Renormalization from Classical to Quantum Physics

by

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To my parents: Mita and Asok Kar

and

in memory of my brother: Kaushik Sinha

BIOGRAPHICAL SKETCH

The author was born in the small town of Durgapur, India in November, 1987. He attended Chennai Mathematical Institute from 2006 to 2009 and graduated with a Bachelor of Science degree with honors in Physics. In Fall, 2009 he began his doctoral studies in mathematical physics at the University of Rochester under the supervision of Professor S. G. Rajeev. He was awarded a Master of Arts degree in Physics in 2011. He also received the Agnes M. and George Messersmith Fellowship in 2013, awarded by the University of Rochester. To add some excitement to the weary winters of Rochester, he taught a course in Nuclear Physics for the undergraduates at Rochester Institute of Technology in Spring, 2014.

Publications

- A. Kar, and S. G. Rajeev, Ann. Phys. **326**, 958 (2011).
- A. Kar, and S. G. Rajeev, Ann. Phys. **327**, 102 (2012).
- A. Kar, and S. G. Rajeev, AMS Contemporary Math. Series **583**, 247 (2012).
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- A. Kar, F. Moolekamp, and S. G. Rajeev, Mod. Phys. Lett. A **28**, 1350139 (2013).

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"Life is full of surprises, but never when you need one."

- BILL WATTERSON

ABSTRACT

The concept of renormalization was first introduced by Dirac to investigate the infinite self energy of an electron classically. This radical theory was probably the first time when an infinity occurring in a physical system was systematically investigated. This thesis presents a new perspective of renormalization by introducing methods from metric geometry to control divergences.

We start by extending Dirac's work and analyzing how the radiation reaction due to the precision of the electron's magnetic moment affects its motion. This is followed by modeling scalar field theory on lattices of various kinds. Scale invariance, which plays a major role in the very few renormalizable theories in nature, is inbuilt in our formalism. We also use Wilson's ideas of effective theory and finite element methods to study continuum systems. Renormalization group transformations form the central theme in this picture. By incorporating finite element methods, an idea borrowed from mechanical engineering, we study scalar fields on triangular lattices in a hierarchal manner. In our case, the cotangent formula turns out to be a fixed point of the renormalization group transformations. We end our thesis by introducing a new metric for space-time which emerges from the scalar field itself. The standard techniques used in the theory of renormalization so far attempt to redefine coupling constants of the theory to remove divergences at short distance scales. In our formalism, we deduce the distance scale itself. In our notion of distance, built from correlation functions of the fields, the divergences disappear.

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LIST OF SYMBOLS

$\mathbb{Z}$	Integers
$\mathbb{R}$	Real Numbers
$\hat{R}_0$	Resolvent Operator
$\hat{H}_0$	Differential Operator
$\mathcal{L}$	Lagrangian of a physical system
S	Action of a physical system
$\Gamma(z)$	Gamma Function
$\psi(z)$	Digamma Function
σ_i	Spin at site i
$F^{\mu\nu}$	Electromagnetic Field Tensor
$\tilde{f}$	Fourier Transform of function f
$f[x]$	Functional in x
$[\mathcal{D}x]$	Functional Path Integral in x
$\phi(x)$	Value of Scalar Field ϕ at x
$\langle . \rangle$	Expectation value
QFT	Quantum Field Theory
FEM	Finite Element Method
RG	Renormalization Group

CHAPTER 1

INTRODUCTION

> *"I must say that I am very dissatisfied with the situation, because this so-called 'good theory' does involve neglecting infinities which appear in its equations, neglecting them in an arbitrary way. This is just not sensible mathematics. Sensible mathematics involves neglecting a quantity when it is small - not neglecting it just because it is infinitely great and you do not want it!"*
>
> — P. A. M. Dirac, *Directions in Physics*

The standard model of particle physics stands out among other theories in physics not only because of its comprehensive explanation of the forces of nature but also for the rigorous experimental verifications that it has undergone. Since ancient times, physicists have been trying to understand the elementary composition of matter and the forces governing them. After all, we now know that Democritus's atomic hypothesis is not accurate. Our knowledge about the basic building blocks of matter has gone a lot of change and a good understanding of it was proposed by the standard model in the 1970s. It captures the fundamental forces of nature- electromagnetic, weak and strong nuclear interactions except gravity.

The recent discovery of the Higgs et al. boson in 2012, followed by the Nobel prize in 2013, provides the missing piece of the puzzle in the standard model. All the theoretically predicted parameters in the Lagrangian of the standard model have now

been verified experimentally over the past fifty years. Does this bring us to an end for the quest of the fundamental forces? Well, apparently not. The discoveries so far open up a chestful of questions to be solved in years to come. For example, attempts are being made to find an unified theory which can explain gravity as well along with the three other fundamental forces of nature.

We shall however take a step back and try to address the mathematical formalism on which the standard model stands, namely quantum field theory (QFT). QFT started off as an offshoot of relativistic quantum mechanics. Such a system can create or destroy particles via interactions. These interactions could not be explained by quantum mechanics as the particle number of the system was no longer conserved. The infinite degrees of freedom involved in the system called for a new theoretical development. Though QFT passed the first barrier to explain particle interactions, it however gave rise to divergences. Theoretical predictions for physically observed quantities were infinite and an elaborate scheme called renormalization had to be introduced to reconcile the theoretical predictions and experimental observations.

Quantum field theory is the calculus of the microworld [1]. It can be viewed as the theory of interacting quantum fields which includes virtual transitions of quantum fields. For example, the electromagnetic fields in nature can be quantized and thought of as particle-like photons. Interactions of the electromagnetic field with matter can be visualized as interactions with these quanta of fields with matter. Each of these interactions has a nice diagrammatic representation which goes by the name of Feynman diagrams. However, the presence of virtual states leads to infinities in the calculations. The emergence of these infinities remained a puzzle for quite sometime.

Following an idea by Dirac, the first step towards solving the puzzle was finally laid by Bethe, Schwinger, Feynman and Dyson in 1940s. They showed that the infinities could be avoided by redefining the coupling constants and masses of the theory. The

process of redefinition of the parameters of the model became famous as renormalization in years to come. Even though it is mathematically quite cumbersome, it led to physical theories of unprecedented accuracy and elegance. The magnetic moment of the electron was calculated using this formalism to an accuracy of 15 decimal places, and agrees with experiments.

Another renormalizable theory is the Yang-Mills theory. This non-abelian gauge theory describes the strong forces in the standard model. The "Higgs" mechanism, which assigns masses to the force carriers in the standard model is also a renormalizable theory. These theories in nature stand on an elegant, yet intricate model. The renormalizable theories have other issues which need to be addressed now. The Yang-Mills theory has the million dollar prize associated with it, to address the mass gap problem. The mass gap problem in the theory deals with the existence and evaluation of the ground state and first excited state energies of the theory. The "Higgs" mechanism on the other hand has a naturalness problem/hierarchy problem associated to it. The mass of the Higgs et al. boson calculated in this theory is lighter than the Planck mass by many orders of magnitude. It is puzzling that the weak forces of nature mediated by the Higgs et al. boson are so much stronger than the gravitational forces.

A major development to the story of renormalization happened in the early 1950s with the introduction of Renormalization Group methods. It became easier to analyze ultraviolet and infrared divergences. Using these algorithms, one could find the changes in the coupling constants with changing energy scales. These ideas were put on a solid theoretical footing by Wilson in 1970s. By discretizing space to solve problems in statistical mechanics, he showed that the renormalization group was a discrete semigroup. He introduced the idea of effective field theory. One could isolate the most important degrees of freedom of a system and average out the other

irrelevant degrees of freedom to find an effective theory. In this process, the coupling constants of the theory scale under the renormalization group transformations. This approach explained phase transitions. Even though space was discretized in the model, phase transitions in critical phenomena turn out to be independent of the microscopic details of the system such as lattice spacings. The fixed points of these transformation have a physical relevance. They are the critical exponents in the theory of phase transitions.

Wilson found the connection between the existing ideas of renormalizability and renormalization group and sewed them into one framework. The two crucial aspects of his formalism were that of understanding a continuum process in finite steps, by discretization and to discard the impertinent degrees of freedom by an averaging process to get an effective model. The renormalization group transformations played a crucial role to execute these ideas.

We try to imbibe Wilson's approach of finite element methods in our work. The other important feature of his work is that of scale invariance. Both Yang-Mills theory and the "Higgs" mechanism in four dimensions have coupling constants which are dimensionless and hence scale invariant. The Wiener process in one dimension, which models Brownian motion, is a scale invariant process. In chapters 3 and 4, we used these ingredients of scale invariance and finite element methods to study scalar fields on lattices. We used a Gaussian distribution similar to the Wiener process in our studies.

In chapter 5, however, we introduce a new perspective for renormalization. So far, we have known that mass influences the metric of space-time. We showed that the correlation functions in scalar fields itself give rise to a new notion of distance. The correlation functions in chapter 4 motivated this study. This metric has the advantage that it is devoid of divergences. Most of the divergences in QFT arise

from the correlation functions blowing up at short distance scales. In this new metric instead of changing the parameters of the theory, the new measure of distance is devoid of such divergences.

Though renormalization has been well accepted at this point, it is still not understood very well. Since it saw a significant development in the context of QFT, a common misconception which prevails is that it arises only in QFT. This is actually not true and regularization methods occur in other areas as well. We list a few examples in Sec. 1.1. In fact, the first example of renormalization was given by Dirac in his study of the motion of an electron in an electromagnetic field, taking into account its self energy. Feynman mentioned this problem in his Nobel lecture in 1965. He mentioned that infinite energy of interaction of the electron with itself existed in the classical theory. Dirac's solution for the problem by redefining the mass of the electron however lead to certain physically inconsistent solutions. In chapter 2, we discuss Dirac's approach and the ideas that followed. The magnetic moment of the electron was included in our work, which was missing in the story.

1.1 Renormalization beyond Field Theory

Infinities do show up in disciplines other than QFT and one has to implement corresponding regularization techniques to give a sensible meaning to divergent quantities. We shall highlight examples from mathematics, quantum mechanics where renormalization techniques play a major role.

The Gamma Function in Mathematics

Divergent sums and integrals occur in mathematics and physics. To avoid them, one has to implement unintuitive methods to deduce finite values for divergent quantities. The Gamma function serves as a good toy model for that [2]. The Euler constant γ which is associated with the Gamma function illustrates how the difference of two divergent quantities can still lead to a finite value.

The Gamma function is defined with the recurrence relation,

$$\Gamma(z+1) = z\,\Gamma(z), \quad \text{and} \quad \Gamma(1) = 1. \tag{1.1.1}$$

$$\Gamma(z) = \frac{1}{z(z+1)\dots(z+k-1)}\Gamma(z+k) \tag{1.1.2}$$

follows from (1.1.1). This leads to the observation that it has poles at non-positive integers ($z = 0, -1, -2, \dots$). An alternative definition of the Gamma function which highlights this fact as well is-

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^z}{z(z+1)\dots(z+n)}. \tag{1.1.3}$$

So its reciprocal ought to have a product representation of the form,

$$\frac{1}{\Gamma(z)} \sim z\prod_{n=1}^{\infty}\Big(1+\frac{z}{n}\Big). \tag{1.1.4}$$

The product (1.1.4) however diverges. It is known that a product of the kind $\prod_{n=1}^{\infty}(1+x_n)$ converges only if $\sum_{n=1}^{\infty}|x_n|$ converges [3]. For the Gamma function $\sum_n \frac{1}{n}$ is not convergent, but $\sum_n \frac{1}{n^2}$ is. This suggests a remedy for the issue. Our product can be fixed by the function $(1+z)\,e^{-z}$ as it has the same zero as $(1+z)$ and to leading

orders it is bounded by [4]

$$\left|1-(1+z)e^{-z}\right| < C\left|z\right|^2. \tag{1.1.5}$$

Hence the product below converges.

$$\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}, \tag{1.1.6}$$

$$\frac{1}{\Gamma(z)} = z\lim_{n\to\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}e^{z\left[\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)-\log n\right]}. \tag{1.1.7}$$

By taking the logarithm of the equation (1.1.7) and by separating the divergent constant as k, we get

$$\log\frac{1}{\Gamma(z)} = \log z+\sum_{n=1}^{\infty}\log\left(e^{-\frac{z}{n}}\left[1+\frac{z}{n}\right]\right)+zk. \tag{1.1.8}$$

The constant k can be determined by comparing the derivative of the logarithm of the Gamma function to leading orders. We find that,

$$\psi(z) = \frac{d}{dz}\log\Gamma(z) \sim -\frac{1}{z}-\gamma+\mathcal{O}(z). \tag{1.1.9}$$

Thus the arbitrary constant k is the Euler constant γ,

$$\begin{aligned}\gamma &= \lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{1}{i}-\int_1^n\frac{dx}{x}\right) \\ &= \lim_{n\to\infty}\left(\frac{1}{n}+\sum_{i=1}^{n-1}\left[\frac{1}{i}-\log\left(1+\frac{1}{i}\right)\right]\right) \\ &= \sum_{i=1}^{\infty}\left[\frac{1}{i}-\log\left(1+\frac{1}{i}\right)\right] \\ &= 0.57721 \\ &= -\Gamma'(1).\end{aligned} \tag{1.1.10}$$

Even though we have a divergent sum $\sum_{i=1}^{n}\frac{1}{i}$ and a divergent integral $\int_1^n\frac{dx}{x}$, the difference- the Euler constant is finite. The result is convergent as $\frac{1}{k}-\log\left(1+\frac{1}{k}\right)$ is bounded as it goes as $\mathcal{O}\left(\frac{1}{k^2}\right)$ to leading orders.

Another noticeable thing is that with higher values of n, accuracy of γ increases. This is similar to the perturbative results in field theory.

n	γ
1	1
10	0.626383
50	0.587182
100	0.582207
1000	0.577716
10000	0.577266

The Gamma function thus illustrates regularization of a logarithmic divergence outside quantum field theory. Another famous technique which is borrowed from mathematics to find the effective action of a theory is the Zeta function regularization

[5]. The "renormalized" product formula for the Gamma function is thus,

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty} e^{-\frac{z}{n}}\left(1+\frac{z}{n}\right). \tag{1.1.11}$$

Delta Potential in Quantum Mechanics

Another instance of renormalization is a simple non-relativistic quantum mechanical system: the attractive delta function potential in two dimensions [6]. The system highlights ideas of cutoff and shows how the coupling constant is associated with the cutoff. This model also highlights how the change in the boundary condition of a system can be viewed as a singular potential such as the delta function potential [7]. The two-dimensional Schrödinger wave equation for the model is,

$$\left[-\nabla^2 - J\,\delta^2(\mathbf{x})\right]\psi(\mathbf{x}) = E\psi(\mathbf{x}).$$

In terms of Fourier components the eigenvalue equation becomes

$$\mathbf{k}^2\psi_k - J\sum_{m\in\mathbb{Z}\times\mathbb{Z}}\psi_m = E\psi_k, \quad \psi(\mathbf{x}) = \sum_k \psi_k e^{i\mathbf{k}.\mathbf{x}}. \tag{1.1.12}$$

The solution is,

$$\psi_k = \frac{J\psi_\bullet}{k^2-E}, \quad \psi(0) = \psi_\bullet = \sum_{m\in\mathbb{Z}\times\mathbb{Z}}\psi_m. \tag{1.1.13}$$

By summing over all values of k and eliminating $\psi_\bullet$, we get an equation for the spectrum as

$$J^{-1} = \sum_{k\in\mathbb{Z}\times\mathbb{Z}}\frac{1}{k^2-E}. \tag{1.1.14}$$

The sum representing J diverges logarithmically. The idea of renormalization translated to this situation is-

- Putting a cutoff in the magnitude of the Fourier index (momentum): $|\mathbf{k}| < \Lambda$.
- Asking how should the "coupling constant" J depend on Λ in order that the lowest energy is some given value $-\mu^2$.
- Showing that if we eliminate J in this way, all the remaining eigenvalues remain finite as $\Lambda \to \infty$.
- We don't care about J. We care about eigenvalues which describe energies, and eigenfunctions which determine probabilities.

In more detail,

$$J^{-1}(\Lambda) = \sum_{|\mathbf{k}|<\Lambda} \frac{1}{k^2 + \mu^2}. \tag{1.1.15}$$

Using (1.1.14), we can rewrite this as

$$\sum_{|\mathbf{k}|<\Lambda} \frac{1}{k^2 + \mu^2} - \sum_{|\mathbf{k}|<\Lambda} \frac{1}{k^2 - E} = 0. \tag{1.1.16}$$

Although each series is divergent, the combined series in the limit as $\Lambda \to \infty$ is,

$$\chi(z) = \sum_{\mathbf{k}\in\mathbb{Z}\times\mathbb{Z}} \left\{ \frac{1}{k^2 + \mu^2} - \frac{1}{k^2 - z} \right\}. \tag{1.1.17}$$

$\chi(z)$ is an analytic function whose zeroes are the eigenvalues. They are all determined in terms of the lowest eigenvalue. The divergent quantity $J(\Lambda)$ has disappeared from the final answer. This is an example of renormalization.

Resolvent Formula

We can reformulate the above example in more satisfactory mathematical terms using the resolvent (Green's function to physicists). Let $\hat{H}_0$ be an operator on $L^2(\mathbb{R})$ self-adjoint in some domain D. In most interesting cases, $\hat{H}_0$ is a second order ordinary differential operator (e.g., Laplacian). We will see how to modify the domain and get an operator with a different spectrum, without changing the effect on differentiable wave-functions. Physically this amounts to changing the boundary conditions on the wave equation, which can also be thought of as singular potentials. The resolvent of $\hat{H}_0$ is defined to be, for λ not in the spectrum

$$\hat{R}_0 = \frac{1}{\lambda - \hat{H}_0}, \tag{1.1.18}$$

which satisfies the equation

$$-\frac{d}{d\lambda}\hat{R}_0(\lambda) = \hat{R}_0^2(\lambda). \tag{1.1.19}$$

Conversely, any operator-valued analytic function satisfying this condition is of the form $(\lambda - \hat{H})^{-1}$; i.e. it is the resolvent of some operator. Given one such solution, we will construct another, as follows. Since

$$[\hat{H}_0 - \lambda]\hat{R}_0(\lambda) = -1, \tag{1.1.20}$$

$R_0(x, y|\lambda)$ satisfies the same differential equation as the eigenfunctions of $\hat{H}_0$ except at $x = y$ where it has a discontinuous derivative. For $\lambda \neq \lambda_n$, $\hat{R}_0$ is a compact operator; the integral kernel is often square summable in each variable.

Proposition. *For μ not in the spectrum of $\hat{H}_0$, the kernel*

$$R(y,x|\lambda) = R_0(y,x|\lambda) + R_0(y,0|\lambda)\frac{1}{\chi(\lambda)-\chi(\mu)}R_0(0,x|\lambda)$$

satisfies the resolvent equation $\frac{d}{d\lambda}R(y,x|\lambda) = -\int R(y,z|\lambda)R(z,x|\lambda)dz$ provided that

$$\frac{d}{d\lambda}\chi(\lambda) = -\frac{d}{d\lambda}R_0(0,0|\lambda).$$

Thus $\hat{R}$ is the resolvent of some operator

$$\hat{R}(\lambda) = \frac{1}{\lambda - \hat{H}}, \tag{1.1.21}$$

with the same form as $\hat{H}_0$ on smooth functions, but with a different domain of self-adjointness in $L^2(\mathbb{R})$. Its spectrum is given by the equation

$$\chi(\lambda) = \chi(\mu). \tag{1.1.22}$$

In particular, μ itself is an eigenvalue which determines the rest of the spectrum $\lambda_n(\mu)$ from the above equation. The orthonormal eigenfunctions of H are

$$\psi_n(x) = \frac{R_0(x,0|\lambda_n(\mu))}{\sqrt{\chi'(\lambda_n(\mu))}}. \tag{1.1.23}$$

In the example of the delta function potential,

$$R_0(x,y|\lambda) = \sum_{\mathbf{k}\in\mathbb{Z}\times\mathbb{Z}} \frac{e^{im\cdot(x-y)}}{\lambda - |\mathbf{k}|^2} \tag{1.1.24}$$

Although $R_0(0,0|\lambda) = \sum_{\mathbf{k}\in\mathbb{Z}\times\mathbb{Z}} \frac{1}{\lambda - |\mathbf{k}|^2}$ is divergent, its derivative

$$-\frac{d}{d\lambda} R_0(0,0|\lambda) = \sum_{\mathbf{k}\in\mathbb{Z}\times\mathbb{Z}} \frac{1}{(\lambda - |\mathbf{k}|^2)^2} \tag{1.1.25}$$

converges. Hence,

$$\chi(\lambda) - \chi(\mu) = \sum_{\mathbf{k}\in\mathbb{Z}\times\mathbb{Z}} \left\{ \frac{1}{\lambda - |\mathbf{k}|^2} - \frac{1}{\mu - |\mathbf{k}|^2} \right\} \tag{1.1.26}$$

is convergent.

1.2 Techniques to address Renormalization

Numerous techniques have been invented by physicists and mathematicians to deal with the problem of renormalization. One such method is the renormalization group transformation, which is explained below by using the one dimensional Ising model [8]. This is followed by a short discussion highlighting the finite element method (FEM) to find an approximation for the self energy of a field on an interval of the real line.

Renormalization Group Transformation

In Sec. 1.1, we got a glimpse of divergent quantities. At times, it is cumbersome yet plausible to deal with those divergences using renormalization techniques. However most theories in nature are not renormalizable. A theory is renormalizable if we can find the right set of renormalization group (RG) transformations of its couplings so that the physically divergent quantities (dependent on couplings) are no more divergent under the redefinition of the coupling constants.

The main idea behind RG transformations is to find an effective theory at a certain length scale by averaging out fluctuations over length scales smaller than that. This decreases the number of degrees in the system. As an analogy, if we are interested in the macroscopic properties of a system we do not pay attention to its microscopic properties. Wilson was the major proponent of this idea [9]. A systematic approach to subdivide length scales in the problem is introduced and an effective action corresponding to each length scale is found. The major steps of the technique are mentioned below,

- *Coarse grain:* All physical systems have an implicit short distance length scale a associated with them. Fluctuations at length scales below a can be averaged out. For spin systems a could be the lattice spacing. Coarse graining involves changing this cutoff scale by a certain factor, say b. It is as if one were zooming out of the system. The action for the system could be represented by this functional integral

$$S = \int_{x>a} d^D x \, \mathcal{L}\left[\phi(x), \lambda\right].$$

- *Rescale:* Due to the coarse graining, the action needs to be evaluated in the new length scale which is

$$S = \int_{x>ba} \frac{d^D x}{b^D} \, \mathcal{L}\left[\phi(bx), \lambda\right].$$

- *Renormalize:* This is the crucial step where we get the RG transformations. In order to preserve the same functional form for the action, we get a set of conditions for the couplings in the theory. Thus the action is implicitly dependent on the scaling b through the coupling constants. What we get is an effective action where few degrees of freedom have been integrated out in the process of

changing the cutoff of the theory to ba.

$$S = \int_{x>ba} d^D x \, \mathcal{L}\left[\phi'(x), \lambda'(b)\right].$$

A theory turns out to be renormalizable if there exists a mapping $\mathcal{G} : \lambda \to \lambda'$. This mapping redefines the couplings of the theory at small length scales such that physically observable quantitates are no more divergent. However we must be aware that these transformations do not form a group actually but a semigroup rather. Even though the scaling parameters $b_1, b_2 \in \mathcal{G}$ satisfy $b_1 \circ b_2 = b_1 b_2$, the inverse may not exist. The information which is lost in the coarse graining process cannot be retrieved by any inverse operation.

Ising Model in One Dimension

We now demonstrate the renormalization group transformation for the Ising model in one dimension. This particular example highlights RG transformation in position space as it involves subdividing the real line on which the spin sites of the Ising model lie. The Ising model is a mathematical construct often used to describe ferromagnetism. In this, the atoms on a lattice are allowed to be in spin up or down state and only interactions between neighboring atoms are considered.

A one dimensional lattice with N sites having nearest neighboring interactions has the following partition function

$$Z = \sum_{\{\sigma_i | i=1,\ldots N\}} e^{-\beta \mathcal{H}[\sigma_i]}, \tag{1.2.1}$$

where

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - K \sum_i \sigma_i, \tag{1.2.2}$$

$\langle ij \rangle$ denote neighboring sites.

The partition function is a quantity of great interest in statistical mechanics. With as many as 2^N configurations, it is difficult to compute the partition function. We use RG transformations to ease this difficulty. We reduce the number of degrees of freedom of the system by summing over the even sites and use the recursive nature of the partition function. We then relabel the sites (Fig. 1.1). The b for rescaling in this turns out to be 2.

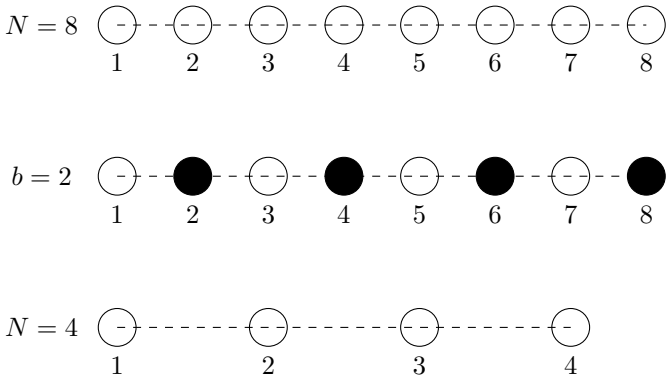

Figure 1.1: Pictorial representation of sites being summed over

In order to sum over the even sites, we let $i = 2j$. By summing over the possible spins at site i,

$$\sum_{\sigma_{2j}=\pm 1} e^{\beta(K + J\sigma_{2j-1}\sigma_{2j} + J\sigma_{2j}\sigma_{2j+1})} = \begin{cases} e^{\beta(K+2J)} + e^{-\beta(K+2J)} & , \sigma_{2j-1} + \sigma_{2j+1} = 2 \\ e^{\beta K} + e^{-\beta K} & , \sigma_{2j-1} + \sigma_{2j+1} = 0 \\ e^{\beta(K-2J)} + e^{-\beta(K-2J)} & , \sigma_{2j-1} + \sigma_{2j+1} = -2 \end{cases} \tag{1.2.3}$$

Once the sites have been relabeled ($j \to i$), we can use the recursive nature of the partition function,

$$e^{\beta(J'\sigma_i\sigma_{i+2}+K'(\sigma_i+\sigma_{i+2}))} = \begin{cases} e^{\beta(J'+2K')} & ,\sigma_i+\sigma_{i+2}=2 \\ e^{\beta J'} & ,\sigma_i+\sigma_{i+2}=0 \\ e^{\beta(J'-2K)} & ,\sigma_i+\sigma_{i+2}=-2 \end{cases} \tag{1.2.4}$$

On comparing the two sets of equations (1.2.3, 1.2.4), we get the following conditions-

$$e^{4\beta K'} = \frac{\cosh\beta(K+2J)}{\cosh\beta(K-2J)}, \tag{1.2.5a}$$

$$e^{4\beta J'} = \frac{\cosh\beta(K+2J)\cosh\beta(K-2J)}{\cosh^2\beta K}. \tag{1.2.5b}$$

These are the RG transformations for the coupling constants in the Ising model. We can now find a fixed point for this map and in turn make the partition function recursive in nature. If the external field strength K is absent, then the fixed points of this map (1.2.6) turn out to be $0, \infty$.

$$\beta J' = \frac{1}{2}\log\cosh 2\beta J. \tag{1.2.6}$$

The two most important features associated with this calculation were-

- The recursive nature of the partition function. For large number of sites N, we can compute the partition function easily with several iterations.

