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CHIRAL SYMMETRY PROPERTIES OF THE WILSON LATTICE ACTION AT STRONG COUPLING*

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ABSTRACT

We study the vacuum structure of lattice group theories with Wilson fermions in the strong-coupling limit, using the Hamiltonian formalism and an alternating-site mean field ansatz for the ground state. For all values of the hopping parameter, we find a unique vacuum which is not chirally invariant. This contrasts with the vacuum degeneracy of a theory with a dynamically broken chiral symmetry. Thus the Wilson theory has no chirally invariant critical point at strong coupling. However, for small values of hopping parameter the mean field vacuum of the Wilson theory coincides with that of the chiral theory with a small quark mass term added.

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1. Introduction

This paper is concerned with the properties of lattice gauge theories with Wilson fermions. Our question is how closely the Wilson theory[1] reproduces the physics of chiral or softly broken chiral theories. Chiral here refers to any theory possessing an explicit continuous chiral invariance, such as the SLAC lattice theory,[2] or the so-called "naive" Dirac lattice theory[1]. As is well known, Wilson's action breaks chiral symmetry explicitly, for all values of the quark mass. This is the price paid for eliminating, in this theory, the unwanted replication of fermion species associated with the "naive" fermion lattice action[1]. However, Wilson's action contains two chiral symmetry breaking terms, with relative magnitude determined by Wilson's "hopping" parameter K. The belief[1] is that there exists a critical value of the hopping parameter, K_c, for which the chiral symmetry breaking effects of these two terms cancel, on distance scales very much greater that the lattice spacing, leading to a chirally symmetric continuum limit. In general the value of K_c will depend on the coupling, g. Thus one hopes for a critical path in the (K, g) plane along which the Wilson theory has a chiral invariance which is, however, broken dynamically. The quark mass is defined to vanish along this path, and one expect the dynamical breaking of chiral invariance to give rise to a massless pion. These properties can be demonstrated explicitly for the case of the free Wilson theory.[1] For the full Wilson theory, many conjectures are required, as has been discussed by Kawamoto.[3] .

We will here study the vacuum structure of the Wilson theory in the strong-coupling limit, using an alternating-site mean field ansatz for the ground state. A dynamically broken chiral invariance would be signalled by the existence of a degeneracy of distinct possible vacuum states, related by chiral transformations. The addition of a small quark mass term merely serves to pick out one of this degenerate set of vacua. For the

the Wilson theory we find the mean field vacuum to be unique for all values of the hopping parameter, indicating the absence of a chirally-invariant critical point at strong coupling. However, in the region of small hopping parameter this vacuum is the same as the mean field vacuum of the chiral theory with a small added mass term. We thus conclude that the Wilson theory probably reproduces well the physics of softly broken chiral theories, but for the case of chiral theories the suitability of the Wilson action is less apparent, since there is never a chiral invariance in the strong-coupling region.

Our analysis employs the Hamiltonian formulation of lattice gauge theories.[4] In the strong-coupling limit, the low-lying states are those with no flux excitations. Acting within this sector, the Hamiltonian is equivalent to an effective Hamiltonian which describes a generalized antiferromagnet. Such effective Hamiltonians were first studied by Svetitsky, Drell, Quinn and Weinstein[2] (hereafter referred to as SDQW), who considered the case of the SLAC lattice theory at strong coupling. The effective Hamiltonian for the Wilson theory has subsequently been derived by Smit.[5] In the work of SDQW, block spin and variational techniques were used to study the realization of chiral symmetry in the SLAC theory. The physics of their analysis was subsequently reproduced by Greensite and Primack,[6] using a simple alternating-site mean field ansatz for the ground state. We therefore employ a similar mean field ansatz, for its simplicity. Though cruder than the methods of SDQW, we do expect it to correctly describe the physics of chiral symmetry.

For simplicity we consider only the single flavour theory. Introducing more flavours does not change the general physical picture. Colour enters the effective Hamiltonian in an essentially trivial fashion, so we initially consider an abelian theory. Within the context of our mean field ansatz, we find the vacuum to be unique for all values of the hopping parameter.

We generalise this result to an arbitrary number of colours, N, by first making the simplifying assumption that the vacuum does not break the lattice rotational invariance, as was found to be the case for the $N = 1$ theory. We are thus able to compare our results with those of Smit[5] who has studied the large N limit of our effective Hamiltonian using a generalisation of the approximate spin-wave methods developed by Anderson[7] for the $SU(2)$ antiferromagnet. In contrast to our result, Smit *does* find a critical value of the hopping parameter, signalled by the vanishing of the pion mass at infinite N. In fact this result is not inconsistent with our result that the mean field vacuum is unique. This is possible due to the special nature of the large N limit. Thus the existence of Smits critical point is intrinsically a property of the large N limit and so we doubt whether it has much relevance to the theory of physical interest, $N = 3$ for QCD. For any finite N the pion mass-squared is of order $1/N$ at the critical point, and this may well not be small for $N = 3$.

