

Study of gravitational deflection of light ray

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Abstract. Gravitational deflection of light ray is one of the famous predictions of Einstein's general theory of relativity. The deflection of light ray, as it passes around a gravitational mass, can be calculated by different methods such as null geodesics method and material medium approach. In this paper a comparative study will be done for gravitational deflection of light ray, calculated by different authors using different methods. In this study, different gravitating body such as static, rotating and charged body will be considered which are represented by Schwarzschild metric, Kerr metric, Reissner-Nordström metric and Janis-Newman-Winicour metric.

1. Introduction

Light around a massive object, such as sun, is curved. This is due to the gravity of that massive body and this phenomenon is known as gravitational bending. The deflection of light ray as it passes around a gravitational mass can be calculated by different methods. Such calculations are generally done by using the null geodesics method under both strong field and weak field approximations. In null geodesics method, based on any of the forms of the line element, either by using the perturbation or by integrating the null geodesic equations the deflection of light ray is calculated. However, there is another method called "Material Medium Approach", through which also one can obtain the gravitational deflection of light ray. In this method the gravitational effect is represented by an equivalent refractive index of the medium. Thus, if a ray of light passes through a material medium, the light ray will deviate due to the variation of the refractive index of the corresponding media. In this paper we will give a comparative study for gravitational deflection of light ray, calculated by different authors using different methods. Einstein first calculated the bending of light ray due to sun as 1.75 arc-sec. After that in 1919 an observation was taken by Arthur Eddington and his collaborators during a total solar eclipse. The deflection angle obtained in Sobral as 1.98 ± 0.12 arc-sec and in Principe as 1.61 ± 0.30 arc-sec. The result was not totally satisfactory but it was accepted that the theory is correct. In this paper we will show that deflection angle of light due to sun can be obtained exactly as Einstein by using material medium approach.

2. Material Medium Approach

In material medium approach, the curved space-time is related to a special optical medium with graded refractive index. Since only vacuum exists between the gravitational matters, it may be considered that vacuum is a special optical medium whose refractive index is influenced by the gravitational field. Thus,



the gravitational effect is represented by an equivalent refractive index of the medium. Therefore, if a ray of light passes through a material medium, the light ray will deviate due to the variation of the refractive index of the corresponding media.

3. Deflection Using Material Medium Approach

The exact solution of Einstein field equation for a static, spherical, uncharged body was found by Schwarzschild in 1916 [1]. Within a year of Schwarzschild solution, Reissner and Nordström independently, obtained the solution for Einstein field equation for a static, charged, spherically symmetric body, known as Reissner-Nordström solution [2-3]. After many years in 1963, a solution for an uncharged rotating body was found by Kerr [4], named as Kerr metric and for a rotating charged body was found by Newman and Janis in 1965 which is known as Kerr-Newman metric [5-6]. Janis, Newman and Winicour in 1968 gave the solution of Einstein field equation which is coupled to a massless scalar field, known as Janis-Newman-Winicour (JNW) space-time [7]. This solution represents the most general spherically symmetric, static and asymptotically flat solution of Einstein field equation. Deflection of light ray have been calculated by many researchers using different approaches and different space-time. Einstein himself calculated the bending angle of light up to first order term as $\alpha = \frac{4GM}{c^2 r}$, assuming null geodesic path of light in space-time manifold.

Different authors have calculated the deflection angle under weak and strong field approximations using null geodesic method considering Schwarzschild space-time [8-12], Kerr space-time [13-19], Reissner-Nordström (RN) space-time [20-25] and Janis-Newman-Winicour (JNW) space-time [23, 25-27]. Material medium approach has been started by Tamm in 1924 [28]. After that Balazs in 1958 [29], Plebanski in 1960 [30], Felice in 1971 [31] used this concept and discussed the optical phenomena for the deflection of electromagnetic wave by gravitational field. In 1980 Fischbach and Freeman [32] derived the effective refractive index of the material medium in Schwarzschild field and calculated the deflection angle up to second order. Evans and his co-workers [33-35] studied this method and used the effective refractive index to calculate the gravitational time delay and trajectories of light rays in Schwarzschild geometry. In 1998, Alsing [36] followed the opto-mechanical analogy of general relativity and extended the Newtonian formalism of Evans et al. [34] to the case of stationary metrics, especial of rotating space-times. He calculated the index of refraction for Kerr field geometry and hence obtained an expression for deflection angle up to 1st order. But his expression of refractive index does not depend on the rotation parameter, depends only on the distance of closest approach and mass of the gravitating body. Sereno [37,38] also used the Fermat's principle to discuss the gravitational lensing and Faraday rotation in the weak field limit. In 2008, Ye and Lin [39] discussed the strong similarities between gravitational lensing and optical lensing using the graded refractive index approach.