- The partial sum over half of the sites at each step to find an effective partition function for the remaining sites.

We will be using the same ideas in the chapter 3 as well for continuous distributions, and not discrete sums like this.

Discrete Approach to Field Theory

We shall now see another instance of an unintuitive method of discretizing space to study fields on them. We get rid of many infinite dimensional integrals in this process, which are not well defined in a rigorous way.

Finite Element Method (FEM) in one dimension

Given a field with values fixed on its boundary points, we want to interpolate the fields to the interior points by dividing space into a mesh. In the continuum limit as the lattice spacing goes to zero, we should obtain results independent of the mesh. We shall demonstrate an example of finite element method in one dimension and estimate the amount of self energy stored in the field.

Let us consider a scalar field defined on an interval of length L and we want to get an approximation for the amount of self energy stored in the field. The action under consideration,

$$S = \frac{1}{2}\int_0^L |\nabla\phi(x)|^2 \, dx. \tag{1.2.7}$$

The equation of motion associated with the action is the Laplace equation

$$\nabla^2\phi(x) = 0 \tag{1.2.8}$$

with boundary values specified as $\phi(0) = \phi_0$, $\phi(L) = \phi_N$.

In order to solve this differential equation by finite element methods, we divide the interval $[0, L]$ into N small pieces of length ϵ such that the field value does not change

much over a small piece. In the limit as $N \to \infty$, we should get a good approximation for the self energy. For simplicity and also to satisfy the Laplace equation we consider our field to be

$$\phi(x_i) = \alpha\, x_i + \beta, \quad \forall x_i \in [0, L] \tag{1.2.9}$$

$$x_i = i\frac{L}{N} = i\,\epsilon, \quad i = 0, \ldots N. \tag{1.2.10}$$

The parameters which satisfy the given boundary conditions are-

$$\alpha = \frac{\phi_N - \phi_0}{L}, \quad \beta = \phi_0 \tag{1.2.11}$$

With these assumptions, the action turns out to be approximately

$$\begin{aligned} S &= \frac{1}{2}\left(\int_0^{\epsilon} |\nabla\phi(x)|^2\, dx + \int_{\epsilon}^{2\epsilon} |\nabla\phi(x)|^2\, dx + \ldots + \int_{(N-1)\epsilon}^{N\epsilon} |\nabla\phi(x)|^2\, dx\right) \\ &= \frac{1}{2}\left(\int_0^{\epsilon} \alpha^2\, dx + \int_{\epsilon}^{2\epsilon} \alpha^2\, dx + \ldots + \int_{(N-1)\epsilon}^{N\epsilon} \alpha^2\, dx\right) \\ &= \frac{N\epsilon}{2}\alpha^2 \\ &= \frac{(\phi_N - \phi_0)^2}{2L}, \end{aligned} \tag{1.2.12}$$

since $\nabla\phi(x) = \alpha$. A dimensional analysis of (1.2.7) shows that our answer has the right dimensions.

We shall see later in Sec. 3.1 that the action (1.2.12) is analogous to the heat kernel for the one dimensional Wiener process. This way of subdividing an interval to find the partition function for the whole interval in a hierarchal manner will be extended to the plane, and is demonstrated in chapter 4. Finite element method had not been used in field theory in such a fashion before.

CHAPTER 2

RADIATING PARTICLE IN ELECTRODYNAMICS

"God runs electromagnetics on Monday, Wednesday, and Friday by the wave theory, and the devil runs it on Tuesday, Thursday, and Saturday by the quantum theory."

— Sir William Henry Bragg, *Dictionary of Scientific Quotations*

Classical electrodynamics doesn't escape divergences as well. Like many other theories, it has certain limitations and gives rise to absurd results in certain domains. In this chapter, when we try to understand the classical motion of an electron in a constant magnetic field, we are faced with lots of complications. Surprisingly, this question does not have a definitive theoretical answer even now. The complications arise when the radiation from the particle is considered. Without it, the answer is quite elementary: a helical path around the magnetic field [10,11] obtained by solving the Lorentz equation of motion,[1]

$$\frac{du^\mu}{d\tau} = \frac{q}{m} F^{\mu\nu} u_\nu, \quad \mu, \nu = 0, \ldots 3 \tag{2.0.1}$$

[1]In units with $c = 1$. q and m are the charge and mass respectively. Also, $u^\mu = \frac{dx^\mu}{d\tau}$ is the four-velocity and τ is proper time. Note the constraint $\eta_{\mu\nu} u^\mu u^\nu = 1$ and its derivative $\eta_{\mu\nu} u^\mu \frac{du^\nu}{d\tau} = 0$.

where

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \tag{2.0.2}$$
$$E_i = -F^{0i}, \quad i, j = 1, 2, 3$$
$$B_i = -\frac{1}{2}\epsilon_{ijk}F^{jk}.$$

E, B being the electric and magnetic fields respectively.

2.1 Orientation

A quantity of interest in electrodynamics is the self-energy for a charge distribution, that is, amount of energy spent to bring the charges from infinity to their current configuration. For a single particle like the electron, the self-energy turns out to be $\frac{1}{2}q\phi$ where ϕ is the potential of the field created by the particle itself at the point of the particle. The fun starts now.

For an electron, since its potential ϕ is $\frac{q}{R}$, its self-energy turns out to be $\frac{q^2}{2R}$. The electron being a point particle itself, the self-energy of the electron at the point of its location diverges to infinity at $R = 0$. Thus the self-energy of the electron turns out to be infinite. This leads to the unreasonable conclusion that its mass (self-energy divided by c^2) would be infinite as well. Classical electrodynamics fails at this point and sets the classical electron radius beyond which it is not applicable.

$$R_0 = \frac{q^2}{mc^2}.$$

This gives rise to the question whether the mass of the electron is of complete electromagnetic origin or not. Dirac [12] answered this question in quite an unintuitive

way and probably set the first example of renormalization while studying the theory of radiating electrons. But we know that reality is not so simple: an accelerated charge radiates, and the radiation emitted by it exerts a force back on it. If the energy is not replenished, the path should be a spiral, the radius of the circle decreasing with the energy.

Determining the force exerted on a point particle by its own field is fraught with conceptual difficulties: the force is infinite. Dirac made a major step forward by showing that this divergence can be removed by a renormalization of the mass. The problem lies in defining the intrinsic mass of point particles like the electron. As discussed earlier, classical electrodynamics leads to an infinite positive mass for the electron. Dirac assigned an infinite negative mass of non-electromagnetic origin to compensate for the infinite mass of electromagnetic origin. Thus renormalization sets in again to give rise to a sensible answer by subtracting one infinity from another.

Since this argument was not based on mathematical rigor, the resulting equation (the Lorentz-Dirac equation of motion)[2]

$$\frac{du_\mu}{d\tau} = \frac{q}{m}F_{\mu\nu}u^\nu + \epsilon\left[\eta_{\mu\nu} - u_\mu u_\nu\right]\frac{d^2u^\nu}{d\tau^2}, \quad \epsilon = \frac{2}{3}\frac{q^2}{m} \tag{2.1.1}$$
$$\epsilon = 6.24 \times 10^{-24}s,$$

is still not the correct answer. It predicts runaway solutions even in the absence of any external fields. A particle can accelerate away to infinity with exponentially growing energy. Part of the problem is that the radiative terms persist even in the absence of external forces. Also, the equation involves the third derivative of position; so it needs a new initial condition. Dirac suggested that this be chosen such that the

[2]The last term includes a projection operator that ensures that the constraint $\eta_{\mu\nu}u^\mu\frac{du^\nu}{d\tau} = 0$ is satisfied.

energy remains finite forever. However such solutions suffer from another problem: a violation of causality; the particle would accelerate before the external field is turned on.

Landau and Lifshitz proposed [13] a way out of this dilemma. Replace the derivative of acceleration by its leading approximation in an expansion in powers of ϵ,

$$\frac{du_\mu}{d\tau} = \frac{q}{m}F_{\mu\nu}u^\nu + \epsilon\left[\eta_{\mu\nu} - u_\mu u_\nu\right]\frac{d}{d\tau}\left[\frac{q}{m}F^\nu_\rho u^\rho\right], \quad \epsilon = \frac{2}{3}\frac{q^2}{m}. \tag{2.1.2}$$

The radiative term now vanishes when the external force is zero; and the equation now involves only the second order derivative of position. The free particle does not run away. Further analysis shows that the orbits in a constant magnetic field [14, 15] and in a Coulomb field [16] decay as expected.

An order of magnitude estimate also tells us the validity of this equation (2.1.2). For a particle in an electric field with frequency of motion of ω, $\dot{\mathbf{E}}$ is roughly proportional to $\omega\mathbf{E}$. Thus the radiative damping is of the order $\frac{q^3\omega|\mathbf{E}|}{m^2}$. The radiation damping force is small in comparison to the force due to the external field if

$$\frac{q^3\omega|\mathbf{E}|}{m} \ll q|\mathbf{E}|.$$

This sets the limit on the wavelength $\lambda = \frac{1}{\omega}$ associated with the radiation as

$$\lambda \gg \frac{q^2}{m}.$$

In units where $c \neq 1$, $\frac{q^2}{mc^2}$ for an electron is its classical radius (2.1). This shows that the radiation damping is pertinent only till the limit where classical electrodynamics is applicable itself else it gives rise to inconsistent results. The wavelength associated with the external force should be large compared to the size of the particle

for the formula (2.1.2) to hold good. But why should we stop at first order in the iteration in powers of ϵ?

Thus we consider the century-old problem of determining the equation of motion of a radiating point charge to be settled finally. However, the electron has a magnetic moment as well as electric charge; radiation arises not only from the acceleration of the charge but also the precession of the magnetic moment. In the relativistic case, these are of comparable magnitude. Therefore, it is necessary to extend the Lorentz-Dirac and Landau-Lifshitz equations to the case of a spinning charged particle.

Bargman, Michel and Telegdi (BMT) proposed [17], more than a half century ago, an equation for a (non-radiating) classical particle with charge and magnetic moment. A theory of radiating spinning particles generalizing this has been proposed some time ago as well [18]. However, the BMT equation does not seem to be the classical limit of the Dirac wave equation of the electron. It would be better to have a classical theory whose canonical quantization leads directly to the Dirac wave equation.

Such an approach was proposed more recently by Barut and Zanghi [19]. After reviewing their ideas, we will find solutions of the Barut-Zanghi equation for a non-radiating charged particle in a constant magnetic field: itself a new result. Barut and Unal [20] also derived the analogue of the Lorentz-Dirac equation, after a mass renormalization. However, it suffers from the same runaway solutions of the free particle, as the spin-less case.

We proposed an equation of motion for a radiating spinning charged particle [21] with gyromagnetic ratio 2 (e.g., the electron or the muon) by applying a prescription analogous to that of Landau and Lifshitz, to the Barut-Zanghi equation. We showed that at least for spin-polarized orbits in a constant magnetic field, the orbits decay as expected physically.

2.2 Non-Radiating Spinning Charged Particle

How can one find the motion of a particle with spin in the classical limit? If we can find a Lagrangian whose canonical quantization is the Dirac wave equation, it can be the basis for such a classical treatment. The Lagrangian proposed by Barut and Zanghi is,

$$L = \frac{i}{2}\left[\dot{\bar{z}}z - \bar{z}\dot{z}\right] + p_\mu \dot{x}^\mu - H, \quad H = \bar{z}\gamma^\nu z\left[p_\nu - qA_\nu\right], \quad \bar{z} = z^\dagger\gamma^0. \tag{2.2.1}$$

γ's are the Dirac matrices or the Gamma matrices.

Here z is a Dirac spinor with four complex numbers as components. The dimensions of z are that of the square root of action (= mass×length) and that of τ is the inverse of mass, in units where $c = 1$ but $\hbar \neq 1$. Then $qF_{\mu\nu}$ has dimensions of mass over length.

The equations of motion are, with $\pi_\mu = p_\mu - qA_\mu$

$$\frac{dx^\mu}{d\tau} = \bar{z}\gamma^\mu z, \tag{2.2.2a}$$

$$-i\frac{dz}{d\tau} = \gamma^\mu \pi_\mu z, \tag{2.2.2b}$$

$$\frac{d\pi_\mu}{d\tau} = qF_{\mu\nu}\bar{z}\gamma^\nu z. \tag{2.2.2c}$$

Note that, τ is not proper time as $\dot{x}^\mu$ is not necessarily of constant length.[3] But we have the orthogonality condition

$$\dot{x}^\mu \dot{\pi}_\mu = 0. \tag{2.2.3}$$

[3]This is unlike the beginning of the chapter, where τ was proper time

These equations follow from the Hamiltonian

$$H = \bar{z}\gamma^{\mu} z \pi_{\mu}, \tag{2.2.4}$$

and Poisson Brackets (P.B.) are obtained using

$$\{f, g\} = i\{\frac{\partial f}{\partial z}\frac{\partial g}{\partial \bar{z}} - \frac{\partial g}{\partial z}\frac{\partial f}{\partial \bar{z}}\} + g_{\mu\nu}\{\frac{\partial f}{\partial p^{\mu}}\frac{\partial g}{\partial x^{\nu}} - \frac{\partial f}{\partial x^{\mu}}\frac{\partial g}{\partial p^{\nu}}\}, \tag{2.2.5}$$

$$\begin{aligned} \{\pi_{\mu}, \pi_{\nu}\} = -qF_{\mu\nu}, \quad \{\pi_{\mu}, x^{\nu}\} = \delta_{\mu}^{\nu} \\ \{z, \bar{z}\} = i, \quad \{\pi_{\mu}, z\} = \{z, x\} = 0. \end{aligned} \tag{2.2.6}$$

All other pairs of P.B. are zero. We can introduce a constraint

$$\bar{z}z = a, \tag{2.2.7}$$

consistent with the P.B. and the equations of motion. The quantity a, with the dimensions of action, sets the scale for the intrinsic angular momentum of the particle.

Upon quantization, the wave function can be thought of as a function of x, $p_{\mu} = -i\hbar\frac{\partial}{\partial x^{\mu}}$; it is also a polynomial in $\bar{z}$, with $z = -\hbar\frac{\partial}{\partial \bar{z}}$ satisfying the constraint

$$-\hbar\bar{z}\frac{\partial}{\partial \bar{z}}\psi = a\psi. \tag{2.2.8}$$

Thus the parameter a is quantized in multiples of $\hbar$. The smallest value is $a = \hbar$ for which $\psi(x, \bar{z}) = \bar{z}\psi(x)$ is a linear function. The Dirac wave equation is the eigenvalue

equation for H, with eigenvalue am

$$i\hbar\gamma^\mu \frac{\partial}{\partial x^\mu}\psi = m\psi. \tag{2.2.9}$$

Other quantizations, for which $\psi(\bar{z})$ is a higher degree polynomial, correspond to the wave equations of Bhabha and Harish-Chandra. They could be interesting as wave equations for composite particles like hadrons [22].

It is convenient to introduce the velocity and spin variables

$$v^\mu = \bar{z}\gamma^\mu z, \quad S^{\mu\nu} = \frac{i}{4}\bar{z}[\gamma^\mu, \gamma^\nu]z, \tag{2.2.10}$$

so that

$$\frac{dx^\mu}{d\tau} = v^\mu, \tag{2.2.11a}$$
$$\frac{dv^\mu}{d\tau} = 4S^{\mu\nu}\pi_\nu, \tag{2.2.11b}$$
$$\frac{d\pi_\mu}{d\tau} = qF_{\mu\nu}v^\nu, \tag{2.2.11c}$$
$$\frac{dS^{\mu\nu}}{d\tau} = v^\nu\pi^\mu - v^\mu\pi^\nu. \tag{2.2.11d}$$

These equations describe a relativistic analogue of the isotropic precessing top of Poinsot [23]. The Hamiltonian

$$H = v^\mu\pi_\mu, \tag{2.2.12}$$

describes a coupling to the external field which causes a precession of the velocity and spin. For a constant electromagnetic field, $K = \pi\cdot\pi - qS^{\mu\nu}F_{\mu\nu}$ is conserved as well.

$$\frac{d}{d\tau}\left[\pi\cdot\pi - qS^{\mu\nu}F_{\mu\nu}\right] = 2qF_{\mu\nu}\pi^\mu v^\nu - 2qF_{\mu\nu}v^\nu\pi^\mu = 0. \tag{2.2.13}$$

This is the classical analogue of the square of the Dirac operator, corresponding to the gyromagnetic ratio 2. We will see later that it is proportional to the angular momentum in the plane of the electromagnetic field.

2.3 Non-Radiating Spinning Particle in a Constant Magnetic Field

Now let us turn to solving the equations of motion in the case of a magnetic field pointed along the third direction: $F_{12} = -F_{21} > 0$; and all other components are zero. Before we solve the problem including radiation damping, we must recover the solution without radiation. Even this appears to be new: Barut and collaborators only solved the case of a free particle.

We consider only orbits that lie in the 12 plane. That is, we restrict to the case where all the tensor indices $\mu, \nu, \rho \ldots$ are in the $0, 1, 2$ subspace in the sections that follow except Sec. 2.4. It is useful to introduce the 'dual' variables

$$qF_{\mu\nu} = \epsilon_{\mu\nu\rho}F^{\rho}, \tag{2.3.1a}$$

$$S^{\mu\nu} = \epsilon^{\mu\nu\rho}S_{\rho}, \tag{2.3.1b}$$

$$S_{\rho} = \frac{1}{2}\epsilon^{\mu\nu\rho}S_{\mu\nu}. \tag{2.3.1c}$$

For a constant magnetic field, only the time component F^0 is non-zero. We can

assume it to be positive without loss of generality. Then,

$$\frac{d\pi_\mu}{d\tau} = \epsilon_{\mu\nu\rho} v^\nu F^\rho, \tag{2.3.2a}$$
$$\frac{dv^\mu}{d\tau} = -4\epsilon^{\mu\nu\rho} S_\nu \pi_\rho, \tag{2.3.2b}$$
$$\frac{dS_\mu}{d\tau} = -\epsilon_{\mu\nu\rho} v^\nu \pi^\rho. \tag{2.3.2c}$$

If we define

$$\lambda_\mu = \frac{1}{2}(S_\mu + \frac{1}{2} v_\mu), \quad \rho_\mu = \frac{1}{2}(S_\mu - \frac{1}{2} v_\mu), \tag{2.3.3}$$

2.3.2 reduce to

$$\frac{d\pi_\mu}{d\tau} = 2\epsilon_{\mu\nu\rho} \left[\lambda^\nu - \rho^\nu\right] F^\rho, \tag{2.3.4a}$$
$$\frac{d\lambda^\mu}{d\tau} = -2\epsilon^{\mu\nu\rho} \lambda_\nu \pi_\rho, \tag{2.3.4b}$$
$$\frac{d\rho_\mu}{d\tau} = 2\epsilon_{\mu\nu\rho} \rho^\nu \pi^\rho. \tag{2.3.4c}$$

Spin-Polarized Orbits

There is a special class of 'left-handed' orbits for which $\rho_\mu = 0$. (Also, the opposite case with $\lambda_\mu = 0$.) For these, the spin and velocity are synchronized in the following way-

$$S_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho} v^\rho. \tag{2.3.5}$$

In the non-relativistic limit, this means the spin is pointed along the magnetic field (times q).

Since these solutions are periodic (not chaotic), orbits of this kind in a synchrotron should be of special interest. If the gyromagnetic ratio is slightly different from two, the spin would depart from this orientation, which is the principle behind a famous

experiment for the measurement of $g-2$ for the muon.

The equations reduce to:

$$\frac{d\pi_\mu}{d\tau} = \epsilon_{\mu\nu\rho} v^\nu F^\rho \tag{2.3.6a}$$

$$\frac{dv^\mu}{d\tau} = -2\epsilon^{\mu\nu\rho} v_\nu \pi_\rho. \tag{2.3.6b}$$

The tensor $\epsilon^\sigma_{\mu\nu} = \epsilon_{\mu\nu\rho}\eta^{\rho\sigma}$ form the structure constants of the three dimensional Lorentz Lie algebra $SO(1,2)$. This is analogous to the way that the Levi-Civita tensor gives the structure constants of the Lie algebra of rotations in Euclidean three dimensional space. Denote the Lie product of $SO(1,2)$ as a relativistic cross product

$$(u \times w)_\mu = \epsilon^{\nu\rho}_\mu u_\nu w_\rho. \tag{2.3.7}$$

This differs from the familiar cross product in Euclidean geometry by some signs,

$$(u \times w)_0 = u_1 w_2 - u_2 w_1 \tag{2.3.8a}$$

$$(u \times w)_1 = -u_2 w_0 + u_0 w_2 \tag{2.3.8b}$$

$$(u \times w)_2 = -u_0 w_1 + u_1 w_0. \tag{2.3.8c}$$

Then our equations are a relativistic version of the equations for a precessing top,

$$\dot{\pi} = v \times F, \quad \dot{v} = -2v \times \pi. \tag{2.3.9}$$

It follows that,

$$v \cdot \dot{v} = 0 \tag{2.3.10}$$

so that $v \cdot v = a^2$ is conserved; a is some number (like Planck's constant) with the dimensions of action; the spin is of magnitude $\frac{a}{2}$. Also, the parameter τ is proportional to proper time in this special case.

The energy $\pi_0 = E$ and

$$K = \pi_\mu \pi^\mu - v^\mu F_\mu \tag{2.3.11}$$

are conserved.[4] Now,

$$\dot{v}_0 = 2(v_2\pi_1 - v_1\pi_2) \tag{2.3.12}$$

and

$$\dot{v}_0^2 = 4\left(\left[v_1^2 + v_2^2\right]\left[\pi_1^2 + \pi_2^2\right] - \left[v_1\pi_1 + v_2\pi_2\right]^2\right). \tag{2.3.13}$$

Re-expressing the r.h.s. in terms of conserved quantities,

$$\dot{v}_0 = 2\sqrt{\left[v_0^2 - a^2\right]\left[E^2 - K + v_0F_0\right] - \left[H - Ev_0\right]^2}. \tag{2.3.14}$$

Weierstrass elliptic function satisfy this differential equation. The remaining components of π, v can be obtained by solving the other constraint equations. It is also possible to understand the solution as an integrable hamiltonian system, a relativistic analogue of the precessing top.

[4] K is proportional to angular momentum.

2.4 Equation of Motion for Radiating Spinning Particles

Barut and Unal followed the method of Dirac to derive the analogue of the Lorentz-Dirac equation of motion including the self-force. Only the equation for π_μ is affected-[5]

$$\frac{d\pi_\mu}{d\tau} = qF_{\mu\nu}v^\nu + q^2\left[\eta_{\mu\nu} - \frac{v_\mu v_\nu}{v\cdot v}\right]\left\{\frac{2}{3}\frac{\ddot{v}^\nu}{v\cdot v} - \frac{9}{4}\frac{v\cdot\dot{v}\dot{v}^\nu}{(v\cdot v)^2}\right\}. \tag{2.4.1}$$

The last term is the contribution from self-interactions, after a renormalization. It is a singular perturbation i.e. q^2 is small and it changes the order of the equations. It has runaway solutions, just like the Lorentz-Dirac equation. Let us apply to it a prescription like that of Landau and Lifshitz, to get a second order equation of motion.

Naively, the Landau-Lifshitz prescription amounts to using

$$\ddot{v}^\mu = 4\dot{S}^{\mu\nu}\pi_\nu + 4S^{\mu\nu}\dot{\pi_\nu}, \tag{2.4.2}$$

and replacing $\dot{\pi}$ by its zeroth order contribution

$$\ddot{v}^\mu \to 4\dot{S}^{\mu\nu}\pi_\nu + 4S^{\mu\nu}qF_{\nu\rho}v^\rho = 4\left[v^\nu\pi^\mu - v^\mu\pi^\nu\right]\pi_\nu + 4S^{\mu\nu}qF_{\nu\rho}v^\rho. \tag{2.4.3}$$

But this can't be right: for a free particle ($F_{\mu\nu} = 0$), the radiation reaction force should be zero. Dropping the terms not proportional to $F_{\mu\nu}$,

$$\frac{d\pi_\mu}{d\tau} = qF_{\mu\nu}v^\nu + \frac{8q^3}{3v\cdot v}\left[\eta_{\mu\nu} - \frac{v_\mu v_\nu}{v\cdot v}\right]S^{\nu\rho}F_{\rho\sigma}v^\sigma. \tag{2.4.4}$$

[5] μ, ν range from $0, \ldots 3$ in this section.

Along with the previous equations for v and S, this system of ordinary differential equations is our proposal for the equation of motion of a spinning radiating charged particle. We must verify that the solutions are physically sensible: there should not be runaway solutions. We will study the case of polarized orbits in a constant magnetic field and show that the energy decreases with time. The ultimate test of our proposal must be experimental, especially since we do not yet have a derivation from first principles.

Radiating Spin-Polarized Particles in a Magnetic Field

Now let us turn to our proposed equations for a radiating spinning charged particle. Again, we will look for solutions in the special case where the spin is polarized

$$S_\mu = \frac{1}{2} v_\mu.$$

Since the radiative terms do not change the equations for v^μ, $S^{\mu\nu}$ this condition is still preserved by time evolution i.e. $\dot{S}^\mu = \frac{1}{2}\dot{v}^\mu$. Unpolarized orbits are chaotic and the radiation damping is likely to be even larger.

