Dualities and Integrability in Low Dimensional AdS/CFT

Antonio Pittelli

Thesis submitted to the University of Surrey
for the degree of Doctor of Philosophy

Department of Mathematics
University of Surrey
Guildford GU2 7XH, United Kingdom

E-mail address: a.pittelli@surrey.ac.uk

Scientific abstract

In this dissertation we perform a series of study concerning dualities and integrability properties underlying the AdS_3/CFT_2 and AdS_2/CFT_1 correspondences. These are particularly interesting because symmetry do not constrain the dynamics of AdS_3 and AdS_3 superstrings in the same way as in the higher dimensional instances of AdS/CFT, allowing for novel phenomena such as the presence of massless worldsheet modes or non-coset fermions.

We will investigate the self-duality of Green–Schwarz supercoset sigma models on $\mathrm{AdS}_d \times S^d \times S^d$ $(d = 2, 3)$, whose isometry supergroups are $(d-1)$ copies of the exceptional Lie supergroup $\mathrm{D}(2,1;\alpha)$. Our main finding is that additional *complex* T-dualities along one of the spheres S^d are needed to map the superstring action to itself. Importantly, this proves dual superconformal symmetry of their CFT duals via AdS/CFT.

Dual superconformal symmetry is strictly related to integrability, which we study in depth for both AdS_3 and AdS_2 superstrings. Indeed, we will derive the exact S-matrix conjectured to be related to the massive modes of type IIB $AdS_2 \times S^2 \times T^6$ superstrings. This S-matrix is $\mathfrak{psu}_c(1|1)$ invariant and it was found to be in perfect agreement with the tree-level result following from string perturbation theory.

We also unveil the Yangian algebra ensuring the integrability of the $\mathrm{AdS}_2 \times S^2 \times T^6$ superstring in the planar limit, $\mathcal{Y}[\mathfrak{psu}(1|1)_c]$, as well as its *secret* symmetries. By using the RTT realisation, we provide two different representations of the Hopf algebra: one is reminiscent of the Yangian underlying AdS_5/CFT_4, but it is not of evaluation type. The other representation, obtained from co-commutativity, is instead of evaluation type.

We explore two limits of the S-matrix for AdS_2/CFT_1: one is the classical r-matrix, which is the first non-trivial order in the $1/g$ expansion, with g being the effective tension in the $\mathrm{AdS}_2 \times S^2 \times T^6$ superstring action. In this limit, corresponding to classical strings, we found that secret symmetries not only are present, but also essential to formulate the classical r-matrix in a universal, representation independent form. On the other hand, the limit $g \to 0$, corresponding to the weakly coupled CFT_1, shows that the dual integrable model is described by an effective theory of free fermions on a periodic spin-chain if $g = 0$, while one obtains a non-trivial spin chain of XYZ type if $g \neq 0$.

Finally, we investigate Yangian and secret symmetries for AdS_3 type IIB superstring backgrounds, verifying the persistence of such structures in AdS_3/CFT_2. Especially, we find that the antipode map, related to crossing symmetry, exchanges in a non-trivial way left and right generators of $\mathcal{Y}[\mathfrak{psu}(1|1)^2_c]$, the Yangian underlying the integrability of AdS_3 superstrings.

Keywords and AMS Classification Codes:

Acknowledgements

I would very much like to thank my supervisors, Dr Alessandro Torrielli and Dr Martin Wolf, for patiently guiding me through the jungle of contemporary theoretical physics and beyond: their advices will stay with me forever.

I am also particularly grateful to them, to Dr Giulia Ghirardello[1], Dr Andrea Prinsloo for several valuable comments regarding this manuscript.

It was a fantastic pleasure to collaborate and share ideas with Michael Abbott, Ben Hoare, Jeff Murugan, Silvia Penati, Dmitri Sorokin, Per Sundin, Justine Tarrant, and Linus Wulff. Their expertise taught me a great deal about physics, suggesting novel exciting research lines.

I am tremendously thankful to Dmitri Sorokin, Alessandro Torrielli and Martin Wolf for supporting me during the past round of applications.

I definitely enjoyed my PhD at the University of Surrey. Especially, it was an extraordinary honour to be part of its *Fields, Strings and Geometry* group, which was a wonderfully pleasant and enthusiastic research environment. Moreover, being a PhD at Surrey allowed me to have intriguing and useful conversations with Fernando Alday, Gleb Arutyunov, Lorenzo Bianchi, Agnese Bissi, Riccardo Borsato, Claude Duhr, Pavel Etingof, Guido Festuccia, Yvonne Geyer, Simone Giombi, Pietro Antonio Grassi, Luca Griguolo, Nikolay Gromov, Jan Gutowski, Amihay Hanany, Martin Heinze, Pietro Longhi, Lionel Mason, Joseph Minahan, Sanefumi Moriyama, Emanuele Moscato, Sara Pasquetti, Jan Plefka, Andrea Prinsloo, Vidas Regelskis, Olof Ohlsson Sax, Christian Saemann, Domenico Seminara, Alessandro Sfondrini, Bogdan Stefanski, Richard Szabo, Yuji Tachikawa, Cesare Tronci, Arkady Tseytlin, Pedro Vieira, Kostantin Zarembo.

Finally, a special thank to the friends I have met during these three years: Sebastian Bartos, Diego Ferruzzo, Andrea Fontanella, Sandra Kabisch, Pier Vittorio Larocca, Bruno Lima de Souza, Fabrizio Nieri, Marzia Petrucci, Michelangelo Preti, Marco Sanchioni, Joakim Stromwall and Roberto Sisca. Thank you so much for making my PhD so colourful as it was, guys.

[1] I especially thank Giulia Ghirardello for lovVing me just as I am.

Declarations

This dissertation is a result of my own work. It is based on my PhD research projects, done at the Department of Mathematics at the University of Surrey between April 2013 and July 2016.

This thesis refers to material that appeared on the electronic print archive http://arXiv.org and led to the following publications:

- A. Pittelli, A. Torrielli, M. Wolf, *Secret Symmetries of Type IIB Superstring Theory on* $AdS_3 \times S^3 \times M^4$, J.Phys. A 47 (2014) 45, 455402
- B. Hoare, A. Pittelli, A. Torrielli, *Integrable S-matrices, Massive and Massless Modes and the* $AdS_2 \times S^2$ *superstring*, JHEP 1411 (2014) 051
- M. Abbott, J. Murugan, S. Penati, A. Pittelli, D. Sorokin, P. Sundin, J. Tarrant, M. Wolf, and L. Wulff, *T-Duality for* $AdS_d \times S^d \times M^{10-2d}$ *Green–Schwarz Superstrings*, JHEP 1512 (2015) 104
- B. Hoare, A. Pittelli, A. Torrielli, *The S-matrix Algebra of the* $AdS_2 \times S^2$ *Superstring*, arXiv: 1509.07587, Phys.Rev. D93 (2016) no.6, 066006

Any ideas, data, pictures or text resulting from the work of others are clearly identified as such within the work and attributed to the authors in the text or bibliography. This thesis has not been submitted for any other academic degree or professional qualification. The University of Surrey reserves the right to require an electronic version of the final document as submitted for assessment as above.

Sincerely yours,

Antonio Pittelli

Contents

1

Introduction

1.1. AdS/CFT Correspondences

1.1.1. $\mathbf{AdS_5/CFT_4}$

There are more symmetries in a theory than those displayed by its lagrangian. For instance, this happens in presence of *dualities*, which map to each other models representing completely different physical systems. As we will see, dualities indicate the existence of hidden symmetries providing a dictionary between the dual pair.

A remarkable example of duality is the AdS/CFT Correspondence, proposed by Maldacena [1] and promptly developed by Gubser, Klebanov, Polyakov [2] and Witten [3]. AdS/CFT claims that string theory on a d-dimensional Anti-de Sitter (AdS) spacetime is equivalent to a specific $(d-1)$-dimensional conformal field theory (CFT) living on the conformal boundary of the corresponding AdS (for reviews, the reader is referred to *e.g.* [4–7]).

This is a holographic duality connecting a gravity theory in the bulk with a gauge theory on the boundary. Moreover, AdS/CFT is a strong/weak coupling duality: the deeply quantum regime of the CFT is given by classical strings, and the other way round: an amazing consequence of this is that we can obtain remarkable insights on quantum gravity by looking at the weakly coupled phase of the dual gauge theory.

Naturally, the best understood instance of the correspondence is the maximally symmetric case: type IIB superstrings on $\mathrm{AdS}_5 \times S^5$ with N units of the self-dual five-form flux through the S^5 is equivalent to four dimensional $\mathcal{N}=4$ super-Yang-Mills theory (SYM) with $\mathrm{SU}(N)$ gauge group. The action for these superstrings is encoded in a non-linear sigma model based on the supercoset [8]

$$\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(1,4)\times\mathrm{SO}(5)} \supset \mathrm{AdS}_5 \times S^5. \tag{1.1}$$

$\mathcal{N}=4$ SYM is the maximally supersymmetric completion of Yang-Mills theory in four dimensions: its field content consists of the gauge bosons A_μ, six massless real scalars, four chiral and four anti-chiral fermions, all transforming in the adjoint representation of the gauge group SU(N). Importantly, the beta function of $\mathcal{N}=4$ SYM vanishes at all loops and the theory is superconformal at the quantum level. The whole AdS/CFT story was born from the intimate connection between these two models, triggered by D3-branes. Indeed, the action for a stack of N D3-branes is made of three pieces:

$$S = S_{\text{bulk}} + S_{\text{brane}} + S_{\text{interactions}}. \tag{1.2}$$

The open strings ending on the D3s realise a U(N) gauge theory, while in the bulk there are closed strings moving in the supergravity background with metric

$$ds^2 = H^{-1/2}(y)\, dx^\mu\, dx_\mu + H^{1/2}(y)\,(dy^2 + y^2\, d\Omega_5^2), \qquad H(y) = 1 + \frac{4\pi g_s N(\alpha')^2}{y^4}, \tag{1.3}$$

with x^μ being the spacetime coordinates parallel to the branes and y measuring the distance from the stack. Close to the branes we have $y^4 \ll 4\pi g_s N(\alpha')^2$ and the metric becomes

$$ds^2 = \frac{y^2}{\sqrt{4\pi g_s N(\alpha')^2}}\, dx_{||}^2 + \sqrt{4\pi g_s N(\alpha')^2}\,\frac{dy^2}{y^2} + \sqrt{4\pi g_s N(\alpha')^2}\,(dy^2 + y^2\, d\Omega_5^2); \tag{1.4}$$

namely, $\text{AdS}_5 \times S^5$ with both AdS and the sphere having the same radius $R^4_{\text{AdS}_5} = 4\pi g_s N(\alpha')^2$. Maldacena's insight was to further take the limit $\alpha' \to 0$, which kills $S_{\text{interactions}}$ decoupling the bulk theory from the worldvolume one. Altogether, in the bulk we have type IIB superstrings propagating in $\text{AdS}_5 \times S^5$; while on the boundary, *i.e.* on the branes, we have a four dimensional SU(N) gauge theory[1] with 16 supersymmetries, which is $\mathcal{N}=4$ SYM.

Consistently, the global symmetries of the two models perfectly match: PSU(2,2|4) is either the isometry supergroup of $\text{AdS}_5 \times S^5$ or the four dimensional superconformal group[2].

Claiming the complete equivalence of the string and the gauge theory is sometimes referred to as the *strong form* of the AdS/CFT correspondence, as it is to hold for all values of N and of $g_s = g^2_{YM}$. This cannot be checked via perturbative techniques only; on the other hand, non-perturbative methods such as Localisation [9, 10], Integrability [11, 12] and lattice computations [13, 14] have allowed for huge progresses.

Another approach consists in comparing asymptotic and perturbative regions. Indeed, one can consider the 't Hooft coupling of $\mathcal{N}=4$ SYM, $\lambda \equiv g^2_{YM}N = g_s N$, and take the limits $N \to \infty, g_s \to 0$ while keeping λ fixed. In Yang-Mills theory, this limit is well defined and yields a topological genus expansion of the field theory's Feynman diagrams: in fact, the leading order of the $1/N$ expansion contains planar diagrams only. On the AdS side, the string coupling can

[1] The U(1) factor in U(N) takes into account the collective motion of the branes, which decouples from the theory living on the worldvolume. The correct gauge group is then SU(N).

[2] Notice also that both theories are invariant under S-duality.

be rexpressed in terms of the ‘t Hooft coupling as $g_s = \lambda/N$. Since λ is kept fixed, the ‘t Hooft limit corresponds to weakly coupled strings. Notice that small g_s implies that strings tend not to split, while the decoupling limit $\alpha' \to 0$ ensures that the worldsheet cannot oscillate wildly.

1.1.2. $\mathbf{AdS_4/CFT_3}$

AdS/CFT holds true in lower dimensions. A well-established example is that of type IIA superstring theory on $\mathrm{AdS}_4 \times \mathbb{C}\mathrm{P}^3$, whose fluxes are the Ramond-Ramond $F^{(4)} \sim N$ through AdS_4 and the Ramond-Ramond $F^{(2)} \sim k$ through a $\mathbb{C}P^1 \subset \mathbb{C}P^3$, admits a dual CFT.

This background is obtained by dimensional reduction from the eleven-dimensional maximally supersymmetric $\mathrm{AdS}_4 \times S^7$ supergravity solution representing the near-horizon limit of a stack of M2-branes. The dimensional reduction is in fact a Hopf projection from S^7 down to $\mathbb{C}\mathrm{P}^3$ [15, 16]. The isometry supergroup of $\mathrm{AdS}_4 \times S^7$ is $\mathrm{OSp}(8|4)$: the projection breaks part of it, leaving a $\mathrm{OSp}(6|4)$ symmetry. The latter consistently reproduces the isometry supergroup of $\mathrm{AdS}_4 \times \mathbb{C}\mathrm{P}^3$ endowed with 24 supercharges. This type IIA solution possesses four-form and two-form RR fluxes. Notice that the $\mathrm{AdS}_4 \times \mathbb{C}\mathrm{P}^3$ action can be expressed as a non-linear sigma model based on a supercoset [17, 18]

$$\frac{\mathrm{OSp}(6|4)}{\mathrm{SO}(1,3) \times \mathrm{U}(3)} \supset \mathrm{AdS}_4 \times \mathbb{C}P^3. \tag{1.5}$$

However, the coset description does not encode the complete superstring dynamics [19].

On the gauge theory side, we have ABJM theory, a $\mathcal{N} = 6$ superconformal Chern-Simons matter theory with gauge group $\mathrm{U}(N) \times \mathrm{U}(N)$ on $\mathbb{R}^{1,2}$ and Chern-Simons levels k and $-k$. Its field content is given by two gauge fields A_μ and $\hat{A}_\mu$, four complex scalar fields Y^A, and four Weyl-spinors ψ_A. The matter fields are $N \times N$ matrices transforming in the bi-fundamental representation of the gauge group.

If $k > 2$, the global symmetry group of ABJM theory is the orthosymplectic supergroup $\mathrm{OSp}(6|4)$, which is exactly the isometry supergroup of $\mathrm{AdS}_4 \times \mathbb{C}\mathrm{P}^3$ [20, 21]. The bosonic components of $\mathrm{OSp}(6|4)$ are the R-symmetry group $\mathrm{SO}(6)_R \cong \mathrm{SU}(4)_R$ and the 3d conformal group $\mathrm{Sp}(4) \cong \mathrm{SO}(2,3)$, while the odd part of $\mathrm{OSp}(6|4)$ generates the $\mathcal{N} = 6$ supersymmetry transformations.

Both the string and the gauge theory are controlled by only two parameters, $k, N \in \mathbb{N}$. These parameters determine all other quantities such as coupling constants and the effective string tension. In ABJM theory, the Chern-Simons level k plays the role of a coupling constant. The fields can be rescaled in such a way that all interactions are suppressed by powers of $1/k$; *i.e.*, taking k large leads to the weakly coupled regime.

The planar limit for ABJM consists in

$$k, N \to \infty, \qquad \lambda \equiv \frac{N}{k} = \text{fixed} . \tag{1.6}$$

On the gravity side, the string coupling constant and effective tension are given by

$$g_s \sim \left(\frac{N}{k^5}\right)^{1/4} = \frac{\lambda^{5/4}}{N} \qquad \frac{R^2}{\alpha'} = 4\pi\sqrt{2\lambda}\,, \tag{1.7}$$

where R is the radius of $\mathbb{C}P^3$ and twice as the radius of AdS$_4$. It is clear to see that the relations (1.7) mimics those found in the context of the AdS$_5$/CFT$_4$ correspondence. In fact, the same way as $\mathcal{N} = 4$SYM is the worldvolume theory of the D-branes generating the AdS$_5 \times S^5$ background in the near-horizon limit, ABJM is the worldvolume theory of a stack of N M2 branes probing $\mathbb{C}^4/\mathbb{Z}_k$ [20]. Since M2 branes are involved, the actual gravity dual of ABJM theory shall be M-theory rather than type IIA superstrings. Indeed, in the case of $N = 2$, the gauge group is SU(2) $\times$ SU(2) $\cong$ SO(4), supersymmetry is enhanced to $\mathcal{N} = 8$ and ABJM coincide with the Bagger-Lambert-Gustavsson higher gauge theory [22–25].

1.1.3. AdS$_3$/CFT$_2$

The AdS/CFT correspondence for superstrings propagating in AdS$_3$ spacetime is much less understood than its higher dimensional counterparts. In this case, the interesting backgrounds are of the form AdS$_3 \times S^3 \times M^4$, with M^4 being either $S^3 \times S^1$, T^4 or $K3$. Especially, type IIA AdS$_3 \times S^3 \times S^3 \times S^1$ can be obtained via dimensional reduction of the eleven-dimensional AdS$_3 \times S^3 \times S^3 \times T^2$ solution representing the near-horizon geometry of two M5-branes intersecting one M2-brane [26]; the corresponding type IIB solution being instead an intersection of D1 and D5 branes [27]. This background, studied *e.g.* in [28], possesses D$(2,1;\alpha) \times$ D$(2,1;\alpha) \times U(1)$ isometry supergroup[3] with 16 supersymmetries and admits either NSNS or RR three-form flux (see also [31–33]). The radii of the two three-spheres $R_{L,R}$ and the AdS radius R_{AdS} are not independent as the supergravity equations impose

$$\frac{1}{R_L^2} + \frac{1}{R_R^2} = \frac{1}{R_{\text{AdS}}^2}\,. \tag{1.8}$$

Indeed, the parameter α into D$(2,1;\alpha)$ is related to the size of the two spheres and AdS via

$$\frac{R_{\text{AdS}_2}^2}{R_L^2} = \alpha; \qquad \frac{R_{\text{AdS}_2}^2}{R_L^2} = 1 - \alpha. \tag{1.9}$$

If $\alpha = 1/2$, the two spheres have equal radii: the exceptional superalgebra $\mathfrak{d}(2,1;\alpha)$ with $\alpha = 1/2$, in fact, coincides with $\mathfrak{osp}(4|2)$. On the other hand, if $\alpha = 0, 1$, the radii of AdS and of one of the two spheres become equal, while the other sphere blows up to an infinite size: by compactifying the latter on a T^3, one obtains the $AdS_3 \times S^3 \times T^4$ background. The $\mathfrak{d}(2,1;\alpha)$ for $\alpha = 0, 1$, in turn, reduces to $\mathfrak{psu}(1,1|2)$: this is sensible as the isometry supergroup of $AdS_3 \times S^3$ is indeed

[3]Please see [29, 30] for more details about the exceptional Lie superalgebra D$(2,1;\alpha)$

PSU(1,1|2) × PSU(1,1|2). Therefore, the $AdS_3 \times S^3 \times S^3 \times S^1$ background with characteristic lengths fulfilling (1.8) encompasses the $AdS_3 \times S^3 \times T^4$ case as well.

$\mathcal{N} = 4$ super-Yang-Mills [34] and ABJM theory [20] are essentially different from The CFT duals of AdS_3 superstrings: in the latter cases, indeed, the couplings of the worldvolume theories are dimensionful and the two dimensional CFTs appear as IR fixed points of non-trivial renormalisation group flows. Furthermore, the field content and the moduli space of these CFTs are reacher than $\mathcal{N} = 4$ and ABJM's [35].

Despite the complications, the dictionary between $AdS_3 \times S^3 \times T^4$ superstrings and two dimensional CFTs was established [36–40] and the $\mathcal{N} = (4, 4)$ CFT dual of the type IIB theory was finally identified by studying the IR dynamics of intersecting D1-D5 branes [41, 42]. In particular, it was found that the 't Hooft coupling of the theory is given by $\lambda \equiv N_c/N_f$, where N_c is the number of colours (coinciding with the number of D1 branes) while N_f is the number of flavours (coinciding with the number of D5 branes). Moreover, the moduli space of such a theory should contain a point described by a symmetric product orbifold CFT, the latter being dual to the tensionless limit of $AdS_3 \times S^3 \times T^4$ superstrings [43]; nonetheless, it not yet clear how the $\mathcal{N} = (4, 4)$ CFT and the $\mathrm{Sym}^N\left(T^4\right)$ orbifold are related to each other. Recently, a candidate for the CFT dual of type IIB $AdS_3 \times S^3 \times S^3 \times S^1$ superstrings was proposed by considering a D1-D5-D5 system [44], but the features of this model are still under investigation.

1.1.4. AdS_2/CFT_1

AdS/CFT should also involve strings propagating in two-dimensional Anti-de Sitter spacetime. Indeed, one can have type IIB $AdS_2 \times S^2 \times T^6$ superstrings supported by self-dual RR 5-form flux: such a background preserves 8 supersymmetries and descends from the near horizon limit of four intersecting D3-branes [45]. By performing three T-duality transformations along the T^6 one obtains the corresponding type IIA background with 4-form and 2-form fluxes, whose brane configuration is given by the superposition of three D4-branes and one D0-brane. Solutions linked to more general compact manifolds [46, 47] or different brane configurations [48, 45] are possible.

In the type IIB version of $AdS_2 \times S^2 \times T^6$, studied *e.g.* in [49], the F_5 flux has non-zero components along the T^6 directions, implying that the T^6 part of the background does not decouple from $AdS_2 \times S^2$, which is elegantly described by a $\mathrm{PSU}(1,1|2)/\mathrm{SO}(1,1) \times U(1)$ supercoset sigma model. This fact radically distinguishes AdS_2 superstrings from the backgrounds discussed previously. In higher dimensions, indeed, kappa-symmetry can be used to either eliminate or decouple the non-coset part of the action. In other words, it is not possible to write down the string action for AdS_2 superstrings as a sort of "direct sum" of coset and non-coset part.

Another remarkable AdS_2 solution is the type IIA $\mathrm{AdS}_2 \times S^2 \times S^2 \times T^4$ background, which can be obtained via dimensional reduction from the eleven-dimensional supergravity solution $\mathrm{AdS}_2 \times S^2 \times S^2 \times T^5$, the latter being the near-horizon geometry of two M2 and four M5 intersecting with each other [50]. The corresponding isometry supergroup is $\mathrm{D}(2,1;\alpha) \times \mathrm{U}(1)^4$, which in the limit $\alpha \to 0, 1$ reduces to type IIA $\mathrm{AdS}_2 \times S^2 \times T^6$. T-duality along the torus of either $\mathrm{AdS}_2 \times S^2 \times S^2 \times T^4$ or $\mathrm{AdS}_2 \times S^2 \times T^6$ provides the related IIB backgrounds.

The one-dimensional CFT duals of the above backgrounds are the most mysterious ones in the whole AdS/CFT story. As yet, there is an unsolved debate about the physical interpretation of these CFT_1, which could be superconformal quantum-mechanical systems, as well as chiral "halves" of underlying two-dimensional CFTs [1, 51–54]. No candidates for the duals of either $\mathrm{AdS}_2 \times S^2 \times S^2 \times T^4$ or $\mathrm{AdS}_2 \times S^2 \times T^6$ have still been formulated.

1.2. T-Self-Duality and Dual Superconformal Symmetry

1.2.1. T-duality and Buscher Rules

One of the most remarkable features of string theory is its *uniqueness*. In fact, only five different perturbative string theories exist: type I, type IIA, type IIB, heterotic SO(32) and heterotic $\mathrm{E}_8 \times \mathrm{E}_8$. These are all connected to each other by maps called *T-duality* and *S-duality*, see Figure 1.1. The former will be one of the main topics of this thesis: we introduce T-duality now by looking at closed free strings moving in flat spacetime.

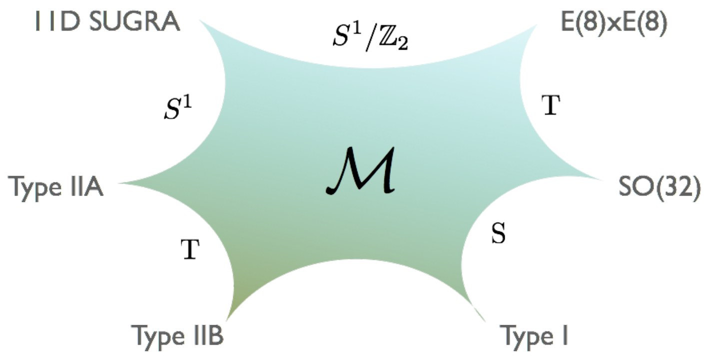

Figure 1.1: Dualities connect to each other the five possible string theories and eleven-dimensional supergravity. They are all encompassed by $\mathcal{M}$-theory, the unique mathematical framework describing the dynamics of strongly coupled membranes.

Let us consider bosonic string theory in 26-dimensional Minkowski spacetime, where one of the 25 spatial directions has been compactified on a circle of radius R. The spectrum for a closed string propagating in this background is

$$M^2 = \frac{n^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} + \dots \tag{1.10}$$

where the dots stand for terms irrelevant to the present discussion, while n and w are the Kaluza-Klein and the winding number of the string respectively. In such a case, a T-duality transformation consists of inverting the radius of the circle by taking $R \to R = \alpha'^2/R$. The important observation here is that (1.10) stays the same provided that the winding number w and the Kaluza - Klein number n are interchanged. In the instance just examined, the spectrum of the system is T-duality invariant and the transformation embodies a genuine symmetry of the model. As for the mode expansion corresponding to the compact coordinate, one finds that the left-moving coordinates are unchanged whereas the right-movers pick up a minus sign:

$$\tilde{X}_L = X_L, \qquad \tilde{X}_R = -X_R, \tag{1.11}$$

with $\tilde{X}_{L,R}$ being the T-dual coordinates.

Two comments are in order. First: T-duality applies also to open strings attached to D-branes: in this case, the duality transformation maps Neumann boundary conditions to Dirichlet ones, and the other way around. Especially, T-duality along (respectively, orthogonally to) a D_p brane will mutate the latter into a D_{p-1} (respectively, D_{p+1}) brane. Second: when T-duality acts on superstrings, the transformations (1.11) changes the relative chirality of the fermionic partners of X_L and X_R. This implies that T-duality maps type IIB superstrings onto type IIA ones, and vice versa. Thus, an even number of T-dualities, is needed to recover the original type of superstrings.

The Polyakov action for bosonic strings in 26-dimensional Minkowski spacetime can be written as the sigma model

$$S[X] = T/2 \int_\Sigma \eta_{\mu\nu}\, dX^\mu \wedge *dX^\nu, \qquad \mu,\nu = 0,\dots,25, \tag{1.12}$$

with T being the string tension, Σ the worldsheet, $\eta_{\mu\nu}$ the Minkowski metric and $X^\mu(\tau,\sigma)$ the string coordinates in target space. In such a setting, the T-duality transformations (1.11) read

$$d\tilde{X}^{25} = *dX^{25}, \tag{1.13}$$

where we supposed that the direction compactified on a circle is the 25th. The relation (1.13) is part of Buscher's procedure for T-duality: its extension to superstrings will be discussed in Chapter 2, while here we shall continue to discuss how it works for the bosonic sigma model.

Let $V^\mu(\tau,\sigma)$ be 26 one-forms defined on Σ, $\tilde{X}^\mu$ as many Lagrange multipliers and $\tilde{S}$ the functional

$$\tilde{S}[V,\tilde{X}] = T/2 \int_\Sigma V^\mu \wedge *V_\mu - 2\,\tilde{X}^\mu\, dV_\mu. \tag{1.14}$$

By varying with respect to $\tilde{X}^\mu$ and assuming that the worldsheet has trivial topology, one obtains

$$dV^\mu = 0 \to V^\mu = dX^\mu, \tag{1.15}$$

giving back (1.12) once (1.15) is substituted into (1.14). From this standpoint, then, (1.14) is equivalent to (1.12) and the one-forms V^μ are interpreted as dX^μ, the differentials of the string coordinates. On the other hand, the variation with respect to V^μ provides

$$V^\mu = *d\tilde{X}^\mu, \tag{1.16}$$

yielding the dual action functional

$$S_T[\tilde{X}] = T/2 \int_\Sigma d\tilde{X}^\mu \wedge *d\tilde{X}_\mu. \tag{1.17}$$

Since V^μ is essentially dX^μ, we see that (1.13) is a subcase of (1.16) with $\mu = 25$. Hence, Buscher's rules naturally generalise T-duality to an arbitrary number of directions. Notice that the latter do not necessarily need to be compact if the theory is defined on a disk. In fact, Buscher rules allow for performing T-duality on *timelike* coordinates as well. Timelike T-Duality is special as it maps type IIA superstrings into type IIB* ones, as well as type IIB superstrings into type IIA* ones. These type II* theories are essentially Euclidean counterparts of ordinary type II superstrings, in the sense that they admit E-branes, which are D-branes with Dirichlet boundary conditions in time. Similar to instantons, E-branes are localised in time; furthermore, the gauge theory living on a stack of N E_4-branes[4] in type IIB* superstring theory is the Euclidean version of $\mathcal{N} = 4$ SYM and is dual to strings propagating in $H_5 \times S^5$, with H_5 being a 5-dimensional hyperboloid describing de Sitter space. In other words, timelike T-duality transforms Maldacena's AdS/CFT correspondence into a dS/ECFT, where ECFT stands for *Euclidean* conformal field theory [55].

The same algorithm straightforwardly applies to the action of strings moving in a curved spacetime with non-trivial B-field: if the original action is

$$S[X, G, B] = T/2 \int_\Sigma G_{\mu\nu}\, dX^\mu \wedge *dX^\nu + B_{\mu\nu}\, dX^\mu \wedge dX^\nu \tag{1.18}$$

and both $G_{\mu\nu}$ and $B_{\mu\nu}$ are independent of, say, X^m, one can T-dualise (1.18) along X^m. Buscher's procedure, therefore, requires one to replace every dX^m with a one-form V and to add a term containing a Lagrange multiplier $\tilde{X}^m$ ensuring the exactness of V:

$$S[X, X^m \mapsto V, \tilde{X}^m, G, B] = T/2 \int_\Sigma G_{\mu\nu}\, dX^\mu \wedge *dX^\nu + B_{\mu\nu}\, dX^\mu \wedge dX^\nu - 2\, \tilde{X}^m\, dV, \tag{1.19}$$

[4] E_p branes are the timelike T-duals of D_{p-1} branes. The worldvolume of a E_p-brane is then p-dimensional.

with $\mu, \nu = 1, \ldots, m, \ldots, 26$. As before, by integrating $\tilde{X}^m$ out, one recovers the original action (1.18), while the equations of motion for the auxiliary one-form V provide $\tilde{G}_{\mu\nu}$ and $\tilde{B}_{\mu\nu}$, the T-dual metric and B-field respectively:

$$\begin{aligned} \tilde{G}_{mm} &= 1/G_{mm}, \qquad \tilde{B}_{m\mu} = G_{m\mu}/G_{mm}, \qquad \tilde{\Phi} = \Phi - (1/2)\log G_{mm}, \\ \tilde{G}_{m\mu} &= B_{m\mu}/G_{mm}, \qquad \tilde{B}_{\mu\nu} = B_{\mu\nu} + \left(g_{m\mu}B_{m\nu} - B_{m\mu}G_{m\nu}\right)/G_{mm}, \\ \tilde{G}_{\mu\nu} &= G_{\mu\nu} + \left(B_{m\mu}B_{m\nu} - G_{m\mu}G_{m\nu}\right)/G_{mm}, \qquad \mu, \nu \neq m, \end{aligned} \tag{1.20}$$

where the dilaton shift is due to the measure factor coming from the integration over V.

Strings moving in flat space are trivially self-dual under T-duality: (1.12) and (1.17) are identical. AdS backgrounds are also T-self-dual; but the invariance is not at all trivial there. In fact, T-self-duality of superstrings on $\mathrm{AdS}_5 \times S^5$ provides an explanation to the dual conformal symmetry of $\mathcal{N} = 4$ SYM, as we will see in the next subsection.

1.2.2. Dual superconformal symmetry from T-duality

Scattering amplitudes in $\mathcal{N} = 4$ SYM display *dual conformal invariance* [56–60]. Indeed, such amplitudes are invariant under the action of two different sets of generators: one is given by Noether charges generating $\mathrm{SO}(2,4)$, the conformal group in four dimensions, which is expected as $\mathcal{N} = 4$ SYM is a conformal field theory. The other set is made of bi-local charges that do not close into a Lie algebra and transform in the adjoint of $\mathrm{SO}(2,4)$. If we denote by J_a the conformal generators and by $J_a^{(1)}$ the dual ones, their Poisson brackets read

$$\{J_a, J_b\} = f_{ab}{}^c\, J_c, \qquad \{J_a, J_b^{(1)}\} = f_{ab}{}^c\, J_c^{(1)}, \qquad \{J_a^{(1)}, J_b^{(1)}\} \notin \mathrm{span}(J_a^{(1)}), \tag{1.21}$$

where $f_{ab}{}^c$ are the structure constants of $\mathrm{SO}(2,4)$. It was found that this hidden symmetry enhances to a dual *superconformal* symmetry [61, 62], which is sensible as $\mathcal{N} = 4$ SYM is a superconformal field theory invariant under the full $\mathrm{PSU}(2,2|4)$ Lie supergroup.

Dual superconformal symmetry is also connected to the Amplitude/Wilson Loop duality, relating a planar scattering amplitude of n gluons to the expectation value of a polygonal lightlike Wilson loop with n edges [63–66]. Indeed, the dual generators behave as ordinary conformal generator when acting on Wilson loops. As we shall see, T-duality is the link between dual and original superconformal symmetry, as well as between amplitudes and Wilson loops [67, 68]. In fact, both are consequences of the invariance of the tree-level superstring theory on $\mathrm{AdS}_5 \times S^5$ under a suitable combination of bosonic and fermionic T-dualities. In a similar way to their bosonic counterparts, fermionic T-dualities are given by Buscher rules applied to Grassmann fields, yielding non-local redefinitions of the fermionic coordinates and changing both the dilaton and the RR fields without modifying the metric. T-duality maps the string sigma model into itself and, on the gauge side of AdS/CFT, scattering amplitudes into Wilson loops.

Let us consider AdS_5 spacetime and write its metric in the Poincare patch:

$$ds^2 = R^2\,\frac{\eta_{\mu\nu}\,dx^\mu\,dx^\nu + dy^2}{y^2}, \tag{1.22}$$

with R being the AdS radius. We want to use AdS/CFT to compute gluon scattering amplitudes at strong coupling[5]. To this end, we place a D3-brane extending along the x^μ coordinates at some fixed large value y_{IR} of y. The asymptotic states are open strings ending on this D-brane. We therefore consider the scattering of these open strings, which will embody the scattering of gluons. The brane at y_{IR} is an infrared regulator, which is needed as the scattering involves massless particles.

The momentum of the strings is $k_{\mathrm{string}} = k\,y_{\mathrm{IR}}/R$, with k being the momentum conjugate to x. The vector k_{string} plays the role of gauge theory momentum and will be kept fixed as we take away the IR cut-off, $y_{\mathrm{IR}} \to \infty$. Let us consider a worldsheet with the topology of a disk, with vertex operator insertions on its boundary, corresponding to the external states. A disk amplitude with a fixed ordering of the open string vertex operators corresponds to a given color ordered amplitude in $\mathcal{N} = 4$ SYM. The open strings are attached to the D-brane; hence, $y = y_{\mathrm{IR}}$ at the boundary.

If we wish to T-dualise along the four dimensional Minkowski boundary, we have to apply Buscher rules to x^μ: after the redefinition $\tilde{y} = R^2/y$, the T-dual metric takes the form

$$d\tilde{s}^2 = R^2\,\frac{\eta_{\mu\nu}\,d\tilde{x}^\mu\,d\tilde{x}^\nu + d\tilde{y}^2}{\tilde{y}^2} \tag{1.23}$$

This metric is equivalent to the original AdS_5 one, but now the worldsheet boundary is located at $\tilde{y} = R^2/y_{\mathrm{IR}}$, which is very small. In addition, T-duality interchanges Neumann with Dirichlet boundary conditions and the worldsheet boundary becomes a polygon with lightlike edges located at $\tilde{y} = R^2/y_{\mathrm{IR}}$. Notice that the polygon is closed due to momentum conservation.

When we remove the IR cut-off by taking $y_{\mathrm{IR}} \to \infty$, the boundary of the worldsheet moves towards the boundary of the T-dual metric, which is at $\tilde{y} = 0$. Computing the area of this dual worldsheet is equivalent to calculating the strong coupling expectation value of a Wilson loop whose contour is given by a sequence of light-like segments [70, 71], see Figure 1.2. In this way, AdS/CFT explains the duality between Wilson loops and scattering amplitudes observed in perturbative computations on the gauge side of the correspondence. This is an astounding breakthrough not only conceptually, but also technically as T-duality reduces the computation of scattering amplitudes at strong coupling to the problem of finding minimal surfaces in AdS spacetime, which is a geometrically clearer and much more manageable task.

In summary, AdS/CFT maps dual superconformal symmetry of $\mathcal{N} = 4$ SYM to T-self-duality of $\mathrm{AdS}_5 \times S^5$ superstrings. In this sense, amplitudes and Wilson loops, as well as dual

[5]Notice that amplitudes in a similar regime were studied by Gross and Mende in the case of flat spacetime [69].

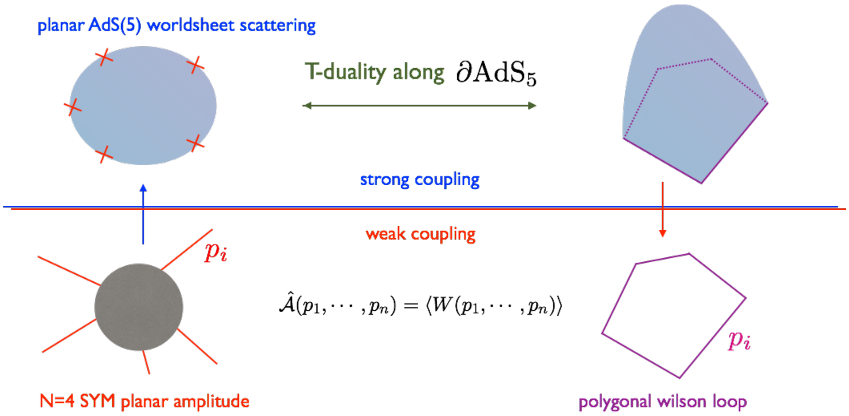

Figure 1.2: At strong coupling, the planar amplitude of five particles scattering among themselves in $\mathcal{N} = 4$ SYM theory is given by a superstring worldsheet with no handles and five punctures in $\mathrm{AdS}_5 \times S^5$. T-duality along the conformal boundary of AdS_5 transforms the worldsheet into the minimal surface providing the strong coupling behaviour of a polygonal Wilson Loops with five edges in $\mathcal{N} = 4$ SYM.

and original superconformal generators, become T-dual pairs.

Dualities are often hints of hidden unifying frameworks. In the case we are investigating, a symmetry structure underlying the duality indeed exists: (1.21) are the commutation relations of the *Yangian* $\mathcal{Y}[\mathfrak{psu}(2,2|4)]$.

A Yangian over a Lie superalgebra $\mathfrak{g}$ is normally denoted by $\mathcal{Y}[\mathfrak{g}]$ and coincides with a specific infinite dimensional extension of $\mathfrak{g}$ endowed with a Hopf algebra structure. Yangian invariance ensures an infinite tower of conserved charges in involution and naturally extends the notion of Liouville integrability to the case of an infinite number of degrees of freedom. Yangians appear in many different systems such as superstring theories realised by coset sigma models; integrable spin chains; Wilson loops, amplitudes and correlation functions in supersymmetric QFTs.

1.3. Yangians and Integrability

1.3.1. Drinfeld realisations for Yangians

If $\mathfrak{g}$ is a Lie superalgebra equipped with a non-degenerate invariant bilinear form κ_{ab}, one first considers the corresponding *loop algebra* $\mathfrak{g}[u]$, which is the algebra of all $\mathfrak{g}$-valued polynomials in the variable $u \in \mathbb{C}$. In turn, the Yangian $\mathcal{Y}[\mathfrak{g}]$ is constructed by deforming the universal enveloping algebra of $\mathfrak{g}[u]$, $\mathcal{U}[\mathfrak{g}[u]]$, and promoting it to a Hopf algebra.

$\mathcal{Y}[\mathfrak{g}]$ is spanned by an infinite tower of generators $J_a^{(m)}$ satisfying the graded commutatators

$$[J_a^{(m)}, J_b^{(n)}\} = \mathfrak{F}_{ab}{}^{c}\, J_c^{(m+n)}, \qquad m, n \in \mathbb{N};\ a, b, c = 1, \ldots, \dim(\mathfrak{g}), \tag{1.24}$$

as well as the Serre relations

$$[J_a^{(1)}, [J_b^{(1)}, J_c^{(0)}\}\} + (-)^{|a|(|b|+|c|)}[J_b^{(1)}, [J_c^{(1)}, J_a^{(0)}\}\}+ \\ + (-)^{|b|(|a|+|c|)}[J_c^{(1)}, [J_a^{(1)}, J_b^{(0)}\}\} = \hbar^2\, \mathfrak{F}_{ag}{}^{d}\, \mathfrak{F}_{bh}{}^{e}\, \mathfrak{F}_{ck}{}^{f}\, \mathfrak{F}^{ghk}\, J_{\{d}^{(0)} J_e^{(0)} J_{f\}}^{(0)}, \quad (1.25)$$

with $\hbar$ being a deformation parameter[6],

$$[X, Y\} := XY - (-)^{|X||Y|}\, YX \quad (1.26)$$

the graded Lie bracket, $|X|$ the grading of X and $\mathfrak{F}_{ab}{}^{c}$ the structure constants of $\mathfrak{g}$. The index m labels the *level* of the generator $J_a^{(m)}$: the zeroth level is the only one closing into a Lie superalgebra and coincides with $\mathfrak{g}$. Notice that, due to (1.24), the Serre relations actually constitute an infinite collection of constraints applying to all levels.

The Hopf algebra structure provides the Yangian with a few more maps:

- Product, $\mu : \mathcal{Y}[\mathfrak{g}] \otimes \mathcal{Y}[\mathfrak{g}] \to \mathcal{Y}[\mathfrak{g}]$; Unit, $\eta : \mathbb{C} \to \mathcal{Y}[\mathfrak{g}]$;
- Counit, $\epsilon : \mathcal{Y}[\mathfrak{g}] \to \mathbb{C}$; Coproduct, $\Delta : \mathcal{Y}[\mathfrak{g}] \to \mathcal{Y}[\mathfrak{g}] \otimes \mathcal{Y}[\mathfrak{g}]$;
- Antipode, $\Sigma : \mathcal{Y}[\mathfrak{g}] \to \mathcal{Y}[\mathfrak{g}]$.

The operators $\mu, \eta, \epsilon, \Delta, \Sigma$ satisfy a number of conditions[7] ensuring the consistency of the Hopf algebra structure, for instance

$$\mu \circ (\Sigma \otimes 1) \circ \Delta = \eta \circ \epsilon. \quad (1.27)$$

Coproduct and antipode have specific physical interpretations. The former naturally encodes how Yangian symmetry acts on two-particles states. As for scattering processes, Yangian symmetry is so constraining that it fixes the S-matrix of a given system up to a scalar pre-factor[8].

The coproduct of the Yangian generators has the form

$$\Delta(J_a^{(0)}) = J_a^{(0)} \otimes 1 + 1 \otimes J_a^{(0)}, \quad \Delta(J_a^{(1)}) = J_a^{(1)} \otimes 1 + 1 \otimes J_a^{(1)} + \hbar\, \mathfrak{F}^{bc}{}_{a}\, J_b^{(0)} \otimes J_c^{(0)}, \quad (1.28)$$

where the higher level coproducts can be obtained by commuting the generators among themselves and exploiting the fact that the coproduct is a linear homomorphism fulfilling

$$\Delta(XY) = \Delta(X)\, \Delta(Y). \quad (1.29)$$

On the other hand, the antipode maps particle states into antiparticle states and vice versa. As far as scattering processes are concerned, the antipode action yields the so-called *crossing*

[6] If $\hbar \to 0$, Serre relations reduce to Jacobi identities and $\mathcal{Y}[\mathfrak{g}]$ to the universal enveloping algebra $\mathcal{U}[\mathfrak{g}[u]]$.

[7] For the complete set of consistency relations, which is quite bulky, we refer the reader to [72, 73].

[8] This scalar pre-factor is the so-called *dressing phase* and carries information about the analyticity properties of the S-matrix.

equations, which determine the aforementioned scalar pre-factor. It is worth stressing that the antipode is a graded linear anti-homomorphism satisfying

$$\Sigma(XY) = (-)^{|X||Y|}\,\Sigma(Y)\,\Sigma(X). \tag{1.30}$$

The graded commutation relations (1.24), together with the Serre relations (1.25) and the underlying Hopf algebra structure, constitute *Drinfeld's First Realisation* of the Yangian $\mathcal{Y}[\mathfrak{g}]$. Other possible representations are *Drinfeld Second Realisation* [74] and Faddeev and Takhtajan's *RTT Realisation* [75, 76].

Drinfeld's second realisation is the Yangian counterpart of Chevalley's basis for finite dimensional Lie algebras; in fact, it is also referred to as the *Chevalley-Serre* realisation. A Yangian written in this fashion solves the recursion relation given in (1.24) and makes immediately accessible all Yangian levels. More concretely, let $\mathfrak{g}$ be a simple Lie algebra with Cartan matrix a_{ij}. Then, the Yangian $\mathcal{Y}[\mathfrak{g}]$ is spanned by the generators $x^{\pm}_{i,m}, h_{i,m}$, with $i = 1, \dots \mathrm{rank}(\mathfrak{g})$, level $m \in \mathbb{N}$ and commutation relations

$$\begin{aligned}
&[h_{i,m}, h_{j,n}] = 0, \qquad [h_{i,0}, x^{\pm}_{j,m}] = \pm\, a_{ij}\, x^{\pm}_{j,m}, \qquad [x^{+}_{i,m}, x^{-}_{j,n}] = \delta_{ij}\, h_{j,m+n},\\
&[h_{i,m+1}, x^{\pm}_{j,n}] - [h_{i,m}, x^{\pm}_{j,n+1}] = \pm\frac{1}{2}\, a_{ij}\, \{h_{i,m}, x^{\pm}_{j,n}\},\\
&[x^{\pm}_{i,m+1}, x^{\pm}_{j,n}] - [x^{\pm}_{i,m}, x^{\pm}_{j,n+1}] = \pm\frac{1}{2}\, a_{ij} \{x^{\pm}_{i,m}, x^{\pm}_{j,n}\},\\
&i \neq j, \quad m_{ij} = 1 + \sqrt{(a_{ij})^2} \quad \Rightarrow \quad \mathrm{Sym}_{\{k\}}[x^{\pm}_{i,k_1}, [x^{\pm}_{i,k_2}, \dots [x^{\pm}_{i,k_{m_{ij}}}, x^{\pm}_{j,l}] \dots]] = 0,\\
&[X, Y] := XY - YX, \qquad \{X, Y\} := XY + YX.
\end{aligned} \tag{1.31}$$

Setting $m = n = 0$ in the first line consistently gives back the Chevalley basis for $\mathfrak{g}$. Drinfeld's first and second realisations are isomorphic: the link between the two realisations of the Yangian level-1, for instance, is made of maps that are quadratic in the generators, see [74, 77, 78].

The RTT realisation, instead, provides the Yangian hidden in an integrable model directly from its scattering matrix. In fact, the RTT realisation of $\mathcal{Y}[g]$ requires the knowledge of the corresponding *quantum R-matrix*: algebraically, this object is an intertwiner between different tensor product representations, while physically it can be associated with the S-matrix of an integrable system. We introduce it in what follows.

1.3.2. R-matrix, Integrability and the RTT realisation

R-matrices and Quantum Yang-Baxter equation. A Yangian $\mathcal{Y}[\mathfrak{g}]$ is qualitatively different from its Lie superalgebra $\mathfrak{g}$. A crucial difference is a non-trivial tensor product structure, which naturally leads to the definition of a *graded permutation operator* acting as

$$P(X \otimes Y) := (-)^{|X|\,|Y|}\, Y \otimes X. \tag{1.32}$$

Especially, one can combine coproduct and permutation to obtain the operator

$$\Delta^{\mathrm{op}} := P \circ \Delta, \tag{1.33}$$

the *opposite coproduct*. In a scattering process, the coproduct Δ acts on *in* states, whereas the opposite coproduct Δ^{op} acts on *out* states. A priori, the coproduct and its opposite give rise to different representations. Nonetheless, it can happen that such representations are related by a similarity transformation encoded in an intertwining operator R:

$$\begin{aligned} &\Delta^{\mathrm{op}}(J_a^{(m)})\, R = R\, \Delta(J_a^{(m)}) \quad \forall J_a^{(m)}, \qquad \mathcal{Y}[\mathfrak{g}] = \mathrm{span}(J_a^{(m)}), \\ &R : \mathbb{C} \times \mathbb{C} \to \mathcal{Y}[\mathfrak{g}] \otimes \mathcal{Y}[\mathfrak{g}], \qquad R \text{ invertible.} \end{aligned} \tag{1.34}$$

The property reported in (1.34) is called *quasi co-commutativity*: full co-commutativity happens when $\Delta \equiv \Delta^{\mathrm{op}}$, which is true for central elements:

$$\Delta^{\mathrm{op}}(C)\, R = R\, \Delta(C) = \Delta(C)\, R \qquad \Rightarrow \qquad \Delta^{\mathrm{op}}(C) = \Delta(C), \quad C \text{ central in } \mathcal{Y}[\mathfrak{g}]. \tag{1.35}$$

Moreover, R can obey the *fusion relations*

$$(\Delta \otimes 1)(R) = R_{13}\, R_{23}, \qquad (1 \otimes \Delta)(R) = R_{13}\, R_{12}, \tag{1.36}$$

representing compatibility conditions between the R-matrix and the coproduct, where the subscripts i, j mean that R_{ij} acts only on the i-th and j-th components of the multiple tensor product of representations. The R-matrix is compatible with the rest of the Hopf algebra as well: for instance,

$$\Sigma(R) = R^{-1}, \qquad \Sigma(R^{-1}) = R, \qquad \epsilon(R) = 1, \tag{1.37}$$

where the relations involving the antipode Σ give rise to the crossing equations for the R-matrix.

If R respects both (1.34) and (1.36), it also satisfies the *Yang-Baxter equation* (YBE)

$$R_{12}\, R_{13}\, R_{23} = R_{23}\, R_{13}\, R_{12} \tag{1.38}$$

and the Yangian $\mathcal{Y}[\mathfrak{g}]$ is named *almost quasi-triangular*. The adverb "almost" signals that $\mathcal{Y}[\mathfrak{g}]$ might not admit a *universal* R-matrix, that is an R-matrix obeying the YBE (1.38) regardless of representation. Typically, Yangians do not possess a universal R-matrix, but only representation-dependent ones. In order for a Yangian $\mathcal{Y}[\mathfrak{g}]$ to become a quasi-triangular Hopf algebra, one would need to show the existence of the corresponding *Yangian Double* $\mathcal{DY}[\mathfrak{g}]$, the infinite dimensional completion of $\mathcal{Y}[\mathfrak{g}]$ with a dual Hopf algebra structure allowing for *negative* Yangian levels. Yangian double provided, one would then adapt Khoroshkin and Tolstoy's prescription [79] to finally write down the universal R-matrix of $\mathcal{DY}[\mathfrak{g}]$. These objects are very important as they compactly encode the most intimate features of the model to which they refer. Sadly, building them is extremely hard, especially for algebras relevant to physics.

YBE and factorised scattering. The YBE is a hallmark of 1+1 dimensional integrable systems. In such a setting, the R-matrix plays the role of the S-matrix[9] and fulfils (1.38), indicating that the scattering is factorised. This is a consequence of the infinite tower of conserved charges provided by the underlying Yangian, which *almost* trivialises the S-matrix[10]. We shall see in a moment how this works: for further details, we refer to the papers [80–82] and the reviews [83, 84].

Let us consider the wavefunction

$$\psi(x) = \int_{-\infty}^{\infty} dp\, e^{-a^2(p-p_0)^2} e^{ip(x-x_0)} \tag{1.39}$$

describing a particle with momentum p_0 localised at $x = x_0$. If we wanted to translate the position of this wave packet by a constant ξ, we should apply on $\psi(x)$ the unitary operator

$$\mathscr{T}_1(\xi) \triangleright \psi(x) := \int_{-\infty}^{\infty} dp\, e^{-a^2(p-p_0)^2} e^{ip(x-x_0)} e^{-ip\,\xi}. \tag{1.40}$$

Invariance of the system under $\mathscr{T}_1$ will correspond to conservation of the total momentum. In fact, one can define a more general translation operator $\mathscr{T}_m(\xi)$ acting as

$$\mathscr{T}_m(\xi) \triangleright \psi(x) = \int_{-\infty}^{\infty} dp\, e^{-a^2(p-p_0)^2} e^{ip(x-x_0)} e^{-ip^m\xi}. \tag{1.41}$$

By using the stationary phase method on the wave packet (1.41), one finds that $\mathscr{T}_m(\xi)$ shifts x_0 by the *momentum-dependent* quantity $(m\, p_0^{m-1}\xi)$. In a process with $a_1, a_2, \ldots$ particle scattering, $\mathscr{T}_m(\xi)$ will translate the position of a_1 by $(m\, p_{a_1}^{m-1}\xi)$, the position of a_2 by $(m\, p_{a_2}^{m-1}\xi)$, and so forth. Summarising: if $m = 1$, one has $\mathscr{T}_1(\xi)$, which is the ordinary translation operator shifting every particle by the same amount ξ. Instead, if $m > 1$, $\mathscr{T}_m(\xi)$ will translate particles with different momenta by different amounts.

Invariance under $\mathscr{T}_m(\xi)$ will be encoded in a higher conserved charge $F_m(p)$, a polynomial of degree m in p: for instance, $F_1(p)$ coincides with the total momentum of the system. Especially, considering a $2 \to n$ scattering process, $F_m(p)$ will commute with the S-matrix leaving unchanged the corresponding amplitude:

$$\begin{aligned}\langle b_1, \ldots, b_n| \, S \, |a_1, a_2\rangle &= \langle b_1, \ldots, b_n| \, e^{i\xi F_m(p)} S e^{-i\xi F_m(p)} \, |a_1, a_2\rangle \\ &= \langle b_1, \ldots, b_n| \, \mathscr{T}_m^\dagger(\xi) S \mathscr{T}_m(\xi) \, |a_1, a_2\rangle \,, \end{aligned} \tag{1.42}$$

where $\{a_1, a_2\}$ and $\{b_1, \ldots, b_n\}$ are the sets of incoming and outgoing particles respectively. Acting with $F_m(p)$ can be used to rearrange initial and final configurations. In particular, the action of $F_m(p)$ is so powerful that, if $\{a_1, a_2\}$ meet at the time $t = t_{\text{in}}$ and $\{b_1, \ldots, b_n\}$

[9]The R-matrix depends on two complex parameters, which physically are interpreted as the rapidities of the incoming particles.

[10]As we will see, Yangians do trivialise the S-matrix of theories living in more than two spacetime dimension.

leave at $t = t_{\text{out}}$, one can actually make $t_{\text{out}} < t_{\text{in}}$. Macroscopically, it is impossible that in a scattering process the outgoing particles are generated *before* the ingoing particles meet: the amplitude (1.42) must therefore vanish. However, (1.42) *might not* vanish if there is no particle production, namely $n = 2$, and initial and final momenta coincide: in such a case, in fact, the swap $t_{\text{in}} \leftrightarrow t_{\text{out}}$ would be a legitimate symmetry of the theory. Consequently, in a system where $F_m(p)$ is conserved for any $m \in \mathbb{N}$, only $n \to n$ processes are allowed.

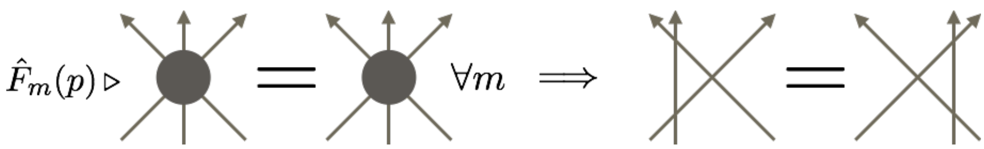

Figure 1.3: Imposing invariance under higher order symmetries provides scattering factorisation (time flows upwards).

Moreover, the fact that $\mathscr{T}_m(\xi)$ shifts particles with different momenta by different amounts implies that multiparticle interactions can always be "disentangled" : as a result, any $n \to n$ scattering is factorised into a sequence of $2 \to 2$ processes. The YBE encodes such a factorisation in the case of $n = 3$, see Figure 1.3 and 1.4.

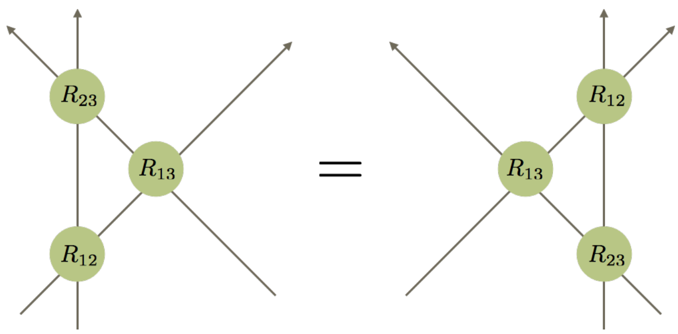

Figure 1.4: Graphical representation of the Yang-Baxter equation.

Scattering amplitudes invariant under higher symmetries, such as $\mathscr{T}_m(\xi)$ or Yangians, are non-trivial in 1+1 dimensions only. In more than one spatial dimension, higher conserved charges can be used to make trajectories miss each other completely, trivialising the corresponding S-matrix. Notice that this is consistent with the Coleman-Mandula theorem.

We saw that a Yangian naturally yields a (representation dependent) object called R-matrix that satisfies the YBE, implying that Yangian invariant systems are automatically integrable. In what follows, we will see that the opposite is also true; namely, that a R-matrix respecting YBE can generate the related Yangian. Such is the essence of the RTT realisation.

