Analysis of fractal events with background

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Abstract. Properties of self-similarity and fractality in the processes of interactions of hadrons and nuclei at high energies are discussed. Different methods of fractal analysis (the box counting *BC*, system of the equations of p-adic coverage *SePaC* methods) are described. Fractal analysis of mixed events was carried out by *BC* and *SePaC* methods. The procedure of separation of fractals and background, estimation of the number of fractals in the original data set and contamination of the extracted data was proposed. The dependence of event contamination on multiplicity and background is studied. Reconstruction of the spectrum of fractal dimensions is found to depend on the method and background.

1 Introduction

The concept of the fractal is widely used in high-energy physics. Among them there are the fractal solution for the Schrödinger equation [1–3], the fractal space-time as a new method of regularization in the quantum field theory [4–7], the topological charge density distribution in the lattice gluodynamics as fractals [8], the intermittency in the spectra of secondary particles [9–11], the fractal structures of thermodynamical functions [12], the fractality of the momentum space as a signal of criticality in nuclear collisions [13], and the fractal properties of nuclei and events [14, 15].

A fractal is defined as a self-similar object characterized by the fractal dimension D_F which is more than the topological one D_T for the same object. The topological dimension determines the measure $M = \lim_{l \to 0} \sum_{i=1}^{N} l^{D_T}$, where N is a number of probes, covering the object with a size l. For the euclidean object this measure is independent of the scale, has a finite limit and determines the size of the object (length, square, volume). For a fractal the measure M tends to infinity, because N increases faster than a size l is reduced. In order to evaluate the rate of the growth the Hausdorff measure $M_H = \lim_{\delta \to 0} \sum_{i=1}^{N} l_i^d$ is introduced. Here N is a number of probes with a size $l < \delta$. The fractal dimension D_F is the value of d which provides the finite value of the measure M_H. Every fractal is characterized by the fractal dimension D_F and the box dimension $D_b = \lim_{\delta \to 0} lnN/lnl$. There are various types of fractals. Among them are fractals with independent, dependent and combined partition. They are analysed by the *SePaC*, *BC* and *PaC* methods in [16–18]. It was found, in particular, that D_F and D_b can be different for some types of fractals.

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2 z-Scaling

The description of inclusive cross sections of hadron production in collisions of protons and nuclei at high energies based on the principles of locality, self-similarity, fractality in the framework of the *z*-scaling concept was proposed in [19]. Authors have concluded that the structure of hadrons, mechanism of constituent interactions and hadronization process reveal self-similar properties over a wide scale range. In this approach an inclusive cross section is described in terms of the dimensionless function $\psi(z)$ depending on the self-similarity parameter *z*. The fractality is reflected in definition of $z = z_0(1 - x_1)^{-\delta_1}(1 - x_2)^{-\delta_2}$. Here z_0 is a slowly changing function of the kinematic variables, x_1 and x_2 are the scale-invariant fractions of incoming four-momenta P_1, P_2 of colliding objects and δ_1, δ_2 are their fractal dimensions. This methods has been applied for analysis of the inclusive spectra of hadrons and jets production at Tevatron and LHC [20, 21]. The dependence of the scaling function ψ on *z* was found to be independent of the collision energy and angle of produced particle. It is also described by the power law $\psi(z) \sim z^{-\beta}$ at high *z*. The value of the slope parameter β is independent of kinematic variables over a wide *z*-range. It confirms self-similarity and fractality of hadron production at various scales. Discontinuity of dimensions δ_1, δ_2 is assumed to be an indication of phase transition which is related to the change of physical sub-processes underlying inclusive particles production. Fractal dimensions have been determined from data analysis of inclusive cross sections and then verified in analysis of other data sets. These dimensions are fundamental characteristics of particles and processes and should be determined from the fractal analysis of the experimental data.

3 Fractal analysis

The fractal analysis is aimed to determine distributions characterizing the fractal properties of particles and processes. First of all the phase space in which the object reveals its fractal properties and an adequate method of analysis should be found. The original data set should be also described.

In the present work we use *BC* [16] and *SePaC* [18] methods to study mixed events $N_{ev} = N_{frac} + N_{bg}$ consisting of fractals N_{frac} and background N_{bg}. The fractals ($N_{frac} = 1857$) of various types are generated by using the procedure of independent, dependent and combined partition. The background is chosen as to be random data set. Portion of background is changed from 0.05 to 0.95. The preliminary analysis includes estimation of number of fractals and extraction of data set of reconstructed fractals. Further the extracted data set is analysed to determine of spectrum of fractal dimensions and contamination.