The equations then reduce to:

$$\frac{d\pi_\mu}{d\tau} = qF_{\mu\nu}v^\nu + \frac{4q^3}{3v\cdot v}\left[\eta_{\mu\nu} - \frac{v_\mu v_\nu}{v\cdot v}\right]\epsilon^{\nu\rho\alpha}v_\alpha F_{\rho\sigma}v^\sigma \tag{2.4.5a}$$

$$\frac{dv^\mu}{d\tau} = -2\epsilon^{\mu\nu\rho}v_\nu\pi_\rho. \tag{2.4.5b}$$

We know that, $v\cdot\dot{v} = 0$ and $v\cdot v = v_0^2 - (v_1^2 + v_2^2) = a^2$.

$$\frac{d\pi_\mu}{d\tau} = \epsilon_{\mu\nu\rho}v^\nu F^\rho + \left[a^2\eta_{\mu\nu} - v_\mu v_\nu\right]\gamma F^\nu, \tag{2.4.6}$$

where

$$\gamma = \frac{4q^2}{3a^2}.$$

In terms of relativistic cross products (2.3.8),

$$\dot{\pi} = v \times [F + \gamma F \times v], \quad \dot{v} = -2v \times \pi. \tag{2.4.7}$$

Thus, even the radiation damping terms have a simple meaning in terms of the relativistic cross product of the Lie algebra $SO(1,2)$.

Note that

$$H = v \cdot \pi \tag{2.4.8}$$

is still conserved and $m = \frac{H}{a}$ continues to have the physical meaning of mass. But energy $E = \pi_0$ and K are no longer conserved. The evolution of v_0 continues to be determined by the equation we derived earlier. Thus we get the system:

$$\frac{dE}{d\tau} = -\gamma F_0(v_0^2 - a^2) \tag{2.4.9a}$$

$$\frac{dK}{d\tau} = 2\gamma F_0 \left[a^2 E - H v_0\right] \tag{2.4.9b}$$

$$\dot{v}_0 = 2\sqrt{\left[v_0^2 - a^2\right]\left[E^2 - K + v_0 F_0\right] - \left[H - E v_0\right]^2}. \tag{2.4.9c}$$

Since E and K are no longer constants, the solution is not an elliptic function. Also, noting that $F_0 > 0$ in our convention and $v_0^2 - a^2 = (v_1^2 + v_2^2) > 0$, we see that energy is monotonically decreasing. The results do match our intuition in this manner.

Summary

Our discussion so far gives an example of a divergence in electrodynamics and shows how the limitations of the subject itself makes it difficult to deal with the divergence. We proposed equations of motion for the electron which gave us answers with the correct physical intuition. We shall now proceed to deal with divergent quantities in field theory from the point of view of renormalization groups.

CHAPTER 3

TOOLBOX TO ELUCIDATE RENORMALIZATION

> *"The smallest particles of matter were said [by Plato] to be right-angled triangles which, after combining in pairs, ... joined together into the regular bodies of solid geometry; cubes, tetrahedrons, octahedrons and icosahedrons. These four bodies were said to be the building blocks of the four elements, earth, fire, air and water ... [The] whole thing seemed to be wild speculation. ... Even so, I was enthralled by the idea that the smallest particles of matter must reduce to some mathematical form ... The most important result of it all, perhaps, was the conviction that, in order to interpret the material world we need to know something about its smallest parts."*
>
> — Werner Heisenberg*, his recollections from Plato's Timaeus*

In this chapter, we will see an application of renormalization group transformations to understand scalar fields on triangular lattices. We will discuss the one dimensional Wiener process (Sec. 3.1) where the idea of subdivision of the interval forms the central theme. An attempt to generalize this to higher dimensions (Sec. 3.3) is presented afterward highlighting some of the unexpected results and the possible flaws with such an extension. We also show an example of renormalization group transformation for non-Gaussian distributions such as the Ising model (Sec. 3.4).

3.1 Recursive Nature of Wiener Process in Field Theory

A view of the streak of light coming in through our window on a bright sunny day makes a whole bunch of dust particles visible to the naked eye. They execute a random motion by traveling in all directions and colliding with each other. The particles have no memory of their motion and their future motion is not influenced by their past motion [24]. Such a motion was noticed by a botanist Robert Brown while observing pollen grains suspended on a liquid. Years later, Einstein explained this random motion (which became famous as Brownian motion) by the differential equation (3.1.1) [25]. The following equation describes the one dimensional Brownian motion, which also goes by the name of diffusion equation or heat equation depending on the context.

$$\frac{\partial u(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2 u(x,t)}{\partial x^2}. \tag{3.1.1}$$

$u(x,t)$ denotes the density of the particles which diffuse.

The differential equation can be solved by using Fourier transform

$$u(x,t) = \frac{1}{2\pi}\int \tilde{u}(x,t)e^{ikx}\,dk, \tag{3.1.2}$$

which reduces it to,

$$\begin{aligned} \partial_t \tilde{u}(k,t) + \frac{k^2}{2}\tilde{u}(k,t) &= 0 \\ \partial_t \left(e^{-\frac{k^2 t}{2}}\tilde{u}(k,t)\right) &= 0. \end{aligned} \tag{3.1.3}$$

The generic solution turns out to be,

$$\tilde{u}(k,t) = f(k)e^{-\frac{k^2 t}{2}}. \tag{3.1.4}$$

If the initial condition for the problem is $u(x,0) = h(x)$ with $\tilde{h}(k)$ as the Fourier transform of $h(x)$, then

$$\tilde{u}(k,t) = \tilde{h}(k)e^{-\frac{k^2 t}{2}}. \tag{3.1.5}$$

The solution $u(x,t)$ is a convolution of the heat kernel and the initial condition for the problem. The heat kernel is

$$\begin{aligned} K(x,t) &= \frac{1}{2\pi}\int e^{-\frac{k^2 t}{2}} e^{ikx}\, dk \\ &= \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2 t}{2}}. \end{aligned} \tag{3.1.6}$$

$$u(x,t) = \int K(x-y,t)h(y)\, dy. \tag{3.1.7}$$

The interesting features of this distribution are:

$$\langle x \rangle = 0, \quad \langle x^2 \rangle = t,$$

which are characteristic of Brownian motion. The mean position of the particle in such a motion is 0 as it has gone in both positive and negative directions with equal probability. However the mean square displacement turns out to be proportional to the time that the particle has undergone the Brownian motion.

The Brownian motion was better explained by Norbert Wiener in 1920 by giving it a sound mathematical footing. He developed a distribution associated to the Brownian motion which captures its essential features. Later it found use in many areas such as finance, economics besides physics [26]. The Wiener distribution is nothing

but the heat kernel (3.1.6) which was derived earlier. If the unit time interval $[0, 1]$ is divided into smaller time steps such that, $0 < t_1 < t_2 < \ldots < t_n \leq 1$, and the position of the particle undergoing Brownian motion at time t_k denoted by $x(t_k)$, then the Wiener distribution is characterized by the following properties-

- All the particles begin at the origin that is $x(0) = 0$.
- The path of the particle is continuous for $0 \leq t \leq 1$ with probability 1.
- The difference of the coordinates of the particle at various time intervals form random variables

$$x(t_1) - x(0), x(t_2) - x(t_3), \ldots, x(t_n) - x(t_{n-1})$$

  which vary independently. They form a distribution with zero mean and variance proportional to $(t_k - t_{k-1})$, $k = 1, \ldots n$.

The Wiener distribution satisfying these properties is given by

$$K(x, y; t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2 t}{2}}. \tag{3.1.8}$$

An interesting property of this function which shall be put to use later is

$$\int K(x, y; t) K(y, z; s)\, dy = K(x, z; t + s). \tag{3.1.9}$$

Wiener constructed this Gaussian measure on functions of one variable by subdividing the interval and imposing the scale invariance of the measure [27]. By subdividing planes into triangles and space into tetrahedra, we can construct analogous distributions for several variables. The real space renormalization group transformations used in this process is similar to the one dimensional Ising Model described in

Sec. 1.2. Our construction of field theory is based on dividing space into simplex. Having assigned random variables at the vertices of these simplex, we look into the correlations between them and their dependence on the spatial distance between the points.

In the continuum limit what we construct is a quantum field theory of fractional scaling dimension, with a correlation function that varies as a power of distance. There are many approaches to constructing quantum field theories in fractional dimensions [28, 29] in part motivated by the success of dimensional regularization. This power is often positive, but we also find a few cases where it is negative. An example of a system with growing correlation function will be that of Brownian motion. The average distance between two points undergoing Brownian motion increases with time as $\langle (x(t_1) - x(t_2))^2 \rangle \sim |t_1 - t_2|$. The quantum nature of the theory sets in as we consider functional integrals over all possible configurations. All possible paths are taken into account with different weights depending on the action associated to the path.

The above correlation is obtained using the Wiener measure which physicists denote by

$$\int [\mathcal{D}x]\, e^{-\frac{1}{2}\int x'^2(t)dt}.$$

Exactly what does this mean? There certainly is no Lebesgue measure which can be called $\mathcal{D}x$ in the space of functions because that space is infinite dimensional [30]. Also, the functions $x(t)$ are not always differentiable but have probability one under the Wiener measure. $x'(t)$ is at best a distribution and does not have a sensible product. $x'(t)^2$ is by itself meaningless.

We recall that the derivative $\frac{dy}{dx}$ in calculus is to be understood as a limit constructed using epsilons and deltas of modern analysis, and not as some infinitesimal

divided by another. Similarly, the path integral is to be understood as a limit of finite dimensional integrals. The action being the integral of a local Lagrangian (a function of the field and its derivative) is not to be taken literally. Indeed, it is a limit of a sequence of local measures (that depends only on nearest neighbors of the triangulation) and may not tend even formally to a local action principle. They can be viewed as quantum analogues of the Laplace's equations studied on fractals by Strichartz [31].

3.2 Partitioning the Real Line

In order to construct our measures over several variables we start off with the simplest case for a measure on the real line. Suppose that ϕ_0 and ϕ_1 are the boundary values of the field on the interval $[x_0, x_1]$ of length $a = x_1 - x_0$. Integrating out the field everywhere on the inside of the interval induces some measure $h(\phi_1, x_1|\phi_0, x_0)$ on the boundary values. The simplest possibility is a Gaussian that is translation invariant in field values,

$$h(\phi_1, x_1|\phi_0, x_0) = C_a e^{-\frac{1}{2}\alpha_a(\phi_1-\phi_0)^2} \tag{3.2.1}$$

for some α_a and C_a which can depend on the length of the interval $|x_1 - x_0|$. If we subdivide the interval into two and integrate over the field at the new point, we should get back the original measure. Let us choose the new point to be the midpoint of the interval $x_{01} = \frac{1}{2}(x_1 + x_0)$.

$$\int h(\phi_1, x_1|\phi_{01}, x_{01})h(\phi_{01}, x_{01}|\phi_0, x_0)d\phi_{01} = h(\phi_1, x_1|\phi_0, x_0) \tag{3.2.2}$$

This leads to the condition

$$C_{\frac{a}{2}}^2 \int e^{-\frac{1}{2}\alpha_{\frac{a}{2}}\left[(\phi_1-\phi_{01})^2+(\phi_{01}-\phi_0)^2\right]} d\phi_{10} = C_a e^{-\frac{1}{2}\alpha_a(\phi_1-\phi_0)^2}, \tag{3.2.3}$$

$$C_a = C_{\frac{a}{2}}^2. \tag{3.2.4}$$

By completing the square, we get

$$C_{\frac{a}{2}}^2 \int e^{-\frac{1}{2}\alpha_{\frac{a}{2}}\left[2\left(\phi_{01}-\frac{\phi_0+\phi_1}{2}\right)^2\right]} e^{-\frac{1}{2}\alpha_{\frac{a}{2}}\frac{(\phi_1-\phi_0)^2}{2}} d\phi_{10} = C_a e^{-\frac{1}{2}\alpha_a(\phi_1-\phi_0)^2}. \tag{3.2.5}$$

Thus,

$$\frac{1}{2}\alpha_{\frac{a}{2}} = \alpha_a \tag{3.2.6a}$$

$$C_{\frac{a}{2}}^2\sqrt{\frac{\pi}{\alpha_{\frac{a}{2}}}} = C_a \tag{3.2.6b}$$

On solving the above set of relations, we get

$$\alpha_a = \frac{\alpha}{a}, \tag{3.2.7a}$$

$$C_a = \sqrt{\frac{\alpha}{2\pi a}}, \tag{3.2.7b}$$

$$h(\phi_1, x_1|\phi_0, x_0) = \frac{\sqrt{\alpha}}{\sqrt{2\pi(x_1-x_0)}} e^{-\frac{\alpha}{2(x_1-x_0)}(\phi_1-\phi_0)^2}. \tag{3.2.7c}$$

By repeated subdivisions of the interval we would get the integral

$$\int \prod_i d\phi_i e^{-\frac{\alpha}{2}\sum_i \frac{(\phi_{i+1}-\phi_i)^2}{x_{i+1}-x_i}}. \tag{3.2.8}$$

This is not to be taken literally: $\lim_{x_{i+1}\to x_i} \frac{\phi_{i+1}-\phi_i}{x_{i+1}-x_i}$ almost never exists and so ϕ'^2 is actually meaningless. In the limit of small size of the interval, the exponent can be

interpreted as

$$\frac{\alpha}{2}\int \phi'^2 dx,$$

justifying the physicists' notation for the Wiener measure.

$$\int \prod_i d\phi_i \, e^{-\frac{\alpha}{2}\sum_i \frac{(\phi_{i+1}-\phi_i)^2}{x_{i+1}-x_i}} \rightarrow \int \mathcal{D}\phi e^{-\frac{\alpha}{2}\int \phi'^2 dx}. \tag{3.2.9}$$

The quantity α may be related to mass divided by $\hbar$ in some physical system or it may represent the inverse of the diffusion coefficient for a different system.

3.3 Higher Dimensional Analogues

A higher dimensional analogue of this process of subdivision would involve triangulating space-time. We would subdivide each triangle (or more generally simplex) to get smaller ones by introducing extra vertices. Integrating out the field at the extra vertices should give back the original measure. Our idea is to find an approximation for the functional integral in quantum field theory by an integration of the scalar values at the vertices of a triangulation:

$$\int \mathcal{D}\phi e^{-S[\phi]} A\left(\phi(x_1), \cdots \phi(x_n)\right) \approx \int \prod_i d\phi_i \prod_f F_a\left(\phi(f)\right) A\left(\phi(x_1), \cdots \phi(x_n)\right).$$

A triangulation is chosen with the points $x_\alpha (\alpha = 1, \cdots, n)$ in the argument of A among the vertices. The measure of integration is assumed to be local; i.e., can be expressed as a product of factors $F_a\left(\phi_0, \phi_1, \phi_2\right)$ each depending only on the field values at the vertices of a single face f of the triangulation. The basic idea is that if we subdivide a triangle by introducing new vertices, and integrate over the field at

these new vertices, we must get back the contribution of the original triangle. These are field theoretic analogues of the hierarchical models [32] commonly studied in spin systems [33].

What kind of triangle can be sub-divided into two equal triangles which are each similar to the original one? A moment's reflection shows that a right isosceles triangle has this property. However, it has been shown that for higher dimensions, one can subdivide a d simplex in n dimensions into n^d similar simplexes [34].

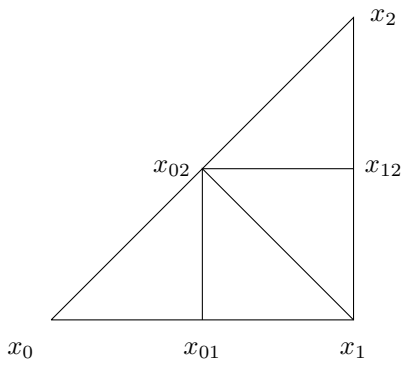

Figure 3.1: Subdivision of right triangle

Let us work out the integral in two dimensions to get a feel for the method before going to the much harder higher dimensional case. We start with the triangle (Fig. 3.1) with vertices $x_0 = (0,0),\ x_1 = (a,0),\ x_2 = (a,a)$. Let

$$F_a(\phi_0, \phi_1, \phi_2) = C_a e^{-Q_a(\phi_0,\phi_1,\phi_2)} \tag{3.3.1}$$

be the factor corresponding to such a triangle, for some normalization constant C_a and quadratic form Q_a. We assume that it is invariant under translations in the field variables for simplicity. It is likely that "massless" scalar fields may have this property which was observed in chapter 4. If the quadratic term was independent of

cross terms between two vertices, it may explain scalar fields in the "infinite mass" limit. The function must be symmetric under the interchange of the vertices x_0, x_2. Thus we have

$$Q_a(\phi_0, \phi_1, \phi_2) = \alpha_a \left((\phi_0 - \phi_1)^2 + (\phi_2 - \phi_1)^2\right) + \beta_a(\phi_0 - \phi_2)^2 \tag{3.3.2}$$

for some constants $\alpha_a,\ \beta_a$. This can also be expressed in matrix form as

$$Q_a(\phi_0, \phi_1, \phi_2) = \phi^T \hat{Q}_a \phi, \quad \hat{Q}_a = \begin{pmatrix} \alpha_a + \beta_a & -\alpha_a & -\beta_a \\ -\alpha_a & 2\alpha_a & -\alpha_a \\ -\beta_a & -\alpha_a & \alpha_a + \beta_a \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \end{pmatrix}. \tag{3.3.3}$$

If we choose the new internal vertices to be midpoints of the sides of the original triangle $x_0x_1x_2$

$$x_{ij} = \frac{1}{2}(x_i + x_j), \quad 0 \leq i < j \leq 2 \tag{3.3.4}$$

it divides the original triangle into the four equal triangles that are similar to the original. The new triangles ($x_0x_{01}x_{02,}\ x_{02}x_{01}x_1,\ x_1x_{12}x_{02},\ x_{02}x_{12}x_2$) have orthogonal edges which are half the length of the original triangle.

The reproducibility condition obtained after integrating out the field variables at the new vertices is

$$e^{-Q_{2a}(\phi_0,\phi_1,\phi_2)} = \int e^{-R_a(\phi_0,\phi_1,\phi_2,\phi_{01},\phi_{02},\phi_{12})} d\phi_{01} d\phi_{02} d\phi_{12}, \tag{3.3.5}$$

where (up to a normalization constant),

$$R_a(\phi_0,\phi_1,\phi_2,\phi_{01},\phi_{02},\phi_{12}) = Q_a(\phi_0,\phi_{01},\phi_{02}) + Q_a(\phi_{02},\phi_{01},\phi_1) + Q_a(\phi_1,\phi_{12},\phi_{02}) + Q_a(\phi_{02},\phi_{12},\phi_2). \tag{3.3.6}$$

This can be expressed in terms of a matrix form as

$$R_a(\phi_0,\phi_1,\phi_2,\phi_{01},\phi_{02},\phi_{12}) = \phi^T \hat{R}_a \phi. \tag{3.3.7}$$

The result of integrating out the new variables $\phi_{01},\phi_{02},\phi_{12}$ is best understood by splitting the matrix R into a block L that acts on the original variables (ϕ_0,ϕ_1,ϕ_2), another K acting on the new variables $(\phi_{01},\phi_{02},\phi_{12})$ and a third matrix J that mixes the two-

$$\hat{R}_a = \begin{pmatrix} L & J \\ J^T & K \end{pmatrix}, \tag{3.3.8a}$$

$$L = \begin{pmatrix} \alpha_a+\beta_a & 0 & 0 \\ 0 & 2\alpha_a+2\beta_a & 0 \\ 0 & 0 & \alpha_a+\beta_a \end{pmatrix}, \quad K = \begin{pmatrix} 4\alpha_a & -2\alpha_a & 0 \\ -2\alpha_a & 4\alpha_a+4\beta_a & -2\alpha_a \\ 0 & -2\alpha_a & 4\alpha_a \end{pmatrix}, \tag{3.3.8b}$$

$$J = -\begin{pmatrix} \alpha_a & \beta_a & 0 \\ \alpha_a & 2\beta_a & \alpha_a \\ 0 & \beta_a & \alpha_a \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_{01} \\ \phi_{02} \\ \phi_{12} \end{pmatrix}. \tag{3.3.8c}$$

The exponent in the above Gaussian can also be written as

$$R_a(\phi_0, \phi_1, \phi_2, \phi_{01}, \phi_{02}, \phi_{12}) = \phi_e^T L \phi_e + 2\phi_e^T J \phi_i + \phi_i^T K \phi_i, \tag{3.3.9}$$

where ϕ_i and ϕ_e denote the set of internal (new) and external (original) vertices of the triangle. On completion of squares we get

$$R_a = \phi_e^T L \phi_e + (\phi_i^T + \phi_e^T J^T (K^{-1})^T) K (\phi_i + K^{-1} J \phi_e) - \phi_e^T J K^{-1} J^T \phi_e. \tag{3.3.10}$$

The result of the integral over the internal vertices then turns out to be

$$\begin{aligned} \hat{Q}_{2a} &= L_a - J_a K_a^{-1} J_a^T \\ &= \begin{pmatrix} \frac{1}{8}(5\alpha_a + 6\beta_a) & \frac{1}{2}(-\alpha_a - \beta_a) & \frac{1}{8}(-\alpha_a - 2\beta_a) \\ \frac{1}{2}(-\alpha_a - \beta_a) & \alpha_a + \beta_a & \frac{1}{2}(-\alpha_a - \beta_a) \\ \frac{1}{8}(-\alpha_a - 2\beta_a) & \frac{1}{2}(-\alpha_a - \beta_a) & \frac{1}{8}(5\alpha_a + 6\beta_a) \end{pmatrix}. \end{aligned} \tag{3.3.11}$$

Fixed Points of the Transformation

The above result should be equal to the quadratic form

$$\alpha_{2a} \left[(\phi_0 - \phi_1)^2 + (\phi_1 - \phi_2)^2 \right] + \beta_{2a} (\phi_0 - \phi_2)^2 = \phi^T Q_{2a} \phi. \tag{3.3.12}$$

On comparing its form we get,

$$\alpha_{2a} = \frac{1}{2}(\alpha_a + \beta_a), \quad \beta_{2a} = \frac{1}{8}(\alpha_a + 2\beta_a). \tag{3.3.13}$$

If we define the ratio $x_a = \frac{\beta_a}{\alpha_a}$ we can write these relations as:

$$x_{2a} = f(x_a), \quad f(x) = \frac{1+2x}{4(1+x)} \tag{3.3.14a}$$

$$\alpha_{2a} = \lambda(x)\alpha_a, \quad \lambda(x) = \frac{1}{2}(1+x). \tag{3.3.14b}$$

We now seek a fixed point of the transformation $x_a \mapsto x_{2a}$. It is a quadratic equation with two roots,

$$\begin{aligned} f(x) &= x \\ x &= \frac{-1 \pm \sqrt{5}}{4}. \end{aligned} \tag{3.3.15}$$

The solution with negative x should be discarded as it does not give an integrable Gaussian. The positive root corresponds to the scaling relation

$$\alpha_{2a} = \lambda\alpha_a, \quad \lambda = \frac{1}{8}\left(3+\sqrt{5}\right),$$

thus we set

$$x = \frac{\sqrt{5}-1}{4}, \quad \lambda = \frac{1}{8}\left(3+\sqrt{5}\right). \tag{3.3.16}$$

If we make the ansatz $\alpha_a = a^{-\mu}\alpha_1$ we get

$$2^{-\mu} = \frac{3+\sqrt{5}}{8},$$

or

$$\mu = \frac{\log(\lambda^{-1})}{\log 2} = \frac{\log\left(\frac{8}{3+\sqrt{5}}\right)}{\log 2} \approx 0.611516. \tag{3.3.17}$$

To summarize, we have a scale covariant solution to the reproducibility condition

$$F_a(\phi_0,\phi_1,\phi_2) = C_a \exp\left(-a^{-\mu}\alpha_1\left\{(\phi_0-\phi_1)^2+(\phi_2-\phi_1)^2+\frac{\sqrt{5}-1}{4}(\phi_0-\phi_2)^2\right\}\right). \tag{3.3.18}$$

We can also determine the normalization constant using the relation

$$C_a^4\sqrt{\frac{\pi^3}{32(\alpha_a^3+2\alpha_a^2\beta_a)}} = C_{2a}, \tag{3.3.19}$$

but that is less interesting. The determinant of K determines the denominator.

Correlation function

Choose two points x_1, x_2 on the plane separated by some distance $L = |x_1 - x_2|$. Draw an iso-right triangle with the line connecting these points as a hypotenuse. We then bisect the right angle to get two iso-right triangles, subdivide them and continue this procedure until after a large number of subdivisions we get triangles with very small sides. We then integrate out all vertices expect the original triangle to calculate the correlation function-

$$G(x_{1,},x_2) = \frac{\int\prod_i d\phi_i \prod_f F_a\left(\phi(f)\right)\left(\phi(x_1)-\phi(x_2)\right)^2}{\int\prod_i d\phi_i\prod_f F_a\left(\phi(f)\right)}. \tag{3.3.20}$$

The integrals can be evaluated using the reproducibility property above. In the end we will end up with an integral over the two vertices of the original large triangle whose side is L.

$$G(x_{1,},x_2) = \frac{\int d\phi_0 d\phi_1 d\phi_2\left(\phi_1-\phi_2\right)^2 e^{-\frac{L^{-\mu}}{2}\alpha_1\left\{(\phi_0-\phi_1)^2+(\phi_2-\phi_1)^2+\frac{\sqrt{5}-1}{4}(\phi_0-\phi_2)^2)\right\}}}{\int d\phi_1 d\phi_2 e^{-\frac{L^{-\mu}}{2}\alpha_1\left\{(\phi_0-\phi_1)^2+(\phi_2-\phi_1)^2+\frac{\sqrt{5}-1}{4}(\phi_0-\phi_2)^2\right\}}}. \tag{3.3.21}$$

Thus, for some positive constant G,

$$G(x_1, x_2) = G|x_1 - x_2|^{\mu} \sim |x_1 - x_2|^{0.611516}. \tag{3.3.22}$$

Note that the magnitude of the correlations increase with distance. This is similar to the behavior of the average distance between two points experiencing a Brownian motion. On the other hand this is not the behavior expected from a two dimensional quantum field theory with action $S = -\frac{1}{2}\int |\nabla\phi|^2 d^2x = \frac{1}{2}(\phi, \Delta\phi)$. The correlations behave logarithmically then

$$\int \frac{d^2k}{(2\pi)^2}\frac{1}{|k|^2}e^{ik\cdot(x_1-x_2)} \sim \log|x_1 - x_2|. \tag{3.3.23}$$

An interpretation is that our formalism leads to a quantum field theory in a space of fractal dimension in between one and two.

Additional Examples on the Plane

In the case above, we get a correlation function that increases with distance. Are these exponents universal? Is it possible to find another subdivision procedure that will produce a correlation that decreases with distance? To answer these questions we worked out several additional examples of subdivision. It turns out that each subdivision of the plane (into squares, triangles of different shapes etc.) gives a different exponent for the correlation function. Also, in most cases other than two examples, we get correlation functions that grow with distance.