RTT Realisation. Given a matrix $R_{12}(u,v)$, which is a function of two spectral parameters and satisfies the YBE, one can extract the symmetries of the system using the RTT realisation of the corresponding Yangian [85, 86]:

$$R_{12}(u,v)\,\mathcal{T}_{13}(u)\mathcal{T}_{23}(v) = \mathcal{T}_{23}(v)\mathcal{T}_{13}(u)\,R_{12}(u,v)\ . \tag{1.43}$$

The matrix $\mathcal{T}(u)$ is called *monodromy matrix* and plays the role of a generating function for the Yangian charges, which in turn are symmetries of R. We shall now summarise the main features of this construction (see [86] for further details). This discussion is tailored to the $\mathfrak{gl}(n|n)$ case.

Let $\{e^A{}_B\}$ be the standard basis of the $\mathfrak{gl}(n|n)$ Lie superalgebra. That is the matrices $e^A{}_B$ are such that their only non-vanishing entry is $(-)^{[B]}$ in row A and column B. The symbol $[A]$ stands for the Graßmann grading of the index A. $\mathcal{T}(u)$ can then be written as follows:

$$\mathcal{T}(u) = \sum_{A,B}(-)^{[B]}\,e^B{}_A \otimes \mathbb{T}^A{}_B(u)\ , \qquad \mathbb{T}^A{}_B : \mathbb{C} \to \mathcal{Y}(\mathfrak{gl}(n|n))\ . \tag{1.44}$$

Assuming that $\mathbb{T}^A{}_B(u)$ is holomorphic in a neighbourhood of $u=\infty$, the asymptotic expansion

$$\mathbb{T}^A{}_B(u) = \sum_{l\in\mathbb{N}} u^{-l}\,\mathbb{T}^A_{l-1B}\ , \tag{1.45}$$

is well defined. At this point, one finds that particular combinations of the $\mathbb{T}^A_{l-1B}$ can be engineered to define Drinfeld's first realisation of $\mathcal{Y}(\mathfrak{gl}(n|n))$. In particular, defining

$$\mathbb{U}^{[B]}\delta^A_B := \mathbb{T}^A_{-1B}\ , \qquad \mathbb{J}^A_{0B} := \mathbb{U}^{-[B]}\,\mathbb{T}^A_{0B}\ , \qquad \mathbb{J}^A_{1B} := \mathbb{U}^{-[B]}\,\mathbb{T}^A_{1B} - \frac{1}{2}\mathbb{J}^A_{0C}\,\mathbb{J}^C_{0B}\ , \tag{1.46}$$

$\mathbb{J}_0$ and $\mathbb{J}_1$ span $\mathfrak{gl}(n|n)$ and the first level of $\mathcal{Y}(\mathfrak{gl}(n|n))$, respectively. The central element $\mathbb{U}$ is the *braiding factor*, and encodes a deformation of the co-algebra structure, with $\mathbb{U}=1$ representing the undeformed case.

The key observation is that the R-matrix is a particular representation of $\mathcal{T}$, namely, $R(u,v) = (1\otimes\pi_v)\,\mathcal{T}(u)$, where π_v indicates a representation depending on the spectral parameter v. Therefore, the coefficients in the Laurent expansion of $R(u,v)$ can be understood as $\mathbb{T}^A_{l-1B}$ for the RTT realisation of the underlying symmetry in the representation of interest. This can then be recast in the form of Drinfeld's first realisation using (1.46). The (representation-independent) graded commutation relations for the $\mathbb{J}^A_{mB}$ are obtained from those for the $\mathbb{T}^A_{l-1B}$, by substituting (1.44) and (1.45) into (1.43) and expanding with respect to both u and v.

Finally, the complete Hopf algebra structure of $\mathcal{Y}(\mathfrak{gl}(n|n))$, in particular, the coproducts and antipodes, can be recovered from the RTT realisation of the Yangian. First, the fusion relation

$$\Delta(\mathbb{T}^A{}_B(u)) = \mathbb{T}^A{}_C(u)\otimes\mathbb{T}^C{}_B(u)\ , \tag{1.47}$$

descending from the R-matrix fusion relations (1.36), provides the coproducts for the individual generators via the same expansion: indeed, expanding $\mathbb{T}^A{}_B(u)$ in inverse powers of u gives

$$\begin{aligned}&\Delta(\mathbb{U}) = \mathbb{U}\otimes\mathbb{U}\ , \qquad \Delta(\mathbb{T}^A_{0B}) = \mathbb{T}^A_{0B}\otimes\mathbb{U}^{[B]} + \mathbb{U}^{[A]}\otimes\mathbb{T}^A_{0B}\ ,\\ &\Delta(\mathbb{T}^A_{1B}) = \mathbb{T}^A_{1B}\otimes\mathbb{U}^{[B]} + \mathbb{U}^{[A]}\otimes\mathbb{T}^A_{1B} + \mathbb{T}^A_{0C}\otimes\mathbb{T}^C_{0B}\ .\end{aligned} \tag{1.48}$$

This can be used to derive the coproducts for $\mathbb{J}_{0,1}$:

$$\begin{aligned}&\Delta(\mathbb{U}) = \mathbb{U}\otimes\mathbb{U}\ , \qquad \Delta(\mathbb{J}^A_{0B}) = \mathbb{J}^A_{0B}\otimes 1 + \mathbb{U}^{[A]-[B]}\otimes\mathbb{J}^A_{0B}\\ &\Delta(\mathbb{J}^A_{1B}) = \mathbb{J}^A_{1B}\otimes 1 + \mathbb{U}^{[A]-[B]}\otimes\mathbb{J}^A_{1B}\\ &\qquad + \frac{1}{2}\mathbb{U}^{[C]-[B]}\mathbb{J}^A_{0C}\otimes\mathbb{J}^C_{0B} - \frac{1}{2}(-)^{([A]+[C])([B]+[C])}\mathbb{U}^{[A]-[C]}\mathbb{J}^C_{0B}\otimes\mathbb{J}^A_{0C}\ .\end{aligned} \tag{1.49}$$

Second, the antipode Σ from RTT correctly fulfils

$$\Sigma\left[\mathbb{T}^A{}_C(u)\right]\mathbb{T}^C{}_B(u) = \mathbb{T}^A{}_C(u)\Sigma\left[\mathbb{T}^C{}_B(u)\right] = \delta^A_B\ , \qquad \Sigma\left[XY\right] = (-)^{[X][Y]}\Sigma\left[Y\right]\Sigma\left[X\right]\ . \tag{1.50}$$

By expanding:

$$\begin{aligned}&\Sigma\left[\mathbb{T}^A_{-1B}\right] = \mathbb{U}^{-[B]}\delta^A{}_B\ , \qquad \Sigma\left[\mathbb{T}^A_{0B}\right] = -\mathbb{U}^{-[A]-[B]}\mathbb{T}^A_{0B}\ ,\\ &\Sigma\left[\mathbb{T}^A_{1B}\right] = -\mathbb{U}^{-[A]-[B]}\mathbb{T}^A_{1B} + \mathbb{U}^{-[A]-[B]-[C]}\mathbb{T}^A_{0C}\mathbb{T}^C_{0B}\ ,\end{aligned} \tag{1.51}$$

which in turn can be used to determine the antipodes for $\mathbb{J}_{0,1}$

$$\begin{aligned}&\Sigma\left[\mathbb{U}\right] = \mathbb{U}^{-1}\ , \qquad \Sigma\left[\mathbb{J}^A_{0B}\right] = -\mathbb{U}^{[B]-[A]}\mathbb{J}^A_{0B}\ ,\\ &\Sigma\left[\mathbb{J}^A_{1B}\right] = -\mathbb{U}^{[B]-[A]}\mathbb{J}^A_{1B} + \frac{1}{2}\mathbb{U}^{[B]-[A]}[\mathbb{J}^A_{0C},\mathbb{J}^C_{0B}\}\ .\end{aligned} \tag{1.52}$$

As a consequence, the whole Hopf algebra structure can be reconstructed in this way.

1.3.3. Classical r-matrices and secret symmetries

Classical r-matrices An essential ingredient of a Yangian $\mathcal{Y}[\mathfrak{g}]$ is the deformation parameter h. Indeed, this constant appears on the right hand side of Serre relations and provides the higher levels coproducts with non-trivial tails, to make two examples. It is then intriguing to see what happens whenever such a parameter is modified. In particular, one can investigate the limit $h\to 0$: this is interesting algebraically, as it should *smoothly* restore the universal enveloping algebra $\mathcal{U}[\mathfrak{g}[u]]$; and physically, as h is present in the scattering matrix and, therefore, is related to the coupling constant of the corresponding integrable system.

The h-expansion of the R-matrix satisfying the YBE reads as follows:

$$R = \mathbb{1}\otimes\mathbb{1} +\ h\,r\ + \mathcal{O}(\,h^2). \tag{1.53}$$

Thus, at first order in h, R reduces to the sum of the identity with a perturbation encoded by a matrix r. The latter is called *classical* r-matrix. It is not difficult to prove that, if R satisfies the YBE, then r fulfils the so-called *classical* YBE (CYBE):

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0, \tag{1.54}$$

where r_{ij} acts on the i-th and j-th copies of the multiple tensor product of representations. The classical r-matrix determines Poisson brackets in the inverse scattering method and plays a major role in Poisson-Lie group theory, see *e.g.* [87,88]. Especially, one can restrict to solutions $r(u_1, u_2)$ to (1.54) that are non-degenerate ($\det(r) \neq 0$) and of difference form, meaning that they depend on the spectral parameters (u_1, u_2) only via $u = u_1 - u_2$. Under this assumptions, Belavin and Drinfeld's theorems[11] strictly constrain the properties of the underlying symmetry algebra, which can be a *quantum group* of elliptic , trigonometric , or rational type[12].

Given a classical r-matrix, there exists a standard method to obtain the related *quantum* R-matrix; namely, to go from classical to ordinary YBE. Such a *quantum deformation* involves the so-called *Manin triples*, see for instance [91] and references therein. In the context of integrable models, this quantum deformation procedure remarkably maps Poisson into Dirac brackets, which is precisely what in physics is meant by "quantisation" [88]. It is definitely worth to see in a little more details how classical r-matrices lead to their quantum counterparts. To this aim, let us take Yang's r-matrix for a Lie algebra $\mathfrak{g}$ [92],

$$r = \frac{C_2}{u_2 - u_1}, \tag{1.55}$$

with $\mathfrak{g} = \mathrm{span}\,(J_A)$, $A = 1, \dots, \dim \mathfrak{g}$, and $C_2 := \sum_A J_A \otimes J_A$ being the quadratic Casimir of $\mathfrak{g}$. By utilising the geometric series, we rephrase r as

$$r = \frac{C_2}{u_2 - u_1} = \frac{\sum_A J_A \otimes J_A}{u_2 - u_1} = \sum_A \sum_{n\geq 0} J_A u_1^n \otimes J_A u_2^{-n-1} = \sum_A \sum_{n\geq 0} J_{A,n} \otimes J_{A,-n-1}\ , \tag{1.56}$$

where, for concreteness, we assumed $|u_1/u_2| < 1$. The new generators $J_{A,n} = u^n J_A$ in (1.56) satisfy

$$[J_{A,m}, J_{B,n}] = f_{AB}^{C}\, J_{C,m+n}, \tag{1.57}$$

reproducing Yangian's commutation relations. However, by computing Serre's relations for $\{J_m^A\}$ one would find a trivial right hand side. Consequently, $\{J_m^A\}$ do not generate the Yangian $\mathcal{Y}[\mathfrak{g}]$; but, rather, $\mathcal{U}[\mathfrak{g}[u]]$. This is actually correct because, from a quantum group perspective, we are

[11] Belavin-Drinfeld theorems involve Lie algebras. The case of graded Lie algebras was studied *e.g.* in [89,90]

[12] Quantum groups generalise ordinary Lie groups by allowing for non-linear commutation relations. If $\mathfrak{g}$ is a Lie algebra, the corresponding quantum group is denoted by $\mathfrak{g}_q$. Yangians are particular rational quantum groups.

considering $O(h)$, whereas Serre relations are $O(h^2)$. Nonetheless, it is remarkable that, from (1.57), one has that

$$r = \sum_A \sum_{n\geq 0} J_n^A \otimes J_{A,-n-1} \tag{1.58}$$

fulfils the CYBE with no reference to any specific representations. The r-matrix (1.58) is therefore the very *universal r-matrix* of $\mathcal{U}[\mathfrak{g}[u]]$. In fact, notice that the sum into (1.58) also involves generators with negative level: as a result, (1.57) can be regarded as commutation relations for the classical limit of the Yangian double $\mathcal{DY}[\mathfrak{g}]$.

A classical r-matrix for AdS_5/ CFT_4. As for the $\mathrm{AdS}_5/\mathrm{CFT}_4$ S-matrix [93], the deformation parameter $\hbar$ ruling the classical limit is the inverse of the effective string tension $g = \lambda/4\pi$:

$$R = \mathbb{1} \otimes \mathbb{1} + \frac{1}{g} r + O\left(\frac{1}{g^2}\right). \tag{1.59}$$

Such a classical r-matrix shall therefore provide the string scattering matrix at tree-level [94]. The matrix r in (1.59) is not of difference form; indeed, the light-cone symmetry algebra $\mathfrak{psu}(2|2)_c$ is non-simple, violating a crucial assumption of the Belavin-Drinfeld theorems discussed above. Nevertheless, the classical r-matrix we are examining presents a simple pole whose residue is the quadratic Casimir of $\mathfrak{gl}(2|2)$: somehow, the additional generator

$$\mathbb{B} = \mathrm{diag}\,(1, 1, -1, -1) \tag{1.60}$$

was borrowed from the $\mathfrak{gl}(2|2)$. The element $\mathbb{B}$, endowed with the trivial coproduct $\Delta(\mathbb{B}) = \mathbb{B} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{B}$, is a symmetry of r, but not of the original $\mathfrak{psu}(2|2)_c$-invariant S-matrix. Furthermore, the classical r-matrix is invariant under $\mathfrak{gl}(2|2)[u]$, whereas one might have expected the smaller $\mathfrak{psu}_c(2|2)[u]$ to be the correct symmetry. The reason for this enhancement is that the full R-matrix takes into account all orders in $\hbar$, while $O(\hbar^2)$ effects are neglected by the classical r-matrix. However, the extra generator $\mathbb{B}$ confirmed the presence of novel, hidden symmetries of the full quantum R-matrix [95] and allowed to write down the universal classical r-matrix of $\mathrm{AdS}_5/\mathrm{CFT}_4$ [96, 97].

Secret symmetries of the AdS_5/ CFT_4 S-matrix. Is there any quantum partner of the element $\mathbb{B}$ discussed previously. In fact, it was proven that the R-matrix of $\mathrm{AdS}_5 \times \mathrm{CFT}_4$ is exactly invariant under the so-called *secret symmetry* [95, 97]

$$\begin{aligned}
\Delta(\hat{\mathbb{B}}) &= \hat{\mathbb{B}} \otimes \mathbb{1} + \mathbb{1} \otimes \hat{\mathbb{B}} + \frac{i}{2g}(\mathbb{S}_a^\alpha \otimes \mathbb{Q}_\alpha^a + \mathbb{Q}_\alpha^a \otimes \mathbb{S}_a^\alpha),\\
\Sigma(\hat{\mathbb{B}}) &= -\hat{\mathbb{B}} + \frac{2i}{g}\mathbb{H},\\
\hat{\mathbb{B}} &= u\,\mathbb{B},
\end{aligned} \tag{1.61}$$

where u is the spectral parameter giving the evaluation representation of the Yangian $\mathcal{Y}[\mathfrak{psu}_c(2|2)]$. The secret symmetry $\hat{\mathbb{B}}$ is indeed the level-1 counterpart of the $\mathbb{B}$ found in the case of the classical r-matrix. Commuting the secret symmetry with the generators of the $\mathfrak{psu}_c(2|2)$ provides the corresponding higher level Yangian charges, under which the S-matrix is invariant . Iterating the procedure, $\hat{\mathbb{B}}$ generates the whole Yangian $\mathcal{Y}[\mathfrak{psu}_c(2|2)]$. Eventually, one discovers that all $\mathbb{B}^{(n)} := u^n\, \mathbb{B}$ are symmetries of the S-matrix as long as $n > 0$; as a result, the Yangian underlying the AdS_5/CFT_4 correspondence is the so-called *indented* $\mathcal{Y}[\mathfrak{gl}(2|2)]$, consisting in $\mathcal{Y}[\mathfrak{gl}(2|2)] \setminus \mathbb{B}$ [95]. The existence of a level-zero secret symmetry is forbidden by the processes that do not preserve the fermionic number. Drinfeld's second realization for the $\mathfrak{psl}(2|2)_c$ Yangian has been obtained in [98], via evaluation representation.

1.4. Yangians and Integrability in AdS/CFT

Superstring theories based on coset sigma models are Yangian invariant and, as such, they are integrable [99]. To be precise, these systems are *classically* integrable as Yangian symmetry is realised via Poisson brackets. One would like to have *quantum* integrability, as the latter would allow for computing observables exactly and at finite coupling. Contemporary techniques are insufficient to quantise superstrings in curved spacetime and, unfortunately, actual proofs of quantum integrability for these models are still missing.

1.4.1. A spin chain/CFT correspondence

In fact, quantum integrability and AdS/CFT first met on the gauge side of the correspondence, when Minahan and Zarembo realised that the one-loop dilatation operator in the SO(6) sector of $\mathcal{N} = 4$ SYM[13] precisely coincides with the Hamiltonian of an integrable spin chain [11], see [105] for a review.

To see what happens, let us consider $Z = \phi^1 + i\phi^2$, with $\phi^I, i = 1, \dots 6$ being the six real scalars of $\mathcal{N} = 4$ SYM. The field Z has spin zero, bare dimension $\Delta_0 = 1$ and R-charge $J = 1$ under one of the three Cartan generators of SO(6); hence, $\Delta_0 = J$. In turn, one can build up the gauge invariant single trace operator $\mathcal{O}_L := \mathrm{Tr}\left(Z^L\right)$, where $L \geq 2$: this will again satisfy the condition $\Delta_0 = J$, meaning that $\mathcal{O}_L$ is a chiral primary operator. Chiral primaries are BPS operators; that is, they commute with half of the supercharges of the corresponding superconformal algebra (PSU(2,2|4) in this case). The BPS condition is a strong constraint; indeed, it implies that supersymmetry protects the relation $\Delta_0 = J$ from quantum corrections. The *anomalous dimension* γ, providing the loop corrections to the bare dimension Δ_0, will then

[13] Integrable structures generalising Heisenberg's XXX spin chain were already noticed in non-supersymmetric Yang-Mills theory and QCD [100–104].

be exactly zero. On the other hand, in the spin chain picture, γ is given by the Hamiltonian

$$\mathrm{H}_{\mathrm{SC}} = \frac{\lambda}{8\pi^2} \sum_{j=1}^{L} \left(1 - P_{j,j+1} + \frac{1}{2}\, K_{j,j+1} \right) , \tag{1.62}$$

proper of an integrable SO(6) spin chain. λ is the 't Hooft coupling of $\mathcal{N} = 4$ SYM , while $P_{j,j+1}$ is the permutation operator swapping the j-th spin of the chain with the $(j+1)$-th one and $K_{j,j+1}$ is the trace operator, contracting the flavour indices of the j-th and the $(j+1)$-th spins. Notice that the cyclicity of the trace into operators such as $\mathcal{O} = \mathrm{tr}\,(X\,Y\,Z\,\ldots)$ imposes *periodic* boundary conditions upon the spin chain, hence the identification $(L+1) \equiv 1$. As for $\mathcal{O}_L$, this is represented by a quantum state $|Z \ldots Z\rangle$ and fulfils $\mathrm{H}_{\mathrm{SC}}\,|Z \ldots Z\rangle = 0$, meaning that $|Z \ldots Z\rangle$ do not evolve in time and corresponds to the spin chain ground state. If we restrict to a SU(2) subsector of SO(6), we find an L-sites XXX Heisenberg spin chain with Hamiltonian

$$\mathrm{H}_{\mathrm{XXX}} = \frac{\lambda}{4\pi^2} \sum_{i=1}^{L} \left(\frac{1}{4} - \vec{S}_i \cdot \vec{S}_{i+1} \right) . \tag{1.63}$$

Here, the Hilbert space is spanned by tensor products of $|Z\rangle = |\!\uparrow\rangle$ and $|X\rangle = |\!\downarrow\rangle$. The operator $\vec{S}_i$ provides the spin of the i-th site according to

$$\begin{aligned} &S^3\,|Z\rangle = S^3\,|\!\uparrow\rangle = (+1/2)\,|\!\uparrow\rangle\,, \quad S^3\,|X\rangle = S^3\,|\!\downarrow\rangle = (+1/2)\,|\!\downarrow\rangle\,, \\ &S^3_1\,|\!\uparrow\downarrow\uparrow\,\ldots\rangle = (+1/2)\,|\!\uparrow\downarrow\uparrow\,\ldots\rangle\,, \quad S^3_2\,|\!\uparrow\downarrow\uparrow\,\ldots\rangle = (-1/2)\,|\!\uparrow\downarrow\uparrow\,\ldots\rangle \end{aligned} \tag{1.64}$$

and so forth. It is plain to see that (1.63) annihilates $|Z\,Z\,\ldots\,Z\rangle = |\!\uparrow\uparrow\,\ldots\,\uparrow\rangle$, so that the latter correctly reproduces the spin chain vacuum state. Excited states correspond to non-BPS operator, *e.g.* $\mathcal{O} := \mathrm{tr}\,(Z\,Z\,X\,Z\,X\,Z\,\ldots\,Z)$. Since $\Delta_0 = L+2$, $\mathcal{O}$ is not protected by supersymmetry and its anomalous dimension will receive quantum corrections. In the spin chain picture, $\mathcal{O}$ is given by an L-sites spin chain with two impurities, also called *magnons.* A priori, these impurities can either propagate along the spin chain or scatter with each other. The system being integrable, the magnon scattering matrix turns out to satisfy the YBE, allowing to solve for the anomalous dimensions of single trace operators in the CFT by solving the Bethe equations [106] of the dual spin chain.

This analysis was finally extended to the other sectors of the gauge theory, to obtain the complete one-loop dilatation operator for $\mathcal{N} = 4$ SYM. Higher loops in the 't Hooft coupling constant were also studied for some sectors of the CFT [107–110]. Similar investigations were performed for ABJM theory [12].

1.4.2. Spin chain and AdS superstrings

The striking success of integrability in $\mathcal{N} = 4$ SYM triggered analogous studies on the gravity side of AdS/CFT. Indeed, Maldacena's conjecture would imply that type IIB $\mathrm{AdS}_5 \times S^5$ superstrings must also display quantum integrability. As already mentioned, the explicit quantisation of the full Green-Schwarz superstring action [111, 112] is a chimera due to its highly non-linearity [8, 113, 114]. Nevertheless, clues of integrability for quantum AdS superstrings were eventually unveiled [115–118]: by looking at specific string solutions, the worldsheet modes were found to exhibit solitonic behaviour; furthermore, their energy was shown to present a dispersion relation of the type $E^2 = m^2 + 4h^2 \sin^2(p/2)$, the same associated with the magnons of an integrable spin chain [119, 120]. The ultimate piece of evidence was given by the derivation of the worldsheet S-matrix for $\mathrm{AdS}_5 \times S^5$ superstrings. In fact, by writing the superstring action in light-cone gauge and decompactifying the worldsheet, one obtains a 1+1 dimensional theory whose scattering matrix is invariant under the centrally extended $\mathfrak{psu}_c(2|2)^2$ algebra [121], an S-matrix that perfectly matches the one descending from $\mathcal{N} = 4$ SYM [93, 122–124]. Let us outline how this works by looking at the sigma model for $\mathrm{AdS}_5 \times S^5$ bosonic strings,

$$S = -\frac{g}{2}\int_{-r/2}^{r/2} d\sigma\, d\tau\, \gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu\, G_{\mu\nu}(X)\,, \tag{1.65}$$

where $g = \sqrt{\lambda}/4\pi = R^2/2\pi\alpha'$ is the effective dimensionless string tension proportional to the 't Hooft coupling λ (and connected to the AdS radius R, as well as to α'). Moreover, $X^\mu = \left(t, \phi, x^i\right)$ the string coordinates, $\gamma^{\alpha\beta} = \sqrt{|h|}h^{\alpha\beta}$ the conformally invariant combination of the worldsheet metric $h^{\alpha\beta}$ and $G_{\mu\nu}(X)$ the target space metric. A crucial assumption is that the latter is independent of t, ϕ: this is the case for $\mathrm{AdS}_5 \times S^5$ superstrings, where t is the global time-coordinate in AdS_5 and ϕ an angle that parametrise the equator of S^5 . The momenta conjugate to $\dot{X}^\mu$ read

$$p_\mu = \frac{\delta S}{\delta \dot{X}^\mu} = -g\, \gamma^{0\beta}\partial_\beta X^\nu G_{\mu\nu}(X)\,, \tag{1.66}$$

with $\dot{X}^\mu = \partial_0 X^\mu$ being the derivative of the string coordinates with respect to τ. The string action can therefore be rephrased in a first-order fashion [113, 125]

$$S = \int_{-r/2}^{r/2} d^2\sigma \left(p_\mu \dot{X}^\mu + \frac{\gamma^{01}}{\gamma^{00}}\mathcal{C}_1 + \frac{1}{2g\,\gamma^{00}}\mathcal{C}_2\right)\,, \tag{1.67}$$

where

$$\mathcal{C}_1 = p_\mu X'^\mu\,, \qquad \mathcal{C}_2 = G^{\mu\nu}p_\mu p_\nu + g^2\, G_{\mu\nu}X'^\mu X'^\nu\,, \tag{1.68}$$

where $X'^\mu = \partial_1 X^\mu$ is the derivative of the string coordinates with respect to σ. The action (1.67) is no longer manifestly covariant on the worldsheet. Furthermore, $\gamma^{\alpha\beta}$ plays the role of a Langrange multiplier imposing the Virasoro constraints

$$\mathcal{C}_1 = 0\,, \qquad \mathcal{C}_2 = 0\,. \tag{1.69}$$

We are interested in $\mathrm{AdS}_5 \times S^5$, whose sigma model is invariant under shifts with respect to the coordinates t, ϕ. As a result, Noether theorem provides the corresponding conserved charges

$$\mathrm{H}_{\mathrm{TS}} = -\int_{-r/2}^{r/2} d\sigma\, p_t\,, \qquad \mathrm{J} = \int_{-r/2}^{r/2} d\sigma\, p_\phi\,, \tag{1.70}$$

where p_t, p_ϕ are the momenta conjugate to t, ϕ. The former Noether charge, H_{TS}, is the target space hamiltonian; whereas J is the angular momentum along S^5. Light-cone coordinates are obtained out of t, ϕ via the linear maps[14] [127]

$$x_- = \phi - t\,, \qquad x_+ = \frac{1}{2}(\phi + t)\,, \tag{1.71}$$

and the gauge-fixing condition is given by[15]

$$x_+ = \tau\,, \qquad p_+ = 1\,. \tag{1.72}$$

These choice of coordinates implies the light-cone momenta

$$P_- = \int_{-r/2}^{r/2} d\sigma\, p_- = J - E, \qquad P_+ = \int_{-r/2}^{r/2} d\sigma\, p_+ = \frac{1}{2}(J + E)\,, \tag{1.73}$$

where E and J are the eigenvalues of H_{TS} and J respectively; namely, the target space energy and angular momentum. By solving the constraints (1.69) one gets $P_+ \propto r$, as well as the relation between the target space and the worldsheet hamiltonian:

$$\mathrm{H}_{\mathrm{WS}} = \mathrm{H}_{\mathrm{TS}} - \mathrm{J}. \tag{1.74}$$

Notice that, in terms of the superconformal algebra, (1.74) reads

$$\mathrm{H}_{\mathrm{WS}} = D - J, \tag{1.75}$$

where J is one of the Cartan of SO(6), the latter being the isometry group of S^5. At this point, two comments are in order. First: the worldsheet energy has got the same expression as the anomalous dimension's, $E_{\mathrm{WS}} = \Delta_0 - J \equiv \gamma$. This is because AdS/CFT maps CFT operators onto string configurations: for instance, the BPS operator $\mathcal{O}_L = \mathrm{tr}\,(Z\,Z\,\ldots Z)$ considered above corresponds to the so-called BMN solution, a point-like string rotating along the equator of S^5 [128, 129]. Second: the symmetries of the worldsheet theory shall be given by a subalgebra of the superconformal one, $\mathfrak{g} \subset \mathrm{PSU}(2,2|4)$. It turns out that the light-cone gauge breaks $\mathrm{PSU}(2,2|4)$ down to $\mathfrak{g} = \mathfrak{psu}(2|2)^2$, which becomes the centrally extended $\mathfrak{g} = \mathfrak{psu}_c(2|2)^2$ once the decompactification limit $P_+ \to \infty$ is taken in order to have proper asymptotic states[16].

[14]See [114, 126] for a more general change of coordinates.

[15]In presence of winding modes, a term linear in σ is to be inserted into the expression for x_+.

[16]$\mathfrak{psu}_c(2|2)^2$ is the *off-shell* symmetry algebra of the system as the central charges vanish on-shell. Indeed, the central charges are given by $C \propto (1 - \exp(ip_{\mathrm{WS}}))$, where p_{WS} is the worldsheet momentum that is zero for physical states (the worldsheet theory is invariant under σ-translations).

Remarkably, imposing that the worldsheet scattering is integrable, as suggested by perturbative string theory computations [124], provides an S-matrix satisfying the YBE (and enjoying $\mathcal{Y}[\mathfrak{psu}_c(2|2)^2]$ symmetry [86]).

Integrability stands also when compact worldsheets are taken into account. This case was studied via Lüscher's procedure [130,131], which lead to the exact spectrum for the finite lenght $\mathrm{AdS}_5 \times S^5/\mathcal{N} = 4$ SYM spin chain via the Thermodynamic Bethe Ansatz[17] (TBA) [134–140].

1.5. Outline of the Thesis

As we saw, the AdS/CFT correspondence for AdS_3 and AdS_2 superstrings is still quite obscure. This thesis is devoted to filling the such a gap by investigating in depth dualities and integrability features of such models. Understanding AdS/CFT for strings propagating in low dimensional Anti-de Sitter space is a fundamental question as it will probe the validity of the AdS/CFT correspondence, showing how much the duality depends on either the dimensionality of the spacetime or the amount of supersymmetry. Furthermore, dualities between 3 gravity and two-dimensional CFTs are particularly remarkable as they identify the central charge of the latter with the Newton constant of the former [141], making G_N constrained by Zamolodchikov's c-theorem [142]. Analogously, supersymmetric quantum mechanics/one-dimensional QFTs are a fascinating field of investigation: these describe the dynamics of certain classes of black holes [51] and provide the quantisation of non-lagrangian QFTs [143], to make two examples.

The thesis is organised as follows:

In Chapter 2 we prove the T-self-duality of Green-Schwarz supercoset sigma models on $\mathrm{AdS}_d \times S^d \times S^d$, described by supercoset sigma models with isometries governed by $\mathrm{D}(2,1;\alpha)^{d-1}$ exceptional Lie supergroups, with $d = 2, 3$. As we will see, whenever $\alpha \neq 0, 1$, the corresponding isometry supergroups are of type OSp, which will imply that such models require additional T-dualities along one of the spheres for the superstring action to transform into themselves.

In Chapter 3 we derive the exact S-matrix for the integrable system associated with light-cone scattering of particular representations of the centrally-extended $\mathfrak{psu}(1|1)^2$ Lie superalgebra, conjectured to be related to the massive modes of the light-cone gauge string theory on $AdS_2 \times S^2 \times T^6$. The S-matrix consists of two copies of a centrally-extended $\mathfrak{psu}(1|1)$ invariant S-matrix and is in agreement with the tree-level result following from perturbation theory. Although the overall factor is left unfixed, the constraints following from crossing symmetry and unitarity are given. The scattering involves long representations of the symmetry algebra, and the relevant representation theory is studied in detail.

[17] A recent alternative to the TBA is the so-called *quantum spectral curve*: for further details please see [132,133].

We also discuss Yangian symmetry in the massless limit. Under the assumption that the massless modes of the light-cone gauge string theory transform in these limiting representations, the resulting S-matrices provide the building blocks of the full S-matrix. Finally, some brief comments are given on the Bethe ansatz.

In Chapter 4 we find the Yangian algebra responsible for the integrability of the $AdS_2 \times S^2 \times T^6$ superstring in the planar limit. We demonstrate the symmetry of the corresponding exact S-matrix in the massive sector, including the presence of the *secret* symmetry. We give two alternative presentations of the Hopf algebra. The first takes the usual canonical form, which, as the relevant representations are long, leads to a Yangian representation that is not of evaluation type.

After investigating the relationship between co-commutativity, evaluation representations and the shortening condition, we find an alternative realisation of the Yangian whose representation is of evaluation type. Finally we explore two limits of the S-matrix. The first is the classical r-matrix, where we re-discover the need for a *secret* symmetry also in this context. The second is the simplifying zero-coupling limit. In this limit, taking the S-matrix as a generating R-matrix for the Algebraic Bethe Ansatz, we obtain an effective model of free fermions on a periodic spin-chain. This limit should provide hints at the *one-loop* anomalous dimension of the mysterious superconformal quantum mechanics dual to the superstring theory in this geometry.

Finally, in Chapter 5 we investigate Yangian and secret symmetries for type IIB superstrings on $\mathrm{AdS}_3 \times S^3 \times T^4$, verifying the persistence of such symmetries to this instance of the AdS/CFT correspondence. Specifically, we find two a priori different classes of secret symmetry generators. One class of generators is more naturally embedded in the algebra governing the integrable scattering problem. The other class of generators is more elusive, and somewhat closer in its form to its higher-dimensional AdS_5 counterpart. All of these symmetries respect left-right crossing.

In addition, by considering the interplay between left and right representations, we gain a new perspective on the AdS_5 case. We also study the RTT realisation of the Yangian related to AdS_3 superstrings.

Concluding remarks are presented in Chapter 6.

Part I

T-Self-Duality for AdS_2 and AdS_3 Superstrings

2

T-Self Duality of AdS_2 and AdS_3 Superstrings

As we saw, the AdS/CFT correspondence connects a string theory on a d-dimensional anti-de Sitter space with a conformal field theory on its $(d-1)$-dimensional conformal boundary, $\partial \mathrm{AdS}_d$. This correspondence is a particular instance of the more general concept of *holography*, linking d-dimensional gravity theories with $(d-1)$-dimensional gauge field theories [144]. In the case of AdS/CFT, a complete dictionary between $\mathrm{AdS}_5 \times S^5$ superstrings with N units of RR 5-flux through S^5 and $\mathcal{N} = 4$ SYM with gauge group $\mathrm{SU}(N)$ was established in the large-N limit. Powerful tools introduced by holographic duality have found intriguing applications not only in string theory but also in nuclear physics, condensed matter systems and several other fields.

Holographic dualities normally relate a perturbative regime of a theory on one side of the correspondence to a strongly coupled regime of a theory on the other side and vice versa. This, in turn, allows one to extract information about the behaviour of theories at strong coupling, to which the perturbative methods do not normally apply.

Instances of holographic dualities include the $\mathrm{AdS}_d/\mathrm{CFT}_{d-1}$ correspondences [1] for $d = 2, 3, 4, 5$. The most investigated example is the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence between type IIB superstring theory in an $\mathrm{AdS}_5 \times S^5$ background and the $\mathrm{SU}(N)$, $\mathcal{N} = 4$ supersymmetric Yang–Mills theory on the four-dimensional conformal boundary of AdS_5. A striking feature of this correspondence, based on the 4-dimensional superconformal group PSU(2,2|4), is the integrability of both the $\mathrm{AdS}_5 \times S^5$ superstring theory and the planar limit $(N \to \infty)$ of the SYM theory [11, 99]. It connects the regime of perturbative gauge theory with the regime of perturbative string theory. Integrability has also been observed in other instances of the AdS/CFT correspondence.

Integrability manifests itself in various features of the theory. For instance, it is believed to be at the core of the relation between planar scattering amplitudes and Wilson loops at strong and weak gauge coupling in the SYM theory, and is related to the existence of a hidden dual superconformal symmetry of gauge theory scattering amplitudes which acts on the momenta as ordinary conformal symmetry acts on coordinates and associates each amplitude to a string worldsheet in a dual AdS space (see [7] for a review and references).

As anticipated in the Introduction, the existence of the dual superconformal symmetry is attributed on the gravity side of AdS/CFT to the self-duality of the superstring sigma model under (Buscher-like) T-duality transformations of fermionic and bosonic string modes on the worldsheet associated with certain (anti-)commuting isometries of the $\mathrm{AdS}_5 \times S^5$ background [67, 68] (see also [145]). In turn, this self-duality is an immediate consequence of the important property that the combined bosonic and fermionic T-dualities do not change the values of the $\mathrm{AdS}_5 \times S^5$ background fields, in particular the Ramond–Ramond flux and the dilaton (see [146] for review and references).

Fermionic T-duality and its relation to the dual superconformal symmetry are pretty well understood and studied in detail in the case of the $\mathrm{AdS}_5 \times S^5$ superstring and corresponding dual $\mathcal{N} = 4$ SYM theory [67, 68, 146]. However, the manifestation and role in the AdS/CFT correspondence of the fermionic T-duality of the sigma models describing superstrings in less supersymmetric integrable[1] AdS backgrounds which give rise to other examples of $\mathrm{AdS}_d/\mathrm{CFT}_{d-1}$ correspondence, such as $\mathrm{AdS}_2 \times S^2 \times M^6$, $\mathrm{AdS}_3 \times S^3 \times M^4$ (where M^{10-2d} is a compact manifold, *e.g.* T^{10-2d} or $S^d \times T^{10-3d}$) and, especially, in $\mathrm{AdS}_4 \times \mathbb{C}P^3$ are much less understood.

In particular, the $\mathrm{AdS}_4 \times \mathbb{C}P^3$ background, which preserves 24 out of 32 supersymmetries of the type IIA superstring theory remains the most challenging case, since it seems to face obstructions in performing the fermionic T-duality of the corresponding superstring sigma model [152, 153] and the supergravity background itself [153–156]. On the other hand, results in the dual field theory indicate that the $\mathrm{AdS}_4 \times \mathbb{C}P^3$ string model should be self-dual under bosonic and fermionic T-duality transformations. In fact, dual superconformal symmetry appears in the planar amplitude sector of the ABJM model both at the tree level [157, 157] and at the loop level [158, 159], Yangian invariance has been observed at the tree level [160], and the amplitudes/Wilson loop duality has been found up to two loops [161, 162]. In [153] it was assumed that an obstruction in performing the fermionic T-duality may be caused by the presence of worldsheet fermionic fields associated with 8 broken supersymmetries [19] in the complete superstring Lagrangian. Indeed, the role of the 'broken supersymmetry' fermions still needs to be better understood and reconciled with other issues caused by a singularity of the bosonic and fermionic T-duality transformations along $\mathbb{C}P^3$ isometries, as observed for example in [152, 156, 163].

We will leave aside the $\mathrm{AdS}_4 \times \mathbb{C}P^3$ case, concentrating rather on the study of remaining issues of the T-duality of superstrings on $\mathrm{AdS}_d \times S^d \times M^{10-2d}$ backgrounds, with the hope

[1]The classical integrability of the full superstring in these backgrounds has been studied in [28, 147, 49, 148–150] and recently a general construction for all symmetric space Ramond–Ramond backgrounds preserving some amount of supersymmetry was given in [151].

that the better understanding of the latter may also provide new insights into the issues of AdS$_4 \times \mathbb{C}P^3$.

So far T-self-duality has been demonstrated for supersymmetric sigma models associated with strings propagating in AdS$_d \times S^d$ ($d = 2, 3, 5$) upon imposing a partial gauge fixing of the kappa symmetry of the sigma model actions by putting to zero a quarter of the supercoset fermionic modes [67, 68, 152]. In [67], the T-self-duality of the AdS$_5 \times S^5$ superstring was demonstrated in the pure spinor formulation, which does not possess kappa symmetry but is instead BRST invariant. The proof used BRST cohomology arguments to extend the kappa-gauge fixed result of the Green–Schwarz formulation to the whole set of the fermionic modes of the pure spinor string. As was mentioned in [67], if the T-dualised Green–Schwarz action could be written in a kappa-invariant form, in order to directly prove the T-self-duality of the pure spinor action, one could use the prescription of [164] which relates the Green–Schwarz kappa symmetry transformations with the pure spinor BRST transformations.

In the cases of the AdS$_d \times S^d$ supercoset models (with $d = 2, 3$) an additional issue arises. It is related to the fact that the supercoset models describe only particular sectors of the complete superstring theories on the AdS$_d \times S^d \times M^{10-2d}$ backgrounds. In the $d = 3$ case, these backgrounds preserve 16 of the 32 supersymmetries in ten dimensions, while in the $d = 2$ case the number of preserved supersymmetries reduces to 8. Therefore, respectively, only 16 and 8 fermionic modes on the string worldsheet can be associated with the fermionic directions of the corresponding coset superspace, while the remaining 16 and 24 fermionic modes correspond to broken supersymmetries. The supercoset sectors of the theory are non-trivially coupled to the non-supercoset directions M^{10-2d} via these fermionic modes.

In the $d = 3$ case, one can use kappa symmetry to put all the 16 off-coset fermions to zero, but this gauge fixing is not admissible for a wide class of classical string configurations (including those when the string moves only in AdS$_3 \times S^3$, [165]). Moreover, though the AdS$_3 \times S^3$ supercoset sigma model with 16 fermions possesses kappa symmetry with 8 independent parameters (see *e.g.* [152]), this kappa symmetry is broken when the supercoset model is coupled via the Virasoro constraints to the T^4 sector of the complete superstring action in AdS$_3 \times S^3 \times T^4$ [28] in which the 16 non-supercoset fermions have already been kappa gauge fixed to zero. In other words, the kappa symmetry of the AdS$_3 \times S^3$ supercoset subsector is part of the kappa symmetry of the complete 10-dimensional superstring and is lost when the latter is completely gauge fixed.

For certain classical string solutions, the $d = 2$ case kappa symmetry allows one to remove only 16 of the 24 non-supersymmetric fermions, so at least 8 non-supercoset fermionic modes are always present in the AdS$_2 \times S^2 \times M^6$ string spectrum (see *e.g.* [28, 49] and references therein for the discussion of these issues). In these cases the self T-duality of the corresponding supercoset models has been proved in a partially fixed kappa symmetry gauge where some of the fermionic coset coordinates are set to zero [152]. However, when the supercoset models are used to describe a gauge-fixed sector of the superstring sigma model where kappa symmetry has already been used to remove part of the non-supersymmetric fermions, one cannot use kappa symmetry anymore for proving the self T-duality of the corresponding supercoset sectors of the $\mathrm{AdS}_d \times S^d \times M^{10-2d}$ superstrings.

In view of the above mentioned issues, it is important to demonstrate explicitly the T-self-duality of superstring theory on the $\mathrm{AdS}_d \times S^d \times M^{10-2d}$ backgrounds without fixing kappa symmetry and taking into account the non-supercoset fermionic modes. This is the main goal of the chapter. Specifically, we verify the combined bosonic and fermionic T-self-duality of Green–Schwarz supercoset sigma models on $\mathrm{AdS}_d \times S^d \times S^d$ backgrounds ($d = 2, 3$) described by supercoset sigma models with the isometries governed by the exceptional supergroups $D(2, 1; \alpha)$ (for $d = 2$) and $D(2, 1; \alpha) \times D(2, 1; \alpha)$ (for $d = 3$). These supercoset models do not possess kappa symmetry and, in order to map the dualised actions to the original ones, the T-dualisation of $d-1$ directions in AdS_d and of $2(d-1)$ fermionic directions should be accompanied by T-dualisation of complex $d-1$ directions of one of the spheres. In [166], the same result was obtained in the corresponding supergravity setup.

2.1. General setup

In this section, we recall some basic facts about superstring sigma models and their T-dualisation.

The conventional form of the Green–Schwarz action describing the propagation of a superstring in a generic 10-dimensional type II background is [112]

$$S = -\frac{T}{2} \int_\Sigma (*\mathcal{E}^A \wedge \mathcal{E}^B \eta_{AB} + 2\kappa B_2) . \tag{2.1}$$

Here, T denotes the string tension and Σ is a 2-dimensional worldsheet with a curved metric $h_{pq}(\tau, \sigma)$ of Lorentz signature so that the corresponding worldsheet Hodge duality operation $*$ squares to one ($*^2 = 1$) when acting on one–forms.[2] The $\mathcal{E}^A = \mathcal{E}^A(X, \Theta)$ with $A, B, \ldots = 0, \ldots, 9$ are vector supervielbeins where (X, Θ) are target space coordinates (10 Graßmann-even (bosonic) coordinates X and 32 Graßmann-odd (fermionic) coordinates Θ) and $(\eta_{AB}) = \mathrm{diag}(-1, 1, \ldots, 1)$ is the 10-dimensional target tangent space Minkowski metric. In addition to $\mathcal{E}^A = \mathcal{E}^A(X, \Theta)$, the description of the geometry also involves spinor supervielbeins $\mathcal{E}^{\hat\alpha} = \mathcal{E}^{\hat\alpha}(X, \Theta)$ with $\hat\alpha, \hat\beta, \ldots = 1, \ldots, 32$. Furthermore, $B_2(X, \Theta)$ is the worldsheet pullback of the Neveu–Schwarz–Neveu–Schwarz 2-form gauge superfield. In the models in which we are interested, it has vanishing field strength at $\Theta = 0$, that is, $\mathrm{d}B_2|_{\Theta=0} = 0$. Kappa symmetry invariance requires the coupling constant κ to be ± 1. In what follows, we shall choose $\kappa = 1$.

[2] Explicitly, in local coordinates (τ, σ) on Σ, $*\mathcal{E}^A \wedge \mathcal{E}^B = \sqrt{-\det(h_{rs})}\, h^{pq} \mathcal{E}^A_p \mathcal{E}^B_q$.

Note that for generic supergravity backgrounds, the action (2.1) is known explicitly up to fourth order in Θ [167].

We will be interested in (bosonic) symmetric space backgrounds of the type $\mathrm{AdS}_d \times S^d \times T^{10-2d}$, $d = 2, 3, 5$ and $\mathrm{AdS}_d \times S^d \times S^d \times T^{10-3d}$, $d = 2, 3$. As shown in [49] for $d = 2$ and in [151] in general, the full type II superspace corresponding to these backgrounds contains a sub-superspace which is a supercoset space $G/H = \{gH \,|\, g \in G\}$, G being the superisometry group and H the isotropy subgroup of the background in question. For a background with no Neveu–Schwarz–Neveu–Schwarz flux, G/H is, in fact, a so-called semi-symmetric superspace meaning that the Lie algebra of G admits a $\mathbb{Z}_4$-automorphism $\Omega : G \to G$ whose fixed point set is H, that is, $\Omega^4 = 1$ and $\Omega(H) = H$. Correspondingly, there exists a truncation of the Green–Schwarz string action to a supercoset sigma model. If the background admits at least 16 supersymmetries, this sigma model can be viewed as a kappa symmetry gauge fixing of the full superstring (for configurations where this gauge fixing is consistent). Below we give the coset superspaces relevant for our discussion.

$\mathbb{Z}_4$-graded coset superspaces. For the $\mathrm{AdS}_d \times S^d \times T^{10-2d}$ backgrounds, we have the following supercosets[3]

$$\begin{aligned}
d = 5: &\quad \frac{PSU(2,2|4)}{SO(1,4) \times SO(5)} \hat{=} \mathrm{AdS}_5 \times S^5 \quad + \quad 32 \text{ fermionic directions}\ , \\
d = 3: &\quad \frac{PSU(1,1|2) \times PSU(1,1|2)}{SU(1,1) \times SU(2)} \hat{=} \mathrm{AdS}_3 \times S^3 \quad + \quad 16 \text{ fermionic directions}\ , \\
d = 2: &\quad \frac{PSU(1,1|2)}{SO(1,1) \times U(1)} \hat{=} \mathrm{AdS}_2 \times S^2 \quad + \quad 8 \text{ fermionic directions}\ .
\end{aligned} \tag{2.2a}$$

while for $\mathrm{AdS}_d \times S^d \times S^d \times T^{10-3d}$, we deal with

$$\begin{aligned}
d = 3: &\quad \frac{D(2,1;\alpha) \times D(2,1;\alpha)}{SO(1,2) \times SO(3) \times SO(3)} \hat{=} \mathrm{AdS}_3 \times S^3 \times S^3 \quad + \quad 16 \text{ fermionic directions}\ , \\
d = 2: &\quad \frac{D(2,1;\alpha)}{SO(1,1) \times SO(2) \times SO(2)} \hat{=} \mathrm{AdS}_2 \times S^2 \times S^2 \quad + \quad 8 \text{ fermionic directions}\ ,
\end{aligned} \tag{2.2b}$$

where $0 \leq \alpha \leq 1$. Note that while for $d = 5$ the coset superspace describes the full superstring theory, for $d = 2, 3$, the listed coset superspaces describe only those subsectors of the full superstring theories in which the non-supersymmetric fermions have been removed by truncation/gauge-fixing and the string does not fluctuate along the torus directions.

[3]Note that since all these coset superspaces are smooth supermanifolds, they naturally fiber over their bosonic part, that is, they are smooth vector bundles with bosonic base and fermionic fibers.

Maurer–Cartan form. The $\mathbb{Z}_4$-automorphism $\Omega : G \to G$ induces a corresponding automorphism on the Lie superalgebra $\mathfrak{g}$ of G, which we shall again denote by $\Omega : \mathfrak{g} \to \mathfrak{g}$ (see *e.g.* [168] for a classification). We therefore have a decomposition $\mathfrak{g} \otimes \mathbb{C} \cong \bigoplus_{m=0}^{3} \mathfrak{g}_{(m)}$ into the eigenspaces of Ω, that is, $\Omega(V_{(m)}) = \mathrm{i}^m V_{(m)}$ for $V_{(m)} \in \mathfrak{g}_{(m)}$. In addition, $[\mathfrak{g}_{(m)}, \mathfrak{g}_{(n)}] \subseteq \mathfrak{g}_{(m+n \bmod 4)}$ and $\mathfrak{g}_{(0)}$ is the Lie algebra of H. Furthermore, $\mathfrak{g}$ comes with a $\mathbb{Z}_2$-grading, and the generators of $\mathfrak{g}_{(0)}$ and $\mathfrak{g}_{(2)}$ are bosonic while the generators of $\mathfrak{g}_{(1)}$ and $\mathfrak{g}_{(3)}$ are fermionic. For various general properties of the Lie superalgebras associated with the Lie supergroups appearing in (2.2), we refer the reader to *e.g.* [30].

Next, we consider maps $g : \Sigma \to G$ from a 2-dimensional worldsheet Riemann surface Σ with an (arbitrarily chosen) Lorentzian metric into G and introduce the (pull-back to Σ via g of the) Maurer–Cartan form

$$J \; := \; g^{-1}\mathrm{d}g \; . \tag{2.3}$$

Here, d denotes the exterior derivative on Σ.[4] By construction, the $\mathfrak{g}$-valued differential 1-form J is invariant under global left G-transformations $g \mapsto g_0 g$ for $g_0 \in G$ and satisfies the Maurer–Cartan equation, $\mathrm{d}J - J \wedge J = 0$. Using the $\mathbb{Z}_4$-automorphism $\Omega : \mathfrak{g} \to \mathfrak{g}$, we may decompose J into the eigenspaces of Ω according to

$$J \; = \; J_{(0)} + J_{(1)} + J_{(2)} + J_{(3)} \quad \text{with} \quad \Omega\left(J_{(m)}\right) \; = \; \mathrm{i}^m J_{(m)}. \tag{2.4}$$

It is then straightforward to check that under local right H-transformations $g \mapsto gh$ for $h \in H$, the part $J_{(0)}$ behaves as a $\mathfrak{g}_{(0)}$-valued connection 1-form, $J_{(0)} \mapsto h^{-1}J_{(0)}h + h^{-1}\mathrm{d}h$, while the $J_{(m)}$s for $m = 1, 2, 3$ transform adjointly, $J_{(m)} \mapsto h^{-1}J_{(m)}h$. Since the physical fields will take values in the coset superspace $G/H = \{gH \,|\, g \in G\}$ for (2.2), the corresponding action must be invariant under such local right H-transformations. This, in turn, implies that the action will involve only the $J_{(m)}$ for $m = 1, 2, 3$. Correspondingly, G/H is parametrised by d_b bosonic local coordinates $\mathbb{X}$ and d_f fermionic local coordinates ϑ, where $d_\mathrm{b} + d_\mathrm{f} := \dim(G/H) = \dim(G) - \dim(H)$, so that we will be dealing with maps $(\mathbb{X}, \vartheta) : \Sigma \to G/H$. Furthermore, $J_{(2)}$ play the role of bosonic supervielbeins while $J_{(1)}$ and $J_{(3)}$ play the role of fermionic supervielbeins.

Supercoset action. The supercoset string action for a $\mathbb{Z}_4$-graded G/H coset superspace is constructed from the 1-forms $J_{(m)}$ for $m = 1, 2, 3$, and it has the following form (see [8, 169–171, 17, 18, 28] and references therein)

$$S \; = \; -T \int_\Sigma \mathcal{L}_{G/H} \; = \; -\frac{T}{2} \int_\Sigma \mathrm{Str}(*J_{(2)} \wedge J_{(2)} + J_{(1)} \wedge J_{(3)}) \; . \tag{2.5}$$

where Str denotes the supertrace compatible with the $\mathbb{Z}_4$-grading,

$$\mathrm{Str}(V_{(m)}V_{(n)}) \; = \; 0 \quad \text{for} \quad V_{(m)} \; \in \; \mathfrak{g}_{(m)} \quad \text{and} \quad m + n \; \neq \; 0 \bmod 4 \; , \tag{2.6}$$

[4] In our conventions, d acts from the right.

As in (2.1), in the non-exceptional cases the relative coefficient of the two terms in (2.5) is fixed by kappa symmetry, while in the exceptional cases the action (2.5) is not kappa symmetry invariant [172, 173]. In the latter cases, the relative coefficient gets fixed by their relation to the original Green–Schwarz action and/or by integrability of the sigma-models. Clearly, the action (2.5) is invariant under rigid left G-transformations and local right H-gauge transformations. The Wess–Zumino term (the second term in this action) was first given in the above form in [171]. Comparison with the Green–Schwarz action (2.1) tells us that $B_2 = \frac{1}{2}\mathrm{Str}(J_{(1)} \wedge J_{(3)})$.

Schematic form of the superconformal algebra. The T-dualisation of the action (2.5) is performed along certain bosonic and fermionic directions of the G/H supercoset which correspond to an (anti-)commuting (that is, Abelian) subgroup of the isometries of the underlying coset superspace. To identify these isometries, one chooses a basis of the Lie superalgebra $\mathfrak{g}$ of G which is associated with the superconformal group on the Minkowski (conformal) boundary $\mathbb{R}^{1,d-2}$ of the AdS$_d$ space. In this basis, $\mathfrak{g}$ is described schematically as follows. The bosonic conformal algebra and the R-symmetry on $\mathbb{R}^{1,d-2}$ are given by (we only display non-vanishing commutators)

$$\begin{gathered} [P,K] \sim D+M\ , \\ [D,P] \sim P\ , \quad [D,K] \sim K\ , \quad [M,P] \sim P\ , \quad [M,K] \sim K\ , \\ [M,M] \sim M\ , \quad [R,R] \sim R\ , \end{gathered} \tag{2.7}$$

where P are the $(d-1)$ translation generators, M are the $\frac{1}{2}(d-1)(d-2)$ Lorentz generators, K are the $(d-1)$ conformal boost generators, and D is the dilatation generator. In the AdS$_d \times S^d$ case the R-symmetry generators R are associated with the $SO(d+1)$ isometries of S^d, while in the case of AdS$_d \times S^d \times S^d$ they correspond to the $SO(d+1)\times SO(d+1)$ isometries of $S^d \times S^d$.

The superconformal extension of the algebra (2.7) contains the fermionic generators Q, $\hat{Q}$, S, and $\hat{S}$ which are the complex supersymmetry and superconformal generators related by Hermitian conjugation (the specific form of the conjugation rules depends on the chosen superalgebra), each being $2(d-1)$-dimensional. The additional non-vanishing (anti-)commutation relations have the following schematic form

$$\begin{gathered} [D,Q] \sim Q\ , \quad [M,Q] \sim Q\ , \quad [K,Q] \sim \hat{S}\ , \quad [R,Q] \sim Q+\alpha\hat{Q}\ , \\ [D,S] \sim S\ , \quad [M,S] \sim S\ , \quad [P,S] \sim \hat{Q}\ , \quad [R,S] \sim S+\alpha\hat{S}\ , \end{gathered} \tag{2.8a}$$

and similarly for $\hat{Q}$ and $\hat{S}$, plus

$$\begin{gathered} \{Q,\hat{Q}\} \sim P\ , \quad \{S,\hat{S}\} \sim K\ , \quad \{Q,\hat{S}\} \sim \alpha R\ , \quad \{\hat{Q},S\} \sim \alpha R \\ \{Q,S\} \sim D+M+R\ , \quad \{\hat{Q},\hat{S}\} \sim D+M+R\ . \end{gathered} \tag{2.8b}$$

In these relations, α is the parameter appearing in the coset superspaces (2.2b). Note that the $d = 2, 3$ coset superspaces in (2.2a) are obtained from those in (2.2b) by taking the limit $\alpha \to 0$. Hence, in the case of the $\mathrm{AdS}_d \times S^d \times T^{10-2d}$ backgrounds we simply set $\alpha = 0$.

In summary, the Lie superalgebra $\mathfrak{g}$ is generated by $\mathfrak{g} = \langle P, K, D, M, R, Q, \hat{Q}, S, \hat{S} \rangle$ and described by the (anti-)commutation relations (2.7) and (2.8).

Choice of $\mathbb{Z}_4$-grading. As we shall see below, the specific choice of a $\mathbb{Z}_4$-grading and its superconformal splitting onto Abelian sub-isometries are crucial when performing the T-duality transformations — an inappropriate choice would make the proof of the self-duality of the complete superstring actions much more complicated if at all possible. Decomposing the R-symmetry generators R as $R = (R_{(0)}, R_{(2)})$ with $R_{(0)} \in \mathfrak{g}_{(0)}$ and $R_{(2)} \in \mathfrak{g}_{(2)}$, the $\mathbb{Z}_4$-grading we shall be using is formally of the form

$$\begin{aligned} \mathfrak{g}_{(0)} &:= \langle P + K, M, R_{(0)} \rangle \ , \quad & \mathfrak{g}_{(2)} &:= \langle P - K, D, R_{(2)} \rangle \ , \\ \mathfrak{g}_{(1)} &:= \langle Q - S, \hat{Q} - \hat{S} \rangle \ , \quad & \mathfrak{g}_{(3)} &:= \langle Q + S, \hat{Q} + \hat{S} \rangle \ . \end{aligned} \tag{2.9}$$

We emphasize that the specific form of the decomposition $R = (R_{(0)}, R_{(2)})$ will depend on the particular form of the superconformal algebra, and we shall say a few things about this in the next paragraph.