The rest of this paper is organised as follows. In Section 2 we derive the effective Hamiltonian of the Wilson theory for strong coupling, using the approach of SDQW. The result has been presented previously by Smit[5], but we include a derivation here for completeness. In Section 3 we describe the construction of our trial, mean field ground state. Our analysis follows closely that of Drell, Gupta and Quinn[8] for the SLAC lattice theory generalised to arbitrary fermion content. In Section 4 we determine the ground state structure, by minimising the expectation value of the effective Hamiltonian in the trial vacuum state. In Section 5 we compare our results with those of Smit for the large N limit[5] and show that they are compatible. Section 6 presents our conclusions.

2. Effective Wilson Hamiltonian for Strong Coupling

Our starting point is the generalised Wilson Hamiltonian, H_ω for a single fermion flavour. This is defined in the $A_0 = 0$ gauge by[1,3]

$$
\begin{aligned}
aH_w(r,K,g,a) = &\sum_{j,\alpha} \psi_j^{+\alpha}\gamma_0\psi_j^\alpha \\
&- \sum_{j,\hat{k},\alpha,\beta} K(\psi_j^{+\alpha}(r\gamma_0 + i\alpha_k)U_{j,\hat{k}}^{\alpha\beta}\psi_{j+\hat{k}}^\beta + h.c.) \\
&+ \frac{g^2}{2}\sum_{j,\hat{k},\alpha}(E_{j,\hat{k}}^\alpha)^2 - \frac{1}{2g^2}\sum_p (Tr\, U_{\partial p} + h.c.)
\end{aligned}
\tag{2.1}
$$

Here a denotes the lattice spacing, g the coupling strength, and K Wilson's hopping parameter. ψ_j^α denotes a fermion field acting on the site j with colour α. $U_{j,\hat{k}}$ denotes the gauge field, and $E_{j,\hat{k}}$ the electric field both acting on the link running from the site j to the site $j+\hat{k}$. We take the gauge group to be $SU(N)$. $U_{\partial p}$ denotes the ordered product of gauge fields over the links bordering the plaquette p. α_k is the Dirac matrix $\gamma_0\gamma_k$. The fields have been scaled so that the only dimensionful quantity is the lattice space, a.

The parameter r, which ranges from zero to one, measures the strength of Wilson's chiral symmetry breaking kinetic term, relative to the chirally symmetric piece. At $r = 0$ one has the "naive" Dirac Hamiltonian, which suffers from fermion doubling, describing sixteen degenerate fermion species for each original fermion flavour.[1] The value $r = 1$ defines the Wilson Hamiltonian, in which the unwanted fermions do not propagate. At intermediate values of r, analysis of the naive continuum limit shows that the unwanted species acquire very large masses, proportional to the inverse lattice spacing, so decouple from the continuum spectrum[5]. It has thus been argued[5] that the value of r is irrelevant to continuum physics, as long as it is non-zero. We will work with general r.

H_ω possesses explicit chiral symmetry only in the limit of infinite hopping parameter, together with $r = 0$, when the massless Dirac Hamiltonian is recovered. This latter Hamiltonian is actually invariant under a very much larger symmetry group, first identified by SDQW. This group is $SU(1) \times SU(4N_f)$, for N_f fermion flavours. We note here the generators of this symmetry for the single flavour case, for later use. These are most simply constructed in terms of the redefined fermion fields

$$\begin{aligned} \tilde{\psi}_j^\alpha &= \alpha_x^{jx} \alpha_y^{jy} \alpha_z^{jz} \psi_j^\alpha \\ &= \begin{pmatrix} b_j^\alpha \\ d_j^{+\alpha} \end{pmatrix} \end{aligned} \tag{2.2}$$

where b, d^+ are two component spinors. The $U(1)$ is generated by the baryon number operator,

$$\begin{aligned} Q^B &= \sum_j Q_j^B \quad . \\ Q_j^B &= \sum_\alpha \left(b_j^{+\alpha} b_j^\alpha - d_j^{+\alpha} d_j^\alpha \right) = \sum_\alpha \left(\tilde{\psi}_j^{+\alpha} \tilde{\psi}_j^\alpha \right) - 2N \quad . \end{aligned} \tag{2.3}$$

The $SU(4)$ generators are given by

$$Q^\eta = \sum_j Q_j^\eta \ , \ \ Q_j^\eta = \sum_\alpha \tilde{\psi}_j^{+\alpha} \Gamma^\eta \tilde{\psi}_j^\alpha \tag{2.4}$$

where the Γ^η are the fifteen Dirac matrices, the unit matrix being disallowed.