Subdividing a Square

We can also subdivide the plane into squares (Fig. 3.2). To each square of side a we can associate a Gaussian weight with quadratic form

$$Q_a(\phi_1, \phi_2, \phi_3, \phi_4) = \alpha_a \left[(\phi_1 - \phi_2)^2 + (\phi_2 - \phi_3)^2 + (\phi_3 - \phi_4) + (\phi_4 - \phi_1)^2\right] + \beta_a \left[(\phi_1 - \phi_3)^2 + (\phi_2 - \phi_4)^2\right]. \tag{3.3.24}$$

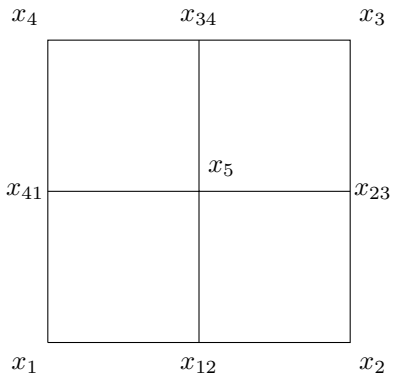

Figure 3.2: Subdivision of square

We subdivide each square by connecting the centroid x_5 to the midpoints of the sides $x_{12}, x_{23}, x_{34}, x_{41}$. If we then integrate out the five new vertices, we will get the transformations as

$$\alpha_{2a} = \frac{1}{4}(2\alpha_a + \beta_a), \quad \beta_{2a} = \frac{2\alpha_a^2 + 4\alpha_a\beta_a + \beta_a^2}{8\alpha_a + 4\beta_a}. \tag{3.3.25}$$

Fixed Points and Correlation Function

We get the condition for a fixed point of by setting $f(x_a) = x_{2a}$, where $x_a = \frac{\beta_a}{\alpha_a}$,

$$\frac{4\,(2+4x+x^2)}{(2+x)(8+4x)} = x. \tag{3.3.26}$$

Of the three solutions, only $x = -1, \sqrt{3}-1$ lead to positive eigenvalues

$$\alpha_{2a} = \lambda\alpha_a, \quad \lambda = \frac{1}{4}, \text{or } \frac{1}{4}\left(1+\sqrt{3}\right). \tag{3.3.27}$$

These two fixed points give growing correlations

$$G(x_1 - x_2) \sim |x_1 - x_2|^{\mu}, \quad \mu = 2, \text{or } 0.550016. \tag{3.3.28}$$

Apollonian Subdivision

Another example of a decreasing correlation function is the iterated Apollonian subdivision of a triangle: we join the vertices to an interior point, thereby subdividing the triangle into three triangles (Fig. 3.3). Another interpretation is in terms of circles: we can regard the plane as being subdivided into three mutually tangent circles centered at the vertices of a triangle. Then we can insert a new circle in the interstitial region that is tangent to the other three circle. This process can then be repeated. Every tangency of a pair of circles corresponds to an edge connecting their centers.

The scaling law of the correlation function can be computed just as before by

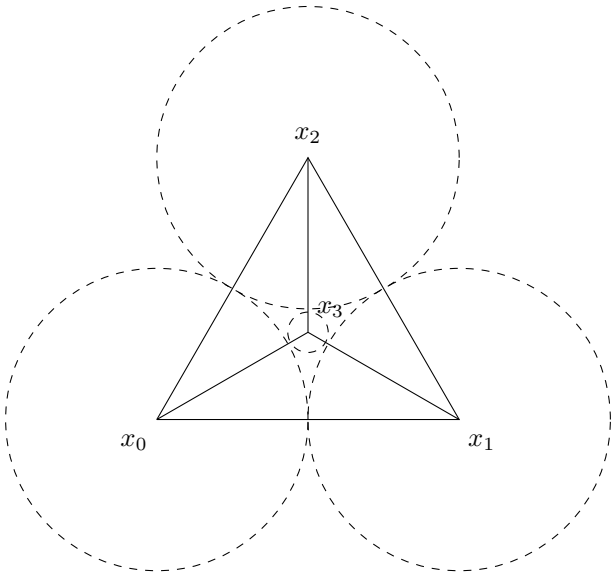

Figure 3.3: Apollonian subdivision

computing the Gaussian integral over the new vertex x_3-

$$e^{-\alpha_{\sqrt{3}a}\left[(\phi_0-\phi_1)^2+(\phi_0-\phi_2)^2+(\phi_1-\phi_2)^2\right]}$$
$$\propto \int e^{-\alpha_a\left[(\phi_0-\phi_1)^2+(\phi_0-\phi_2)^2+(\phi_1-\phi_2)^2+2(\phi_0-\phi_3)^2+2(\phi_1-\phi_3)^2+2(\phi_2-\phi_3)^2\right]}d\phi_3. \tag{3.3.29}$$

Each side emanating from vertex x_3 is shared by two triangles, which explains the factor of 2 in those terms. The terms are proportional as the scaling is not done rigorously here. We say that one third of the vertices are integrated out roughly after each step and we get $\sqrt{3}$ as the scale factor in two dimensions.

Fixed Points and Correlation Function

This leads to the transformation

$$\alpha_{\sqrt{3}a} = \frac{5}{3}\alpha_a, \tag{3.3.30}$$

and a correlation function with exponent

$$G(x_1, x_2) \sim |x_1 - x_2|^{\mu}, \quad \mu = \frac{\log \frac{3}{5}}{\log \sqrt{3}} \approx -0.929947. \tag{3.3.31}$$

Finally we see an example of a correlation decreasing with distance.

Division of a triangle into six similar triangles

Another example of a decreasing correlation function is the subdivision of an equilateral triangle, divided up into six equilateral triangles (Fig. 3.4). Topologically this is the same as the barycentric subdivision, or the bisection of the angles of the triangles [35]. We add a new point at the centroid (on the incenter) and connect it to the original vertices and midpoints of the sides. As long as the six smaller triangles occupy the same plane as the original one, it is not possible for them all to be equilateral. But we can assign equal lengths to all the edges as long as we do not insist that the graph has an isometric embedding in Euclidean space. By iterating the subdivision process, we will get a fractal which makes sense as a metric space, but not as a sub-manifold of Euclidean space.

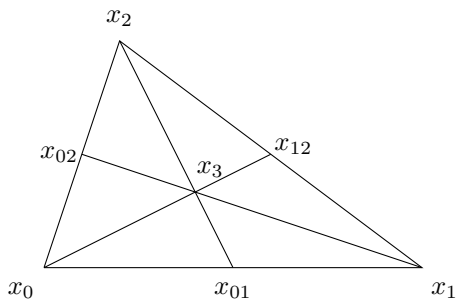

Figure 3.4: Six fold division of triangle

To each of the equilateral triangle we associate the Gaussian weight

$$Q_a(\phi_0, \phi_1, \phi_2) = \alpha_a \left[(\phi_0 - \phi_1)^2 + (\phi_1 - \phi_2)^2 + (\phi_2 - \phi_0)^2\right]. \tag{3.3.32}$$

The field variables at the vertices (x_{01}, x_{02}, x_{12}) and the center of the original triangle (x_3) are integrated over to give a larger equilateral triangle with the vertices (x_0, x_1, x_2). Using the same scaling argument from the previous case we say that the side of the smaller triangles is $\sqrt{6}$ smaller than the side of the triangle which is subdivided

$$e^{-Q_{\sqrt{6}a}(\phi_0,\phi_1,\phi_2)} = \int e^{-R_a(\phi_0,\phi_1,\phi_2,\phi_3,\phi_{01},\phi_{02},\phi_{12})} d\phi_3 d\phi_{01} d\phi_{12} d\phi_{12}, \tag{3.3.33}$$

with

$$\begin{aligned} R_a(\phi_0, \phi_1, \phi_2, \phi_3, \phi_{01}, \phi_{02}, \phi_{12}) = {} & Q_a(\phi_0, \phi_{01}, \phi_3) + Q_a(\phi_0, \phi_3, \phi_{02}) + Q_a(\phi_1, \phi_3, \phi_{01}) \\ & + Q_a(\phi_1, \phi_{12}, \phi_3) + Q_a(\phi_2, \phi_3, \phi_{12}) + Q_a(\phi_2, \phi_{02}, \phi_3). \end{aligned} \tag{3.3.34}$$

Fixed Points and Correlation Function

On computing the Gaussian integral and comparing it with the Gaussian form expected we get the following transformation

$$\alpha_{\sqrt{6}a} = \frac{5}{4}\alpha_a. \tag{3.3.35}$$

The new quadratic form for the triangle will be

$$Q_{\sqrt{6}a}(\phi_0, \phi_1, \phi_2) = \frac{5}{4}\alpha_a \left[(\phi_1 - \phi_0)^2 + (\phi_0 - \phi_2)^2 + (\phi_1 - \phi_2)^2\right], \tag{3.3.36}$$

thus the correlation function scales with exponent as

$$G(x_1 - x_2) \sim |x_1 - x_2|^{\mu}, \quad \mu = \frac{\log \frac{4}{5}}{\log \sqrt{6}} \approx -0.249078, \tag{3.3.37}$$

giving us another example of a correlation function that decreases with distance.

Subdividing Tetrahedrons

We extend this procedure of subdivision from the plane to three dimensional space. It is not possible to subdivide a tetrahedron into two that are similar to itself by a simple bisection. That would be equivalent to the ancient problem of doubling the cube, which is impossible by ruler-compass methods (i.e., the cube root of 2 cannot be found by such "Platonic" constructions). But it is possible to subdivide a tetrahedron into eight tetrahedrons, each similar to the original. The trick is to choose right isosceles tetrahedra, whose faces are all right triangles, based on three mutually orthogonal edges of equal length [36]. Fig. 3.5 shows an example of such a tetrahedron, the three mutually orthogonal edges being shown in bold lines.

To be more specific, the vertices

$$x_0 = (0,0,0), \quad x_1 = (a,0,0), \quad x_2 = (a,a,0), \quad x_3 = (a,a,a), \tag{3.3.38}$$

give a right isosceles tetrahedron of smallest side a (Fig. 3.6) where every face is a right triangle. There are three kinds of edges- of squared lengths $a^2, 2a^2, 3a^2$. In fact, the squared length of the edge x_{ij} is just $|i-j|a^2$, for $i,j = 0,1,2,3$. To each such tetrahedron we associate a function $T_a(\phi_0, \phi_1, \phi_2, \phi_3)$ which represents the result of integrating out the field in its interior.

The internal vertices of the subdivision are the six points $x_{ij} = \frac{1}{2}(x_i + x_j)$ for

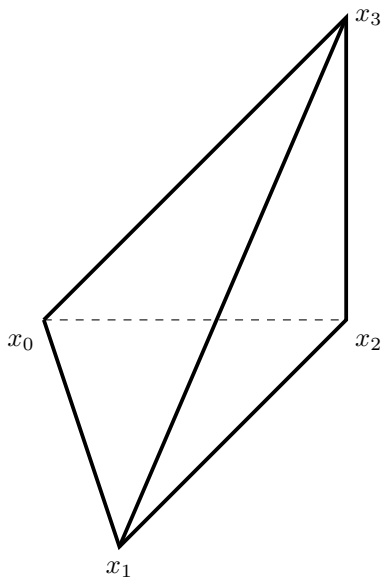

Figure 3.5: Right regular tetrahedron

$0 \leq i < j \leq 3$.

The reproducibility condition is the integral equation

$$\begin{aligned} T_{2a}(\phi_0, \phi_1, \phi_2, \phi_3) = \int & T_a(\phi_0, \phi_{01}, \phi_{02}, \phi_{03}) T_a(\phi_1, \phi_{01}, \phi_{02}, \phi_{03}) \\ & T_a(\phi_1, \phi_{12}, \phi_{02}, \phi_{03}) T_a(\phi_1, \phi_{12}, \phi_{13}, \phi_{03}) T_a(\phi_2, \phi_{12}, \phi_{02}, \phi_{03}) \\ & T_a(\phi_2, \phi_{12}, \phi_{13}, \phi_{03}) T_a(\phi_2, \phi_{23}, \phi_{13}, \phi_{03}) T_a(\phi_3, \phi_{23}, \phi_{13}, \phi_{03}) \\ & d\phi_{01} d\phi_{02} d\phi_{03} d\phi_{12} d\phi_{13} d\phi_{23}. \end{aligned} \tag{3.3.39}$$

Remark. *It is important to have the correct order of arguments in the T_a as we are not dealing with a regular tetrahedron. We have verified that with the given order above, the distance matrices are exactly half the distance matrix of the original tetrahedron.*

We assume a translation symmetry in field space (a global symmetry we expect

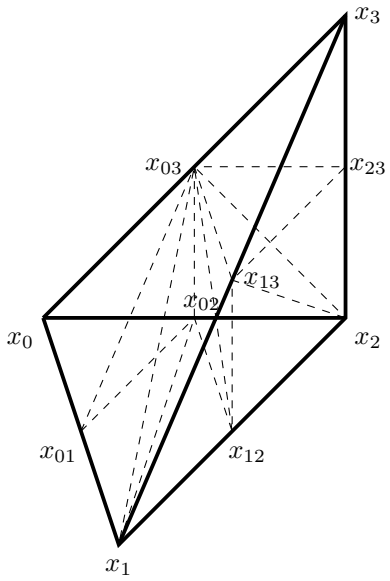

Figure 3.6: Subdivision of right regular tetrahedron

for massless fields) that is

$$T_a(\phi_0+\phi,\phi_1+\phi,\phi_2+\phi,\phi_3+\phi) = T_a(\phi_0,\phi_1,\phi_2,\phi_3).$$

Also, we assume a Gaussian ansatz for T_a,

$$T_a(\phi_0,\phi_1,\phi_2,\phi_3) = e^{-Q_a(\phi_0,\phi_1,\phi_2,\phi_3)} \tag{3.3.40}$$

$$\begin{aligned} Q_a(\phi_0,\phi_1,\phi_2,\phi_3) &= \alpha\left((\phi_0-\phi_1)^2+(\phi_3-\phi_2)^2\right)+\beta\left((\phi_0-\phi_2)^2+(\phi_3-\phi_1)^2\right) \\ &\quad +\gamma(\phi_0-\phi_3)^2+\delta(\phi_1-\phi_2)^2. \end{aligned} \tag{3.3.41}$$

This builds in the translation invariance as well as the reflection symmetry (0123) $\rightleftarrows$ (3210) of the right isosceles tetrahedron.

The exponent of the integral (3.3.39) is a quadratic function of ten variables-

$$\begin{aligned} R_a(\phi_0,\phi_1,\phi_2,\phi_3,\phi_{01},\phi_{02},\phi_{03},\phi_{12},\phi_{13},\phi_{23}) = {} & Q_a(\phi_0,\phi_{01},\phi_{02},\phi_{03}) + Q_a(\phi_1,\phi_{01},\phi_{02},\phi_{03}) \\ & + Q_a(\phi_1,\phi_{12},\phi_{02},\phi_{03}) + Q_a(\phi_1,\phi_{12},\phi_{13},\phi_{03}) \\ & + Q_a(\phi_2,\phi_{12},\phi_{02},\phi_{03}) + Q_a(\phi_2,\phi_{12},\phi_{13},\phi_{03}) \\ & + Q_a(\phi_2,\phi_{23},\phi_{13},\phi_{03}) + Q_a(\phi_3,\phi_{23},\phi_{13},\phi_{03}). \end{aligned} \tag{3.3.42}$$

This is equal to $\frac{1}{2}\phi^T\hat{R}\phi$ for some 10×10 symmetric matrix $\hat{R}$, which can be calculated by evaluating the second derivatives (Hessian), a straightforward procedure in Mathematica. This matrix can be split into blocks

$$\hat{R} = \begin{pmatrix} L & J \\ J^T & K \end{pmatrix} \tag{3.3.43}$$

where K is a matrix in the six dimensional subspace of the internal variables $(\phi_{01},\cdots,\phi_{23})$, L is a matrix in the four dimensional subspace of the external variables $(\phi_0,\cdots,\phi_3)$ and J is a 4×6 matrix that mixes them. The result of evaluating the integral over the internal variables by completing the squares is again a Gaussian with the Hessian matrix

$$L - JK^{-1}J^T. \tag{3.3.44}$$

Fixed Points of the Transformation

Thus we can write the renormalization group transformation as

$$\hat{Q}_{2a} = L_a - J_aK_a^{-1}J_a^T. \tag{3.3.45}$$

On comparing with the form of Q_a, the parameters $\alpha_{2a} \cdots \delta_{2a}$ will be determined as some homogeneous rational functions of $\alpha_a \cdots \delta_a$. The ratios

$$\frac{\beta_{2a}}{\alpha_{2a}}, \frac{\gamma_{2a}}{\alpha_{2a}}, \frac{\delta_{2a}}{\alpha_{2a}} \tag{3.3.46}$$

depend only on the variables $x = \frac{\beta_a}{\alpha_a}, y = \frac{\gamma_a}{\alpha_a}, z = \frac{\delta_a}{\alpha_a}$; i.e. α_a and α_{2a} can be scaled out.

Thus we get a map of the projective space P^3 to itself. Each fixed point (x^*, y^*, z^*) of this map is a Gaussian that is mapped into itself under our transformation, except for an overall multiplication of the quadratic form by $\frac{\alpha_{2a}}{\alpha_a}$ (the "eigenvalue" λ) evaluated at (x^*, y^*, z^*). A sensible solution must have a positive eigenvalue and also the parameters (x^*, y^*, z^*) must define a positive quadratic form Q_a. The equations for the fixed points can be reduced to a polynomial for x-

$$\begin{aligned} &-2 + 22x + 132x^2 + 380x^3 + 699x^4 + 188x^5 - 1639x^6 - 2488x^7 - 948x^8 \\ &+ 594x^9 + 608x^{10} + 152x^{11} = 0. \end{aligned} \tag{3.3.47}$$

The fixed point values for the other variables are determined in terms of this root-

$$\begin{aligned} y = \frac{1}{107606064063}(&-88214437948 - 533552485049x - 1626406118465x^2 - 3995440895007x^3 \\ &- 4531514030431x^4 + 6599814153205x^5 + 18490138345312x^6 + 10764883382910x^7 \\ &- 3123153862430x^8 - 5175977687624x^9 - 1495324502248x^{10}) \end{aligned} \tag{3.3.48}$$

$$\begin{aligned} z = \frac{1}{35868688021}(&-16477061906 - 533552485049x - 1626406118465x^2 - 3995440895007x^3 \\ &- 4531514030431x^4 + 6599814153205x^5 + 18490138345312x^6 + 10764883382910x^7 \\ &- 3123153862430x^8 - 5175977687624x^9 - 1495324502248x^{10}). \end{aligned} \tag{3.3.49}$$

Of the eleven solutions, only five are real and among them exactly one leads to a positive quadratic form Q_a

$$x \approx 0.699787174329122, \quad y \approx 0.182495561519144, \quad z \approx 2.547486684557433. \tag{3.3.50}$$

The eigenvalue i.e. the ratio for this solution is $\frac{\alpha_{2a}}{\alpha_a} = \lambda \approx 0.7958942$.

As for the right triangles, the correlation function can now be calculated. We can choose any two vertices on the original tetrahedron for this purpose and find the correlation between them. The choice of vertices is not as important as the distance between them, which is very large compared to the sides of the tetrahedrons obtained by subdivision. The field variables in those vertices obtained by subdivision are integrated out. The correlations will scale as $\frac{\log(\lambda^{-1})}{\log 2}$ when the distance is doubled. That is,

$$G(x_1, x_2) \sim |x_1 - x_2|^{\frac{\log \lambda}{\log 2}} \sim |x_1 - x_2|^{0.329351}. \tag{3.3.51}$$

3.4 Extension to Discrete Distributions

We now extend our ideas to non Gaussian measures in order to find non trivial fixed points. To do so, we consider the same simplexes from the previous examples. The field values (ϕ_i) are no more assigned random values but only discrete values ($\sigma_i = \pm 1$) similar to the Ising model.

Ising Model in One Dimension

We carry out the idea of subdivision of the real line once more, only instead of integrals we end up calculating discrete sums. Let the function associated with an

interval $[x_0, x_1]$ be-

$$F_a(\sigma_0, \sigma_1) = e^{Q_a(\sigma_0, \sigma_1)} \tag{3.4.1a}$$

$$Q_a(\sigma_0, \sigma_1) = \alpha_a(\sigma_0\sigma_1) + \gamma_a. \tag{3.4.1b}$$

There will not be any higher order terms of σ_i since $\sigma_i^2 = 1$ and $\sigma_i\sigma_j = i\epsilon_{ijk}\sigma_k$. We divide the interval into two halves now and sum over the new variables.

$$\begin{aligned}\sum_{\sigma_{01}} e^{Q_a(\sigma_0,\sigma_{01})} e^{Q_a(\sigma_{01},\sigma_1)} &= \sum_{\sigma_{01}} e^{\alpha_a\sigma_{01}(\sigma_0+\sigma_1)+2\gamma_a} \\ &= \left[e^{\alpha_a(\sigma_0+\sigma_1)} + e^{-\alpha_a(\sigma_0+\sigma_1)}\right] e^{2\gamma_a}. \end{aligned} \tag{3.4.2}$$

Fixed Points and Correlation Function

On comparing with the function associated with the whole interval

$$F_{2a}(\sigma_0, \sigma_1) = e^{\alpha_{2a}(\sigma_0\sigma_1)+\gamma_{2a}}, \tag{3.4.3}$$

we get the following pair of equations considering all the possible values taken by the pair (σ_0, σ_1),

$$2e^{2\gamma_a} = e^{\gamma_{2a}-\alpha_{2a}} \tag{3.4.4a}$$

$$e^{2\gamma_a}\left(e^{2\alpha_a} + e^{-2\alpha_a}\right) = e^{\gamma_{2a}+\alpha_{2a}}. \tag{3.4.4b}$$

The transformation obtained from these equations is

$$e^{2\alpha_{2a}} = \frac{e^{2\alpha_a} + e^{-2\alpha_a}}{2}. \tag{3.4.5}$$

The trivial fixed point $\alpha_a = 0$ is obtained by setting $\alpha_{2a} = \alpha_a$. The normalization constant can be determined from the relation between γ_a and γ_{2a}.

Ising Model on a Right Triangle in the Plane

We carry out the idea of sub division for a triangle now (Fig. 3.7). Let the function associated with the triangle be

$$F_a(\sigma_0, \sigma_1, \sigma_2) = e^{Q_a(\sigma_0,\sigma_1,\sigma_2)} \tag{3.4.6a}$$

$$Q_a(\sigma_0, \sigma_1, \sigma_2) = \alpha_a(\sigma_0\sigma_1 + \sigma_1\sigma_2) + \beta_a(\sigma_0\sigma_2) + \gamma_a. \tag{3.4.6b}$$

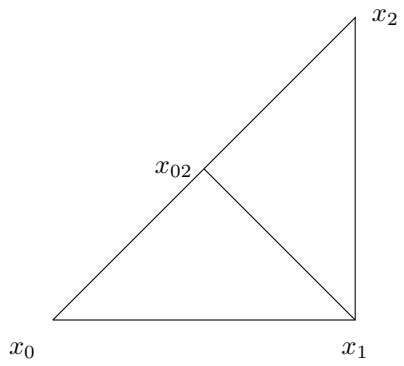

Figure 3.7: Subdivision of right triangle for Ising model

We now divide the original triangle into two new similar triangles. The new vertex is summed over and the total contribution from the two triangles is computed:

$$\begin{aligned}\sum_{\sigma_{02}} e^{Q_a(\sigma_1,\sigma_{02},\sigma_0)} e^{Q_a(\sigma_2,\sigma_{02},\sigma_1)} &= \sum_{\sigma_{02}} e^{\alpha_a\sigma_{02}(\sigma_0+\sigma_2+2\sigma_1)+\beta_a\sigma_1(\sigma_0+\sigma_2)+2\gamma_a} \\ &= \left[e^{\alpha_a(\sigma_0+\sigma_2+2\sigma_1)} + e^{-\alpha_a(\sigma_0+\sigma_2+2\sigma_1)}\right] e^{\beta_a\sigma_1(\sigma_0+\sigma_2)+2\gamma_a}.\end{aligned} \tag{3.4.7}$$

Fixed Points and Correlation Function

On comparing it with the form of the energy associated with the bigger triangle

$$F_{\sqrt{2}a}(\sigma_0,\sigma_1,\sigma_2) = e^{\alpha_{\sqrt{2}a}\sigma_1(\sigma_0+\sigma_2)+\beta_{\sqrt{2}a}(\sigma_0\sigma_2)+\gamma_{\sqrt{2}a}}, \tag{3.4.8}$$

we get the following pairs of equations:

$$e^{2(\gamma_a+\beta_a)}\left(e^{4\alpha_a}+e^{-4\alpha_a}\right) = e^{2\alpha_{\sqrt{2}a}+\beta_{\sqrt{2}a}+\gamma_{\sqrt{2}a}} \tag{3.4.9a}$$

$$e^{2\gamma_a}\left(e^{2\alpha_a}+e^{-2\alpha_a}\right) = e^{-\beta_{\sqrt{2}a}+\gamma_{\sqrt{2}a}} \tag{3.4.9b}$$

$$2e^{2(\gamma_a-\beta_a)} = e^{-2\alpha_{\sqrt{2}a}+\beta_{\sqrt{2}a}+\gamma_{\sqrt{2}a}}. \tag{3.4.9c}$$

In order to obtain these equations we look into all the tuples formed by assigning values to $(\sigma_0,\sigma_1,\sigma_2)$. Out of the set of eight equations we get only three independent equations. The transformations associated with these equations are,

$$e^{4\alpha_{\sqrt{2}a}} = \frac{e^{4\beta_a}\left(e^{4\alpha_a}+e^{-4\alpha_a}\right)}{2} \tag{3.4.10a}$$

$$e^{4\beta_{\sqrt{2}\alpha}} = \frac{2\left(e^{4\alpha_a}+e^{-4\alpha_a}\right)}{\left(e^{2\alpha_a}+e^{-2\alpha_a}\right)^2}. \tag{3.4.10b}$$

The trivial fixed points $\alpha_a = 0$, $\beta_a = 0$ are obtained by setting $\alpha_{\sqrt{2}a} = \alpha_a$, $\beta_{\sqrt{2}a} = \beta_a$. γ_a and $\gamma_{\sqrt{2}a}$ again determine the normalization factor.

Ising Model on sixfold subdivision of a triangle

Having found the obvious fixed points $\alpha_a = 0$, we now investigate the sixfold subdivision of the triangle to see if it has a non trivial fixed point, and indeed it has one.