Coset representative and associated current. In the $\mathrm{AdS}_d \times S^d \times T^{10-2d}$ case, the form of the superalgebra (2.7) and (2.8) (with $\alpha = 0$) implies that the $(d-1)$ generators P and $2(d-1)$ complex supercharges Q are in involution, and, hence, a maximal Abelian subalgebra of $\mathfrak{g}$ is simply $\langle P, Q \rangle$. Thus, the (anti-)commuting isometries of the G/H Green–Schwarz sigma model can be associated with $\langle P, Q \rangle$.

In the $\mathrm{AdS}_d \times S^d \times S^d$ case, the situation is somewhat more complicated, and as we shall see in Section 2.2., a maximal Abelian subalgebra of $\mathfrak{g}$ is again generated by P and Q but also by some of the R-symmetry generators which we denote formally by L_+. To jump ahead of our story a bit, we will have one complex generator $L_+ \equiv L_+^1$ for $d = 2$ and two complex generators $L_+ \equiv L_+^{1,2}$ for $d = 3$. Hence, the (anti-)commuting isometries are associated with $\langle P, Q, L_+ \rangle$ in this case. In the following, we shall denote the Hermitian conjugate of L_+ by L_- and we have $[L_+^1, L_-^1] \sim L_3 \sim [L_+^2, L_-^2]$. In view of the $\mathbb{Z}_4$-grading (2.9), it turns out that $L_+^1 + L_-^1 \in \mathfrak{g}_{(0)}$, $L_+^1 - L_-^1 \in \mathfrak{g}_{(2)}$, $L_+^2 - L_-^2 \in \mathfrak{g}_{(0)}$, $L_+^2 + L_-^2 \in \mathfrak{g}_{(2)}$, and $L_3 \in \mathfrak{g}_{(2)}$ will be the appropriate choice. See Section 2.2. for details.

Motivated by this discussion, to perform the T-dualisation of the action (2.5) along these isometries, it is convenient to take the supercoset representative g in a form similar to that of [67, 68, 173]

$$g := \mathrm{e}^{xP + \theta Q + \sqrt{\alpha}\,\lambda_+ L_+} \mathrm{e}^B \mathrm{e}^{\xi S} \ , \quad \mathrm{e}^B := \mathrm{e}^{\hat{\theta}\hat{Q} + \hat{\xi}\hat{S}} |y|^D \mathrm{e}^{-\sqrt{\alpha}\,\lambda_3 L_3} \Lambda_\alpha(y) \ , \tag{2.10}$$

where x are the coordinates of the Minkowski boundary and $|y|$ is associated with the radial (bulk) direction in AdS_d. In the $\mathrm{AdS}_d \times S^d \times T^{10-2d}$ case ($\alpha = 0$) the coordinates y parametrize S^d, whereas in the $\mathrm{AdS}_d \times S^d \times S^d$ background ($\alpha \neq 0$) one S^d is parametrized by y and the second one is described by λ_+ and λ_3. The latter coordinates are assumed to be complex (we will explain this in more detail in Section 2.2.). Moreover, the specific form of $\Lambda_\alpha = \Lambda_\alpha(y)$ will depend on the chosen background. The set of $2(d-1)$ complex conjugate fermionic coordinates $(\theta, \hat{\theta}, \xi, \hat{\xi})$ parametrize the Graßmann-odd directions of the coset superspace. In order to achieve the form (2.10) of the representative, we have employed local right H-transformations, and since P, Q, and L_+ are in involution, this choice of the representative ensures that the action (2.5) will depend on x, θ, and λ_+ only through their derivatives $\mathrm{d}x$, $\mathrm{d}\theta$, and $\mathrm{d}\lambda_+$.

So far, the proof of self-duality of supercoset sigma models (2.5) under bosonic and fermionic T-duality has been performed in a fixed kappa symmetry gauge, the most convenient choice being $\xi = 0$ [67, 68, 173]. However, as already explained, if the supercoset model describes the gauge-fixed version of the corresponding superstring action, the kappa symmetry has been already used to (partially) gauge away the non-supersymmetric fermions, and cannot be used once again in the T-dualisation procedure. Moreover, for sigma models based on the exceptional Lie supergroups (2.2b), the rank of the kappa symmetry is zero [28, 172, 173] and one cannot put any of the fermionic coordinates to zero. Therefore, in what follows we are not going to (partially) fix kappa symmetry to get rid of some of the fermionic coordinates. All the fermionic coordinates in (2.10) will be taken into account.

In the realization (2.10) of the coset element, the current (2.3) has the following form

$$J \;=\; g^{-1}\mathrm{d}g \;=\; \mathrm{e}^{-\xi S} J^{(0)} \mathrm{e}^{\xi S} + \mathrm{d}\xi S\ , \tag{2.11}$$

where $J^{(0)}$ is the current at $\xi = 0$. Decomposing the currents J and $J^{(0)}$ along the generators as

$$J \;=\; J_P P + J_K K + \cdots + J_{\mathfrak{J}} \mathfrak{J} + \cdots \quad \text{and} \quad J^{(0)} \;=\; J^{(0)}_P P + J^{(0)}_K K + \cdots + J_{\mathfrak{J}} \mathfrak{J} + \cdots\ , \tag{2.12}$$

$J_{\mathfrak{J}}$ being the component of J along $\mathfrak{J}$. Accordingly, the components of $J^{(0)}$ are given by

$$\begin{gathered}
J^{(0)}_P \;=\; \big[\mathrm{e}^{-B}(\mathrm{d}xP + \mathrm{d}\theta Q + \sqrt{\alpha}\,\mathrm{d}\lambda_+ L_+)\mathrm{e}^{B}\big]_P\ , \quad J^{(0)}_K \;=\; 0\ , \\
J^{(0)}_D \;=\; \big[\mathrm{e}^{-B}\mathrm{d}\mathrm{e}^{B}\big]_D\ , \quad J^{(0)}_M \;=\; \big[\mathrm{e}^{-B}\mathrm{d}\mathrm{e}^{B}\big]_M\ , \\
J^{(0)}_R \;=\; \big[\mathrm{e}^{-B}\mathrm{d}\mathrm{e}^{B}\big]_R\ , \\
J^{(0)}_{L_+} \;=\; \big[\mathrm{e}^{-B}(\mathrm{d}xP + \mathrm{d}\theta Q + \sqrt{\alpha}\,\mathrm{d}\lambda_+ L_+)\mathrm{e}^{B}\big]_{L_+}\ , \quad J^{(0)}_{L_-} \;=\; 0\ , \quad J^{(0)}_{L_3} \;=\; \big[\mathrm{e}^{-B}\mathrm{d}\mathrm{e}^{B}\big]_{L_3}\ , \\
J^{(0)}_Q \;=\; \big[\mathrm{e}^{-B}(\mathrm{d}xP + \mathrm{d}\theta Q + \sqrt{\alpha}\,\mathrm{d}\lambda_+ L_+)\mathrm{e}^{B}\big]_Q\ , \quad J^{(0)}_{\hat{Q}} \;=\; \big[\mathrm{e}^{-B}\mathrm{d}\mathrm{e}^{B}\big]_{\hat{Q}}\ , \\
J^{(0)}_S \;=\; 0\ , \quad J^{(0)}_{\hat{S}} \;=\; \big[\mathrm{e}^{-B}\mathrm{d}\mathrm{e}^{B}\big]_{\hat{S}}\ ,
\end{gathered} \tag{2.13a}$$

where $[\cdots]_P$ etc. indicates the projection onto the generators P etc., while the components of J read schematically as

$$\begin{aligned} J_P &= J_P^{(0)}\,, \quad J_Q = J_Q^{(0)}\,, \quad J_{L_+} = J_{L_+}^{(0)}\,, \\ J_D &= J_D^{(0)} + J_Q^{(0)}\xi\,, \quad J_M = J_M^{(0)} + J_Q^{(0)}\xi\,, \\ J_R &= J_R^{(0)} + J_Q^{(0)}\xi\,, \quad J_{L_3} = J_{L_3}^{(0)} + \alpha J_Q^{(0)}\xi\,, \\ J_{\hat{Q}} &= J_{\hat{Q}}^{(0)} + J_P^{(0)}\xi\,, \quad J_{\hat{S}} = J_{\hat{S}}^{(0)} + \alpha J_{L_+}^{(0)}\xi\,, \\ J_K &= J_{\hat{S}}^{(0)}\xi + \alpha J_{L_+}^{(0)}\xi^2\,, \quad J_{L_-} = \alpha J_{\hat{Q}}^{(0)}\xi + \alpha J_P^{(0)}\xi^2\,, \\ J_S &= \mathrm{d}\xi + (J_D^{(0)} + J_M^{(0)} + J_R^{(0)} + \alpha J_{L_3}^{(0)})\xi + J_Q^{(0)}\xi^2\,. \end{aligned} \tag{2.13b}$$

We note that thanks to the appropriate choice of the coset representative (2.10), the current J depends on ξ at most quadratically. This will drastically simplify the T-dualisation procedure.

Finally, decomposing the current J according to the $\mathbb{Z}_4$-grading (2.9), the supercoset sigma model action (2.5) takes the following schematic form

$$\begin{aligned} S = -\frac{T}{2}\int_\Sigma \Big[&* (J_P - J_K)\wedge(J_P - J_K) + *J_D\wedge J_D + *J_{R_{(2)}}\wedge J_{R_{(2)}} + \\ &+\sum_{k=1}^{d-1} *(J_{L_+^k} + (-)^k J_{L_-^k})\wedge(J_{L_+^k} + (-)^k J_{L_-^k}) + *J_{L_3}\wedge J_{L_3} + \\ &+ \big(J_Q\wedge\gamma J_Q - J_{\hat{S}}\wedge\gamma J_{\hat{S}} - J_S\wedge\gamma J_S + J_{\hat{Q}}\wedge\gamma J_{\hat{Q}}\big)\Big]\,. \end{aligned} \tag{2.14}$$

Here, γ in the last four terms stands for a constant symmetric matrix being part of the G-invariant bilinear form contracting the spinor indices of the fermionic currents. The form of this matrix is related to the value of Ramond–Ramond fluxes supporting the corresponding $\mathrm{AdS}_d \times S^d \times M^{10-2d}$ background.

T-duality procedure. In the $\alpha = 0$ case, one performs T-duality upon the action (2.14) along x and θ, following the discussion of [67,68]. For the $\alpha \neq 0$ case, we dualise also along λ_+, following ideas of [173].

According to the standard procedure [174–176], starting from the action (2.14) we first make the substitution $(\mathrm{d}x, \mathrm{d}\theta, \mathrm{d}\lambda_+) \mapsto (A_\mathrm{b}, A_\mathrm{f}, A_+)$ and modify it according to

$$S \;\mapsto\; S_{\text{f.o.}} = S[(\mathrm{d}x, \mathrm{d}\theta, \mathrm{d}\lambda_+) \;\mapsto\; (A_\mathrm{b}, A_\mathrm{f}, A_+)] + \int_\Sigma \big(\tilde{x}\mathrm{d}A_\mathrm{b} + \tilde{\theta}\mathrm{d}A_\mathrm{f} + \sqrt{\alpha}\,\tilde{\lambda}_+\mathrm{d}A_+\big) \tag{2.15}$$

Here, $\{A_\mathrm{b}, A_\mathrm{f}, A_+\}$ are auxiliary differential 1-forms and $\{\tilde{x}, \tilde{\theta}, \tilde{\lambda}_+\}$ are Lagrange multipliers. The latter enforce the constraints $\mathrm{d}A_\mathrm{b} = 0$, $\mathrm{d}A_\mathrm{f} = 0$, and $\mathrm{d}A_+ = 0$ or, equivalently, $A_\mathrm{b} = \mathrm{d}x$, $A_\mathrm{f} = \mathrm{d}\theta$, and $A_+ = \mathrm{d}\lambda_+$. Consequently, upon integrating $\{\tilde{x}, \tilde{\theta}, \tilde{\lambda}_+\}$ out, we recover the original action (2.14).

To derive the dualised action $\tilde{S}$, we instead need to integrate out the differential 1-forms $\{A_{\rm b}, A_{\rm f}, A_+\}$. Once done, the Lagrange multipliers $\{\tilde{x}, \tilde{\theta}, \tilde{\lambda}_+\}$ shall be interpreted as T-dual coordinates. In order to perform this operation, we make a simplification by noticing that

$$\mathrm{e}^{-B}\big(A_{\rm b}P + A_{\rm f}Q + \sqrt{\alpha}\,A_+L_+\big)\mathrm{e}^{B} \;=\; A'_{\rm b}P + A'_{\rm f}Q + \sqrt{\alpha}\,A'_+L_+ \tag{2.16}$$

since the Abelian algebra $\langle P, Q, L_+\rangle$ is invariant under conjugation by the group element e^B. Equivalently,

$$A_{\rm b}P + A_{\rm f}Q + \sqrt{\alpha}\,A_+L_+ \;=\; \mathrm{e}^{B}\big(A'_{\rm b}P + A'_{\rm f}Q + \sqrt{\alpha}\,A'_+L_+\big)\mathrm{e}^{-B}\ . \tag{2.17}$$

Thus, we may consider the field re-definition $(A_{\rm b}, A_{\rm f}, A_+) \mapsto (A'_{\rm b}, A'_{\rm f}, A'_+)$. Note that the on-shell relations $\mathrm{d}x = A_{\rm b}$, $\mathrm{d}\theta = A_{\rm f}$, and $\mathrm{d}\lambda_+ = A_+$ together with (2.13) imply the on-shell relations $A'_{\rm b} = J_P$, $A'_{\rm f} = J_Q$, and $A'_+ = J_{L_+}$.

Upon substituting

$$\begin{aligned}
A_{\rm b} \;&=\; \big[\mathrm{e}^{B}\big(A'_{\rm b}P + A'_{\rm f}Q + \sqrt{\alpha}\,A'_+L_+\big)\mathrm{e}^{-B}\big]_P\ ,\\
A_{\rm f} \;&=\; \big[\mathrm{e}^{B}\big(A'_{\rm b}P + A'_{\rm f}Q + \sqrt{\alpha}\,A'_+L_+\big)\mathrm{e}^{-B}\big]_Q\ ,\\
A_+ \;&=\; \big[\mathrm{e}^{B}\big(A'_{\rm b}P + A'_{\rm f}Q + \sqrt{\alpha}\,A'_+L_+\big)\mathrm{e}^{-B}\big]_{L_+}
\end{aligned} \tag{2.18}$$

into the action (2.15) and integrating out $\{A'_{\rm b}, A'_{\rm f}, A'_+\}$, one obtains the dualised action $\tilde{S}$. The main goal is to show that the action $\tilde{S}$ (upon certain field re-definitions) is again of the Green–Schwarz form (2.5), however, in a coordinate system which is associated with a different choice of the coset representative

$$\tilde{g} \;:=\; \mathrm{e}^{\tilde{x}K + \tilde{\theta}M^{-1}S + \sqrt{\alpha}\,\tilde{\lambda}_+L_-}\mathrm{e}^{B}\mathrm{e}^{F(\xi)}\ , \quad \mathrm{e}^{B} \;:=\; \mathrm{e}^{\hat{\theta}\hat{Q}+\hat{\xi}\hat{S}}|y|^{D}\mathrm{e}^{-\sqrt{\alpha}\,\lambda_3L_3}\Lambda_\alpha(y)\ , \tag{2.19}$$

where $M := \mathrm{Str}(QS)$. Note that e^B in the representative (2.19) is the same as given in (2.10). Furthermore, in the $\mathrm{AdS}_5 \times S^5$ case, $F(\xi)$ is of the schematic form [166]

$$F(\xi) \;\sim\; -\big[\xi + \xi^5\big]Q + \big[\xi^3 + \xi^7\big]S\ , \tag{2.20}$$

while for $\mathrm{AdS}_2 \times S^2$ and $\mathrm{AdS}_2 \times S^2 \times S^2$, $F(\xi)$ contains only the first linear term in ξ and for $\mathrm{AdS}_3 \times S^3$ and $\mathrm{AdS}_3 \times S^3 \times S^3$ it consists of both linear and cubic terms. Because of the presence of $F(\xi)$, the current $\tilde{J} = \tilde{g}^{-1}\mathrm{d}\tilde{g}$ arising from the representative (2.19), will, in general, not be quadratic in the fermionic coordinates ξ. However, as we will show, upon further complicated field re-definitions $(X, \Theta) \to (X', \Theta')$, the dual coset element (2.19) can be nevertheless brought to a form similar to that of (2.10), *i.e.*

$$\tilde{g} \;=\; \mathrm{e}^{\tilde{x}'K + \tilde{\theta}'M^{-1}S + \sqrt{\alpha}\,\tilde{\lambda}'_+L_-}\mathrm{e}^{B'}\mathrm{e}^{-\xi'Q}\,. \tag{2.21}$$

Let us discuss the T-duality procedure for the superstring sigma models on $\mathrm{AdS}_d \times S^d \times S^d$ for $d = 2, 3$.

Comment on the self-duality at the quantum level. Since the duality transformations can be performed via a Gaußian path integral, they can be promoted to a duality of the quantum sigma model. A priori, the path integral measure could change upon integrating out the auxiliary fields $\{A'_{\mathrm{b}}, A'_{\mathrm{f}}, A'_{+}\}$. However, this is not the case, since the corresponding Berezinian is equal to one provided one also regularises the bosonic and fermionic determinants in the same way (*e.g.* by using heat kernel methods as in [175, 177]). Therefore, there will be no shift in the dilaton and we may thus conclude that the self-duality of the Green–Schwarz sigma models under consideration also holds at the quantum level.

2.2. Self-duality of $\mathrm{AdS}_d \times S^d \times S^d \times T^{10-3d}$ superstrings

In this section, we will extend the previous discussion to the cases of superstrings on $\mathrm{AdS}_d \times S^d \times S^d \times T^{10-3d}$ ($d = 2, 3$). As their $\mathrm{AdS}_d \times S^d \times T^{10-2d}$ counterparts, these backgrounds preserve 1/4 and 1/2 of the 10-dimensional supersymmetry and can be supported by either Neveu–Schwarz–Neveu–Schwarz or Ramond–Ramond fluxes [27, 26, 31–33]. Here we will consider the latter ones. For instance, a type IIB $\mathrm{AdS}_3 \times S^3 \times S^3 \times S^1$ background can be supported by the following F_3 flux

$$F_3 \;=\; \frac{1}{3}\Big(\varepsilon_{cba} e^a \wedge e^b \wedge e^c + \frac{R_{AdS}}{R_+}\varepsilon_{\hat{c}\hat{b}\hat{a}} e^{\hat{a}} \wedge e^{\hat{b}} \wedge e^{\hat{c}} + \frac{R_{AdS}}{R_-}\varepsilon_{c'b'a'} e^{a'} \wedge e^{b'} \wedge e^{c'}\Big), \tag{2.22}$$

where $\hat{a}$ and a' are, respectively the tangent space indices of the two three-spheres and $R_\pm$ are their radii.

Upon the T-dualization of the above background along the S^1 one gets the type IIA $\mathrm{AdS}_3 \times S^3 \times S^3 \times S^1$ with the F_4-flux

$$F_4 \;=\; \mathrm{d}\varphi^9 \wedge \frac{1}{3}\Big(\varepsilon_{cba} e^a \wedge e^b \wedge e^c + \frac{R_{AdS}}{R_+}\varepsilon_{\hat{c}\hat{b}\hat{a}} e^{\hat{a}} \wedge e^{\hat{b}} \wedge e^{\hat{c}} + \frac{R_{AdS}}{R_-}\varepsilon_{c'b'a'} e^{a'} \wedge e^{b'} \wedge e^{c'}\Big). \tag{2.23}$$

Because of the technical complexity, in these cases we will put the off-coset fermionic modes of the string to zero by fixing a kappa symmetry gauge in the $d = 3$ case and by hand in the $d = 2$ case. Modulo Virasoro constraints, the T^{10-3d}-sector decouples and we may concentrate on the $\mathrm{AdS}_d \times S^d \times S^d$ sectors described by supercoset sigma models with the isometries governed by the exceptional Lie supergroups $D(2, 1; \alpha)$ (for $d = 2$) and $D(2, 1; \alpha) \times D(2, 1; \alpha)$ (for $d = 3$). In particular, we shall show that they are also T-self-dual under combined bosonic and fermionic T-dualities, provided that T-dualisation involves one of the spheres S^d, the latter causing some additional technical difficulties.

2.2.1. Self-duality for $\mathrm{AdS}_2 \times S^2 \times S^2$

Supercoset structure. The sigma model on $\mathrm{AdS}_2 \times S^2 \times S^2$ is based on the supercoset

$$\frac{\mathrm{D}(2, 1; \alpha)}{SO(1, 1) \times SO(2) \times SO(2)}\,. \tag{2.24}$$

To construct the corresponding action and analyse its T-duality properties, let us discuss the Lie superalgebra $\mathfrak{d}(2,1;\alpha)$ of $D(2,1;\alpha)$. For general properties of the exceptional Lie superalgebra $\mathfrak{d}(2,1;\alpha)$ see *e.g.* [30, 178]. For the 10-dimensional supergravity solutions under consideration, the values of the parameter α are restricted to the interval $[0,1]$.[5] They determine the relation between the radii of $\mathrm{AdS}_2 \times S^2_+ \times S^2_-$,

$$\alpha \;=\; \frac{R^2_{\mathrm{AdS}}}{R^2_-} \quad \text{and} \quad 1-\alpha \;=\; \frac{R^2_{\mathrm{AdS}}}{R^2_+}\ . \tag{2.25}$$

In order to avoid confusion between the parameter α and the spinor index α, in what follows we will set $\alpha := \cos^2(\tau) := c^2$ and $1-\alpha := \sin^2(\tau) := s^2$, respectively.

Lie superalgebra $\mathfrak{d}(2,1;c^2)$. The maximal Graßmann-even subalgebra of the Lie superalgebra $\mathfrak{d}(2,1;c^2)$ is $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, and we set $\mathfrak{sl}(2,\mathbb{R}) := \langle P,K,D\rangle$, $\mathfrak{su}(2) := \langle L_a\rangle$, and $\mathfrak{su}(2) := \langle R^\alpha{}_\beta\rangle$, respectively, for $a,b,\ldots = 1,2,3$ and $\alpha,\beta,\ldots = 1,2$. The corresponding commutation relations are[6]

$$\begin{aligned}
[D,P] \;&=\; P\ , \quad [D,K] \;=\; -K\ , \quad [P,K] \;=\; 2D\ ,\\
[L_+,L_-] \;&=\; -2\mathrm{i}L_3\ , \quad [L_3,L_\pm] \;=\; \pm\mathrm{i}L_\pm\ , \quad L_\pm \;:=\; \mathrm{i}L_1 \pm L_2\ ,\\
[R^\alpha{}_\beta, R^\gamma{}_\delta] \;&=\; \mathrm{i}(\delta^\gamma{}_\beta R^\alpha{}_\delta - \delta^\alpha{}_\delta R^\gamma{}_\beta)\ .
\end{aligned} \tag{2.26a}$$

Furthermore, $\mathfrak{d}(2,1;c^2)$ contains eight fermionic generators which we denote by Q_α, $\hat{Q}_\alpha$, S_α, and $\hat{S}_\alpha$, respectively. Letting $\sigma^{1,2,3}_{\alpha\beta}$ be the Pauli matrices[7], the remaining non-vanishing (anti-)commutation relations of $\mathfrak{d}(2,1;c^2)$ are given by

$$\begin{aligned}
\{Q_\alpha,\hat{Q}_\beta\} \;&=\; -\sigma^2_{\alpha\beta}P\ , \quad \{S_\alpha,\hat{S}_\beta\} \;=\; -\sigma^2_{\alpha\beta}K\ ,\\
\{Q_\alpha,\hat{S}_\beta\} \;&=\; -c^2\sigma^2_{\alpha\beta}L_+\ , \quad \{\hat{Q}_\alpha,S_\beta\} \;=\; c^2\sigma^2_{\alpha\beta}L_-\ ,\\
\{Q_\alpha,S_\beta\} \;&=\; -\sigma^2_{\alpha\beta}(D+\mathrm{i}c^2L_3) - \mathrm{i}s^2\sigma^2_{\alpha\gamma}R^\gamma{}_\beta\ ,\\
\{\hat{Q}_\alpha,\hat{S}_\beta\} \;&=\; \sigma^2_{\alpha\beta}(D-\mathrm{i}c^2L_3) + \mathrm{i}s^2\sigma^2_{\alpha\gamma}R^\gamma{}_\beta\ ,\\
[P,S_\alpha] \;=\; -\hat{Q}_\alpha\ , \quad [P,\hat{S}_\alpha] \;&=\; -Q_\alpha\ , \quad [K,Q_\alpha] \;=\; -\hat{S}_\alpha\ , \quad [K,\hat{Q}_\alpha] \;=\; -S_\alpha\ ,\\
[D,Q_\alpha] \;=\; \tfrac{1}{2}Q_\alpha\ , \quad [D,\hat{Q}_\alpha] \;&=\; \tfrac{1}{2}\hat{Q}_\alpha\ , \quad [D,S_\alpha] \;=\; -\tfrac{1}{2}S_\alpha\ , \quad [D,\hat{S}_\alpha] \;=\; -\tfrac{1}{2}\hat{S}_\alpha\ ,\\
[L_3,Q_\alpha] \;=\; \tfrac{\mathrm{i}}{2}Q_\alpha\ , \quad [L_3,\hat{Q}_\alpha] \;&=\; -\tfrac{\mathrm{i}}{2}\hat{Q}_\alpha\ , \quad [L_3,S_\alpha] \;=\; -\tfrac{\mathrm{i}}{2}S_\alpha\ , \quad [L_3,\hat{S}_\alpha] \;=\; \tfrac{\mathrm{i}}{2}\hat{S}_\alpha\ ,\\
[L_+,S_\alpha] \;=\; \hat{S}_\alpha\ , \quad [L_-,\hat{S}_\alpha] \;&=\; -S_\alpha\ , \quad [L_-,Q_\alpha] \;=\; \hat{Q}_\alpha\ , \quad [L_+,\hat{Q}_\alpha] \;=\; -Q_\alpha\ ,\\
[R^\alpha{}_\beta,T_\gamma] \;&=\; -\mathrm{i}(\delta^\alpha{}_\gamma T_\beta - \tfrac{1}{2}\delta^\alpha{}_\beta T_\gamma)\ , \quad \text{for} \;\; T_\alpha \;\in\; \{Q_\alpha,\hat{Q}_\alpha,S_\alpha,\hat{S}_\alpha\}\ .
\end{aligned} \tag{2.26b}$$

[5] For superbackgrounds whose isometries are governed by $\mathfrak{d}(2,1;\alpha)$ with other values of α see *e.g.* [179, 180].

[6] One could also start from the 10-dimensional form of the algebra analogous to the $\mathfrak{psu}(2,2|4)$ case. This form follows directly from the general construction of the symmetric space superisometry algebras of [151] upon inserting the form of the fluxes.

[7] We lower and raise Greek indices using $\epsilon_{\alpha\beta} = \mathrm{i}\sigma^2_{\alpha\beta}$ and $\epsilon^{\alpha\beta} = \mathrm{i}\sigma^{2\,\alpha\beta}$ with $\epsilon^{\alpha\gamma}\epsilon_{\gamma\beta} = \delta^\alpha{}_\beta$, so that *e.g.* $\sigma^{1\,\alpha}{}_\beta := \mathrm{i}\sigma^{2\,\alpha\gamma}\sigma^1_{\gamma\beta} = -\mathrm{i}\sigma^{1\,\alpha\gamma}\sigma^2_{\gamma\beta}$.

The bosonic generators P, K, D, and L_a are skew-Hermitian while $(R^1{}_1)^\dagger = R^2{}_2$ and $(R^1{}_2)^\dagger = -R^2{}_1$. The fermionic generators enjoy the reality conditions $Q_1^\dagger = \hat{Q}_2$, $Q_2^\dagger = -\hat{Q}_1$ and $S_1^\dagger = -\hat{S}_2$, $S_2^\dagger = \hat{S}_1$. It is straightforward to check that the superalgebra (2.26a) is invariant under these reality conditions. Notice also that in the limit $c^2 \to 0$, we recover the isometry superalgebra of $\mathrm{AdS}_2 \times S^2$, $\mathfrak{psu}(1,1|2)$, and the generators $L_\pm$ and L_3 decouple. This is consistent with the fact that $\mathrm{AdS}_2 \times S^2$ is T-self-dual without performing T-duality along S^2.

Furthermore, the non-vanishing components of the invariant form of $\mathfrak{d}(2,1;c^2)$ that is compatible with the above choice of the basis are

$$\begin{aligned} \mathrm{Str}(PK) &= 2\ , \quad \mathrm{Str}(DD) = 1\ , \\ \mathrm{Str}(L_+L_-) &= -\frac{2}{c^2}\ , \quad \mathrm{Str}(L_3L_3) = \frac{1}{c^2}\ , \\ \mathrm{Str}(R^\alpha{}_\beta R^\gamma{}_\delta) &= \frac{2}{s^2}\big(\delta^\alpha{}_\delta\delta^\gamma{}_\beta - \frac{1}{2}\delta^\alpha{}_\beta\delta^\gamma{}_\delta\big)\ , \\ \mathrm{Str}(Q_\alpha S_\beta) &= -2\sigma^2_{\alpha\beta}\ , \quad \mathrm{Str}(\hat{Q}_\alpha\hat{S}_\beta) = 2\sigma^2_{\alpha\beta}\ . \end{aligned} \tag{2.27}$$

$\mathbb{Z}_4$-grading and order-4 automorphism. In order to formulate the supercoset action based on (2.24), we need to fix a $\mathbb{Z}_4$-grading of the superalgebra $\mathfrak{d}(2,1;c^2)\otimes\mathbb{C} \cong \bigoplus_{m=0}^3 \mathfrak{g}_{(m)}$. In view of (2.9) we choose the following decomposition

$$\begin{aligned} \mathfrak{g}_{(0)} &:= \big\langle P+K, L_+ + L_-, \sigma^1_{\gamma[\alpha}R^\gamma{}_{\beta]}\big\rangle\ , \\ \mathfrak{g}_{(1)} &:= \big\langle Q_\alpha - \sigma^{1\,\beta}{}_\alpha S_\beta, \hat{Q}_\alpha - \sigma^{1\,\beta}{}_\alpha\hat{S}_\beta\big\rangle\ , \\ \mathfrak{g}_{(2)} &:= \big\langle P-K, D, L_+ - L_-, L_3, \sigma^1_{\gamma(\alpha}R^\gamma{}_{\beta)}\big\rangle\ , \\ \mathfrak{g}_{(3)} &:= \big\langle Q_\alpha + \sigma^{1\,\beta}{}_\alpha S_\beta, \hat{Q}_\alpha + \sigma^{1\,\beta}{}_\alpha\hat{S}_\beta\big\rangle\ , \end{aligned} \tag{2.28a}$$

where brackets (respectively, parentheses) indicate normalised anti-symmetrisation (respectively, symmetrisation) of the enclosed indices. Notice that we have indeed $\mathfrak{g}_{(0)} \cong \mathfrak{so}(1,1)\oplus\mathfrak{so}(2)\oplus\mathfrak{so}(2)$. The order-4 automorphism $\Omega : \mathfrak{d}(2,1;c^2) \to \mathfrak{d}(2,1;c^2)$ associated with this $\mathbb{Z}_4$-grading is given explicitly by

$$\begin{aligned} \Omega(P) &= K\ , \quad \Omega(K) = P\ , \quad \Omega(D) = -D\ , \\ \Omega(L_3) &= -L_3\ , \quad \Omega(L_\pm) = L_\mp\ , \quad \Omega(R^\alpha{}_\beta) = \sigma^{1\,\alpha\gamma}\sigma^1_{\beta\delta}R^\delta{}_\gamma\ , \\ \Omega(Q_\alpha) &= -\mathrm{i}\sigma^{1\,\beta}{}_\alpha S_\beta\ , \quad \Omega(\hat{Q}_\alpha) = -\mathrm{i}\sigma^{1\,\beta}{}_\alpha\hat{S}_\beta\ , \\ \Omega(S_\alpha) &= -\mathrm{i}\sigma^{1\,\beta}{}_\alpha Q_\beta\ , \quad \Omega(\hat{S}_\alpha) = -\mathrm{i}\sigma^{1\,\beta}{}_\alpha\hat{Q}_\beta\ . \end{aligned} \tag{2.29}$$

Furthermore,

$$
\begin{aligned}
\mathrm{Str}\big[(P\pm K)(P\pm K)\big] \;&=\; \pm 4\;, \quad \mathrm{Str}(DD) \;=\; 1\;,\\
\mathrm{Str}\big[(L_+\pm L_-)(L_+\pm L_-)\big] \;&=\; \mp\frac{4}{c^2}\;, \quad \mathrm{Str}(L_3L_3) \;=\; \frac{1}{c^2}\;,\\
\mathrm{Str}\big[(\sigma^1_{\mu[\alpha}R^\mu{}_{\beta]})(\sigma^1_{\nu[\gamma}R^\nu{}_{\delta]})\big] \;&=\; -\frac{1}{s^2}\sigma^2_{\alpha\beta}\sigma^2_{\gamma\delta}\;,\\
\mathrm{Str}\big[(\sigma^1_{\mu(\alpha}R^\mu{}_{\beta)})(\sigma^1_{\nu(\gamma}R^\nu{}_{\delta)})\big] \;&=\; -\frac{1}{s^2}(\sigma^1_{\alpha\beta}\sigma^1_{\gamma\delta}-\sigma^1_{\alpha\gamma}\sigma^1_{\beta\delta}-\sigma^1_{\alpha\delta}\sigma^1_{\gamma\beta})\;,\\
\mathrm{Str}\big[(Q_\alpha\pm\sigma^{1\,\gamma}{}_\alpha S_\gamma)(Q_\beta\mp\sigma^{1\,\delta}{}_\beta S_\delta)\big] \;&=\; \mp 4\mathrm{i}\sigma^1_{\alpha\beta}\;,\\
\mathrm{Str}\big[(\hat Q_\alpha\pm\sigma^{1\,\gamma}{}_\alpha \hat S_\gamma)(\hat Q_\beta\mp\sigma^{1\,\delta}{}_\beta \hat S_\delta)\big] \;&=\; \pm 4\mathrm{i}\sigma^1_{\alpha\beta}
\end{aligned}
\tag{2.30}
$$

which follow from (2.27).

Coset representative and associated current. Next, we need to choose a coset representative g for the supercoset space (2.24). In view of (2.26b), the generators P, Q_α, and L_+ are in involution[8] and, consequently, are associated with the directions along which we will perform T-dualisation. Following our general discussion in Section 2.1., an appropriate form of the coset representative is[9]

$$
\begin{aligned}
g \;&:=\; \mathrm{e}^{xP+\theta^\alpha Q_\alpha+\lambda_+L_+}\,\mathrm{e}^{B}\,\mathrm{e}^{\xi^\alpha S_\alpha}\;,\\
\mathrm{e}^B \;&:=\; \mathrm{e}^{\hat\theta^\alpha\hat Q_\alpha+\hat\xi^\alpha\hat S_\alpha}\,|y|^D\,\mathrm{e}^{-\lambda_3L_3}\,\mathrm{e}^{-\rho^\beta{}_\alpha R^\alpha{}_\beta}\;.
\end{aligned}
\tag{2.31}
$$

Here, we assume that both λ_+ and λ_3 are complex. This is merely a technical assumption which will facilitate the T-duality transformations below. Hence, we are essentially dealing with the complexification $SL(2,\mathbb{C})/\mathbb{C}^*$ of the coset $SO(3)/SO(2)\cong SU(2)/U(1)\cong S^2$, and from the point of view of fermionic T-duality, such a complexification is rather natural (see [67] for a similar case in $\mathrm{AdS}_5\times S^5$). Note that the resulting line element on $SL(2,\mathbb{C})/\mathbb{C}^*$ is

$$
(\mathrm{d}s)^2 \;=\; \frac{1}{4c^2}\big[(\mathrm{d}\lambda_3)^2+\mathrm{e}^{2\mathrm{i}\lambda_3}(\mathrm{d}\lambda_+)^2\big]\;. \tag{2.32}
$$

Upon performing the change of coordinates $(\lambda_+,\lambda_3)\mapsto(\varphi,\vartheta)$,

$$
\begin{aligned}
\lambda_+ \;&=\; \frac{2\tan(\frac{\vartheta}{2})\sin(\varphi)}{1+2\mathrm{i}\tan(\frac{\vartheta}{2})\cos(\varphi)-\tan^2(\frac{\vartheta}{2})}\;,\\
\mathrm{e}^{-\mathrm{i}\lambda_3} \;&=\; \frac{1+\tan^2(\frac{\vartheta}{2})}{1+2\mathrm{i}\tan(\frac{\vartheta}{2})\cos(\varphi)-\tan^2(\frac{\vartheta}{2})}
\end{aligned}
\tag{2.33}
$$

for $\varphi,\vartheta\in\mathbb{C}$, we find the line element

$$
(\mathrm{d}s)^2 \;=\; \frac{1}{4c^2}\big[(\mathrm{d}\vartheta)^2+\sin^2(\vartheta)\,(\mathrm{d}\varphi)^2\big]\;, \tag{2.34}
$$

[8] Note that the maximal Abelian subalgebra of $\mathfrak{d}(2,1;c^2)$ has two bosonic and two fermionic generators.

[9] To obtain the coset representative for $\mathrm{AdS}_2\times S^2\times T^2$ from the representative (2.31) in the limit $c\to 0$, one first needs to re-scale the coordinates $\lambda_+\to c\lambda_+$, $\lambda_3\to c\lambda_3$, and $\rho^\alpha{}_\beta\to s\rho^\alpha{}_\beta$ and then perform the limit. In this limit, the second sphere S^2, whose metric becomes flat, decouples from the $\mathrm{AdS}_2\times S^2$ supercoset and re-compactifies into T^2 which is part of T^6 of the backgrounds discussed in Section 5.

which, upon considering the real slice $\varphi^* = \varphi$ and $\vartheta^* = \vartheta$, becomes the standard line element on the two-sphere S^2.

The Maurer–Cartan form $J = g^{-1}\mathrm{d}g$ corresponding to the coset representative (2.31) is of the form

$$\begin{aligned} J &= \mathrm{e}^{-\xi^\alpha S_\alpha} J^{(0)} \mathrm{e}^{\xi^\alpha S_\alpha} + \mathrm{d}\xi^\alpha S_\alpha \\ &= J^{(0)} - \xi^\alpha \big[S_\alpha, J^{(0)}\big] + \frac{\mathrm{i}}{4}\xi^2 \sigma^{2\,\alpha\beta}\big\{S_\alpha, \big[S_\beta, J^{(0)}\big]\big\} + \mathrm{d}\xi^\alpha S_\alpha \ , \end{aligned} \tag{2.35}$$

where, as before, $J^{(0)}$ does not depend on the fermionic coordinate ξ^α, and we have set $\xi^2 := \mathrm{i}\sigma^2_{\alpha\beta}\xi^\alpha\xi^\beta$. The explicit form of the components of the current J is given in [166].

Using the $\mathbb{Z}_4$-grading (2.28a), the coset current J decomposes according to $J = J_{(0)} + J_{(1)} + J_{(2)} + J_{(3)}$ with

$$\begin{aligned} J_{(0)} &= \frac{1}{2}(J_P + J_K)(P + K) + \frac{1}{2}(J_{L_+} + J_{L_-})(L_+ + L_-) - J_{R^\alpha{}_\beta}\sigma^{1\,\alpha\gamma}\sigma^1_{\delta[\gamma} R^\delta{}_{\beta]} \ , \\ J_{(1)} &= \frac{1}{2}(J_{Q_\alpha} - \sigma^{1\,\alpha}{}_\beta J_{S_\beta})(Q_\alpha - \sigma^{1\,\beta}{}_\alpha S_\beta)\frac{1}{2}(J_{\hat{Q}_\alpha} - \sigma^{1\,\alpha}{}_\beta J_{\hat{S}_\beta})(\hat{Q}_\alpha - \sigma^{1\,\beta}{}_\alpha \hat{S}_\beta) \ , \\ J_{(2)} &= \frac{1}{2}(J_P - J_K)(P - K) + J_D D + \frac{1}{2}(J_{L_+} - J_{L_-})(L_+ - L_-) + J_{L_3}L_3 - \\ &\quad - J_{R^\alpha{}_\beta}\sigma^{1\,\alpha\gamma}\sigma^1_{\delta(\gamma} R^\delta{}_{\beta)} \ , \\ J_{(3)} &= \frac{1}{2}(J_{Q_\alpha} + \sigma^{1\,\alpha}{}_\beta J_{S_\beta})(Q_\alpha + \sigma^{1\,\beta}{}_\alpha S_\beta) + \frac{1}{2}(J_{\hat{Q}_\alpha} + \sigma^{1\,\alpha}{}_\beta J_{\hat{S}_\beta})(\hat{Q}_\alpha + \sigma^{1\,\beta}{}_\alpha \hat{S}_\beta) \ . \end{aligned} \tag{2.36}$$

Supercoset action. Upon using the $\mathbb{Z}_4$-grading (2.28a) together with the invariant form (2.27) and the currents (2.36), the sigma model action (2.14) becomes

$$\begin{aligned} S = -\frac{T}{2}\int_\Sigma \Big\{ & - *(J_P - J_K)\wedge(J_P - J_K) + *J_D \wedge J_D + \\ & + \frac{1}{c^2} * (J_{L_+} - J_{L_-})\wedge(J_{L_+} - J_{L_-}) + \frac{1}{c^2} *J_{L_3}\wedge J_{L_3} + \\ & + \frac{1}{s^2}(*J_{R^\alpha{}_\beta}\wedge J_{R^\beta{}_\alpha} - \sigma^{1\,\alpha\gamma}\sigma^1_{\beta\delta} *J_{R^\alpha{}_\beta}\wedge J_{R^\gamma{}_\delta}) - \\ & - \mathrm{i}\sigma^1_{\alpha\beta}\big(J_{Q_\alpha}\wedge J_{Q_\beta} + J_{S_\alpha}\wedge J_{S_\beta} - J_{\hat{Q}_\alpha}\wedge J_{\hat{Q}_\beta} - J_{\hat{S}_\alpha}\wedge J_{\hat{S}_\beta}\big)\Big\} \ . \end{aligned} \tag{2.37}$$

Note that in the $\mathrm{AdS}_2 \times S^2$ case, the matrix $(-\sigma^1)$ can be identified with the matrix $\Gamma^4\mathbb{P}$ along the AdS_2 radial direction with $\mathbb{P}$ being the projector which singles out eight unbroken supersymmetries of the background under consideration.

T-dualisation. Now, performing the T-dualisation of the action (2.37) following the general procedure described in Section 2.1., upon some technically involved algebra, a field re-definition and using the Maurer–Cartan equations one can check that the resulting dual action has the same form as the initial one but with the currents constructed with the different coset element

$$\tilde{g} := \mathrm{e}^{\tilde{x}K - \mathrm{i}\sigma^{2\,\alpha\beta}\tilde{\theta}_\alpha S_\beta + \tilde{\lambda}_+ L_-}\, \mathrm{e}^B\, \mathrm{e}^{\sigma^{1\beta}{}_\alpha \xi^\alpha Q_\beta} \ , \tag{2.38}$$

where, e^B is the same as in (2.31). Therefore, the supercoset sigma model on $\mathrm{AdS}_2 \times S^2 \times S^2$ is self-dual under the combined T-dualities along x, θ^α, and λ_+.

In the limit $c^2 \to 0$, upon an appropriate re-scaling of the J_L-currents, the action reduces to the $\frac{PSU(1,1|2)}{SO(1,1)\times U(1)}$ supercoset sigma model. In this limit, the dualised sphere S^2 gets 'decompactified' into a T^2 torus which completely decouples from the $\mathrm{AdS}_2 \times S^2$ and fermionic sector.

2.2.2. Self-duality for $\mathrm{AdS}_3 \times S^3 \times S^3$

Considering the subsector of the $\mathrm{AdS}_3 \times S^3 \times S^3 \times S^1$ theory in which the string moves only in $\mathrm{AdS}_3 \times S^3 \times S^3$ while its non-coset fermionic modes are gauge fixed to zero and the S^1-fluctuations decouple from the rest (modulo the Virasoro constraints), the T-dualisation process is almost identical to the just-presented discussion in the $\mathrm{AdS}_2 \times S^2 \times S^2$ case and we refrain from giving any details here.

The supercoset sigma model on $\mathrm{AdS}_3 \times S^3 \times S^3$ is based on the supercoset

$$\frac{\mathrm{D}(2,1;c^2)\times \mathrm{D}(2,1;c^2)}{SO(1,2)\times SO(3)\times SO(3)}\ . \tag{2.39}$$

The Lie superalgebra $\mathfrak{d}(2,1;c^2)\oplus\mathfrak{d}(2,1;c^2)$ (whose 10-dimensional form can be found in [149]) has $\{P_m, D, M, K_m, L^\pm_a, R^{\pm\, i}{}_j\}$ for $m = 0, 1$, $a = 1, 2, 3$, and $i, j = 1, 2$ as its bosonic generators and $\{Q_{i\alpha}, S_{i\alpha}, \hat{Q}_{i\alpha}, \hat{S}_{i\alpha}\}$ for $\alpha = 1, 2$ as its fermionic generators, respectively. Here, the $L^\pm_a$ and $R^{\pm\, i}{}_j$ are the generators of $\mathfrak{so}(3)\oplus\mathfrak{so}(3)\oplus\mathfrak{so}(3)\oplus\mathfrak{so}(3)$. Furthermore, the generators $\{P_m, Q_{i\alpha}, L^\pm := \mathrm{i}L^\pm_1 + L^\pm_2\}$ are in involution[10] so that the coset representative (2.10) will have the left factor of the form $\mathrm{e}^{x^m P_m+\theta^{i\alpha}Q_{i\alpha}+\lambda_+L^+ +\lambda_- L^-}$.[11] The coordinates x^m parametrize the 2-dimensional Minkowski boundary of AdS_3. Furthermore, as in the $\mathrm{AdS}_2\times S^2\times S^2$ case, we shall work with the complexification $SO(4,\mathbb{C})/SO(3,\mathbb{C})$ of $SO(4)/SO(3) \cong [SU(2)\times SU(2)]/SU(2) \cong SU(2) \cong S^3$ and consequently, the coordinates $\lambda_\pm$ are assumed to be complex. The resulting line element on $SO(4,\mathbb{C})/SO(3,\mathbb{C})$ will be of the form

$$(\mathrm{d}s)^2 \;=\; \frac{1}{4c^2}\left[(\mathrm{d}\lambda_3)^2 + \mathrm{e}^{2\mathrm{i}\lambda_3}(\mathrm{d}\lambda_+)^2 + \mathrm{e}^{2\mathrm{i}\lambda_3}(\mathrm{d}\lambda_-)^2\right]\ . \tag{2.40}$$

Upon choosing an appropriate $\mathbb{Z}_4$-grading for (2.39), T-duality is then performed along the bosonic directions x^m and $\lambda_\pm$ and the fermionic directions $\theta^{i\alpha}$ following the same steps as in the previous subsection. The T-self-duality of the supercoset sigma model on $\mathrm{AdS}_3 \times S^3 \times S^3$ then follows. We have explicitly checked this up to the second order in the four-component fermions $\xi^{i\alpha}$, like in the $\mathrm{AdS}_2 \times S^2 \times S^2$ case. We believe that the invariance holds to the highest (4th-order) in $\xi^{i\alpha}$. This is supported by the fact that at $\alpha = 0$, the model reduces to the $\mathrm{AdS}_3 \times S^3$ supercoset sigma model times the torus sector, which have proved to be duality invariant.

[10] Note that the maximal Abelian subalgebra of $\mathfrak{d}(2,1;c^2)\oplus\mathfrak{d}(2,1;c^2)$ has four bosonic and four fermionic generators.

[11] See also our general discussion given in Section 2.1..

Part II

Integrable S-matrices and AdS_2 Superstrings

3

The AdS$_2 \times S^2$ Worldsheet S-matrix

3.1. Symmetries of the massive modes of AdS$_2 \times S^2 \times T^6$

The BMN light-cone gauge $AdS_2 \times S^2 \times T^6$ superstring action describes $2+2$ massive and $6+6$ massless modes. The algebra underlying the scattering of the massive modes is expected to be $\mathfrak{psu}(1|1)^2 \ltimes \mathbb{R}^3$, which is found by considering the subalgebra of $\mathfrak{psu}(1,1|2)$ that is preserved by the BMN geodesic. We expect two additional central extensions to appear, by analogy with the $AdS_5 \times S^5$ case, in the decompactification limit and relaxing the level-matching condition.

Although a full off-shell analysis, as in [121], would be necessary (and is planned for future work) to confirm the nature of the central extensions, in this Chapter we construct the massive S-matrix on the basis of certain assumptions. The first assumption is the analogy with higher dimensional AdS/CFT integrable systems, and in particular the way the central extensions manifest themselves. The second crucial assumption is integrability itself. On the one hand, integrability should work to complete the perturbative results into the structure of classified representations of superalgebras. On the other hand, it should maintain the tree-level factorized form of the S-matrix at higher string loops.

With these assumptions in mind, we will nevertheless pursue a broad approach and explore the most general central extension based on the available kinematical algebra. We denote the massive boson associated to the transverse direction of S^2 as y and the corresponding boson for AdS_2 as z. The two massive fermions will be represented as two real Grassmann fields ζ and χ. We can then formally define the following tensor product states

$$\begin{aligned} |y\rangle &= |\phi\rangle \otimes |\phi\rangle \ , \qquad & |z\rangle &= |\psi\rangle \otimes |\psi\rangle \ , \\ |\zeta\rangle &= |\phi\rangle \otimes |\psi\rangle \ , \qquad & |\chi\rangle &= |\psi\rangle \otimes |\phi\rangle \ , \end{aligned} \tag{3.1}$$

where ϕ is bosonic and ψ is fermionic, such that we expect one of the factors of $\mathfrak{psu}(1|1)$ to act on each of the two entries. Furthermore, as a consequence of the form of the symmetry algebra and the integrability of the theory [49,181] we expect that the S-matrix for y, z, ζ and χ can be constructed as a graded tensor product of an S-matrix for ϕ and ψ, with each factor S-matrix invariant under the symmetry $\mathfrak{psu}(1|1) \ltimes \mathbb{R}^3$.

In this section we will construct the relevant massive representation of $\mathfrak{psu}(1|1) \ltimes \mathbb{R}^3$. This representation has an obvious massless limit, and, by analogy with the construction for $AdS_3 \times S^3 \times T^4$ [182], one may expect the massless modes to also transform in representations of $\mathfrak{psu}(1|1) \ltimes \mathbb{R}^3$ in the light-cone gauge-fixed theory. The massless limit is discussed in detail in section 3.4..

Let us also briefly mention that there is an additional $U(1)$ outer automorphism symmetry [183] of the S-matrix (3.56), under which the $\mathfrak{psu}(1|1)$ factors transform in the vector representation. The origin of this $U(1)$ symmetry is the T^6 compact space that is required for a consistent 10-d superstring theory. Under this symmetry $(\zeta, \chi)^T$ also transforms as a vector, while the bosons are uncharged. It is worth noting that taking the tensor product of two copies of any S-matrix for ϕ and ψ preserving the value of $(-1)^F$, where F is the fermion number operator, we find that the $U(1)$ symmetry is present so long as a certain quadratic relation between the parametrizing functions is satisfied (see [184]). In the case of interest, this quadratic identity turns out to be true just from demanding invariance under the $\mathfrak{psu}(1|1) \ltimes \mathbb{R}^3$ symmetry and satisfaction of the Yang-Baxter equation. The $U(1)$ does not act in a well-defined way on the individual factor S-matrices and hence for now we will ignore it. We will reconsider it in section 3.5., where it will play a role in defining a pseudovacuum, an important first step in the algebraic Bethe ansatz.

3.1.1. The $\mathfrak{gl}(1|1)$ Lie superalgebra and its representations

Let us start by summarizing the relevant information from [185] regarding the Lie superalgebra $\mathfrak{gl}(1|1)$ and its representations. There are two bosonic generators $\mathfrak{N}$ and $\mathfrak{C}$, with $\mathfrak{C}$ central, and two fermionic generators $\mathfrak{Q}$ and $\mathfrak{S}$. The commutation relations read

$$[\mathfrak{N}, \mathfrak{Q}] = -\mathfrak{Q} , \qquad [\mathfrak{N}, \mathfrak{S}] = \mathfrak{S} , \qquad \{\mathfrak{Q}, \mathfrak{S}\} = 2\mathfrak{C} . \tag{3.2}$$

The typical (long) irreps are the 2-dimensional *Kac modules* $\langle C, \nu\rangle$, defined by the following non-zero entries on a boson-fermion $(|\phi\rangle, |\psi\rangle)$ pair of states:

$$\begin{aligned} &\mathfrak{Q}\,|\phi\rangle = |\psi\rangle \ , \qquad \mathfrak{S}\,|\psi\rangle = 2\,C\,|\phi\rangle \ , \qquad \mathfrak{N}\,|\phi\rangle = \nu\,|\phi\rangle \ , \qquad \mathfrak{N}\,|\psi\rangle = (\nu-1)\,|\psi\rangle \ , \\ &\mathfrak{C}\,|\Phi\rangle = C\,|\Phi\rangle \qquad \forall\ |\Phi\rangle \in \{|\phi\rangle, |\psi\rangle\}, \qquad C, \nu \in \mathbb{C}, \qquad C \neq 0. \end{aligned} \tag{3.3}$$

We have summarized the generator action in figure 3.1. As long as $C \neq 0$, this module is isomorphic to the *anti-Kac module* $\overline{\langle C, \nu\rangle}$

$$\begin{aligned} &\mathfrak{Q}\,|\psi\rangle = 2\,C\,|\phi\rangle \ , \qquad \mathfrak{S}\,|\phi\rangle = |\psi\rangle \ , \qquad \mathfrak{N}\,|\phi\rangle = (\nu-1)\,|\phi\rangle \ , \qquad \mathfrak{N}\,|\psi\rangle = \nu\,|\psi\rangle \ , \\ &\mathfrak{C}\,|\Phi\rangle = C\,|\Phi\rangle \qquad \forall\ |\Phi\rangle \in \{|\phi\rangle, |\psi\rangle\}, \qquad C, \nu \in \mathbb{C}, \qquad C \neq 0. \end{aligned} \tag{3.4}$$

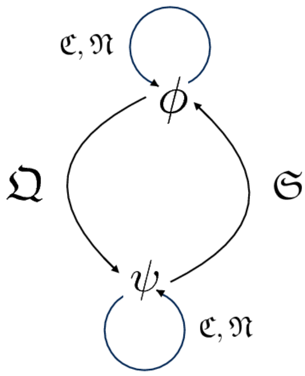

Figure 3.1: The Kac module $\langle C, \nu \rangle$.

However, if $C = 0$, the two modules are not isomorphic and they are no longer irreducible. Rather they become reducible but indecomposable.

To elucidate further we introduce the 1-dimensional modules $\langle \mu \rangle$, which form the atypical (short) irreps of $\mathfrak{gl}(1|1)$. These irreps are characterized by the vanishing of all generators except $\mathfrak{N}$, which acts with eigenvalue μ. We then see that for the Kac module, $\langle 0, \nu \rangle$, the fermion $|\psi\rangle$ spans a sub-representation $\langle \nu - 1 \rangle$, and the indecomposable is denoted as

$$\langle \nu - 1 \rangle \longleftarrow \langle \nu \rangle \ , \tag{3.5}$$

where the arrow represents module homomorphism and the diagram indicates that $\langle \nu - 1 \rangle$ is an irreducible sub-representation of the Kac module [186, 187]. On the other hand, the anti-Kac module $\overline{\langle 0, \nu \rangle}$ is also reducible but indecomposable and is denoted as

$$\langle \nu - 1 \rangle \longrightarrow \langle \nu \rangle \ , \tag{3.6}$$

with the fermion $|\psi\rangle$ once again spanning the sub-representation $\langle \nu \rangle$. This indecomposable is *not* isomorphic to $\langle 0, \nu \rangle$. Let us mention that modding out the indecomposable representations by their sub-representations one obtains the *factor* representations, which in this case are isomorphic to the short 1-dimensional $\langle \mu \rangle$ modules and are spanned by the boson $|\phi\rangle$.

If we take the tensor product of two typical modules, we get

$$\begin{aligned} \langle C_1, \nu_1 \rangle \otimes \langle C_2, \nu_2 \rangle &= \langle C_1 + C_2, \nu_1 + \nu_2 - 1 \rangle \oplus \langle C_1 + C_2, \nu_1 + \nu_2 \rangle \qquad \text{if } C_1 + C_2 \neq 0 \ , \\ \langle C_1, \nu_1 \rangle \otimes \langle -C_1, \nu_2 \rangle &= P_{\nu_1 + \nu_2} \ , \end{aligned} \tag{3.7}$$

where P_ν is the so-called *projective module*

$$\langle \nu \rangle \longrightarrow \langle \nu + 1 \rangle \oplus \langle \nu - 1 \rangle \longrightarrow \langle \nu \rangle \ , \tag{3.8}$$

on which $\mathfrak{C}$ acts identically as zero. The rightmost 1-dimensional short sub-module $\langle \nu \rangle$ is known as the *socle* of P_ν and is the only irreducible one-dimensional sub representation of the module. The multiple arrows indicate that, by taking the quotient with respect to $\langle \nu \rangle$, the module is again reducible.

Since $\mathfrak{N}$ does not appear on the r.h.s. of the commutation relations, the algebra $\mathfrak{gl}(1|1)$ has a non-trivial ideal generated by $\mathfrak{Q}$, $\mathfrak{S}$ and $\mathfrak{C}$. This ideal is the superalgebra $\mathfrak{sl}(1|1)$. Furthermore, this algebra is also not simple, as $\mathfrak{C}$, being central, is a non-trivial ideal. Additionally modding out $\mathfrak{C}$ gives the algebra $\mathfrak{psl}(1|1)$, which is still not simple, as the two remaining anti-commuting fermionic generators each form a separate ideal. The fact that $\mathfrak{psl}(1|1)$ is not simple sets this algebra outside the classification of the possible central extensions of basic classical Lie superalgebras.