We now proceed with the derivation of the effective Wilson Hamiltonian for strong coupling, following the approach of SDQW. In this limit, states containing colour electric flux excitations are highly energetic, since they have energy of order g^2. As our interest is in the vacuum structure of the theory, we limit our attention to the low-lying, flux free sector of states. The physical states are those satisfying the non-abelian equivalent of Gauss' law, which is not an equation of motion in the $A_0 = 0$ gauge. In the flux-free sector, this requires gauge invariant fermionic structure at every site. To zeroth order in $1/g^2$, these states are all degenerate. At higher orders in $1/g^2$, the Hamiltonian acts to lift this degeneracy by introducing mixing with intermediate states of non-zero flux. Second order degenerate perturbation theory in the sector of flux-free states thus leads to the effective Hamiltonian,

$$H_{eff} = H_M + H^{(2)} \quad ,$$

$$H_M = \sum_{j,\alpha} \frac{1}{a} \; \psi_j^{+,\alpha} \gamma_0 \psi_j^\alpha \quad ,$$

$$H^{(2)} = \frac{-2K^2}{ag^2 C_2(F) N} \times \sum_{j,\hat{k},\alpha,\beta} \psi_j^{+,\alpha}(r\gamma_0 - i\alpha_k)\psi_{j+\hat{k}}^\beta \times \psi_{j+\hat{k}}^\beta (r\gamma_0 + i\alpha_k)\psi_j^\alpha \quad . \tag{2.5}$$

where $C_2(F)$ is the quadratic Casimir for the fundamental representation of $SU(N)$, and we have used the relation

$$U_{\alpha\beta} U_{\gamma\delta}^+ = \frac{1}{N} \delta_{\alpha\delta}\delta_{\gamma\beta} + non - singlet \;\; piece \quad .$$

Following SDQW, we rewrite $H^{(2)}$ using a Fierz transformation, obtaining

$$H^{(2)} = \frac{K^2}{2ag^2 C_2(F) N} \times \sum_{j,\hat{k},\eta} \Bigg\{ \left(\sum_\alpha \psi_j^{+\alpha}(r\gamma_0 - i\alpha_k)\Gamma^\eta (r\gamma_0 + i\alpha_k)\psi_j^\alpha \right) \times \left(\sum_\beta \psi_{j+\hat{k}}^{+\beta} \Gamma^\eta \psi_{j+\hat{k}}^\beta \right) - 4N \sum_\alpha \psi_j^{+\alpha}(1 + r^2 + 2ir\gamma_\mu)\psi_j^\alpha \Bigg\} \tag{2.6}$$

where the Γ^η are the sixteen Dirac matrices, in a hermitian basis.

It is now straightforward to express H_{eff} entirely in terms of the generators of the $SU(4) \times U(1)$ symmetry defined by Eq. (2.2-4). To simplify the Dirac matrix structure we define the signs $s(\eta, \hat{k})$, $t(\eta, \hat{k})$ by

$$\begin{aligned} \alpha_k \Gamma^\eta \alpha_k &= s(\eta, \hat{k}) \Gamma^\eta \\ \gamma_k \Gamma^\eta \gamma_k &= t(\eta, \hat{k}) \Gamma^\eta \end{aligned} \tag{2.7}$$

in terms of which

$$(r\gamma_0 - i\alpha_k)\Gamma^\eta(r\gamma_0 + i\alpha_k) = s(\eta, \hat{k}) \times ((1 - r^2 t(\eta, \hat{k}))\Gamma^\eta + \{\Gamma^\eta, i\gamma_k\}) \ . \tag{2.8}$$

We now introduce the redefined fermion fields $\tilde{\psi}_j$ (see Eq. (2.2), which eliminates the overall sign factor $s(\eta, k)$ in Eq. (2.8). Finally the Dirac matrix algebra is used to simplify the anticommutator term. We thus find

$$\begin{aligned} H_{eff} = &\sum_j (-)^{j_x + j_y + j_z} Q_j^{\gamma_0} \\ &+ \frac{\tilde{K}}{N} \sum_{j,\hat{k}} \left(\frac{(1 + r^2)}{4} A_{j,\hat{k}} + \frac{(1 - r^2)}{4} B_{j+\hat{k}} + \frac{r}{2} C_{j,\hat{k}} \right) \ , \end{aligned} \tag{2.9}$$