To each of the equilateral triangles we associate a function-

$$F_a(\sigma_0,\sigma_1,\sigma_2) = e^{Q_a(\sigma_0,\sigma_1,\sigma_2)}, \tag{3.4.11a}$$

$$Q_a(\sigma_0,\sigma_1,\sigma_2) = \alpha_a(\sigma_0\sigma_1+\sigma_1\sigma_2+\sigma_2\sigma_0)+\gamma_a. \tag{3.4.11b}$$

A sum is now carried over the vertices $(\sigma_{01},\sigma_{02},\sigma_{12},\sigma_3)$ to form a bigger equilateral triangle with the vertices $(\sigma_0,\sigma_1,\sigma_2)$,

$$\begin{aligned}
&\sum_{\sigma_{01}}\sum_{\sigma_{02}}\sum_{\sigma_{12}}\sum_{\sigma_3} e^{Q_a(\sigma_1,\sigma_{12},\sigma_3)}e^{Q_a(\sigma_2,\sigma_{12},\sigma_3)}e^{Q_a(\sigma_2,\sigma_{02},\sigma_3)}e^{Q_a(\sigma_0,\sigma_{02},\sigma_3)}e^{Q_a(\sigma_0,\sigma_{01},\sigma_3)}e^{Q_a(\sigma_1,\sigma_{01},\sigma_3)}\\
&= e^{-6\alpha_a+6\gamma_a}\big(2+e^{-2(-2+\sigma_2+\sigma_0)\alpha_a}+e^{2(2+\sigma_2+\sigma_0)\alpha_a}+e^{-2(-2+\sigma_2+\sigma_1)\alpha_a}+e^{2(2+\sigma_2+\sigma_1)\alpha_a}\\
&+e^{-2(-2+\sigma_0+\sigma_1)\alpha_a}+e^{2(2+\sigma_0+\sigma_1)\alpha_a}+e^{-4(-3+\sigma_2+\sigma_0+\sigma_1)\alpha_a}+e^{4(3+\sigma_2+\sigma_0+\sigma_1)\alpha_a}\\
&+e^{-2(-4+2\sigma_2+\sigma_0+\sigma_1)\alpha_a}+e^{2(4+2\sigma_2+\sigma_0+\sigma_1)\alpha_a}+e^{-2(-4+\sigma_2+2\sigma_0+\sigma_1)\alpha_a}+e^{2(4+\sigma_2+2\sigma_0+\sigma_1)\alpha_a}\\
&+e^{-2(-4+\sigma_2+\sigma_0+2\sigma_1)\alpha_a}+e^{2(4+\sigma_2+\sigma_0+2\sigma_1)\alpha_a}\big).
\end{aligned} \tag{3.4.12}$$

Fixed Points and Correlation Function

The function associated with the bigger triangle should be of the form

$$F_{\sqrt{6}a}(\sigma_0,\sigma_1,\sigma_2) = e^{\alpha_{\sqrt{6}a}(\sigma_1\sigma_0+\sigma_0\sigma_2+\sigma_2\sigma_1)+\gamma_{\sqrt{6}a}}. \tag{3.4.13}$$

We get the following pairs of equations when comparing the previous results by assigning all admissible values to $(\sigma_0,\sigma_1,\sigma_2)$:

$$e^{-\alpha_{\sqrt{6}a}+\gamma_{\sqrt{6}a}} = e^{-6(\alpha_a-\gamma_a)}\left(3+6e^{4\alpha_a}+4e^{8\alpha_a}+2e^{12\alpha_a}+e^{16\alpha_a}\right) \tag{3.4.14a}$$

$$e^{3\alpha_{\sqrt{6}a}+\gamma_{\sqrt{6}a}} = e^{-6(\alpha_a-\gamma_a)}\left(9+3e^{8\alpha_a}+3e^{16\alpha_a}+e^{24\alpha_a}\right). \tag{3.4.14b}$$

Thus the renormalization group transformation on α is the iteration of the function

$$f(\alpha_a) = \frac{-1}{4} \log \left[\frac{3 + 6e^{4\alpha_a} + 4e^{8\alpha_a} + 2e^{12\alpha_a} + e^{16\alpha_a}}{9 + 3e^{8\alpha_a} + 3e^{16\alpha_a} + e^{24\alpha_a}} \right]. \tag{3.4.15}$$

There is a trivial fixed point for the equation $\alpha_a = f(\alpha_a)$ at $\alpha_a = 0$ and a non trivial fixed point at $\alpha_{a*} = 0.159579$. The normalization constants are determined by γ_a and $\gamma_{\sqrt{6}a}$.

To find the critical exponents we linearize the recursion relation at the fixed point.

$$f'(\alpha_{a*}) = 1.93411. \tag{3.4.16}$$

This shows that the fixed point is unstable. The scaling factor for this transformation is $\sqrt{6}$. Using these numbers we can calculate the thermal exponent $y_t \simeq \frac{\log(1.93411)}{\log(\sqrt{6})} = 0.736312$. It leads to the critical exponents for the correlation length and specific heat to be $\nu = \frac{1}{y_t} = 1.35812$, $\alpha = 2(1 - \nu) = -0.716239$.

Summary

We list the shortcomings of the subdivisions of simplex used in this chapter-

- The correlation functions obtained vary as irrational power of the distance in the continuum limit. This suggests that the theory corresponds to field theories in spaces of fractional dimensions even though the simplex lies in integer dimensions. The strange fact is that the irrational power is always less than 2.

- The edge weights of the simplex are always considered to be positive real numbers.

- Some of the internal edges are over counted in the action. This may be the

cause for the weird power law.

- Different subdivisions lead to different powers and no universality is obtained.
- The size of the geometrical object was not taken into account.

We attempt to address these issues and find a different subdivision scheme for the two dimensional simplex in chapter 4.

CHAPTER 4

FINITE ELEMENT METHOD IN FIELD THEORY

> *"Philosophy is written in that great book which ever lies before our eyes I mean the universe but we cannot understand it if we do not first learn the language and grasp the symbols, in which it is written. This book is written in the mathematical language, and the symbols are triangles, circles and other geometrical figures, without whose help it is impossible to comprehend a single word of it; without which one wanders in vain through a dark labyrinth."*
>
> — Galileo Galilei, *The Metaphysical Foundations of Modern Science*

In the previous chapter 3, we came across a method to construct the partition function of the fields in a recursive manner. This however has certain limitations which we try to avoid in this chapter by using the finite element method. We show how the partition function for the self energy of a field defined on a triangular lattice can be calculated recursively using ideas of renormalization group transformations. Our formalism is based on a nice geometric construction.

4.1 Orientation

The most successful regularization method in understanding non-perturbative Quantum Field Theory (QFT) is the lattice method [37, 38], which replaces space-time by a periodically arranged finite set of points. Numerical simulations based on this are becoming increasingly accurate, therefore any attempt at a mathematical formulation of quantum field theory must build on this success and aim to improve upon it.

The classical analogue of the problem would be the solution of Partial Differential Equations (PDEs). In the early days a lattice with identically shaped fundamental regions was used for numerical solutions of PDEs. Later it was realized that using meshes adapted to the boundary conditions makes more economical use of computing resources by adding more points where the field varies rapidly and fewer where it varies slowly. The Finite Element Method [39, 40] was developed in the seventies: it allows fundamental cells to have different shapes and sizes and use sophisticated interpolation methods to model the field in the interior of each cell. This Discrete Differential Geometry is useful not only to solve PDEs, but also to model shapes for use in computer graphics [41, 42].

The analogue in Quantum Field Theory is to replace the periodic lattice with a mesh that contains different length scales. This approach has been looked at by groups the in past and met with varying degrees of success. The first approach in this direction was by Christ, Friedberg and Lee [43] (except they proposed to average over all locations of lattice points as a way to restore rotation invariance, which did not turn out to be helpful). There is also some early work by Bender, Guralnik and Sharp [44]. Patodi's FEM to solve the eigenvalue problem for Laplacians was not noticed by physicists at this time. Since then much of the work on Lattice Gauge Theories has been computational along with some analytic work [45].

We propose to adapt existing methods of QFT and develop new Finite Element Methods to understand the essential problem from Wilson's point of view: how to integrate out some variables and get an effective theory for the remaining degrees of freedom (for a recent review, see the volume [46]). The simplest case is the one dimensional lattice (the set of integers), which has a natural subdivision into even and odd numbered elements. By integrating out the odd sites and leaving only the even sites we are left with an identical lattice with a different separation between field points (Sec. 1.2). Unfortunately there is no simple procedure to extend this into higher dimensions.

A natural idea would be to divide space into triangles (simplices in higher dimensions) and to fit them together to form larger ones, allowing us to integrate out the interior vertices and obtain an effective large scale theory. An advantage of our regularization method is that the renormalization map can be calculated exactly. The transformation between the angles of the triangles from subsequent generations is obtained at each stage of the subdivision. The first examples [47] we constructed this way ignored the shape (information contained in the angles) of the triangles. The Finite Element Method used by engineers leads to a "cotangent formula" [48]. It approximates the Laplacian in two dimensions on one hand and also happens to be a fixed point [49] of the renormalization dynamics. We determine this dynamics explicitly [50].

We expect this fixed point to be a continuum limit on a fractal, analogous to the Bethe lattice for which the renormalization group can be exactly calculated. Such QFTs can serve as approximations to theories on Euclidean spaces. Or perhaps at short distances, space-time really is not Euclidean.

If generalized to the Ising model, nonlinear sigma models or to four dimensional field theories, we could get interesting examples of Discrete Conformal Field Theory.

In our approach we do not average over triangulations. Such an average has been proposed as an approach to quantum gravity [51] and as a way to restore translation invariance.

4.2 Cotangent Formula

In the early days of computational engineering, Duffin derived a formula for the discrete approximation for the energy of an electrostatic field on a planar domain. In this Finite Element Method the plane is divided into triangles where the field is specified at each vertex and the energy of the field is the sum of contributions from each triangle. An approximation for the energy of a triangle is obtained by linear interpolation of the field to the interior.

Suppose the vertices x_0, x_1, x_2 correspond to field values ϕ_0, ϕ_1, ϕ_2. Each point x in the interior of the triangle divides it into three sub-triangles with vertices $\{x, x_0, x_1\}$, $\{x, x_1, x_2\}$ and $\{x, x_2, x_0\}$ respectively.

If the ratio of the area of a sub-triangle opposite to x_0 to the larger triangle is

$$u_0 = \frac{\Delta(x, x_1, x_2)}{\Delta(x_0, x_1, x_2)}, \tag{4.2.1}$$

then

$$x = u_0 x_0 + u_1 x_1 + u_2 x_2, \quad u_0 + u_1 + u_2 = 1, \quad u_0, u_1, u_2 > 0. \tag{4.2.2}$$

We can use the pair u_0, u_1 as coordinates instead of the cartesian components of x. The linear interpolation of the field values to the point x is then

$$\phi(x) = u_0 \phi_0 + u_1 \phi_1 + u_2 \phi_2. \tag{4.2.3}$$

The energy of the interpolated field inside a triangle on calculation turns out to be

$$S = \frac{1}{4}\left[a_2\left(\phi_0 - \phi_1\right)^2 + a_1\left(\phi_2 - \phi_0\right)^2 + a_0\left(\phi_1 - \phi_2\right)^2\right] \tag{4.2.4}$$

where a_0, a_1, a_2 are the cotangents of the angles at the vertices.

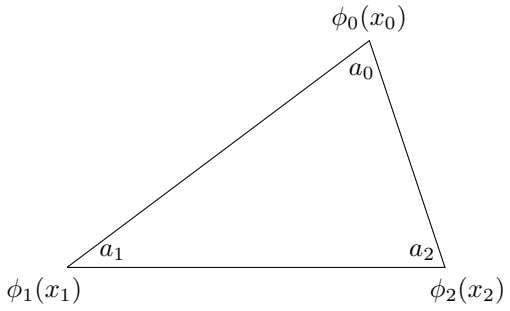

Figure 4.1: Triangle with labelled vertices and cotangents of the angles

Proof. Define the vectors along the sides of the triangle (see Fig. 4.1),

$$e_1^\mu = x_1^\mu - x_0^\mu, \quad e_2^\mu = x_2^\mu - x_0^\mu. \tag{4.2.5}$$

Using u^a for $a = 1, 2$ as coordinates,

$$x^\mu = u^a e_a^\mu \implies \partial_a x^\mu = e_a^\mu \tag{4.2.6}$$

Then the metric tensor of the plane in these coordinates has as components the dot products of the sides:

$$g_{ab} = e_a^\mu e_b^\nu \delta_{\mu\nu}, \quad g = \begin{pmatrix} |e_1|^2 & e_1 \cdot e_2 \\ e_1 \cdot e_2 & |e_2|^2 \end{pmatrix}. \tag{4.2.7}$$

Also, $\sqrt{\det g} = e_1 \times e_2$ is twice the area of the triangle. The cotangents are

$$a_0 = \frac{e_1 \cdot e_2}{e_1 \times e_2}, \quad a_1 = \frac{(e_2 - e_1) \cdot e_1}{(e_2 - e_1) \times e_1}, \quad a_2 = \frac{e_2 \cdot (e_2 - e_1)}{e_2 \times (e_2 - e_1)}. \tag{4.2.8}$$

Then,

$$a_0 + a_1 = \frac{|e_1|^2}{e_1 \times e_2}, \quad a_0 + a_2 = \frac{|e_2|^2}{e_1 \times e_2}, \tag{4.2.9}$$

and

$$\sqrt{\det g} g^{ab} = \begin{pmatrix} \frac{|e_2|^2}{e_1 \times e_2} & -\frac{e_1 \cdot e_2}{e_1 \times e_2} \\ -\frac{e_1 \cdot e_2}{e_1 \times e_2} & \frac{|e_1|^2}{e_1 \times e_2} \end{pmatrix} = \begin{pmatrix} a_0 + a_2 & -a_0 \\ -a_0 & a_0 + a_1 \end{pmatrix}. \tag{4.2.10}$$

Thus, using $\int d^2u = \frac{1}{2}$,

$$\begin{aligned} S &= \frac{1}{2} \int \sqrt{\det g} g^{ab} \partial_a \phi \partial_b \phi d^2u \\ &= \frac{1}{4} \left[(a_0 + a_2)(\phi_1 - \phi_0)^2 - 2a_0(\phi_1 - \phi_0)(\phi_2 - \phi_0) + (a_0 + a_1)(\phi_2 - \phi_0)^2 \right]. \end{aligned} \tag{4.2.11}$$

This can be rewritten as

$$S(\phi_0, \phi_1, \phi_2 | a_0, a_1, a_2) = \frac{1}{4} \left[a_0(\phi_1 - \phi_2)^2 + a_1 (\phi_2 - \phi_0)^2 + a_2 (\phi_0 - \phi_1)^2 \right] \tag{4.2.12}$$

as claimed. □

4.3 Triangles to Special Relativity

The space $\mathcal{S}$ of similarity classes of triangles (with marked vertices) is a hyperboloid [52]. This can be understood in several ways. A pair of sides of a triangle forms a

basis, thus the space of marked triangles may be identified with $GL(2,\mathbb{R})$: this group acts transitively and without a fixed point on the space of bases. Quotienting by rotation, scaling and reflection around a side gives

$$\mathcal{S} = GL(2,\mathbb{R})/\left(SO(2,\mathbb{R}) \times \mathbb{R}^+ \times \mathbb{Z}_2\right) = SL(2,\mathbb{R})/SO(2,\mathbb{R}).$$

which is a hyperboloid. This argument generalizes to n dimensions: the similarity classes of marked simplices is $GL(n,\mathbb{R})/\mathbb{R}^+ \times SO(n,\mathbb{R}) \times \mathbb{Z}_2 = SL(n,\mathbb{R})/SO(n,\mathbb{R})$.

An equivalent point of view is that $\mathcal{S}$ is the space of symmetric tensors of determinant one: a pair of sides of the triangles define a symmetric tensor through their inner products. By scaling we can choose this symmetric tensor to have determinant one. It is clear that $SL(2,\mathbb{R})$ acts on the space of such tensors transitively, with $SO(2,\mathbb{R})$ as the isotropy group at one point. Again this generalizes to n dimensions.

A more explicit point of view will be useful in what follows. A similarity class of marked triangles is determined by the angles at the vertices (or, for convenience, the cotangents of the angles). Since the angles $(\theta_0, \theta_1, \theta_2)$ of a triangle add to up π, the cotangents satisfy

$$a_0a_1 + a_1a_2 + a_2a_0 = 1, \quad a_i = \cot\theta_i \tag{4.3.1}$$

This can be written as

$$a^T\eta a = 1, \qquad a = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}, \quad \eta = \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}. \tag{4.3.2}$$

Since η has signature $(1,-1,-1)$, this is the equation for a time-like hyper-surface in

Minkowski space $\mathbb{R}^{1,2}$. Setting

$$p_0 = \frac{a_1 + a_2 + a_0}{\sqrt{3}}, \quad p_1 = \frac{a_2 - a_1}{2}, \quad p_2 = \frac{2a_0 - a_1 - a_2}{2\sqrt{3}}, \tag{4.3.3}$$

the "cotangent identity" (4.3.1) becomes the equation for a hyperboloid

$$p_0^2 - p_1^2 - p_2^2 = 1. \tag{4.3.4}$$

The quantity $4\,(a_0 + a_1 + a_2)$ is the ratio of the sum of squares of the sides to the area of the triangle. It is a minimum for an equilateral triangle and becomes large for a flat triangle (one with small area or large perimeter).

So far we discussed triangles with marked vertices but we should also consider invariant transformations of the vertices. The group S_3 of permutations of vertices is generated by the cyclic permutation

$$\sigma : 012 \mapsto 120$$

and the interchange of a pair of vertices

$$\tau : 012 \mapsto 021$$

$$S_3 = \langle \sigma, \tau | \sigma^3 = 1, \tau^2 = 1, \tau\sigma\tau = \sigma^2 \rangle. \tag{4.3.5}$$

These permutations act on the cotangents through the matrices

$$\sigma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{4.3.6}$$

We can also parametrize $\mathcal{S}$ by the complex number

$$z = \frac{a_1 + i}{a_1 + a_2}. \tag{4.3.7}$$

By a translation, we can choose the first vertex $x_1 = 0$ and by a rotation and scaling we may choose $x_2 = 1$. z is then the coordinate of the remaining vertex.

Then the permutation of the vertices becomes

$$\sigma(z) = \frac{1}{1-z}, \quad \tau(z) = 1 - \bar{z}. \tag{4.3.8}$$

By a reflection around the side 12, we can choose $a_0 + a_1 + a_2 > 0$; equivalently $\mathrm{Im}(z) > 0$. Note that τ is the reflection around the perpendicular from vertex 0 to the opposite side 12 of the triangle.

Subdivision of a triangle

We can subdivide a triangle into three sub-triangles of equal area by connecting the centroid $x_3 = \frac{x_0+x_1+x_2}{3}$ to the vertices x_0, x_1, x_2 by straight lines. (If we subdivide at some other interior point, we get similar results).

The cotangents of the angles of the sub-triangle opposite vertex 0 are given by

$$\cot(x_2 x_1 x_3) = 2a_1 + a_2 \tag{4.3.9a}$$

$$\cot(x_3 x_2 x_1) = 2a_2 + a_1 \tag{4.3.9b}$$

$$\cot(x_1 x_3 x_2) = \frac{a_0 - 2a_1 - 2a_2}{3} \tag{4.3.9c}$$

as shown in Fig. 4.2.

To see this, choose a coordinate system with $x_1 = (0, 0)$, $x_2 = (1, 0)$, $x_0 = (x, y)$

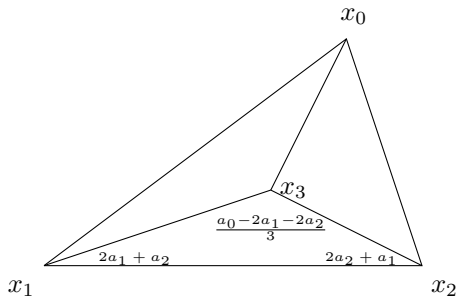

Figure 4.2: Triangle with few cotangents of the angles labelled after subdivision

so that $x_3 = (\frac{1+x}{3}, \frac{y}{3})$. Then,

$$a_1 = \frac{x}{y}, \quad a_2 = \frac{1-x}{y}, \quad a_0 = \frac{1-a_1a_2}{a_1+a_2}. \tag{4.3.10}$$

By dropping a perpendicular from x_3 to the side x_1x_2 we get

$$\cot(x_2x_1x_3) = \frac{\frac{1+x}{3}}{\frac{y}{3}} = 2a_1 + a_2, \quad \cot(x_3x_2x_1) = \frac{1-\frac{1+x}{3}}{\frac{y}{3}} = 2a_2 + a_1. \tag{4.3.11}$$

The remaining angle is given by solving the cotangent formula:

$$\cot(x_1x_3x_2) = \frac{1-(2a_1+a_2)(2a_2+a_1)}{(2a_1+a_2)+(2a_1+a_2)} = \frac{1-a_1a_2-2(a_1+a_2)^2}{3(a_1+a_2)} = \frac{a_0-2a_1-2a_2}{3}. \tag{4.3.12}$$

We can thus express the cotangents of this sub-triangle as Λa where

$$\Lambda = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}. \tag{4.3.13}$$

It is noted that

$$\Lambda^T \eta \Lambda = \eta,$$

due to the cotangent identity. Thus subdivisions are represented by Lorentz transformations in $\mathbb{R}^{1,2}$. The symmetry under the interchange of 1 and 2 leads to

$$\Lambda\tau = \tau\Lambda.$$

The cotangents of the remaining sub-triangles are given by cyclic permutations $\Lambda\sigma$ and $\Lambda\sigma^2$ (see Fig. 4.3). In this convention, the central angle is listed first.

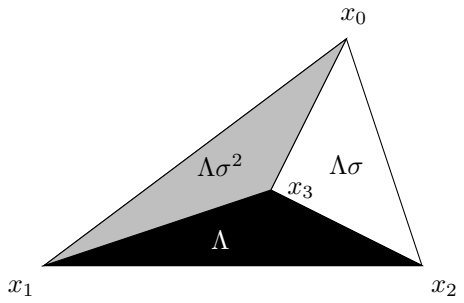

Figure 4.3: Action of Λ and σ matrices to produce subdivision of a triangle

In the complex parametrization $z = x + iy = \frac{a_1+i}{a_1+a_2}$, the subdivision Λ corresponds to

$$\Lambda(z) = \frac{1+z}{3}, \tag{4.3.14}$$

which is the complex co-ordinate of the centroid when $x_1 = 0$, $x_2 = 1$, $x_0 = z$. Recall that in this parametrization $\sigma(z) = \frac{1}{1-z}$. Clearly, both Λ and σ map the upper half plane to itself.

The semigroup generated by $\langle \Lambda, \Lambda\sigma, \Lambda\sigma^2 \rangle$ describe repeated subdivisions of a triangle. After many iterations, most of the triangles are flat: they have small area and

large perimeter [53]. The dynamics generated by this semigroup is the renormalization group of real space decimations.

4.4 Renormalization Dynamics

Consider a Gaussian scalar field with values ϕ_0, ϕ_1, ϕ_2 at the vertices of a triangle with cotangents a_0, a_1, a_2. The most general quadratic form for the discrete approximation to the action will be

$$S(\phi_0, \phi_1, \phi_2|a) = P(a)\phi_0^2 + Q(a)\phi_1\phi_2 + P(\sigma a)\phi_1^2 + Q(\sigma a)\phi_2\phi_0 + P(\sigma^2 a)\phi_2^2 + Q(\sigma^2 a)\phi_0\phi_1 \tag{4.4.1}$$

The coefficients $P(a), Q(a)$ are functions of the cotangents satisfying the symmetry

$$P(a) = P(\tau a), \quad Q(a) = Q(\tau a). \tag{4.4.2}$$

For example, the cotangent formula corresponds to the choice

$$P(a) = \frac{a_1 + a_2}{4}, \quad Q(a) = -\frac{a_0}{2}. \tag{4.4.3}$$

If we subdivide the triangle and associate a field ϕ_3 at the central vertex, the action will be the sum of contributions from each triangle.

$$\begin{aligned} S_{\text{sub}}(\phi_0, \phi_1, \phi_2, \phi_3|a) &= S(\phi_3, \phi_1, \phi_2|\Lambda a) + S(\phi_3, \phi_2, \phi_0|\Lambda\sigma a) + S(\phi_3, \phi_0, \phi_1|\Lambda\sigma^2 a) \\ &= A\phi_3^2 + B\phi_3 + C \end{aligned} \tag{4.4.4}$$

where

$$A = P(\Lambda a) + P(\Lambda\sigma a) + P(\Lambda\sigma^2 a)$$

$$B = \phi_0 \left\{Q(\sigma\Lambda\sigma a) + Q(\sigma^2\Lambda\sigma^2 a)\right\} + \phi_1 \left\{Q(\sigma^2\Lambda a) + Q(\sigma\Lambda\sigma^2 a)\right\} + \phi_2 \left\{Q(\sigma\Lambda a) + Q(\sigma^2\Lambda\sigma a)\right\}$$

$$C = \phi_0^2 \left[P(\sigma^2\Lambda\sigma a) + P(\sigma\Lambda\sigma^2 a)\right] + \phi_1^2 \left[P(\sigma\Lambda a) + P(\sigma^2\Lambda\sigma^2 a)\right] + \phi_2^2 \left[P(\sigma^2\Lambda a) + P(\sigma\Lambda\sigma a)\right]$$
$$+ \phi_1\phi_2 Q(\Lambda a) + \phi_2\phi_0 Q(\Lambda\sigma a) + \phi_0\phi_1 Q(\Lambda\sigma^2 a) \tag{4.4.5}$$

The effective action after integrating out the central field variable is given by

$$e^{-\tilde{S}(\phi_0,\phi_1,\phi_2|a)} = Z \int e^{-S_{\text{sub}}(\phi_0,\phi_1,\phi_2,\phi_3|a)} d\phi_3 \tag{4.4.6}$$

where $Z = \sqrt{\frac{3(a_0+a_1+a_2)}{2\pi}}$ is a normalization constant.

$$\tilde{S}(\phi_0, \phi_1, \phi_2|a) = C - \frac{B^2}{4A} \tag{4.4.7}$$

On comparing coefficient of ϕ_0^2, we get

$$\tilde{P}(a) = P(\sigma^2\Lambda\sigma a) + P(\sigma\Lambda\sigma^2 a) - \frac{\left[Q(\sigma\Lambda\sigma a) + Q(\sigma^2\Lambda\sigma^2 a)\right]^2}{4\left[P(\Lambda a) + P(\Lambda\sigma a) + P(\Lambda\sigma^2 a)\right]}. \tag{4.4.8}$$

On comparing coefficient of $\phi_1\phi_2$, we get

$$\tilde{Q}(a) = Q(\Lambda a) - \frac{\left[Q(\sigma^2\Lambda a) + Q(\sigma\Lambda\sigma^2 a)\right]\left[Q(\sigma\Lambda a) + Q(\sigma^2\Lambda\sigma a)\right]}{2\left[P(\Lambda a) + P(\Lambda\sigma a) + P(\Lambda\sigma^2 a)\right]}. \tag{4.4.9}$$

Using $\Lambda\tau = \tau\Lambda, \tau\sigma\tau = \sigma^2$ we can verify that $\tilde{P}(\tau a) = \tilde{P}(a), \tilde{Q}(\tau a) = \tilde{Q}(a)$ as needed for symmetry. The denominator

$$A(a) = P(\Lambda a) + P(\Lambda\sigma a) + P(\Lambda\sigma^2 a) \tag{4.4.10}$$

is invariant under σ, τ and hence, under all permutations.