3.1.2. The centrally-extended $\mathfrak{psu}(1|1)$ Lie superalgebra

We are now ready to introduce the centrally-extended version of the algebra we discussed above, which, as anticipated by the discussion at the beginning of section 3.1., we conjecture to be relevant for the scattering of the massive modes of the $AdS_2 \times S^2 \times T^6$ superstring. The algebra $\mathfrak{psu}(1|1) \ltimes \mathbb{R}^3$ is defined by the commutation relations

$$\{\mathfrak{Q}, \mathfrak{Q}\} = 2\mathfrak{P} \ , \qquad \{\mathfrak{S}, \mathfrak{S}\} = 2\mathfrak{K} \ , \qquad \{\mathfrak{Q}, \mathfrak{S}\} = 2\mathfrak{C} \ . \tag{3.9}$$

The states $|\phi\rangle$ and $|\psi\rangle$, introduced in (3.1), then transform in the following representation:

$$\begin{aligned} &\mathfrak{Q}\,|\phi\rangle = a\,|\psi\rangle \ , \qquad &&\mathfrak{Q}\,|\psi\rangle = b\,|\phi\rangle \ , \qquad &&\mathfrak{S}\,|\phi\rangle = c\,|\psi\rangle \ , \qquad &&\mathfrak{S}\,|\psi\rangle = d\,|\phi\rangle \ , \\ &\mathfrak{C}\,|\Phi\rangle = C\,|\Phi\rangle \ , \qquad &&\mathfrak{P}\,|\Phi\rangle = P\,|\Phi\rangle \ , \qquad &&\mathfrak{K}\,|\Phi\rangle = K\,|\Phi\rangle \ . \end{aligned} \tag{3.10}$$

Here a, b, c, d, C, P and K are the representation parameters that will eventually be functions of the energy and momentum of the states. For the supersymmetry algebra to close the following conditions should be satisfied

$$ab = P \ , \qquad cd = K \ , \qquad ad + bc = 2C \ . \tag{3.11}$$

This representation corresponds to the typical (long) Kac module $\langle C, \nu \rangle$ discussed in the previous section. We have summarized the generator action in figure 3.2. We will be interested in a particular real form of the algebra (3.9), which is given by

$$\mathfrak{Q}^\dagger = \mathfrak{S} \ , \qquad \mathfrak{P}^\dagger = \mathfrak{K} \ , \qquad \mathfrak{C}^\dagger = \mathfrak{C} \ . \tag{3.12}$$

These relations further constrain the representation parameters as follows

$$a^* = d \ , \qquad b^* = c \ , \qquad C^* = C \ , \qquad P^* = K \ . \tag{3.13}$$

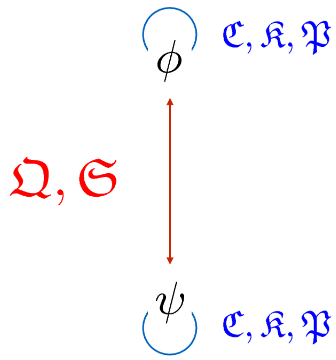

Figure 3.2: The 2-dimensional module of the centrally-extended algebra (several lines are superimposed).

The closure conditions (3.11) imply that

$$C^2 = \frac{(ad-bc)^2}{4} + PK \ . \tag{3.14}$$

Unlike the $AdS_5 \times S^5$ case, with the larger symmetry algebra $\mathfrak{psu}(2|2)^2 \ltimes \mathbb{R}^3$, here we are scattering long representations and hence there is no shortening condition – that is, $ad-bc$ is free to take any value, which we denote

$$m \equiv ad - bc \ . \tag{3.15}$$

The reality conditions (3.13) imply that m is real. From (3.14) we then have

$$(C+\frac{m}{2})(C-\frac{m}{2}) = PK > 0 \ , \tag{3.16}$$

also as a consequence of the reality conditions (3.13) . Motivated by the fact that C will later be associated to an energy, we will take it to be positive. However, let us point out that the algebraic analysis we perform in this Chapter is largely insensitive to this choice, and hence it does not represent a loss of generality. If we make this positivity assumption, it immediately follows that both $(C+\frac{m}{2})$ and $(C-\frac{m}{2})$ are also positive. The analogy with the higher dimensional AdS/CFT cases suggests that we should associate (the absolute value of) m with the mass of the scattering particle. Later it will be useful to solve the set of equations (3.11) for a, b, c and d in terms of m, C, P and K

$$\begin{aligned} a &= \alpha\, e^{-\frac{i\pi}{4}}\big(C+\frac{m}{2}\big)^{\frac{1}{2}} \ , & b &= \alpha^{-1} e^{\frac{i\pi}{4}}\big(C+\frac{m}{2}\big)^{-\frac{1}{2}} P \ , \\ c &= \alpha\, e^{-\frac{i\pi}{4}}\big(C+\frac{m}{2}\big)^{-\frac{1}{2}} K \ , & d &= \alpha^{-1} e^{\frac{i\pi}{4}}\big(C+\frac{m}{2}\big)^{\frac{1}{2}} \ . \end{aligned} \tag{3.17}$$

Here α is a phase parametrizing the normalization of the fermionic states with respect to the bosonic states and can be a function of the central extensions.

To define the action of this symmetry on the two-particle states we need to introduce the coproduct

$$\begin{aligned}
&\Delta(\mathfrak{Q}) = \mathfrak{Q} \otimes \mathbf{1} + \mathfrak{U} \otimes \mathfrak{Q} \,, && && \Delta(\mathfrak{S}) = \mathfrak{S} \otimes \mathbf{1} + \mathfrak{U}^{-1} \otimes \mathfrak{S} \,, \\
&\Delta(\mathfrak{P}) = \mathfrak{P} \otimes \mathbf{1} + \mathfrak{U}^2 \otimes \mathfrak{P} \,, && \Delta(\mathfrak{C}) = \mathfrak{C} \otimes \mathbf{1} + \mathbf{1} \otimes \mathfrak{C} \,, && \Delta(\mathfrak{K}) = \mathfrak{K} \otimes \mathbf{1} + \mathfrak{U}^{-2} \otimes \mathfrak{K} \,,
\end{aligned} \tag{3.18}$$

and the opposite coproduct, defined as

$$\Delta^{\mathrm{op}}(\mathfrak{J}) = \mathcal{P}\, \Delta(\mathfrak{J}) \,, \tag{3.19}$$

where $\mathfrak{J}$ is an arbitrary abstract generator (prior to considering a representation), and $\mathcal{P}$ defines the graded permutation of the tensor product.

The coproduct differs from the trivial one by the introduction of a new abelian generator $\mathfrak{U}$, with $\Delta(\mathfrak{U}) = \mathfrak{U} \otimes \mathfrak{U}$ [188]. This is done according to a $\mathbb{Z}$-grading of the algebra, whereby the charges $-2, -1, 1$ and 2 are associated to the generators $\mathfrak{K}$, $\mathfrak{S}$, $\mathfrak{Q}$ and $\mathfrak{P}$ respectively, while $\mathfrak{C}$ remains uncharged. The action of $\mathfrak{U}$ on the single-particle states is given by

$$\mathfrak{U}\,|\phi\rangle = U\,|\phi\rangle \;, \qquad \mathfrak{U}\,|\psi\rangle = U\,|\psi\rangle \;. \tag{3.20}$$

This braiding allows for the existence of a non-trivial S-matrix.

One important consequence of the non-trivial braiding (3.18) is that it leads to a constraint between U and the eigenvalues of the central charges. This follows from the requirement that, to admit an S-matrix, the coproduct of any central element should be equal to its opposite.[1] This implies

$$\mathfrak{P} \propto (1 - \mathfrak{U}^2) \;, \qquad \mathfrak{K} \propto (1 - \mathfrak{U}^{-2}) \;. \tag{3.21}$$

We fix the normalization of $\mathfrak{P}$ relative to $\mathfrak{K}$ by taking both constants of proportionality to be equal to $\frac{1}{2}\mathrm{h}$ where the reality conditions (3.13) require that h is real.[2] The parameter h is a coupling constant and eventually should be fixed in terms of the string tension, which we will return to in section 3.2.2.. Acting on the single-particle states then gives us the relations

$$P = \frac{\mathrm{h}}{2}\,(1 - U^2) \;, \qquad K = \frac{\mathrm{h}}{2}\,(1 - U^{-2}) \;, \tag{3.22}$$

[1] If $\Delta(\mathfrak{c})$ is central, then

$$\Delta^{op}(\mathfrak{c})\, R \;=\; R\, \Delta(\mathfrak{c}) = \Delta(\mathfrak{c})\, R \;,$$

which, for an invertible R-matrix, necessarily implies $\Delta^{op}(\mathfrak{c}) = \Delta(\mathfrak{c})$. This is expressed by saying that the coproduct of $\mathfrak{c}$ is *co-commutative.*

[2] The reality conditions (3.13) do allow for the introduction of an additional phase into the constants of proportionality, *i.e.* $\frac{1}{2}\mathrm{h}e^{i\varphi}$ and $\frac{1}{2}\mathrm{h}e^{-i\varphi}$. However, this phase does not appear in the S-matrix and thus we set $\varphi = 0$.

where U should satisfy, as a consequence of (3.13) , the following reality condition

$$U^* = U^{-1} \ . \tag{3.23}$$

The relation (3.14) in terms of C, U and m is then given by

$$C^2 = \frac{m^2 - \mathrm{h}^2(U - U^{-1})^2}{4} \ . \tag{3.24}$$

While this is a single equation for three undetermined parameters, we will later still attempt to interpret it as a dispersion relation with C, U and m defined in terms of just two kinematic variables, the energy and momentum. These precise definitions are not fixed by symmetry considerations, and hence should be found from direct string computations.

It is now useful to introduce the Zhukovsky variables $x^\pm$, in terms of which we will write the S-matrix, in place of the central extensions, C and U. These are defined as

$$U^2 = \frac{x^+}{x^-} \ , \qquad 2C + m = i\mathrm{h}(x^- - x^+) \ , \tag{3.25}$$

In these variables the dispersion relation (3.24) takes the following familiar form

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{2im}{\mathrm{h}} \ . \tag{3.26}$$

The representation parameters a, b, c and d in (3.17) and (3.32) are then given by

$$\begin{aligned} a &= \alpha\, e^{-\frac{i\pi}{4}} \sqrt[4]{\frac{x^+}{x^-}} \sqrt{\frac{\mathrm{h}}{2}}\, \eta \ , & b &= \alpha^{-1} e^{-\frac{i\pi}{4}} \sqrt[4]{\frac{x^-}{x^+}} \sqrt{\frac{\mathrm{h}}{2}}\, \frac{\eta}{x^-} \ , \\ c &= \alpha\, e^{\frac{i\pi}{4}} \sqrt[4]{\frac{x^+}{x^-}} \sqrt{\frac{\mathrm{h}}{2}}\, \frac{\eta}{x^+} \ , & d &= \alpha^{-1} e^{\frac{i\pi}{4}} \sqrt[4]{\frac{x^-}{x^+}} \sqrt{\frac{\mathrm{h}}{2}}\, \eta \ , \end{aligned} \tag{3.27}$$

where

$$\eta \equiv \sqrt{i(x^- - x^+)} \ . \tag{3.28}$$

Here we clearly see that the advantage of these variables is that the parameters a, b, c and d do not depend on m and hence, written as a function of $x^\pm$ and m, neither will the S-matrix. Finally, let us note that for the reality conditions (3.13) we have the usual $(x^\pm)^* = x^\mp$.

We could also eliminate the central extensions, C and U, in terms of two variables that will later be identified with the energy and momentum. Motivated by the $AdS_5 \times S^5$ case we write

$$C = \frac{\mathrm{e}}{2} \ , \qquad U = e^{\frac{i}{2}\mathrm{p}} \ , \tag{3.29}$$

where e is the energy and p is the spatial momentum. While the identification of e with the energy and p with the spatial momentum is at present only motivated by analogy with the

$AdS_5 \times S^5$ case, a posteriori it will be further justified by matching with perturbative results in section 3.2.2.. Solving for $x^{\pm}$ in terms of e and p we find

$$x^{\pm} = r\, U^{\pm 1}\ , \qquad r = \frac{\mathrm{e}+m}{2\mathrm{h}\,\sin\frac{\mathrm{p}}{2}} = \frac{2\mathrm{h}\,\sin\frac{\mathrm{p}}{2}}{\mathrm{e}-m}\ , \qquad U = e^{\frac{i\mathrm{p}}{2}}\ . \tag{3.30}$$

Using (3.22) and (3.29) we can substitute in for C, P and K in terms of the energy and the momentum in (3.24) to find the following familiar dispersion relation

$$\mathrm{e}^2 = m^2 + 4\,\mathrm{h}^2\,\sin^2\frac{\mathrm{p}}{2}\ . \tag{3.31}$$

It is important to emphasize that here m is algebraically a free parameter. However, for (3.31) to really be interpreted as a dispersion relation m should be fixed by the spectral analysis of the theory. In terms of the energy and the momentum the representation parameters a, b, c and d (3.17) are given by

$$\begin{aligned} a &= \frac{\alpha\, e^{\frac{i\mathrm{p}}{4}-\frac{i\pi}{4}}}{\sqrt{2}}\sqrt{\mathrm{e}+m}\ , & b &= \frac{\alpha^{-1} e^{-\frac{i\mathrm{p}}{4}+\frac{i\pi}{4}}}{\sqrt{2}}\frac{\mathrm{h}(1-e^{i\mathrm{p}})}{\sqrt{\mathrm{e}+m}}\ , \\ c &= \frac{\alpha\, e^{\frac{i\mathrm{p}}{4}-\frac{i\pi}{4}}}{\sqrt{2}}\frac{\mathrm{h}(1-e^{-i\mathrm{p}})}{\sqrt{\mathrm{e}+m}}\ , & d &= \frac{\alpha^{-1} e^{-\frac{i\mathrm{p}}{4}+\frac{i\pi}{4}}}{\sqrt{2}}\sqrt{\mathrm{e}+m}\ . \end{aligned} \tag{3.32}$$

In the $AdS_5 \times S^5$ and $AdS_3 \times S^3 \times M^4$ models, the choice of the phase factor α that is appropriate for the light-cone gauge-fixed string theory is

$$\alpha = 1\ . \tag{3.33}$$

As we will see, this is also a natural choice for α in the $AdS_2 \times S^2$ theory.

3.1.3. Tensor product of irreps and scattering theory

In this section we consider the tensor product of two of the irreps we discussed in the previous section, with the aim of constructing the relevant scattering theory. In particular, we want to investigate the persistence of the phenomenon observed for $\mathfrak{gl}(1|1)$ modules in section 3.1.1., namely complete reducibility of the tensor product of two 2-dimensional irreps, for *generic* values of the momenta, into two 2-dimensional irreps of the same type.

Let us proceed by constructing a 4-dimensional representation of the algebra (3.9). To do this we start with the bosonic state

$$|w_0\rangle\ . \tag{3.34}$$

Let us assume that the action of the central elements on this state is given by

$$(\mathfrak{P}, \mathfrak{K}, \mathfrak{C})|w_0\rangle = (P, K, C)|w_0\rangle\ . \tag{3.35}$$

This assumption will be justified by the concrete example we will consider later in our treatment of the scattering theory. We can then construct two fermionic states by considering the action of $\mathfrak{Q}$ and $\mathfrak{S}$

$$|w_1\rangle \equiv \mathfrak{Q}|w_0\rangle \ , \qquad |\tilde{w}_1\rangle \equiv \mathfrak{S}|w_0\rangle \ . \tag{3.36}$$

Using the definitions (3.36) and the fact that $\mathfrak{P}, \mathfrak{K}, \mathfrak{C}$ commute with the supercharges, the action of the central elements on the fermionic states reads

$$(\mathfrak{P}, \mathfrak{K}, \mathfrak{C})|w_1\rangle = (P, K, C)|w_1\rangle \ , \qquad (\mathfrak{P}, \mathfrak{K}, \mathfrak{C})|\tilde{w}_1\rangle = (P, K, C)|\tilde{w}_1\rangle \ . \tag{3.37}$$

We can then look at the action of $\mathfrak{Q}$ and $\mathfrak{S}$ on $|w_1\rangle$ and $|\tilde{w}_1\rangle$

$$\begin{aligned} \mathfrak{Q}|w_1\rangle &= P|w_0\rangle \ , \qquad \mathfrak{Q}|\tilde{w}_1\rangle = C|w_0\rangle + \frac{1}{2}[\mathfrak{Q}, \mathfrak{S}]|w_0\rangle \ , \\ \mathfrak{S}|\tilde{w}_1\rangle &= K|w_0\rangle \ , \qquad \mathfrak{S}|w_1\rangle = C|w_0\rangle - \frac{1}{2}[\mathfrak{Q}, \mathfrak{S}]|w_0\rangle \ . \end{aligned} \tag{3.38}$$

Here we see that we have generated one additional new state

$$|\tilde{w}_0\rangle \equiv \frac{1}{M}[\mathfrak{Q}, \mathfrak{S}]|w_0\rangle \ , \tag{3.39}$$

where we have chosen a normalization depending on

$$M \equiv 2\sqrt{C^2 - PK} \ . \tag{3.40}$$

Given the real form we are interested in, see eq. (3.12), and the assumption that $C^2 > PK$, or equivalently that M is real and non-zero (we will briefly discuss the case when M vanishes at the end of this section), the above normalization implies that $|\tilde{w}_0\rangle$ has the same norm as $|w_0\rangle$. Therefore, the action of $\mathfrak{Q}$ and $\mathfrak{S}$ on $|w_1\rangle$ and $|\tilde{w}_1\rangle$ is given by

$$\begin{aligned} \mathfrak{Q}|w_1\rangle &= P|w_0\rangle \ , \qquad \mathfrak{Q}|\tilde{w}_1\rangle = C|w_0\rangle + \frac{M}{2}|\tilde{w}_0\rangle \ , \\ \mathfrak{S}|\tilde{w}_1\rangle &= K|w_0\rangle \ , \qquad \mathfrak{S}|w_1\rangle = C|w_0\rangle - \frac{M}{2}|\tilde{w}_0\rangle \ . \end{aligned} \tag{3.41}$$

Again it is clear that the action of the central elements on $|\tilde{w}_0\rangle$ is given by

$$(\mathfrak{P}, \mathfrak{K}, \mathfrak{C})|\tilde{w}_0\rangle = (P, K, C)|\tilde{w}_0\rangle \ . \tag{3.42}$$

Finally, using the definitions of $|w_0\rangle, |\tilde{w}_0\rangle, |w_1\rangle, |\tilde{w}_1\rangle$ and the superalgebra commutation relations, the action of $\mathfrak{Q}$ and $\mathfrak{S}$ on $|\tilde{w}_0\rangle$ turns out to be

$$\mathfrak{Q}|\tilde{w}_0\rangle = \frac{2P}{M}|\tilde{w}_1\rangle - \frac{2C}{M}|w_1\rangle \ , \qquad \mathfrak{S}|\tilde{w}_0\rangle = -\frac{2K}{M}|w_1\rangle + \frac{2C}{M}|\tilde{w}_1\rangle \ . \tag{3.43}$$

Summarising, we have constructed the following 4-dimensional representation:

$$(\mathfrak{P}, \mathfrak{K}, \mathfrak{C})|\Phi\rangle = (P, K, C)|\Phi\rangle \ , \qquad \forall \ |\Phi\rangle \ \in \ \{|w_0\rangle, |w_1\rangle, |\tilde{w}_1\rangle, |\tilde{w}_0\rangle)\} \ ,$$

$$\begin{aligned}
\mathfrak{Q}|w_0\rangle &= |w_1\rangle\ , & \mathfrak{S}|w_0\rangle &= |\tilde{w}_1\rangle\ ,\\
\mathfrak{Q}|w_1\rangle &= P|w_0\rangle\ , & \mathfrak{S}|\tilde{w}_1\rangle &= K|w_0\rangle\ ,\\
\mathfrak{Q}|\tilde{w}_1\rangle &= C|w_0\rangle + \frac{M}{2}|\tilde{w}_0\rangle\ , & \mathfrak{S}|w_1\rangle &= C|w_0\rangle - \frac{M}{2}|\tilde{w}_0\rangle\ ,\\
\mathfrak{Q}|\tilde{w}_0\rangle &= \frac{2P}{M}|\tilde{w}_1\rangle - \frac{2C}{M}|w_1\rangle\ , & \mathfrak{S}|\tilde{w}_0\rangle &= -\frac{2K}{M}|w_1\rangle + \frac{2C}{M}|\tilde{w}_1\rangle\ .
\end{aligned} \tag{3.44}$$

We have summarized the situation in figure 3.3.

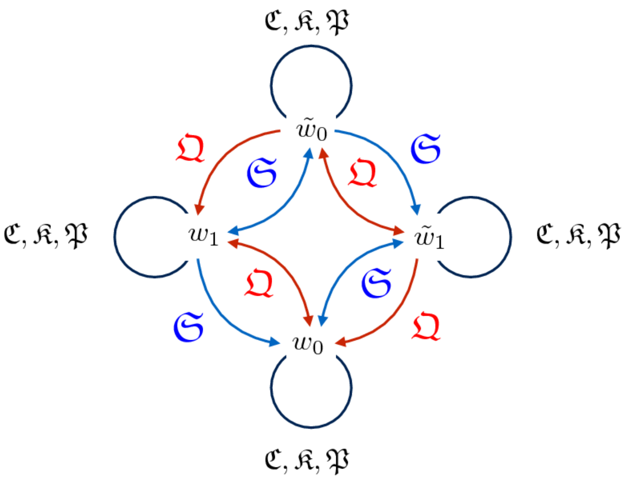

Figure 3.3: The 4-dimensional module of the centrally-extended algebra.

However, using the fact that

$$\mathfrak{Q}\mathfrak{S}|\tilde{w}_0\rangle = C|\tilde{w}_0\rangle + \frac{M}{2}|w_0\rangle\ , \qquad \mathfrak{Q}\mathfrak{S}|w_0\rangle = C|w_0\rangle + \frac{M}{2}|\tilde{w}_0\rangle\ , \tag{3.45}$$

$$\mathfrak{S}\mathfrak{Q}|\tilde{w}_0\rangle = C|\tilde{w}_0\rangle - \frac{M}{2}|w_0\rangle\ , \qquad \mathfrak{S}\mathfrak{Q}|w_0\rangle = C|w_0\rangle - \frac{M}{2}|\tilde{w}_0\rangle\ , \tag{3.46}$$

we see that defining the linear combinations

$$|\Phi_\pm\rangle = |w_0\rangle \pm |\tilde{w}_0\rangle\ , \tag{3.47}$$

implies

$$\mathfrak{Q}\mathfrak{S}|\Phi_\pm\rangle = \big(C + \frac{M}{2}\big)|\Phi_\pm\rangle\ , \qquad \mathfrak{S}\mathfrak{Q}|\Phi_\pm\rangle = \big(C - \frac{M}{2}\big)|\Phi_\pm\rangle\ . \tag{3.48}$$

Furthermore,

$$\mathfrak{Q}|\Phi_\pm\rangle = \mp\frac{2C \mp M}{M}|w_1\rangle \pm \frac{2P}{M}|\tilde{w}_1\rangle\ , \qquad \mathfrak{S}|\Phi_\pm\rangle = \pm\frac{2C \pm M}{M}|\tilde{w}_1\rangle \mp \frac{2K}{M}|w_1\rangle\ . \tag{3.49}$$

Using the definition of M (3.40) one can easily see that

$$\mathfrak{Q}|\Phi_\pm\rangle \propto \mathfrak{S}|\Phi_\pm\rangle \propto |\Psi_\pm\rangle \ , \tag{3.50}$$

and hence the 4-dimensional representation we constructed is actually reducible and is formed of two 2-dimensional representations

$$\{|\Phi_\pm\rangle, |\Psi_\pm\rangle\} \ . \tag{3.51}$$

To conclude, let us briefly mention orthogonality. Here we will make use of the real form of the algebra given in eq. (3.12), and the assumption that M is real. We then have

$$\langle\Phi_\mp|\Phi_\pm\rangle = \langle w_0|(\mathbf{1} + \frac{1}{M}([\mathfrak{Q},\mathfrak{S}] - [\mathfrak{Q},\mathfrak{S}]^\dagger) - \frac{1}{M^2}[\mathfrak{Q},\mathfrak{S}]^\dagger[\mathfrak{Q},\mathfrak{S}]|w_0\rangle \ . \tag{3.52}$$

Using the conjugation relations we find that $[\mathfrak{Q},\mathfrak{S}]^\dagger = [\mathfrak{Q},\mathfrak{S}]$. Furthermore, as $[\mathfrak{Q},\mathfrak{S}] = 2\mathfrak{C} - 2\mathfrak{S}\mathfrak{Q} = -2\mathfrak{C} + 2\mathfrak{Q}\mathfrak{S}$ we find

$$\begin{aligned}\langle\Phi_\mp|\Phi_\pm\rangle &= \langle w_0|\mathbf{1} + \frac{1}{M^2}(2\mathfrak{C} - 2\mathfrak{S}\mathfrak{Q})(2\mathfrak{C} - 2\mathfrak{Q}\mathfrak{S})|w_0\rangle = \langle w_0|\mathbf{1} - \frac{4}{M^2}(\mathfrak{C}^2 - \mathfrak{P}\mathfrak{K})|w_0\rangle \\ &= (1 - \frac{4(C^2 - PK)}{M^2})\langle w_0|w_0\rangle = 0 \ .\end{aligned} \tag{3.53}$$

Therefore, the two representations are orthogonal.

This construction can then be straightforwardly applied to the 4-dimensional representation arising as the tensor product of two of the 2-dimensional irreps of section 3.1.2.. Explicit details of this construction are given in [184] and will be particularly relevant for the scattering theory discussed in section 3.2.. In particular, it implies that the S-matrix for the scattering of two of the 2-dimensional irreps is not completely fixed by symmetries up to an overall factor.

Let us finally make the important observation that the arguments of this section cannot be applied for the $M = 0$ case (such as, for instance, the scattering of two massless particles with the momenta taken at the bound-state point [3]). In this case what we find is the analog of the projective indecomposable representation of section 3.1.1.. In particular, one can check that, at $M = 0$, the state $|\tilde{w}_0^{(0)}\rangle \equiv [\mathfrak{Q},\mathfrak{S}]|w_0\rangle$ is such that

$$\mathfrak{Q}\mathfrak{S}\,|\tilde{w}_0^{(0)}\rangle \;=\; \mathfrak{S}\mathfrak{Q}\,|\tilde{w}_0^{(0)}\rangle \;=\; C\,|\tilde{w}_0^{(0)}\rangle, \qquad \mathfrak{Q}\,|\tilde{w}_0^{(0)}\rangle \;\propto\; \mathfrak{S}\,|\tilde{w}_0^{(0)}\rangle, \tag{3.54}$$

where we have used $M^2 = 4(C^2 - PK) = 0$ to derive the last proportionality statement. However, this is the only state which satisfies these properties, meaning we do not have two solutions to these conditions (as we did in the $M \neq 0$ case above). Therefore, there is only one irreducible 2-dimensional block, containing the states $\{|\tilde{w}_0^{(0)}\rangle, \mathfrak{Q}|\tilde{w}_0^{(0)}\rangle\}$, and the 4-dimensional representation is reducible but not fully reducible (*i.e.* it is indecomposable).

[3] Here by *bound-state point* we simply mean the value of momenta such that $\Delta^2(C) - \Delta(P)\Delta(K) = (m_1+m_2)^2 = 0$, namely $\ell_{ac} = 0$ or $\ell_{bd} = 0$ (see [184] for details). In fact, it is not clear if there is a meaning of bound states for massless scattering [189].

3.2. S-matrix for the massive modes of $\mathrm{AdS}_2 \times S^2 \times T^6$

In this section we study the S-matrix for the massive modes of the light-cone gauge $AdS_2 \times S^2 \times T^6$ superstring. As mentioned in section 3.1. from the structure of the symmetry algebra and the integrability of the theory we expect the S-matrix for the massive fields y, z, ζ and χ to be constructed from the graded tensor product of two copies of an S-matrix describing the scattering of $1+1$ massive modes, ϕ and ψ. The former are defined in terms of the latter in (3.1).

The excitations ϕ and ψ should transform in the massive representation of $\mathfrak{psu}(1|1) \ltimes \mathbb{R}^3$ discussed in section 3.1.2.. Their S-matrix is then fixed by demanding invariance under this symmetry

$$\Delta^{op}(\mathfrak{J})\mathbb{S} = \mathbb{S}\Delta(\mathfrak{J}) \ . \tag{3.55}$$

Accounting for conservation of the value of $(-1)^F$, where F is the fermion number, the most general form for the S-matrix is

$$\begin{aligned} \mathbb{S}\left|\phi\phi'\right\rangle &= S_1\left|\phi\phi'\right\rangle + Q_1\left|\psi\psi'\right\rangle \ , \qquad & \mathbb{S}\left|\psi\psi'\right\rangle &= S_2\left|\psi\psi'\right\rangle + Q_2\left|\phi\phi'\right\rangle \ , \\ \mathbb{S}\left|\phi\psi'\right\rangle &= T_1\left|\phi\psi'\right\rangle + R_1\left|\psi\phi'\right\rangle \ , \qquad & \mathbb{S}\left|\psi\phi'\right\rangle &= T_2\left|\psi\phi'\right\rangle + R_2\left|\phi\psi'\right\rangle \ , \end{aligned} \tag{3.56}$$

where $x^\pm$, m are the kinematic variables associated to the first particle and $x'^\pm$, m' to the second particle, that is

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{2im}{\mathrm{h}} \ , \qquad x'^+ + \frac{1}{x'^+} - x'^- - \frac{1}{x'^-} = \frac{2im'}{\mathrm{h}} \ . \tag{3.57}$$

As a consequence of the discussion in section 3.1.3. this symmetry will only fix the S-matrix up to two arbitrary functions. One of these functions can be found by requiring the S-matrix also satisfies the Yang-Baxter equation along with additional physical requirements. There are four solutions to the Yang-Baxter equation, two of which we ignore as they violate crossing symmetry. The other two are related by a sign. To fix the sign, we demand that in the BMN limit (for details see section 3.2.2.) the S-matrix reduces to the identity operator. The functions parametrizing the exact S-matrix (3.56) are then given by[4]

$$\begin{aligned} S_1 &= \sqrt{\frac{x^+x'^-}{x^-x'^+}}\frac{x^- - x'^+}{x^+ - x'^-}\frac{1+s_1}{2}\tilde{\mathcal{P}}_0 \ , & S_2 &= \frac{1+s_2}{2}\tilde{\mathcal{P}}_0 \ , \\ T_1 &= \sqrt{\frac{x'^-}{x'^+}}\frac{x^+ - x'^+}{x^+ - x'^-}\frac{1+t_1}{2}\tilde{\mathcal{P}}_0 \ , & T_2 &= \sqrt{\frac{x^+}{x^-}}\frac{x^- - x'^-}{x^+ - x'^-}\frac{1+t_2}{2}\tilde{\mathcal{P}}_0 \ , \\ \frac{Q_1}{\alpha\alpha'} &= \alpha\alpha' Q_2 = -\frac{i}{2}\sqrt[4]{\frac{x^-x'^+}{x^+x'^-}}\frac{\eta\eta'}{x^+ - x'^-}\frac{f}{x^-x'^+}\tilde{\mathcal{P}}_0 \ , & \frac{\alpha'}{\alpha}R_1 &= \frac{\alpha}{\alpha'}R_2 = -\frac{i}{2}\sqrt[4]{\frac{x^+x'^-}{x^-x'^+}}\frac{\eta\eta'}{x^+ - x'^-}\tilde{\mathcal{P}}_0 \ , \end{aligned} \tag{3.58}$$

[4]Note that here we are choosing the branch so that $\left(\frac{x^-}{x^+}\right)^{\#} = \left(\frac{x^+}{x^-}\right)^{-\#}$ for $\# = \frac{1}{2}, \frac{1}{4}$ and similarly for $x'^\pm$. For $\mathrm{p} \in [-\pi, \pi]$ this corresponds to taking the branch cut on the negative real axis.

where

$$f = \frac{\sqrt{\frac{x^+}{x^-}}(x^- - \frac{1}{x^+}) - \sqrt{\frac{x'^+}{x'^-}}(x'^- - \frac{1}{x'^+})}{1 - \frac{1}{x^+x^-x'^+x'^-}} , \qquad s_1 = \frac{1 - \frac{1}{x^+x'^-}f}{x^- - x'^+} f , \qquad s_2 = \frac{1 - \frac{1}{x^-x'^+}}{x^+ - x'^-} f , \tag{3.59}$$

$$t_1 = \frac{1 - \frac{1}{x^-x'^-}}{x^+ - x'^+} f , \qquad t_2 = \frac{1 - \frac{1}{x^+x'^+}}{x^- - x'^-} f . \tag{3.60}$$

$\tilde{\mathcal{P}}_0$ is an overall factor that sits outside the matrix structure and is not fixed by symmetries or the Yang-Baxter equation. Let us emphasize that, as discussed beneath eq. (3.27), when written in these variables the S-matrix is independent of m and m', which can take any value. The limits $m \to 0$ and $m' \to 0$ are subtle however, and will be discussed in detail in section 3.4.. Let us also note that if we take α to be given by (3.33), which is the choice suitable for string theory, then $Q_1 = Q_2$ and $R_1 = R_2$. From now on we will take α to be given by this value.

The S-matrix (3.56) can be thought of as a 4×4 block diagonal matrix

$$\begin{pmatrix} S_1 & Q_1 & 0 & 0 \\ Q_2 & S_2 & 0 & 0 \\ 0 & 0 & T_1 & R_1 \\ 0 & 0 & R_2 & T_2 \end{pmatrix} . \tag{3.61}$$

One can then check that each of the two 2×2 blocks have equal trace and determinant,

$$S_1 + S_2 = T_1 + T_2 , \qquad S_1 S_2 - Q_1 Q_2 = T_1 T_2 - R_1 R_2 . \tag{3.62}$$

The second of these equations is particularly important as it implies the tensor product of two copies of the S-matrix possesses an additional $U(1)$ symmetry, which will be discussed further in Section 3.5.

For completeness let us note that the two solutions that violate crossing symmetry are given by $f = 0$ and $f \to \infty$ (for the latter one should first rescale $\tilde{\mathcal{P}}_0$ by f^{-1} and then take $f \to \infty$). As ϕ and ψ are real and the charge conjugation matrix diagonal, which will be demonstrated in the next section (cf. (3.77)), the two processes

$$\phi\,\phi \to \psi\,\psi \qquad \text{and} \qquad \phi\,\psi \to \psi\,\phi , \tag{3.63}$$

should be related by a crossing transformation. However, if f vanishes then so does the amplitude for the first of these processes, but not for the second. Similarly, if $f \to \infty$ then the amplitude for the second process vanishes, but not for the first. Consequently, in both cases the two processes cannot be related by a crossing transformation and hence there is a violation of crossing symmetry as claimed.

It is interesting to note that taking $f = 0$ and $f \to \infty$ we recover the massive S-matrices of the $AdS_3 \times S^3 \times T^4$ light-cone gauge superstring [190]. The symmetry is enhanced accordingly from $\mathfrak{psu}(1|1) \ltimes \mathbb{R}^3$ to $[\mathfrak{u}(1) \subset \mathfrak{psu}(1|1)^2] \ltimes \mathfrak{u}(1) \ltimes \mathbb{R}^3$. For the $AdS_3 \times S^3 \times T^4$ light-cone gauge-fixed theory there is no issue with crossing symmetry as the fields are complex. Therefore, the individual S-matrices do not map to themselves under the crossing transformation, rather to a different S-matrix with the crossed particle replaced by its antiparticle. Finally let us also point out that the S-matrix relevant for the $AdS_2 \times S^2 \times T^6$ light-cone gauge superstring, see eqs. (3.56) and (3.58), is a linear combination, with coefficients depending on $x^\pm$ and $x'^\pm$, of the $f = 0$ and $f \to \infty$ S-matrices. It is non-trivial that such a combination exists with unitarity, crossing symmetry and the Yang-Baxter equation all satisfied.

3.2.1. The overall factor and crossing symmetry

As currently written the factor $\tilde{\mathcal{P}}_0$ is neither a phase factor or antisymmetric. Indeed, given the reality conditions $(x^\pm)^* = x^\mp$ and $(x'^\pm)^* = x^\mp$, the functions f, $s_{1,2}$ and $t_{1,2}$ satisfy the following relations:

$$f^* = f \,, \qquad s^*_{1,2} = s_{2,1} \,, \qquad t^*_{1,2} = t_{2,1} \,, \tag{3.64}$$

$$f(x', x) = -f(x', x) \,, \qquad s_{1,2}(x', x) = s_{2,1}(x, x') \,, \qquad t_{1,2}(x', x) = t_{1,2}(x, x') \,. \tag{3.65}$$

Notice that, if we consider the $m = m'$ case, then on-shell (*i.e.* when the dispersion relations (3.57) are satisfied) we have $t_1 \approx t_2$. Given the reality conditions, this means in particular that t_1, t_2 are real.

Based on this, and as a consequence of braiding and QFT unitarity, the overall factor should satisfy [5]

$$\tilde{\mathcal{P}}_0 \tilde{\mathcal{P}}^*_0 = \tilde{\mathcal{P}}_0(x, x')\tilde{\mathcal{P}}_0(x', x) = \frac{4(x^- - x'^+)(x^+ - x'^-)}{(x^+ - x'^+)(x^- - x'^-)(1 + t_1)(1 + t_2) - (x^+ - x^-)(x'^+ - x'^-)} \equiv N(x, x') \,. \tag{3.66}$$

To isolate an antisymmetric phase factor, we can define $\mathcal{P}_0$ as follows:

$$\mathcal{P}_0 = \det \begin{pmatrix} S_1 & Q_1 \\ Q_2 & S_2 \end{pmatrix} = \det \begin{pmatrix} T_1 & R_1 \\ R_2 & T_2 \end{pmatrix} \equiv \exp i\theta(x, x'), \tag{3.67}$$

where $\theta(x, x')$ is an antisymmetric *phase shift*, *i.e.* $\theta(x, y) = -\theta(y, x)$, and the second equality follows from eq. (3.62). We then have that $\mathcal{P}_0$ is proportional to $\tilde{\mathcal{P}}^2_0$, and hence is a natural phase

[5]Note that the $AdS_5 \times S^5$ S-matrix contains copies of the 2×2 block:

$$\left.\begin{pmatrix} 2T_1 & 2R_1 \\ 2R_2 & 2T_2 \end{pmatrix}\right|_{t_i = 0} .$$

Taking into account the factor of 2, when $t_i = 0$ (3.66) simplifies to $\tilde{\mathcal{P}}_0 \tilde{\mathcal{P}}^*_0 = \tilde{\mathcal{P}}_0(x, x')\tilde{\mathcal{P}}_0(x', x) = 1$ so that $\tilde{\mathcal{P}}_0$ is an antisymmetric phase factor. This is the familiar $AdS_5 \times S^5$ story.

to consider recalling that the full S-matrix for the massive modes is given by the tensor product of two of the factor S-matrices (3.56). As claimed the unitarity conditions for $\mathcal{P}_0$ are then

$$\mathcal{P}_0 \mathcal{P}_0^{\,*} = \mathcal{P}_0(x,x')\mathcal{P}_0(x',x) = 1 \ . \tag{3.68}$$

Crossing symmetry provides an additional constraint on the overall factor $\tilde{\mathcal{P}}_0$, which takes the form

$$\tilde{\mathcal{P}}_0(x',\bar{x}) = s_2(x,x')\,\tilde{\mathcal{P}}_0(x,x') \ , \tag{3.69}$$

where the "crossed" Zhukovsky variables $\bar{x}^{\pm}$ are, as usual, given by

$$\bar{x}^{\pm} = \frac{1}{x^{\pm}} \ , \tag{3.70}$$

corresponding to $\bar{e} = -e$ and $\bar{\mathrm{p}} = -\mathrm{p}$. It is useful to note that we have the following identities

$$s_{1,2}(x',\bar{x}) = s_{1,2}^{-1}(x,x') \ , \qquad t_{1,2}(x',\bar{x}) = t_{2,1}^{-1}(x,x') \ . \tag{3.71}$$

Using the braiding unitarity relation (3.66) it is simple to recast (3.69) in the more familiar form

$$\tilde{\mathcal{P}}_0(x,x')\tilde{\mathcal{P}}_0(\bar{x},x') = \frac{N(\bar{x},x')}{s_2(x,x')} \ . \tag{3.72}$$

This relation then translates to the following rather complicated constraint for the antisymmetric phase factor $\mathcal{P}_0$

$$\mathcal{P}_0(x,x')\mathcal{P}_0(\bar{x},x') = \frac{S_1S_2 - Q_1Q_2}{S_1S_2 + R_1R_2} = \frac{T_1T_2 - R_1R_2}{T_1T_2 + Q_1Q_2} \equiv f_2(x,x') \ , \tag{3.73}$$

and hence it appears that we either have a simple crossing relation or simple unitarity relations.

Using Hopf algebra arguments, we have checked that crossing symmetry is present for the representation of interest for any value of m and m'. Denoting the symmetry algebra as $\mathcal{A}$, the antipode Σ is found from the defining rule

$$\mu\,(\Sigma \otimes \mathbf{1})\,\Delta = \eta\,\epsilon \ , \tag{3.74}$$

where μ is the multiplication map, $\eta : \mathbb{C} \to \mathcal{A}$ is the unit and $\epsilon : \mathcal{A} \to \mathbb{C}$ is the counit, which annihilates all generators apart from $\mathbf{1}$ and $e^{i\mathrm{p}}$ (acting on which, it returns 1). The antipode being a Lie algebra anti-homomorphism, we simply need to derive

$$\Sigma(\mathfrak{S}) = -e^{-i\frac{\mathrm{p}}{2}}\mathfrak{Q}, \qquad \Sigma(\mathfrak{Q}) = -e^{i\frac{\mathrm{p}}{2}}\mathfrak{S}, \qquad \Sigma(\mathbf{1}) = \mathbf{1}, \qquad \Sigma(e^{i\mathrm{p}}) = e^{-i\mathrm{p}} \ . \tag{3.75}$$

This map is idempotent and therefore equal to its inverse. We impose

$$\Sigma\big(\mathfrak{J}(x^{\pm})\big) = \mathscr{C}^{-1}\left[\mathfrak{J}\left(\frac{1}{x^{\pm}}\right)\right]^{st}\mathscr{C} \ , \tag{3.76}$$

where $\mathscr{C}$ is the charge conjugation matrix

$$\mathscr{C} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} , \tag{3.77}$$

and the label st denotes supertransposition. The fundamental crossing relation for an abstract R-matrix[6] is then given by (*cf.* [191])

$$(\Sigma \otimes \mathbf{1})R = R^{-1} = (\mathbf{1} \otimes \Sigma^{-1})R , \tag{3.78}$$

which projects into the representation of interest as

$$(\mathscr{C}^{-1} \otimes \mathbf{1})\mathbb{S}^{st_1}(\bar{x}, x')(\mathscr{C} \otimes \mathbf{1})\mathbb{S}(x, x') = \mathbf{1} \otimes \mathbf{1} , \tag{3.79}$$

and an analogous equation for the second factor. Here ^{st_i} denotes the supertranspose for factor i, and we are using the Hopf algebra convention for the S-matrix crossing [191] . The S-matrix (3.56) with parametrizing functions (3.58) satisfies this relation provided the overall factor satisfies the crossing equation given in (3.69).

It is important to note that the crossing equations given above are somewhat formal as we have not specified a path on the rapidity plane. To specify such a path we would need to know the precise form of the dispersion relation, and hence its uniformization. In particular, there is still the logical possibility that m and m' are themselves momentum-dependent functions (which should be invariant under crossing). This possibility would not alter the analysis we have performed so far. In the scenario that m and m' are non-vanishing and constant the dispersion relation becomes the same as in the $AdS_5 \times S^5$ light-cone gauge string theory and the analytic continuation should be the same as in that case [191–193].

In spite of our lack of knowledge of the complete dispersion relation, one thing we can investigate is double crossing [191].[7] In particular, the left-hand sides of (3.72) and (3.73) are symmetric under $x \leftrightarrow \bar{x}$, however the right-hand sides are not. This asymmetry encodes the fact that the overall factor should not be a meromorphic function of the parameters that uniformize the dispersion relation (generalized rapidities). Furthermore, as a consistency check, one can confirm that the following equality holds true:

$$\Omega(x, x') \equiv \frac{\tilde{\mathcal{P}}_0^2(x, x')}{\tilde{\mathcal{P}}_0^2(\bar{\bar{x}}, x')} \cdot \frac{\mathcal{P}_0(\bar{\bar{x}}, x')}{\mathcal{P}_0(x, x')} = \left(\frac{N(\bar{x}, x')}{s_2(x, x')}\right)^2 \cdot \left(\frac{s_2(\bar{x}, x')}{N(x, x')}\right)^2 \cdot \frac{f_2(\bar{x}, x')}{f_2(x, x')} = 1 . \tag{3.80}$$

This is obtained by comparing the ratio of the right-hand side of (3.72) (squared) to the same quantity with $x \to \bar{x}$, against the corresponding ratio for the right-hand side of (3.73). The fact

[6] For our purposes, S-matrices will be representations of abstract R-matrices.

[7] We would like to thank the referee for suggesting the consideration of double crossing.

that $\Omega = 1$ confirms that $\tilde{\mathcal{P}}_0^2$ and $\mathcal{P}_0$ differ only by a factor that behaves like a rational function under double crossing, as is expected.

It is easy to convince oneself that the ratio $\frac{f_2(\bar{x},x')}{f_2(x,x')}$ encodes the discontinuity of the overall S-matrix factor across branch cuts in the, as yet unknown, rapidity plane. It is of interest to note that this ratio differs from the corresponding one in the $AdS_5 \times S^5$ case, suggesting that the analytic structure of the $AdS_2 \times S^2$ light-cone gauge-fixed theories is not the same. To understand crossing symmetry and the phase in more detail clearly requires a deeper knowledge of the dispersion relation, which, as it is not entirely fixed by symmetries, we leave for future investigation.

3.2.2. Comparison with perturbation theory

Defining the effective string tension

$$h = \frac{R^2}{2\pi\alpha'} \ , \tag{3.81}$$

the tree-level S-matrix for the scattering of massive modes in the light-cone gauge $AdS_2 \times S^2 \times T^6$ superstring following from near-BMN perturbation theory can be be found by suitably truncating the corresponding result for $AdS_5 \times S^5$ or $AdS_3 \times S^3 \times T^4$ (various components were also computed in [183]). This gives

$$\begin{aligned}
S_1 &= 1 + \frac{i}{4h}\big[(1-2\mathrm{a})(e'p - ep') + l_1\big] + \mathcal{O}(\frac{1}{h^2}) \ , \\
S_2 &= 1 + \frac{i}{4h}\big[(1-2\mathrm{a})(e'p - ep') - l_1\big] + \mathcal{O}(\frac{1}{h^2}) \ , \\
T_1 &= 1 + \frac{i}{4h}\big[(1-2\mathrm{a})(e'p - ep') - l_2\big] + \mathcal{O}(\frac{1}{h^2}) \ , \\
T_2 &= 1 + \frac{i}{4h}\big[(1-2\mathrm{a})(e'p - ep') + l_2\big] + \mathcal{O}(\frac{1}{h^2}) \ , \\
Q_1 &= Q_2 = \frac{i}{2h} l_3 + \mathcal{O}(\frac{1}{h^2}) \ , \qquad R_1 = R_2 = -\frac{i}{2h} l_4 + \mathcal{O}(\frac{1}{h^2}) \ ,
\end{aligned} \tag{3.82}$$

where the functions l_i are defined as

$$\begin{aligned}
l_1(p,p') &= \frac{p^2 + p'^2}{e'p - ep'} \ , \qquad\qquad l_2(p,p') = \frac{p^2 - p'^2}{e'p - ep'} \ , \\
l_3(p,p') &= -\frac{pp'}{2(e'p - ep')}\big[\sqrt{(e+p)(e'-p')} - \sqrt{(e-p)(e'+p')}\big] \ , \\
l_4(p,p') &= -\frac{pp'}{2(e'p - ep')}\big[\sqrt{(e+p)(e'-p')} + \sqrt{(e-p)(e'+p')}\big] \ .
\end{aligned}$$

The parameter a is the standard gauge-fixing parameter of the uniform light-cone gauge . In [183] it was shown that to one-loop the near-BMN dispersion relation is given by

$$e^2 = 1 + p^2 + \mathcal{O}(h^{-2}) \ . \tag{3.83}$$

The one-loop near-BMN result can be constructed via unitarity methods following [194]. As expected from unitarity methods, this will certainly give the correct logarithmic terms in the one-loop S-matrix, and indeed this has already been argued in [183]. However, the prescription given in [194] is also conjectured to give the correct rational terms for integrable theories. Under this assumption we find that the one-loop S-matrix takes the following form

$$\begin{aligned}
S_1 =&\ \exp\big\{\frac{i}{4h}(1-2\mathrm{a})(e'p-ep')\big\}\ \sigma_{AdS_2}\ \big[1+\frac{i}{4h}l_1-\frac{\ell}{32h^2}\big]+\mathcal{O}(\frac{1}{h^3})\ ,\\
S_2 =&\ \exp\big\{\frac{i}{4h}(1-2\mathrm{a})(e'p-ep')\big\}\ \sigma_{AdS_2}\ \big[1-\frac{i}{4h}l_1-\frac{\ell}{32h^2}\big]+\mathcal{O}(\frac{1}{h^3})\ ,\\
T_1 =&\ \exp\big\{\frac{i}{4h}(1-2\mathrm{a})(e'p-ep')\big\}\ \sigma_{AdS_2}\ \big[1-\frac{i}{4h}l_2-\frac{\ell}{32h^2}\big]+\mathcal{O}(\frac{1}{h^3})\ ,\\
T_2 =&\ \exp\big\{\frac{i}{4h}(1-2\mathrm{a})(e'p-ep')\big\}\ \sigma_{AdS_2}\ \big[1+\frac{i}{4h}l_2-\frac{\ell}{32h^2}\big]+\mathcal{O}(\frac{1}{h^3})\ ,\\
Q_1 = Q_2 =&\ \exp\big\{\frac{i}{4h}(1-2\mathrm{a})(e'p-ep')\big\}\ \sigma_{AdS_2}\ \big[\frac{i}{2h}l_3\big]+\mathcal{O}(\frac{1}{h^3})\ ,\\
R_1 = R_2 =&\ \exp\big\{\frac{i}{4h}(1-2\mathrm{a})(e'p-ep')\big\}\ \sigma_{AdS_2}\ \big[-\frac{i}{2h}l_4\big]+\mathcal{O}(\frac{1}{h^3})\ ,
\end{aligned} \tag{3.84}$$

where the expansion of the phase factor σ_{AdS_2} is given by

$$\sigma_{AdS_2} = \exp\big\{\frac{i}{8\pi\, h^2}\frac{p^2p'^2\big((e'p-ep')-(ee'-pp')\,\mathrm{arsinh}[e'p-ep']\big)}{(e'p-ep')^2}+\mathcal{O}(\frac{1}{h^3})\big\}\ , \tag{3.85}$$

while

$$\ell = \frac{p^4+p'^4+2p^2p'^2(ee'-pp')}{(e'p-ep')^2}\ , \tag{3.86}$$

is fixed by the requirement of unitarity. As observed in [183] the one-loop logarithms are consistent with the one-loop phase being related to the Hernandez-Lopez phase.

We define the near-BMN expansion of the exact result as follows

$$\begin{aligned}
&\mathrm{e} = e\ , \qquad m = \rho_3+\rho_4 h^{-1}+\mathcal{O}(h^{-2})\ , \qquad \mathrm{p} = \frac{p}{h\big(\rho_5+\rho_6 h^{-1}+\mathcal{O}(h^{-2})\big)}\ ,\\
&\mathrm{h} = h\big(\rho_1+\rho_2 h^{-1}+\mathcal{O}(h^{-2})\big)\ ,
\end{aligned} \tag{3.87}$$

and similarly for e', p' and m'. Here for generality we have allowed for various rescalings, however, for simplicity we will assume that the ρ_i are constants.[8]

Let us remark that in this Chapter we are considering the $AdS_2\times S^2\times T^6$ background supported by Ramond-Ramond fluxes [49], and hence the light-cone gauge-fixed theory should be parity invariant [181,183]. Therefore, if it were the case that m receives quantum corrections

[8] To be completely general, one could in principle let e, m and p be arbitrary functions of e and p. However, naively truncating the classical/tree-level results for $AdS_5\times S^5$ and $AdS_3\times S^3\times T^4$, for example [121,182], to the massive sector of $AdS_2\times S^2\times T^6$ the ansatz (3.87) seems reasonable. Of course to check this claim one should construct the light-cone gauge symmetry algebra explicitly.

depending on the momentum they should respect the corresponding constraint.[9] This is in contrast to backgrounds partially (or wholly) supported by Neveu-Schwarz flux, for which m may have a dependence on p that breaks parity (see, for example, for discussions of the dispersion relation of the $AdS_3 \times S^3 \times T^4$ light-cone gauge-fixed theory supported by a mix of fluxes). It is worth noting that the $AdS_2 \times S^2 \times T^6$ background can also be supported by a mixture of Ramond-Ramond and Neveu-Schwarz fluxes and it would be interesting to see how the presence of the latter affects the representations discussed in this Chapter.

Expanding the exact dispersion relation (3.31) in the near-BMN regime, we recover (3.83) if we take

$$\rho_5 = \rho_1 \ , \qquad \rho_6 = \rho_2 \ , \qquad \rho_3 = 1 \ , \qquad \rho_4 = 0 \ . \tag{3.88}$$

Further expanding the exact S-matrix (3.58) in the near-BMN regime, taking α given by (3.33), $a = 1/2$ and fixing the overall factor $\tilde{\mathcal{P}}_0$ such that any one of the eight amplitudes agrees with perturbation theory, we find that, so long as

$$\rho_1 = 1 \ , \tag{3.89}$$

the remaining seven also agree with perturbation theory, (3.82) and (3.84), while ρ_2 cancels out of the equations.

3.3. Yangian symmetry

3.3.1. Massive case

In this section we would like to discuss the issue of Yangian symmetry. The first observation is that, in the massive case (we can fix $m = m' = 1$ for the purposes of this section), we could not apply the same standard Yangian symmetry of the R-matrix which works for the massless case (see section 3.3.2.). The massive representation is a long one (*cf.* section 3.1.2.), and a similar result was found for long representations of $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ [196]. The long representations studied in [196] bear a strong resemblance to the ones in this Chapter, up to the different dimensionality.

We proceed by postulating the commutation relations of the standard $\mathfrak{sl}(1|1)$ Yangian in Drinfeld's second realization [74, 98] (with central extensions)

$$\{\mathfrak{e}_m, \mathfrak{f}_n\} = -\mathfrak{h}_{m+n} \ , \qquad \{\mathfrak{e}_m, \mathfrak{e}_n\} = \mathfrak{p}_{m+n} \ , \qquad \{\mathfrak{f}_m, \mathfrak{f}_n\} = \mathfrak{p}^\dagger_{m+n} \ , \qquad [\mathfrak{h}_m, \cdot] = [\mathfrak{p}_m, \cdot] = [\mathfrak{p}^\dagger_m, \cdot] = 0 \ . \tag{3.90}$$

[9]In section 3.4. we will study the $m \to 0$ limit as a massless regime, with the *proviso* that if it were the case that m becomes momentum-dependent at a quantum level, this limit would no longer be relevant for the massless modes of the superstring. This issue should be addressed through a more detailed study of the off-shell symmetry algebra of the theory and its representations [121, 195].

One can check that the coproducts obtained from

$$\begin{aligned}\Delta(\mathfrak{e}_1) &= \mathfrak{e}_1 \otimes \mathbf{1} + e^{i\frac{\mathrm{p}}{2}} \otimes \mathfrak{e}_1 + \mathfrak{h}_0\, e^{i\frac{\mathrm{p}}{2}} \otimes \mathfrak{e}_0 \ , \\ \Delta(\mathfrak{f}_1) &= \mathfrak{f}_1 \otimes \mathbf{1} + e^{-i\frac{\mathrm{p}}{2}} \otimes \mathfrak{f}_1 + \mathfrak{f}_0 \otimes \mathfrak{h}_0 \ ,\end{aligned} \tag{3.91}$$

and their opposites satisfy the defining relations (5.26) and hence provide homomorphisms of the Yangian. The antipode Σ can be easily found from (3.91) using the defining property

$$\mu\,(\Sigma \otimes \mathbf{1})\,\Delta = \eta\,\epsilon \ , \tag{3.92}$$

where ϵ annihilates all level 1 generators. Combined, this defines the Hopf algebra structure of the standard Yangian.

One can construct a family of representations of the Yangian (3.91) starting from a slightly simpler level-zero (Lie algebra) representation compared to the one we use in section 3.1.2.. Determining the level 1 generators in this representation, we can obtain all the central elements up to and including level 2, together with their coproducts and opposite coproducts.[10] Following the strategy of [196], one can check whether all the central coproducts are co-commutative, as this is a necessary condition for the existence of an R-matrix scattering two such representations (see footnote 1). We found that

$$\Delta^{op}(\mathfrak{p}_2) \neq \Delta(\mathfrak{p}_2) \ , \tag{3.93}$$

for all members of the family of representations. This implies that at least one representation of the standard Yangian does not admit an R-matrix, excluding the existence of a universal R-matrix.

However, it is likely that the massive R-matrix may admit a coproduct which is not precisely the same as for massless representations, but still of the type found in [197]. Moreover, considerations as in footnote 3 of [196] are likely to apply. We leave this investigation for future work.

3.3.2. Massless case

The situation is different for the massless limit $m = m' = 0$ (see the discussion at the beginning of section 3.4.). In this case, in the absence of the central extensions ($b = c = 0$, *i.e.* considering again the $\mathfrak{gl}(1|1)$ algebra), the representation would become one of the reducible but indecomposable modules of section 3.1.1.. In fact, in that case the condition $m = ad - bc = ad = 0$ would force one of the fermionic generators to be identically zero. The indecomposable would

[10] In the absence of non-central Cartan elements, we cannot mechanically generate the level 2 and higher supercharges and they would have to be guessed. However we do not need them for the sake of this argument.

then be made up of *short* 1-dimensional $\mathfrak{gl}(1|1)$ irreps. This suggests that the Yangian might now be straightforwardly derived from the standard one.

The fact that $m = 0$ effectively works as a shortening condition, and the consequence that this allows for the existence of a Yangian representation, gives us significant encouragement that $m = 0$ might be protected against quantum corrections in the full theory. This is also corroborated by explicit perturbative results, which have not yet found any evidence for a quantum lift of this condition (see, for instance, [49, 181]). Moreover, the subgroup of $SO(6)$ controlling the symmetry of the massless sector might allow one to construct a mechanism protecting the $m = 0$ condition, analogous to the one described in [195] for $AdS_3 \times S^3 \times T^4$.

