^{(1,0;0)}$ with highest weight $\mu = \omega_0$, the weights are all of the form $\sigma = (1-2m)\omega_0 + 2m\omega_1 - p\delta$ with $m \in \mathbf{Z}$ and $p \in \mathbf{Z}_+$. The top ten rows of the infinite weight diagram are shown below. Ignoring for the moment the foot of the table, the Weyl reflection planes are the vertical lines through $\theta_0 = (0,1)$ and $\theta_1 = (1,0)$. All weights $\sigma = (1-2m, 2m; p)$ on any vertical string in a column labelled by m are Weyl equivalent to those on the one string in the dominant sector labelled by $m = 0$. In fact $\sigma^+ = (1,0;k)$ with $k = p - m^2$, since $d(\sigma) = \frac{1}{4}(\sigma_1^2 - \mu_1^2) = m^2$. The weight multiplicities themselves are given by $m^\mu_\sigma = a_{p-m^2}$ where $a_n = p(n)$, the number of partitions of n, as can be shown through the use of Racah's recurrence relation [5] or otherwise [1].

$m=$	$\cdots$	-3	-2	-1	0	1	2	3	$\cdots$
(σ_0,σ_1)	$\cdots$	$(7,-6)$	$(5,-4)$	$(3,-2)$	$(1,0)$	$(-1,2)$	$(-3,4)$	$(-5,6)$	$\cdots$
$d(\sigma)$		9	4	1	0	1	4	9	
					θ_1 ↓ θ_0 ↓				
$p=0$					1				
$p=1$				1	1	1			
$p=2$				1	2	1			
$p=3$				2	3	2			
$p=4$			1	3	5	3	1		
$p=5$			1	5	7	5	1		
$p=6$			2	7	11	7	2		
$p=7$			3	11	15	11	3		
$p=8$			5	15	22	15	5		
$p=9$		1	7	22	30	22	7	1	
				↑ ϕ_1			↑ ϕ_0		
$d(\zeta)$		–	0	0	0	0	–	2	
(ζ_0,ζ_1)	$\cdots$	$(10,-5)$	$(8,-3)$	$(6,-1)$	$(4,1)$	$(2,3)$	$(0,5)$	$(-2,7)$	$\cdots$

The above weight diagram may then be used to calculate, for example, the tensor product multiplicities for $V^{(1,0;0)} \otimes V^{(2,0;0)}$. The procedure based directly on (3) involves shifting the weight diagram of $V^{(1,0;0)}$ through $\nu+\rho = (3,1;0)$ so that $\sigma = (1-2m, 2m; p)$ goes to $\zeta = (4-2m, 1+2m; p)$ with $L(\zeta) = 5$. This can be effected by the relabelling given at the foot of the above diagram. The reflection planes are now at the positions $\phi_0 = (0,5)$ and $\phi_1 = (5,0)$. All weights ζ on any vertical string either lie on a Weyl reflection plane or are such that $\zeta^+ = (4,1;k)$ or $(2,3;k)$ for some k, with $d(\zeta) = \frac{1}{5}m(m+1)$ or $\frac{1}{5}(m^2+m-2)$, respectively. Carrying out the Weyl reflections for each vertical string, taking signatures into account and subtracting $\rho = (1,1;0)$ gives the following tensor product multiplicities $g_{(1,0;0)(2,0;0)}^{(\lambda_0,\lambda_1;k)}$:

	$(\lambda_0,\lambda_1) = (3,0)$	$(\lambda_0,\lambda_1) = (1,2)$
$k=0$	1	
$k=1$	$1-1=0$	1
$k=2$	$2-1=1$	1
$k=3$	$3-2=1$	2
$k=4$	$5-3=2$	$3-1=2$
$k=5$	$7-5=2$	$5-1=4$
$k=6$	$11-7=4$	$7-2=5$
$k=7$	$15-11=4$	$11-3-1=7$

Since tensor products are commutative, the same multiplicities must arise if the problem is approached in the same way but starting from the weight diagram of the module $V^{(2,0;0)}$. This takes the following form in which reflections in the planes signified by $\psi_0 = (0,2)$ and $\psi_1 = (2,0)$ have been used to parametrise the weight multiplicities in terms of those in the dominant sector. The weights are all of the form $\tau = (2-2m, 2m; p)$, with $L(\tau) = 2$ and $d(\tau) = \frac{1}{2}m^2$ or $\frac{1}{2}(m^2-1)$, according as m is even or odd.