Very recently, in 2010 Sen [40] also used the material medium approach to calculate the light deflection angle for a static non rotating mass in Schwarzschild geometry without using any approximation. The author expressed the line element in an isotropic form, obtained the refractive index and hence the light deflection angle in terms of elliptic integral to determine the trajectory of the light ray. Following the same, Roy and Sen in 2015 [41], obtained the refractive index in terms of rotation parameter and an analytical expression of deflection angle for a rotating mass in Kerr geometry. They have evaluated the numerical value of the deflection angle of light ray due to sun and for some milli seconds pulsars, to show the effect of rotation. In this paper, it is totally clear that the refractive indices are different for different gravitating body and due to pro-grade and retro-grade orbits of the light ray. Refractive index is greater for pro-grade direction and smaller for retro-grade direction as compared to Schwarzschild one. Thus, as compared to the Schwarzschild case, the deflection angle for light ray should be also greater for pro-grade and smaller for retro-grade orbits of light ray. The deflection angle due to sun is 1.7520 arc-sec for prograde whereas 1.7519 arc-sec for retrograde position of the light ray. The results are exactly matching with the Einstein's theoretical value of the deflection of light ray. In 2017 [42], the authors applied this approach for RN and JNW space-time geometry. In this paper also, they have evaluated the refractive index and hence the deflection angle of light ray due to both the space-

time geometry. They have concluded one interesting result that the bending angle decreases with the increase of scalar charge in RN space-time and increases with charge in JNW space-time.

4. Comparison with Others work

4.1. Schwarzschild metric

Fischbach and Freeman [32] defined the index of refraction for static field as an infinite convergent series as

$$n(r) = 1 + A/r + B/r^2 + \dots$$

with $A = r_g$ and $B = f(r_g)$, where r_g is the Schwarzschild radius.

Nandi and Islam [35] defined the index of refraction for Schwarzschild metric as

$$n(r) = \left(1 + \frac{GM}{2rc^2}\right)^3 \left(1 - \frac{GM}{2rc^2}\right)^{-1} \cong \left(1 + \frac{GM}{2rc^2}\right)^4 \cong 1 + \left(\frac{r_g}{r}\right) + \dots$$

Ye and Lin [39] expressed in exponent form as

$$n(r) = \exp\left(\frac{2GM}{rc^2}\right) = \exp\left(\frac{r_g}{r}\right) = 1 + (r_g/r) + (r_g/r)^2 + (r_g/r)^3 + \dots$$

Refractive index expression represented by Sen [40] can also be represented by the infinite convergent series as

$$n(r) = \frac{r}{r - r_g} = 1 + (r_g/r) + (r_g/r)^2 + (r_g/r)^3 + \dots$$

This shows that the refractive index expressions derived by different authors for Schwarzschild metric are the same in the weak field limit.

Ye and Lin [39] and Sen [40] both expressed the deflection angle in terms of refractive index as

$$\Delta\psi = 2 \int_{r_\odot}^{\infty} \frac{dr}{r \sqrt{\left(\frac{n(r) \cdot r}{n(r_\odot) \cdot r_\odot}\right)^2 - 1}} - \pi$$

Ye and Lin [39] had used a value of the refractive index $n(r)$ that was approximated and considered terms only up to second order. However, Sen [40] didn't make any such approximation to use refractive index in the expression of deflection angle. Consequently, these authors [39] did not derive higher-order terms for gravitational deflections. But with the exact value of refractive index, Sen [40] was able to derive the expression for deflection for a strong field which coincides with the standard expression under the weak field limit. Sen [40] derived the expression for deflection in terms of the elliptic integral of the first kind and an incomplete elliptic integral of the third kind. Iyer and Petters [12] have calculated the deflection term in the strong field with an expression containing complete and incomplete elliptical integrals by using null geodesic method. From the general expression, the authors [12,40] could calculate the first-order strong deflection term, calculated earlier by Darwin [43]. Thus, it may be concluded that the strong field deflection term calculated by Iyer and Petters [12] and Sen [40], using different methods, both contain same linear combination of elliptical integrals of the same kind.

4.2. Kerr metric

Alsing [36] calculated the index of refraction for Kerr field geometry as

$$n(r) = 1 + 2m/\rho$$

where $2m = r_g$ and ρ is the impact parameter.