Indeed, this time we construct an evaluation representation of the Yangian (3.91)

$$\mathfrak{e}_1 = u\,\mathfrak{e}_0 = u\,\mathfrak{Q}\ , \qquad \mathfrak{f}_1 = u\,\mathfrak{f}_0 = u\,\mathfrak{G}\ , \qquad u = \frac{i\mathrm{h}}{x^-}\ , \tag{3.94}$$

starting from the level 0 one we consider in section 3.1.2., specializing to $m = 0$. Due to the additional parameters compared to the $\mathfrak{gl}(1|1)$ case, the representation remains generically irreducible. Nevertheless, the obstruction encountered in the massive case is no longer present, *i.e.* all central charges we can build are co-commutative and in fact the R-matrix (for $m = m' = 0$) can be shown to be invariant under the standard Yangian. This is reminiscent of the $AdS_5 \times S^5$ case, where the Yangian for short representations does not directly transfer to long ones as it stands [197, 91].

The crossing symmetry transformation reveals an interesting property, related to what was observed in [190] for the case of $AdS_3 \times S^3 \times T^4$, namely the existence of two different Yangian spectral (evaluation) parameters for the particle and the anti-particle representations. Here, the difference is superficial, as the massless condition makes the two spectral parameters coincide. In fact, the antipode obtained from applying (3.92) reads

$$\Sigma(\mathfrak{e}_1) = -e^{-i\frac{\mathrm{p}}{2}}\,(\mathfrak{e}_1 + \mathfrak{e}_0\,\mathfrak{h}_0)\ , \qquad \Sigma(\mathfrak{f}_1) = -e^{i\frac{\mathrm{p}}{2}}\,(\mathfrak{f}_1 + \mathfrak{f}_0\,\mathfrak{h}_0)\ . \tag{3.95}$$

This effectively amounts to a shift in the spectral parameter u by one of the central elements. When plugging this into the relation

$$\Sigma\big(\mathfrak{j}_1(x^{\pm})\big) = \mathscr{C}^{-1}\left[\mathfrak{j}^{\mathfrak{a}}{}_1\left(\frac{1}{x^{\pm}}\right)\right]^{st}\mathscr{C}\ , \tag{3.96}$$

and postulating that the anti-particle representation is also of evaluation type, that is

$$\mathfrak{e}^{\mathfrak{a}}{}_1 = u_a\,\mathfrak{Q}\ , \qquad \mathfrak{f}^{\mathfrak{a}}{}_1 = u_a\,\mathfrak{G}\ , \tag{3.97}$$

we see that the conditions (3.96) and (3.95) reduce to the same equation that holds true for the level 0 charges, *i.e.* (5.11a), provided that the anti-particle spectral parameter is chosen to be

$$u_a = i\mathrm{h}\,x^+\ . \tag{3.98}$$

For massless particles,

$$u = u_a \ . \tag{3.99}$$

3.4. S-matrix for massless modes

In this section we investigate the $m \to 0$ and $m' \to 0$ limits of the S-matrix constructed in section 3.2.. From the dispersion relation (3.31) and under the assumption of constant m and m', we may consider it natural to interpret these as massless limits.

While in principle these limits are already of interest in their own right, given the Yangian symmetry discussed in section 3.3.2., the resulting S-matrices may also be relevant for the scattering of massless modes in the $AdS_2 \times S^2 \times T^6$ light-cone gauge string theory. Indeed, for the $AdS_3 \times S^3 \times T^4$ light-cone gauge-fixed theory the massless modes transformed in the same type of representations as the massive modes (with vanishing mass and up to a suitable identification of highest weight states) [195]. Motivated by this, one may conjecture that the S-matrices constructed below can be used to build the S-matrices describing scattering processes involving massless modes (under the assumption that they remain massless and m and m' remain zero at a quantum level – see the discussion below (3.87) and footnote 9) in the $AdS_2 \times S^2 \times T^6$ light-cone gauge superstring.

3.4.1. Derivation from Yangian invariance

The S-matrix describing the scattering of two massless excitations can be directly obtained by imposing Lie algebra and Yangian invariance for two $m = 0$ representations of section 3.1.2., or as an $m, m' \to 0$ limit of the massive S-matrix. In the latter case, one has to treat various $\frac{0}{0}$ limiting expressions, which come from the function f in eq. (3.59).[11] Taking care when resolving these singular limits we find agreement with the result from imposing Yangian invariance. In the massless limit the dispersion relation in terms of the Zhukovsky variables takes the form [195][12]

$$x^+ = \frac{1}{x^-} \ . \tag{3.100}$$

In terms of the energy and momenta this translates to

$$\mathrm{e}^2 = 4\mathrm{h}^2 \sin^2 \frac{\mathrm{p}}{2} \qquad \Rightarrow \qquad \mathrm{e} = 2\mathrm{h} \left| \sin \frac{\mathrm{p}}{2} \right| \ , \tag{3.101}$$

[11] This is somehow reminiscent of the relativistic case [198].

[12] There is a second solution $x^+ = x^-$, however, this corresponds to $\mathrm{p} = 0$ and therefore is not physically sensible.

and hence there are two branches of the dispersion relation depending on the sign of $\sin\frac{\mathrm{p}}{2}$ [195]

$$x^+ = \sigma e^{i\,\frac{\mathrm{p}}{2}}\ , \quad x^- = \frac{1}{x^+}\ , \quad \sigma = \pm 1\ , \qquad\qquad x'^+ = \sigma' e^{i\,\frac{\mathrm{p}'}{2}}\ , \quad x'^- = \frac{1}{x'^+}\ , \quad \sigma' = \pm 1\ . \tag{3.102}$$

In the following we will use the convention that $\sigma = +1$ corresponds to a particle moving from left spatial infinity to right spatial infinity, *i.e.* right-moving, while $\sigma = -1$ corresponds to a left-moving particle.

Although the doubly-branched dispersion relation $\mathrm{e} = 2\mathrm{h}|\sin\frac{\mathrm{p}}{2}|$ is non-relativistic, there are some similarities with the kinematics of massless relativistic scattering. Following [198], in the relativistic case one has

$$e = \frac{m_0}{2}\, e^u\ , \qquad p = \pm\frac{m_0}{2}\, e^u, \qquad m_0,\, u \in \mathbb{R}\ . \tag{3.103}$$

A boost sends the rapidity $u \to u + \lambda$, with $\lambda \in \mathbb{R}$, hence the two branches can never be connected by such a transformation. In the non-relativistic case we have the two branches

$$\begin{aligned} \frac{i\,\mathrm{e}}{\mathrm{h}} &= \left[x^+ - \frac{1}{x^+}\right], \qquad \mathrm{p} = -2i\log x^+ \in [0,\pi]\ , \\ \frac{i\,\mathrm{e}}{\mathrm{h}} &= \left[x^+ - \frac{1}{x^+}\right], \qquad \mathrm{p} = -2i\log(-x^+) \in [-\pi,0]\ , \end{aligned} \tag{3.104}$$

with x^+ a pure phase for real momentum and energy. As the S-matrix is not of difference form there is a priori no notion of boosts and hence it is not clear if the presence of two branches represents an obstruction to interpreting the $\sigma = \sigma' = \pm 1$ scattering. However, as pointed out in [195], while the small momentum dispersion relation is relativistic, for the exact non-relativistic dispersion relation, the group velocity $\mathrm{v} = \frac{\partial \mathrm{e}}{\partial \mathrm{p}}$ is a non-trivial function of p and hence one may hope to give a physical interpretation to the $\sigma = \sigma' = \pm 1$ scattering.

For $\sigma = \sigma' = +1$, the Yangian invariance fixes the S-matrix up to *two* undetermined functions $\chi^{++}_{1,2}$:

$$\begin{aligned} S_1 = -S_2 &= \frac{1}{\sin\frac{1}{4}(\mathrm{p}+\mathrm{p}')}\left[\chi_1^{++}\,\sin\frac{1}{4}(\mathrm{p}-\mathrm{p}') + \chi_2^{++}\sqrt{\sin\frac{\mathrm{p}}{2}}\sqrt{\sin\frac{\mathrm{p}'}{2}}\right]\ , \\ T_1 = -T_2 &= -\chi_1^{++}\ , \\ Q_1 = Q_2 &= \frac{1}{\sin\frac{1}{4}(\mathrm{p}+\mathrm{p}')}\left[\chi_2^{++}\,\sin\frac{1}{4}(\mathrm{p}-\mathrm{p}') - \chi_1^{++}\sqrt{\sin\frac{\mathrm{p}}{2}}\sqrt{\sin\frac{\mathrm{p}'}{2}}\right]\ , \\ R_1 = R_2 &= \chi_2^{++}\ . \end{aligned}$$

We have checked that the Yangian representation with the coproducts taken in the appropriate branches – and away from the bound-state point (see footnote 3) – is fully reducible simultaneously at level zero and one, which is consistent with the appearance of two undetermined functions in the scattering matrix. In order to match the limit from the massive S-matrix, the functions $\chi_{1,2}^{++}$ should be chosen as follows:

$$\chi_2^{++} = -\frac{\sqrt{\sin\frac{\mathrm{p}}{2}}\sqrt{\sin\frac{\mathrm{p}'}{2}}}{2\sin\frac{1}{4}(\mathrm{p}+\mathrm{p}')}\tilde{\mathcal{P}}_0^{++}, \qquad \chi_1^{++} = \Big(\frac{f^{++}}{2} - \frac{\sin\frac{1}{4}(\mathrm{p}-\mathrm{p}')}{2\sin\frac{1}{4}(\mathrm{p}+\mathrm{p}')}\Big)\tilde{\mathcal{P}}_0^{++}\,, \tag{3.105}$$

where f^{++} is the limit of f. The limit of f is not fixed by the comparison with the Yangian S-matrix. However, imposing the Yang-Baxter equation

$$\mathbb{S}_{12}^{++}\,\mathbb{S}_{13}^{++}\,\mathbb{S}_{23}^{++} = \mathbb{S}_{23}^{++}\,\mathbb{S}_{13}^{++}\,\mathbb{S}_{12}^{++}\,. \tag{3.106}$$

requires that

$$f^{++} = \pm 1\,,\, 0\,. \tag{3.107}$$

The Yang-Baxter equation for $\sigma = \sigma' = +1$ scattering (3.106) does not allow for non-constant limits of the function f. In particular, the condition it imposes reads (we denote $\lim_{m,m'\to 0} f(\mathrm{p}_i,\mathrm{p}_j) \equiv f_{ij}^{++}$)

$$f_{13}^{++} - f_{23}^{++} + f_{12}^{++}\,(f_{13}^{++}f_{23}^{++} - 1) = 0\,. \tag{3.108}$$

If $f_{13}^{++}f_{23}^{++} = 1$, we immediately get $f^{++} = \pm 1$. If $f_{13}^{++}f_{23}^{++} \neq 1$, we find

$$f_{12}^{++} = \frac{f_{13}^{++} - f_{23}^{++}}{1 - f_{13}^{++}f_{23}^{++}}\,. \tag{3.109}$$

However, the l.h.s. of (3.109) does not depend on p_3, and hence we should impose that the derivative of the r.h.s. with respect to p_3 is zero. Doing so, we find that either once again $f^{++} = \pm 1$, or, if $f^{++} \neq \pm 1$, then

$$\frac{\partial_3 f_{13}^{++}}{1 - (f_{13}^{++})^2} = -\frac{1}{2}\,\partial_3 \log\Big(\frac{1 - f_{13}^{++}}{1 + f_{13}^{++}}\Big) \tag{3.110}$$

should be independent of p_1. Let us call this function $\omega(\mathrm{p}_3)$. This implies that

$$f_{13}^{++} = \frac{1 - \bar{\omega}(\mathrm{p}_1)\tilde{\omega}(\mathrm{p}_3)}{1 + \bar{\omega}(\mathrm{p}_1)\tilde{\omega}(\mathrm{p}_3)}\,, \qquad \tilde{\omega}(\mathrm{p}_3) = \exp\Big[-2\int^{\mathrm{p}_3}\omega(\mathrm{p}_3')d\mathrm{p}_3'\Big]\,. \tag{3.111}$$

Plugging this expression back into (3.109) we find that either $\bar{\omega}(\mathrm{p}) = 0$, in which case $f^{++} = 1$ and we are done, or $\bar{\omega}(\mathrm{p}) = \tilde{\omega}^{-1}(\mathrm{p})$. Finally, substituting into (3.108) we find that $\tilde{\omega}(\mathrm{p})$ is a constant and hence $f^{++} = 0$. This then demonstrates that the solutions of (3.108) are $f^{++} = \pm 1, 0$.

As in the relativistic case [198], a different situation applies for $\sigma = +1$, $\sigma' = -1$. The Yangian invariance again fixes the S-matrix up to two undetermined functions $\chi^{+-}_{1,2}$:

$$
\begin{aligned}
&S_1 = S_2 = \frac{1}{\cos\frac{1}{4}(\mathrm{p}+\mathrm{p}')}\left[\chi^{+-}_1 \, \cos\frac{1}{4}(\mathrm{p}-\mathrm{p}') + i\chi^{+-}_2 \sqrt{\sin\frac{\mathrm{p}}{2}}\sqrt{-\sin\frac{\mathrm{p}'}{2}}\right] ,\\
&T_1 = T_2 = \chi^{+-}_1 \ ,\\
&Q_1 = Q_2 = \frac{1}{\cos\frac{1}{4}(\mathrm{p}+\mathrm{p}')}\left[\chi^{+-}_2 \, \cos\frac{1}{4}(\mathrm{p}-\mathrm{p}') + i\chi^{+-}_1 \sqrt{\sin\frac{\mathrm{p}}{2}}\sqrt{-\sin\frac{\mathrm{p}'}{2}}\right] ,\\
&R_1 = R_2 = \chi^{+-}_2 \ .
\end{aligned}
$$

Again one can check that the Yangian representation with the coproducts taken in the appropriate branches – and away from the bound-state point (see footnote 3) – is fully reducible simultaneously at level zero and one, which is as before consistent with the appearance of two undetermined functions in the scattering matrix. In order to match the limit from the massive S-matrix, the functions $\chi^{+-}_{1,2}$ should be chosen as follows:

$$
\chi^{+-}_2 = -i\frac{\sqrt{\sin\frac{\mathrm{p}}{2}}\sqrt{-\sin\frac{\mathrm{p}'}{2}}}{2\cos\frac{1}{4}(\mathrm{p}+\mathrm{p}')}\tilde{\mathcal{P}}^{+-}_0 \ , \qquad \chi^{+-}_1 = \Big(\frac{f^{+-}}{2} + \frac{\cos\frac{1}{4}(\mathrm{p}-\mathrm{p}')}{2\cos\frac{1}{4}(\mathrm{p}+\mathrm{p}')}\Big)\tilde{\mathcal{P}}^{+-}_0 \ , \tag{3.112}
$$

where f^{+-} is the limit of f. For this mixed case the limit of f is also not fixed by the comparison with the Yangian S-matrix. Once again, the Yang-Baxter equation fixes this limiting value. In order to write down the Yang-Baxter equation for the mixed case, we need to first calculate the S-matrix for $\sigma = \sigma' = -1$, as schematically it is given by

$$
\mathbb{S}^{+-}_{12}\,\mathbb{S}^{+-}_{13}\,\mathbb{S}^{--}_{23} = \mathbb{S}^{--}_{23}\,\mathbb{S}^{+-}_{13}\,\mathbb{S}^{+-}_{12} \ . \tag{3.113}
$$

The Yangian invariance again fixes the $\sigma = \sigma' = -1$ S-matrix up to two undetermined functions $\chi^{--}_{1,2}$:

$$
\begin{aligned}
&S_1 = -S_2 = \frac{1}{\sin\frac{1}{4}(\mathrm{p}+\mathrm{p}')}\left[\chi^{--}_1 \, \sin\frac{1}{4}(\mathrm{p}-\mathrm{p}') - \chi^{--}_2 \sqrt{-\sin\frac{\mathrm{p}}{2}}\sqrt{-\sin\frac{\mathrm{p}'}{2}}\right] ,\\
&T_1 = -T_2 = -\chi^{--}_1 \ ,\\
&Q_1 = Q_2 = \frac{1}{\sin\frac{1}{4}(\mathrm{p}+\mathrm{p}')}\left[-\chi^{--}_2 \, \sin\frac{1}{4}(\mathrm{p}-\mathrm{p}') - \chi^{--}_1 \sqrt{-\sin\frac{\mathrm{p}}{2}}\sqrt{-\sin\frac{\mathrm{p}'}{2}}\right] ,\\
&R_1 = R_2 = \chi^{--}_2 \ .
\end{aligned}
$$

In order to match the limit from the massive S-matrix, the functions $\chi^{--}_{1,2}$ have to be chosen as follows:

$$
\chi^{--}_2 = \frac{\sqrt{-\sin\frac{\mathrm{p}}{2}}\sqrt{-\sin\frac{\mathrm{p}'}{2}}}{2\sin\frac{1}{4}(\mathrm{p}+\mathrm{p}')}\tilde{\mathcal{P}}^{--}_0 \ , \qquad \chi^{--}_1 = \Big(-\frac{f^{--}}{2} - \frac{\sin\frac{1}{4}(\mathrm{p}-\mathrm{p}')}{2\sin\frac{1}{4}(\mathrm{p}+\mathrm{p}')}\Big)\tilde{\mathcal{P}}^{--}_0 \ , \tag{3.114}
$$

where f^{--} is the limit of f. The Yang-Baxter equation

$$\mathbb{S}_{12}^{--}\,\mathbb{S}_{13}^{--}\,\mathbb{S}_{23}^{--} = \mathbb{S}_{23}^{--}\,\mathbb{S}_{13}^{--}\,\mathbb{S}_{12}^{--}\ . \tag{3.115}$$

fixes this limiting value to

$$f^{--} = \pm 1\,,\, 0\ . \tag{3.116}$$

Taking this result into account, the mixed Yang-Baxter equation (3.113) fixes $f^{+-} = \pm 1$ if one chooses either $f^{--} = 1$ or $f^{--} = -1$, or f^{+-} to any constant if one chooses $f^{--} = 0$.

To exhaust all possibilities, the $\sigma = -1$, $\sigma' = +1$ S-matrix is given by

$$\begin{aligned}
S_1 = S_2 &= \frac{1}{\cos\frac{1}{4}(\mathrm{p}+\mathrm{p}')}\left[\chi_1^{-+}\cos\frac{1}{4}(\mathrm{p}-\mathrm{p}') - i\chi_2^{-+}\sqrt{-\sin\frac{\mathrm{p}}{2}}\sqrt{\sin\frac{\mathrm{p}'}{2}}\right]\ ,\\
T_1 = T_2 &= \chi_1^{-+}\ ,\\
Q_1 = Q_2 &= \frac{1}{\cos\frac{1}{4}(\mathrm{p}+\mathrm{p}')}\left[-\chi_2^{-+}\cos\frac{1}{4}(\mathrm{p}-\mathrm{p}') + i\chi_1^{-+}\sqrt{-\sin\frac{\mathrm{p}}{2}}\sqrt{\sin\frac{\mathrm{p}'}{2}}\right]\ ,\\
R_1 = R_2 &= \chi_2^{-+}\ .
\end{aligned}$$

In order to match the limit from the massive S-matrix, the functions $\chi_{1,2}^{-+}$ have to be chosen as follows:

$$\chi_2^{-+} = i\frac{\sqrt{-\sin\frac{\mathrm{p}}{2}}\sqrt{\sin\frac{\mathrm{p}'}{2}}}{2\cos\frac{1}{4}(\mathrm{p}+\mathrm{p}')}\tilde{\mathcal{P}}_0^{-+}\ ,\qquad \chi_1^{-+} = \Big(-\frac{f^{-+}}{2} + \frac{\cos\frac{1}{4}(\mathrm{p}-\mathrm{p}')}{2\cos\frac{1}{4}(\mathrm{p}+\mathrm{p}')}\Big)\tilde{\mathcal{P}}_0^{-+}\ , \tag{3.117}$$

where f^{-+} is the limit of f.

By imposing the Yang-Baxter equation for all possible remaining sequences of scattering processes we find the following possibilities for the limits of f:

$$\begin{aligned}
&f^{++} = \pm 1,\, 0\ , && f^{+-} = \pm 1\ , && f^{-+} = \pm 1\ , && f^{--} = \pm 1,\, 0\ ,\\
&f^{++} = 0\ , && f^{+-} = \mu_1, && f^{-+} = \mu_2\ , && f^{--} = 0\ ,
\end{aligned} \tag{3.118}$$

where μ_1 and μ_2 are arbitrary constants. Note that we have not included the following two Yang-Baxter equations:

$$\mathbb{S}_{12}^{+-}\,\mathbb{S}_{13}^{++}\,\mathbb{S}_{23}^{-+} = \mathbb{S}_{23}^{-+}\,\mathbb{S}_{13}^{++}\,\mathbb{S}_{12}^{+-}\ ,\qquad \mathbb{S}_{12}^{-+}\,\mathbb{S}_{13}^{--}\,\mathbb{S}_{23}^{+-} = \mathbb{S}_{23}^{+-}\,\mathbb{S}_{13}^{--}\,\mathbb{S}_{12}^{-+}\ , \tag{3.119}$$

as they do not correspond to physically realizable scattering processes. If particles 1 and 3 are both right- or left-moving then they have to scatter with each other before scattering with an excitation travelling in the opposite direction. If we formally include them then the possibilities for the limits of f are reduced to

$$(f^{++}, f^{+-}, f^{-+}, f^{--}) \in \{(1,1,1,1), (-1,-1,-1,-1), (0,\mu,-\mu,\tilde{\mu}), (\tilde{\mu},\mu,-\mu,0)\}\ , \tag{3.120}$$

with μ any constant for $\tilde{\mu} = 0$, $\mu = \pm 1$ for $\tilde{\mu} = 1$, and $\mu = \pm 1$ for $\tilde{\mu} = -1$.

The various choices for $f^{++}, f^{+-}, f^{-+}, f^{--}$ can be further restricted by considering crossing symmetry. Although in the massless case there is no clear physical interpretation of crossing, see, for example, [198], one may nevertheless demand that it is still present. Let us recall that the crossing transformation simultaneously changes the sign of the energy and momentum, therefore the crossing of a $+$ $(-)$ particle is still a $+$ $(-)$ particle. Consequently in the crossing relation (5.15) we should consider two massless S-matrices of the same type. Considering the various possible limits of f, we find that the choices $f^{++} = 0$ and $f^{--} = 0$ are incompatible with crossing. Indeed, before taking the massless limit, the function f satisfies the following crossing transformation with respect to the first particle:

$$f \to \frac{x'^{+}\, x'^{-}}{f} \;, \tag{3.121}$$

which is is clearly problematic for $f \to 0$. We are then left with the following choices for the limits of f

$$f^{++} = \pm 1 \;, \qquad f^{+-} = \pm 1 \;, \qquad f^{-+} = \pm 1 \;, \qquad f^{--} = \pm 1 \;. \tag{3.122}$$

It is worth noting that for the crossing relation to be satisfied for these choices we should not only consider two massless S-matrices of the same type, but also with the same limit of f.

Now that we are left with the choices in eq. (3.122), let us recall that in the massive case the sign of f is not determined by symmetry or the Yang-Baxter equation, rather from comparing with perturbation theory. This is consistent with the residual ambiguity we are finding in this limit.

If we look at the BMN limit (see section 3.2.2.) for the $\sigma = \sigma' = \pm 1$ S-matrices, we don't necessarily expect to (and indeed we do not) find the identity. This expectation comes from the fact that the quadratic Lagrangian of the light-cone gauge-fixed theory is relativistic and it is not clear how one should perform a perturbative computation for the scattering of two massless relativistic particles on the same branch, or if there should be a perturbative expansion at all.

For the $\sigma = -\sigma' = \pm 1$ S-matrices one may expect the limit to be better behaved as perturbative computations can be carried out. Indeed, assuming that the phase goes like one plus corrections, then for the $\sigma = -\sigma' = +1$ case we find that if $f^{+-} = 1$ the S-matrix is the identity at leading order, while for the $\sigma = -\sigma' = -1$ case the same is true, but with $f^{-+} = -1$. Therefore, we end up with the following choices for the limits of f

$$f^{++} = \pm 1 \;, \qquad f^{+-} = 1 \;, \qquad f^{-+} = -1 \;, \qquad f^{--} = \pm 1 \;. \tag{3.123}$$

We may attribute some physical meaning to this result by considering the group velocities

$$\mathrm{v} = \frac{\partial \mathrm{e}}{\partial \mathrm{p}} \;, \qquad \mathrm{v}' = \frac{\partial \mathrm{e}'}{\partial \mathrm{p}'} \;. \tag{3.124}$$

Let us remark that our considerations (especially those referring to the ordering of velocities) will only apply when trying to attach a physical interpretation of real time scattering to these amplitudes. In general, for a complete analysis, one should also consider the possibility of analytically continuing the S-matrices as functions of the kinematical variables. With this in mind, for a physically realizable scattering process with $\sigma = -\sigma' = +1$ the group velocities satisfy $\mathrm{v} > \mathrm{v}'$, while for a scattering process with $\sigma = -\sigma' = -1$ we have $\mathrm{v}' > \mathrm{v}$. Therefore, we may associate $\lim_{m,m'\to 0} f \to 1$ with $\mathrm{v} > \mathrm{v}'$ and $\lim_{m,m'\to 0} f \to -1$ with $\mathrm{v} < \mathrm{v}'$. This is consistent with the crossing symmetry discussed above as the group velocity is invariant under the crossing transformation. Furthermore, one may expect the $\sigma = -\sigma' = +1$ and $\sigma = -\sigma' = -1$ S-matrices to be related upon interchanging the arguments. Indeed, the following equation is satisfied for real momenta[13]

$$\mathcal{S}^{\pm\mp}{}_{ab}^{cd}(\mathrm{p},\mathrm{p}')\big|_{f\to\pm1} = (-1)^{[a][b]+[c][d]}\mathcal{S}^{\mp\pm}{}_{ba}^{dc}(\mathrm{p}',\mathrm{p})^*\big|_{f\to\mp1} \ . \tag{3.125}$$

The corresponding relation for the $\sigma = \sigma' = \pm 1$ S-matrices is given by

$$\mathcal{S}^{\pm\pm}{}_{ab}^{cd}(\mathrm{p},\mathrm{p}')\big|_{f\to\pm1} = (-1)^{[a][b]+[c][d]}\mathcal{S}^{\pm\pm}{}_{ba}^{dc}(\mathrm{p}',\mathrm{p})^*\big|_{f\to\mp1} \ . \tag{3.126}$$

To conclude, let us briefly comment on unitarity. Motivated by the physical interpretation outlined above, one may expect that braiding unitarity for the massless S-matrix will involve one S-matrix with $f \to 1$ and one with $f \to -1$, and indeed, one can explicitly check that braiding unitarity relations can be constructed in this way. They are given by

$$\begin{aligned}(-1)^{[c][d]+[e][f]}\mathcal{S}^{\pm\pm}{}_{ab}^{ef}(\mathrm{p},\mathrm{p}')\big|_{f\to\pm1}\mathcal{S}^{\pm\pm}{}_{fe}^{dc}(\mathrm{p}',\mathrm{p})\big|_{f\to\mp1} \propto \delta_a^c\delta_b^d \ .\\ (-1)^{[c][d]+[e][f]}\mathcal{S}^{\pm\mp}{}_{ab}^{ef}(\mathrm{p},\mathrm{p}')\big|_{f\to\pm1}\mathcal{S}^{\mp\pm}{}_{fe}^{dc}(\mathrm{p}',\mathrm{p})\big|_{f\to\mp1} \propto \delta_a^c\delta_b^d \ .\end{aligned} \tag{3.127}$$

These relations can also be found by taking the massless limit of the braiding unitarity relation for the massive S-matrix. Finally, one can see that by combining (3.125), (3.126) and (3.127), all the four massless S-matrices are also QFT unitary so long as the overall factors satisfy appropriate constraints.

3.4.2. Massless limits and symmetry enhancement

Let us now consider taking the various massless limits of the parametrizing functions of the massive S-matrix, *i.e.* one massless and one massive or two massive particles. Here we work in terms of the variables $x^\pm$, $x'^\pm$ as it allows us to consider the four cases of section 3.4.1. at the same time. For convenience we introduce the following notation for the massless Zhukovsky variables

$$\mathrm{x} = x^+ = \frac{1}{x^-} \ , \qquad \mathrm{x}' = x'^+ = \frac{1}{x'^-} \ . \tag{3.128}$$

[13]Here we are defining $\mathbb{S}|\Phi_a\Phi_b'\rangle = \mathcal{S}_{ab}^{cd}(\mathrm{p},\mathrm{p}')|\Phi_c\Phi_d'\rangle$, $\Phi_0 = \phi$, $\Phi_1 = \psi$ and $[a] = a$.

The parametrizing functions are then given by

Massive-Massless $\qquad f \to x^- \sqrt{\frac{x^+}{x^-}}$

$$S_1 = T_1 = -\frac{\mathrm{x}'}{\sqrt{\mathrm{x}'^2}} \frac{(x^+ - \mathrm{x}') + \sqrt{\frac{x^+}{x^-}}(x^- - \mathrm{x}')}{2(1 - x^+\mathrm{x}')} \tilde{\mathcal{P}}_0 \,, \qquad S_2 = T_2 = \frac{(1 - x^+\mathrm{x}') + \sqrt{\frac{x^+}{x^-}}(1 - x^-\mathrm{x}')}{2(1 - x^+\mathrm{x}')} \tilde{\mathcal{P}}_0 \,,$$
$$Q_1 = Q_2 = i \sqrt[4]{\frac{x^+}{x^-}\frac{1}{\mathrm{x}'^2}} \frac{\mathrm{x}'}{\sqrt{\mathrm{x}'^2}} \frac{\mathrm{x}'\eta\eta'}{2(1 - x^+\mathrm{x}')} \tilde{\mathcal{P}}_0 \,, \qquad R_1 = R_2 = i \sqrt[4]{\frac{x^+}{x^-}\frac{1}{\mathrm{x}'^2}} \frac{\mathrm{x}'\eta\eta'}{2(1 - x^+\mathrm{x}')} \tilde{\mathcal{P}}_0 \,, \tag{3.129}$$

Massless-Massive $\qquad f \to -x'^- \sqrt{\frac{x'^+}{x'^-}}$

$$S_1 = T_2 = \frac{\sqrt{\mathrm{x}^2}}{\mathrm{x}} \frac{(1 - \mathrm{x}x'^-) + \sqrt{\frac{x'^-}{x'^+}}(1 - \mathrm{x}x'^+)}{2(\mathrm{x} - x'^-)} \tilde{\mathcal{P}}_0 \,, \qquad S_2 = T_1 = \frac{(\mathrm{x} - x'^-) + \sqrt{\frac{x'^-}{x'^+}}(\mathrm{x} - x'^+)}{2(\mathrm{x} - x'^-)} \tilde{\mathcal{P}}_0 \,,$$
$$Q_1 = Q_2 = i \sqrt[4]{\mathrm{x}^2 \frac{x'^-}{x'^+}} \frac{\sqrt{\mathrm{x}^2}}{\mathrm{x}} \frac{\eta\eta'}{2(\mathrm{x} - x'^-)} \tilde{\mathcal{P}}_0 \,, \qquad R_1 = R_2 = -i \sqrt[4]{\mathrm{x}^2 \frac{x'^-}{x'^+}} \frac{\eta\eta'}{2(\mathrm{x} - x'^-)} \tilde{\mathcal{P}}_0 \,, \tag{3.130}$$

Massless-Massless $\qquad f \to \pm 1$

$$S_1 = -\frac{\sqrt{\mathrm{x}^2}}{\mathrm{x}} \frac{\mathrm{x}'}{\sqrt{\mathrm{x}'^2}} \frac{1 - \mathrm{x}\mathrm{x}' \pm (\mathrm{x} - \mathrm{x}')}{2(1 - \mathrm{x}\mathrm{x}')} \tilde{\mathcal{P}}_0 \,, \qquad S_2 = \frac{1 - \mathrm{x}\mathrm{x}' \pm (\mathrm{x} - \mathrm{x}')}{2(1 - \mathrm{x}\mathrm{x}')} \tilde{\mathcal{P}}_0 \,,$$
$$T_1 = -\frac{\mathrm{x}'}{\sqrt{\mathrm{x}'^2}} \frac{\mathrm{x} - \mathrm{x}' \pm (1 - \mathrm{x}\mathrm{x}')}{2(1 - \mathrm{x}\mathrm{x}')} \tilde{\mathcal{P}}_0 \,, \qquad T_2 = \frac{\sqrt{\mathrm{x}^2}}{\mathrm{x}} \frac{\mathrm{x} - \mathrm{x}' \pm (1 - \mathrm{x}\mathrm{x}')}{2(1 - \mathrm{x}\mathrm{x}')} \tilde{\mathcal{P}}_0 \,,$$
$$Q_1 = Q_2 = \pm i \frac{\sqrt{\mathrm{x}^2}}{\mathrm{x}} \frac{\mathrm{x}'}{\sqrt{\mathrm{x}'^2}} \sqrt[4]{\frac{\mathrm{x}^2}{\mathrm{x}'^2}} \frac{\mathrm{x}'\eta\eta'}{2(1 - \mathrm{x}\mathrm{x}')} \tilde{\mathcal{P}}_0 \,, \qquad R_1 = R_2 = i \sqrt[4]{\frac{\mathrm{x}^2}{\mathrm{x}'^2}} \frac{\mathrm{x}'\eta\eta'}{2(1 - \mathrm{x}\mathrm{x}')} \tilde{\mathcal{P}}_0 \,. \tag{3.131}$$

Given that $\frac{\sqrt{\mathrm{x}^2}}{\mathrm{x}}$ and $\frac{\mathrm{x}'}{\sqrt{\mathrm{x}'^2}}$ are equal to ± 1 one can see that the limit of the function f is well-defined if we just take one of the two masses to zero. In particular, taking $m \to 0$ we have $f \to x^- \sqrt{\frac{x^+}{x^-}}$ while for $m' \to 0$ we have $f \to -x'^- \sqrt{\frac{x'^+}{x'^-}}$.

The factors of $\frac{\sqrt{\mathrm{x}^2}}{\mathrm{x}}$ and $\frac{\mathrm{x}'}{\sqrt{\mathrm{x}'^2}}$ in (3.131) are the origin of the various expressions for the different choices of σ and σ' in section 3.4.1.. For example, to recover the results of section 3.4.1. we should take $\frac{\sqrt{\mathrm{x}^2}}{\mathrm{x}} = 1$ for $\sigma = +1$ and $\frac{\sqrt{\mathrm{x}^2}}{\mathrm{x}} = -1$ for $\sigma = -1$, and similarly for x'. For $\mathrm{p} \in [-\pi, \pi]$, this again corresponds to taking the branch cut on the negative real axis.

We may also consider taking the massless limit of the S-matrices for one massive and one massless excitation. Following the same set of rules as above, *i.e.* setting $\frac{\sqrt{\mathrm{x}^2}}{\mathrm{x}}$ equal to 1 for $\sigma = +1$ and -1 for $\sigma = -1$, and similarly for $\frac{\mathrm{x}'}{\sqrt{\mathrm{x}'^2}}$, the following table gives the expressions we find for the limits of f

Before limit	After limit	Limit of f
Massive - Massless ($\sigma' = +1$)	Massless-Massless ($\sigma = +1$, $\sigma' = +1$)	$f^{++} = 1$
Massive - Massless ($\sigma' = +1$)	Massless-Massless ($\sigma = -1$, $\sigma' = +1$)	$f^{-+} = -1$
Massive - Massless ($\sigma' = -1$)	Massless-Massless ($\sigma = +1$, $\sigma' = -1$)	$f^{+-} = 1$
Massive - Massless ($\sigma' = -1$)	Massless-Massless ($\sigma = -1$, $\sigma' = -1$)	$f^{--} = -1$
Massless - Massive ($\sigma = +1$)	Massless-Massless ($\sigma = +1$, $\sigma' = +1$)	$f^{++} = -1$
Massless - Massive ($\sigma = +1$)	Massless-Massless ($\sigma = +1$, $\sigma' = -1$)	$f^{+-} = 1$
Massless - Massive ($\sigma = -1$)	Massless-Massless ($\sigma = -1$, $\sigma' = +1$)	$f^{-+} = -1$
Massless - Massive ($\sigma = -1$)	Massless-Massless ($\sigma = -1$, $\sigma' = -1$)	$f^{--} = 1$

Therefore we find the same set of possible limits of f as found from the analysis in section 3.4.1., the result of which is given in eq. (3.123).

Finally, from eqs. (3.129)–(3.131) we can see that taking the various massless limits results in many of the parametrizing functions (or products thereof) coinciding and there will then be additional $U(1)$ symmetries of the S-matrix acting on both the bosons and fermions. This is surely required for these S-matrices to describe the scattering of the massless modes of the light-cone gauge $AdS_2 \times S^2 \times T^6$ superstring as they (the bosons and fermions) will transform under various $U(1)$ symmetries originating from the T^6 compact space [183]. The precise construction of the S-matrices involving massless modes from the building blocks described above requires the knowledge of the full light-cone gauge symmetry algebra and its action on all the states, as was done for $AdS_3 \times S^3 \times T^4$ in [195] and $AdS_5 \times S^5$ in [121].

3.5. Bethe Ansatz

As discussed at the beginning of section 3.1. the tensor product of two copies of any S-matrix of the form (3.56) satisfying (3.62) possesses an additional $U(1)$ symmetry, which does not have a well-defined action on the individual factor S-matrices. This symmetry is expected from string theory as a consequence of the additional compact space T^6 required for a consistent 10-d superstring theory [183].[14] Under this symmetry the bosons y and z are uncharged, while the fermions $(\zeta, \chi)^T$ form an $SO(2)$ vector. Furthermore $(\mathfrak{Q}_2, \mathfrak{Q}_1)^T$ and $(\mathfrak{S}_2, \mathfrak{S}_1)^T$ are also charged as $SO(2)$ vectors under the symmetry.[15]

Here we will summarize the relevant details of this symmetry. Explicit details (including the

[14] We are grateful to O. Ohlsson Sax and P. Sundin for pointing out to us the existence of this symmetry in the superstring theory.

[15] Here the subscripts on the supercharges $\mathfrak{Q}$ and $\mathfrak{S}$ refer to the two copies of $\mathfrak{psu}(1|1)$ in the full symmetry algebra. In particular the charges with the label 1 act on the first entry in the tensor product (3.1), while the charges with the label 2 on the second entry.

expansion of the tensor product) are given in [184]. Defining

$$|\theta_\pm\rangle = \frac{1}{\sqrt{2}}(|\zeta\rangle \pm i|\chi\rangle) \ , \qquad \mathfrak{G}_{\mathfrak{q}\pm} = \frac{1}{\sqrt{2}}(\mathfrak{Q}_2 \pm i\mathfrak{Q}_1) \ , \qquad \mathfrak{G}_{\mathfrak{s}\pm} = \frac{1}{\sqrt{2}}(\mathfrak{S}_2 \pm i\mathfrak{S}_1) \ , \tag{3.132}$$

and their conjugates, we have the following actions of the $U(1)$ generator, $\mathfrak{J}_{U(1)}$,

$$\mathfrak{J}_{U(1)}|\theta_\pm\rangle = \pm i|\theta_\pm\rangle \ , \qquad [\mathfrak{J}_{U(1)}, \mathfrak{G}_{\mathfrak{q},\mathfrak{s}\pm}] = \pm i\mathfrak{G}_{\mathfrak{q},\mathfrak{s}\pm} \ . \tag{3.133}$$

To proceed with the algebraic Bethe ansatz (ABA) technique one constructs the monodromy matrix as a string of R-matrices acting on an auxiliary space a and on N physical spaces

$$T_a(\lambda) = R_{a,1} \cdot \ \dots \ \cdot R_{a,N} \ = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \ , \tag{3.134}$$

where $\cdot$ denotes multiplication in the auxiliary space. $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ and $D(\lambda)$ are operators on N-particle physical space, while the 2×2 matrix acts on the auxiliary space. As a consequence of the Yang-Baxter equation one has

$$R_{a_1,a_2}(\lambda_1 - \lambda_2)\, T_{a_1}(\lambda_1)\, T_{a_2}(\lambda_1) \ = \ T_{a_2}(\lambda_2)\, T_{a_1}(\lambda_1)\, R_{a_1,a_2}(\lambda_1 - \lambda_2) \ . \tag{3.135}$$

Taking the trace $tr_{a_1} \otimes tr_{a_2}$ on both sides of (3.135), one finds that the *transfer matrix* $T(\lambda) \equiv tr\, T_a(\lambda) = A(\lambda) + D(\lambda)$ satisfies:

$$[T(\lambda), T(\lambda')] = 0 \ . \tag{3.136}$$

As $T(\lambda)$ is an Nth order polynomial in λ (with the highest-power coefficient chosen equal to 1), we see that (3.136) implies that $T(\lambda)$ generates N non-trivial independent commuting operators.

To find the simultaneous eigenvectors of all the commuting charges (which include the Hamiltonian), one assumes that $B(\lambda)$ is a creation operator acting on a pseudo-vacuum $|vac\rangle$, which is annihilated by $C(\lambda)$:

$$|\Psi(\lambda_1, \dots, \lambda_M)\rangle = B(\lambda_1)\dots B(\lambda_M)\,|vac\rangle \ . \tag{3.137}$$

The pseudo-vacuum should be a highest-weight $T(\lambda)$-eigenstate, whether or not that is the true ground state of the Hamiltonian. The vectors (3.137) are not immediately eigenstates of $T(\lambda)$ because of unwanted terms obtained when acting with $T(\lambda)$. These unwanted terms are cancelled by imposing the Bethe equations, providing the quantization condition for the momenta of excitations.

Let us now give some initial observations on applying the ABA procedure to the S-matrix for the light-cone gauge $AdS_2 \times S^2 \times T^6$ superstring. We can immediately remark that a single copy of the centrally-extended S-matrix does not seem to admit a pseudovacuum on which to construct the ABA procedure. However, when we take the tensor product of two copies there is a pseudovacuum. This is given by a uniform sequence of either all $|\theta_+\rangle$ states or, alternatively, $|\theta_-\rangle$. In fact, thanks to the conservation of the additional $U(1)$ charge discussed above these states are the only ones with maximal (minimal) such charge, and therefore have to be eigenvalues of the transfer matrix. By a similar logic they are also annihilated by some of the lower-corner entries of the (now 4-dimensional) transfer matrix. This in principle could allow the ABA procedure to be applied. However, this still remains technically challenging given the complexity of the parametrizing functions of the S-matrix.

Part III

Yangians and Secret Symmetries for $\mathbf{AdS_2/CFT_1}$ and $\mathbf{AdS_3/CFT_2}$

4 Yangian Symmetry of AdS_2 Superstrings

The remarkable impact integrability has had on the solution of string theory in the $AdS_5 \times S^5$ background motivates trying to apply the same strategy to the other less supersymmetric string backgrounds which are still integrable. One of these is indeed the $AdS_2 \times S^2 \times T^6$ background with Ramond-Ramond fluxes in Type II superstring theory, which preserves 8 supersymmetries. One way to generate this background is by taking the near-horizon limit of various intersecting brane configurations in Type IIA/B supergravity, related by T-dualities in the T^6 directions. The dual "field" theory is thought to be either a superconformal quantum mechanics, or a chiral two-dimensional CFT [51–53, 199]. AdS_2 holography, and $AdS_2 \times S^2$ in particular, is an interesting open problem that can be approached from several distinct directions. Given the reduced dimensionality, it would be tempting to regard it as the simplest example where one can test the AdS/CFT duality, but instead it turns out to be one of the most mysterious.

The $AdS_2 \times S^2$ (coset) part of the background is conveniently encoded into a Metsaev-Tseytlin [8, 171] type action , based on the quotient

$$\frac{PSU(1,1|2)}{SO(1,1) \times SO(2)} .$$

The algebra $\mathfrak{psu}(1,1|2)$ admits a $\mathbb{Z}_4$ automorphism, which is traditionally the key to the super-coset model being classically integrable. Indeed, this is what happens in the $AdS_5 \times S^5$ [99] and $AdS_3 \times S^3 \times M^4$ [28] cases. In the AdS_2 case one can truncate the Green-Schwarz action [200] to the coset degrees of freedom, however there is no choice of κ-symmetry gauge which decouples the coset from the remaining fermions [49]. The integrability of the Green-Schwarz action for the full ten-dimensional background has been shown up to quadratic order in fermions [49, 148, 151].

In the previous Chapter, based on [184], we have used the symmetries of the system and its conjectured quantum integrability to determine the exact S-matrix for the worldsheet scattering of magnon excitations, taking the light-cone gauge-fixed $AdS_2 \times S^2 \times T^6$ action to infinite length.

This S-matrix describes the scattering above the BMN vacuum [128], which is a point-like string travelling at the speed of light on a great circle of S^2. The light-cone gauge-fixed Lagrangian is highly non-trivial and breaks two-dimensional Lorentz symmetry. Only the quadratic action preserves the Lorentz group, and describes $2+2$ (bosons+fermions) massive plus $6+6$ massless modes. The massive bosonic excitations are associated to the transverse directions in $AdS_2 \times S^2$, while the massless ones are associated to the T^6 directions.

By following a procedure which has been successful in AdS_5 and AdS_3, in [184] we fixed (up to an overall factor) the S-matrix for the excitations transforming under the $\mathfrak{psu}(1|1)^2 \ltimes \mathbb{R}$ symmetry of the BMN vacuum. In order to do that, we relaxed the level-matching condition and postulated the presence of two central extensions, while simultaneously deforming the coproduct in the standard fashion . The resulting massive S-matrix satisfies the Hopf-algebra crossing relation [191], and is unitarity so long as the overall factor (dressing phase) satisfies a certain constraint. We also studied the near-BMN expansion under certain assumptions for the dressing phase, finding consistency with the perturbative computations.

The main difference with the AdS_5 and AdS_3 cases is that the representations which scatter are *long*, and there is no shortening condition to be interpreted as the magnon dispersion relation. Furthermore, because of reducibility of the tensor product representation, the S-matrix depends on an undetermined function, which we fixed by imposing the Yang-Baxter equation. Similar features were observed in [196] for long representations in AdS_5, and in the Pohlmeyer reduction of $AdS_2 \times S^2$ superstrings. Finally, the S-matrix enjoys an accidental $U(1)$ symmetry under which only the fermionic excitations are charged, and which is connected to the presence of T^6. This $U(1)$ allows for the existence of a pseudo-vacuum state, and could be instrumental to derive the Bethe equations conjectured in [49] from our S-matrix.

Although it is not completely clear what representation one should adopt for the massless modes, in [184] we assumed that the massless representations and the corresponding S-matrix are the *zero-mass / finite h* limit of the corresponding massive ones, at least as far as the $\mathfrak{psu}(1|1)^2 \ltimes \mathbb{R}$ building block is concerned. We obtained in this way the limiting S-matrices for all the choices of left and right chiralities, and discovered that there exists a canonical Yangian for the massless sector.

In this Chapter, we obtain several results on the algebraic structure of the exact S-matrix of the system, therefore deepening our understanding of the associated spectral problem. Our aim is to explore the Yangian symmetry for the massive sector. The Yangian relevant to the AdS_5 S-matrix was found in [197], while for AdS_3 it was found in [190] in separate sectors, and a larger version encompassing both left and right algebras was discovered in [201]. The approach we will follow is based on the RTT formulation [85], which was first applied to the

AdS_5 S-matrix in [86], and later to the AdS_3 S-matrix in [202].

As discussed above, the crucial new feature in the case of interest is that the representations are long. As a result the canonical realisation of the Yangian, similar in spirit to those in [197, 190], results in a representation that is not of evaluation type. This is also a feature for long representations in the AdS_5 case [196]. After investigating this realisation, we find a new alternative realisation of the Yangian that does lead to a representation of evaluation type. This gives more control over the symmetry and its action on one-particle states. Indeed the evaluation representation is the most natural physical manifestation of Yangians in integrable scattering problems. Furthermore, this new realisation and the original one are contained within a larger family of realisations originating from a symmetry of the restricted Yangian algebra.

Besides the $\mathcal{Y}(\mathfrak{su}_c(2|2))$ Yangian, the S-matrix of the $AdS_5 \times S^5$ superstring admits an additional infinite tower of conserved charges, which constitute the so called *secret symmetry* of the model [95, 97]. Such symmetries are present in several other parts of the correspondence, for example in the pure spinor sigma model [203], scattering amplitudes [204]. Recently, secret symmetries were also found for the AdS_3 superstring [202], providing further evidence of their universal nature in the AdS/CFT framework.

We will begin in section 4.1. with a brief summary of the key properties of Yangians that will be necessary for the following exposition. In section 4.2. we use the RTT formulation to construct the Yangian algebra underlying the integrability of the massive sector, including the *secret* symmetry. Two distinct realisations of this symmetry are given, the first of which is close in spirit to that used in the AdS_5 and AdS_3 cases [86,202] and in general leads to a representation not of evaluation type, while the second is a new realisation leading to a representation that is of evaluation type. We then discuss the issue of evaluation representations in detail, demonstrating a relation to shortening (massless) condition.

In section 4.3. we explore the strong- and weak-coupling limits. First, we perform a study of the classical r-matrix, and discover that the need for the *secret* symmetry, based on the residue-analysis at the simple pole in the spectral-parameter plane, is present also in this context. Second, we study an effective Bethe ansatz in the simplifying limit of zero coupling, in which the problem reduces to a standard rational spin-chain in each copy of the symmetry algebra. In this way we obtain the spectrum of free fermions on a periodic chain. This should represent an entry-point to the leading-order (traditionally dubbed *one-loop*) anomalous dimension for the composite operators of the mysterious superconformal quantum mechanics, supposed to be dual to the superstring theory.

4.1. Yangians and integrability

We start by reviewing some of the underlying formalism of Yangians and their various realisations. Here we focus on the details that will be relevant for us when investigating the Yangian of the massive sector of the $AdS_2 \times S^2 \times T^6$ superstring.

Yangians and integrable systems. Quantum systems exhibiting Yangian symmetry are integrable, as a consequence of the fact that the Yangian allows to construct a solution to the Yang-Baxter equation which controls the inverse scattering problem (and often the S-matrix of the excitations) [205]. Indeed, let $R \in A \otimes A$ be a scattering matrix, with $A = \mathcal{Y}(\mathfrak{g})$ being the Hopf superalgebra whose coproduct $\Delta : A \to A \otimes A$ defines the action of the conserved charges on two-particle states. If A and R are such that

$$\begin{aligned} &\Delta^{\rm op}(a)\, R = R\, \Delta(a) && \forall a \in A\ , && \text{(quasi co-commutativity)},\\ &(\Delta \otimes \mathbb{1})(R) = R_{13}R_{23}, && (\mathbb{1} \otimes \Delta)(R) = R_{13}R_{12}\ , && \text{(quasi-triangularity)}, \end{aligned} \tag{4.1}$$

where $\Delta^{\rm op} = (\sigma \circ \Delta)$ is the opposite coproduct, σ the graded permutation operator and the subscripts 1,2,3 indicate the copy of A in the triple tensor product, then R obeys the *quantum Yang-Baxter equation* (QYBE)

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{4.2}$$

(the hallmark of integrability) and the *crossing symmetry* equations. This is the content of a famous theorem of Drinfeld, proving the crucial role played by quantum groups in producing solutions to the QYBE with the desired symmetry properties [91].

4.2. RTT realisation for the $\mathcal{Y}(\mathfrak{gl}_c(1|1))$ Yangian

In this section we employ the techniques of [85, 86] to construct the RTT realisation for the $\mathcal{Y}(\mathfrak{gl}_c(1|1))$ Yangian. The starting point is the $\mathfrak{su}_c(1|1)$ R-matrix of [184], which describes the scattering of one bosonic state $|\phi\rangle$ and one fermionic state $|\psi\rangle$ transforming in a long representation of the centrally-extended algebra $\mathfrak{su}_c(1|1)$

$$\{\mathbb{Q},\mathbb{Q}\} = 2\,\mathbb{P}\ , \qquad \{\mathbb{S},\mathbb{S}\} = 2\,\mathbb{K}\ , \qquad \{\mathbb{Q},\mathbb{S}\} = 2\,\mathbb{H}\ . \tag{4.3}$$

The explicit form of the representation is given by

$$\begin{aligned} &\mathbb{Q}|\phi\rangle = a|\psi\rangle\ , && \mathbb{Q}|\psi\rangle = b|\phi\rangle\ , && \mathbb{S}|\phi\rangle = c|\psi\rangle\ , && \mathbb{S}|\psi\rangle = d|\phi\rangle\ ,\\ &\mathbb{P}|\Phi\rangle = P|\Phi\rangle\ , && \mathbb{K}|\Phi\rangle = K|\Phi\rangle\ , && \mathbb{H}|\Phi\rangle = H|\Phi\rangle\ , && |\Phi\rangle \in \{|\phi\rangle, |\psi\rangle\}\ . \end{aligned} \tag{4.4}$$

where the eigenvalues of the central elements $\mathbb{P}$, $\mathbb{K}$ and $\mathbb{H}$ are given by

$$P = ab \ , \qquad K = cd \ , \qquad 2H = ad + bc \ , \tag{4.5}$$

as a consequence of the algebra relations (4.3). There are no further conditions on the central elements for the algebra to close and hence this 2-dimensional representation is long.

For generic values of P, K and H the tensor product of two of these representations gives a 4-dimensional representation that is fully reducible into two 2-dimensional representations.[1] Therefore, the R-matrix acting on the tensor product is fixed up to two functions by demanding invariance under the symmetry

$$\Delta^{\mathrm{op}}(\mathbb{J})\, R = R\, \Delta(\mathbb{J}) \ , \qquad \mathbb{J} = \{\mathbb{Q}, \mathbb{S}, \mathbb{P}, \mathbb{K}, \mathbb{H}\} \ . \tag{4.6}$$

Here Δ is the coproduct, while Δ^{op} is the opposite coproduct defined in (4.1). As usual for integrable systems arising in the context of the AdS/CFT correspondence the coproduct is deformed through the introduction of an abelian generator $\mathbb{U}$

$$\begin{aligned} &\Delta(\mathbb{Q}) = \mathbb{Q} \otimes \mathbb{1} + \mathbb{U} \otimes \mathbb{Q} \ , \quad \Delta(\mathbb{S}) = \mathbb{S} \otimes \mathbb{1} + \mathbb{U}^{-1} \otimes \mathbb{S} \ , \quad \Delta(\mathbb{U}) = \mathbb{U} \otimes \mathbb{U} \ , \\ &\Delta(\mathbb{P}) = \mathbb{P} \otimes \mathbb{1} + \mathbb{U}^2 \otimes \mathbb{P} \ , \quad \Delta(\mathbb{K}) = \mathbb{K} \otimes \mathbb{1} + \mathbb{U}^{-2} \otimes \mathbb{K} \ , \quad \Delta(\mathbb{H}) = \mathbb{H} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{H} \ . \end{aligned} \tag{4.7}$$

To admit an R-matrix the coproducts of the central elements should be co-commutatitve, i.e. $\Delta^{\mathrm{op}}(\mathbb{C}) = \Delta(\mathbb{C})$. This relates the central charges $\mathbb{P}$ and $\mathbb{K}$ to the braiding factor $\mathbb{U}$:

$$\mathbb{P} = \frac{h}{2}(1 - \mathbb{U}^2) \ , \qquad \mathbb{K} = \frac{h}{2}(1 - \mathbb{U}^{-2}) \ , \tag{4.8}$$

where without loss of generality we have taken the constants of proportionality to be equal. In the following we will refer to the algebra (4.3) with these relations (4.8) imposed as the restricted algebra. While the symmetry only constrains the R-matrix up to two functions, demanding that the QYBE is solved (along with imposing various physical requirements such as crossing symmetry and a sensible strong coupling limit) fixes the R-matrix up to a single overall factor.

For reference we will quote the necessary details of the $\mathfrak{su}_c(1|1)$ R-matrix from [184]. The R-matrix in terms of the usual Zhukovsky variables

$$\frac{x^+}{x^-} = U^2 \ , \qquad x^+ - \frac{1}{x^+} - x^- + \frac{1}{x^-} = \frac{iH}{h} \ , \qquad x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{iM}{h} \ . \tag{4.9}$$

Here

$$M = \frac{ad - bc}{2} = \sqrt{H^2 - PK} \ , \tag{4.10}$$

[1] At the special (massless) point $H_{\mathrm{tot}}^2 - P_{\mathrm{tot}} K_{\mathrm{tot}} = 0$ the tensor product is still reducible but no longer decomposable. Here the subscript "tot" indicates that these are the eigenvalues of the central charges acting on the tensor product state.

is unconstrained as the representation (4.4) is long. The representation parameters a, b, c and d are given in terms of the Zhukovsky variables in [184] – we set the parameter α used there, which controls the normalisation of the bosonic state relative to the fermionic state, to one. The R-matrix is then given by

$$\begin{aligned} R\,|\phi_x\phi_y\rangle &= S_1\,|\phi_x\phi_y\rangle + Q_1\,|\psi_x\psi_y\rangle\ , \qquad & R\,|\psi_x\psi_y\rangle &= S_2\,|\psi_x\psi_y\rangle + Q_2\,|\phi_x\phi_y\rangle\ , \\ R\,|\phi_x\psi_y\rangle &= T_1\,|\phi_x\psi_y\rangle + R_1\,|\psi_x\phi_y\rangle\ , \qquad & R\,|\psi_x\phi_y\rangle &= T_2\,|\psi_x\phi_y\rangle + R_2\,|\phi_x\psi_y\rangle\ , \end{aligned} \tag{4.11}$$

with $x^\pm$ the kinematic variables associated to the first representation and $y^\pm$ to the second. The parameterising functions are given by [184]

$$\begin{aligned} S_1 &= \sqrt{\frac{x^+y^-}{x^-y^+}}\frac{x^- - y^+}{x^+ - y^-}\frac{1+s_1}{2}P_0\ , \qquad & S_2 &= \frac{1+s_2}{2}P_0\ , \\ T_1 &= \sqrt{\frac{y^-}{y^+}}\frac{x^+ - y^+}{x^+ - y^-}\frac{1+t_1}{2}P_0\ , \qquad & T_2 &= \sqrt{\frac{x^+}{x^-}}\frac{x^- - y^-}{x^+ - y^-}\frac{1+t_2}{2}P_0\ , \\ Q_1 = Q_2 &= -\frac{i}{2}\sqrt[4]{\frac{x^-y^+}{x^+y^-}}\frac{\eta_x\eta_y}{x^+ - y^-}\frac{f}{x^-x'^+}P_0\ , \qquad & R_1 = R_2 &= -\frac{i}{2}\sqrt[4]{\frac{x^+y^-}{x^-y^+}}\frac{\eta_x\eta_y}{x^+ - y^-}P_0\ , \end{aligned} \tag{4.12}$$

where

$$\begin{aligned} f &= \frac{\sqrt{\frac{x^+}{x^-}}(x^- - \frac{1}{x^+}) - \sqrt{\frac{y^+}{y^-}}(y^- - \frac{1}{y^+})}{1 - \frac{1}{x^+x^-y^+y^-}}P_0\ , \quad & s_1 &= \frac{1 - \frac{1}{x^+y^-}}{x^- - y^+}f\ , \quad & s_2 &= \frac{1 - \frac{1}{x^-y^+}}{x^+ - y^-}f\ , \\ \eta_x &= \sqrt{i(x^- - x^+)}\ , \quad \eta_y = \sqrt{i(y^- - y^+)}\ , \quad & t_1 &= \frac{1 - \frac{1}{x^-y^-}}{x^+ - y^+}f\ , \quad & t_2 &= \frac{1 - \frac{1}{x^+y^+}}{x^- - y^-}f\ . \end{aligned} \tag{4.13}$$

As discussed above, there is an overall factor P_0 that is not fixed by the considerations of symmetry and as such will not be relevant for the following analysis. For concreteness we will fix P_0 such that $S_1 = 1$ and the R-matrix is normalised as in [86].