$m =$	-4	-3	-2	-1	0	1	2	3	4
(τ_0, τ_1)	$\cdots$	$(8,-6)$	$(6,-4)$	$(4,-2)$	$(2,0)$	$(0,2)$	$(-2,4)$	$(-4,6)$	$(-6,8)$
$d(\tau)$	8	4	2	0	0	0	2	4	8
					ψ_1 $\downarrow$	ψ_0 $\downarrow$			
$p=0$					a_0				
$p=1$				b_1	a_1	b_1			
$p=2$			a_0	b_2	a_2	b_2	a_0		
$p=3$			a_1	b_2	a_3	b_3	a_1		
$p=4$			a_2	b_4	a_4	b_4	a_2		
$p=5$		b_1	a_3	b_5	a_5	b_5	a_3	b_1	
$p=6$		b_2	a_4	b_6	a_6	b_6	a_4	b_2	
$p=7$		b_3	a_5	b_7	a_7	b_7	a_5	b_3	
$p=8$	a_0	b_4	a_6	b_8	a_8	b_8	a_6	b_4	a_0
				$\uparrow$ ϕ_1			$\uparrow$ ϕ_0		
$d(\zeta)$	2	$-$	0	0	0	0	$-$	2	4
(ζ_0, ζ_1)	$\cdots$	$(10,-5)$	$(8,-3)$	$(6,-1)$	$(4,1)$	$(2,3)$	$(0,5)$	$(-2,7)$	$(-4,9)$

Proceeding as in the previous example on the basis of (3) with μ and ν interchanged, now involves adding $\mu + \rho = (2,1;0)$ to the weights τ to give ζ. Of course the reflection planes signified by ϕ_0 and ϕ_1 and $d(\zeta)$ are exactly as before. Carrying out the reflections, taking signatures into account and subtracting ρ leads to expressions for the tensor product multiplicities which may be solved recursively for the weight multiplicities a_k and b_k of the dominant weights $(2,0;k)$ and $(0,2;k)$ of $V^{(2,0;0)}$ as shown below.

	$(\lambda_0, \lambda_1) = (3,0)$	$(\lambda_0, \lambda_1) = (1,2)$		
$k=0$	$1 = a_0$		$a_0 = 1$	
$k=1$	$0 = a_1 - b_1$	$1 = b_1$	$a_1 = 1$	$b_1 = 1$
$k=2$	$1 = a_2 - b_2$	$1 = b_2 - b_0 - a_0$	$a_2 = 3$	$b_2 = 2$
$k=3$	$1 = a_3 - b_3$	$2 = b_3 - b_1 - a_1$	$a_3 = 5$	$b_3 = 4$
$k=4$	$2 = a_4 - b_4 - a_0$	$2 = b_4 - b_2 - a_2$	$a_4 = 10$	$b_4 = 7$
$k=5$	$2 = a_5 - b_5 - a_1$	$4 = b_5 - b_3 - a_3$	$a_5 = 16$	$b_5 = 13$

REFERENCES

[1] V.G.Kac, *Infinite dimensional Lie algebras* (Boston, Mass.: Birkhauser, 1983)

[2] A.U. Klimyk, *Amer. Math. Soc. Transl. Series 2 Vol 76* (Providence, RI.: Amer. Math. Soc., 1968)

[3] G. Racah, in *Group theoretical concepts and methods in elementary particle physics*, Ed. F. Gursey (New York, NY.: Gordon and Breach, 1964) pp1-36

[4] D. Speiser, in *Group theoretical concepts and methods in elementary particle physics*, Ed. F. Gursey (New York, NY.: Gordon and Breach, 1964) pp237-246

[5] A.J. Feingold and J. Lepowsky, *Adv. Math.* **29**, 271-309 (1978)