4 *BC* method

In the section we briefly describe the *BC* method [16]. It consists of the following steps:
1. Reading out the data *X*.
2. Construction of the set of distributions of *X*. The number of bins M_i in distributions changes as multiplicatively $M_i = (P)^i$ or additively $M_{i+1} = M_i + step$.
3. Counting the number of non-zero *N* bins for each distribution.
4. Finding the value of the slope parameter D_b of the power function $N \sim M^{D_F}$ and corresponding χ^2 of the linear approximation of this function in a double-log scale.
6. Verification of the accuracy condition: if $\chi^2 < \chi^2_{lim}$ then the set of particles is considered to be a fractal.

Notice, that the *BC* method determines box dimension D_b and has one parameter χ^2_{lim}. The dimension errors are strongly dependent of the set of distributions and region of linear approximation.

Further we present the results of analysis of various events. The dependence of the portion *Prtn* of the events reconstructed as fractals on χ^2 for fractal, background and mixed events with different multiplicity $N = 64, 27$ is shown in figure 1.
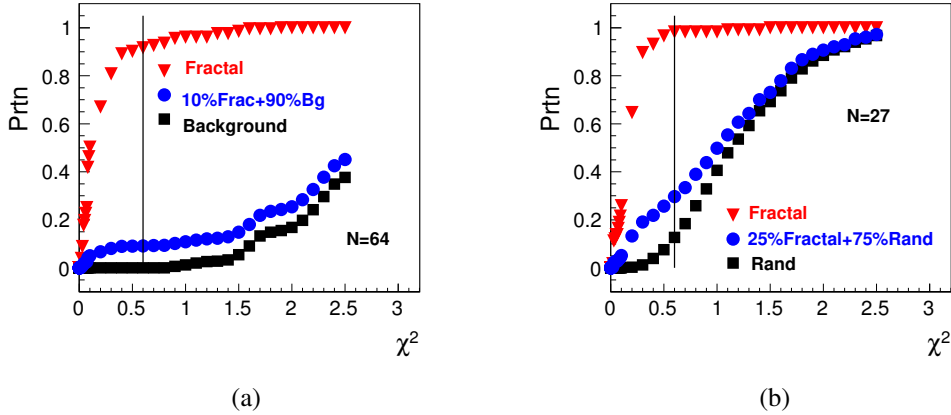
Figure 1. *BC*: the dependence of the portion *Prtn* of the events reconstructed as fractals on χ^2_{lim} for fractal, background and mixed events with different multiplicity $N = 64$ (a), $N = 27$ (b).

One can see from figure 1 that for fractals (red symbols) the value of $min\chi^2$ at which $Prtn \approx 0.01$ does not exceed 0.1. The portion of all fractals which are reconstructed as fractals at $\chi^2 = 0.6$ reaches 0.9-0.96. For background (black symbols) the value of $min\chi^2$ increases and $Prtn$ at $\chi^2 = 0.6$ decreases with increasing N. Notice, that fractals and background with multiplicity $N = 8$ are found to be not separable. The condition $min\chi^2 \leq 0.1$ corresponds to the existence of fractals in the original data set. The value 0.6 of χ^2 was used to extract fractals from the mixed events. We found that the extracted data $N_{ext} = P_{frac} \cdot N_{frac} + P_{bg} \cdot N_{bg}$ contain a significant portion of fractals ($P_{frac} = 0.9 \div 0.98$). For multiplicity $N \geq 64$ there is no background ($P_{bg} = 0$). For other multiplicities the extracted data have some portion ($P_{bg} \neq 0$) of the background. Thus, we can estimate N_{frac} in the original data set and contamination *Cont* of the extracted data as follows:

$$N_{frac} = \frac{N_{ext} - P_{bg} \cdot N_{ev}}{P_{frac} - P_{bg}}, \qquad Cont = \frac{P_{bg}}{N_{ext}} \cdot \frac{P_{frac} \cdot N_{ev} - N_{ext}}{P_{frac} - P_{bg}}. \tag{1}$$

5 *SePaC* method

In the section we briefly describe the *SePaC* method [18]. It consists of the following steps:
1. Reading out the data *X*.
2. Construction of *P*-adic coverages: $P = 3, ..P_{Max}$. Each coverage is a set of distributions of variable *X*. The number of bins M_i in distributions changes as a degree of basis *P*: $M_i = (P)^i$.
3. Counting the number of non-zero bins $N(lev, P)$.
4. Analysis of the system of the equations for verification of hypothesis of independent/dependent partition:
a. Construction of the system of the equations for all levels: $\sum_{i=1}^{N_{lev}}(d_{lev})^{D_F^{lev}} = 1$. Here N_{lev} is a number of levels, and d_{lev} is the length of non-zero ranges for each level.
b. Finding the solution D_F^{lev} of equations for each level by using the dichotomy method.

