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Structure and Solution of the Massive Thirring Model

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ABSTRACT

The Hamiltonian of the massive Thirring model is explicitly diagonalized by formulating a Bethe ansatz for the eigenstates. The physical states are described by many-body wave functions representing the vacuum as a filled Fermi-Dirac sea and particle states as excitations built upon it. The spectrum of states is determined by imposing periodic boundary conditions on the wave functions. Energies are calculated by reducing the periodic boundary conditions to linear integral equations. For fermion-antifermion bound states the Dashen-Hasslacher-Neveu spectrum is obtained. It is shown that the solution to the massive Thirring model can be understood as the critical limit of Baxter's solution of the eight-vertex model.

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I. INTRODUCTION[1]

The massive Thirring model is the theory of a self-coupled fermion field ψ in two dimensions described by the Lagrangian

$$\mathscr{L} = \overline{\psi}(i\partial\!\!\!/ - m_o)\psi - \tfrac{1}{2}g j_\mu j^\mu \tag{1.1}$$

where $j_\mu = \tfrac{1}{2}[\overline{\psi},\gamma_\mu\psi]$. The massless model[2] ($m_o = 0$) has been extensively analyzed. It is exactly soluble, and possesses a conserved axial vector current j^μ_5 in addition to the conserved vector current. This is an important ingredient in the construction of the solution. In addition, the massless model is scale invariant with anomalous dimensions. Thus, the bare mass appearing in Eq. 1.1 is related to the physical mass by some power of a cutoff.

The massive Thirring model does not have a conserved axial current and cannot be solved by the same techniques used in the solution of the massless model. Recently, however, considerable evidence has gathered to support the belief that the massive model is also exactly soluble. This evidence comes from two major approaches to the theory. These are the equivalence of the massive Thirring model to the quantized sine-Gordon equation,[3] and the equivalence of a lattice version of the model to the XYZ Heisenberg spin chain.[4-6]

The sine-Gordon theory is exactly integrable at the classical level.[7] Within the classical theory it is possible to construct an infinite family of conserved currents.[8] It has been shown that these currents can be consistently defined in the quantum theory.[9] They have been used to construct the exact S-matrix for the theory.[10] In the context of a simpler theory, the non-linear Schrodinger equation, it has been argued that the existence of such a family of conserved currents in the quantum theory is intimately related to the success of a Bethe ansatz as a means of diagonalizing the Hamiltonian.[11] In addition, Dashen, Hasslacher, and Neveu[12] (DHN) have used semiclassical methods to compute the mass spectrum of the sine-Gordon theory, and these results are found to be exact.

The second approach suggesting that the massive Thirring model is exactly soluble is the study of the XYZ spin chain.[4-6] This is a one-dimensional model described by the Hamiltonian

$$\mathscr{H} = -\tfrac{1}{2} \sum_n \left[J_x \sigma_n^1 \sigma_{n+1}^1 + J_y \sigma_n^2 \sigma_{n+1}^2 + J_z \sigma_n^3 \sigma_{n+1}^3 \right] \quad . \tag{1.2}$$

where the σ^i's are Pauli matrices. In a pioneering work, Baxter[4] succeeded in computing the ground state energy of the system. Johnson, Krinsky, and McCoy[5] then computed the exact excitation spectrum. Luther[6] pointed out that via a Jordan Wigner transformation, the XYZ Hamiltonian could be regarded as a lattice version of the massive Thirring model. He showed that in an appropriate limit, the excitation spectrum computed by Johnson, Krinsky, and McCoy (JKM) reduced to the DHN spectrum. The elegant methods of Baxter and Johnson, Krinsky, and McCoy can be described as a generalization of Bethe's ansatz.

Bethe's ansatz was first used[13] to solve the isotropic Heisenberg chain ($J_x = J_y = J_z$). Bethe found that the eigenstates of the Hamiltonian could be described in terms of interacting spin waves. The only effect of the interaction was to cause the spin waves to scatter elastically, with a nontrivial phase shift. He wrote down exact eigenstates with an arbitrary number of spin waves labeled by momenta k_i. To construct the ground state, it was necessary to fill all negative energy modes,[14] using periodic boundary conditions to determine the density of states. Since then, the Bethe ansatz has been used in various forms to solve the quantum nonlinear Schrodinger equation,[15] several ice models,[16] and the XYZ spin chain[17] ($J_x = J_y = J_z$).