The semigroup generated by the map $\mathcal{R} : (P, Q) \mapsto (\tilde{P}, \tilde{Q})$ on the space of pairs of functions on the hyperboloid is the renormalization dynamics ("renormalization group"). This explicit example should help us understand such dynamics. For example, is there is a notion of entropy that increases monotonically? Its fixed points correspond to some sort of continuum limit (which could be fractals) [54].

Fixed Points

An obvious fixed point consists of constant P, Q. This corresponds to the "Apollonian subdivisions" considered in an earlier paper.

We now show that the cotangent formula of the FEM,

$$P(a) = \frac{a_1 + a_2}{4}, \quad Q(a) = -\frac{a_0}{2} \tag{4.4.11}$$

is also a fixed point of the above dynamics. It is not hard to verify that

$$A = P(\Lambda a) + P(\Lambda\sigma a) + P(\Lambda\sigma^2 a) = \frac{3}{2}(a_0 + a_1 + a_2) \tag{4.4.12a}$$

$$Q(\sigma\Lambda\sigma a) + Q(\sigma^2\Lambda\sigma^2 a) = -(a_0 + a_1 + a_2) \tag{4.4.12b}$$

$$P(\sigma^2\Lambda\sigma a) + P(\sigma\Lambda\sigma^2 a) = \frac{1}{12}\left[2a_0 + 5(a_1 + a_2)\right] \tag{4.4.12c}$$

so that $\tilde{P}(a) = \frac{a_1+a_2}{4}$.

Similarly,

$$Q(\Lambda a) = \frac{-a_0 + 2a_1 + 2a_2}{6} \tag{4.4.13a}$$

$$Q(\sigma^2\Lambda a) + Q(\sigma\Lambda\sigma^2 a) = -(a_0 + a_1 + a_2) \tag{4.4.13b}$$

$$Q(\sigma\Lambda a) + Q(\sigma^2\Lambda\sigma a) = -(a_0 + a_1 + a_2) \tag{4.4.13c}$$

from which $\tilde{Q}(a) = -\frac{a_0}{2}$ follows.

This fixed point describes some sort of continuum limit of two dimensional scalar field theory. As in the examples of Ref. [47] it is likely to be a fractal of dimension less than two; but we have not been able to determine this dimension yet.

4.5 Correlation Function

The correlation functions of the fields is a quantity of great interest. We use our formalism to find the correlation for the fixed points of the action. The spatial dependence of the correlation functions turns out to have a very nice geometric interpretation.

We find the correlation function for the following action-

$$S(\phi_0, \phi_1, \phi_2|a) = \frac{1}{4}\left[a_2\left(\phi_0 - \phi_1\right)^2 + a_1\left(\phi_2 - \phi_0\right)^2 + a_0\left(\phi_1 - \phi_2\right)^2\right]. \tag{4.5.1}$$

By setting the field value at $\phi_0 = 0$, the expectation values of interest can be evaluated by this integral-

$$\langle\phi_i\phi_j\rangle = \frac{\int d\phi_1 d\phi_2 e^{-S(\phi_0,\phi_1,\phi_2|a)}\phi_i\phi_j}{\int d\phi_1 d\phi_2 e^{-S(\phi_0,\phi_1,\phi_2|a)}}, \quad i,j = 1,\,2 \tag{4.5.2}$$

For the choice of the fixed points as coefficients in the action,

$$\langle \phi_1^2 \rangle = 2\,(a_1 + a_0) \tag{4.5.3a}$$

$$\langle \phi_2^2 \rangle = 2\,(a_2 + a_0) \tag{4.5.3b}$$

$$\langle \phi_1 \phi_2 \rangle = 2a_0 \tag{4.5.3c}$$

If we choose two points x, y inside the triangle and set x_0 to be origin-

$$\begin{aligned} \mathbf{x} &= u_0\mathbf{x_0} + u_1\mathbf{x_1} + u_2\mathbf{x_2} \\ &= u_1\mathbf{x_1} + u_2\mathbf{x_2} \end{aligned} \tag{4.5.4}$$

$$u_i = \frac{\tau_{ix}}{\tau},\ i = 0, 1, 2, \quad \text{and} \quad \tau = \tau_{0x} + \tau_{1x} + \tau_{2x}. \tag{4.5.5}$$

Similarly,

$$\begin{aligned} \mathbf{y} &= v_0\mathbf{x_0} + v_1\mathbf{x_1} + v_2\mathbf{x_2} \\ &= v_1\mathbf{x_1} + v_2\mathbf{x_2} \end{aligned} \tag{4.5.6}$$

$$v_i = \frac{\tau_{iy}}{\tau},\ i = 0, 1, 2, \quad \text{and} \quad \tau = \tau_{0y} + \tau_{1y} + \tau_{2y}. \tag{4.5.7}$$

By our linear interpolation, the field values at those points turn out to be-

$$\phi(x) = u_1\phi_1 + u_2\phi_2 \tag{4.5.8a}$$

$$\phi(y) = v_1\phi_1 + v_2\phi_2. \tag{4.5.8b}$$

Thus,

$$|\mathbf{x}-\mathbf{y}|^2 = (u_1-v_1)^2\,|\mathbf{x_1}|^2 + (u_2-v_2)^2\,|\mathbf{x_2}|^2 + 2\,(u_1-v_1)\,(u_2-v_2)\,|\mathbf{x_1}.\,\mathbf{x_2}|$$
$$\frac{|\mathbf{x}-\mathbf{y}|^2}{\tau} = (u_1-v_1)^2\,\frac{|\mathbf{x_1}|^2}{\tau} + (u_2-v_2)^2\,\frac{|\mathbf{x_2}|^2}{\tau} + 2\,(u_1-v_1)\,(u_2-v_2)\,\frac{|\mathbf{x_1}.\,\mathbf{x_2}|}{\tau}. \tag{4.5.9}$$

However we know the following relations as well for this case -

$$\frac{|\mathbf{x_1}|^2}{\tau} = 2(a_0+a_1), \quad \frac{|\mathbf{x_2}|^2}{\tau} = 2(a_0+a_2), \quad 2\frac{|\mathbf{x_1}.\,\mathbf{x_2}|}{\tau} = 4a_0. \tag{4.5.10}$$

We use this to simplify (4.5.9)-

$$\begin{aligned}\frac{|\mathbf{x}-\mathbf{y}|^2}{\tau} &= (u_1-v_1)^2\,2(a_0+a_1) + (u_2-v_2)^2\,2(a_0+a_2) + (u_1-v_1)\,(u_2-v_2)\,4a_0 \\ &= (u_1-v_1)^2\,\langle\phi_1^2\rangle + (u_2-v_2)^2\,\langle\phi_2^2\rangle + 2\,(u_1-v_1)\,(u_2-v_2)\,\langle\phi_1\phi_2\rangle \\ &= \langle(\phi(x)-\phi(y))^2\rangle. \end{aligned} \tag{4.5.11}$$

Thus the correlation function for two interior points to be-

$$\langle(\phi(x)-\phi(y))^2\rangle = \frac{|\mathbf{x}-\mathbf{y}|^2}{\tau} \tag{4.5.12}$$

The correlation function between two points turns out to be the ratio of the Euclidean distance between the points and the area of the triangle. The result has an elegant geometrical description associated to it. An extension of this method to higher dimensions would be interesting.

Summary

The discrete approach to field theory developed in this chapter helps one to find an effective action. The striking features of this approach were-

- The action has a recursive nature at each level of barycentric subdivision of the triangles.
- The functional form of the action has surprising connections with the cotangent formula used by electrical engineers to solve differential equations by finite element method.
- The correlation of the fields between two points can be used to find the spatial distance between the points.

The interdisciplinary nature of the result made us curious about the other tools used by engineers. We shall show in chapter 5, how electrical resistor networks lead to a new perspective of distance where we evade divergences associated with correlation functions.

CHAPTER 5

METRIC FROM FIELD THEORY

"An experiment is a question which science poses to Nature, and a measurement is the recording of Nature's answer."
— Max Planck, *Scientific Autobiography and other papers*

The last few chapters have dwelt in explaining the regimes in which divergences occur in nature and the tools used to remove those divergences. Divergences are classified as ultraviolet divergences or infrared depending on whether they occur at short or large distances. The prevalent renormalization techniques attempt at redefining the coupling constants of the system so that one gets finite values for the observables in the short or long distance scales. In this chapter, we tried developing a new perspective of distance solely from the scalar field itself [55]. This notion of distance arises [56] from the correlation functions of the field and divergences are avoided in this. The key idea for this work comes from resistance distance used in electrical resistor networks with applications ranging from chemistry [57] to computer science.

5.1 Orientation

In view of the discovery of the Higgs et al. boson [58–60], it is timely to reconsider the fundamental implications of a plain scalar field of the standard model: one that

is not composite, associated to supersymmetry, or to extra dimensions of space-time. It is of interest to study a quantum theory of scalar fields in any case, as it describes many other phenomena as well; such as phase transitions.

Such a field is not expected to have an effect on the geometry of space-time. This is unlike the other bosonic fields: according to General Relativity, gravity modifies the metric of space-time from Euclidean to Riemannian. And gauge fields have a geometric meaning in terms of parallel transport. A fundamental scalar field is thought to have a geometrical meaning only as a remnant of dimensional reduction: if space time has extra dimensions (whether continuous or discrete, as in non-commutative geometry), the extra components of the gauge or gravitational field would be scalars in four dimensional space-time.

In our work, we will show that a scalar quantum field defines a metric on space-time as well. But to understand this metric we must go beyond Riemannian geometry. In recent years, the study of such general metric spaces has emerged [61, 62] as a fundamental branch of mathematics, touching on topology, geometry and analysis.

We know from classical mechanics that a free particle moves along a straight-line. The length of this line is the distance between points. In quantum mechanics, the propagator is the sum over all paths,[1] with a weight proportional to e^{-S}, where $S = \frac{1}{2}\int \dot{x}^2\, dt$ is the action [63]. Thus, a quantum notion of distance should involve the propagator itself rather than a property of a particular path.

However in quantum field theory, we should seek a notion of distance based on the correlations of fields at two points, which is the analogue of the propagator. We

[1] In this paper we will study quantum theories formulated in terms of a path integral $\int e^{-S}\,[\mathcal{D}\phi]$ where S is a real positive function of the field, the action. That is why the signature of the metric is positive instead of being Lorentzian. To get physical answers we must do an analytical continuation in time.

will show that the quantity

$$\sigma(x, x') = \sqrt{\langle [\phi(x) - \phi(x')]^2 \rangle}$$

(defined with a regularization) in a scalar quantum field theory satisfies the axioms of a metric (see the Appendix A for a summary of metric geometry). In particular, the triangle inequality

$$\sigma(x, x') \leq \sigma(x, x'') + \sigma(x'', x')$$

holds. This notion of distance between two points is very different though, from the Riemannian notion. It is worth exploring on its own right, even if we continue to use the Euclidean length for most physical purposes.

For example, the triangle inequality above cannot be saturated if x, x', x'' are all distinct. By contrast, for the Euclidean distance, as long as x'' lies on the straight line connecting x to x', we would have $|x - x'| = |x - x''| + |x'' - x'|$. More generally, in Riemannian geometry, we can saturate the triangle inequality by choosing x'' to be any point on a shortest geodesic connecting x to x'.

This means that $\sigma(x, x')$ is not the length of geodesics in any Riemannian geometry: it defines a non-Riemannian metric geometry.

So why would space-time look Euclidean classically? It turns out that the length of a curve defined by $\sigma(x, x')$ (in a super-renormalizable or asymptotically free theory; we do not know the general answer yet) is the same as the Euclidean length. The reason is that for small distances and small interactions, $\sigma(x, x') \propto |x - x'|$. (The proportionality constant depends on the regularization.) The length of a curve is defined by breaking it up into small segments (see Appendix A). For small enough segments we will get (up to a constant) their Euclidean length. Classical measure-

ments of distance always involve lengths of curves. So even σ will give the Euclidean answer (up to the proportionality constant) in these measurements.

According to σ, the shortest curve connecting two points is still the straight line. But this shortest length is not the same as $\sigma(x, x')$. We will see that $\sigma(x, x')$ is a monotonically increasing function of $|x - x'|$ which, for dimensions $n = 3, 4$ tend to a constant for large $|x - x'|$; space-time has *finite diameter* according to σ. For the case of a massless free field in four dimensions using the heat kernel regularization, we have an explicit formula-

$$\sigma(x, x') = \left[\frac{1}{8\pi^2 a^2} - \frac{1}{2\pi^2 (x - x')^2} + \frac{e^{-\frac{(x-x')^2}{4a^2}}}{2\pi^2 (x - x')^2} \right]^{\frac{1}{2}} . \tag{5.1.1}$$

The story can be different for a scalar quantum field theory that does not tend to free field theory at short distances. For a $\lambda\phi^4$ theory in four dimensions, perturbation theory breaks down at short distances. We are not able to determine analytically the relationship of $\sigma(x, x')$ to $|x - x'|$ for short distances. This case is of great importance, as it describes the Higgs boson of the standard model. Large scale computer simulations are needed to study this relationship.

Since our metric depends on the cutoff a, can it still have a physical significance? We will see in the continuum regularization schemes, a change of the cutoff does not change the Lipschitz equivalence class of the metric. Thus the Lipschitz class of space-time should have a physical significance: instead of differentiable functions we would talk of Lipschitz functions for example. This equivalence class does change if we let the cutoff go to zero; it is different from that of Euclidean space. Thus, the functions that are Lipschitz with respect to σ are not the same as those with respect to $|x - x'|$.

When non-Euclidean geometry was still new, it was useful to understand a curved metric in terms of an embedding into Euclidean space. In the same way, it is useful to understand a non-Riemannian metric such as ours by embedding into a Riemannian manifold. We will show that our metric $\sigma(x, x')$ can be thought of as the length of the geodesic in a Riemannian manifold with one extra dimension: the geodesic does not lie in the submanifold, so has a shorter length than any curve that stays within the submanifold (in particular the Euclidean straight line).

In the following section, we describe scalar quantum field theory on a lattice and show how $\sigma(x, x')$ can be defined for it (Sec. 5.2). We find the metric explicitly for the case of free massless scalar field on a lattice in terms of certain discrete Fourier series. This special case is related to the resistance metric of Kigami [64]. We show that the metric cannot be induced by any Riemannian geometry. In Sec. 5.3, we consider other regularization schemes, in particular the heat kernel method. This allows us to get an explicit form in terms of elementary functions for free field theory. It is shown that the Lipschitz equivalence class of the metric is independent of the cutoff. We also show that it is possible to embed space-time in a Riemannian manifold (Sec. 5.4) of one higher dimension such that our σ is the induced metric. We have an application of our result in network theory, mainly collaboration graphs (Sec. 5.5).

Lattice Scalar Field Theory

A scalar field on a graph Γ is a function $\phi : \Gamma \to \mathbb{R}^N$. The action (or energy, depending on the physical application) is a function of the scalar field

$$S(\phi) = \frac{a^{n-2}}{2} \sum_{x \sim x'} \left[\phi(x) - \phi(x')\right]^2 + a^n \sum_x V(\phi(x)). \tag{5.1.2}$$

The first sum is over nearest neighbors in the graph and a is the distance between them.[2] V is a polynomial whose coefficients are the "bare coupling constants". Free field theory is the special case where V is a quadratic function. Massless free field theory is the case $V = 0$. The expectation value of any function of the field is defined to be

$$\langle f \rangle = \frac{\int f(\phi) e^{-S(\phi)} d\phi}{\int e^{-S(\phi)} d\phi}. \tag{5.1.3}$$

In particular, the correlation functions are the expectation values

$$G(x_1, \cdots x_n) = \langle \phi(x_1) \cdots \phi(x_n) \rangle. \tag{5.1.4}$$

Sometimes it will be more convenient to work with quantities such as

$$R(x, x') = \langle [\phi(x) - \phi(x')]^2 \rangle \tag{5.1.5}$$

related to the correlation function by-

$$R(x, x') = G(x, x) + G(x', x') - G(x, x') - G(x', x). \tag{5.1.6}$$

The case of greatest interest is a cubic lattice $\Omega_{a,L} = a\,(\mathbb{Z}/\Lambda\mathbb{Z})^n$ with period $L = \Lambda a$. The aim of quantum field theory is to construct the continuum limit $a \to 0, L \to \infty$ such that the correlation functions have a sensible limit. In taking this limit, the coefficients of the polynomial V are to be varied as functions of the cutoff a. This program is essentially complete [65] in the case $n = 2$. Constructing non-trivial examples (i.e., with a V of degree higher than two) is very difficult in the physically interesting cases of dimensions three (for the theory of phase transitions) and four

[2] Obviously, we can absorb a into ϕ or V but we will find it convenient not to do so.

(for particle physics). Wilson and Fischer used ingenious approximations [66, 67] to understand the three dimensional case. In dimensions higher than four, such a continuum limit does not exist except for the case of a free field [68]. The case of four dimensions is marginal and a non-trivial continuum limit cannot be constructed by standard methods [69].

5.2 Standard Deviation Metric

The mean deviation

$$D(x, x') = \langle |\phi(x) - \phi(x')| \rangle$$

satisfies the triangle inequality since, for each instance of ϕ.

$$|\phi(x) - \phi(x')| \leq |\phi(x) - \phi(x'')| + |\phi(x'') - \phi(x')| \tag{5.2.1}$$

holds. So it will hold in the average as well.

More generally,

$$D_p(x, x') = [\langle |\phi(x) - \phi(x')|^p \rangle]^{\frac{1}{p}} \tag{5.2.2}$$

for $p \geq 1$ is a metric on space-time.[3] The most interesting is the case $p = 2$ of the standard deviation

$$\sigma(x, x') = \left[\langle |\phi(x) - \phi(x')|^2 \rangle\right]^{\frac{1}{2}} . \tag{5.2.3}$$

$\sigma(x, x')$ is obviously positive and symmetric. Also, $\sigma(x, x') > 0$ if $x \neq x'$ because otherwise, every instance of a scalar field would have to take the same value at x and x'.

Each theory of matter field will define a metric on space-time. The distance is

[3] Due to translation invariance, $\langle |\phi(x) - \phi(x')| \rangle = 0$.

a simple concept for scalar fields. For gauge fields, it is more subtle, but gauge invariant notions do exist [70]. Reflection positivity seems to imply such a metric even for fermion fields. When the scalar field takes values in a curved target space $\phi : \Omega^n \to M$ (e.g., the nonlinear sigma model) we would use the metric d_M of the target to define $\sigma(x, x') = \sqrt{\langle | \left[d_M(\phi(x), \phi(x')) \right]^2 | \rangle}$.

Why could we not have defined a metric using the average of the square of the distance

$$R(x, x') = \langle (\phi(x) - \phi(x'))^2 \rangle$$

itself? It is more closely related to the correlation functions (5.1.6). The point is that the square of a metric, such as $(\phi(x) - \phi(x'))^2$ does not in general satisfy the triangle inequality. (By contrast, the square root of a metric always does.) In some special cases (e.g., massless free field) the expectation value $R(x, x')$ itself is a metric. This particular case was discovered by Kigami in the context of fractals [64, 71]. But there are other probability distributions (that are not Gaussians) for which $R(x, x')$ does not satisfy the triangle inequality. For example, the exponential distribution in certain parameter regimes. Also, with regularizations other than the lattice (e.g., heat kernel method, see Sec. 5.3) $R(x, x')$ does not satisfy the triangle inequality. But $D_1(x, x')$ and $\sigma(x, x')$ always do (Appendix A). Of the two, the standard deviation is $\sigma(x, x')$ is easier to calculate, as usual.

Free Scalar Field

In the special case (5.1.2), of a free field

$$V(\phi) = \frac{1}{2} m^2 \sum_x \phi(x)^2, \tag{5.2.4}$$

$\phi(x) - \phi(x')$ is a Gaussian random variable with variance $R(x, x')$ and zero mean. So we have the relation

$$\sigma = \sqrt{\frac{\pi}{2}} D. \tag{5.2.5}$$

This relation is universal for free fields: it does not depend on the cutoff procedure used (e.g., square vs triangular lattice). But the range of values of σ will depend on the cutoff.[4]

Explicit Formula for R

The expectation values are Gaussian integrals which we can evaluate explicitly in terms of the Green's function [72]-

$$G(x, x') = \langle \phi(x)\phi(x') \rangle.$$

It is the solution of the lattice Helmholtz equation,

$$[\Delta_x + m^2] G(x, x') = \delta_{\Omega_{a,L}}(x, x'). \tag{5.2.6}$$

The lattice laplacian is the sum over nearest neighbors y, for fixed x of [73]-

$$\Delta\psi(x) = \frac{1}{a^2} \sum_{\langle yx \rangle} [\psi(x) - \psi(y)] \tag{5.2.7}$$

(According to this definition, the eigenvalues of the operator are positive.)

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[4] Note that $I_k = \int_{-\infty}^{\infty} |\phi|^k e^{-\frac{1}{2}\frac{\phi^2}{R}} d\phi = 2R^{\frac{1+k}{2}} \int_0^{\infty} e^{-u} (2u)^{\frac{k-1}{2}} du = 2^{\frac{k+1}{2}} \Gamma\left(\frac{k+1}{2}\right) R^{\frac{1+k}{2}}$.
Thus $\langle |\phi| \rangle = \frac{I_1}{I_0} = \frac{2}{\sqrt{2\pi}} \sqrt{R} = \sqrt{\frac{2}{\pi}} R$, $\langle \phi^2 \rangle = \frac{I_2}{I_0} = \frac{2^{\frac{3}{2}} \frac{1}{2} \sqrt{\pi}}{\sqrt{2\pi}} R = R$

The lattice delta function depends on L through periodicity;

$$\delta_{\Omega_{a,L}}(x,x') = \begin{cases} a^{-n}, & \text{if } x = x' \bmod L \\ 0, & \text{otherwise.} \end{cases} \tag{5.2.8}$$

The discrete Fourier transform of a function is given by

$$\tilde{\psi}(p) = a^n \sum_{x \in \Omega_{a,L}} e^{-ip\cdot x}\psi(x), \tag{5.2.9}$$

where the wavenumber p belongs to the dual lattice

$$p \in \tilde{\Omega}_{a,L} = \Omega_{\frac{2\pi}{L},\frac{2\pi}{a}} \tag{5.2.10}$$

for which a, L are exchanged for their reciprocals. Note the identity

$$L^{-n} \sum_{p \in \tilde{\Omega}_{a,L}} e^{ip\cdot(x-x')} = \delta_{\Omega_{a,L}}(x,x'). \tag{5.2.11}$$

The inverse discrete Fourier transform is then-

$$\psi(x) = L^{-n} \sum_{p} e^{ip\cdot x}\tilde{\psi}(p). \tag{5.2.12}$$

Then,

$$\widetilde{\Delta\psi}(p) = \tilde{\Delta}(p)\tilde{\psi}(p), \quad \tilde{\Delta}(p) = \frac{4}{a^2}\sum_{r=1}^{n} \sin^2\frac{ap_r}{2}. \tag{5.2.13}$$

Thus

$$G(x,x') = L^{-n} \sum_{p \in \tilde{\Omega}_{a,L}} \frac{e^{ip\cdot(x-x')}}{\tilde{\Delta}(p) + m^2}. \tag{5.2.14}$$

It follows that

$$R(x,x') = L^{-n} \sum_{p\in\tilde{\Omega}_{a,L}} \frac{4\sin^2\frac{p\cdot(x-x')}{2}}{\tilde{\Delta}(p)+m^2}. \tag{5.2.15}$$

Resistance Metric

In the limit $m \to 0$ of a free massless scalar field, $R(x,x')$ itself (and not only its square root) is a metric. Explicitly,

$$R(x,x') = L^{-n} \sum_{p\in\tilde{\Omega}_{a,L}}{}' \frac{4\sin^2\frac{p\cdot(x-x')}{2}}{\tilde{\Delta}(p)} \tag{5.2.16}$$

the sum being over terms with $\tilde{\Delta}(p) \neq 0$. This is the resistance metric of Kigami, evaluated for the cubic lattice. There is a simple physical interpretation for this quantity. Imagine a network, each pair of nearest neighbors being connected by a resistor of same resistance. Then $R(x,x')$ is the effective resistance between the pair of points x, x' after all the others have been eliminated using Kirchoff's laws of current conduction. A method to compute this effective resistance for any arbitrary configuration of resistances has been done in Appendix B. It is obvious that $R(x,x')$ is positive and symmetric. See Ref. [71] for an ingenious proof that it satisfies the triangle inequality.

Infinite Lattice

The resistance metric is often studied on sequences of graphs that tend to a fractal. Not much has been studied as a metric on the more familiar graph of a cubic lattice. Perhaps the reason is that it is totally different from the Euclidean metric. Physicists have already calculated the properties of resistance on cubic lattices [74] without noting that it satisfies the triangle inequality. In the limit of a lattice of infinite

period $L \to \infty$, the momentum space becomes a torus of period $\frac{2\pi}{a}$. Then we have the integral representation[5] with $m = 0$,

$$R(x,x') = \left(\frac{1}{2\pi}\right)^n \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} d^n p \, \frac{4\sin^2 \frac{p\cdot(x-x')}{2}}{\frac{4}{a^2}\sum_{r=1}^{n} \sin^2 \frac{p_r a}{2}}$$

$$R(x,x') = \frac{a^{2-n}}{n}, \quad |x-x'| = a, \quad m = 0. \tag{5.2.17}$$

If we consider $x - x' = (0, \cdots a, \cdots 0)$ to be along the i th direction alone and consider the sum over all the n integrals, the trigonometric terms in the numerator and denominator would cancel and the integral evaluates to a^{2-n}. Also, the nearest neighbors have the same resistance.