c. Determination of the averaged value $< D_F^{lev} >$ and deviation ΔD_F^{lev}.

d. Verification of the accuracy condition: if $\Delta D_F^{lev} < Dev$, then the set of particles is considered to be a fractal (P is the base and $D_F = < D_F^{lev} >$ is the fractal dimension).

Notice, that the $SePaC$ method determines fractal dimension D_F and has two parameters: maximal base P_{Max} and deviation Dev.

Further we present the results of analysis of various events. The dependence of the portion $Prtn$ of the events reconstructed as fractals on P_{Max} at $Dev = 0.18$ for fractal, background and mixed events with different multiplicity $N = 64, 27$ is shown in figure 2. One can see from figure 2, that for

(a)   (b)
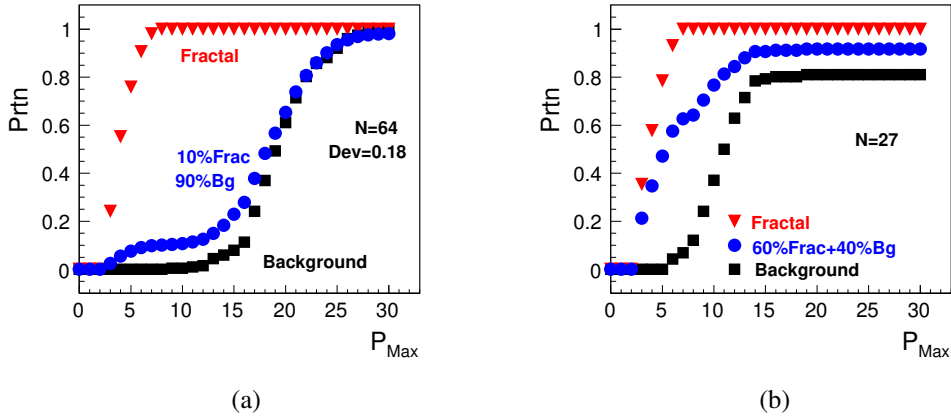
Figure 2. $SePaC$: the dependence of the portion $Prtn$ of the events reconstructed as fractal on Dev for fractal, background and mixed events with different multiplicity $N = 64$ (a), $N = 27$ (b).

fractals (red symbols) the value $minP_{Max}$ at which $Prtn \neq 0$ is equal to 3. Full reconstruction of the fractals takes place when P_{Max} is equal to the minimal value on the plateau. For background (black symbols) the value of $minP_{Max}$ increases with increasing N. The condition $minP_{Max} = 3$ corresponds to the existence of fractals in the original data set. The difference in the values of $minP_{Max}$ for fractals and background was used to extract fractals from the mixed events. The value of P_{Max} is taken to be minimal value at the first plateau or the value on the shoulder. We found that the extracted data $N_{ext} = N_{frac} + P_{bg} \cdot N_{bg}$ contain all fractals. For multiplicity $N \geq 64$ there are not background ($P_{bg} = 0$). For other multiplicities the extracted data have some portion ($P_{bg} \neq 0$) of the background. Thus, we can determine N_{frac} in the original data set and contamination $Cont$ of the extracted data as follows:

$$N_{frac} = \frac{N_{ext} - P_{bg} \cdot N_{ev}}{1 - P_{bg}}, \qquad Cont = \frac{P_{bg}}{N_{ext}} \cdot \frac{P_{frac} \cdot N_{ev} - N_{ext}}{1 - P_{bg}}. \qquad (2)$$

6 Fractal dimension

We found that the $SePaC$-method separates the fractals and the background for all multiplicities and BC - only for events with $N \geq 16$. For high multiplicity events ($N \geq 64$) both methods completely suppress background. For events with smaller multiplicities contamination $Cont$ is found to be approximately equal for both methods. For mixed events with multiplicity $N = 32, 27$ and background less than 75% the value of $Cont$ for extracted data is estimated to be less than 30%. The similar result is obtained for events with $N = 16, 8$ and background less than 50%.

