Categories, sheaves and other algebraic tools

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1 Introduction

Students of physics at our University are very well aware of the amount of mathematical skills needed in today's theoretical physics. In some areas, they are even better educated than our average students of mathematics, say in group representations or in vector bundles. However, the enormous growth of connections to algebraic geometry, which occurred during the last twenty years, seems to be underestimated on the part of physicists. The aim of my lectures at the Summer School in Modern Mathematical Physics or $M \cap \Phi$ for short, held in Sokobanja, in August 2001[1], was to stress this point of view, through presentation of some classical, as well as contemporary tools and results in algebraic geometry. The present text[2] describes the three main parts of my lectures.

[1]The Summer School took place in the beautiful Serbian spa resort Sokobanja, and I am indebted to Prof. Branko Dragovich (Institute of Physics, Belgrade, and Steklov Mathematical Institute, Moscow), for his invitation

[2]The text was written under different circumstances, in the beautiful surroundings of the University Guest House in the old picturesque town of Pécs, Hungary, for which opportunity I am indebted to Prof. Judit Nagy (University of Pécs, and Hungarian Society of Nephrology)

2 Viète, Maxwell, Arnol'd, and $M \cap \Phi$

2.1 Viète's theorem and symmetric powers

Let's start with the simplest and perhaps the oldest algebraic formula - the high-school formula

$$x_{1,2} = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

for the roots of quadratic equation $x^2 + px + q = 0$. Since the equation is uniquely determined by the pair of its coefficients $(p, q) \in \mathbb{C}^2$, we can think of equations as points in $\mathbb{C}^2$. The formula then gives rise to mapping $\mathbb{C}^2 \longrightarrow \mathbb{C}^2$, $(p, q) \mapsto (x_1, x_2)$. Does this mapping have an inverse? The high-school Viète's formulae

$$\begin{aligned} x_1 + x_2 &= -p \\ x_1 x_2 &= q \end{aligned}$$

give the answer to this question: yes, it does, but unfortunately, it is 2-to-1 almost everywhere, since $(x_1, x_2) \mapsto (p, q)$ and $(x_2, x_1) \mapsto (p, q)$. In other words, if we consider the action of the symmetric group S_2 on $\mathbb{C}^2$ by interchanging the coordinates, then we obtain a bijection $\mathbb{C}^2/S_2 \cong \mathbb{C}^2$.

This reasoning may be generalised in two ways. The first is to projectivise. Put simply, it means to take the infinite point ∞ into account. So, x becomes $x = \frac{u}{v}$ or $(u : v)$ and the equation transforms into a homogeneous quadratic form $au^2 + buv + cv^2$. The mapping is thus $((u_1 : v_1), (u_2 : v_2)) \mapsto (a : b : c)$. In this way we obtain a projective version of the classical Viete's theorem: $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}/S_2 \cong \mathbb{P}^2_{\mathbb{C}}$.

Another way to generalise is to increase the dimension. This gives rise to generalised Viète's formulae associated with the equality

$$x^n + a_1 x^{n-1} + \ldots + a_n = (x - x_1)(x - x_2) \ldots (x - x_n)$$

and to bijection $\mathbb{C}^n/S_n \cong \mathbb{C}^n$. Finally, the combination of the two processes gives us $\mathbb{P}^1_{\mathbb{C}} \times \ldots \times \mathbb{P}^1_{\mathbb{C}}/S_n \cong \mathbb{P}^n_{\mathbb{C}}$. The object on the left-hand-side is known as the n-th symmetric power $Sym^n(X)$ of the manifold $X = \mathbb{P}^1_{\mathbb{C}}$. So, the general projective Viète's theorem can be expressed in the following form

$$Sym^n\left(\mathbb{P}^1_{\mathbb{C}}\right) \cong \mathbb{P}^n_{\mathbb{C}}.$$

2.2 Maxwell's theorem and spherical harmonics

What happens over the reals? Topologically, $\mathbb{P}^1_{\mathbb{C}}$ is a sphere. If we use real numbers instead of complex ones, would it be possible to describe the symmetric powers of the real projective plane $\mathbb{P}^2_{\mathbb{R}}$, which is a non-orientable surface? In the brilliant paper of V.I. Arnol'd [1], a connection is established between symmetric powers of $\mathbb{P}^2_{\mathbb{R}}$ on one hand, and the famous Maxwell's theorem on multipole representation of spherical functions on the other. This connection brings to attention an interesting and strange algebraic fact.

It is elementary to verify that

$$x^2 + 2y^2 + 3z^2 = (z - x)(z + x) + 2\left(x^2 + y^2 + z^2\right)$$

or, more generally,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = (\alpha z - \gamma x)(\alpha z + \gamma x) + \frac{1}{b^2}\left(x^2 + y^2 + z^2\right)$$

where the coefficients are $\alpha = \frac{\sqrt{b^2-c^2}}{bc}$ and $\gamma = \frac{\sqrt{a^2-b^2}}{ab}$, and $a \geq b \geq c$.

Slightly more difficult will it be to show that, if $a = \left(1+\sqrt{2}\right)^{\frac{1}{3}} - \left(1+\sqrt{2}\right)^{-\frac{1}{3}}$ is the real root of the cubic equation $x^3 + 3x - 2 = 0$, then the following formula holds:

$$\begin{aligned} x^3 + y^3 + z^3 \quad = \quad & \frac{1}{a^2+1}(ax - y - z)(-x + ay - z)(-x - y + az) + \\ & + \frac{a^2 - a + 1}{a^2+1}(x + y + z)\left(x^2 + y^2 + z^2\right) \end{aligned}$$

The formula was found with a little help of my computer and Mathematica program package.