In the opposite limit of large Euclidean distance, the answer depends more dramatically on the dimension.

For $n = 1$, it is easy to see that the resistance metric is simply the Euclidean distance

$$R(x,x') = |x - x'|.$$

We have

$$R(x,x') = \frac{1}{2\pi}\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dp \, \frac{\sin^2 \frac{p(x-x')}{2}}{a^{-2}\sin^2 \frac{pa}{2}}. \tag{5.2.18}$$

Let $p \to \frac{2\pi}{a}p$ and $r = \frac{|x-x'|}{a}$,

$$R(x,x') = a\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\sin^2 \pi r}{\sin^2 \pi p} dp = ra = |x - x'|.$$

[5]Use the Euler-MacLauren formula $\lim_{L\to\infty} L^{-n} \sum_{k=1}^{\Lambda} f\left(\frac{2\pi}{L}k\right) = \int_0^{\frac{2\pi}{a}} f(p)\frac{d^n p}{(2\pi)^n}$

For $n = 2$,

$$R(x, x') \to \frac{1}{2\pi}\left[\log\frac{|x-x'|}{a} + \gamma + \frac{1}{2}\log 8 + \cdots\right], \quad |x-x'| \to \infty. \tag{5.2.19}$$

For $n > 2$, and $|x - x'| >> a$ we can approximate

$$R(x, x') \approx \frac{C_n}{a^{n-2}} - 2G_n(x - x'), \tag{5.2.20}$$

where $G_n(x)$ is the continuum Green's function of the Laplace operator. The constant C_n is independent of a but depends on the method of regularization. For the lattice regularization [75, 76],

$$C_n = 2\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{4\left[\sin^2\pi p_1 + \sin^2\pi p_2 + \cdots \sin^2\pi p_n\right]}\, d^n p. \tag{5.2.21a}$$

$$R(x, x') \approx \frac{C_n}{a^{n-2}} - \frac{C'_n}{|x-x'|^{n-2}} + \mathrm{O}\left(|x-x'|^{n-3}\right). \tag{5.2.21b}$$

The constant C'_n is universal: it is the same in every regularization scheme, being simply related to the continuum Green's function.

$$C'_n = \frac{1}{2}\pi^{-\frac{n}{2}}\Gamma\left(\frac{n}{2} - 1\right) \tag{5.2.22}$$

When $n = 3, 4$

$$C_3 \approx 0.505462, \quad C'_3 = \frac{1}{2\pi}, \quad n = 3 \tag{5.2.23a}$$

$$C_4 \approx 0.309867, \quad C'_4 = \frac{1}{2\pi^2}, \quad n = 4 \tag{5.2.23b}$$

The plot (Fig. 5.1) shows that $\sigma(x, x') = \sqrt{R(x, x')}$ always grows with the Euclidean distance.

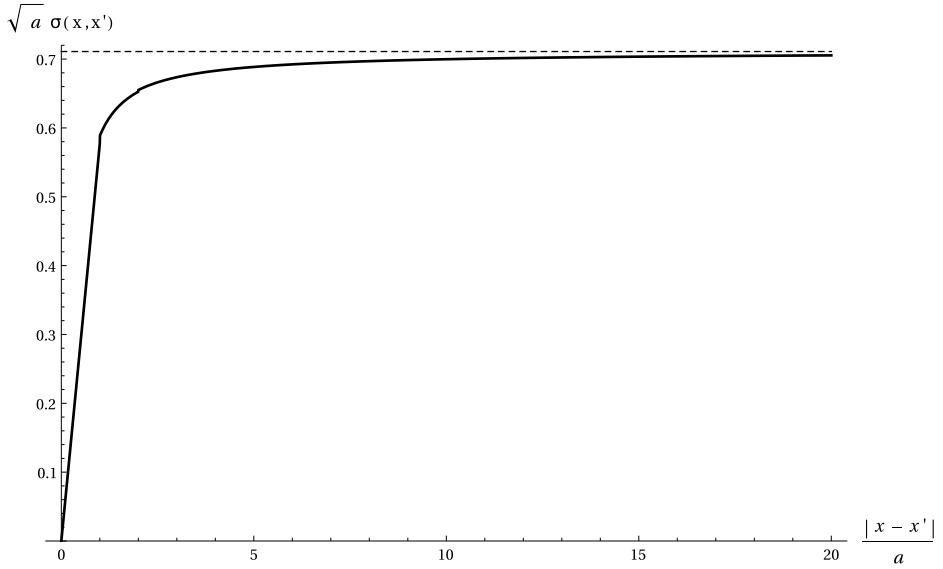

Figure 5.1: $\sqrt{a}\sigma(x,x')$ is plotted as a function of Euclidean distance scaled by lattice length

But the rate of growth is very slow for large distances. For nearest neighbors

$$\sigma(x,x') = a^{1-\frac{n}{2}}\sqrt{\frac{1}{n}}, \quad |x-x'| = a. \tag{5.2.24}$$

And for large distances,

$$\sigma(x,x') \approx \sigma_n a^{1-\frac{n}{2}} - \frac{\sigma'_n}{|x-x'|^{n-2}} + \mathrm{O}\left(|x-x'|^{n-3}\right). \tag{5.2.25}$$

$$\sigma_n = \sqrt{C_n}, \quad \sigma'_n = a^{\frac{n}{2}-1}\frac{C'_n}{2\sqrt{C_n}}. \tag{5.2.26}$$

Note that the above equations (5.2.23) is consistent with $C_n > \frac{1}{n}$.

We plot (Fig. 5.1) the distance in dimension three, in units of the lattice spacing a. Note that the length of a path is the sum of distances between nearest neighbors along the path. Hence the geodesic distance (the length of the shortest path connecting

two points) is proportional to the Euclidean distance

$$\sigma_l(x,x') = a^{-\frac{n}{2}}\sqrt{\frac{1}{n}}|x-x'| \geq \sigma(x,x'). \tag{5.2.27}$$

The equality holds only for nearest neighbors. For large $|x-x'|$, the geodesic distance is much greater: $\sigma_l(x,x') >> \sigma(x,x')$. It is easy to understand why using the resistance model: the shortest curve is just one among possibly many paths that connect the pair of points. When resistances corresponding to the paths are combined in parallel, the effective resistance obtained is smaller than all of them.

No Intermediate Point

We now show that when $n \geq 3$, the triangle inequality

$$\sigma(x,x') \leq \sigma(x,x'') + \sigma(x'',x') \tag{5.2.28}$$

can *never* be saturated unless one of the distances is zero. The distance between nearest neighbors is

$$\frac{1}{\sqrt{n}}a^{1-\frac{n}{2}}.$$

So the smallest value for the r.h.s. (5.2.28), being the sum of two non-zero distances, is twice this,

$$\frac{2}{\sqrt{n}}a^{1-\frac{n}{2}}.$$

On the other hand, the distance between any pair of points is bounded (5.2.25),

$$\sigma(x,x') < \sqrt{C_n}a^{1-\frac{n}{2}}. \tag{5.2.29}$$

Now, its easy to check that

$$C_n < \frac{4}{n}. \tag{5.2.30}$$

Instead of an analytic proof, we can simply calculate the values in the two interesting cases numerically,

$$C_3 \approx 0.505462 < \frac{4}{3} \tag{5.2.31a}$$

$$C_4 \approx 0.309867 < 1 \tag{5.2.31b}$$

Thus, the minimum value of the r.h.s. (5.2.29) is greater than the maximum value of the l.h.s. and the inequality can never be saturated.

It follows that $\sigma(x, x')$ cannot be induced by any Riemannian metric.

Massive Scalar Field

In the case of a massive free scalar field, the asymptotic dependence on the Euclidean distance is given by the Yukawa potential between points charges exchanging a massive particle: it vanishes exponentially.

$$\sigma(x, x') \approx \sigma_n a^{1-\frac{n}{2}} - \sigma'_n \frac{e^{-m|x-x'|}}{|x-x'|^{n-2}} + \mathrm{O}\left(|x-x'|^{n-3}\right), \quad n > 2 \tag{5.2.32}$$

The constants σ_n, σ'_n are as above.

5.3 Other Regularization Schemes

Although we have used the lattice definition of scalar field theories, universality implies that other regularization schemes will suffice. For example, we could use a sharp momentum cutoff or a smooth momentum cutoff or a heat kernel method.

We saw that in the limit $L \to \infty$, the momentum variables takes values on a torus of period $\frac{2\pi}{a}$ in each direction. Thus, the lattice regularization amounts to replacing a potentially divergent integral by

$$\int d^n p f(p) \to \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^n p}{(2\pi)^n} f'(p),$$

where $f'(p)$ agrees with $f(p)$ for $|p| << \frac{1}{a}$. A typical example we encountered above is

$$\frac{1}{p^2} \to \frac{1}{a^{-2} \sum_{r=1}^{n} 4 \sin^2 \frac{a p_r}{2}}. \tag{5.3.1}$$

Another method, commonly used in scalar QFT is

$$\int d^n p f(p) \to \int d^n p K\left(a|p|\right) f(p), \tag{5.3.2}$$

where $K(\xi) \approx 1$ for $\xi << 1$ and is zero for $\xi >> 1$. The advantage is that this preserves rotation invariance which the lattice breaks. Examples are-

$$\begin{aligned} K(a|p|) &= \frac{1}{1 + a^2 p^2}, && \text{Pauli} - \text{Villars} \\ K(a|p|) &= \Theta(a|p| < 1), && \text{Sharpcutoff} \\ K(a|p|) &= e^{-a^2 p^2}, && \text{HeatKernel} \end{aligned}$$

Polchinski [77], among others, advocated for a smooth function that is one for $|p| < a^{-1}$ and zero for $|p| > a^{-1}$.

The advantage of these schemes is that the underlying space-time continues to be $\mathbb{R}^n$, but with a possibly different measure of integration on its dual space (momentum space). Our proposal would be to determine a metric on space-time from the standard

deviation, computed using this regularized measure in momentum space. Again we begin with the free field,

$$G(x,x') = \int \frac{d^n p}{(2\pi)^n} K(a|p|) \frac{1}{p^2+m^2} e^{ip\cdot(x-x')}. \tag{5.3.3}$$

$$R(x,x') = \int \frac{d^n p}{(2\pi)^n} K(a|p|) \frac{4\sin^2 \frac{p\cdot(x-x')}{2}}{p^2+m^2}. \tag{5.3.4}$$

The explicit answer seems simplest for the heat kernel regularization. With $m=0$, we get an answer in terms of the incomplete Gamma function, $\Gamma(\nu,z) = \int_z^\infty t^{\nu-1}e^{-t}dt$

$$G(x,x') = \frac{\pi^{\frac{n}{2}}}{4|x-x'|^{n-2}} \left[\Gamma\left(\frac{n-2}{2}\right) - \Gamma\left(\frac{n-2}{2}, \frac{(x-x')^2}{4a^2}\right)\right]. \tag{5.3.5}$$

To see this,

$$\begin{aligned} G(x,x') &= \int \frac{d^n p}{(2\pi)^n} e^{-a^2p^2} \frac{1}{p^2} e^{ip\cdot(x-x')} \\ &= \int_{a^2}^\infty dt \int \frac{d^n p}{(2\pi)^n} e^{-tp^2} e^{ip\cdot(x-x')} \\ &= \int_{a^2}^\infty dt \frac{e^{-\frac{(x-x')^2}{4t}}}{(4\pi t)^{\frac{n}{2}}}, \end{aligned} \tag{5.3.6}$$

which we evaluate. a plays the same role as the nearest neighbor distance in the lattice regularization. It is the short distance cutoff.

In particular,

$$G(x,x) = \frac{2^{1-n}\pi^{-\frac{n}{2}} a^{2-n}}{n-2}, \tag{5.3.7}$$

$$R(x,x') = \frac{2^{2-n}\pi^{-\frac{n}{2}}}{n-2} a^{2-n} - \frac{1}{2}\frac{|x-x'|^{2-n}}{\pi^{\frac{n}{2}}} \left[\Gamma\left(\frac{n-2}{2}\right) - \Gamma\left(\frac{n-2}{2}, \frac{(x-x')^2}{4a^2}\right)\right] \tag{5.3.8}$$

For even n the expression is more elementary:

$$R(x,x') = \frac{1}{8\pi^2 a^2} - \frac{1}{2\pi^2 (x-x')^2} + \frac{e^{-\frac{(x-x')^2}{4a^2}}}{2\pi^2 (x-x')^2}, \quad n = 4. \tag{5.3.9}$$

It follows that our metric $\sigma(x,x') = \sqrt{R(x,x')}$ is proportional to the Euclidean distance for small $|x-x'|$:

$$R(x,x') = \frac{2^{-n}\pi^{-\frac{n}{2}}(x-x')^2}{na^n} + O\left(|x-x'|^4\right). \tag{5.3.10a}$$

$$\sigma(x,x') = \frac{2^{-\frac{n}{2}}\pi^{-\frac{n}{4}}}{\sqrt{n}a^{\frac{n}{2}}}|x-x'| + O\left(|x-x'|^2\right). \tag{5.3.10b}$$

We plot (Fig. 5.2) the metric as a function of Euclidean distance, in units with $a = 1$ for $n = 3, 4$.

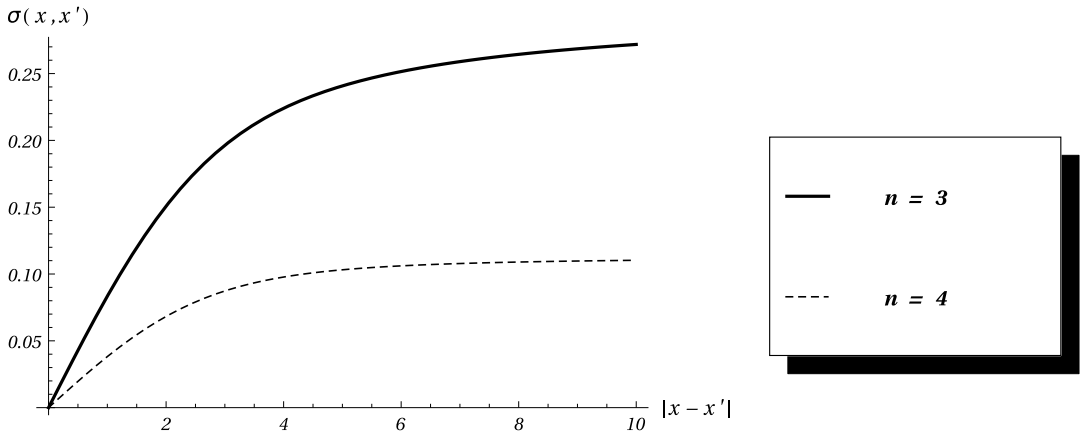

Figure 5.2: Plot showing the dependence of σ with Euclidean distance in dimensions 3 and 4

Note that $R(x,x')$ would not satisfy the triangle inequality, being proportional to the *square* of the Euclidean distance for small $|x-x'|$. This confirms that the correct choice of metric is the standard deviation, not the variance of the scalar field.

For large $|x - x'|$,

$$R(x,x') = \frac{2^{2-n}\pi^{-\frac{n}{2}}}{(n-2)a^{n-2}} - \frac{\pi^{-\frac{n}{2}}}{2}\Gamma\left(\frac{n}{2}-1\right)\frac{1}{|x-x'|^{n-2}} + \cdots \tag{5.3.11}$$

Thus, in the heat kernel regularization

$$C_n = \frac{2^{2-n}\pi^{-\frac{n}{2}}}{n-2}, \quad C_n' = \frac{1}{2}\pi^{-\frac{n}{2}}\Gamma\left(\frac{n}{2}-1\right). \tag{5.3.12}$$

As noted earlier C_n is not universal but C_n' is. To compare the numerical values

$$C_3 \approx 0.0897936 \quad \text{vs } 0.505462 \text{ for lattice} \tag{5.3.13}$$

$$C_4 \approx 0.0126651 \quad \text{vs } 0.309867 \text{ for lattice.} \tag{5.3.14}$$

Lipschitz Equivalence

How does the change of the cutoff affect the geometry defined by σ? We now show that for the free field, there are constants k_1, k_2 such that

$$0 < k_1(a,b) \leq \frac{\sigma_a(x,x')}{\sigma_b(x,x')} \leq k_2(a,b) < \infty.$$

That is, k_1, k_2 depend on the cutoffs a, b but not on the points $x, x' \in \mathbb{R}^n$. This means that the two metrics σ_a and σ_b on $\mathbb{R}^n$ are Lipschitz equivalent. The proof for the free massless theory uses the explicit formula to show that when $a < b$, the ratio $\frac{\sigma_a(x,x')}{\sigma_b(x,x')}$ takes its largest value for $x = x'$ and its smallest value as $|x - x'| \to \infty$. Fig. 5.3 illustrates this fact. Thus,

$$\left(\frac{b}{a}\right)^{\frac{n}{2}-1} \leq \frac{\sigma_a(x,x')}{\sigma_b(x,x')} \leq \left(\frac{b}{a}\right)^{\frac{n}{2}}, \quad a < b, \quad n > 2. \tag{5.3.15}$$

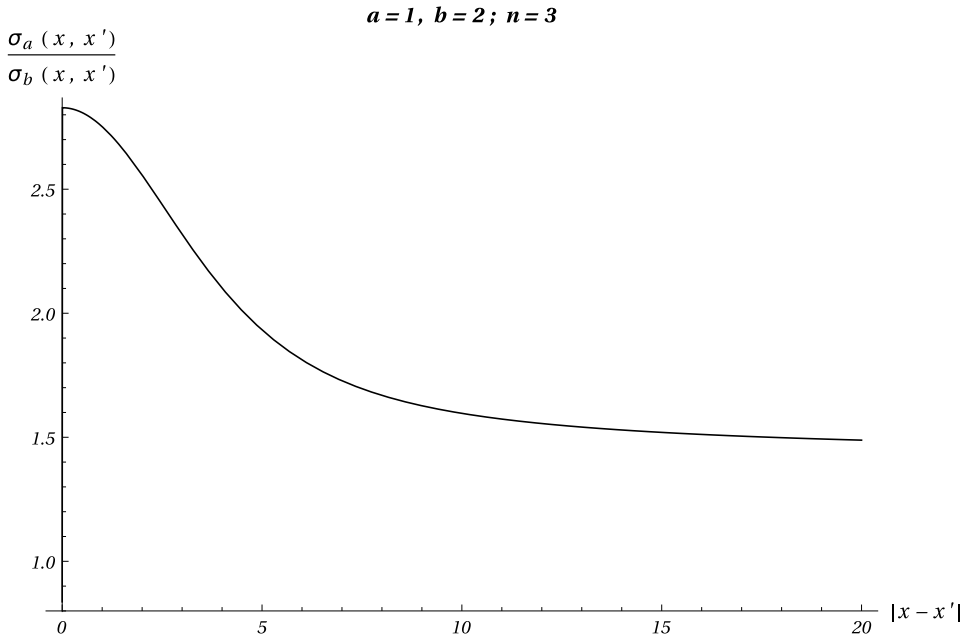

Figure 5.3: Plot showing the ratio of distance using two cutoffs on the Euclidean distance

We conjecture that this bi-Lipschitz inequality (5.3) holds for all renormalizable scalar QFT in the continuum regularization schemes. The actual values of the Lipschitz constants might change however. Thus, we propose that although the metric itself depends on the cutoff, its Lipschitz equivalence class is universal. This makes some sense as Lipschitz equivalence for metric spaces is analogous to diffeomorphisms for manifolds.

Beyond Free Fields: Wilson-Fisher

When $3 < n < 4$, the scalar field theory with potential

$$V(\phi) = \frac{1}{2}m^2|\phi|^2 + \frac{1}{4}\lambda|\phi|^4, \tag{5.3.16}$$

has a fixed point of the renormalization group. The momentum integrals defining the scalar theory make sense even for fractional values of n, even though the case, $n = 3$ is the case of physical interest. The case $n = 3, N = 1$ (N being the number of components of ϕ) for example, describes the critical point of a liquid and a gas. The connection of this to fractals remains mysterious. A more detailed study of quantum field theory on fractals is called for. We took a first step in this direction ourselves.

At this Wilson-Fisher fixed point, the Green's function is

$$G(x - x') = \int \frac{e^{ip\cdot(x-x')}}{p^{2-\eta}} K\left(a|p|\right) \frac{d^3p}{(2\pi)^3}, \tag{5.3.17}$$

where the critical exponent η can be calculated in the ϵ-expansion (Sec. 25.5 of Ref. [67]).

$$\eta = \frac{N+2}{2(N+8)^2}\epsilon^2 + \mathrm{O}\left(\epsilon^3\right), \quad \epsilon = 4 - n. \tag{5.3.18}$$

This quantity is universal and has been calculated to much higher precision. (See [78] for the result up to order ϵ^5). In this example, the metric

$$\sigma^2(x, x') = 2G(0) - 2G(x - x') \tag{5.3.19}$$

becomes for $|x - x'| >> a$,

$$\sigma(x, x') = \frac{\sigma_3}{a^{\frac{1+\eta}{2}}} - \frac{\sigma_3'}{|x - x'|^{1+\eta}} + \cdots \tag{5.3.20}$$

That is, even when $n = 3$, it scales as if the dimension of space were a little bit higher than three. Again, the diameter of space is finite and the next-to-leading order correction contains the Green's function of physical interest. Also, for small $|x - x'|$ our $\sigma(x, x')$ is proportional to the Euclidean metric.

5.4 Embedding in a Riemannian Manifold

Our standard deviation is *not* a geodesic metric. The length metric of σ is proportional to the Euclidean metric, which is typically larger than σ. In the lattice regularization, the length of any curve is just the number of edges along it: the supremum above is achieved when each segment connects nearest neighbors. In the heat kernel regularization, we saw that when x, x' are close enough, $\sigma(x, x')$ is proportional to $|x-x'|$. So the length of any curve according to σ will be, up to a constant multiple, its Euclidean length.

Thus, the distance perceived by a quantum model of propagation is drastically different from the classical model. Classically, the particle simply takes the shortest path (which is also the path of least action). Quantum mechanically, we must sum over all the paths; longer paths are simply less probable. Long paths can dominate the sum, if the sheer number of long path makes up for their smaller probability. This is what happens on cubic lattices of dimension $n \geq 2$ as is easy to verify using the explicit form we found above. Again, this illustrates how far from being a geodesic metric σ is.

We will now show that we can embed $\mathbb{R}^n$ into a Riemannian manifold of one dimension higher such that the length of the geodesic connecting (x, x') in the ambient space is equal to $\sigma(x, x')$. The extra dimension provides a "short-cut" that allows us to realize our metric as a geodesic distance. In the Appendix A, we give an example involving the chord length of circles that illustrates this situation.

Consider a Riemannian metric on $\mathbb{R}^n \times \mathbb{R}^+$

$$ds^2 = h^2(\rho)d\rho^2 + \rho^2 dx^i dx^i. \tag{5.4.1}$$

These coordinates are chosen to make later expressions simpler.

As an example, $n = 1$ and $h(\rho) = \frac{1}{\rho}$ is one description of the metric of constant negative curvature on a hyperboloid. The substitution $y = \frac{1}{\rho}$ turns this into the familiar Poincare' metric

$$ds^2 = \frac{dy^2 + dx^2}{y^2}. \tag{5.4.2}$$

To continue with this example, (which we will not use directly, but is similar enough to those we will use) the real line is a sub-manifold, the line of constant $y = y_0$. There are two induced distances on this sub-manifold: we can ask for the minimum length of curves that lie on the sub-manifold: this is just the Euclidean metric on the real line. Or the minimum over all curves that start and end at points (y_0, x) and (y_0, x') on the line, but in between can lie anywhere on the plane (Fig. 5.4). It is clear that the latter can be smaller than the Euclidean distance. It is in fact a non-geodesic metric on the real line, whose length metric is the Euclidean metric.

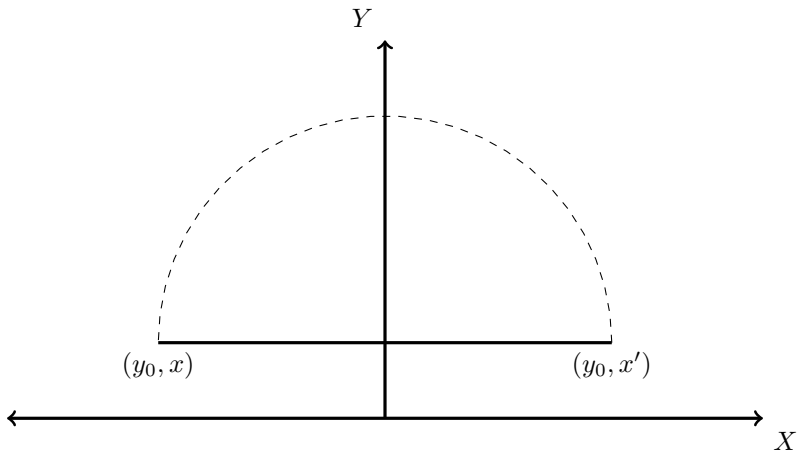

Figure 5.4: Trajectory in the $x\,y$ plane shown by dotted line

The geodesic (5.4.1) minimizes

$$\int \sqrt{h^2(\rho) + \rho^2 \left(\frac{dx}{d\rho}\right)^2}\, d\rho. \tag{5.4.3}$$

We need to find the geodesic that connects the (ρ_0, x') with (ρ_0, x). By a rotation and translation we can choose $x = (\frac{r}{2}, 0, \cdots 0)$ and $x' = (-\frac{r}{2}, 0, \cdots 0)$, where $r = |x - x'|$ is the Euclidean distance. The geodesic will then lie in the (ρ, x^1) plane. The Euler-Lagrange equation implies that

$$\frac{\rho^2}{\sqrt{h^2(\rho) + \rho^2 \left(\frac{dx}{d\rho}\right)^2}} \left(\frac{dx}{d\rho}\right) = \rho_1, \tag{5.4.4}$$

for some constant ρ_1. A moments thought will show that the geodesic is reflection symmetric around $x = 0$ and that $\frac{d\rho}{dx} = 0$ at $x = 0$. Thus $\frac{dx}{d\rho} = \infty$ at $x = 0$, and we conclude from the above equation that ρ_1 is simply the point at which $x = 0$.

$$\frac{dx}{d\rho} = \pm \frac{\rho_1 h(\rho)}{\rho\sqrt{\rho^2 - \rho_1^2}}. \tag{5.4.5}$$

Thus

$$x(\rho) = \pm \int_{\rho_1}^{\rho} \frac{\rho_1 h(\rho)}{\rho\sqrt{\rho^2 - \rho_1^2}}\, d\rho. \tag{5.4.6}$$

It follows from

$$h^2(\rho) + \rho^2 \left(\frac{dx}{d\rho}\right)^2 = \frac{h^2(\rho)\rho^2}{\rho^2 - \rho_1^2}, \tag{5.4.7}$$

that

$$r = 2\int_{\rho_1}^{\rho_0} \frac{\rho_1 h(\rho)}{\rho\sqrt{\rho^2-\rho_1^2}}\, d\rho. \tag{5.4.8}$$

$$L = 2\int_{\rho_1}^{\rho_0} \frac{\rho h(\rho)}{\sqrt{\rho^2-\rho_1^2}}\, d\rho. \tag{5.4.9}$$

where r is the Euclidean distance and L is the length of geodesic.