We would like to notice that the distribution of fractal dimension D_F for the extracted data is distorted due to systematic errors of the method and the presence of background events. Figure 3 shows the distribution of the error Er_{D_F} of fractal dimension found by the $SePaC$ and BC methods for fractals with independent, dependent and combined partition.
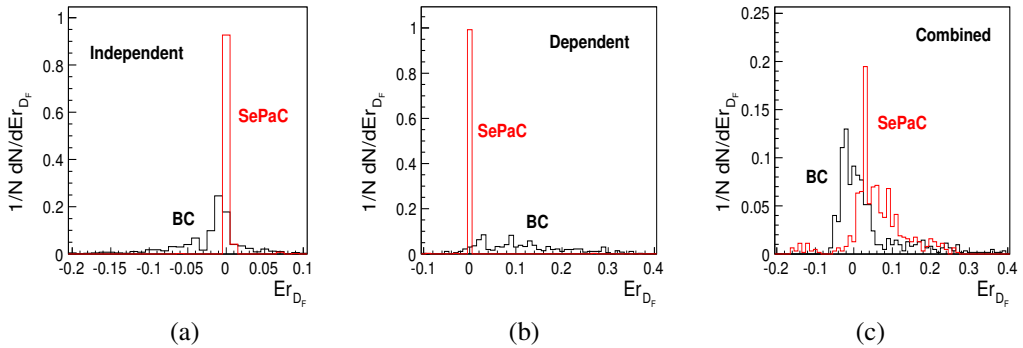
Figure 3. The distribution of the error Er_{D_F} of fractal dimension found by $SePaC$ (red line) and BC (black line) methods for fractals with independent (a), dependent (b) and combine (c) partition.

One can see from figure 3, that the $SePaC$ method exactly reconstructs dimension D_F for fractals with independent and dependent partition. For fractals with combined partition the distribution is found to be wide and shifted to positive values of Er_{D_F}. The distributions of errors obtained by the BC method demonstrate the asymmetric shape for all types of fractals. They are shifted to negative (a), positive (b) values of Er_{D_F} for fractals with independent and dependent partition, respectively. For fractals with combined partition the distribution (c) has positive and negative values of Er_{D_F}.

The distribution of fractal dimension D_F for background events with different multiplicity is shown in figure 4.
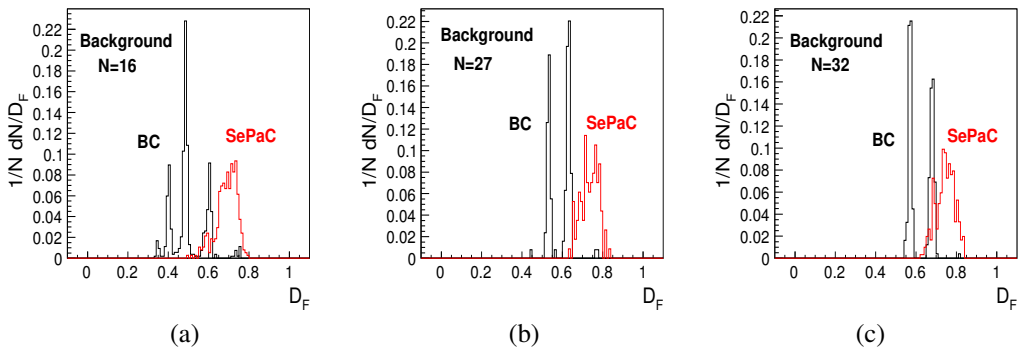
Figure 4. The distribution of fractal dimension D_F for background events with multiplicity $N = 16$ (a), 27 (b), 32 (c).

As it is seen from figure 4 the spectra of fractal dimensions reconstructed by the BC and $SePaC$ methods from the background events are different. Notice that D_F changes in the range 0.6-0.8 for distribution obtained by the $SePaC$ method. The spectrum of D_F reconstructed by the BC method has a more complex structure.

7 Conclusions

Fractal analysis of mixed events was carried out by the *BC* and *S ePaC* methods. The procedure of separation of fractals and background, estimation of the number of fractals (N_{frac}) in the original data set and contamination (*Cont*) in the extracted data was proposed. We found that the *S ePaC*-method separates the fractals and the background for all multiplicities and *BC* - only for events with $N \geq 16$. For high multiplicity events ($N \geq 64$) both methods completely suppress background. For events with smaller multiplicities contamination *Cont* was found to be approximately equal for both methods. The value of *Cont* depends on event multiplicity and background. It was shown that the distribution of fractal dimension D_F for the extracted data is distorted due to systematic errors of the used method and the presence of background events.

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