The general algebraic fact, of which the above are examples, was discovered by Arnol'd: for any real homogeneous polynomial $f(x, y, z)$ of degree n in three variables, there exists a unique representation in the form

$$f(x, y, z) = g(x, y, z) + \left(x^2 + y^2 + z^2\right) \cdot h(x, y, z)$$

where $g(x, y, z) = l_1 \cdot l_2 \cdot \ldots \cdot l_n$ is the product of n real linear forms and $h(x, y, z)$ is real homogeneous of degree $n - 2$. The proof involves some beautiful classical methods of algebraic geometry: projection in the space of all real algebraic curves, and dimensional reasoning.

What does this have in common with symmetric powers and spherical harmonics? Now, Maxwell's theorem appears to be the physical counterpart of that algebraic fact: every spherical function of degree n on the unit sphere in $\mathbb{R}^3$ (i.e. the restriction of a homogeneous harmonic polynomial of degree n) can be uniquely represented in the form $g(X, Y, Z)\left(\frac{1}{r}\right)$ where $X = \frac{\partial}{\partial x}$, $Y = \frac{\partial}{\partial y}$, $Z = \frac{\partial}{\partial z}$ and $g(X, Y, Z)$ is the product of n real linear forms $g = l_1 \cdot \ldots \cdot l_n$ (the so-called multipole representation). On the other hand, this was used by Arnol'd to show that

$$Sym^n\left(\mathbb{P}^2_{\mathbb{R}}\right) \cong \mathbb{P}^{2n}_{\mathbb{R}}$$

which is even a diffeomorphism. What a nice example of $M \cap \Phi$!

3 General abstract nonsense

Category theory has been developing since the middle of the XXth century. In this section of my talk, a brief exposition of basic notions such as categories, functors, natural transformations, universal arrows, limits and adjoints, was presented following the excellent textbook of MacLane

[2]. It included many examples, such as interpreting abelian groups as colimits of their finitely generated subgroups, or interpreting p-adic numbers as limits of groups $\mathbb{Z}/p^k$. Special attention was payed to adjoints and to connection between the existence of adjoints and the corresponding behavior of functors. Let's recall a definition of an adjoint pair. If C and D are categories and $\mathcal{F} : C \longrightarrow D$ and $\mathcal{G} : D \longrightarrow C$ is a pair of functors, then we say that $\mathcal{F}$ is a left adjoint of $\mathcal{G}$ (and $\mathcal{G}$ is a right adjoint of $\mathcal{F}$) if for all objects $c \in C$ and $d \in D$ there is a bijection of morphisms $Hom_C(c, \mathcal{G}d) \cong Hom_D(\mathcal{F}c, d)$, natural in both c and d. The adjoint pairs occur everywhere in mathematics: free constructions in algebra and forgetful functors, limits or colimits and diagonal functors. A functor which possesses a left (or right) adjoint must preserve limits (or colimits), i.e. $\longleftarrow \lim \mathcal{G}(d_i) = \mathcal{G}(\longleftarrow \lim d_i)$(or $\longrightarrow \lim \mathcal{F}(c_i) = \mathcal{F}(\longrightarrow \lim c_i)$). This can be very helpful in the interpretation of behaviour of many constructions in mathematics and physics. Say, categorial product of groups is a group on the cartesian product of their underlying sets. More examples of this philosophy were shown, such as the comment on different behaviour of the direct and the inverse image of a function [3].

4 Why sheaves?

In the last part of my lectures, the notion of sheaf was introduced and explained with some basic examples and properties. The following points are concerned as the advantages of such approach:

- sheaves enable us to introduce different geometries (differentiable, complex or algebraic) in a uniform way;

- sheaves make it possible to treat singularities in a convenient way;

- sheaves form an abelian category which gives rise to extensive use of algebraic categorical tools (exact sequences, cohomology, etc.).

Then a short description of (Čech) cohomology theory with sheaf coefficients was given. As a good motivation for the sheaf theory for students of physics, the interpretation of classes of line bundles as elements of the first cohomology $H^1(X, \mathcal{O}^*)$ of the multiplicative sheaf $\mathcal{O}^*$ of functions on the manifold X was explained.

The expository paper [4], which was written as an introduction and a digest for students, covers some of the topics mentioned, and contains references to many excellent textbooks which do exist in the area of algebraic geometry.

5 References

[1] V.I. Arnol'd: Topological content of the Maxwell theorem on multipole representation of spherical functions. Topological Methods in Nonlinear Analysis, Journal of the Juliusz Schauder Center, vol. 7, 1996, 205-217

[2] S. MacLane: Categories for the Working Mathematician. GTM 5, Springer-Verlag, New York-Heidelberg-Berlin, 1971

[3] A. Lipkovski: The Direct and the Inverse Image - a Categorical Viewpoint. Nieuw Archief voor Wiskunde, no. 1-2, 1998, 23-26

[4] A. Lipkovski: Algebraic Geometry (selected topics). Zbornik Radova (Matematički Institut SANU), no. 7(15), 1997, 5-69