Together eqns. (5.4.8, 5.4.9) give parametrically via ρ_1, the length of the geodesic in terms of the Euclidean distance r. When $r \to 0$, we have also $\rho_0 \approx \rho \approx \rho_1$ and

$$L \approx \rho_0 r. \tag{5.4.10}$$

The inverse problem of determining $h(\rho)$ given $L(r)$ looks hard. But recall that only the asymptotic behavior of $\sigma(r)$ as $r \to \infty$ is universal. So we should be able to find a $h(\rho)$ within the same universality class by looking at the asymptotic behavior. What should $h(\rho)$ be in order that

$$L(r) \sim C - \frac{C'}{r^{n-2+\eta}} + \dots \tag{5.4.11}$$

This is the behavior of the metric $\sigma(x, x')$ in the continuum regularization we derived in the last Sec. 5.3.

The equation (5.4.10) determines ρ_0 in terms of the cutoff.

$$\rho_0 = \frac{2^{\frac{-n}{2}}\pi^{-\frac{n}{4}}}{\sqrt{n}a^{\frac{n}{2}}}. \tag{5.4.12}$$

Suppose $h(\rho) \sim h_1 \rho^{-\mu}$ as $\rho \to 0$. Then we get

$$r \approx 2h_1\rho_1 \frac{\rho_1^{-1-\mu} - \rho_0^{-1-\mu}}{\mu+1} \tag{5.4.13}$$

$$r \approx \frac{2h_1}{\mu+1}\rho_1^{-\mu}. \tag{5.4.14}$$

Also,

$$L \approx 2h_1 \frac{\rho_0^{1-\mu} - \rho_1^{1-\mu}}{1-\mu}, \tag{5.4.15}$$

so that

$$L \approx \frac{2h_1}{1-\mu}\rho_0^{1-\mu} - \frac{2h_1}{1-\mu}\left[\frac{\mu+1}{2h_1}\right]^{\frac{\mu-1}{\mu}} r^{1-\frac{1}{\mu}}. \tag{5.4.16}$$

This gives us what we want if

$$\begin{aligned} 1 - \frac{1}{\mu} &= -(n-2+\eta), \\ \mu &= \frac{1}{n-1+\eta}. \end{aligned} \tag{5.4.17}$$

The resulting metric

$$ds^2 \approx h_1^2 \rho^{-\frac{2}{n-1+\eta}} d\rho^2 + \rho^2 dx^i dx^i \tag{5.4.18}$$

has curvature going to $-\infty$ as $\rho \to 0$ when $(n+\eta-1) \geq 1$. The case $n = 2, \eta = 0$ is marginal. In that we get a metric of constant negative curvature asymptotically.

5.5 Application

The motivation for the standard deviation metric which we have obtained in Sec. 5.2 comes from the resistance metric. Resistance metric which also goes by the name

of resistance distance was probably introduced by D. J. Klein [57] in chemistry. He wanted to associate a notion of a chemical distance which would be dependent on the multiplicity of the bonds between atoms in a molecule. Atoms having multiple bonds between them are in a sense closer as compared to a pair with lower number of bonds.

To give an insight into this abstract notion of distance, we use collaboration graphs. These graphs arise in network theory based on the team work of the people involved. The nodes in the graphs are representative of the actors/authors and a link exists between a pair of nodes depending on the mutual collaborations of the actors/authors. Using this notion of resistance distance, we can form clusters of people based on their collaborations. Intuitively, a pair of authors, who have collaborated on ten papers (say) are closer than a pair of authors who have only one joint paper.

We shall demonstrate our ideas of resistance distance for four movie actors. We list the movies in which each pair of actors have acted together and find a notion of closeness between them based on their collaborations. The movies in which the pairs have acted together are-

- Brad Pitt, Gary Oldman- *True Romance*
- Morgan Freeman, Brad Pitt- *Se7en*
- Harvey Keitel, Brad Pitt- *Thelma & Louise, Inglourious Basterds*
- Harvey Keitel, Gary Oldman- *None*
- Gary Oldman, Morgan Freeman- *Batman Begins, The Dark Knight, The Dark Knight Rises*
- Morgan Freeman, Harvey Keitel- *None*

Based on this we find the adjacency matrix for these actors. An entry is made if a pair of actors have a collaboration. We also find a collaboration matrix based on the number of collaborations between the authors. The diagonal entries for the collaboration matrix denote the number of the movies the actor has acted in this set of movies.

Adjacency Matrix

	Pitt	Oldman	Morgan	Keitel
Pitt		✓	✓	✓
Oldman	✓		✓	×
Morgan	✓	✓		×
Keitel	✓	×	×	

Collaboration Matrix

	Pitt	Oldman	Freeman	Keitel
Pitt	4	−1	−1	−2
Oldman	−1	4	−3	0
Freeman	−1	−3	4	0
Keitel	−2	0	0	2

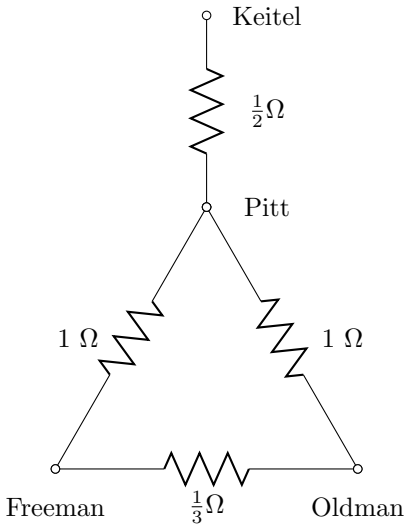

Figure 5.5: Resistor network with resistances weighted by number of collaborations

We now construct an electrical circuit (Fig. 5.5) with resistances between nodes

based on the inverse of the number of collaborations. On computing the effective resistance between each node we can classify the closeness between the actors. The effective resistance between all the pairs are listed below-

- Gary Oldman, Morgan Freeman- 0.285
- Harvey Keitel, Brad Pitt- 0.500
- Brad Pitt, Gary Oldman- 0.571
- Morgan Freeman, Brad Pitt- 0.571
- Harvey Keitel, Gary Oldman- 1.071
- Morgan Freeman, Harvey Keitel- 1.071

The results indeed show that the pairs (Gary Oldman, Harvey Keitel) and (Morgan Freeman, Harvey Keitel) are farthest from other pairs since they have *not* acted in movies together. However Gary Oldman and Morgan Freeman are closest to each other since they have most number of collaborations between them. We did similar demonstrations over large number of actors using Mathematica [79, 80]. We sort actors based on their affinity (Fig. 5.6) to Robert De Niro. The corresponding graph showing mutual collaborations between those actors is also drawn (Fig. 5.7).

Summary

In this chapter, we show that a non-Riemannian metric on space-time emerges from scalar quantum field theory. In dimensions $n > 2$ even the free field induces a very different metric from Euclidean space: space-time has finite diameter for example. Yet, the length of any curve as defined by this metric is (up to a constant) the usual

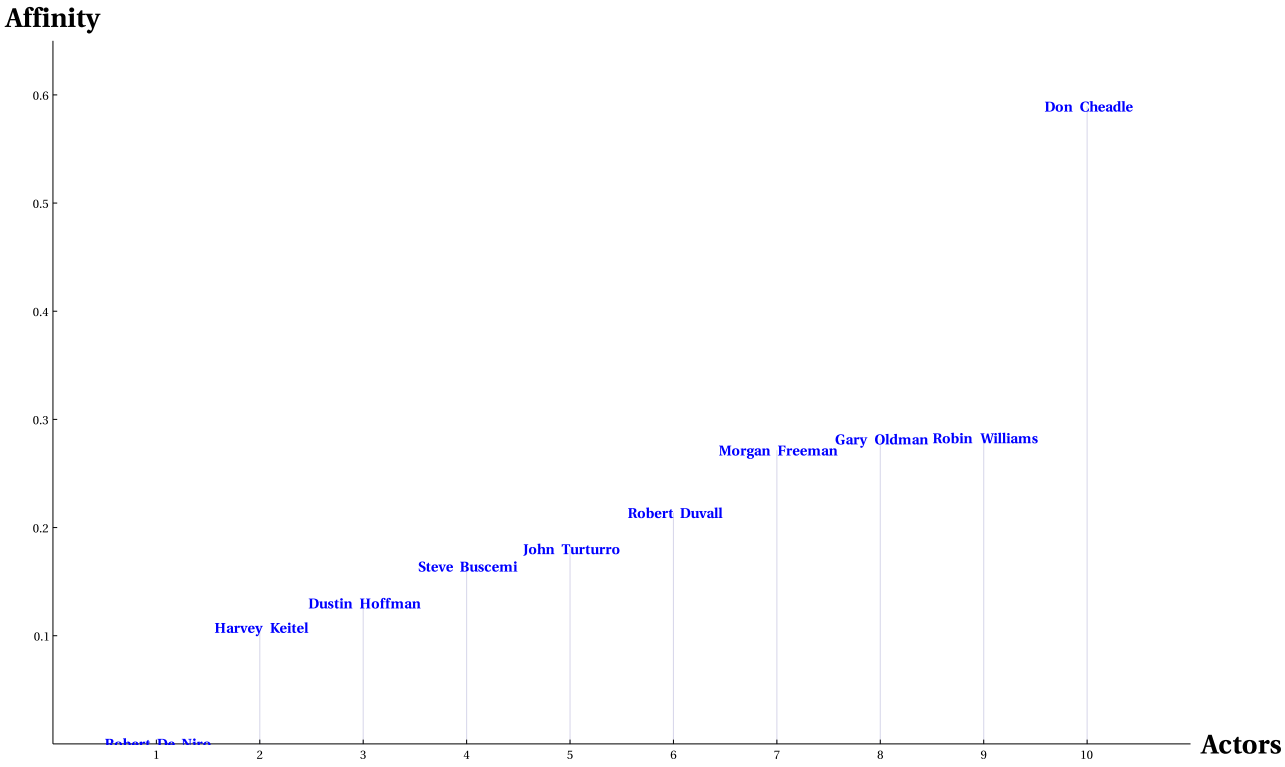

Figure 5.6: Plot of effective resistance with Robert De Niro as central character

Euclidean length. Thus, classical measurements are unaffected. We calculated the metric explicitly in free field theory and also took a step towards understanding the interactions by calculating it for the Wilson-Fischer fixed point.

It is of great interest to calculate the metric (perhaps exactly) for the case of asymptotically free scalar quantum field theories in two dimensions. We would expect that there are logarithmic corrections to the length of a curve, a first indication of non-Riemannian geometry. Also, there are many two dimensional scalar field theories that are exactly solvable [81]; can we get an exact formula for the metric in some of them?

But by far the question of greatest interest is that of $\lambda\phi^4$ theory in four dimen-

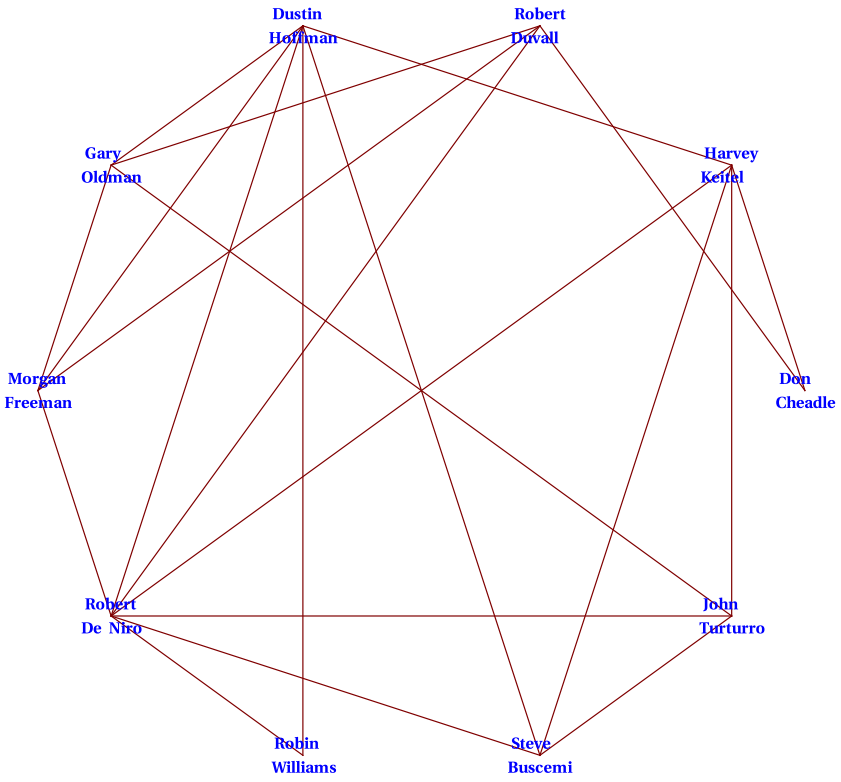

Figure 5.7: Graph with edges between actors who have acted in same movie

sions. Since perturbation theory breaks down at short distances, it is not possible to study this question analytically. A numerical simulation is needed to understand how $\sigma(x, x')$ depends on, or differs from, the Euclidean distance $|x - x'|$. How does the embedding in Sec. 5.4 of two dimensional space-time into three dimensional hyperbolic space change in the presence of interactions?

Another interesting offshoot of our work could be to extend this notion of distance beyond Gaussian distributions. There are many distributions in nature which do not fall under this. They may be discrete as well. Can one give a notion of proximity for

such arbit distributions by studying the data associated with them? For example, if one were to study the amount of rainfall all over the world, will the deserts be easily clustered under one group based on the data.

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APPENDIX A

METRIC GEOMETRY

A metric on a set X is a function $d : X \times X \to \mathbb{R}$ such that

- $d(x, x) = 0$
- $d(x, x') > 0$ if $x \neq x'$ (separation)
- $d(x, x') = d(x', x)$ (symmetry)
- $d(x, x') \leq d(x, x'') + d(x'', x')$ (triangle inequality)

The most familiar example is the Euclidean metric on $\mathbb{R}^n$.

$$|x - x'| = \sqrt{\sum_{i=1}^{n} (x^i - x'^i)^2}. \tag{A.1}$$

This metric is so ingrained in us that we might forget that the actual metric of space-time should be deduced by physical measurements and is not self-evidently Euclidean. Often (e.g., numerical simulations of scalar field theory, solution of PDEs) we have to approximate space by a discrete lattice $\Omega^n_{a,L} = a\,(\mathbb{Z}/\Lambda\mathbb{Z})^n$ with nearest neighbor spacing a and period $L = \Lambda a$ in each direction. Then the Euclidean metric is approximated by the length of the shortest path connecting two points on the

lattice

$$l(x,x') = \sqrt{\sum_{i=1}^{n}(x^i - x'^i \bmod L)^2}. \tag{A.2}$$

The square root of a metric is again a metric. More generally, if $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a concave function $f''(x) < 0$ with $f(0) = 0$, then we can construct from a given metric d a new one $\tilde{d}(x,x') = f((d(x,x'))$. But the square of a metric is not always a metric. For example, the square of the Euclidean metric is not a metric: the sum of the sides of an obtuse triangle is not greater than the square of the side opposite.

The length of the shortest curve (geodesic) connecting two points in a Riemannian manifold is a metric. Any metric arising as the length of geodesics has the intermediate property: given any pair of points (x,x') there is another x'' (a midpoint) that saturates the triangle inequality:

$$d(x,x') = d(x,x'') + d(x'',x').$$

Any point x'' lying along the shortest geodesic connecting x, x' would suffice. There are metrics that do not have this property. According to them, the distance between two points can be shorter than the length of every curve connecting them. Obviously, such metrics are non-Riemannian. This is precisely the case of interest to us.

Although the concept of derivative does not make sense in general on a metric space (for that we would need a differential manifold), Lipschitz functions are the analogue of differentiable functions. A function $f : X \to Y$ between metric spaces is said to be k-Lipschitz if

$$\frac{d_Y(f(x), f(x'))}{d_X(x,x')} < k,$$

for all $x, x' \in X$. Roughly speaking, the magnitude of the derivative is less than k. Two metric spaces are Lipschitz equivalent if there are continuous, one-to-one Lipschitz maps in each direction which are inverses of each other. Lipschitz equivalence is roughly analogous to diffeomorphisms between manifolds.

Length of Curves

Given a metric we can define the length of a curve as the largest sum of the length of line segments. In more detail, a curve $\gamma : [0, T] \to X$ can be broken up into segments

$$0 \equiv t_0 < t_1 < t_2 < \cdots t_k < T \equiv t_{k+1}.$$

The sum of the chord lengths

$$\sum_{i=1}^{k+1} d\left(\gamma(t_{i-1}), \gamma(t_i)\right)$$

can be thought of as an approximation to its length. The actual length of the curve is the least upper bound of all such approximations; i.e., avoiding all the "short-cuts" made by the chords:

$$l[\gamma] = \sup_{0<t_1<\cdots t_k<T} \sum_{i=1}^{k+1} d(\gamma(t_{i-1}), \gamma(t_i)).$$

The length of a continuous curve can be infinite. There are well known examples of continuous curves (e.g., Koch curve) with an infinite length in the Euclidean metric. Also, suppose we define $d(x, x') = |x - x'|^{\frac{1}{2}}$, the square root of the Euclidean distance. Then the length of every straight-line segment is infinite!

For the familiar case of a differentiable curve in a Riemannian manifold, it is not

hard to verify that this agrees with the usual definition

$$l[\gamma] = \int_0^T \sqrt{g_{\gamma_t}\left(\gamma(\dot{t}), \gamma(\dot{t})\right)} dt.$$

The reason is that, for short enough segments, the chord length is approximated by the length of the tangent vector. It is not hard to come up with continuous but not differentiable curves of infinite length.

Length Metric

Given a metric d, we can often construct from it a (possibly distinct) length metric $d_l(x, x')$ as the greater lower bound of the lengths of all the curves that connect x to x'. (This construction could fail if the length of every continuous curve is infinite, or if there is no greater lower bound.)

A metric is said to be *geodesic* (also called a interior space or intrinsic metric) if this the one we started with: $d_l(x, x') = d(x, x')$.

The Euclidean distance is an example of a geodesic metric. Any length metric is itself a geodesic metric. That is, $(d_l)_l = d_l$ for any d.

An example of a non-geodesic metric (Fig. A.1) is the length of a chord connecting two points on a circle:

$$d(\theta, \theta') = 2 \sin \frac{|\theta - \theta'|}{2} \tag{A.3}$$

where $0 \leq \theta, \theta' \leq 2\pi$ is the usual polar co-ordinate. For small angles, this agrees with the arc-length

$$d(\theta, \theta') \approx |\theta - \theta'|. \tag{A.4}$$

Therefore if we break-up an arc into small segments and add up the lengths we

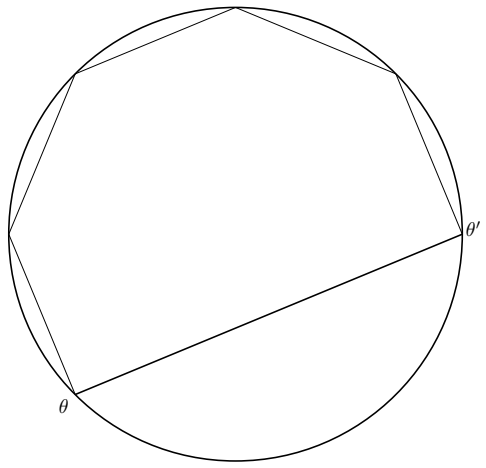

Figure A.1: Diagram shows how arcs are approximated by chord lengths

will get the arc-length; i.e., the length metric of d is just the arc-length

$$d_l(\theta, \theta') = |\theta - \theta'|. \tag{A.5}$$

But in general

$$d(\theta, \theta') < d_l(\theta, \theta').$$

Note that although the chord length is not a geodesic metric on the circle, it is the length of a geodesic in the plane in which the circle is embedded. We showed that the standard deviation metric of a scalar field theory can be similarly realized as the geodesic length in a space of one dimension higher.

Triangle Inequality for Standard Deviation Metric

Suppose that a_i, b_i, c_i (for some finite range of the index i) are positive numbers satisfying the inequality $a_i \leq b_i + c_i$. Then it is obvious that the weighted averages

$\langle a \rangle = \dfrac{\sum\limits_i a_i w_i}{\sum\limits_i w_i}$ also satisfy $\langle a \rangle \leq \langle b \rangle + \langle c \rangle$. More generally, the L^p-averages for $p \geq 1$,

$$\langle a \rangle_p = \left[\frac{\sum\limits_i a_i^p w_i}{\sum\limits_i w_i} \right]^{\frac{1}{p}}$$

satisfy

$$\langle a \rangle_p \leq \langle b \rangle_p + \langle c \rangle_p.$$

To see this, simply note that $\langle a \rangle_p \leq \langle b + c \rangle_p$ by monotonicity; the rest follows by the fact that the L^p-norm satisfies the triangle inequality.

If we replace the discrete average above by an integral with respect to a probability measure $e^{-S(\phi)} d\phi$, the inequality continues to hold. For positive functions,

$$a(\phi) \leq b(\phi) + c(\phi) \implies \langle a \rangle_p \leq \langle b \rangle_p + \langle c \rangle_p. \tag{A.6}$$

$$\langle a \rangle_p = \left[\frac{\int e^{-S(\phi)} a^p(\phi) d\phi}{\int e^{-S(\phi)} d\phi} \right]^{\frac{1}{p}}. \tag{A.7}$$

These facts are useful for us because they show that the L^p average of a metric is also a metric. We just have to choose

$$a(\phi) = |\phi(x) - \phi(x')|, b(\phi) = |\phi(x) - \phi(x'')|, c(\phi) = |\phi(x'') - \phi(x')|. \tag{A.8}$$

Thus $\sigma(x, x') = \sqrt{\langle (\phi(x) - \phi(x'))^2 \rangle}$ satisfies the triangle inequality.

APPENDIX B

EFFECTIVE RESISTANCE FROM GAUSSIAN INTEGRALS

The effective resistance in an electrical resistor circuit can be obtained by doing Gaussian integrals. Though it is not the conventional way to find the effective resistance, the technique comes in handy for complicated circuits. They are both minimization problems in two different domains.

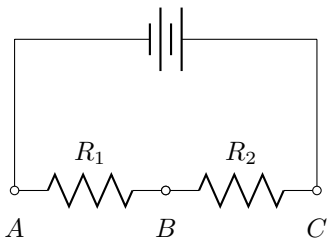

Figure B.1: Resistors in series

Given an electrical circuit of resistors in series (Fig. B.1) we can find the total dissipation energy. If the voltages ϕ_A, ϕ_C at A, C are fixed, the problem amounts to

finding the variable voltage χ at B. In order to minimize the energy ($\frac{V^2}{R}$),

$$E(\chi) = \frac{(\phi_A - \chi)^2}{R_1} + \frac{(\chi - \phi_C)^2}{R_2}, \tag{B.1}$$

we get the optimal voltage χ_0 by setting $\frac{dE}{d\chi} = 0$ as below-

$$\chi_{\min} = \frac{R_2\phi_A + R_1\phi_C}{R_1 + R_2}. \tag{B.2}$$

The minimum energy is obtained for $E(\chi_0)$, i.e.

$$E_{\min} = \frac{(\phi_A - \phi_C)^2}{R_1 + R_2}. \tag{B.3}$$

This shows that the effective resistance in the circuit is $R_{\text{net}} = R_1 + R_2$. One can obtain the same result by doing the following Gaussian integral.

$$\int d\chi\, e^{-\frac{(\phi_A-\chi)^2}{R_1} - \frac{(\chi-\phi_C)^2}{R_2}} = \sqrt{\pi \frac{R_1 R_2}{R_1 + R_2}}\, e^{-\frac{(\phi_A-\phi_C)^2}{R_1+R_2}} \tag{B.4}$$

The result of doing the integral amounts to minimizing the exponent in the integrand.

The same principle can be carried over to other complicated circuits which amounts to doing multi-dimensional Gaussian integrals. The steps involved in finding the effective resistance across nodes A, B have been listed below-

- The dissipation energy for the electrical circuit is computed by fixing the voltages ϕ_A, ϕ_B at two nodes A, B and varying the voltages $\phi_1, \ldots, \phi_n$ at n other nodes.

- The dissipation energy $E(\phi_A, \phi_B|\phi_1, \ldots, \phi_n)$ turns out to be-

$$\begin{pmatrix} \phi_A & \phi_B & \phi_1 & \phi_2 & \ldots & \phi_n \end{pmatrix} \left(\begin{array}{c|c} L_{2\times 2} & J_{2\times n} \\ \hline J^T_{n\times 2} & K_{n\times n} \end{array}\right) \begin{pmatrix} \phi_A \\ \phi_B \\ \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{pmatrix}. \tag{B.5}$$

  L involves the resistances between the edges A, B; J involves the resistances which join edges A or B to node $1, 2, \ldots n$; K includes the resistances between the nodes 1 to n alone. The block structure divides the vertices where voltages is fixed, from the ones where it isn't.

- The dissipation energy can be minimized by computing the n dimensional Gaussian integral-

$$\int d\phi_1 \ldots d\phi_n \, e^{-E(\phi_A,\phi_B|\phi_1,\ldots,\phi_n)} = e^{-E_{\min}(\phi_A,\phi_B)}. \tag{B.6}$$

- The exponent in the result is-

$$E_{\min}(\phi_A, \phi_B) = \begin{pmatrix} \phi_A & \phi_B \end{pmatrix} \begin{pmatrix} Q_{2\times 2} \end{pmatrix} \begin{pmatrix} \phi_A \\ \phi_B \end{pmatrix}, \quad Q = L - JK^{-1}J^T. \tag{B.7}$$

The effective resistance is $R_{AB} = 2/\mathrm{Tr}(\mathbf{Q})$.

The above methodology can be used to compute the resistance metric for any arbitrary configuration of resistors easily, which are hard to find in general. A lot of this can be programmed into Mathematica and computed.