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## **Momentum Compaction and Slippage Factor**

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### 0.0.1 Momentum Compaction and Slippage Factor

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The slippage factor  $\eta$  is the relative slip in revolution period  $T$  for a particle with fractional off-momentum  $\delta = \Delta p/p_0$ , or

$$\frac{\Delta T}{T_0} = \eta \delta.$$

where the subscript zero stands for on-momentum. The various orders of momentum-compaction factor  $\alpha_i$  give the relative increase in closed-orbit length  $C$  for an off-momentum particle, or  $C = C_0[1 + \alpha_0\delta(1 + \alpha_1\delta + \alpha_2\delta^2 + \dots)]$ . With the slippage factor expanded as  $\eta = \eta_0 + \eta_1\delta + \eta_2\delta^2 + \dots$ , we have [1]

$$\begin{aligned} \eta_0 &= \alpha_0 - \frac{1}{\gamma_0^2}, \quad \eta_1 = \alpha_0\alpha_1 - \frac{\eta_0}{\gamma_0^2} + \frac{3\beta_0^2}{2\gamma_0^2}, \\ \eta_2 &= \alpha_0\alpha_2 - \frac{\eta_1}{\gamma_0^2} + \frac{3\beta_0^2\eta_0}{2\gamma_0^2} + \frac{(1 - 5\beta_0^2)\beta_0^2}{2\gamma_0^2}, \\ \eta_3 &= \alpha_0\alpha_3 - \frac{\eta_2}{\gamma_0^2} + \frac{3\beta_0^2\eta_1}{2\gamma_0^2} \\ &\quad + \frac{(1 - 5\beta_0^2)\beta_0^2\eta_0}{2\gamma_0^2} - \frac{5(3 - 7\beta_0^2)\beta_0^4}{8\gamma_0^2}, \end{aligned}$$

where  $\beta_0$  and  $\gamma_0$  are the Lorentz factors of the on-momentum particle. The transition gamma is defined as  $\gamma_t = \sqrt{1/\alpha_0}$ . To lowest order, all off-momentum particles have the same transition gamma when  $\alpha_1 \approx -\frac{1}{2}$ , and cross transition at the same time when  $\alpha_1 \approx -\frac{3}{2}$ .

For a FODO lattice with *thin* quadrupoles of integrated strength  $B'\ell/(B\rho) = S/L$ , where  $L$  is the half cell length with dipole bending angle  $\theta$ , we have approximately [2, 3]

$$\begin{aligned} \alpha_0 &\approx 1 - \frac{S(\hat{D}_0 - \check{D}_0)}{L\theta}, \quad \alpha_0\alpha_1 \approx -\frac{S(\hat{D}_1 - \check{D}_1)}{L\theta}, \\ \alpha_0\alpha_2 &\approx -\frac{S(\hat{D}_2 - \check{D}_2)}{L\theta} - \frac{S^3(\hat{D}_0^3 - \check{D}_0^3)}{6L^3\theta}, \end{aligned}$$

where the dispersions at the F- and D-quadrupoles have been expanded, respectively, as  $\hat{D} = \hat{D}_0 + \hat{D}_1\delta + \hat{D}_2\delta^2 + \dots$ , and  $\check{D} = \check{D}_0 + \check{D}_1\delta + \check{D}_2\delta^2 + \dots$ . When  $S \ll 12$ ,  $\alpha_1 \rightarrow +\frac{3}{2}$  and reduces to  $+\frac{1}{2}$  when chromaticities are corrected by sextupoles.

For an isochronous or quasi-isochronous ring, we must require the spread in  $\eta$  for off-momentum particles to be small also. Therefore,  $\alpha_1$  and  $\alpha_2$  need to be controlled in addition to  $\alpha_0$ . In fact, first-order effect of sextupoles can alter  $\alpha_1$ , that of octupoles can alter  $\alpha_2$ , etc. For example, when a *thin* quadrupole of integrated strength  $S_1 = B'\ell/(B\rho)$ , a *thin* sextupole of integrated strength

$S_2 = B''\ell/(B\rho)$ , or an octupole of integrated strength  $S_3 = B'''\ell/(B\rho)$  is placed at a location where the horizontal and vertical dispersions are, respectively  $D_x$  and  $D_y$ , their first-order effects are [4]

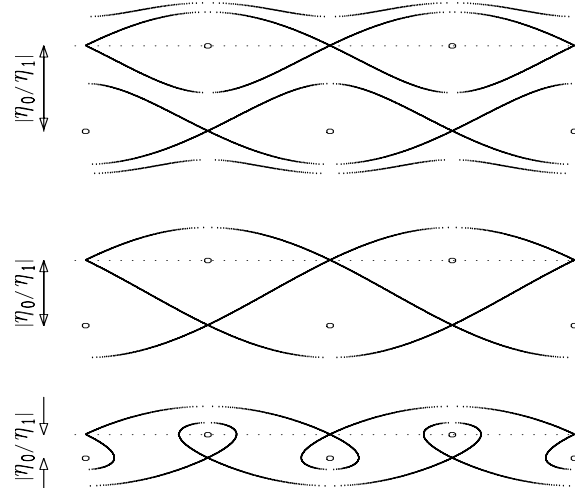
$$\begin{aligned} \Delta\alpha_0 &= -\frac{S_1}{C_0}(D_x^2 - D_y^2), \\ \Delta\alpha_0\alpha_1 &= -\frac{S_2}{C_0}(D_x^3 - 3D_xD_y^2), \\ \Delta\alpha_0\alpha_2 &= -\frac{S_3}{C_0}(D_x^4 - 6D_x^2D_y^2 + D_y^4). \end{aligned}$$

The Hamiltonian describing the longitudinal phase space can be written as [5]

$$\begin{aligned} H &= h \left( \frac{\eta_0\delta^2}{2} + \frac{\eta_1\delta^3}{3} + \frac{\eta_2\delta^4}{4} + \dots \right) + \\ &\quad \frac{eV}{2\pi\beta^2 E} [\cos(\phi_s + \Delta\phi) + \Delta\phi \sin \phi_s], \end{aligned}$$

where  $V$  is the rf voltage with synchronous phase  $\phi_s$  and harmonic  $h$  while  $E$  the particle energy. If only the  $\eta_0$  and  $\eta_1$  terms are considered, the two series of distorted pendulum-like buckets in the top figure ( $\phi_s$  is set to zero) begin to merge to the middle figure when  $|\eta_0/\eta_1|$  is lowered to

$$\left| \frac{\eta_0}{\eta_1} \right| = \left\{ \frac{6eV}{\pi\beta^2 h \eta_0 E} \left[ \left( \frac{\pi}{2} - \phi_s \right) \sin \phi_s - \cos \phi_s \right] \right\}^{1/2}.$$



With further reduction of  $|\eta_0/\eta_1|$ , the buckets become  $\alpha$ -like (lower figure), which shrink to zero when  $|\eta_0/\eta_1| = 0$ . The  $\alpha$ -like bucket of total height  $|3\eta_0/(2\eta_1)|$  is small. It is asymmetric with momentum spread and is susceptible to longitudinal head-tail instability. If the  $\eta_1$  term is eliminated, the Hamiltonian will be dominated by  $\eta_0$  and  $\eta_2$  and the bucket becomes pendulum-like again [3]. If the Hamiltonian is dominated by the  $\eta_2$  term alone, the kinetic term is similar to

a quartic potential providing maximal amount of synchrotron-frequency spread and therefore Landau damping.

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