where we have defined

$$\begin{aligned} A_{j,\hat{k}} &= \left(\sum_{\eta, t(\eta,\hat{k}) = -1} Q_j^\eta Q_{j+\hat{k}}^\eta \right) + Q_j^B Q_{j+\hat{k}}^B \ , \\ B_{j+\hat{k}} &= \sum_{\eta, t(\eta,\hat{k}) = +1} Q_j^\eta Q_{j+\hat{k}}^\eta \ , \\ C_{j,\hat{k}} &= (-)^{jk} \left\{ \left(Q_j^B Q_{j+\hat{k}}^{i\gamma_k} - Q_j^{i\gamma_5\gamma_0} Q_{j+\hat{k}}^{\sigma_k} - \epsilon_{ik\ell} Q_j^{\gamma_5} Q_{j+\hat{k}}^{\alpha_i} \right) + (j \leftrightarrow j + \hat{k}) \right\} . \end{aligned} \tag{2.10}$$

and

$$\tilde{K} = \frac{K^2}{2ag^2 C_2(F)} \tag{2.11}$$

In this form the $SU(4) \times U(1)$ symmetry of the $r = 0$, $K \to \infty$ limit is explicit. At $r = 0$ H_{eff} is a generalised antiferromagnet with the charges Q_j^B, Q_j^η replacing the $SU(2)$ spins of a simple antiferromagnet, and the mass term acting as a symmetry breaking magnetic field. Here it breaks the $SU(4)$ symmetry to the $SU(2) \times SU(2) \times U(1)$ subgroup generated by the charges which commute with Q^{γ_0}. The r-dependent Wilson term, however, is invariant only under the $U(1)$ subgroup of $SU(4)$ generated by $Q^{i\gamma_5\gamma_0}$. Thus the $SU(4)$ is completely broken by the full H_{eff}. The $U(1)$ of baryon number of course remains a good symmetry.

Note that the structure of H_{eff} is essentially unchanged if we generalise to N_f fermion flavours. The charges Q^η are replaced by the generators of $SU(4N_f)$, which have an additional flavour degree of freedom, but this enters H_{eff} in a trivial fashion apart from introducing a more general mass term. Thus we expect the physics to be independent of the number of fermion flavours, and so consider only the single flavour case.

3. Mean Field Analysis of H_{eff}

In this section we will study the chiral properties of our effective Hamiltonian, which takes the form of a generalised antiferromagnet. We investigate the vacuum structure using the alternating-site mean field ansatz of Greensite and Primack.[6] They have shown this simple ansatz to reproduce correctly the chiral physics revealed by the more sophisticated analysis of SDQW for similar generalised antiferromagnets.

The alternating-site mean field ansatz assumes a ground state of the form

$$|\psi> = \prod_{\substack{jx+jy+jz \\ even}} |\psi_e>_j \prod_{\substack{jx+jy+jz \\ odd}} |\psi_0>_j \tag{3.1}$$

Thus we have two intersecting sub-lattices, each with its own single-site mean field. The structure of $|\psi_e>$, $|\psi_0>$ is to be determined by minimizing $<\psi|H_{eff}|\psi>$, which takes the form

$$\frac{<H_{eff}>}{N_\ell} = \frac{<\psi|H_{eff}|\psi>}{N_\ell} = \frac{1}{6}(<Q_e^{\gamma_0}> - <Q_0^{\gamma_0}>)$$

$$+\frac{\tilde{K}}{N}\left(\left(\sum_\eta C^\eta(r) <Q_e^\eta><Q_0^\eta>\right)\right. \tag{3.2}$$

$$\left. + \frac{(1+r^2)}{4} <Q_e^B> <Q_0^B>\right) ,$$

where

$$<Q_e^\eta> =_j <\psi_e|Q_j^\eta|\psi_e>_j \ , \quad <Q_0^\eta> =_j <\psi_0|Q_j^\eta|\psi_0>_j \ ,$$

and the $C^\eta(r)$ are given in Table 1. N_ℓ is the number of links in the lattice. We see here the antiferromagnetic nature of the ground state, in the tendency for $<Q_0^\eta>$, $<Q_e^\eta>$ to take opposite signs.

It is convenient to classify the single-site states according to their $SU(4) \times U(1)$ representation content. This has been done by Drell, Gupta and Quinn[8] for theories with more general fermion content. In brief, the states of definite representation content are constructed as follows:

(a) Define the trivial "vacuum" $|0>$, which satisfies $b_s^\alpha|0> = d_s^\alpha|0> = 0$.

(b) Define the state of minimal baryon number,

$$\prod_{\alpha,s} d_s^{+\alpha} \mid 0> \quad .$$

This is an $SU(4)$ singlet, with baryon number $B = -2N$.

(c) Generate states of higher baryon number by acting on the singlet state with the baryon creation operator, $\prod_{i=1}^N b_{s_i}^{+\alpha_i}$, antisymmetrised with respect to colour and symmetrised with respect to spin indices. This operator has $B = N$, and belongs to the representation of $SU(4)$ specified by the Young diagram shown in Fig. 1. This result is obtained by noting that the fermion field belongs to the fundamental representation of $SU(4)$.