Finally, let us observe that the $AdS_2 \times S^2$ worldsheet S-matrix underlying the scattering of the massive modes is built from the tensor product of two copies of this centrally-extended $\mathfrak{su}(1|1)$ R-matrix.

4.2.1. The $\mathcal{Y}(\mathfrak{su}_c(1|1))$ and $\mathcal{Y}(\mathfrak{gl}_c(1|1))$ Yangians and their RTT realisation

The centrally extended $\mathcal{Y}(\mathfrak{su}_c(1|1))$ Yangian is defined by the graded commutation relations

$$\{\mathbb{Q}_m, \mathbb{Q}_n\} = 2\,\mathbb{P}_{m+n}\ , \qquad \{\mathbb{S}_m, \mathbb{S}_n\} = 2\,\mathbb{K}_{m+n}\ , \qquad \{\mathbb{Q}_m, \mathbb{S}_n\} = 2\,\mathbb{H}_{m+n}\ , \tag{4.14}$$

which extend the algebra (4.3). Here, $m, n \geq 0$ indicate the level of the corresponding generator, with $\mathbb{Q}_0$, $\mathbb{S}_0$, $\mathbb{P}_0$, $\mathbb{K}_0$ and $\mathbb{H}_0$ playing the role of the original generators in (4.3).

It is worth noting that, in contrast to ordinary situations, the infinite-dimensional algebra (4.14) contains infinitely many finite-dimensional $\mathfrak{su}_c(1|1)$ subalgebras. Indeed, the set of generators $\mathbb{Q}_{\hat{m}}$, $\mathbb{S}_{\hat{n}}$, $\mathbb{P}_{2\hat{m}}$, $\mathbb{K}_{2\hat{n}}$ and $\mathbb{H}_{\hat{m}+\hat{n}}$ forms a subalgebra for all $\hat{m}, \hat{n} \geq 0$. This is due to the fact that the generators on the right-hand side of the relations (4.14) are central.

Another consequence of this, again in contrast to usual, is that for an arbitrary representation one cannot generate the whole infinite dimensional algebra by (anti) commuting a finite set of generators. However, considering the natural lift of the 2-dimensional representation (4.4)

$$\begin{aligned} &\mathbb{Q}_m|\phi\rangle = a_m|\psi\rangle \ , \quad \mathbb{Q}_m|\psi\rangle = b_m|\phi\rangle \ , \quad \mathbb{S}_m|\phi\rangle = c_m|\psi\rangle \ , \quad \mathbb{S}_m|\psi\rangle = d_m|\phi\rangle \ , \\ &\mathbb{P}_m|\Phi\rangle = P_m|\Phi\rangle \ , \quad \mathbb{K}_m|\Phi\rangle = K_m|\Phi\rangle \ , \quad \mathbb{H}_m|\Phi\rangle = H_m|\Phi\rangle \ , \quad |\Phi\rangle \in \{|\phi\rangle, |\psi\rangle\} \ , \end{aligned} \tag{4.15}$$

it is relatively easy to see that the relations

$$\begin{aligned} 2H_m &= a_0 d_m + b_0 c_m = a_1 d_{m-1} + b_1 c_{m-1} \ , \quad 2P_m = a_0 b_m + b_0 a_m = a_1 b_{m-1} + b_1 a_{m-1} \ , \\ 2H_m &= c_0 b_m + d_0 a_m = c_1 b_{m-1} + d_1 a_{m-1} \ , \quad 2K_m = c_0 d_m + d_0 c_m = c_1 d_{m-1} + d_1 c_{m-1} \ , \end{aligned} \tag{4.16}$$

which follow from the graded commutation relations (4.14), can be used to solve recursively for the higher-level representation parameters given their values at level 0 and 1.

As we will see, one property that does carry down from the higher-dimensional cases is that the $\mathfrak{su}_c(1|1)$ R-matrix (4.11), (4.12), (4.13) exhibits an additional family of symmetries, $\mathbb{B}_n$, $n \geq 1$, known as *bonus* or *secret*. These symmetries enhance the $\mathcal{Y}(\mathfrak{su}_c(1|1))$ Yangian to some *indented* Yangian-like quantum group we call $\mathcal{Y}(\mathfrak{gl}_c(1|1))$, which contains all the generators in (4.14) along with $\mathbb{B}_n$ (not including $\mathbb{B}_0$).

To construct the RTT realisation of the Yangian we introduce a spectral parameter u defined in terms of $x^\pm$, and similarly v for $y^\pm$. We shall use two different definitions for the spectral parameter, but in both cases the expansion of $x^\pm$ in powers of u^{-1} will take the form

$$x^\pm = u + \mathcal{O}(1) \ . \tag{4.17}$$

Once this expansion has been specified, we expand the R-matrix $R(x^\pm, y^\pm)$ in inverse powers of one of the two spectral parameters, say u,[2] and from the resulting Laurent coefficients extract a series of generators J_m whose graded commutation relations reproduce those of the underlying infinite-dimensional symmetry algebra. We can then define abstract generators $\mathbb{J}_m$, of which J_m are a representation. At this point we can construct generators satisfying (4.14) along with their

[2]For definiteness we will assume that $M(x^+, x^-) = 2$. For the second set of kinematical variables $y_\pm$, in terms of which we find the symmetry generators, we will not assume anything and consequently the symmetries we find hold for any M in (4.10). If we leave $M(x^+, x^-)$ unfixed it simply appears as an overall factor in the generators, for example Q_m , $S_m \sim M(x^+, x^-)^{\frac{1}{2}}$, and hence gives no new information.

coproducts and antipodes. It should be noted that this construction automatically leads to the restricted Yangian, for which, in addition to (4.8), the higher-level central charges $\mathbb{P}_m$ and $\mathbb{K}_m$ are defined in terms of lower-level central elements. We refer the reader to [86, 202] for a more complete discussion.

4.2.2. Canonical representation

Let us start by considering the canonical spectral parameter and a Hopf algebra structure that is close in spirit to that used in the AdS_5 and AdS_3 cases [86, 202]. The spectral parameter is given by

$$u = \frac{1}{2}\big(x^+ + \frac{1}{x^+} + x^- + \frac{1}{x^-}\big) , \tag{4.18}$$

such that (assuming $M(x^+, x^-) = 2$) the expansions of $x^\pm$ are

$$x^\pm = u \pm \frac{i}{h} - \frac{1}{u} \pm \frac{i}{hu^2} + \mathcal{O}(u^{-3}) . \tag{4.19}$$

Following the procedure outlined above we then identify the following combinations[3]

$$\begin{aligned}
&\mathbb{Q}_0 = \sqrt{i}h\, \mathbb{J}^2_{01} , && \mathbb{Q}_1 = \sqrt{i}h\, \mathbb{J}^2_{11} - \frac{i}{2}\left(\mathbb{1} + \mathbb{U}^2\right) \sqrt{-i}h\, \mathbb{J}^1_{02} , \\
&\mathbb{S}_0 = \sqrt{-i}h\, \mathbb{J}^1_{02} , && \mathbb{S}_1 = \sqrt{-i}h\, \mathbb{J}^1_{12} + \frac{i}{2}\left(\mathbb{1} + \mathbb{U}^{-2}\right) \sqrt{i}h\, \mathbb{J}^2_{01} , \\
&\mathbb{H}_0 = \frac{ih}{2}\left(\mathbb{J}^1_{01} - \mathbb{J}^2_{02}\right) , && \mathbb{H}_1 = \frac{ih}{2}\left(\mathbb{J}^1_{11} - \mathbb{J}^2_{12}\right) - \frac{ih}{4}\left(\mathbb{U}^2 - \mathbb{U}^{-2}\right) , \\
&\mathbb{P}_0 = \frac{h}{2}\left(\mathbb{1} - \mathbb{U}^2\right) , && \mathbb{P}_1 = -i\left(\mathbb{1} + \mathbb{U}^2\right) \mathbb{H}_0 , \\
&\mathbb{K}_0 = \frac{h}{2}\left(\mathbb{1} - \mathbb{U}^{-2}\right) , && \mathbb{K}_1 = i\left(\mathbb{1} + \mathbb{U}^{-2}\right) \mathbb{H}_0 ,
\end{aligned} \tag{4.20}$$

which satisfy the defining commutation relations (4.14). Evaluated in the representation arising from the expansion of R-matrix, the level-0 generators coincide with those used in [184].

Coproducts. The level-0 coproducts are given in (4.7), while the level-1 coproducts can be constructed from (1.49) and read

$$\begin{aligned}
\Delta(\mathbb{Q}_1) &= \mathbb{Q}_1 \otimes \mathbb{1} + \mathbb{U} \otimes \mathbb{Q}_1 \\
&\quad - \frac{i}{h}\mathbb{Q}_0 \otimes \mathbb{H}_0 + \frac{i}{h}\mathbb{U}\mathbb{H}_0 \otimes \mathbb{Q}_0 + \frac{i}{h}\mathbb{U}^2\mathbb{S}_0 \otimes \mathbb{P}_0 - \frac{i}{h}\mathbb{U}^{-1}\mathbb{P}_0 \otimes \mathbb{S}_0 , \\
\Delta(\mathbb{S}_1) &= \mathbb{S}_1 \otimes \mathbb{1} + \mathbb{U}^{-1} \otimes \mathbb{S}_1 \\
&\quad + \frac{i}{h}\mathbb{S}_0 \otimes \mathbb{H}_0 - \frac{i}{h}\mathbb{U}^{-1}\mathbb{H}_0 \otimes \mathbb{S}_0 - \frac{i}{h}\mathbb{U}^{-2}\mathbb{Q}_0 \otimes \mathbb{K}_0 + \frac{i}{h}\mathbb{U}\mathbb{K}_0 \otimes \mathbb{Q}_0 , \\
\Delta(\mathbb{H}_1) &= \mathbb{H}_1 \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{H}_1 + \frac{i}{h}\mathbb{U}^{-2}\mathbb{P}_0 \otimes \mathbb{K}_0 + \frac{i}{h}\mathbb{U}^2\mathbb{K}_0 \otimes \mathbb{P}_0 ,
\end{aligned} \tag{4.21}$$

[3] Here the branch cut is chosen such that $\sqrt{i} = e^{\frac{i\pi}{4}}$ and $\sqrt{-i} = e^{-\frac{i\pi}{4}}$.

where $\mathbb{P}_0$ and $\mathbb{K}_0$ are defined in terms of lower-level central elements in (4.20). Indeed inserting these definitions, the coproduct for $\mathbb{H}_1$ becomes manifestly co-commutative as expected.

The coproducts for $\mathbb{P}_1$ and $\mathbb{K}_1$ can be obtained from the graded commutation relations

$$\Delta(\mathbb{P}_1) = \frac{1}{2}\{\Delta(\mathbb{Q}_0), \Delta(\mathbb{Q}_1)\} , \qquad \Delta(\mathbb{K}_1) = \frac{1}{2}\{\Delta(\mathbb{S}_0), \Delta(\mathbb{S}_1)\} . \tag{4.22}$$

If $\mathbb{P}_1$ and $\mathbb{K}_1$ are defined in terms of lower-level central elements as in (4.20) we find that these coproducts are also co-commutative as required. Moreover, we can compute the coproducts for the level-2 central charges

$$\Delta(\mathbb{P}_2) = \frac{1}{2}\{\Delta(\mathbb{Q}_1), \Delta(\mathbb{Q}_1)\} , \quad \Delta(\mathbb{K}_2) = \frac{1}{2}\{\Delta(\mathbb{S}_1), \Delta(\mathbb{S}_1)\} , \quad \Delta(\mathbb{H}_2) = \frac{1}{2}\{\Delta(\mathbb{Q}_1), \Delta(\mathbb{S}_1)\} . \tag{4.23}$$

Doing so, we find that for co-commutativity of $\Delta(\mathbb{P}_2)$ and $\Delta(\mathbb{K}_2)$ we require

$$\begin{aligned} \mathbb{P}_2 &= -i \left(\mathbb{1} + \mathbb{U}^2\right) \mathbb{H}_1 - \frac{1}{2h} \left(\mathbb{1} - \mathbb{U}^2\right) (\mathbb{H}_0^2 - \mathbb{P}_0\mathbb{K}_0) , \\ \mathbb{K}_2 &= i \left(\mathbb{1} + \mathbb{U}^{-2}\right) \mathbb{H}_1 - \frac{1}{2h} \left(\mathbb{1} - \mathbb{U}^{-2}\right) (\mathbb{H}_0^2 - \mathbb{P}_0\mathbb{K}_0) , \end{aligned} \tag{4.24}$$

where the normalisation is fixed by matching with the expansion of the R-matrix,[4] while the coproduct $\Delta(\mathbb{H}_2)$ is automatically co-commutative upon using the definitions of $\mathbb{P}_{0,1}$ and $\mathbb{K}_{0,1}$ in (4.20).

With these definitions of the generators, one can show that the representation of the Yangian arising from the expansion of the R-matrix is in general not evaluation. However, if the eigenvalues of the central elements satisfy

$$H_0^2 - P_0 K_0 = 0 , \tag{4.25}$$

which has the interpretation as a (massless) shortening condition, we find that the representation does become of evaluation type, in agreement with [184]. We will return to this issue in the following sections.

Crossing. Using the general expression for the antipodes of (1.52) we can derive the antipode for the generators of interest

$$\begin{aligned} &\Sigma\left[\mathbb{Q}_m\right] = -\mathbb{U}^{-1}\mathbb{Q}_m , && \Sigma\left[\mathbb{S}_m\right] = -\mathbb{U}\mathbb{S}_m , \\ &\Sigma\left[\mathbb{H}_m\right] = -\mathbb{H}_m , && \Sigma\left[\mathbb{P}_m\right] = \mathbb{K}_m , \qquad m = 0, 1 , \end{aligned} \tag{4.26}$$

i.e. it is involutive for the generators of $\mathcal{Y}_{0,1}(\mathfrak{su}_c(1|1))$.

[4] It is worth recalling that the coproducts for the central charges arising from the RTT realisation of the Yangian are co-commutative by construction, and hence, up to a normalisation, $\mathbb{P}_2$ and $\mathbb{K}_2$ have to take this form when evaluated in the representation arising from the expansion of the R-matrix.

Secret symmetry. While $\mathbb{J}^1_{01} + \mathbb{J}^2_{02}$ is central, the combination

$$\mathbb{B}_1 = -\frac{ih}{2}\left(\mathbb{J}^1_{11} + \mathbb{J}^2_{12}\right) \ , \tag{4.27}$$

satisfies

$$[\mathbb{B}_1, \mathbb{Q}_0] = \mathbb{Q}_1 + i\left(\mathbb{1} + \mathbb{U}^2\right)\mathbb{S}_0 \ , \qquad [\mathbb{B}_1, \mathbb{S}_0] = -\mathbb{S}_1 + i\left(\mathbb{1} + \mathbb{U}^{-2}\right)\mathbb{Q}_0 \ , \tag{4.28}$$

The coproduct reads

$$\Delta(\mathbb{B}_1) = \mathbb{B}_1 \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{B}_1 - \frac{i}{2h}\mathbb{U}^{-1}\mathbb{Q}_0 \otimes \mathbb{S}_0 - \frac{i}{2h}\mathbb{U}\mathbb{S}_0 \otimes \mathbb{Q}_0 \ , \tag{4.29}$$

while the antipode for the secret symmetry is

$$\Sigma\left[\mathbb{B}_1\right] = -\mathbb{B}_1 - \frac{i}{h}\mathbb{H}_0 \ . \tag{4.30}$$

As in the AdS_5 and AdS_3 case, the antipode is not an involution when acting upon the secret symmetry.

4.2.3. Co-commutativity and shortening condition

Let us now investigate what happens if we try to impose evaluation representation onto the Hopf algebra structure described in section (4.2.2.). In particular, we will show that this demonstrates the existence of representations for which one of the higher-level central charges is not co-commutative and hence does not admit an R-matrix.

In evaluation representation we have

$$J_n = u^n J_0 \qquad \forall \ J \in \{Q, S, H, P, K\} \ , \tag{4.31}$$

which manifestly satisfies the algebra relations (4.14). As discussed above, in order to have a co-commutative coproduct for all the level-0 central charges one can impose

$$P_0 = \frac{h}{2}(1 - U^2) \ , \qquad K_0 = \frac{h}{2}(1 - U^{-2}) \ , \tag{4.32}$$

where we recall that the constant h is independent on the representation space of the coproduct. Similarly, for the level-1 central charges for co-commutativity one can impose

$$P_1 = -i(1 + U^2)H_0 \ , \qquad K_1 = i(1 + U^{-2})H_0 \ , \tag{4.33}$$

It then follows that for a representation of evaluation type the spectral parameter is given by

$$u = -\frac{2i}{h}\frac{1 + U^2}{1 - U^2}H_0 \ , \tag{4.34}$$

where we use H_m, P_m, K_m and U to denote both the generator in evaluation representation and its eigenvalue, as it is always clear from context which is meant.

At this point, we compute the coproduct of the level-2 central charge P_2 using

$$\Delta(P_2) = \frac{1}{2}\{\Delta(Q_1), \Delta(Q_1)\} \ . \tag{4.35}$$

The expression one obtains is rather lengthy, however it simplifies considerably if one takes the antisymmetric combination

$$\delta P_2 \equiv \Big(\Delta(P_2) - \Delta^{\text{op}}(P_2)\Big) \ , \tag{4.36}$$

which is precisely the quantity that determines whether the coproduct is co-commutative or not.

There are two notable contributions to δP_2, coming from two separate pieces of the coproduct of P_2. The first contribution comes from the part of $\Delta(P_2)$ arising when the S_0 generators in $\Delta(Q_1)$ meet among themselves in the anti-commutator (4.35). Upon antisymmetrisation these terms contribute

$$-\frac{1}{2h}\big((1-U^2)\otimes(1-U^2)\big)\big(P_0K_0\otimes 1 - 1\otimes P_0K_0\big) \ , \tag{4.37}$$

to δP_2. In fact this would be the only surviving term had we set $H_0 = u = 0$ in both representation spaces. In principle this could already be enough to conclude that there exist representations with non co-commutative P_2. Nevertheless, it is instructive to continue.

The remaining terms reduce to

$$\frac{1}{2h}\big((1-U^2)\otimes(1-U^2)\big)\big(H_0^2\otimes 1 - 1\otimes H_0^2\big) \tag{4.38}$$

and, combining all contributions together, we obtain

$$\delta P_2 = \frac{1}{2h}\big((1-U^2)\otimes(1-U^2)\big)\big((H_0^2 - P_0K_0)\otimes 1 - 1\otimes(H_0^2 - P_0K_0)\big) \ . \tag{4.39}$$

This means that we can achieve co-commutativity if we demand that

$$H_0^2 - P_0K_0 = \text{constant} \ , \tag{4.40}$$

where the constant does not depend on the representation space. The relation (4.40) is nothing else than the known shortening condition, which is in this way reinterpreted as the condition that makes the central charges' coproduct co-commutative at higher levels (similarly to what the Serre relations do for the canonical part of the Yangian).

4.2.4. Evaluation representation

The Hopf algebra structure discussed in section 4.2.2., which was motivated by similar constructions in the AdS_5 and AdS_3 cases, turned out to give a representation of the Yangian that was not of evaluation type. It turns out that there is an alternative Hopf algebra structure we can put on the same infinite-dimensional algebra, such that the representation arising from the expansion of the R-matrix is evaluation.

To do this we introduce a new spectral parameter

$$u = \frac{1}{4}\Big(1+\sqrt{\frac{x^+}{x^-}}\Big)^2\Big(x^- + \frac{1}{x^+}\Big) = \frac{1}{4}\Big(1+\sqrt{\frac{x^-}{x^+}}\Big)^2\Big(x^+ + \frac{1}{x^-}\Big) \ , \tag{4.41}$$

such that the expansions of $x^\pm$ (again assuming that $M(x^+, x^-) = 2$) are given by

$$x^\pm = u \pm \frac{i}{h} - \Big(1+\frac{1}{4h^2}\Big)\frac{1}{u} \pm \frac{i}{h\,u^2} + \mathcal{O}(u^{-3}) \ , \tag{4.42}$$

Let us now define the following (as an alternative to (4.20)) combinations of generators

$$\begin{aligned}
&\mathbb{Q}_0 = \sqrt{ih}\,\mathbb{J}^2_{01} \ , && \mathbb{Q}_1 = \sqrt{ih}\,\mathbb{J}^2_{11} - i\mathbb{U}\sqrt{-ih}\,\mathbb{J}^1_{02} \ , \\
&\mathbb{S}_0 = \sqrt{-ih}\,\mathbb{J}^1_{02} \ , && \mathbb{S}_1 = \sqrt{-ih}\,\mathbb{J}^1_{12} + i\mathbb{U}^{-1}\sqrt{ih}\,\mathbb{J}^2_{01} \ , \\
&\mathbb{H}_0 = \frac{ih}{2}\left(\mathbb{J}^1_{01} - \mathbb{J}^2_{02}\right) \ , && \mathbb{H}_1 = \frac{ih}{2}\left(\mathbb{J}^1_{11} - \mathbb{J}^2_{12}\right) - \frac{ih}{2}\left(\mathbb{U} - \mathbb{U}^{-1}\right) \ , \\
&\mathbb{P}_0 = \frac{h}{2}\left(\mathbb{1} - \mathbb{U}^2\right) \ , && \mathbb{P}_1 = -\frac{i}{2}\left(\mathbb{1} + \mathbb{U}\right)^2\mathbb{H}_0 \ , \\
&\mathbb{K}_0 = \frac{h}{2}\left(\mathbb{1} - \mathbb{U}^{-2}\right) \ , && \mathbb{K}_1 = \frac{i}{2}\left(\mathbb{1} + \mathbb{U}^{-1}\right)^2\mathbb{H}_0 \ ,
\end{aligned} \tag{4.43}$$

which also satisfy the defining commutation relations (4.14). Again, evaluated in the representation arising from the expansion of R-matrix, the level-0 generators coincide with those used in [184].

Coproducts. The level-0 coproducts are given in (4.7), while the level-1 coproducts can be constructed from (1.49) and read

$$\begin{aligned}
\Delta(\mathbb{Q}_1) &= \mathbb{Q}_1 \otimes \mathbb{1} + \mathbb{U} \otimes \mathbb{Q}_1 \\
&\quad - \frac{i}{h}\mathbb{Q}_0 \otimes \mathbb{H}_0 + \frac{i}{h}\mathbb{U}\mathbb{H}_0 \otimes \mathbb{Q}_0 + i\mathbb{U}\mathbb{S}_0 \otimes (\mathbb{1} - \mathbb{U}) - i(\mathbb{1} - \mathbb{U}) \otimes \mathbb{U}\mathbb{S}_0 \ , \\
\Delta(\mathbb{S}_1) &= \mathbb{S}_1 \otimes \mathbb{1} + \mathbb{U}^{-1} \otimes \mathbb{S}_1 \\
&\quad + \frac{i}{h}\mathbb{S}_0 \otimes \mathbb{H}_0 - \frac{i}{h}\mathbb{U}^{-1}\mathbb{H}_0 \otimes \mathbb{S}_0 - i\mathbb{U}^{-1}\mathbb{Q}_0 \otimes (\mathbb{1} - \mathbb{U}^{-1}) + i(\mathbb{1} - \mathbb{U}^{-1}) \otimes \mathbb{U}^{-1}\mathbb{Q}_0 \ , \\
\Delta(\mathbb{H}_1) &= \mathbb{H}_1 \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{H}_1 \\
&\quad - \frac{ih}{2}\big(\mathbb{U} \otimes \mathbb{U} - \mathbb{U}^{-1} \otimes \mathbb{U}^{-1} - (\mathbb{U} - \mathbb{U}^{-1}) \otimes \mathbb{1} - \mathbb{1} \otimes (\mathbb{U} - \mathbb{U}^{-1})\big) \ .
\end{aligned} \tag{4.44}$$

Here $\Delta(\mathbb{H}_1)$ is written in a manifestly co-commutative form. To check that these coproducts, along with those in (4.7), obey the graded commutation relations (4.14) one needs to use the definitions of $\mathbb{P}_0$ and $\mathbb{K}_0$ in (4.43).

The coproducts for $\mathbb{P}_1$ and $\mathbb{K}_1$ can be obtained from the graded commutation relations (4.22). If $\mathbb{P}_1$ and $\mathbb{K}_1$ are defined in terms of lower-level central elements as in (4.43) we find that these coproducts are also co-commutative as required. Furthermore, we can compute the coproducts for the level-2 central charges using (4.23). Co-commutativity of $\Delta(\mathbb{P}_2)$ and $\Delta(\mathbb{K}_2)$ then requires

$$\mathbb{P}_2 = -2i\mathbb{U}\mathbb{H}_1 - \frac{1}{2h}\left(\mathbb{1} - \mathbb{U}^2\right)\mathbb{H}_0^2 \; ,$$
$$\mathbb{K}_2 = 2i\mathbb{U}^{-1}\mathbb{H}_1 - \frac{1}{2h}\left(\mathbb{1} - \mathbb{U}^{-2}\right)\mathbb{H}_0^2 \; , \tag{4.45}$$

where normalisations are fixed by matching with the expansion of the R-matrix, while the coproduct $\Delta(\mathbb{H}_2)$ is automatically co-commutative by the definitions of $\mathbb{P}_{0,1}$ and $\mathbb{K}_{0,1}$ in (4.43).

From (4.43) we find that the representation of the Yangian arising from the R-matrix expansion is indeed of evaluation type with spectral parameter (4.41)

$$u = -\frac{i}{h}\frac{1+U}{1-U}H_0 \; . \tag{4.46}$$

Crossing. Using the general expression for the antipodes of (1.52) we can derive the antipode for the generators of interest. These turn out to be the same as for the canonical case, i.e. (4.26).

Secret symmetry. While $\mathbb{J}^1_{01} + \mathbb{J}^2_{02}$ is central, the combination

$$\mathbb{B}_1 = -\frac{ih}{2}\left(\mathbb{J}^1_{11} + \mathbb{J}^2_{12}\right) \; , \tag{4.47}$$

satisfies

$$[\mathbb{B}_1, \mathbb{Q}_0] = \mathbb{Q}_1 + \frac{i}{2}\left(\mathbb{1} + \mathbb{U}\right)^2 \mathbb{S}_0 \; , \qquad [\mathbb{B}_1, \mathbb{S}_0] = -\mathbb{S}_1 + \frac{i}{2}\left(\mathbb{1} + \mathbb{U}^{-1}\right)^2 \mathbb{Q}_0 \; . \tag{4.48}$$

The coproduct reads

$$\Delta(\mathbb{B}_1) = \mathbb{B}_1 \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{B}_1 - \frac{i}{2h}\mathbb{U}^{-1}\mathbb{Q}_0 \otimes \mathbb{S}_0 - \frac{i}{2h}\mathbb{U}\mathbb{S}_0 \otimes \mathbb{Q}_0 \; , \tag{4.49}$$

while the antipode for the secret symmetry is

$$\Sigma\left[\mathbb{B}_1\right] = -\mathbb{B}_1 - \frac{i}{h}\mathbb{H}_0 \; . \tag{4.50}$$

As before, the antipode is not an involution when acting upon the secret symmetry.

It is worth highlighting that the modification of the combinations in (4.43) has altered the commutation relations involving $\mathbb{B}_1$, changing the tail.

4.2.5. Freedom in the realisation of the Yangian

The two different Hopf algebra structures described in sections 4.2.2. and 4.2.4. are indicative of a larger possible freedom, which we will now describe. Let us consider the defining graded commutation relations (4.14), but in particular focus on the restricted form in which $\mathbb{P}_m$ and $\mathbb{K}_m$ are defined in terms of lower-level central charges.

Motivated by the definitions of $\mathbb{P}_{0,1,2}$ and $\mathbb{K}_{0,1,2}$ in (4.20), (4.24), (4.43), (4.45), we postulate that the following relation is true for all levels:

$$\mathbb{U}^{-1}\mathbb{P}_m = -\mathbb{U}\mathbb{K}_m \; . \tag{4.51}$$

If we now consider the following redefinitions:

$$\begin{aligned}
&\tilde{\mathbb{Q}}_0 = \mathbb{Q}_0 \; , \qquad \tilde{\mathbb{S}}_0 = \mathbb{S}_0 \; , \qquad \tilde{\mathbb{P}}_0 = \mathbb{P}_0 \; , \qquad \tilde{\mathbb{K}}_0 = \mathbb{K}_0 \; , \qquad \tilde{\mathbb{H}}_0 = \mathbb{H}_0 \; , \\
&\tilde{\mathbb{Q}}_m = \mathbb{Q}_m + \sum_{k=0}^{m-1} y_{q_{m,k}} \mathbb{Q}_{m-k} + z_{q_{m,k}} \mathbb{S}_{m-k} \; , \qquad \tilde{\mathbb{S}}_m = \mathbb{S}_m + \sum_{k=0}^{m-1} y_{s_{m,k}} \mathbb{S}_{m-k} + z_{s_{m,k}} \mathbb{Q}_{m-k} \; , \\
&\tilde{\mathbb{P}}_m = \mathbb{P}_m + \delta^{\mathbb{P}}_m \; , \qquad \tilde{\mathbb{K}}_m = \mathbb{K}_m + \delta^{\mathbb{K}}_m \; , \qquad \tilde{\mathbb{H}}_m = \mathbb{H}_m + \delta^{\mathbb{H}}_m \; , \qquad m > 1 \; ,
\end{aligned} \tag{4.52}$$

where $y_{q,s}$ and $z_{q,s}$ are functions of the braiding factor $\mathbb{U}$. We are interested in finding a set of these functions such that the algebra relations (4.14) and (4.51) are still satisfied by the new generators. Indeed such a solution exists and is given by

$$\begin{aligned}
y_{m,k} &= y_{q_{m,k}} = y_{s_{m,k}} = \frac{1}{2}\binom{m}{k}\left((y+z)^{m-k} + (y-z)^{m-k}\right) \; , \\
z_{m,k} &= -i\mathbb{U}^{-1} z_{q_{m,k}} = i\mathbb{U} z_{s_{m,k}} = \frac{1}{2}\binom{m}{k}\left((y+z)^{m-k} - (y-z)^{m-k}\right) \; , \\
\delta^{\mathbb{P}}_m &= \sum_{k=0}^{m-1} y_{m,k}\mathbb{P}_k + i\mathbb{U} z_{m,k}\mathbb{H}_k \; , \qquad \delta^{\mathbb{K}}_m = \sum_{k=0}^{m-1} y_{m,k}\mathbb{K}_k - i\mathbb{U}^{-1} z_{m,k}\mathbb{H}_k \; , \\
\delta^{\mathbb{H}}_m &= \sum_{k=0}^{m-1} y_{m,k}\mathbb{H}_k + i\mathbb{U} z_{m,k}\mathbb{K}_k = \sum_{k=0}^{m-1} y_{m,k}\mathbb{H}_k - i\mathbb{U}^{-1} z_{m,k}\mathbb{P}_k \; .
\end{aligned} \tag{4.53}$$

I.e. the freedom is parameterised by two functions, y and z, of the braiding factor $\mathbb{U}$. For a representation of evaluation type, its effect is to shift the spectral parameter by $y(U)$.

The redefinitions (4.52) will modify many of the relations underlying the Hopf algebra structure, including the relations between $\mathbb{P}_m$ and $\mathbb{K}_m$ and the lower-level central charges, the coproducts of the generators and the commutation relations involving the secret symmetry $\mathbb{B}_1$. If we demand that the antipode structure (4.26) is preserved, we find

$$\Sigma(y(\mathbb{U})) = y(\mathbb{U}) \; , \qquad \Sigma(z(\mathbb{U})) = z(\mathbb{U}) \; . \tag{4.54}$$

These relations are solved by functions symmetric in $\mathbb{U}$ and $\mathbb{U}^{-1}$.

Observing that mapping between the Hopf algebra structures in sections 4.2.2. and 4.2.4. precisely takes the form given above in (4.52) and (4.53), we investigate what happens if we take a more general ansatz for the level-1 Yangian supercharges

$$\begin{aligned} &\mathbb{Q}_0 = \sqrt{ih}\,\mathbb{J}^2_{01} \ , && \mathbb{S}_0 = \sqrt{-ih}\,\mathbb{J}^1_{02} \ , \\ &\mathbb{Q}_1 = \sqrt{ih}\,\mathbb{J}^2_{11} + i\mathbb{U}z(\mathbb{U})\,\sqrt{-ih}\mathbb{J}^1_{02} \ , && \mathbb{S}_1 = \sqrt{-ih}\,\mathbb{J}^1_{12} - i\mathbb{U}^{-1}z(\mathbb{U})\,\sqrt{ih}\mathbb{J}^2_{01} \ , \end{aligned} \tag{4.55}$$

with

$$z(\mathbb{U}) = z(\mathbb{U}^{-1}) \ , \tag{4.56}$$

to preserve the antipode structure. By anti-commuting $\mathbb{Q}_1$ and $\mathbb{S}_1$ we obtain the central charges

$$\begin{aligned} &\mathbb{P}_1 = -\frac{i}{2}\left(\mathbb{1} - 2\mathbb{U}z(\mathbb{U}) + \mathbb{U}^2\right)\mathbb{H}_0 \ , && \mathbb{P}_2 = 2i\mathbb{U}z(\mathbb{U})\,\mathbb{H}_1 - h^{-2}\left(\mathbb{H}_0^2 - h^2(1-z(\mathbb{U})^2)\right)\mathbb{P}_0 \ , \\ &\mathbb{K}_1 = \frac{i}{2}\left(\mathbb{1} - 2\mathbb{U}^{-1}z(\mathbb{U}) + \mathbb{U}^{-2}\right)\mathbb{H}_0 \ , && \mathbb{K}_2 = -2i\mathbb{U}^{-1}z(\mathbb{U})\,\mathbb{H}_1 - h^{-2}\left(\mathbb{H}_0^2 - h^2(1-z(\mathbb{U})^2)\right)\mathbb{K}_0 \ , \\ &\mathbb{H}_1 = \frac{ih}{2}\left(\mathbb{J}^1_{11} - \mathbb{J}^2_{12}\right) + \frac{ih}{2}\left(\mathbb{U} - \mathbb{U}^{-1}\right)z(\mathbb{U}) \ . \end{aligned} \tag{4.57}$$

We can now ask for what choices of the function $z(\mathbb{U})$ we can have an evaluation type representation of the Yangian. In particular, this would imply the following two relations

$$P_0\,P_2 = P_1^2 \ , \qquad P_0\,H_1 = P_1\,H_0 \ . \tag{4.58}$$

Combining (4.57) with the conditions just above reveals that a necessary requirement for an evaluation type representation is

$$U^2(1-z(U)^2)(H_0^2 - P_0K_0) = 0 \ . \tag{4.59}$$

There are two cases solutions of interest to this condition. The first is

$$H_0^2 - P_0K_0 = 0 \ , \tag{4.60}$$

which can be interpreted as a (massless) shortening conditon. This is indeed consistent with our findings in section 4.2.2.. If we admit long representations, as in the context of the $\mathfrak{su}_c(1|1)$ R-matrix (4.11), (4.12), (4.13), a consistent evaluation representation demands

$$z(\mathbb{U}) = \pm\mathbb{1} \ . \tag{4.61}$$

Indeed, the choice $z(\mathbb{U}) = -\mathbb{1}$ was the representation analysed in section 4.2.4..

Finally, let us observe that the generator

$$\mathbb{B}_1 = -\frac{ih}{2}\left(\mathbb{J}^1_{11} + \mathbb{J}^2_{12}\right) \ , \tag{4.62}$$

now satisfies

$$[\mathbb{B}_1, \mathbb{Q}_0] = \mathbb{Q}_1 + \frac{i}{2}\left(\mathbb{1} - 2\mathbb{U}z(\mathbb{U}) + \mathbb{U}^2\right)\mathbb{S}_0 \ ,$$
$$[\mathbb{B}_1, \mathbb{S}_0] = -\mathbb{S}_1 + \frac{i}{2}\left(\mathbb{1} - 2\mathbb{U}^{-1}z(\mathbb{U}) + \mathbb{U}^{-2}\right)\mathbb{Q}_0 \ . \tag{4.63}$$

Choosing

$$z(\mathbb{U}) = \frac{1}{2}\left(\mathbb{U} + \mathbb{U}^{-1}\right) \ , \tag{4.64}$$

we see that $\mathbb{P}_1$ and $\mathbb{K}_1$ in (4.57), along with the tails in (4.63) vanish. It is therefore natural to ask if the existence of this choice (4.64) is related to the existence of the secret symmetry.

4.3. Strong and weak coupling expansions

4.3.1. Strong coupling expansion and the classical r-matrix

As was done in the AdS_5 case [206, 97] it is instructive to study the so-called *classical r-matrix* of the system. This can be obtained by expanding the quantum R-matrix

$$R = \mathbb{1} \otimes \mathbb{1} + h^{-1} r + O\left(h^{-2}\right) \ , \tag{4.65}$$

at strong coupling. In standard quantum group theory, the knowledge of the classical r-matrix and of its Lie bi-algebra structure allows one to reconstruct the quantum group underlying the exact problem. This is still an open problem for AdS superstrings, nevertheless much can be learnt from this exercise.

4.3.2. Parameterisation and loop algebra

Following [207] we introduce $\zeta = h^{-1}$ and the spectral paramter z

$$x^{\pm} = z\left(\sqrt{1 - \frac{\zeta^2}{\left(z - \frac{1}{z}\right)^2}} \pm \frac{i\zeta}{z - \frac{1}{z}}\right) \ , \tag{4.66}$$

where as before we assume $M(x^+, x^-) = 2$. Expanding the representations of the generators we find

$$\begin{aligned}
&U = \exp\left(i\zeta\mathfrak{D}\right) = \mathbb{1} + i\zeta\mathfrak{D} + \mathcal{O}(\zeta^2) \ , \\
&P_0 = -i\mathfrak{D} + \mathcal{O}(\zeta) \ , \qquad K_0 = i\mathfrak{D} + \mathcal{O}(\zeta), \qquad H_0 = \mathfrak{H}_0 + \mathcal{O}(\zeta) \ , \\
&Q_0 = \mathfrak{Q}_0 + \mathcal{O}(\zeta) \ , \qquad S_0 = \mathfrak{S}_0 + \mathcal{O}(\zeta) \ , \\
&Q_1 = \zeta^{-1}Q_1 + \mathcal{O}(\zeta) \ , \qquad S_1 = \zeta^{-1}\mathfrak{S}_1 + \mathcal{O}(\zeta) \ , \qquad B_1 = \zeta^{-1}\mathfrak{B}_1 + \mathcal{O}(1) \ .
\end{aligned} \tag{4.67}$$

The $\zeta \to 0$ limit of the spectral parameter (4.41) is

$$u = \frac{1}{4}\big(1 + \sqrt{\frac{x^+}{x^-}}\big)^2\big(x^- + \frac{1}{x^+}\big) = \frac{1}{4}\big(1 + \sqrt{\frac{x^-}{x^+}}\big)^2\big(x^+ + \frac{1}{x^-}\big) \to z + z^{-1} \ . \tag{4.68}$$

In what follows, it is convenient to perform the rescaling

$$u \to \frac{u}{2i} \ , \tag{4.69}$$

such that the limiting generators $\mathfrak{J}$ are in the evaluation representation with

$$\mathfrak{J}_m = u^m \mathfrak{J}_0 \ , \qquad u = \frac{1}{2i}(z + z^{-1}) \equiv -i\mathfrak{H}_0\mathfrak{D}^{-1} \ . \tag{4.70}$$

The non-trivial commutation relations for these $\mathfrak{J}_m$ read

$$\begin{aligned} &\{\mathfrak{Q}_m, \mathfrak{Q}_n\} = -\{\mathfrak{S}_m, \mathfrak{S}_n\} = 2\mathfrak{H}_{m+n-1} \ , && \{\mathfrak{Q}_m, \mathfrak{S}_n\} = 2\mathfrak{H}_{m+n} \ , \\ &[\mathfrak{B}_m, \mathfrak{Q}_n] = \mathfrak{Q}_{m+n} + \mathfrak{S}_{m+n-1}, && [\mathfrak{B}_m, \mathfrak{S}_n] = -\mathfrak{S}_{m+n} + \mathfrak{Q}_{m+n-1} \ . \end{aligned} \tag{4.71}$$

4.3.3. Classical r-matrix for the deformed $\mathfrak{gl}(1|1)_{u,u^{-1}}$

The classical limit of the R-matrix gives the classical r-matrix, whose non-trivial entries are[5]

$$\begin{aligned} &r_{14} = r_{41} = -\frac{i\sqrt{\frac{z_1{}^2}{z_1{}^2-1}}\sqrt{\frac{z_2{}^2}{z_2{}^2-1}}}{z_1 z_2 - 1} \ , && r_{23} = r_{32} = \frac{z_1 z_2 - 1}{z_1 - z_2} r_{14} \ , \\ &r_{22} = \frac{i z_1{}^2\left(z_2{}^2 - 1\right)}{\left(z_1{}^2-1\right)(z_1 - z_2)(z_1 z_2 - 1)} \ , && r_{33} = \frac{i\left(z_1{}^2 - 1\right) z_2{}^2}{\left(z_2{}^2-1\right)(z_1 - z_2)(z_1 z_2 - 1)} \ , \\ &r_{44} = \frac{i\left(z_1{}^2\left(\left(z_1{}^2 - 4\right) z_2{}^2 + z_2{}^4 + 1\right) + z_2{}^2\right)}{\left(z_1{}^2-1\right)\left(z_2{}^2-1\right)(z_1 - z_2)(z_1 z_2 - 1)} \ . \end{aligned} \tag{4.72}$$

The residue at $z_2 = z_1$ is

$$\mathrm{Re}\, r_{|z_2 \to z_1} = f(z_1)\,(\mathbb{1}\otimes\mathbb{1} - \mathfrak{C}) = \frac{i z_1^2}{1 - z_1^2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} , \tag{4.73}$$

where $\mathfrak{C}$ is the Casimir operator for the $\mathfrak{gl}(1|1)$ tensor algebra.

The classical r-matrix admits the following expression in terms of the generators of the $\mathfrak{gl}(1|1)$ algebra

$$\begin{aligned} r = \frac{1}{u_2 - u_1}\Big(&\frac{1}{2}\mathfrak{Q}_0 \otimes \mathfrak{S}_0 - \frac{1}{2}\mathfrak{S}_0 \otimes \mathfrak{Q}_0 - \frac{u_1}{u_2}\mathfrak{B}_0 \otimes \mathfrak{H}_0 - \frac{u_2}{u_1}\mathfrak{H}_0 \otimes \mathfrak{B}_0 \\ &+ \frac{1 + u_1^2 + u_2^2}{u_1 u_2}\,\mathfrak{H}_0 \otimes \mathfrak{H}_0\Big) \ . \end{aligned} \tag{4.74}$$

The same matrix can then be rewritten as an element in the tensor product of two copies of the loop algebra $\mathfrak{gl}(1|1)_{u,u^{-1}}$

$$r = r_{\mathfrak{psu}(1|1)} - \sum_{n=0}^{\infty}\left(\tilde{\mathfrak{B}}_{n+1} \otimes \mathfrak{H}_{-n-2} + \mathfrak{H}_{n-1} \otimes \tilde{\mathfrak{B}}_{-n} - \mathfrak{H}_{n-1} \otimes \mathfrak{H}_{-n-2}\right) \ , \tag{4.75}$$

[5] In this section we label the spectral parameters with integers, for example z_1, z_2, z_3 and so on.

where

$$r_{\mathfrak{psu}(1|1)} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\mathfrak{Q}_n \otimes \mathfrak{S}_{-n-1} - \mathfrak{S}_n \otimes \mathfrak{Q}_{-n-1}\right) \ , \qquad \tilde{\mathfrak{B}}_n = \mathfrak{B}_n - \mathfrak{H}_n \ . \tag{4.76}$$

The peculiarity of (4.75) is that it is representation independent, and can therefore be taken as a candidate for the *universal* classical r-matrix in the AdS_2 case.

4.3.4. Weak coupling limit and Bethe equations

In this section we study the leading-order weak-coupling ($h \to 0$) term in the R-matrix, extracting from it a set of Bethe equations. These equations should relate to the leading-order first-level nested Bethe equations one would in principle obtain from the spin-chain Hamiltonian of the putative dual superconformal quantum mechanics that is meant to live on the boundary of AdS_2. In AdS_5 parlance, this would be called the *one-loop* nearest-neighbour spin-chain Hamiltonian [11].

The advantage of restricting to this limit is that we can avoid one crucial complication present when dealing with the full R-matrix. To admit a pseudo-vacuum we need to take the tensor product of two copies of the centrally-extended $\mathfrak{su}(1|1)$ R-matrix. (This tensor product is the one relevant for building up the $AdS_2 \times S^2$ worldsheet S-matrix [184]). The corresponding pseudo-vacuum is a specific fermionic linear combination of the states in the two copies, with a definite charge under a certain $U(1)$ quantum number. The corresponding $U(1)$ symmetry does not act in a well-defined way on the individual copies. Performing the algebraic Bethe-ansatz procedure starting from the full pseudo-vacuum is at the moment an open issue.

Dealing with the individual copies, which do not admit a pseudo-vacuum in their own right, could in principle be approached by adapting alternative methods (such as, for instance, the one of Baxter operators). However, the limit $h \to 0$ switches off the most unconventional entries of the R-matrix and allows for the existence of a pseudo-vacuum separately in each copy. Moreover, it drastically simplifies all the remaining entries, allowing for an almost straightforward treatment.

4.3.5. Weak coupling R-matrix

The R-matrix up to order h has the form

$$R_{12} = R_{12}^{(0)} + h\, R_{12}^{(1)} \ , \tag{4.77}$$

where

$$R_{12}^{(0)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & B_{12} & C_{12} & 0 \\ 0 & C_{12} & D_{12} & 0 \\ 0 & 0 & 0 & E_{12} \end{pmatrix} \ , \qquad R_{12}^{(1)} = \begin{pmatrix} 0 & 0 & 0 & A_{12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ A_{12} & 0 & 0 & 0 \end{pmatrix} \ , \tag{4.78}$$

with the parameterising functions given by

$$
\begin{aligned}
A_{12} &= \frac{4i(u_1-u_2)(u_1u_2-1)}{(1+u_1^2)(1+u_2^2)(u_1-u_2-2i)} e^{-\frac{i}{4}(p_1-p_2)} \,, \\
B_{12} &= \frac{u_1-u_2}{u_1-u_2-2i} e^{-\frac{i}{2}p_1} \,, \\
C_{12} &= \frac{2i}{u_1-u_2-2i} e^{-\frac{i}{4}(p_1-p_2)} \,, \\
D_{12} &= \frac{u_1-u_2}{u_1-u_2-2i} e^{\frac{i}{2}p_2} \,, \\
E_{12} &= \frac{u_1-u_2+2i}{u_1-u_2-2i} e^{-\frac{i}{2}(p_1-p_2)} \,.
\end{aligned} \tag{4.79}
$$

The maps between $x^\pm$, u and p are

$$
x^\pm = \frac{1}{2h}\Big(\frac{\mathrm{cn}(\frac{p}{2},-4h^2)}{\mathrm{sn}(\frac{p}{2},-4h^2)} \pm i\Big)\Big(1+\mathrm{dn}(\frac{p}{2},-4h^2)\Big) \,, \qquad u = \cot\frac{p}{4} \,. \tag{4.80}
$$

The factors of e^{ip_k} in (4.78) and (4.79) are the result of a Drinfeld twist [208]. The presence of such a twist was already observed in the same scaling limit of the AdS_5 R-matrix. Indeed, R_{12} can be written as

$$
R_{12} = T_{21}\tilde{R}_{12}T_{12}^{-1} \,, \tag{4.81}
$$

where

$$
T_{12} = \mathrm{diag}\big(e^{\alpha(p_1+p_2)},\, e^{-\frac{i}{4}p_2+\beta(p_1+p_2)},\, e^{(\beta-\frac{i}{4})p_1+(\beta-\frac{i}{2})p_2},\, e^{(\alpha+\frac{i}{4})p_1+(\alpha-\frac{i}{4})p_2}\big) \,,
$$

with α and β arbitrary coefficients. The entries of $\tilde{R}_{12}$ are given by

$$
\begin{aligned}
\tilde{A}_{12} &= \frac{4i(u_1-u_2)(u_1u_2-1)}{(1+u_1^2)(1+u_2^2)(u_1-u_2-2i)} \,, \\
\tilde{B}_{12} &= \frac{u_1-u_2}{u_1-u_2-2i} \,, \\
\tilde{C}_{12} &= \frac{2i}{u_1-u_2-2i} \,, \\
\tilde{D}_{12} &= \frac{u_1-u_2}{u_1-u_2-2i} \,, \\
\tilde{E}_{12} &= \frac{u_1-u_2+2i}{u_1-u_2-2i} \,,
\end{aligned} \tag{4.82}
$$

with the same associations of letters to entries as in (4.77) and (4.78).

Taking $h=0$, the entries of $\tilde{R}_{12}$ involving $\tilde{A}_{12}$ drop out, and we recover the $\mathfrak{gl}(1|1)$ R-matrix, written down in its canonical rational form.

4.3.6. Bethe ansatz and twist

As we have seen, at leading order the R-matrix reduces to the canonical rational (Yangian) R-matrix $R_{\rm can}$ of $\mathfrak{gl}(1|1)$, decorated by a Drinfeld twist. Let us rewrite the twist as

$$
T_{12} = t_{12}^{ij}\, E_{ii}\otimes E_{jj} \;=\; e^{i[\underline{i}\,p_1+\underline{j}\,p_2]}\, E_{ii}\otimes E_{jj} \,, \tag{4.83}
$$

where E_{ij} are unit matrices, i.e. 1 in row i, column j and zero everywhere else, and we denote by $\underline{i}$ the numerical coefficient multiplying the momentum in the respective spaces of the T_{12} matrix. Similarly, we write the canonical R-matrix as

$$R_{\text{can}\,12} = r_{ij}^{kl}(p_1, p_2)\, E_{ki} \otimes E_{jl} \;\equiv\; r_{12}{}_{ij}^{kl}\, E_{ki} \otimes E_{jl} \ , \tag{4.84}$$

such that at leading order the R-matrix reads

$$R_{12} = T_{21}\, R_{\text{can}\,12}\, T_{12}^{-1} \ . \tag{4.85}$$

We are now ready to write down the monodromy matrix. Denoting the auxiliary space with the label 0, we have

$$\begin{aligned} M &= R_{01}\, R_{02} \cdots R_{0N} \\ &= T_{10}\, R_{\text{can}01}\, T_{01}^{-1}\, T_{20}\, R_{\text{can}02}\, T_{02}^{-1} \cdots T_{N0}\, R_{\text{can}0N}\, T_{0N}^{-1} \ . \end{aligned} \tag{4.86}$$

Plugging the explicit expressions (4.83) and (4.84) into (4.86) we obtain

$$\begin{aligned} M = t_{10}^{i_1 j_1}\, t_{20}^{i_3 k_1}\, t_{30}^{i_5 k_2} \cdots r_{01}{}_{k_1 j_2}^{j_1 i_1}\, r_{02}{}_{k_2 j_4}^{k_1 i_3}\, r_{03}{}_{k_3 j_6}^{k_2 i_5} \cdots (t_{01}^{k_1 j_2})^{-1} (t_{02}^{k_2 j_4})^{-1} (t_{03}^{k_3 j_6})^{-1} \cdots \\ E_{j_1 k_N} \otimes E_{i_1 j_2} \otimes E_{i_3 j_4} \otimes E_{i_5 j_6} \cdots \ . \end{aligned} \tag{4.87}$$

Now we insert the dependence of the twist on the momentum. Most of the factors appearing in the auxiliary space cancel, such that we are left with

$$\begin{aligned} M = e^{i[\underline{j_1} - \underline{k_N}]p_0}\, e^{i[\underline{i_1} - \underline{j_2}]p_1}\, e^{i[\underline{i_3} - \underline{j_4}]p_2}\, e^{i[\underline{i_5} - \underline{j_6}]p_3} \cdots r_{01}{}_{k_1 j_2}^{j_1 i_1}\, r_{02}{}_{k_2 j_4}^{k_1 i_3}\, r_{03}{}_{k_3 j_6}^{k_2 i_5} \cdots \\ E_{j_1 k_N} \otimes E_{i_1 j_2} \otimes E_{i_3 j_4} \otimes E_{i_5 j_6} \cdots \ . \end{aligned} \tag{4.88}$$

We can therefore define a new set of states and matrices

$$\widetilde{E}_{ab} \equiv e^{i[\underline{a} - \underline{b}]p}\, E_{ab} \ , \qquad |\underline{a}\rangle \equiv e^{i\underline{a}p}\, |a\rangle \ , \tag{4.89}$$

such that we still have

$$\widetilde{E}_{ab}\, \widetilde{E}_{cd} \;=\; \delta_{bc}\, \widetilde{E}_{ad} \ , \qquad \widetilde{E}_{ab}\, |\underline{c}\rangle = \delta_{bc}\, |\underline{a}\rangle \ , \qquad \langle \underline{a} | \underline{b} \rangle = \delta_{ab} \ . \tag{4.90}$$

Using these new vectors and unit matrices, the expression for M becomes indistinguishable from the canonical one. Consequently the Bethe ansatz reduces to the standard one, except with the vectors $|\underline{a}\rangle$ now appearing in the wave functions: in particular, the Bethe equations for M magnons will be

$$\left(\frac{u_k - i}{u_k + i} \right)^L = (-1)^{M-1} \ . \tag{4.91}$$

In the effective model obtained from this limit, excitations propagate as free fermions on a periodic one-dimensional lattice.

4.3.7. Bethe equations via the algebraic Bethe ansatz

To conclude this discussion of the weak coupling limit let us recall how the algebraic Bethe ansatz procedure works for the standard $\mathfrak{gl}(1|1)$ rational R-matrix, which, as we have just shown, is relevant in the $h \to 0$ limit. We define $\mathcal{T}(u)$ as in [86], namely

$$\mathcal{T}(u) = \sum_{A,B} (-)^{[B]} \, e^{B}{}_{A} \otimes T^{A}{}_{B}(u) \ , \tag{4.92}$$

with

$$\begin{aligned} A(u) &= T^{1}{}_{1}(u) \ , \qquad & B(u) &= -T^{2}{}_{1}(u) \ , \\ C(u) &= -T^{1}{}_{2}(u) \ , \qquad & D(u) &= T^{2}{}_{2}(u) \ . \end{aligned} \tag{4.93}$$

The RTT equations determine the commutation relations for A, B, C and D. In particular, we will need

$$\begin{aligned} A(\lambda)B(\mu) &= f(\mu,\lambda)B(\mu)A(\lambda) + g(\mu,\lambda)B(\lambda)A(\mu) \ , \\ D(\lambda)B(\mu) &= h(\lambda,\mu)B(\mu)D(\lambda) + k(\lambda,\mu)B(\lambda)D(\mu) \ , \\ B(\lambda)B(\mu) &= -q(\lambda,\mu)B(\mu)B(\lambda) \ , \end{aligned} \tag{4.94}$$

where

$$\begin{aligned} f(\mu,\lambda) &= R_{11}(\mu,\lambda)/R_{33}(\mu,\lambda) \ , \\ g(\mu,\lambda) &= R_{23}(\mu,\lambda)/R_{33}(\mu,\lambda) \ , \\ h(\lambda,\mu) &= R_{44}(\lambda,\mu)/R_{33}(\lambda,\mu) \ , \\ k(\lambda,\mu) &= -R_{23}(\lambda,\mu)/R_{33}(\lambda,\mu) \ , \\ q(\lambda,\mu) &= R_{44}(\lambda,\mu)/R_{33}(\lambda,\mu) \ . \end{aligned} \tag{4.95}$$

The standard rational monodromy matrix admits a pseudo-vacuum state Ω, such that

$$A(\lambda)\Omega = \alpha(\lambda)\Omega \ , \qquad C(\lambda)\Omega = 0 \ , \qquad D(\lambda)\Omega = \delta(\lambda)\Omega \ . \tag{4.96}$$

This implies that Ω is an eigenstate of the transfer matrix

$$t(\lambda) = (-)^{[I]} T^{I}{}_{I}(u) = A(\lambda) - D(\lambda) \ . \tag{4.97}$$

The Bethe equations arise from requiring that the M-magnon state

$$\Phi(\mu_1, \ldots, \mu_M) = \prod_{i=1}^{M} B(\mu_i) \, \Omega \ , \tag{4.98}$$

is an eigenstate of $t(\lambda)$. For instance, for $M = 2$ one gets

$$\frac{\alpha(\mu_1)}{\delta(\mu_1)} = \frac{h(\mu_1, \mu_2)\, k(\lambda, \mu_1)}{f(\mu_2, \mu_1)\, g(\mu_1, \lambda)} \,. \tag{4.99}$$

Substituting in the entries of the rational (weak coupling) R-matrix, we see that (4.99) is the same as (4.91). This is the expected result for the $\mathfrak{gl}(1|1)$ R-matrix[6] (see for instance [209]).

[6] Let us note that this would actually be true for both the twisted and the untwisted R-matrix.

5

Yangian Symmetry of AdS_3 Superstrings

5.1. Secret symmetries and AdS_3 Superstrings

5.1.1. Secret symmetries

The integrable structure permeating the AdS/CFT correspondence (see *e.g.* [7, 210] for reviews) keeps revealing new surprising features that extend our algebraic understanding of scattering problems and uncover new structures in supersymmetric quantum groups.