Roy and Sen [41] evaluated the index of refraction in terms of rotation parameter (α) and frame dragging ($\frac{d\phi}{cdt}$) as

$$n(x) = \frac{x}{x-1} \left(1 + 2 \frac{\alpha}{c} \frac{1}{(x-1)} \frac{d\phi}{dt} \right)^{-1/2}$$

and hence with the value of frame dragging, refractive index in equatorial plane as

$$n\left(x, \frac{\pi}{2}\right) = \frac{x}{x-1} \left(1 + 2u \frac{xv + u - v}{(x-1)(x^3 - uv)} \right)^{-1/2}$$

where $x = \frac{r}{r_g}$, $u = \frac{\alpha}{r_g}$, $v = \frac{b}{r_g}$ and b is the impact parameter.

Thus, rotation parameter (α) is included in the expression of refractive index of Roy and Sen [41], by which only one can separate the Kerr geometry from the Schwarzschild geometry.

Alsing [36] and Werner [44] obtained the deflection angle only up to 1st order as

$$\delta = -\frac{4GM}{c^2 R} \left(1 + \frac{a}{R} \right)$$

a is the rotation parameter.

But Roy and Sen [41] obtained the higher order terms in the expression of deflection angle for Kerr geometry which are in good agreement with those other similar calculations done using null geodesic method [18]. Roy and Sen [41] also evaluated numerical values due to sun and for some millisecond pulsars without any approximation. The first experimental confirmation of the prediction of Einstein was given by Eddington in 1919 during total solar eclipse as 1.98 arc-sec (in Sobral) and 1.61 arc-sec (in Principe). After that also, many times similar experiments were done for the confirmation of deflection of light ray. But in every experiment, the bending of star light showed a slight but significant error with the predicted value. The scatter is the indication of the difficulty in the measurement process. The measurements were taken up to 2nd digit after decimal. Now a days, the technology is developed and it is possible to take the measurement up to 6th digit after decimal. Roy and Sen [41] also calculated the deflection of light ray due to Sun is 1.7520 arc-sec and 1.7519 arc-sec considering pro- grade and retro-grade orbit of light ray respectively. These findings of Roy and Sen [41], may be taken for consideration to compare with the observed value of the deflection of the light ray due to Sun.

4.3. RN and JNW metric

Sereno [38] studied the gravitational lensing by RN black hole in the weak field limit in quasi-Minkovskian co-ordinate following the Fermat's principle and obtained the approximated value of refractive index in quasi-Minkovskian co-ordinate. But Roy and Sen [42] studied the charged gravitating body considering both RN and JNW space-time geometry. They have evaluated the exact value of refractive index, without any approximation, in terms of charge radius ($q = \frac{Q}{r_g}$) as

$$n(x, q) = \frac{1}{\left(1 - \frac{1}{x} + \frac{q^2}{x^2} \right)} \quad (RN)$$

and

$$n(x, q) = \frac{1}{\left(1 - \frac{1}{x} \sqrt{1 + 4q^2} \right) \sqrt{1 + 4q^2}} \quad (JNW)$$

Eiroa et al. [20] studied the strong gravitational lensing by RN black hole using null geodesic method in Boyer-Lindquist co-ordinate up to 2nd order of charge. Chakraborty and Sen [45] also studied the deflection of light ray by Kerr-Newman geometry, using null geodesic method in Boyer-Lindquist co-ordinate up to 4th order of charge and mass. By putting rotation parameter equal to zero, they obtained the deflection angle due to RN geometry. Virbhadra et al. [26] calculated the deflection angle up to second order with JNW space-time using null geodesic method.

Roy and Sen [42] also obtained the expression for deflection angle corresponding to both the space-time geometry using material medium approach. But in this paper, it is observed that the refractive index and hence the deflection angle with the increase of charge radius (q) decreases in case of RN space-time geometry whereas increases in case of JNW space-time geometry. This is a noticeable feature of Roy and Sen [42]. This may appear to be a strange phenomenon. The deflection angles as a function of charge radius (q) for the two individual geometries seem to match with the works reported by other authors [20,26,45] for the two geometries separately. In all the figures reported by Roy and Sen [42], the set of curves follow similar pattern, but the difference lies in the magnitude of deflection values. The reason could be due to the fact that, Eiroa et al. [20] calculated up to 2nd order of charge, Chakraborty and Sen [45] calculated up to 4th order of charge and mass and Virbhadra et al. [26] calculated only up to 2nd order. But no approximation has been considered by Roy and Sen [42] to calculate the deflection angle due to both RN and JNW space-time geometry. So, the obtained values are claimed to be most exact so far.

5. Results and Discussions

From the above discussions, it may be concluded that the material medium approach gives similar values of deflection as compared to that obtained by other most conventional method of null geodesic. This has been verified by taking sun as a test case. Moreover, material medium approach gives most exact value than that of null geodesic. It is also noticed that the RN space-time and JNW space-time both are not similar kind of space-time although both represents the static solution of Einstein-Maxwell field equation for a charged, non-rotating and spherically symmetric gravitating body.

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