The single-site states are thus classified according to $B = (\ell - 2)N$ for $SU(4)$ content given by the Young diagram of Fig. 2, with ℓ not greater than four. Note that the representations of baryon number $B, -B$ are conjugate, with the $B = 0$ representation self-conjugate.

We now choose the representation content of the trial state. Let us assign $|\psi_0>$ to the $SU(4)$ representation of baryon number B. Then since the ground state must have net baryon number zero, $|\psi_e>$ must have baryon number $-B$. Since these representation are conjugate, we can further impose the condition

$$< Q_e^\eta > = - < Q_0^\eta > \tag{3.3}$$

which minimises $< H_{eff} >$ for a given $|\psi_0>$, realising the antiferromagnetic nature of H_{eff}. Henceforth we discard the odd and even labels. $|\psi>$ will refer to $|\psi_0>$

To proceed further, recall that our aim is to investigate the existence of a critical value of-the hopping parameter, for which the Wilson theory is chirally invariant. In the region of such a critical point, the two $SU(4)$ breaking terms are at least partially cancelled. It is therefore reasonable to treat the $SU(4)$ as an approximate symmetry of the full effective Hamiltonian. So, following Drell, Gupta and Quinn[8], we first examine the $SU(4)$ symmetric part of the mean field expectation value:

$$< H_{eff}^D > = -\frac{\tilde{K}}{4N}\left(< Q^B >^2 + \sum_\eta < Q^\eta >^2\right) \ . \tag{3.4}$$

Within a given representation, this is maximised by choosing $|\psi>$ to be one of the maximal weight states, which are defined by

$$\begin{array}{l} <Q^\lambda> = maximal, \quad for \;\; any \;\; one \;\; \lambda, \\ <Q^\eta> = 0, \quad \eta \neq \lambda \;\; . \end{array} \tag{3.5}$$

Due to the $SU(4)$ symmetry, all fifteen of these states are degenerate. Returning now to the full Wilson expectation value the two $SU(4)$ breaking terms lift this degeneracy, introducing preferred directions in $SU(4)$ space. Treating the $SU(4)$ breaking as a small perturbation, we first analyse $< H_{eff} >$ in terms of the maximal weight states. This will enable us to determine the preferred baryon number and representation content for $|\psi>$. We will subsequently allow $|\psi>$ to take up more general group orientations, within the preferred representation. We thus have to minimise

$$\begin{aligned} \frac{<H_{eff}>}{N_L} = & -\frac{<Q^{\gamma_0}>}{3} \\ & -\frac{\tilde{K}}{N}\left(\frac{1+r^2}{4} <Q^B>^2 + \sum_\eta C^\eta(r) <Q^\eta>^2\right) \;, \end{aligned} \tag{3.6}$$

with the $C^\eta(r)$ given in Table 1. We see immediately a competition between the two $SU(4)$ breaking terms, the mass term wanting to point the maximal weight in the γ_0 direction, and Wilson's r term favouring the $i\gamma_5\gamma_0$ direction, since this has maximum $C^\eta(r)$. These states are related by chiral transformations. Which state has the lower energy depends on the value of the hopping parameter. The degeneracy of such states is the signal for chiral symmetry restoration to the Wilson theory.

To find the preferred representation for $|\psi>$ we explicitly evaluate $< H_{eff} >$ for $|\psi>$ oriented in the γ_0 and $i\gamma_5\gamma_0$ directions, and for each representation we find that in the γ_0 direction the $B = 0$ representation would be favoured. However, in the $i\gamma_5\gamma_0$ direction there is a degeneracy amongst representations. A similar degeneracy was found by Drell, Gupta and Quinn in their analysis of the SLAC lattice Theory.[8] They note that such a degeneracy is largely an artefact of the mean field ansatz for the ground state. In a more general ground state, one expects the $SU(4)$ symmetric part of $< H_{eff} >$ to depend on $< \sum_\eta (Q^\eta)^2 >$, rather than $\sum_\eta < Q^\eta >^2$, which will tend to favour the $B = 0$ representation. This has been verified by SDQW for the SLAC lattice theory, using more accurate block-spin techniques.[2] We will thus treat this degeneracy as spurious, and henceforth take $|\psi >$ to lie in the $B = 0$, meson representation, with the Young diagram as in Fig. 2, ℓ taking the value 2.

It remains to find the preferred orientation of $|\psi >$ within this representation, allowing for states more general than the maximal weight states. This will be the subject of the next section.