For instance, the Hopf superalgebra which controls the AdS_5/CFT_4 integrable system is a rather unconventional infinite-dimensional Yangian-type symmetry [79, 211], whose level-0 Lie algebra sector is Beisert's three-fold centrally-extended $\mathfrak{psu}(2|2)$ superalgebra [93, 121]. At level 1, the Yangian generators come each as partners to the level-0 ones, except for the presence of an additional symmetry [95, 97], which has no analog at level 0. This *secret* generator corresponds to a hypercharge, and acts as a fermion number on the scattering particles, counting the total number of fermions. Specific entries in Beisert's S-matrix [93] of the form $|\text{boson}\rangle \otimes |\text{boson}\rangle \mapsto |\text{fermion}\rangle \otimes |\text{fermion}\rangle$ and vice versa, for example, break the fermion number $(-)^F \otimes \mathbb{1} + \mathbb{1} \otimes (-)^F$; here, F denotes the fermion number operator. This is restored at level 1 by means of the non-trivial coproduct characteristic of the secret symmetry.

The presence of the hypercharge at level 0 would automatically extend the algebra, which transforms the scattering excitations, to $\mathfrak{gl}(2|2)$, but this is not straightforwardly compatible with the central extension. Nevertheless, recent progress [86] based on the so-called $R\mathcal{T}\mathcal{T}$-formulation of the Yangian has revealed how the secret symmetry is non-trivially embedded in the algebra. However, it is still a challenge to obtain a translation to Drinfeld's picture of this discovery. Moreover, the issue of crossing symmetry is particularly delicate.

It is also not clear how much of this secret symmetry is accidental to AdS_5 and to the special features of the Lie superalgebra $\mathfrak{psu}(2|2)$ [212, 196, 213]. The fact that a similar effect has been observed by now in a variety of (*a priori* unrelated) sectors of the AdS/CFT correspondence is an indication that this is not an isolated feature of the spectral problem. Nevertheless, all these other sectors still lie within AdS_5 or close relatives. To see if this phenomenon is a truly universal feature of integrability within the AdS/CFT framework, it is crucial to extend the analysis beyond AdS_5 to other dimensions. At the same time, the deeper physical nature of the secret symmetry remains quite mysterious, and we hope that investigating it in other instances of the AdS/CFT correspondence might shed light on some of its most elusive properties.

This is the direction we wish to pursue in this Chapter. Specifically, we find secret symmetries in the AdS_3 spectral problem and analyse their features. Our present analysis also yields interesting information about the AdS_5 case, when looking back in perspective, as we will explain next.

5.1.2. AdS_3 integrable scattering.

Recently, a new example of the AdS/CFT correspondence has become amenable to integrability methods. This involves type IIB backgrounds with an AdS_3 factor in the metric, and the two most studied examples with 16 supersymmetries are $AdS_3 \times S^3 \times T^4$ and $AdS_3 \times S^3 \times S^3 \times S^1$. The latter background is characterised by a continuum parameter α related to the radii of the two 3-spheres, and which has a reflection in the appearance of the exceptional Lie superalgebra $\mathfrak{D}(2,1;\alpha) \times \mathfrak{D}(2,1;\alpha)$ as a superconformal algebra. Notice that in the contracting limit $\alpha \to 0$, the algebra reduces to $\mathfrak{psu}(1,1|2) \times \mathfrak{psu}(1,1|2)$, which, in turn, corresponds to the aforementioned background with $\mathbb{R}^3 \times S^1$.

Such configurations provide instances of the AdS_3/CFT_2 correspondence. This was transferred into the framework of integrable systems by [28], in which classical integrability was demonstrated (see also [149]), a set of semiclassical finite gap equations for the spectrum were formulated, and a conjecture was put forward for an all-loop quantum Bethe ansatz. The latter was also fully elaborated into a spin chain picture [214]. The initial focus has been primarily on the massive BMN modes [128], leaving aside the massless modes that now appear in contrast with the AdS_5 case (see however [215, 216]). The problem of a fully consistent treatment of such massless modes has recently been investigated from a world-sheet perspective, paving the way to incorporate these excitations into the scaffolding of integrability [217, 195, 218]. Our motivation for pursuing an analysis of the quantum group symmetry algebra is dictated by the desire of eventually elucidating the role the massless modes in the full quantum group behind the scattering problem, building up on the Hopf algebra analysis of [195]. One is interested in seeing if the algebra can help achieving the full description of the massless modes at the level of the universal R-matrix. We hope this Chapter's findings can provide a further step in that direction.

To this end, we resort to the scattering theory developed in [219, 220]. Specifically, the authors of [219, 220] derived an S-matrix and Bethe ansatz for the $AdS_3 \times S^3 \times S^3 \times S^1$ case from a centrally-extended algebra of the Beisert type [93, 121], adapted to a much smaller residual symmetry transforming the excitations. In a way, the smaller components of the algebra proliferate into several factors, only transferring the complication of a large single multiplet to many smaller ones instead. In particular, one has left and right components to worry about in the present case. We will describe this symmetry at length in the main text of the Chapter.

In [190], both the exact S-matrix and the Bethe ansatz for the $AdS_3 \times S^3 \times T^4$ case were constructed. This S-matrix received a series of confirmations from perturbative computations performed using the string sigma model [165,149,221–223,194,224,225]. Suitable dressing phases to supplement the S-matrix and making it into a solution of the crossing equation were found in [226–228]. More studies have followed addressing various aspects of the spectral problem [229]. Furthermore, $AdS_3 \times S^3 \times T^4$ theories with mixed R-R and NS-NS flux have been analysed, and they provide in principle an interesting setup for the study of the conformal limit of a combined massive-massless integrable structure [230–233],David:2014qta, [234], cf. [198]. As a remark, we should say that it is quite crucial to test the dressing-phase solutions with new methods, and the algebraic one—via the universal R-matrix—could be a potential tool, as we will comment upon in the conclusions. The present treatment may therefore have a bearance on the problem of the dressing phases as well, as the inclusion of all symmetries of the system is essential to understand how to formulate the universal R-matrix for the complete centrally-extended algebra.

Our algebraic treatment is general enough to encompass both backgrounds, hence we study the sphere case from the beginning. In fact, the only difference is that one is dealing with two copies of the fundamental $\mathfrak{gl}(1|1)_l \times \mathfrak{gl}(1|1)_r$ rather than only one copy, and having a parameter $s \in \mathbb{R}$ ($s = \alpha$ for the modes we study in this Chapter) equal to 1 in the torus case ($\alpha \to 1$). The two copies factorise anyway, and all our formulæ just go through for arbitrary s (we will only need to remember to keep s non-zero to remain inside the massive sector for the current concern of this Chapter).

5.2. Yangian of $\mathfrak{gl}(1|1)_l \times \mathfrak{gl}(1|1)_r$

In this section, we first review the features of Hopf superalgebras salient to the description of integrable scattering problems. For reviews, we refer the reader to [77,91] and references therein. We then move on and summarise the relevant findings of [190] for the reader's convenience. Finally, we provide the details of the secret symmetries.

5.2.1. Hopf superalgebra generalities

One first has to fix an algebra of symmetries of the system which one wishes to describe. In the superstring case, this turns out to be quite systematically a certain Lie superalgebra $\mathfrak{g}$.

To be able to deal with multiparticle states, one needs two additional maps which turn the algebra into a bi-superalgebra A. One is the coproduct, $\Delta : A \to A \otimes A$, that encodes how the symmetry acts on two-particle states. The other map is the counit, $\epsilon : A \to \mathbb{C}$. A series of compatibility relations, most of which of immediate physical intuition, guarantee the consistency of the mathematical structure.

To go from a bi-superalgebra to a Hopf superalgebra, one equips the former with an antipode map, $\mathscr{S} : A \to A$, which is used to define the antiparticle (conjugated) representation to any given representation of the Lie superalgebra. The antipode is also subject to compatibility with the other maps, as we will have a chance to revisit in the main text. For its nature, the antipode is an anti-morphism, that is, $\mathscr{S}(ab) = (-)^{|a||b|}\mathscr{S}(b)\mathscr{S}(a)$ for all $a, b \in A$; here $|a|$ denotes the Graßmann-parity of $a \in A$. Moreover, one can prove that, if a bi-superalgebra has an antipode then the latter is unique.

To obtain the two-particle S-matrix, one first defines the action of symmetry on *in* states by means of the coproduct, as we stated above. On the other hand, the permuted map $P \circ \Delta =: \Delta^{\text{op}}$, with P the graded permutation map, will be declared the action on *out* states. These two actions can differ, in general, extending standard textbook quantum mechanics where the Leibniz rule $\Delta(a) = a \otimes \mathbb{1} + \mathbb{1} \otimes a$ for all $a \in A$ guarantees the cocommutativity of the Hopf superalgebra, that is, $\Delta^{\text{op}} = \Delta$. Since, nevertheless, Δ and Δ^{op} generate tensor product representations with the same dimension, the two may be related by conjugation via an invertible element in the tensor product algebra, and this is the S-matrix or the R-matrix in mathematical literature:

$$R \in A \otimes A \quad \text{with} \quad \Delta^{\text{op}}(a)\, R = R\, \Delta(a) \quad \text{for all} \quad a \in A \, . \tag{5.1}$$

When this happens, the Hopf superalgebra is dubbed quasi-cocommutative, and, if the R-matrix satisfies an additional property which we will loosely relate to the physical bootstrap principle [235, 83], it is also called quasi-triangular. The S-matrix needs to be compatible with the antipode, ensuring the physical crossing symmetry. A theorem of Drinfeld's shows that quasi-triangularity implies the Yang–Baxter equation and the crossing condition.

Hopf superalgebras provide a particularly suitable framework to formulate integrable scattering problems. Moreover, they unify the treatment of arbitrary representations of the symmetry algebra (not only the fundamental particles but also the bound states transforming in higher irreducible representations) in one single language. The so-called universal R-matrix $\mathcal{R}$ (that is, the abstract solution to the quasi cocommutativity condition) has a special importance and may sometimes be seen as an alternative to other approaches to the inverse scattering method.

5.2.2. Lie algebra, representations, and R-matrices

Let us begin by writing the action of the symmetry generators on the elementary scattering excitations, by focusing on one of the two copies of $\mathfrak{gl}(1|1)_l \times \mathfrak{gl}(1|1)_r$, say the left copy $\mathfrak{gl}(1|1)_l$.

The right copy can be studied in complete analogy [190], and we will connect the two in section 5.3.2.. Each of these copies is further split into left and right representations that are related by crossing symmetry. The non-vanishing (anti-)commutation relations of the generators of $\mathfrak{gl}(1|1)_l := \langle \mathfrak{B}, \mathfrak{H}, \mathfrak{Q}, \mathfrak{S} \rangle$ read as

$$[\mathfrak{B}, \mathfrak{Q}] = -2\mathfrak{Q} , \quad [\mathfrak{B}, \mathfrak{S}] = 2\mathfrak{S} , \quad \text{and} \quad \{\mathfrak{Q}, \mathfrak{S}\} = -\mathfrak{H} . \tag{5.2}$$

Here, $\mathfrak{B}$ and $\mathfrak{H}$ are Graßmann-parity even (bosonic) and $\mathfrak{Q}$ and $\mathfrak{S}$ are Graßmann-parity odd (fermionic), respectively. Moreover, $[\cdot,\cdot]$ denotes the commutator and $\{\cdot,\cdot\}$ the anti-commutator. We will directly put ourselves into what was dubbed the *most symmetric frame* in [190]. The coproduct is obtained as (cf. [188, 236] for the AdS_5 case)

$$\begin{aligned}
\Delta(\mathfrak{B}) &:= \mathfrak{B} \otimes \mathbb{1} + \mathbb{1} \otimes \mathfrak{B} \quad \text{and} \quad \Delta(\mathfrak{H}) := \mathfrak{H} \otimes \mathbb{1} + \mathbb{1} \otimes \mathfrak{H} , \\
\Delta(\mathfrak{Q}) &:= \mathfrak{Q} \otimes \mathrm{e}^{-\mathrm{i}\frac{p}{4}} + \mathrm{e}^{\mathrm{i}\frac{p}{4}} \otimes \mathfrak{Q} \quad \text{and} \quad \Delta(\mathfrak{S}) := \mathfrak{S} \otimes \mathrm{e}^{\mathrm{i}\frac{p}{4}} + \mathrm{e}^{-\mathrm{i}\frac{p}{4}} \otimes \mathfrak{S} , \\
\Delta(\mathrm{e}^{\mathrm{i}p}) &:= \mathrm{e}^{\mathrm{i}p} \otimes \mathrm{e}^{\mathrm{i}p} \quad \text{and} \quad \Delta(\mathbb{1}) := \mathbb{1} \otimes \mathbb{1} ,
\end{aligned} \tag{5.3}$$

where $\mathrm{i} := \sqrt{-1}$ and $p \in \mathbb{R}$. Due to the centrality of $\mathrm{e}^{\mathrm{i}p}$, this coproduct is a Lie superalgebra homomorphism.

Left representation. The left representation of $\mathfrak{gl}(1|1)_l$ is described by a left doublet (ϕ, ψ) with symmetry action given by

$$\begin{aligned}
\mathfrak{B}_L &:= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \mathfrak{H}_L := -\gamma^2 \frac{h}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \\
\mathfrak{Q}_L &:= \gamma \sqrt{\frac{h}{2}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \qquad \mathfrak{S}_L := \gamma \sqrt{\frac{h}{2}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} ,
\end{aligned} \tag{5.4a}$$

where

$$\gamma := \sqrt{\mathrm{i}(x^- - x^+)} , \quad \frac{2\mathrm{i}s}{h} =: x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} , \quad \text{and} \quad \mathrm{e}^{\mathrm{i}p} = \frac{x^+}{x^-} , \tag{5.4b}$$

with $x^\pm \in \mathbb{C}$ and $s, h \in \mathbb{R}$. Notice that h is a function of the 't Hooft coupling. There is no need to specify the momentum generator p as being left or right, as it will be common to the two representations. If $\Phi_{LL} \in \mathbb{C}$ is an overall scalar factor (determined shortly), it can be checked that the left-left R-matrix denoted by R_{LL} and defined by

$$\begin{aligned}
R_{LL}(\phi \otimes \phi) &:= \Phi_{LL} \frac{x_2^+ - x_1^-}{x_2^- - x_1^+} e^{i\frac{(p_1 - p_2)}{4}} \phi \otimes \phi, \\
R_{LL}(\phi \otimes \psi) &:= \Phi_{LL} \frac{x_2^+ - x_1^+}{x_2^- - x_1^+} e^{-i\frac{(p_1 + p_2)}{4}} \phi \otimes \psi + \Phi_{LL} \frac{x_2^+ - x_2^-}{x_2^- - x_1^+} \frac{\gamma_1}{\gamma_2} \psi \otimes \phi, \\
R_{LL}(\psi \otimes \phi) &:= \Phi_{LL} \frac{x_2^- - x_1^-}{x_2^- - x_1^+} e^{i\frac{(p_1 + p_2)}{4}} \psi \otimes \phi + \Phi_{LL} \frac{x_1^+ - x_1^-}{x_2^- - x_1^+} \frac{\gamma_2}{\gamma_1} \phi \otimes \psi, \\
R_{LL}(\psi \otimes \psi) &:= \Phi_{LL} \, \mathrm{e}^{i\frac{(p_2 - p_1)}{4}} \psi \otimes \psi,
\end{aligned} \tag{5.5}$$

where the indices 1 and 2 refer to the two scattering particles (first and second factor of the tensor product), indeed satisfies (5.1), that is,

$$\Delta^{\mathrm{op}}_{LL}(\mathfrak{J})\, R_{LL} \;=\; R_{LL}\, \Delta_{LL}(\mathfrak{J}) \quad \text{for all} \quad \mathfrak{J} \;\in\; \mathfrak{gl}(1|1)_l \;. \tag{5.6}$$

The subscript in $\Delta^{\mathrm{op}}_{LL}$ and Δ_{LL} means taking both factors of the coproduct (5.3) in the left representation (5.4a).

Right representation. Crossing symmetry relates left and right representations of $\mathfrak{gl}(1|1)_l$. The action of the right representation on the right doublet of excitations $(\bar{\phi}, \bar{\psi})$ is described by

$$\begin{aligned} \mathfrak{B}_R \;&:=\; -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \qquad & \mathfrak{H}_R \;&:=\; -\frac{\gamma^2}{x^+x^-}\frac{h}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \\ \mathfrak{Q}_R \;&:=\; -\frac{\gamma}{\sqrt{x^+\,x^-}}\sqrt{\frac{h}{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \qquad & \mathfrak{S}_R \;&:=\; -\frac{\gamma}{\sqrt{x^+\,x^-}}\sqrt{\frac{h}{2}}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{aligned} \tag{5.7}$$

with γ and h as in (5.4b). The antipode $\mathscr{S}$ performing the connection is easily found applying the defining rule

$$\mu \circ (\mathscr{S} \otimes \mathbb{1}) \circ \Delta \;=\; \eta \circ \epsilon \tag{5.8}$$

to the coproduct (5.3); the map μ multiplies together two generators of the symmetry algebra, while ϵ is the counit and η the unit map. Both η and ϵ have to satisfy certain Hopf algebra consistency conditions with the multiplication and the coproduct map. In our case, these conditions amount to

$$\epsilon(\mathfrak{J}) \;=\; 0 \quad \text{for all} \quad \mathfrak{J} \;\in\; \mathfrak{gl}(1|1)_l \quad \text{and} \quad \epsilon(\mathbb{1}) \;=\; 1 \;, \tag{5.9}$$

and the antipode acts simply as

$$\mathscr{S}(\mathfrak{J}) \;=\; -\mathfrak{J} \quad \text{for all} \quad \mathfrak{J} \;\in\; \mathfrak{gl}(1|1)_l \;, \quad \mathscr{S}(\mathrm{e}^{\mathrm{i}p}) \;=\; \mathrm{e}^{-\mathrm{i}p} \;, \quad \text{and} \quad \mathscr{S}(\mathbb{1}) \;=\; \mathbb{1} \;. \tag{5.10}$$

Hence, the antipode map is idempotent and thus the same as its inverse. It was found in [190] that the left-right relation for any generator $\mathfrak{J}$ can be written as

$$\mathscr{S}\big(\mathfrak{J}_L(x^{\pm})\big) \;=\; \mathscr{C}^{-1}\left[\mathfrak{J}_R\left(\frac{1}{x^{\pm}}\right)\right]\mathscr{C} \;, \tag{5.11a}$$

where $\mathscr{C}$ is the matrix of charge conjugation

$$\mathscr{C} \;:=\; \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{pmatrix} , \tag{5.11b}$$

and the apex denotes supertransposition. Charge conjugation allows to convert the left moving basis states into the right moving ones by means of $\phi \mapsto \bar{\phi}$ and $\psi \mapsto \mathrm{i}\bar{\psi}$.

Letting $\Phi_{RR} \in \mathbb{C}$ be an overall scalar factor (determined shortly), it can be checked that the right-right R-matrix denoted by R_{RR} and defined by

$$\begin{aligned}
R_{RR}(\bar\phi\otimes\bar\phi) &:= \Phi_{RR}\frac{x_2^+ - x_1^-}{x_2^- - x_1^+}e^{3i\frac{(p_1-p_2)}{4}}\bar\phi\otimes\bar\phi\,,\\
R_{RR}(\bar\phi\otimes\bar\psi) &:= \Phi_{RR}\frac{x_2^+ - x_1^+}{x_2^- - x_1^+}e^{i\frac{(p_1-3p_2)}{4}}\bar\phi\otimes\bar\psi + \Phi_{RR}\frac{i\gamma_1\gamma_2}{x_2^- - x_1^+}e^{i\frac{(p_1-p_2)}{2}}\bar\psi\otimes\bar\phi\,,\\
R_{RR}(\bar\psi\otimes\bar\phi) &:= \Phi_{RR}\frac{x_2^- - x_1^-}{x_2^- - x_1^+}e^{i\frac{(3p_1-p_2)}{4}}\bar\psi\otimes\bar\phi + \Phi_{RR}\frac{i\gamma_1\gamma_2}{x_2^- - x_1^+}e^{i\frac{(p_1-p_2)}{2}}\bar\phi\otimes\bar\psi\,,\\
R_{RR}(\bar\psi\otimes\bar\psi) &:= \Phi_{RR}\,e^{i\frac{(p_1-p_2)}{4}}\bar\psi\otimes\bar\psi
\end{aligned} \tag{5.12}$$

satisfies the right-version of (5.1).

Mixed representations. One can now project the coproduct (5.3) into a mixed combination of right representation in the first factor and left representation in the second, namely Δ_{RL}, and solve the equation for the scattering of a left mover with a right mover:

$$\Delta^{\text{op}}_{RL}(\mathfrak{J})\,R_{RL} \;=\; R_{RL}\,\Delta_{RL}(\mathfrak{J}) \quad \text{for all} \quad \mathfrak{J} \;\in\; \mathfrak{gl}(1|1)_l\,. \tag{5.13}$$

Letting $\Phi_{RL} \in \mathbb{C}$ be an overall scalar factor (determined shortly), the corresponding R-matrix reads

$$\begin{aligned}
R_{RL}(\bar\phi\otimes\phi) &:= \Phi_{RL}\frac{x_2^- x_1^+ - 1}{x_2^+ x_1^+ - 1}e^{i\frac{(p_1+p_2)}{4}}\bar\phi\otimes\phi + \Phi_{RL}\frac{i\gamma_1\gamma_2}{(x_2^+ x_1^+ - 1)}e^{i\frac{p_1}{2}}\bar\psi\otimes\psi\,,\\
R_{RL}(\bar\phi\otimes\psi) &:= \Phi_{RL}\frac{x_2^- x_1^- - 1}{x_2^+ x_1^+ - 1}e^{i\frac{(3p_1+p_2)}{4}}\bar\phi\otimes\psi\,,\\
R_{RL}(\bar\psi\otimes\phi) &:= \Phi_{RL}e^{i\frac{(p_1-p_2)}{4}}\bar\psi\otimes\phi\,,\\
R_{RL}(\bar\psi\otimes\psi) &:= \Phi_{RL}\frac{x_2^+ x_1^- - 1}{x_2^+ x_1^+ - 1}e^{i\frac{(3p_1-p_2)}{4}}\bar\psi\otimes\psi + \Phi_{RL}\frac{i\gamma_1\gamma_2}{x_2^+ x_1^+ - 1}e^{i\frac{p_1}{2}}\bar\phi\otimes\phi\,,
\end{aligned} \tag{5.14}$$

and satisfies the crossing equation (cf. [191])

$$(\mathscr{C}^{-1}\otimes\mathbb{1})R_{RL}^{1}\Big(\frac{1}{x_1^\pm},x_2^\pm\Big)(\mathscr{C}\otimes\mathbb{1})R_{LL}(x_1^\pm,x_2^\pm) \;=\; \mathbb{1}\otimes\mathbb{1} \tag{5.15}$$

($_i$ meaning supertransposition in the factor i), provided the scalar factors are related by a specific condition. Such a condition was the object of the analysis in [227].

The other possible combination of mixed scattering is obtained in [190] by solving the analog of the coproduct relations (5.6) and (5.13) projected into the appropriate representations, the result being

$$\begin{aligned} R_{LR}(\phi\otimes\bar\phi) &:= \Phi_{LR}\frac{x_2^-x_1^+-1}{x_2^-x_1^--1}e^{-i\frac{(p_1+p_2)}{4}}\phi\otimes\bar\phi+\Phi_{LR}\frac{i\gamma_1\gamma_2}{x_2^-x_1^--1}e^{-i\frac{p_2}{2}}\psi\otimes\bar\psi\ ,\\ R_{LR}(\phi\otimes\bar\psi) &:= \Phi_{LR}\,e^{i\frac{(p_1-p_2)}{4}}\phi\otimes\bar\psi\ ,\\ R_{LR}(\psi\otimes\bar\phi) &:= \Phi_{LR}\frac{x_1^+x_2^+-1}{x_1^-x_2^--1}e^{-i\frac{(p_1+3p_2)}{4}}\psi\otimes\bar\phi,\\ R_{LR}(\psi\otimes\bar\psi) &:= \Phi_{LR}\frac{x_2^+x_1^--1}{x_2^-x_1^--1}e^{i\frac{(p_1-3p_2)}{4}}\psi\otimes\bar\psi+\Phi_{LR}\frac{i\gamma_1\gamma_2}{x_2^-x_1^--1}e^{-i\frac{p_2}{2}}\phi\otimes\bar\phi\ , \end{aligned} \tag{5.16}$$

where $\Phi_{LR}\in\mathbb{C}$ is an overall scalar factor (determined shortly).

The remaining crossing equations analogous to (5.15), and similarly dealt with in [227], read

$$\begin{aligned} (\mathscr{C}^{-1}\otimes\mathbb{1})\,R_{RR}^{\bar 1}\Big(\frac{1}{x_1^\pm},x_2^\pm\Big)\,(\mathscr{C}\otimes\mathbb{1})\,R_{LR}(x_1^\pm,x_2^\pm) &= \mathbb{1}\otimes\mathbb{1}\ ,\\ (\mathbb{1}\otimes\mathscr{C}^{-1})\,R_{LR}^{\bar 2}\Big(x_1^\pm,\frac{1}{x_2^\pm}\Big)\,(\mathbb{1}\otimes\mathscr{C})\,R_{LL}(x_1^\pm,x_2^\pm) &= \mathbb{1}\otimes\mathbb{1}\ ,\\ (\mathbb{1}\otimes\mathscr{C}^{-1})\,R_{RR}^{\bar 2}\Big(x_1^\pm,\frac{1}{x_2^\pm}\Big)\,(\mathbb{1}\otimes\mathscr{C})\,R_{RL}(x_1^\pm,x_2^\pm) &= \mathbb{1}\otimes\mathbb{1}\ . \end{aligned} \tag{5.17}$$

Let us notice the the above R-matrices have the following property:

$$R_{RR} = \frac{\Phi_{RR}}{\Phi_{LL}}e^{i\frac{(p_1-p_2)}{2}}R_{LL}\quad\text{and}\quad R_{LR} = \frac{\Phi_{LR}}{\Phi_{RL}}e^{-i\frac{(p_1+p_2)}{2}}\frac{x_2^+x_1^+-1}{x_2^-x_1^--1}R_{RL}\ . \tag{5.18}$$

Scalar factors. We report for completeness the equations imposed on the scalar factors by the crossing equations, together with the requirements of unitarity. In what follows, $R_{21} = \big[P\circ R\big](x_2^\pm,x_1^\pm)$ when we deal with R-matrices, while $\Phi_{21}=\Phi(x_2^\pm,x_1^\pm)$ when we deal with scalar factors.

$$\begin{aligned} [R_{LL}]_{12}\,[R_{LL}]_{21} &= \mathbb{1}\otimes\mathbb{1}\ , & [R_{RL}]_{12}\,[R_{LR}]_{21} &= \mathbb{1}\otimes\mathbb{1}\ ,\\ [R_{LR}]_{12}\,[R_{RL}]_{21} &= \mathbb{1}\otimes\mathbb{1}\ , & [R_{RR}]_{12}\,[R_{RR}]_{21} &= \mathbb{1}\otimes\mathbb{1}\ . \end{aligned} \tag{5.19}$$

We will not describe the solution proposed in [227], since we will not need such a solution for the present algebraic purposes. The conditions are

$$\begin{aligned} \Phi_{LL}\,[\Phi_{RL}]_{\bar 1} &= \frac{x_2^+-x_1^+}{x_2^+-x_1^-}\ , & \Phi_{LR}\,[\Phi_{RR}]_{\bar 1} &= \frac{x_1^+-\frac{1}{x_2^-}}{x_1^+-\frac{1}{x_2^+}}\ ,\\ \Phi_{LL}\,[\Phi_{LR}]_{\bar 2} &= \frac{x_2^--x_1^-}{x_2^+-x_1^-}\ , & \Phi_{RL}\,[\Phi_{RR}]_{\bar 2} &= \frac{x_2^--\frac{1}{x_1^+}}{x_2^--\frac{1}{x_1^-}}\ ,\\ \Phi_{LL}\,[\Phi_{LL}]_{21} = 1\ , \qquad \Phi_{LR}\,[\Phi_{RL}]_{21} &= 1\ , & \Phi_{RR}\,[\Phi_{RR}]_{21} &= 1\ . \end{aligned} \tag{5.20}$$

In the above, $[\Phi]_{\bar i}$ denotes the antiparticle map in the variable i, namely $x_i^\pm\mapsto 1/x_i^\pm$.

5.2.3. Yangian secret symmetries

In this section, we display the R-matrix Yangian symmetry as found in [190], completing the information that was missing there regarding the type $\mathfrak{B}$ hypercharge generator at Yangian level 1. We shall focus on the Yangian $\mathcal{Y}(\mathfrak{gl}(1|1)_l)$ of $\mathfrak{gl}(1|1)_l$.

Secret symmetry I. It can be shown that the left representation of the level-1 Yangian generators of $\mathcal{Y}(\mathfrak{gl}(1|1)_l)$

$$\mathfrak{e}_{1L} := u_L \mathfrak{Q}_L \ , \quad \mathfrak{f}_{1L} := u_L \mathfrak{S}_L \ , \quad \mathfrak{h}_{1L} := u_L \mathfrak{H}_L \ , \quad \text{and} \quad \mathfrak{b}_{1L} := u_L \mathfrak{B}_L \ , \tag{5.21a}$$

with the left spectral parameter

$$u_L := \mathrm{i}\tfrac{h}{2}\, x^+ \tag{5.21b}$$

and the corresponding right representation of the level-1 Yangian generators of $\mathcal{Y}(\mathfrak{gl}(1|1)_l)$

$$\mathfrak{e}_{1R} := u_R \mathfrak{Q}_R \ , \quad \mathfrak{f}_{1R} := u_R \mathfrak{S}_R \ , \quad \mathfrak{h}_{1R} := u_R \mathfrak{H}_R \ , \quad \text{and} \quad \mathfrak{b}_{1R} := \begin{pmatrix} \omega_{11} & 0 \\ 0 & \omega_{22} \end{pmatrix} \tag{5.22a}$$

with

$$\omega_{11} := 1 + \mathrm{i}\frac{h}{2x^+} - \mathrm{i}\frac{h}{x^-} \ , \quad \omega_{22} := 1 + \mathrm{i}\frac{h}{2x^+} \ , \quad \text{and} \quad u_R := \mathrm{i}\frac{h}{2x^-} \ , \tag{5.22b}$$

are symmetries of the R-matrices which we displayed in the previous sections; here $x^\pm$ and h are as in (5.4b). The novelty with respect to [190] is the expression for the right level-1 hypercharge. Notice that the generators of type $\mathfrak{Q}$, $\mathfrak{S}$, and $\mathfrak{H}$ form an ideal inside the Yangian, since type $\mathfrak{B}$ is never produced by commuting any of the elements of the ideal. In view of this fact, in principle, one might like to disregard a multiple of the identity added to $\mathfrak{b}_1$. However, this would not agree with the crossing symmetry to be described shortly. These generators being symmetries amounts to saying that by projecting one and the same universal expression for each level-1 Yangian coproduct $\Delta(\mathfrak{j}_1)$ each time in the appropriate combination of representations, one satisfies all the relations

$$\Delta^{\mathrm{op}}_{kl}(\mathfrak{j}_1)\, R_{kl} = R_{kl}\, \Delta_{kl}(\mathfrak{j}_1), \tag{5.23}$$

with $k, l \in \{\text{L}, \text{R}\}$ and $\mathfrak{j}_1 \in \{\mathfrak{e}_1, \mathfrak{f}_1, \mathfrak{h}_1, \mathfrak{b}_1\}$. After defining the level-0 generators simply as

$$\mathfrak{e}_{0R} := \mathfrak{Q}_R \ , \quad \mathfrak{f}_{0R} := \mathfrak{S}_R \ , \quad \mathfrak{h}_{0R} := \mathfrak{H}_R \ , \quad \text{and} \quad \mathfrak{b}_{0R} := \mathfrak{B}_R \ , \tag{5.24}$$

the coproducts for the generators of the $\mathfrak{sl}(1|1)$ ideal, already given in [190], look like

$$\begin{aligned} \Delta(\mathfrak{e}_1) &:= \mathfrak{e}_1 \otimes \mathrm{e}^{-\mathrm{i}\frac{p}{4}} + \mathrm{e}^{\mathrm{i}\frac{p}{4}} \otimes \mathfrak{e}_1 + \mathrm{e}^{\mathrm{i}\frac{p}{4}}\, \mathfrak{h}_0 \otimes \mathfrak{e}_0 \ , \\ \Delta(\mathfrak{f}_1) &:= \mathfrak{f}_1 \otimes \mathrm{e}^{\mathrm{i}\frac{p}{4}} + \mathrm{e}^{-\mathrm{i}\frac{p}{4}} \otimes \mathfrak{f}_1 + \mathfrak{f}_0 \otimes e^{i\frac{p}{4}}\, \mathfrak{h}_0 \ , \\ \Delta(\mathfrak{h}_1) &:= \mathfrak{h}_1 \otimes \mathbb{1} + \mathbb{1} \otimes \mathfrak{h}_1 + \mathfrak{h}_0 \otimes \mathfrak{h}_0 \ , \end{aligned} \tag{5.25a}$$

where we have here made explicit the braiding by the e^{ip}-type central element in the frame we are using. The novelty is that we now have a level-1 canonical coproduct for the hypercharge given by

$$\Delta(\mathfrak{b}_1) \;:=\; \mathfrak{b}_1 \otimes \mathbb{1} + \mathbb{1} \otimes \mathfrak{b}_1 \;-\; 2\, \mathfrak{f}_0\, e^{i\frac{p}{4}} \otimes \mathfrak{e}_0\, e^{i\frac{p}{4}} \;+\; \mathfrak{b}_0 \otimes \mathfrak{b}_0 \;. \tag{5.25b}$$

With these formulæ, one can prove that the level-0 and level-1 generators (both in the left and the right representation), as well their coproducts (in all possible combinations of left and right choices), are compatible with the so-called Drinfeld's second realisation [74] of the $\mathfrak{gl}(1|1)$ Yangian:

$$\begin{aligned}
&[\mathfrak{b}_0, \mathfrak{e}_n] \;=\; -2\,\mathfrak{e}_n \;, \qquad [\mathfrak{b}_0, \mathfrak{f}_n] \;=\; 2\,\mathfrak{f}_n \;, \qquad \{\mathfrak{e}_m, \mathfrak{f}_n\} \;=\; -\mathfrak{h}_{m+n} \;,\\
&[\mathfrak{b}_m, \mathfrak{b}_n] \;=\; 0 \;, \qquad [\mathfrak{h}_m, \cdot] \;=\; 0 \;, \qquad \{\mathfrak{e}_m, \mathfrak{e}_n\} \;=\; \{\mathfrak{f}_m, \mathfrak{f}_n\} \;=\; 0 \;,\\
&[\mathfrak{b}_{m+1}, \mathfrak{e}_n] - [\mathfrak{b}_m, \mathfrak{e}_{n+1}] + \{\mathfrak{b}_m, e_n\} \;=\; 0 \;, \qquad [\mathfrak{b}_{m+1}, \mathfrak{f}_n] - [\mathfrak{b}_m, \mathfrak{f}_{n+1}] - \{\mathfrak{b}_m, \mathfrak{f}_n\} \;=\; 0
\end{aligned} \tag{5.26}$$

for $m, n \in \mathbb{N}_0$.

Furthermore, all these symmetries, including the hypercharge generator, satisfy the crossing symmetry. This proceeds as already described in [190]. Firstly, one derives from the coproduct the expression for the antipode utilising (5.8) (with ϵ annihilating all level-1 Yangian generators), obtaining

$$\begin{aligned}
&\mathscr{S}(\mathfrak{e}_1) \;=\; -\mathfrak{e}_1 + \mathfrak{e}_0\,\mathfrak{h}_0 \;, \qquad \mathscr{S}(\mathfrak{f}_1) \;=\; -\mathfrak{f}_1 + \mathfrak{f}_0\,\mathfrak{h}_0 \;, \qquad \mathscr{S}(\mathfrak{h}_1) \;=\; -\mathfrak{h}_1 + \mathfrak{h}_0^2 \;,\\
&\mathscr{S}(\mathfrak{b}_1) \;=\; -\mathfrak{b}_1 - 2\,\mathfrak{f}_0\,\mathfrak{e}_0 + \mathfrak{b}_0^2 \;.
\end{aligned} \tag{5.27}$$

At this point, one can verify that the equation

$$\mathscr{S}\big(\mathfrak{j}_{1\,L}(x^\pm)\big) \;=\; \mathscr{C}^{-1}\left[\mathfrak{j}_{1\,R}\left(\frac{1}{x^\pm}\right)\right]\mathscr{C} \;, \tag{5.28}$$

indeed holds with the same charge conjugation matrix (5.11b), and for all the generators including the hypercharge, that is, for $\mathfrak{e}_1$, $\mathfrak{f}_1$, $\mathfrak{h}_1$, and $\mathfrak{b}_1$.

Secret symmetry II. In the previous paragraph, we have found a secret symmetry (hypercharge generator) which, in contrast with the AdS_5 case [237, 98], is embedded in Drinfeld's second realisation of the relevant S-matrix Yangian. In fact, there is a whole class $[\mathfrak{b}_1] := \mathfrak{b}_1 + s\mathbb{1}$ for $s \in \mathbb{R}$ of secret generators preserving all desired relations. The coproduct on equivalence classes $[\mathfrak{j}] := \mathfrak{j} + s\mathbb{1}$ is consistently defined up to multiples of $\mathbb{1} \otimes \mathbb{1}$. Next, we would like to show that, in the case of AdS_3, there exists another class of secret symmetry generators, not embedded in Drinfeld's second realisation of the Yangian, and related to the class $[\mathfrak{b}_1]$ by performing a quadratic Drinfeld map of the form

$$\widehat{\mathfrak{j}} \;=\; c\,\mathfrak{j}_1 + c_{AB}\,\mathfrak{j}_0^A\,\mathfrak{j}_0^B \;. \tag{5.29}$$

Here, $\widehat{\mathfrak{j}}$ is the level-1 Yangian generator in Drinfeld's first realisation that is associated with $\mathfrak{j}_1$, (c, c_{AB}) are some constant coefficients, and $\mathfrak{j}_0^A$ are the level-0 generators including $\mathfrak{b}_0$. Such map is used to switch between Drinfeld's first and second realisations.

The generator we will now present is a symmetry of the R-matrix in all possible combinations of left and right representations, and, similarly to the hypercharge in the previous section, satisfies crossing symmetry. The expression for this alternative secret symmetry (which we call $\widehat{\mathfrak{b}}$) is much closer to its AdS_5 analog, and it may be the true AdS_3 correspondent of that general phenomenon.

The symmetry is given by

$$\widehat{\mathfrak{b}}_L \;:=\; \delta\,\mathfrak{B}_L \quad \text{and} \quad \widehat{\mathfrak{b}}_R \;:=\; \begin{pmatrix} \tau_{11} & 0 \\ 0 & \tau_{22} \end{pmatrix}, \tag{5.30a}$$

where

$$\delta \;:=\; -\mathrm{i}\frac{h}{4}\,(x^+ + x^-)\;, \quad \tau_{11} \;:=\; \mathrm{i}\frac{h}{4}\left(\frac{3}{x^-} - \frac{1}{x^+}\right), \quad \text{and} \quad \tau_{22} \;:=\; -\mathrm{i}\frac{h}{4}\left(\frac{3}{x^+} - \frac{1}{x^-}\right) \tag{5.30b}$$

and $x^\pm$ and h as in (5.4b). Its universal coproduct reads

$$\Delta(\widehat{\mathfrak{b}}) \;:=\; \widehat{\mathfrak{b}} \otimes \mathbb{1} + \mathbb{1} \otimes \widehat{\mathfrak{b}} + \mathfrak{e}_0\, \mathrm{e}^{-\mathrm{i}\frac{p}{4}} \otimes \mathfrak{f}_0\, \mathrm{e}^{-\mathrm{i}\frac{p}{4}} + \mathfrak{f}_0\, \mathrm{e}^{\mathrm{i}\frac{p}{4}} \otimes \mathfrak{e}_0\, \mathrm{e}^{\mathrm{i}\frac{p}{4}}\;, \tag{5.31}$$

implying an antipode

$$\mathscr{S}(\widehat{\mathfrak{b}}) \;=\; -\widehat{\mathfrak{b}} - \mathfrak{h}_0\;. \tag{5.32}$$

Crossing reads the same way as for all the other generators we have in this Chapter, that is,

$$\mathscr{S}\big(\widehat{\mathfrak{b}}_L(x^\pm)\big) \;=\; \mathscr{C}^{-1}\left[\widehat{\mathfrak{b}}_R\left(\frac{1}{x^\pm}\right)\right]\mathscr{C}\;, \tag{5.33}$$

where $\mathscr{C}$ was given in (5.11b). Finally, the quadratic map (5.29) is given by

$$[\widehat{\mathfrak{b}}] \;=\; -[\mathfrak{b}_1] + \tfrac{1}{2}(\mathfrak{e}_0\mathfrak{f}_0 - \mathfrak{f}_0\mathfrak{e}_0 + \mathfrak{b}_0\mathfrak{b}_0)\;. \tag{5.34}$$

The relation between the parameters s and t in $[\mathfrak{b}_1] = \mathfrak{b}_1 + s\mathbb{1}$ and $[\widehat{\mathfrak{b}}] = \widehat{\mathfrak{b}} + t\mathbb{1}$ is fixed by the choice of matrix representation of the generators. Note that we are making explicit use of the level-0 generator $\mathfrak{b}_0$ to establish the relation between the two classes of secret generators. Because of this, it remains to be seen whether the two secret symmetries can, in fact, be identified. We emphasise that this is in contrast with the AdS_5 case where no such embedding exists.

AdS_5 versus AdS_3 secret symmetries. Our above analysis in the AdS_3 case yields a new perspective on the AdS_5 problem. In particular, our analysis provides a re-interpretation of the observation made in [86] concerning how to incorporate the secret symmetry within crossing (a problem that was observed in [95]). In fact, even in the AdS_5 case we may accept that the crossed generator is some sort of *right* generator, to a *left* one being the secret symmetry originally found. Most of the AdS_5 magnon multiplet is crossing self-dual, apart from the secret generator, which indeed satifies a formula perfectly analogous to (5.32). Therefore, as noticed in [86], the secret symmetry respects (5.33) by defining (in the notation of [95])

$$\mathfrak{J}_R(\mathrm{AdS}_5) \;=\; \Big[-\mathfrak{J}(\mathrm{AdS}_5) + \frac{2\mathrm{i}}{g}\mathfrak{C} \Big]_{x^\pm \mapsto 1/x^\pm} , \tag{5.35}$$

where $\mathfrak{C}$ is the central charge associated to the magnon energy, and g is the square root of the AdS_5 't Hooft coupling divided by 4π.

5.3. RTT-realisation for the deformed $\mathfrak{gl}(1|1)_l \times \mathfrak{gl}(1|1)_r$ Yangian

In this section, we construct the RTT-realisation for the Yangian $\mathcal{Y}(\mathfrak{gl}(1|1)_l \times \mathfrak{gl}(1|1)_r)$ originating from the full R-matrix for $\mathrm{AdS}_3 \times S^3 \times M^4$.

5.3.1. Algebraic formulation and representations

Let us now move on and specialise to the Yangian $\mathcal{Y}(\mathfrak{gl}(1|1)_l)$. We shall first analyse the algebraic formulation of the $\mathbb{T}_{\mathbf{B}}{}^{\mathbf{A}}$s and then discuss explicit representations.

Algebraic formalism. Let us first focus on the left representation π_{L,u_L}, and, consequently, on the left-left R-matrix R_{LL} as given in (5.5). In order to be able to derive abstract commutation relations of the generators $\mathbb{T}_{m\mathbf{B}}{}^{\mathbf{A}}$, we need to set (see (5.4b))

$$\begin{aligned} u_L &:= \mathrm{i}\frac{h}{2}x_1^+ \quad \Rightarrow \quad \mathrm{i}\frac{h}{2}x_1^- \;=\; u_L + s - \frac{h^2 s}{4u_L^2} + \mathcal{O}(u_L^{-3}) , \\ v_L &:= \mathrm{i}\frac{h}{2}x_2^+ \quad \Rightarrow \quad \mathrm{i}\frac{h}{2}x_2^- \;=\; v_L + s - \frac{h^2 s}{4v_L^2} + \mathcal{O}(v_L^{-3}) \end{aligned} \tag{5.36}$$

and expand R_{LL} for large u_L. One finds that

$$\mathbb{T}_{-1\mathbf{B}}{}^{\mathbf{A}} \;=\; \delta_{\mathbf{B}}{}^{\mathbf{A}} \mathbb{U}^{|\mathbf{B}|} . \tag{5.37}$$

The RTT-relations involving $\mathbb{T}_{-1\mathbf{B}}{}^{\mathbf{A}}$ show it is central. This, in turn, implies the centrality of $\mathbb{U}$. Furthermore, the RTT-relations involving $\mathbb{T}_{0\mathbf{B}}{}^{\mathbf{A}}$ yield the (anti-)commutation relations of a certain deformation of the Lie superalgebra $\mathfrak{gl}(1|1)_l$. In fact, to obtain the (anti-)commutation relations of $\mathfrak{gl}(1|1)_l$, one should re-define the generators and work with

$$\mathbb{J}_{0\mathbf{B}}{}^{\mathbf{A}} \;:=\; \mathbb{U}^{-\frac{1}{2}(|\mathbf{A}|+|\mathbf{B}|)} \mathbb{T}_{0\mathbf{B}}{}^{\mathbf{A}} \tag{5.38}$$

instead of $\mathbb{T}_{0\mathbf{B}}{}^{\mathbf{A}}$. Moreover, from level 1 upwards, the $R\mathcal{T}\mathcal{T}$-relations neither yield directly the (anti-)commutation relations of $\mathfrak{gl}(1|1)_l$ nor those of its Yangian $\mathcal{Y}(\mathfrak{gl}(1|1)_l)$. For instance, one may check that

$$\{\mathbb{T}_{12}{}^1, \mathbb{T}_{01}{}^2\} \;=\; s\big(-\mathbb{U}\,\mathbb{T}_{11}{}^1 + \mathbb{T}_{12}{}^2 + \tfrac{1}{4}\mathbb{T}_{02}{}^1\mathbb{T}_{01}{}^2 - \tfrac{1}{4}\mathbb{T}_{01}{}^2\mathbb{T}_{02}{}^1\big) \;. \tag{5.39}$$

Hence, to bring the (anti-)commutation relations into Yangian form we shall follow [86] and, in addition to (5.38), consider

$$\mathbb{J}_{1\mathbf{B}}{}^{\mathbf{A}} \;:=\; \mathbb{U}^{-\frac{1}{2}(|\mathbf{A}|+|\mathbf{B}|)}\mathbb{T}_{1\mathbf{B}}{}^{\mathbf{A}} - \tfrac{1}{2}(-)^{(|\mathbf{A}|+|\mathbf{C}|)(\mathbf{B}|+|\mathbf{C}|)}\mathbb{U}^{-\frac{1}{2}(|\mathbf{A}|+|\mathbf{B}|+2|\mathbf{C}|)}\mathbb{T}_{0\mathbf{B}}{}^{\mathbf{C}}\mathbb{T}_{0\mathbf{C}}{}^{\mathbf{A}} \;. \tag{5.40}$$

The $\mathbb{J}_{m\mathbf{B}}{}^{\mathbf{A}}$ we defined are slightly different from those reported in [86] as we want the corresponding coproducts to be in the most symmetric frame.

Since the symmetry of the R-matrix for $\mathrm{AdS}_3 \times S^3 \times M^4$ is a deformation of the Yangian $\mathcal{Y}(\mathfrak{gl}(1|1)_l \times \mathfrak{gl}(1|1)_r)$, we expect identifications that resemble the $\mathrm{AdS}_5 \times S^5$ ones. Indeed, a short calculation shows that the combinations

$$\begin{aligned}
\mathbb{B}_0 \;&:=\; \tfrac{2}{s}(\mathbb{J}_{01}{}^1 + \mathbb{J}_{02}{}^2)\;, \qquad & \mathbb{B}_1 \;&:=\; \tfrac{1}{s}\big(\mathbb{J}_{11}{}^1 + \mathbb{J}_{12}{}^2\big) + \mathbb{J}_{01}{}^1 + \tfrac{1}{2}\big(\mathbb{Q}_0\mathbb{S}_0 - \mathbb{S}_0\mathbb{Q}_0\big) + \tfrac{1}{2}\mathbb{B}_0\mathbb{B}_0\;,\\
\mathbb{H}_0 \;&:=\; -\mathbb{J}_{01}{}^1 + \mathbb{J}_{02}{}^2\;, & \mathbb{H}_1 \;&:=\; -\mathbb{J}_{11}{}^1 + \mathbb{J}_{12}{}^2 + \tfrac{1}{2}\mathbb{H}_0\mathbb{H}_0 + \tfrac{s}{2}\mathbb{H}_0\;,\\
\mathbb{Q}_0 \;&:=\; \tfrac{1}{\sqrt{s}}\mathbb{J}_{01}{}^2\;, & \mathbb{Q}_1 \;&:=\; \tfrac{1}{\sqrt{s}}\mathbb{J}_{11}{}^2 + \tfrac{1}{2}\mathbb{Q}_0\mathbb{H}_0 + \tfrac{s}{2}\mathbb{Q}_0\;,\\
\mathbb{S}_0 \;&:=\; -\tfrac{1}{\sqrt{s}}\mathbb{J}_{02}{}^1\;, & \mathbb{S}_1 \;&:=\; -\tfrac{1}{\sqrt{s}}\mathbb{J}_{12}{}^1 + \tfrac{1}{2}\mathbb{S}_0\mathbb{H}_0 + \tfrac{s}{2}\mathbb{S}_0,
\end{aligned} \tag{5.41}$$

for the left copy $\mathfrak{gl}(1|1)_l$ in terms of the $\mathbb{J}_{0\mathbf{B}}{}^{\mathbf{A}}$ and $\mathbb{J}_{1\mathbf{B}}{}^{\mathbf{A}}$ as given in (5.38) and (5.40) obey $(m, n = 0, 1)$

$$\begin{aligned}
&[\mathbb{B}_0, \mathbb{Q}_n] \;=\; -2\,\mathbb{Q}_n\;, \quad [\mathbb{B}_0, \mathbb{S}_n] \;=\; 2\,\mathbb{S}_n\;, \quad \{\mathbb{Q}_m, \mathbb{S}_n\} \;=\; -\mathbb{H}_{m+n}\;,\\
&[\mathbb{B}_m, \mathbb{B}_n] \;=\; 0\;, \quad [\mathbb{H}_m, \cdot] \;=\; 0\;, \quad \{\mathbb{Q}_m, \mathbb{B}_n\} \;=\; \{\mathbb{S}_m, \mathbb{S}_n\} \;=\; 0\;,\\
&[\mathbb{B}_{m+1}, \mathbb{Q}_n] - [\mathbb{B}_m, \mathbb{Q}_{n+1}] + \{\mathbb{B}_m, \mathbb{Q}_n\} \;=\; 0\;, \quad [\mathbb{B}_{m+1}, \mathbb{S}_n] - [\mathbb{B}_m, \mathbb{S}_{n+1}] - \{\mathbb{B}_m, \mathbb{S}_n\} \;=\; 0\;.
\end{aligned} \tag{5.42}$$

The (anti-)commutation relations coincide precisely with (5.26). Here, $\mathbb{B}_1$ corresponds the first secret generator embedded in the Yangian we have found in the first part of the Chapter: see (5.21a) and (5.22a).

We would like to emphasise that the above (anti-)commutation relations are inherited from the $R\mathcal{T}\mathcal{T}$-relations: consequently, they are truly universal. Indeed, the generators given in (5.4a) and (5.21a) are simply representations (by means of π_{L,u_L}) of these abstract ones, as we will demonstrate shortly.

The coproducts for the generators (5.41) correctly match the expected ones (see e.g. (5.25a) and (5.25b)). To verify this, one needs to make use of the identity

$$\Delta\left(\mathbb{T}_{\mathbf{B}}{}^{\mathbf{A}}(u)\right) \;=\; \mathbb{T}_{\mathbf{B}}{}^{\mathbf{C}}(u) \otimes \mathbb{T}_{\mathbf{C}}{}^{\mathbf{A}}(u) \tag{5.43}$$

induced by the R-matrix fusion relations.

The coproduct and commutation rules for

$$ß \;:=\; -\tfrac{1}{s}\big(\mathbb{J}_{11}{}^{1} + \mathbb{J}_{12}{}^{2}\big) - \mathbb{J}_{01}{}^{1} \tag{5.44}$$

are the same as those for the second secret generator, however, the representations $\pi_{L,u_L}(ß)$ and $\pi_{L,u_L}(\hat{\mathfrak{b}})$ do not coincide. On the other hand,

$$\hat{\mathfrak{b}}_L \;:=\; -\tfrac{1}{s}\big(\mathbb{J}_{11}{}^{1} + \mathbb{J}_{12}{}^{2}\big) - \mathbb{J}_{01}{}^{1} - \tfrac{1}{2}\mathbb{1} \tag{5.45}$$

gives the correct representation, but not the right coproduct. The key to resolving this issue is to consider equivalence classes of generators as in (5.29). This enables us to reproduce all the results from Section 5.2.3..

Representations. By projecting the second leg of the R-matrix we obtain a representation for the abstract generators we found in the previous section. In particular, upon Laurent-expanding the left-left R-matrix (5.5) in the spectral parameter u_L around infinity, we obtain immediately the following expressions for $T_{m\mathbf{B}}{}^{\mathbf{A}}(v_L)$ for $m = -1, 0$:

$$\begin{aligned} T_{L\,-11}{}^{1}(v_L) \;&=\; \mathbb{1}\;, \qquad T_{L\,-12}{}^{2}(v_L) \;=\; U(v_L)\mathbb{1}\;, \\ T_{L\,-11}{}^{2}(v_L) \;&=\; 0 \;=\; T_{L\,-12}{}^{1}(v_L) \end{aligned} \tag{5.46a}$$

and

$$\begin{aligned} T_{L\,01}{}^{1}(v_L) \;&= \tfrac{h}{4}\gamma^2(v_L)\mathbb{1} + \tfrac{s}{4}\big(E_1{}^{1} + E_2{}^{2}\big)\;, \quad T_{L\,02}{}^{1}(v_L) \;=\; \gamma(v_L)\sqrt{\tfrac{h}{2}\,s\,U(v_L)}\,E_2{}^{1}\;, \\ T_{L\,01}{}^{2}(v_L) \;&=\; \gamma(v_L)\sqrt{\tfrac{h}{2}\,s\,U(v_L)}\,E_1{}^{2}, \quad T_{L\,02}{}^{2}(v_L) \;= U(v_L)\big[-\tfrac{h}{4}\gamma^2(v_L)\mathbb{1} + \tfrac{s}{4}\big(E_1{}^{1} + E_2{}^{2}\big)\big] \end{aligned} \tag{5.46b}$$

with

$$U(v_L) \;:=\; \sqrt{\frac{x_2^+}{x_2^-}} \quad \text{and} \quad \gamma(v_L) \;:=\; \sqrt{\mathrm{i}(x_2^- - x_2^+)} \tag{5.46c}$$

All higher level generators can be found in the same spirit. By using (5.41), this precisely reproduces the representation of the first part of the Chapter, with the level-1 Yangian generators indeed given in the evaluation representation with spectral parameter v_L, that is, $\mathfrak{J}_{L\,1\mathbf{A}}{}^{\mathbf{B}} = v_L\mathfrak{J}_{L\,0\mathbf{A}}{}^{\mathbf{B}}$. Here, $\mathfrak{J}_{L\,0,1\mathbf{A}}{}^{\mathbf{B}}$ denotes the left representation of the generators in (5.41).

5.3.2. Central Extensions

Finally, we would like to comment on central extensions. We can show that, if now we take the right-left R-matrix (5.14) and proceed with the $RT\mathcal{T}$-formalism in that case, we are actually capable of reproducing the right copy $\mathfrak{gl}(1|1)_r$ of the algebra $\mathfrak{gl}(1|1)_l \times \mathfrak{gl}(1|1)_r$. This means that the Laurent expansion of the right-left R-matrix (5.14) in u_R around zero yields the appropriate

${T_{R\,m\mathbf{B}}}^{\mathbf{A}}(v_L)$ generators for the right copy of the factor algebra. We choose an expansion around $u_R = 0$ as this is the point at which the right-left R-matrix becomes almost the identity needed to eventually get the Yangian charges. Let us notice that either large u_L or small u_R correspond to large u_{AdS_5} as utilised in [86]. The explicit calculation confirms that, as anticipated, they behave as their left partner. For the right copy $\mathfrak{gl}(1|1)_r$, we shall use the very same combinations (5.41) apart from $\mathbb{B}_1$ and $\hat{\beta}$ which we re-define as $\mathbb{B}_1 \to \mathbb{B}_1 - \mathbb{H}_0 + \mathbb{1}$ and $\hat{\beta} \to \hat{\beta} + \mathbb{H}_0$. This (central) shift is necessary to match the evaluation representation for the right copy. We therefore refrain from repeating the whole procedure. Instead, we shall rather derive the central extensions of the algebra by means of the RTT-relations via the expansion of R_{RL}:

$$\begin{aligned} \{\mathbb{Q}_{l0}, \mathbb{Q}_{r0}\} &= \mathbb{P}_0 \ , & \{\mathbb{Q}_{l1}, \mathbb{Q}_{r0}\} &= \mathbb{P}_{l1} \ , & \{\mathbb{Q}_{l0}, \mathbb{Q}_{r1}\} &= \mathbb{P}_{r1} \ , \\ \{\mathbb{S}_{l0}, \mathbb{S}_{r0}\} &= \mathbb{K}_0 \ , & \{\mathbb{S}_{l1}, \mathbb{S}_{r0}\} &= \mathbb{K}_{l1} \ , & \{\mathbb{S}_{l0}, \mathbb{S}_{r1}\} &= \mathbb{K}_{r1} \ , \end{aligned} \tag{5.47a}$$

with

$$\mathbb{P}_0 \;=\; \mathbb{K}_0 \;=\; \mathrm{i}\tfrac{h}{2}\left(\mathbb{U} - \mathbb{U}^{-1}\right) \quad \text{and} \quad \mathbb{P}_{l,r\,1} \;=\; \mathbb{K}_{l,r\,1} \;=\; \mathrm{i}\tfrac{h}{2}\mathbb{U}\,\mathbb{H}_{l,r\,0} \ . \tag{5.47b}$$

In summary, the RTT-formulation not only gives back the whole deformed Yangian but also puts left and right algebras on the same footing.