4. Minimisation of $< H_{eff} >$

Since colour does not appear explicitly in H_{eff}, but merely serves to determine the $SU(4)$ representation content of the single-site states, we expect the physics of H_{eff} to be independent of the number of colours N. We therefore first consider the simplest case, the $N = 1$, Abelian theory, and allow the mean field state to take up arbitrary orientations within the $B = 0$ representation. We subsequently generalise this result to arbitrary N, introducing the additional assumption, justified by the $N = 1$ case, that only Lorentz invariant group orientations are allowed.

(a) Abelian Theory For $N = 1$ the zero baryon number, mean field state lies in the $\underline{6}$ of $SU(4)$. In terms of creation operators, these states are given by

$$|1> = b_\uparrow^+ d_\uparrow^+ |0>$$
$$|2> = \frac{1}{\sqrt{2}}\left(1 + b_\uparrow^+ b_\downarrow^+ d_\uparrow^+ d_\downarrow^+\right)|0> \tag{4.1a}$$

$$
\begin{aligned}
|3> &= b_\downarrow^+ d_\downarrow^+ |0> \\
|4> &= b_\uparrow^+ d_\downarrow^+ |0> \\
|5> &= \frac{1}{\sqrt{2}}\left(1 - b_\uparrow^+ b_\downarrow^+ d_\uparrow^+ d_\downarrow^+\right)|0> \\
|6> &= b_\downarrow^+ d_\uparrow^+ |0>
\end{aligned}
\tag{4.1b}
$$

where the $\uparrow$, $\downarrow$ labels refer to spin components. We find it convenient to work in a basis with γ_0 diagonal.

As discussed by Greensite and Primack[6], these states group into two $SU(2)$ triplets, one $SU(2)$ generated by chiral transformations, and the other by spin. Here the first three states form the chiral triplet, and the latter three the spin triplet. The chiral states are spin singlets, and vice versa for the spin states. All other $SU(4)$ operators act between the two triplets.

In the γ_0 diagonal basis the state $|1>$ is the "spin" up state of the chiral triplet, corresponding to maximal $< Q^{\gamma_0} >$. Thus, recalling the discussion of the previous section, we expect this to be the preferred direction of the mean field in the limit of small hopping parameter, when the mass term is dominant. For large hopping parameter, we expect the mean field to maximise $< Q^{i\gamma_5\gamma_0} >$. Whether in the intermediate region there is a degeneracy amongst the chiral triplet states, which would be indicative of broken chiral symmetry, is the question we now investigate.

We thus define the trial, single-site mean field state

$$
|\psi> = \sum_{i=1}^{6} a_i |i> \quad , \quad \sum_i |a_i|^2 = 1 \ . \tag{4.2}
$$

The a_i's are complex, variational parameters, constrained only by the normalisation condition. We use the overall arbitrariness of phase to fix a_1, the coefficient of the chiral "up" state to be real. Now the only part of $< H_{eff} >$ which is sensitive to the sign of $< Q^{\gamma_0} >$ is the mass term, which is diagonal and so phase independent. We therefore fix the coefficient of the chiral "down" state, a_3, to be real also. We then minimise $< H_{eff} >$ with respect to the remaining variational parameters, subject to the normalisation constraint.

For $r \neq 0$, $\tilde{K}$ finite, we find two distinct regions of solution, the changeover occurring at the value of hopping parameter $\tilde{K}_0 = 1/6r^2$. The solutions are

$\underline{\tilde{K} \leq \tilde{K}_0}$

$$|\psi> = |1> \ ,$$
$$< Q^{\gamma_0} > = 2 \ ,$$
$$< H_{eff} > = -\frac{2}{3} - \tilde{K}(1 - r^2) \ ,$$

$\underline{\tilde{K} \geq \tilde{K}_0}$

$$|\psi> = \frac{1}{2}\left\{\left(1 + \frac{\tilde{K}_0}{\tilde{K}}\right)|1> + \left(1 - \frac{\tilde{K}_0}{\tilde{K}}\right)|3> + i\sqrt{\left(1 - \left(\frac{\tilde{K}_0}{\tilde{K}}\right)^2\right)}|2>\right\}$$

$$< Q^{\gamma_0} > = 2\frac{\tilde{K}_0}{\tilde{K}} \ ,$$

$$< Q^{i\gamma_5\gamma_0} > = 2\sqrt{\left(1 - \left(\frac{\tilde{K}_0}{\tilde{K}}\right)^2\right)} \ ,$$

$$< H >_{eff} = -\frac{1}{3}\frac{\tilde{K}_0}{\tilde{K}} - \tilde{K}(1 + r^2) \ . \tag{4.3}$$

With all other $< Q^{\eta} > = 0$.

Thus, as expected, the mass term determines the ground state structure in the small $\tilde{K}$ region, and the vacuum here is the same as that of the chiral theory with a small mass term added. Above the value $\tilde{K}_0$, the Wilson kinetic term has some effect. For all values of $\tilde{K}$ the vacuum breaks chiral invariance but not Lorentz invariance. Note that the vacuum is unique for every value of the hopping parameter. There is no degeneracy amongst the chiral triplet states, indicating that the underlying theory has no chiral symmetry, for any value of the hopping parameter. The only exception is in the chirally symmetric limit $r = 0$, $\tilde{K} \to \infty$, when the full $SU(4)$ symmetry is restored, together with fermion doubling. Then the maximal weight states (Eq. (3.5)) form a degenerate set of possible vacuum states.

(b) Non-Abelian Theory We now generalise this result to an arbitrary number of colours, N. The single site mean field state now belongs to the representation specified by the Young diagram shown in Fig. 2, with ℓ taking the value 2. We make the simplifying assumption that the ground state does not break the invariance of the lattice under 90^o rotations about the x, y or z axis as was the case for $N = 1$. Thus $< Q^\eta >= 0$ for all but the chiral generators Q^{γ_0}, Q^{γ_5} and $Q^{i\gamma_5\gamma_0}$. In terms of the states, this implies a restriction to the $(2N+1)$-plet of state which forms a representation of the $SU(2)$ generated by the chiral operators.

Within this multiplet we parametrise the non-vanishing expectation values of the trial, single-site state according to

$$\begin{aligned} < Q^{\gamma_0} > &= 2Nk \ \ cos\phi \\ < Q^{\gamma_5} > &= 2Nk \ \ sin\phi \ cos\theta \qquad 0 < k \leq 1 \\ < Q^{i\gamma_5\gamma_0} > &= 2Nk \ \ sin\phi \ sin\theta \ . \end{aligned} \tag{4.4}$$

Substitution into $< H_{eff} >$ (Eq. (3.21)) gives

$$< H_{eff} >= -\frac{2}{3} Nk \ cos\phi - \tilde{K} \ Nk^2 \, (1 - r^2(cos^2\phi + sin^2\phi \ cos2\theta)) \tag{4.5}$$

which is minimised for $k = 1$, $\cos 2\phi = -1$, independently of $\tilde{K}$. Minimising with respect to ϕ we find again that the mean field begins to turn away from its large $\tilde{K}$ value at the point $\tilde{K}_0 = 1/6r^2$. The minimum is given by

$$
\begin{aligned}
cos\phi &= 1 \ , \quad \tilde{K} \leq \tilde{K}_0 \\
cos\phi &= \frac{\tilde{K}_0}{\tilde{K}} \ , \quad \tilde{K} \geq \tilde{K}_0
\end{aligned}
\tag{4.6}
$$

which leads to expectation vlues exactly as for the Abelian theory, up to an overall normalisation factor. Thus in the non-Abelian case also, there is no degeneracy amongst the chiral states, indicating that the Wilson theory is not chirally invariant at strong-coupling, for any value of hopping parameter.

5. Discussion of the Large-N Limit

We have found, assuming an alternating-site mean field vacuum structure, that the strongly coupled Wilson theory does not show the vacuum degeneracy associated with a broken chiral invariance for any value of the hopping parameter. This result is independent of the number of colours, N. However, the large N limit of this theory has been previously analysed by Smit,[5] using a generalisation of the approximate spin-wave techniques developed by Anderson for the $SU(2)$ antiferromagnet.[7] These methods allow a computation of the pion mass as a function of hopping parameter. In contrast to our result, Smit finds a critical value of the hopping parameter at which the pion mass vanishes, to leading order in N, indicating a broken chiral invariance at infinite N. The results of our mean field calculation strongly suggest the existence of this critical point to be a very special property of the large N limit. A brief consideration of Smits approach clarifies this point, and also shows why his result is not inconsistent with ours.

Smit assumes an alternating-site mean field ground state with the mean field pointing in the γ_0 direction, as we found that it did for small enough $\tilde{K}$ (see Eq. (4.6)). Thus

$$
\begin{aligned}
&< Q^{\gamma_0} = 2N \\
&< Q^{\alpha} > = 0 \quad , \quad \alpha \neq \gamma_0
\end{aligned}
\tag{5.1}
$$

The crux of Smits analysis is the assumption that the low-lying excitations of the mean field ground state involve only those states for which the Q^{γ_0} eigenvalue remains of order $2N$. Acting amongst these states the $SU(4)$ generators satisfy

$$
\begin{aligned}
&Q_j^{\gamma_0} \sim 2N - 0(1) \\
&Q_j^{\alpha} \sim 0\left(\sqrt{N}\right) \quad \textit{for} \quad [Q^{\alpha}, Q^{\gamma_0}] \neq 0 \\
&Q_j^{\alpha} \sim 0(1) \quad \textit{otherwise},
\end{aligned}
\tag{5.2}
$$

where we have introduced site labels since these states are not in general mean field states. Thus at infinite N the $SU(4)$ orientation of these states is indistinguishable from that of the mean field ground state, and any degeneracy amongst these states would not be apparent in our mean field calculation. With the above assumptions the effective Hamiltonian reduces to a sum of independent harmonic oscillators, and the pion identified with the oscillation generated by the axial charge. Its mass, m_π, satisfies

$$
m_\pi^2 \sim 0(1) \times (\tilde{K}_c - \tilde{K}) + 0(\frac{1}{N}) \qquad \tilde{K}_c = \frac{1}{6r^2} \tag{5.3}
$$