Finally, we wish to emphasise that one should be capable of deriving these relations from the universal R-matrix $\mathcal{R}$ of the full algebra by means of

$$\mathcal{T}(u) \;=\; {E_{\mathbf{A}}}^{\mathbf{B}} \otimes {\mathbb{T}_{\mathbf{B}}}^{\mathbf{A}}(u) \;=\; {E_{\mathbf{A}}}^{\mathbf{B}} \otimes \sum_{m\geqslant 0} u^{-m} {\mathbb{T}_{m-1\,\mathbf{B}}}^{\mathbf{A}}(u) \;=\; (\pi_u \otimes \mathbb{1})\,\mathcal{R} \ . \tag{5.48}$$

Here, π_u denotes the suitable spectral-parameter-dependent representation onto $\mathrm{End}(V)$. Turning the argument around, our treatment can give important insight into the issue of formulating a universal R-matrix for the current problem.

Part IV

Concluding Remarks

6 Conclusions and Outlook

6.1. T-self-duality of AdS_2 and AdS_3 Superstrings

In Chapter 2, we have proved the self-duality of the supercoset sigma models describing strings in $AdS_d \times S^d \times S^d$ ($d = 2, 3$) under a combined T-duality along Abelian bosonic and fermionic isometries of these backgrounds without gauge-fixing kappa symmetry of the sigma model actions. The main finding of our investigation is that T-duality along complex directions along one of the spheres S^d is needed in order to map the original superstring action onto itself.

When $d = 2$ and $d = 3$, the corresponding sigma models describe only subsectors of the complete superstring theories in $AdS_d \times S^d \times M^{10-2d}$ backgrounds which also include $8(5-d)$ non-supercoset fermionic modes associated with the 10-dimensional supersymmetries that are broken in these backgrounds. In the $d = 3$ case, the 16 non-supercoset fermions can be put to zero by gauge fixing kappa symmetry, though this gauge is not admissible for all classical string configurations. In the $d = 2$ case, there are not enough kappa symmetries to remove all the 24 non-supercoset fermions. Thus, one should prove the invariance of the $AdS_d \times S^d \times M^{10-2d}$ superstring actions under the bosonic and fermionic T-dualities in the presence of the non-supercoset fermions υ.

It would be of interest to see whether and how the presence of these modes determine the combined bosonic-fermionic T-dualisation of the $AdS_4 \times \mathbb{C}P^3$ background and corresponding string sigma model. In particular, whether T-duality along complex $\mathbb{C}P^3$ directions is required.

It would also be important to address an important problem of the combined fermionic and bosonic T-duality of $AdS_d \times S^d \times M^{10-2d}$ backgrounds for $d = 2, 3$ in the presence of Neveu–Schwarz–Neveu–Schwarz flux, as well as to find a manifestation of this T-duality on the CFT sides of the correspondences for the Ramond–Ramond (and Neveu–Schwarz–Neveu–Schwarz) $d = 2, 3$ backgrounds.

6.2. The $AdS_2 \times S^2$ Worldsheet S-matrix

In Chapter 3 we have constructed the S-matrix describing the scattering of particular representations of the centrally-extended $psu_c(1|1)^2$ Lie superalgebra, conjectured to be related to the massive modes of the $AdS_2 \times S^2 \times T^6$ light-cone gauge superstring. A significant difference with the $AdS_5 \times S^5$ and $AdS_3 \times S^3 \times T^4$ light-cone gauge superstrings is that the massive excitations are taken to transform in long representations of the symmetry algebra $psu(1|1)^2 \ltimes \mathbb{R}^3$. Consequently, under these assumptions there is no shortening condition and the dispersion relation is not entirely fixed by symmetry. Furthermore, the symmetry only fixes the S-matrix up to an overall phase, for which we have given the crossing and unitarity relations, which appear to be more complicated than those in the $AdS_5 \times S^5$ case. The exact form of both the dispersion relation and the phase remain to be determined.

We have identified a natural way to take the massless limit on these representations, and have analyzed in detail the limits (one massive and one massless or two massless particles) of the massive S-matrix. The resulting expressions should play the role of building blocks for the S-matrices of the massless modes of the $AdS_2 \times S^2 \times T^6$ superstring. As for the $AdS_3 \times S^3 \times T^4$ case the precise nature of this construction requires the knowledge of how all the states transform under the full light-cone gauge symmetry algebra including any additional bosonic symmetries originating from the T^6 compact directions.

In the massless limit the light-cone gauge symmetry $psu(1|1)^2 \ltimes \mathbb{R}^3$ can be extended to a Yangian of the standard form. However this does not generalize in an obvious way to the massive S-matrix. It would be interesting to see if there exists a non-standard Yangian in this case. We are also currently investigating the presence of the secret symmetry [95, 97] and the RTT realization of the symmetry algebra [86, 202]. Finally, we gave some initial considerations to the Bethe ansatz for the massive S-matrix, in particular highlighting the existence of a pseudovacuum. Due to the complexity of the parametrizing functions of the S-matrix and the fact that we are considering long representations of the symmetry algebra the completion of the algebraic Bethe ansatz remains an open problem.

6.3. Yangian Symmetry of AdS_2 Superstrings

In Chapter 4 we performed a series of studies on the conjectured exact S-matrix for the massive excitations of the $AdS_2 \times S^2 \times T^6$ superstring. This S-matrix encodes the integrability of the quantum problem, and is supposed to be the first step towards the complete solution of the theory in the “planar” limit (no joining or splitting of strings). This in turn is expected to provide information on the spectrum of the elusive superconformal quantum mechanics, which should be holographically related to the superstring in this background.

By employing the technique of the RTT realisation, we have found the presence of Yangian symmetry for the massive sector, and given two alternative presentations – both in the spirit of Drinfeld's second realisation [74] – along with the map relating them. We have studied the Yangian coproduct, and found the conditions under which we can have a consistent evaluation representation. In order to ascertain these requirements, we studied the co-commutativity of the higher central charges, which is a necessary condition for the existence of an R-matrix. We discovered that shortening is one way to have a consistent evaluation representation, exactly as it was noticed in AdS_5 [196]. However, we demonstrated explicitly that there is a second route, which crucially for the $AdS_2 \times S^2$ superstring holds for long representations.

We also found, as in the higher dimensional cases, a secret symmetry, which is present only at level 1 of the Yangian and at higher levels. This confirms the ubiquitous presence of this symmetry in all the known manifestations of integrability in AdS/CFT.

We have studied the classical r-matrix of the problem, and rediscovered from its analytic structure the need for the extra gl type (secret) generator. This is also similar to the situation in the AdS_5 case.

We have taken the first steps towards a derivation of the Bethe equations starting from our S-matrix (inverse scattering method). At zero coupling, we discovered that our S-matrix becomes (up to a twist which is easily dealt with) two copies of the standard rational $gl(1|1)$ R-matrix. This allows one to define a pseudo-vacuum in each copy individually, and enormously simplifies the problem. Taking the zero-coupling limit of the S-matrix as a generating R-matrix for the Algebraic Bethe Ansatz, we obtain an effective model of free fermions on a periodic spin-chain. Let us note that this should be related to the would-be *one-loop* result of the first nested level in the Bethe ansatz of the putative spin-chain, describing the superconformal quantum mechanics dual to the superstring.

It seems that we are re-discovering many of the features of the AdS_5 (and, to a certain extent, the AdS_3) Yangian. In the AdS_2 case, however, the algebra is small enough that we are able to say more, especially in terms of alternative presentations. This means that we may hope to find the complete Drinfeld second realisation and derive the universal R-matrix through a suitable ansatz. This would also help us to understand the higher dimensional cases. In turn, we would then be able to finally construct the much sought after exotic quantum group, which should quantise the classical r-matrix algebra, and would prove the algebraic integrability of the system.

The most urgent challenge is probably to derive the full set of Bethe (Beisert-Staudacher) equations for the spectral problem. They encode the information of the planar anomalous dimensions in the dual theory, and would hence provide decisive information regarding the nature of the holographic dual to the AdS_2 superstring theory. The simplifying assumption of zero coupling of course eliminates those entries which are responsible for the full pseudo-vacuum being a mixed state in the two copies. Therefore, a more sophisticated technique, rather than the simple algebraic Bethe ansatz computation we have performed here, might be required. This should tie in with a thorough off-shell worldsheet analysis in the spirit of [121, 195].

Further directions include studying D-branes in this background, and performing a boundary integrability analysis as recently done in [238]. Also, it would be illuminating to continue the perturbative and unitarity analyses of [181, 239, 94, 194], obtaining further information on the dressing phase and the dispersion relation.

6.4. Yangian Symmetry of AdS_3 Superstrings

In Chapter 5, we found a complete realisation of the secret Yangian symmetries for AdS_3 backgrounds, including their crossing symmetry condition. We obtained two different classes of secret generators: one is embedded in the $gl(1|1)$ Yangian; while the other, more reminiscent of its higher-dimensional AdS_5 analog, is not . Due to this embedding we are able to relate these two classes abstractly by a quadratic map in the generators. Because of their Yangian-like coproducts with the typical quadratic tail, a quadratic map is the most one would be allowed to have, to map not only the generators to each other but their coproducts as well. For the *embedded* level-1 secret symmetry, an interpretation within the superconformal algebra of the theory (before the symmetry-breaking implemented by the choice of the spin-chain vacuum) is likely to occur in the T^4 case, where the level zero counterpart preserves the vacuum [190]. This is, however, not the case for the secret generator. By discussing the crossing symmetry relation of the latter, we gain a new perspective that might be useful in interpreting a recent observation of [86]. The secret symmetry was accommodated within crossing by allowing a shift in the multiplying parameter. Our results seem to suggest that even in the AdS_5 case one might find it useful to think of the crossed secret generator as a right generator, to a *left* one being the secret symmetry originally found.

We also found an incarnation of the $R\mathcal{T}\mathcal{T}$-construction of Beisert and de Leeuw's [86] in the AdS_3 case. We checked that it is possible to reproduce the Yangian from the R-matrix via the $R\mathcal{T}\mathcal{T}$-construction, both for the left and the right scattering problem, extending the validity of their framework to the lower-dimensional case at hand.

Let us point out a few open problems deserving further investigation and serving as future directions for research.

Firstly, all our conclusions are extrapolated from specific representations. Therefore, one should study general representations to assess the universality of our results.

Secondly, the universal R-matrix for the complete centrally-extended algebra is still unknown. The same problem still plagues the AdS_5 case, although the recent result of [86] opened up the problem to a new promising approach. It also became clear that a significant re-interpretation of the Khoroshkin–Tolstoy formula has to occur for superalgebras with vanishing Killing form [240], and we hope that our analysis will provide further input to tackle the problem.

Finally, a world-sheet realisation of the secret symmetry in terms of non-local charges is still missing in any dimension, and we believe it is absolutely crucial to close this gap. This would be particularly important in view of the off-shell symmetry algebra approach of [121, 217]. We leave this fascinating problem to future investigations.

References

[1] J. M. Maldacena, *The large N limit of superconformal field theories and supergravity,* Adv. Theor. Math. Phys. **2** (1998) 231 [hep-th/9711200].

[2] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, *Gauge theory correlators from non-critical string theory,* Phys. Lett. B **428** (1998) 105 [hep-th/9802109].

[3] E. Witten, *Anti-de Sitter space and holography,* Adv. Theor. Math. Phys. **2** (1998) 253 [hep-th/9802150].

[4] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, *Large N field theories, string theory and gravity,* Phys. Rept. **323** (2000) 183 [hep-th/9905111].

[5] J. M. Maldacena, *Lectures on AdS/CFT,* hep-th/0309246 [hep-th].

[6] H. Nastase, *Introduction to AdS-CFT,* 0712.0689 [hep-th].

[7] N. Beisert and others, *Review of AdS/CFT Integrability: An Overview,* Lett.Math.Phys. **99** (2010) 3 [1012.3982 [hep-th]].

[8] R. R. Metsaev and A. A. Tseytlin, *Type IIB superstring action in* $\mathrm{AdS}_5 \times S^5$ *background,* Nucl. Phys. B **533** (1998) 109 [hep-th/9805028 [hep-th]].

[9] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops,* Commun. Math. Phys. **313** (2012) 71 [0712.2824 [hep-th]].

[10] F. Benini and S. Cremonesi, *Partition Functions of* $\mathcal{N} = (2,2)$ *Gauge Theories on* S^2 *and Vortices,* Commun. Math. Phys. **334** (2015) 1483 [1206.2356 [hep-th]].

[11] J. A. Minahan and K. Zarembo, *The Bethe-ansatz for* $\mathcal{N} = 4$ *super Yang-Mills,* JHEP **0303** (2003) 013 [hep-th/0212208].

[12] J. A. Minahan and K. Zarembo, *The Bethe ansatz for superconformal Chern-Simons,* JHEP **0809** (2008) 040 [0806.3951 [hep-th]].

[13] R. W. McKeown and R. Roiban, *The quantum $AdS_5 \times S^5$ superstring at finite coupling,* 1308.4875 [hep-th].

[14] L. Bianchi, M. S. Bianchi, V. Forini, B. Leder, and E. Vescovi, *Green-Schwarz superstring on the lattice,* JHEP **07** (2016) 014 [1605.01726 [hep-th]].

[15] B. E. W. Nilsson and C. N. Pope, *Hopf Fibration of Eleven-dimensional Supergravity,* Class. Quant. Grav. **1** (1984) 499.

[16] D. P. Sorokin, V. I. Tkach, and D. V. Volkov, *ON THE RELATIONSHIP BETWEEN COMPACTIFIED VACUA OF $D = 11$ AND $D = 10$ SUPERGRAVITIES,* Phys. Lett. B **161** (1985) 301.

[17] G. Arutyunov and S. Frolov, *Superstrings on* $\mathrm{AdS}_4 \times \mathrm{CP}^3$ *as a Coset Sigma-model,* JHEP **0809** (2008) 129 [0806.4940 [hep-th]].

[18] B. Stefański, jr, *Green-Schwarz action for Type IIA strings on* $\mathrm{AdS}_4 \times \mathrm{CP}^3$, Nucl. Phys. B **808** (2008) 80 [0806.4948 [hep-th]].

[19] J. Gomis, D. Sorokin, and L. Wulff, *The complete* $\mathrm{AdS}_4 \times \mathrm{CP}^3$ *superspace for the type IIA superstring and D-branes,* JHEP **0903** (2009) 015 [0811.1566 [hep-th]].

[20] O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, *$\mathcal{N} = 6$ superconformal Chern-Simons-matter theories, M2-branes and their gravity duals,* JHEP **0810** (2008) 091 [0806.1218 [hep-th]].

[21] M. A. Bandres, A. E. Lipstein, and J. H. Schwarz, *Studies of the ABJM Theory in a Formulation with Manifest* SU(4) *R-Symmetry,* JHEP **0809** (2008) 027 [0807.0880 [hep-th]].

[22] J. Bagger and N. Lambert, *Modeling multiple M2's,* Phys. Rev. D **75** (2007) 045020 [hep-th/0611108].

[23] J. Bagger and N. Lambert, *Gauge Symmetry and Supersymmetry of Multiple M2-Branes,* Phys. Rev. D **77** (2008) 065008 [0711.0955 [hep-th]].

[24] J. Bagger and N. Lambert, *Comments On Multiple M2-branes,* JHEP **0802** (2008) 105 [0712.3738 [hep-th]].

[25] M. Van Raamsdonk, *Comments on the Bagger-Lambert theory and multiple $M2$-branes,* JHEP **0805** (2008) 105 [0803.3803 [hep-th]].

[26] H. J. Boonstra, B. Peeters, and K. Skenderis, *Brane intersections, anti-de Sitter spacetimes and dual superconformal theories,* Nucl. Phys. B **533** (1998) 127 [hep-th/9803231].

[27] P. M. Cowdall and P. K. Townsend, *Gauged supergravity vacua from intersecting branes,* Phys. Lett. B **429** (1998) 281 [hep-th/9801165].

[28] A. Babichenko, B. Stefański, jr., and K. Zarembo, *Integrability and the* AdS_3/CFT_2 *correspondence,* JHEP **1003** (2010) 058 [0912.1723 [hep-th]].

[29] V. G. Kac, *A Sketch of Lie Superalgebra Theory,* Commun. Math. Phys. **53** (1977) 31.

[30] L. Frappat, P. Sorba, and A. Sciarrino, *Dictionary on Lie superalgebras,* hep-th/9607161 [hep-th].

[31] J. P. Gauntlett, R. C. Myers, and P. K. Townsend, *Supersymmetry of rotating branes,* Phys. Rev. D **59** (1999) 025001 [hep-th/9809065].

[32] J. de Boer, A. Pasquinucci, and K. Skenderis, *AdS/CFT dualities involving large 2d* $\mathcal{N} = 4$ *superconformal symmetry,* Adv. Theor. Math. Phys. **3** (1999) 577 [hep-th/9904073].

[33] S. Gukov, E. Martinec, G. W. Moore, and A. Strominger, *The search for a holographic dual to* $\mathrm{AdS}_3 \times S^3 \times S^3 \times S^1$, Adv. Theor. Math. Phys. **9** (2005) 435 [hep-th/0403090].

[34] L. Brink, J. H. Schwarz, and J. Scherk, *Supersymmetric Yang-Mills Theories,* Nucl. Phys. B **121** (1977) 77.

[35] F. Larsen and E. J. Martinec, U(1) *charges and moduli in the D1-D5 system,* JHEP **9906** (1999) 019 [hep-th/9905064].

[36] J. M. Maldacena and A. Strominger, *AdS(3) black holes and a stringy exclusion principle,* JHEP **12** (1998) 005 [hep-th/9804085 [hep-th]].

[37] E. J. Martinec, *Matrix models of AdS gravity,* hep-th/9804111 [hep-th].

[38] C. Vafa, *Puzzles at large N,* hep-th/9804172 [hep-th].

[39] J. de Boer, *Six-dimensional supergravity on S**3 x AdS(3) and 2-D conformal field theory,* Nucl. Phys. B **548** (1999) 139 [hep-th/9806104 [hep-th]].

[40] O. Aharony and M. Berkooz, *IR dynamics of D = 2, N=(4,4) gauge theories and DLCQ of 'little string theories',* JHEP **10** (1999) 030 [hep-th/9909101 [hep-th]].

[41] N. Seiberg and E. Witten, *The D1/D5 system and singular CFT,* JHEP **9904** (1999) 017 [hep-th/9903224].

[42] O. Ohlsson Sax, A. Sfondrini, and B. Stefanski, *Integrability and the Conformal Field Theory of the Higgs branch,* JHEP **06** (2015) 103 [1411.3676 [hep-th]].

[43] M. R. Gaberdiel and R. Gopakumar, *Higher Spins & Strings,* JHEP **11** (2014) 044 [1406.6103 [hep-th]].

[44] D. Tong, *The holographic dual of* $AdS_3 \times S^3 \times S^3 \times S^1$, JHEP **1404** (2014) 193 [1402.5135 [hep-th]].

[45] I. R. Klebanov and A. A. Tseytlin, *Intersecting M-branes as four-dimensional black holes,* Nucl. Phys. B **475** (1996) 179 [hep-th/9604166 [hep-th]].

[46] J. M. Maldacena, A. Strominger, and E. Witten, *Black hole entropy in M theory,* JHEP **12** (1997) 002 [hep-th/9711053 [hep-th]].

[47] A. Simons, A. Strominger, D. M. Thompson, and X. Yin, *Supersymmetric branes in AdS(2) x S**2 x CY(3),* Phys. Rev. D **71** (2005) 066008 [hep-th/0406121 [hep-th]].

[48] A. A. Tseytlin, *Harmonic superpositions of M-branes,* Nucl. Phys. B **475** (1996) 149 [hep-th/9604035 [hep-th]].

[49] D. Sorokin, A. Tseytlin, L. Wulff, and K. Zarembo, *Superstrings in* $\mathrm{AdS}_2 \times S^2 \times T^6$, J.Phys. A **44** (2011) 275401 [1104.1793 [hep-th]].

[50] J. P. Gauntlett, G. W. Gibbons, G. Papadopoulos, and P. K. Townsend, *Hyper-Kahler manifolds and multiply intersecting branes,* Nucl. Phys. B **500** (1997) 133 [hep-th/9702202 [hep-th]].

[51] G. W. Gibbons and P. K. Townsend, *Black holes and Calogero models,* Phys. Lett. B **454** (1999) 187 [hep-th/9812034 [hep-th]].

[52] A. Strominger, *AdS(2) quantum gravity and string theory,* JHEP **01** (1999) 007 [hep-th/9809027 [hep-th]].

[53] J. M. Maldacena, J. Michelson, and A. Strominger, *Anti-de Sitter fragmentation,* JHEP **02** (1999) 011 [hep-th/9812073 [hep-th]].

[54] J. Michelson and A. Strominger, *The Geometry of (super)conformal quantum mechanics,* Commun. Math. Phys. **213** (2000) 1 [hep-th/9907191 [hep-th]].

[55] C. M. Hull, *Timelike T duality, de Sitter space, large N gauge theories and topological field theory,* JHEP **07** (1998) 021 [hep-th/9806146 [hep-th]].

[56] J. M. Drummond, J. Henn, V. A. Smirnov, and E. Sokatchev, *Magic identities for conformal four-point integrals,* JHEP **01** (2007) 064 [hep-th/0607160 [hep-th]].

[57] J. M. Drummond, G. P. Korchemsky, and E. Sokatchev, *Conformal properties of four-gluon planar amplitudes and Wilson loops,* Nucl. Phys. B **795** (2008) 385 [0707.0243 [hep-th]].

[58] A. Brandhuber, P. Heslop, and G. Travaglini, *MHV amplitudes in N=4 super Yang-Mills and Wilson loops,* Nucl. Phys. B **794** (2008) 231 [0707.1153 [hep-th]].

[59] J. M. Drummond, J. Henn, G. P. Korchemsky, and E. Sokatchev, *On planar gluon amplitudes/Wilson loops duality,* Nucl. Phys. B **795** (2008) 52 [0709.2368 [hep-th]].

[60] J. M. Drummond, J. Henn, G. P. Korchemsky, and E. Sokatchev, *Conformal Ward identities for Wilson loops and a test of the duality with gluon amplitudes,* Nucl. Phys. B **826** (2010) 337 [0712.1223 [hep-th]].

[61] J. M. Drummond, J. Henn, G. P. Korchemsky, and E. Sokatchev, *Dual superconformal symmetry of scattering amplitudes in N=4 super-Yang-Mills theory,* Nucl. Phys. B **828** (2010) 317 [0807.1095 [hep-th]].

[62] A. Brandhuber, P. Heslop, and G. Travaglini, *A Note on dual superconformal symmetry of the N=4 super Yang-Mills S-matrix,* Phys. Rev. D **78** (2008) 125005 [0807.4097 [hep-th]].

[63] L. F. Alday and J. M. Maldacena, *Gluon scattering amplitudes at strong coupling,* JHEP **06** (2007) 064 [0705.0303 [hep-th]].

[64] L. F. Alday and J. Maldacena, *Comments on gluon scattering amplitudes via AdS/CFT,* JHEP **11** (2007) 068 [0710.1060 [hep-th]].

[65] J. M. Drummond, J. Henn, G. P. Korchemsky, and E. Sokatchev, *Hexagon Wilson loop = six-gluon MHV amplitude,* Nucl. Phys. B **815** (2009) 142 [0803.1466 [hep-th]].

[66] L. F. Alday and R. Roiban, *Scattering Amplitudes, Wilson Loops and the String/Gauge Theory Correspondence,* Phys. Rept. **468** (2008) 153 [0807.1889 [hep-th]].

[67] N. Berkovits and J. Maldacena, *Fermionic T-Duality, Dual Superconformal Symmetry, and the Amplitude/Wilson Loop Connection,* JHEP **09** (2008) 062 [0807.3196 [hep-th]].

[68] N. Beisert, R. Ricci, A. A. Tseytlin, and M. Wolf, *Dual Superconformal Symmetry from AdS(5) x S**5 Superstring Integrability,* Phys. Rev. D **78** (2008) 126004 [0807.3228 [hep-th]].

[69] D. J. Gross and P. F. Mende, *String Theory Beyond the Planck Scale,* Nucl. Phys. B **303** (1988) 407.

[70] S.-J. Rey and J.-T. Yee, *Macroscopic strings as heavy quarks in large N gauge theory and anti-de Sitter supergravity,* Eur.Phys.J. C **22** (2001) 379 [hep-th/9803001 [hep-th]].

[71] J. M. Maldacena, *Wilson loops in large N field theories,* Phys.Rev.Lett. **80** (1998) 4859 [hep-th/9803002 [hep-th]].

[72] A. Pressley and V. Chari, *Notes on quantum groups,* Nucl. Phys. Proc. Suppl. **18A** (1990) 207.

[73] C. Kassel, *Quantum groups,* New York, USA: Springer (1995) 531 p. (Graduate text in mathematics, 155)(1995).

[74] V. Drinfeld, *A New realization of Yangians and quantized affine algebras,* Sov.Math.Dokl. **36** (1988) 212.

[75] L. Takhtajan and L. Faddeev, *The Quantum method of the inverse problem and the Heisenberg XYZ model,* Russ.Math.Surveys **34** (1979) 11.

[76] V. Drinfeld, *Quantum groups,* J.Sov.Math. **41** (1988) 898.

[77] A. Torrielli, *Review of AdS/CFT Integrability, Chapter VI.2: Yangian Algebra,* Lett.Math.Phys. **99** (2010) 547 [1012.4005 [hep-th]].

[78] F. Loebbert, *Lectures on Yangian Symmetry,* J. Phys. A **49** (2016) 323002 [1606.02947 [hep-th]].

[79] S. Khoroshkin and V. Tolstoi, *Yangian double and rational R matrix,* Lett.Math.Phys **36** (1994) 373 [hep-th/9406194 [hep-th]].

[80] A. M. Polyakov, *Hidden Symmetry of the Two-Dimensional Chiral Fields,* Phys. Lett. B **72** (1977) 224.

[81] R. Shankar and E. Witten, *The S Matrix of the Supersymmetric Nonlinear σ Model,* Phys. Rev. D **17** (1978) 2134.

[82] S. J. Parke, *Absence of Particle Production and Factorization of the S Matrix in (1+1)-dimensional Models,* Nucl. Phys. B **174** (1980) 166.

[83] P. Dorey, *Exact S matrices,* hep-th/9810026 [hep-th].

[84] D. Bombardelli, *S-matrices and integrability,* J. Phys. A **49** (2016) 323003 [1606.02949 [hep-th]].

[85] A. Molev, M. Nazarov, and G. Olshansky, *Yangians and classical Lie algebras,* Russ. Math. Surveys **51** (1996) 205 [hep-th/9409025 [hep-th]].

[86] N. Beisert and M. de Leeuw, *The RTT realization for the deformed $gl(2|2)$ Yangian,* J.Phys. A **47** (2014) 305201 [1401.7691 [math-ph]].

[87] L. D. Faddeev and L. A. Takhtajan, *HAMILTONIAN METHODS IN THE THEORY OF SOLITONS,* , 1987.

[88] E. K. Sklyanin, *Quantum version of the method of inverse scattering problem,* J. Sov. Math. **19** (1982) 1546.

[89] D. A. Leites and V. V. Serganova, *SOLUTIONS OF THE CLASSICAL YANG-BAXTER EQUATION FOR SIMPLE SUPERALGEBRAS,* Theor. Math. Phys. **58** (1984) 16.

[90] R.-b. Zhang, M. D. Gould, and A. J. Bracken, *Solutions of the graded classical Yang-Baxter equation and integrable models,* J. Phys. A **24** (1991) 1185.

[91] A. Torrielli, *Yangians, S-matrices and* AdS/*CFT,* J.Phys. A **44** (2011) 263001 [1104.2474 [hep-th]].

[92] C.-N. Yang, *Some exact results for the many body problems in one dimension with repulsive delta function interaction,* Phys. Rev. Lett. **19** (1967) 1312.

[93] N. Beisert, *The* su(2|2) *dynamic S-matrix,* Adv. Theor. Math. Phys. **12** (2008) 945 [hep-th/0511082].

[94] T. Klose, T. McLoughlin, R. Roiban, and K. Zarembo, *Worldsheet scattering in* $\mathrm{AdS}_5 \times S^5$, JHEP **0703** (2007) 094 [hep-th/0611169].

[95] T. Matsumoto, S. Moriyama, and A. Torrielli, *A Secret Symmetry of the AdS/CFT S-matrix,* JHEP **0709** (2007) 099 [0708.1285 [hep-th]].

[96] S. Moriyama and A. Torrielli, *A Yangian double for the AdS/CFT classical r-matrix,* JHEP **06** (2007) 083 [0706.0884 [hep-th]].

[97] N. Beisert and F. Spill, *The Classical r-matrix of AdS/CFT and its Lie Bialgebra Structure,* Commun. Math. Phys. **285** (2009) 537 [0708.1762 [hep-th]].

[98] F. Spill and A. Torrielli, *On Drinfeld's second realization of the AdS/CFT su(2—2) Yangian,* J. Geom. Phys. **59** (2009) 489 [0803.3194 [hep-th]].

[99] I. Bena, J. Polchinski, and R. Roiban, *Hidden symmetries of the* $\mathrm{AdS}_5 \times S^5$ *superstring,* Phys. Rev. D **69** (2004) 046002 [hep-th/0305116].

[100] L. N. Lipatov, *Reggeization of the Vector Meson and the Vacuum Singularity in Nonabelian Gauge Theories,* Sov. J. Nucl. Phys. **23** (1976) 338.

[101] E. A. Kuraev, L. N. Lipatov, and V. S. Fadin, *Multi - Reggeon Processes in the Yang-Mills Theory,* Sov. Phys. JETP **44** (1976) 443.

[102] E. A. Kuraev, L. N. Lipatov, and V. S. Fadin, *The Pomeranchuk Singularity in Nonabelian Gauge Theories,* Sov. Phys. JETP **45** (1977) 199.

[103] V. S. Fadin, E. A. Kuraev, and L. N. Lipatov, *On the Pomeranchuk Singularity in Asymptotically Free Theories,* Phys. Lett. B **60** (1975) 50.

[104] I. I. Balitsky and L. N. Lipatov, *The Pomeranchuk Singularity in Quantum Chromodynamics,* Sov. J. Nucl. Phys. **28** (1978) 822.

[105] J. A. Minahan, *Review of AdS/CFT Integrability, Chapter I.1: Spin Chains in* $\mathcal{N} = 4$ *Super Yang-Mills,* Lett.Math.Phys. **99** (2010) 33 [1012.3983 [hep-th]].

[106] H. Bethe, *On the theory of metals. 1. Eigenvalues and eigenfunctions for the linear atomic chain,* Z. Phys. **71** (1931) 205.

[107] N. Beisert, C. Kristjansen, and M. Staudacher, *The dilatation operator of conformal $\mathcal{N} = 4$ super Yang-Mills theory,* Nucl. Phys. B **664** (2003) 131 [hep-th/0303060 [hep-th]].

[108] N. Beisert, *The complete one-loop dilatation operator of $\mathcal{N} = 4$ super Yang-Mills theory,* Nucl. Phys. B **676** (2004) 3 [hep-th/0307015].

[109] N. Beisert and M. Staudacher, *The $\mathcal{N} = 4$ SYM integrable super spin chain,* Nucl. Phys. B **670** (2003) 439 [hep-th/0307042].

[110] N. Beisert, V. Dippel, and M. Staudacher, *A novel long range spin chain and planar $\mathcal{N} = 4$ super Yang- Mills,* JHEP **0407** (2004) 075 [hep-th/0405001 [hep-th]].

[111] M. B. Green and J. H. Schwarz, *Covariant Description of Superstrings,* Phys. Lett. B **136** (1984) 367.

[112] M. T. Grisaru, P. S. Howe, L. Mezincescu, B. Nilsson, and P. K. Townsend, *$\mathcal{N} = 2$ Superstrings in a Supergravity Background,* Phys. Lett. B **162** (1985) 116.

[113] R. R. Metsaev, C. B. Thorn, and A. A. Tseytlin, *Light cone superstring in AdS space-time,* Nucl. Phys. B **596** (2001) 151 [hep-th/0009171 [hep-th]].

[114] G. Arutyunov and S. Frolov, *Integrable Hamiltonian for classical strings on* $\mathrm{AdS}_5 \times S^5$*,* JHEP **0502** (2005) 059 [hep-th/0411089].

[115] G. Arutyunov, S. Frolov, J. Russo, and A. A. Tseytlin, *Spinning strings in AdS(5) x S**5 and integrable systems,* Nucl. Phys. B **671** (2003) 3 [hep-th/0307191 [hep-th]].

[116] A. A. Tseytlin, *Spinning strings and AdS / CFT duality,* hep-th/0311139 [hep-th].

[117] V. A. Kazakov, A. Marshakov, J. A. Minahan, and K. Zarembo, *Classical/quantum integrability in AdS/CFT,* JHEP **5** (2004) 24 [hep-th/0402207].

[118] G. Arutyunov, S. Frolov, and M. Staudacher, *Bethe ansatz for quantum strings,* JHEP **0410** (2004) 016 [hep-th/0406256].

[119] S. Frolov, J. Plefka, and M. Zamaklar, *The* $\mathrm{AdS}_5 \times S^5$ *superstring in light-cone gauge and its Bethe equations,* J.Phys. A **39** (2006) 13037 [hep-th/0603008 [hep-th]].

[120] T. Klose, T. McLoughlin, J. A. Minahan, and K. Zarembo, *World-sheet scattering in* $\mathrm{AdS}_5 \times S^5$ *at two loops,* JHEP **0708** (2007) 051 [0704.3891 [hep-th]].

[121] G. Arutyunov, S. Frolov, J. Plefka, and M. Zamaklar, *The off-shell symmetry algebra of the light-cone* $\mathrm{AdS}_5 \times S^5$ *superstring,* J. Phys. A **40** (2007) 3583 [hep-th/0609157].

[122] N. Beisert and M. Staudacher, *Long-range* $\mathrm{PSU}(2,2|4)$ *Bethe ansaetze for gauge theory and strings,* Nucl. Phys. B **727** (2005) 1 [hep-th/0504190 [hep-th]].

[123] N. Beisert, B. Eden, and M. Staudacher, *Transcendentality and crossing,* J. Stat. Mech. **0701** (2007) P021 [hep-th/0610251].

[124] G. Arutyunov, S. Frolov, and M. Zamaklar, *The Zamolodchikov-Faddeev algebra for* $\mathrm{AdS}_5 \times S^5$ *superstring,* JHEP **0704** (2007) 002 [hep-th/0612229].

[125] A. A. Tseytlin, *Superstrings in AdS in light cone gauge,* Int. J. Mod. Phys. A **16** (2001) 900 [hep-th/0009226 [hep-th]].

[126] G. Arutyunov, S. Frolov, and M. Zamaklar, *Finite-size effects from giant magnons,* Nucl. Phys. B **778** (2007) 1 [hep-th/0606126].

[127] G. Arutyunov and S. Frolov, *Uniform light-cone gauge for strings in* $\mathrm{AdS}_5 \times S^5$*: Solving* su(1|1) *sector,* JHEP **0601** (2006) 055 [hep-th/0510208 [hep-th]].

[128] D. E. Berenstein, J. M. Maldacena, and H. S. Nastase, *Strings in flat space and pp waves from* $\mathcal{N} = 4$ *super Yang Mills,* JHEP **0204** (2002) 013 [hep-th/0202021].

[129] T. McLoughlin, *Review of AdS/CFT Integrability, Chapter II.2: Quantum Strings in* $\mathrm{AdS}_5 \times S^5$*,* Lett.Math.Phys. **99** (2010) 127 [1012.3987 [hep-th]].

[130] M. Lüscher, *Volume Dependence of the Energy Spectrum in Massive Quantum Field Theories. 1. Stable Particle States,* Commun. Math. Phys. **104** (1986) 177.

[131] M. Lüscher, *Volume Dependence of the Energy Spectrum in Massive Quantum Field Theories. 2. Scattering States,* Commun. Math. Phys. **105** (1986) 153.

[132] N. Gromov, V. Kazakov, S. Leurent, and D. Volin, *Quantum Spectral Curve for Planar* $\mathcal{N} =$ *Super-Yang-Mills Theory,* Phys. Rev. Lett. **112** (2014) 011602 [1305.1939 [hep-th]].

[133] N. Gromov, V. Kazakov, S. Leurent, and D. Volin, *Quantum spectral curve for arbitrary state/operator in* AdS_5/CFT_4*,* JHEP **09** (2015) 187 [1405.4857 [hep-th]].

[134] G. Arutyunov and S. Frolov, *String hypothesis for the* $\mathrm{AdS}_5 \times S^5$ *mirror,* JHEP **0903** (2009) 152 [0901.1417 [hep-th]].

[135] N. Gromov, V. Kazakov, and P. Vieira, *Integrability for the Full Spectrum of Planar AdS/CFT,* Phys. Rev. Lett. **103** (2009) 131601 [0901.3753 [hep-th]].

[136] G. Arutyunov and S. Frolov, *Thermodynamic Bethe Ansatz for the* $\mathrm{AdS}_5 \times S^5$ *Mirror Model,* JHEP **0905** (2009) 068 [0903.0141 [hep-th]].

[137] D. Bombardelli, D. Fioravanti, and R. Tateo, *Thermodynamic Bethe Ansatz for planar AdS/CFT: a proposal,* J. Phys. A **42** (2009) 375401 [0902.3930].

[138] N. Gromov, V. Kazakov, A. Kozak, and P. Vieira, *Integrability for the Full Spectrum of Planar AdS/CFT II,* Lett. Math. Phys. **91** (2010) 265 [0902.4458 [hep-th]].

[139] A. Cavaglia, D. Fioravanti, and R. Tateo, *Extended Y-system for the AdS_5/CFT_4 correspondence,* Nucl. Phys. B **843** (2011) 302 [1005.3016 [hep-th]].

[140] J. Balog and A. Hegedus, *$AdS_5 \times S^5$ mirror TBA equations from Y-system and discontinuity relations,* JHEP **08** (2011) 095 [1104.4054 [hep-th]].

[141] J. D. Brown and M. Henneaux, *Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity,* Commun. Math. Phys. **104** (1986) 207.

[142] A. B. Zamolodchikov, *Irreversibility of the Flux of the Renormalization Group in a 2D Field Theory,* JETP Lett. **43** (1986) 730.

[143] O. Aharony, M. Berkooz, S. Kachru, N. Seiberg, and E. Silverstein, *Matrix description of interacting theories in six-dimensions,* Adv. Theor. Math. Phys. **1** (1998) 148 [hep-th/9707079 [hep-th]].

[144] G. 't Hooft, *A Planar Diagram Theory for Strong Interactions,* Nucl. Phys. B **72** (1974) 461.

[145] R. Ricci, A. A. Tseytlin, and M. Wolf, *On T-Duality and Integrability for Strings on AdS Backgrounds,* JHEP **12** (2007) 082 [0711.0707 [hep-th]].

[146] E. O Colgain, *Fermionic T-duality: A snapshot review,* Int. J. Mod. Phys. A **27** (2012) 1230032 [1210.5588 [hep-th]].

[147] D. Sorokin and L. Wulff, *Evidence for the classical integrability of the complete* $\mathrm{AdS}_4 \times \mathrm{CP}^3$ *superstring,* JHEP **1011** (2010) 143 [1009.3498 [hep-th]].

[148] A. Cagnazzo, D. Sorokin, and L. Wulff, *More on integrable structures of superstrings in* $\mathrm{AdS}_4 \times \mathrm{CP}^3$ *and* $\mathrm{AdS}_2 \times S^2 \times T^6$ *superbackgrounds,* JHEP **1201** (2012) 004 [1111.4197 [hep-th]].

[149] P. Sundin and L. Wulff, *Classical integrability and quantum aspects of the* $\mathrm{AdS}_3 \times S^3 \times S^3 \times S^1$ *superstring,* JHEP **1210** (2012) 109 [1207.5531 [hep-th]].

[150] L. Wulff, *Superisometries and integrability of superstrings,* JHEP **05** (2014) 115 [1402.3122 [hep-th]].

[151] L. Wulff, *On integrability of strings on symmetric spaces,* JHEP **09** (2015) 115 [1505.03525 [hep-th]].

[152] I. Adam, A. Dekel, and Y. Oz, *On Integrable Backgrounds Self-dual under Fermionic T-duality,* JHEP **04** (2009) 120 [0902.3805 [hep-th]].

[153] P. A. Grassi, D. Sorokin, and L. Wulff, *Simplifying superstring and D-brane actions in* $\mathrm{AdS}_4 \times \mathrm{CP}^3$ *superbackground,* JHEP **0908** (2009) 060 [0903.5407 [hep-th]].

[154] I. Bakhmatov and D. S. Berman, *Exploring Fermionic T-duality,* Nucl. Phys. B **832** (2010) 89 [0912.3657 [hep-th]].

[155] H. Godazgar and M. J. Perry, *Real fermionic symmetry in type II supergravity,* JHEP **01** (2011) 032 [1008.3128 [hep-th]].

[156] I. Bakhmatov, *On $AdS_4xCP^3T-duality$*, *Nucl.Phys.B***847**(2011) 38[1011.0985[hep – th]]

[157] *D. Gang, Y.-t. Huang, E. Koh, S. Lee, and A. E. Lipstein,* Tree-level Recursion Relation and Dual Superconformal Symmetry of the ABJM Theory, *JHEP* **03** *(2011) 116 [*1012.5032 [hep-th]*].*

[158] *W.-M. Chen and Y.-t. Huang,* Dualities for Loop Amplitudes of N=6 Chern-Simons Matter Theory, *JHEP* **11** *(2011) 057 [*1107.2710 [hep-th]*].*

[159] *M. S. Bianchi, M. Leoni, A. Mauri, S. Penati, and A. Santambrogio,* Scattering in ABJ theories, *JHEP* **12** *(2011) 073 [*1110.0738 [hep-th]*].*

[160] *T. Bargheer, F. Loebbert, and C. Meneghelli,* Symmetries of Tree-level Scattering Amplitudes in $\mathcal{N} = 6$ Superconformal Chern-Simons Theory, *Phys.Rev. D* **82** *(2010) 045016 [*1003.6120 [hep-th]*].*

[161] *M. S. Bianchi, M. Leoni, A. Mauri, S. Penati, C. A. Ratti, and others,* From Correlators to Wilson Loops in Chern-Simons Matter Theories, *JHEP* **1106** *(2011) 118 [*1103.3675 [hep-th]*].*

[162] *M. S. Bianchi, M. Leoni, A. Mauri, S. Penati, and A. Santambrogio,* Scattering Amplitudes/Wilson Loop Duality In ABJM Theory, *JHEP* **01** *(2012) 056 [*1107.3139 [hep-th]*].*

[163] *I. Adam, A. Dekel, and Y. Oz,* On the fermionic T-duality of the AdS_4xCP^3 sigma-model, *JHEP* **10** *(2010) 110 [*1008.0649 [hep-th]*].*

[164] *I. Oda and M. Tonin,* On the Berkovits covariant quantization of GS superstring, *Phys. Lett. B* **520** *(2001) 398 [*hep-th/0109051 [hep-th]*].*

[165] *N. Rughoonauth, P. Sundin, and L. Wulff,* Near BMN dynamics of the $\mathrm{AdS}^3 \times S^3 \times S^3 \times S^1$ superstring, *JHEP* **1207** *(2012) 159 [*1204.4742 [hep-th]*].*

[166] *M. C. Abbott, J. Murugan, S. Penati, A. Pittelli, D. Sorokin, P. Sundin, J. Tarrant, M. Wolf, and L. Wulff,* T-duality of Green-Schwarz superstrings on $\mathrm{AdS}_d \times \mathrm{S}^d \times \mathrm{M}^{10-2d}$, *JHEP* **12** *(2015) 104 [*1509.07678 [hep-th]*].*

[167] *L. Wulff,* The type II superstring to order θ^4, *JHEP* **07** *(2013) 123* [1304.6422 [hep-th]].

[168] *V. V. Serganova,* Classification of real simple Lie superalgebras and symmetric superspaces, *Funct. Anal. Appl.* **17** *(1983) 200.*

[169] *J. Rahmfeld and A. Rajaraman,* The GS string action on $\text{AdS}_3 \times S^3$ with Ramond-Ramond charge, *Phys.Rev. D* **60** *(1999) 064014* [hep-th/9809164 [hep-th]].

[170] *J.-G. Zhou,* Super 0-brane and GS superstring actions on AdS(2) x S**2, *Nucl. Phys. B* **559** *(1999) 92* [hep-th/9906013 [hep-th]].

[171] *N. Berkovits, M. Bershadsky, T. Hauer, S. Zhukov, and B. Zwiebach,* Superstring theory on $\text{AdS}_2 \times S^2$ as a coset supermanifold, *Nucl. Phys. B* **567** *(2000) 61* [hep-th/9907200].

[172] *K. Zarembo,* Strings on Semisymmetric Superspaces, *JHEP* **1005** *(2010) 002* [1003.0465 [hep-th]].

[173] *A. Dekel and Y. Oz,* Self-Duality of Green-Schwarz Sigma-Models, *JHEP* **03** *(2011) 117* [1101.0400 [hep-th]].

[174] *T. H. Buscher,* A Symmetry of the String Background Field Equations, *Phys. Lett. B* **194** *(1987) 59.*

[175] *T. H. Buscher,* Path Integral Derivation of Quantum Duality in Nonlinear Sigma Models, *Phys. Lett. B* **201** *(1988) 466.*

[176] *J. Simon,* T duality and effective D-brane actions, *Phys. Rev. D* **61** *(2000) 047702* [hep-th/9812095 [hep-th]].

[177] *A. S. Schwarz and A. A. Tseytlin,* Dilaton shift under duality and torsion of elliptic complex, *Nucl. Phys. B* **399** *(1993) 691* [hep-th/9210015 [hep-th]].

[178] *E. Ivanov, S. Sidorov, and F. Toppan,* Superconformal mechanics in SU(2—1) superspace, *Phys. Rev. D* **91** *(2015) 085032* [1501.05622 [hep-th]].

[179] *I. A. Bandos, E. Ivanov, J. Lukierski, and D. Sorokin,* On the superconformal flatness of AdS superspaces, *JHEP* **06** *(2002) 040* [hep-th/0205104 [hep-th]].

[180] *D. Butter, G. Inverso, and I. Lodato,* Rigid 4D $\mathcal{N} = 2$ supersymmetric backgrounds and actions, *JHEP* **09** *(2015) 088* [1505.03500 [hep-th]].

[181] *J. Murugan, P. Sundin, and L. Wulff,* Classical and quantum integrability in AdS_2/CFT_1, *JHEP* **01** *(2013) 047* [1209.6062 [hep-th]].

[182] *R. Borsato, O. Ohlsson Sax, A. Sfondrini, and B. Stefanski,* The $AdS_3 \times S^3 \times S^3 \times S^1$ worldsheet S matrix, *J. Phys. A* **48** *(2015) 415401* [1506.00218 [hep-th]].

[183] *M. C. Abbott, J. Murugan, P. Sundin, and L. Wulff,* Scattering in AdS(2)/CFT(1) and the BES Phase, *JHEP* **10** *(2013) 066* [1308.1370 [hep-th]].

[184] *B. Hoare, A. Pittelli, and A. Torrielli,* S-matrix for the massive and massless modes of the $AdS_2 \times S^2$ superstring, 1407.0303 [hep-th].

[185] *G. Gotz, T. Quella, and V. Schomerus,* Representation theory of sl(2—1), *J. Algebra* **312** *(2007) 829* [hep-th/0504234 [hep-th]].

[186] *V. G. Kac,* Lie Superalgebras, *Adv. Math.* **26** *(1977) 8*.

[187] *V. G. Kac.* Representations of Classical Lie Superalgebras, . *In* 2nd Conference on Differential Geometrical Methods in Mathematical Physics. Bonn, Germany, July 13-16, 1977 *5971977*.

[188] *C. Gómez and R. Hernández,* The magnon kinematics of the AdS/CFT correspondence, *JHEP* **0611** *(2006) 021* [hep-th/0608029].

[189] *A. B. Zamolodchikov,* Thermodynamics of imaginary coupled sine-Gordon: Dense polymer finite size scaling function, *Phys. Lett. B* **335** *(1994) 436*.

[190] *R. Borsato, O. O. Sax, A. Sfondrini, B. Stefanski, jr., and A. Torrielli,* The all-loop integrable spin-chain for strings on $AdS_3 \times S^3 \times T^4$: the massive sector, 1303.5995 [hep-th].

[191] *R. A. Janik,* The $AdS_5 \times S^5$ superstring worldsheet S-matrix and crossing symmetry, *Phys. Rev. D* **73** *(2006) 086006* [hep-th/0603038].

[192] *G. Arutyunov and S. Frolov,* On String S-matrix, Bound States and TBA, *JHEP* **0712** *(2007) 024* [0710.1568 [hep-th]].

[193] *G. Arutyunov and S. Frolov,* The Dressing Factor and Crossing Equations, *J. Phys. A* **42** *(2009) 425401* [0904.4575 [hep-th]].

[194] *L. Bianchi, V. Forini, and B. Hoare,* Two-dimensional S-matrices from unitarity cuts, *JHEP* **1307** *(2013) 088* [1304.1798 [hep-th]].

[195] *R. Borsato, O. Ohlsson Sax, A. Sfondrini, and B. Stefanski,* The complete $AdS_3 \times S^3 \times T^4$ worldsheet S matrix, *JHEP* **10** *(2014) 66* [1406.0453 [hep-th]].

[196] *G. Arutyunov, M. de Leeuw, and A. Torrielli,* On Yangian and Long Representations of the Centrally Extended su(2|2), *JHEP* **1006** *(2010) 033* [0912.0209 [hep-th]].

[197] *N. Beisert,* The S-matrix of AdS / CFT and Yangian symmetry, *PoS S* **OLVAY** *(2006) 002 [*0704.0400 [nlin.SI]*].*

[198] *A. B. Zamolodchikov and A. B. Zamolodchikov,* Massless factorized scattering and sigma models with topological terms, *Nucl. Phys. B* **379** *(1992) 602.*

[199] *C. Chamon, R. Jackiw, S.-Y. Pi, and L. Santos,* Conformal quantum mechanics as the CFT_1 dual to AdS_2, *Phys. Lett. B* **701** *(2011) 503 [*1106.0726 [hep-th]*].*

[200] *M. B. Green and J. H. Schwarz,* Anomaly Cancellation in Supersymmetric $D = 10$ Gauge Theory and Superstring Theory, *Phys. Lett. B* **149** *(1984) 117.*

[201] *V. Regelskis,* Yangian of AdS3/CFT2 and its deformation, *J. Geom. Phys.* **106** *(2016) 213 [*1503.03799 [math-ph]*].*

[202] *A. Pittelli, A. Torrielli, and M. Wolf,* Secret Symmetries of Type IIB Superstring Theory on AdS3 x S3 x M4, 1406.2840 [hep-th].

[203] *N. Berkovits and A. Mikhailov,* Nonlocal Charges for Bonus Yangian Symmetries of Super-Yang-Mills, *JHEP* **07** *(2011) 125 [*1106.2536 [hep-th]*].*

[204] *N. Beisert and B. U. Schwab,* Bonus Yangian Symmetry for the Planar S-Matrix of N=4 Super Yang-Mills, *Phys.Rev.Lett.* **106** *(2011) 231602 [*1103.0646 [hep-th]*].*

[205] *V. Drinfeld,* Hopf algebras and the quantum Yang-Baxter equation, *Sov.Math.Dokl.* **32** *(1985) 254.*

[206] *A. Torrielli,* Classical r-matrix of the su(2|2) SYM spin-chain, *Phys. Rev. D* **75** *(2007) 105020 [*hep-th/0701281*].*

[207] *G. Arutyunov and S. Frolov,* On $\mathrm{AdS}_5 \times S^5$ string S-matrix, *Phys. Lett. B* **639** *(2006) 378 [*hep-th/0604043*].*

[208] *V. Drinfeld,* Quasi Hopf algebras, *Alg.Anal.* **1N6** *(1989) 114.*

[209] *V. Kazakov, A. S. Sorin, and A. Zabrodin,* Supersymmetric Bethe ansatz and Baxter equations from discrete Hirota dynamics, *Nucl. Phys. B* **790** *(2008) 345 [*hep-th/0703147*].*

[210] *G. Arutyunov and S. Frolov,* Foundations of the $\mathrm{AdS}_5 \times S^5$ Superstring. Part I, *J.Phys.A A* **42** *(2009) 254003 [*0901.4937 [hep-th]*].*

[211] *N. J. MacKay,* Introduction to Yangian symmetry in integrable field theory, *Int. J. Mod. Phys. A* **20** *(2005) 7189 [*hep-th/0409183*].*

[212] *N. Beisert,* The Analytic Bethe Ansatz for a Chain with Centrally Extended su(2|2) Symmetry, *J. Stat. Mech.* **0701** *(2007) P017* [nlin/0610017].

[213] *A. Torrielli,* The Hopf superalgebra of AdS/CFT, *J. Geom. Phys.* **61** *(2011) 230.*

[214] *O. Ohlsson Sax and B. Stefański, jr.,* Integrability, spin-chains and the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence, *JHEP* **1108** *(2011) 029* [1106.2558 [hep-th]].

[215] *O. Ohlsson Sax, j. Stefanski, Bogdan, and A. Torrielli,* On the massless modes of the AdS3/CFT2 integrable systems, *JHEP* **1303** *(2013) 109* [1211.1952 [hep-th]].

[216] *T. Lloyd and B. Stefanski, AdS_3/CFT_2,* finite-gap equations and massless modes, 1312.3268 [hep-th].

[217] *R. Borsato, O. O. Sax, A. Sfondrini, and B. Stefanski,* All-loop worldsheet S matrix for $AdS_3 \times S^3 \times T^4$, 1403.4543 [hep-th].

[218] *A. Sfondrini,* Towards integrability for $\mathrm{AdS}_\mathbf{3}/\mathrm{CFT}_\mathbf{2}$, *J. Phys. A* **48** *(2015) 023001* [1406.2971 [hep-th]].

[219] *R. Borsato, O. Ohlsson Sax, and A. Sfondrini,* A dynamic $\mathrm{su}(1|1)^2$ S-matrix for $\mathrm{AdS}_3/\mathrm{CFT}_2$, 1211.5119 [hep-th].

[220] *R. Borsato, O. Ohlsson Sax, and A. Sfondrini,* All-loop Bethe ansatz equations for $\mathrm{AdS}_3/\mathrm{CFT}_2$, 1212.0505 [hep-th].

[221] *M. C. Abbott,* Comment on Strings in $\mathrm{AdS}_3 \times S^3 \times S^3 \times S^1$ at One Loop, 1211.5587 [hep-th].

[222] *M. Beccaria and G. Macorini,* Quantum corrections to short folded superstring in $\mathrm{AdS}_3 \times S^3 \times M^4$, 1212.5672 [hep-th].

[223] *P. Sundin and L. Wulff,* Worldsheet scattering in $\mathrm{AdS}_3/\mathrm{CFT}_2$, 1302.5349 [hep-th].

[224] *O. T. Engelund, R. W. McKeown, and R. Roiban,* Generalized unitarity and the worldsheet S matrix in $AdS_n \times S^n \times M^{10-2n}$, *JHEP* **1308** *(2013) 023* [1304.4281 [hep-th]].

[225] *L. Bianchi and B. Hoare, $AdS_3 \times S^3 \times M^4$* string S-matrices from unitarity cuts, *JHEP* **08** *(2014) 097* [1405.7947 [hep-th]].

[226] *M. Beccaria, F. Levkovich-Maslyuk, G. Macorini, and A. A. Tseytlin,* Quantum corrections to spinning superstrings in $\mathrm{AdS}_3 \times S^3 \times M^4$: determining the dressing phase, 1211.6090 [hep-th].

[227] *R. Borsato, O. O. Sax, A. Sfondrini, J. Stefanski, Bogdan, and A. Torrielli,* Dressing phases of AdS3/CFT2, *Phys.Rev. D* **88** *(2013) 066004* [1306.2512 [hep-th]].

[228] *M. C. Abbott,* The AdS_3 S^3 S^3 S^1 Hernandez - Lopez phases: a semiclassical derivation, *J. Phys. A* **46** *(2013) 445401* [1306.5106 [hep-th]].

[229] *P. Sundin and L. Wulff,* The low energy limit of the AdS(3) x S(3) x M(4) spinning string, *JHEP* **10** *(2013) 111* [1306.6918 [hep-th]].

[230] *A. Cagnazzo and K. Zarembo,* B-field in AdS_3/CFT_2 Correspondence and Integrability, 1209.4049 [hep-th].

[231] *B. Hoare and A. A. Tseytlin,* On string theory on $AdS_3 \times S^3 \times T^4$ with mixed 3-form flux: tree-level S-matrix, 1303.1037 [hep-th].

[232] *B. Hoare and A. Tseytlin,* Massive S-matrix of $AdS_3 x S^3 x T^4$ superstring theory with mixed 3-form flux, *Nucl.Phys. B* **873** *(2013) 395* [1304.4099 [hep-th]].

[233] *C. Ahn and P. Bozhilov,* String solutions in $AdS_3 x S^3 x T^4 with NS - NSB -$ $field,$ *Phys.Rev.D*90(2014)066010[1404.7644[hep − th]].*J. R. DavidandA. Sadhukhan, Spinningstring* with mixed 3-form fluxes, *JHEP* **10** *(2014) 49* [1405.2687 [hep-th]].

[234] *A. Babichenko, A. Dekel, and O. Ohlsson Sax,* Finite-gap equations for strings on AdS_3 x S^3 x T^4 with mixed 3-form flux, *JHEP* **11** *(2014) 122* [1405.6087 [hep-th]].

[235] *A. B. Zamolodchikov and A. B. Zamolodchikov,* Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field models, *Annals Phys.* **120** *(1979) 253.*

[236] *J. Plefka, F. Spill, and A. Torrielli,* On the Hopf algebra structure of the AdS/CFT S-matrix, *Phys. Rev. D* **74** *(2006) 066008* [hep-th/0608038].

[237] *M. de Leeuw, T. Matsumoto, S. Moriyama, V. Regelskis, and A. Torrielli,* Secret Symmetries in AdS/CFT, *Phys. Scripta* **02** *(2012) 028502* [1204.2366 [hep-th]].

[238] *A. Prinsloo,* D1 and D5-brane giant gravitons on $AdS_3 \times S^3 \times S^3 \times S^1$, *JHEP* **12** *(2014) 094* [1406.6134 [hep-th]].

[239] *R. Roiban, P. Sundin, A. Tseytlin, and L. Wulff,* The one-loop worldsheet S-matrix for the AdS_n x S^n x T^{10-2n} superstring, *JHEP* **08** *(2014) 160* [1407.7883 [hep-th]].

[240] *L. Ferro, T. Lukowski, C. Meneghelli, J. Plefka, and M. Staudacher,* Harmonic R-matrices for Scattering Amplitudes and Spectral Regularization, 1212.0850 [hep-th].