Thus there is a critical point at infinite N. Note that the critical value of hopping parameter, $\tilde{K}_c$, coincides with the point $\tilde{K}_0$, found in our mean field calculation, above which the mean field vacuum is turned away from the γ_0 direction (see Eq. (4.6)). Thus the tachyonic nature of Smit's pion for values of $\tilde{K}$ greater than the critical value is consistent with the instability of Smit's vacuum in that region.

We conclude that the existence of Smit's large-N critical point is not inconsistent with a unique mean field vacuum. However for any finite N we find that the pion mass is always finite for any value of $\tilde{K}$. The fact that the correct mean field vacuum is not Smit's assumed vacuum for $\tilde{K} > \tilde{K}_c$ means that it is not possible simply to choose $\tilde{K} > \tilde{K}_c$ in Smit's formula for the mass to achieve zero mass at finite N.

6. Conclusion

We have studied the Wilson lattice gauge theory in the limit of strong-coupling for a single fermion flavour and both Abelian and non-Abelian gauge groups. Using an alternating-site mean-field ansatz for the ground state, we have found the vacuum to be unique and exhibit no broken chiral invariance for all value of the Wilson hopping parameter. Thus the physics of the Wilson theory at strong-coupling is never that of a theory with an underlying chiral invariance. Whether there exists a chiral continuum limit is, of course, an open question. Thus there is some doubt as to the suitability of Wilson's lattice action for the study of chiral theories such as massless QCD. However, for the study of softly broken chiral theories, with a small fermion mass term, our results indicate that the Wilson theory with small hopping parameter is probably adequate, since the two theories have the same mean field vacuum at strong coupling.

It should be mentioned here that the pion mass has been computed for the continuum limit of the Wilson theory by Shigemitsu,[9] using strong-coupling perturbation theory followed by Padé extrapolation to the zero coupling continuum point. She finds that it is indeed possible to set the continuum pion mass to zero by a suitable choice of hopping parameter. However, this choice is not obviously related to any restoration of chiral symmetry to the underlying theory.

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Table 1

η	$C^{\eta}(r)$
γ_0, γ_5	$\frac{1-r^2}{4}$
$i\gamma_5\gamma_0$	$\frac{1+r^2}{4}$
$\sigma^{\ell}, i\gamma^{\ell}$	$\frac{3-r^2}{12}$
$\alpha^{\ell}, \gamma^5\gamma^{\ell}$	$\frac{3+r^2}{12}$

REFERENCES

1. K. Wilson, Phys. Rev. D10, 2445 (1974) and in New Phenomena in Subnuclear Physics, ed. A. Zichichi (Plenum Press, N.Y. 1977).
2. B. Svetitsky, S. D. Drell, H. R. Quinn and M. Weinstein, Phys. Rev. D22, 490 (1980).
3. N. Kawamoto, Nucl. Phys. (FS3) B190, 617 (1981)
4. J. Kogut and L. Susskind, Phys. Rev. 11, 395 (1975).
5. J. Smit, Nucl. Phys. B175, 307 (1980).
6. J. Greensite and J. Primack, Nucl. Phys. B180, 170 (1981).
7. P. W. Anderson, Phys. Rev. 130, 76 (1963).
8. H. R. Quinn, S. D. Drell and S. Gupta, SLAC-PUB-2920, May 1982.
9. J. Shigemitsu, Phys. Rev. D18, 1709 (1978).

FIGURE CAPTIONS

1. Young diagram for the $SU(4)$ representation content of the baryon creation operator.

2. Young diagram for the $SU(4)$ representation of the single-site state of baryon number $B = (\ell - 2)N$.

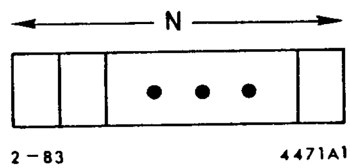

Fig. 1

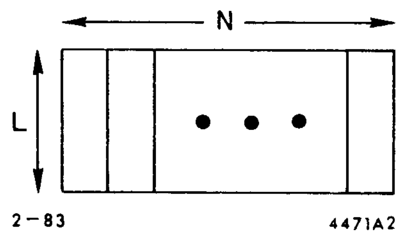

Fig. 2