In section II of this paper, we will demonstrate the diagonalization of the massive Thirring model Hamiltonian via a Bethe ansatz. This will be accomplished on the "unfilled" Dirac sea. In section III, we discuss implications of the periodic boundary conditions for all states, and in particular, excited states. The spectrum of the massive Thirring model will be computed in section IV. Finally, in section V, we discuss the relationship between our methods and the techniques of Baxter and Johnson, Krinsky, and McCoy in their analysis of the lattice theory.

II. DIAGONALIZATION OF THE HAMILTONIAN

The diagonalization of the Hamiltonian will be accomplished in an unphysical Hilbert space built on a reference state $|0>$. This state is defined by $\psi_1(x)|0> = \psi_2(x)|0> = 0$. Of course, this Hilbert space is very far from the physical one. Even in free field theory, it has an unbounded negative energy spectrum. We will see that this feature persists in the interacting theory. The physical vacuum is formally constructed from the reference state by filling all negative energy modes, i.e. filling the Dirac sea. It is in this phase of our investigation that we deal with the non-trivial renormalization properties known[2] to be present in the Thirring model (mass renormalization and anomalous dimensions for operators). The spectrum of the Hamiltonian on the filled Dirac sea (true vacuum) is profoundly different from the spectrum in the unphysical Hilbert space, unlike the free field theory case where the only spectral effect is to eliminate negative energy states and replace them by antiparticle states. By taking careful account of the nature of the interacting Dirac sea, we will be able to compute the physical spectrum of H.

We choose a basis in which γ_5 is diagonal. The Hamiltonian is

$$H = \int dx \left\{ -i(\psi_1^\dagger \frac{\partial}{\partial x}\psi_1 - \psi_2^\dagger \frac{\partial}{\partial x}\psi_2) + m_o(\psi_1^\dagger \psi_2 + \psi_2^\dagger \psi_1) + 2g_o \psi_1^\dagger \psi_2^\dagger \psi_2 \psi_1 \right\} \quad . (2.1)$$

We have actually dropped a term proportional to the operator

$$N = \int dx \psi^\dagger \psi \quad .$$

This operator commutes with H, and can be used to classify superselection sectors of the theory. Motivated by the character of Bethe's ansatz, we want to find modes which are preserved by the interaction, suffering only a phase shift. We will accomplish this in two steps.

Consider first the free Hamiltonian ($g_o = 0$). We introduce the momentum space operators

$$a_{ik} = \int \frac{dx}{\sqrt{2\pi}} e^{-ikx} \psi_i(x) \tag{2.2}$$

and find

$$H_o = \int dk \left\{ k \left(a^\dagger_{1k} a_{1k} - a^\dagger_{2k} a_{2k} \right) + m_o \left(a^\dagger_{1k} a_{2k} + a^\dagger_{2k} a_{1k} \right) \right\} . \tag{2.3}$$

Since H_o is quadratic in the a's, we can diagonalize it by a canonical transformation,

$$A_{1k} = \cos\theta_k a_{1k} + \sin\theta_k a_{2k}$$

$$A_{2k} = -\sin\theta_k a_{1k} + \cos\theta_k a_{2k} \quad . \tag{2.4}$$

If $\tan 2\theta_k = (m_o/k)$, the Hamiltonian is diagonal in the A's, becoming

$$H_o = \int dk\, E_k \left(A^\dagger_{1k} A_{1k} - A^\dagger_{2k} A_{2k}\right)$$

$$E_k = \sqrt{k^2 + m_o^2} \quad . \tag{2.5}$$

From Equations (2.2), (2.4) and the definition of $|0\rangle$ it follows that

$$A_{1k}|0\rangle = A_{2k}|0\rangle = 0 \quad . \tag{2.6}$$

Then, from Eq. (2.5), we see that the spectrum of H_o in a Hilbert space built on $|0\rangle$ has two types of excitations. There are "1" particles with a dispersion relation $E_{1k} = +\sqrt{k^2 + m_o^2}$, and "2" particles with $E_{2k} = -\sqrt{k^2 + m_o^2}$. Eigenstates are constructed by applying creation operators to the state $|0\rangle$

$$|k_1, \ldots k_n; p_1, \ldots p_m\rangle = \prod_{i=1}^{n} A^\dagger_{1k_i} \prod_{j=1}^{m} A^\dagger_{2p_j} |0\rangle \quad .$$

The energies of "1" and "2" excitations are positive and negative, respectively,

$$H\,|k_1, \ldots k_n; p_1, \ldots p_m\rangle = \left(\sum_i E_{k_i} - \sum_j E_{p_j}\right) |k_1, \ldots k_n; p_1, \ldots p_m\rangle \quad . \tag{2.7}$$

The full Hamiltonian is diagonalized by a generalization of this procedure. The remaining step is to include phase shifts in the wavefunctions. A two-body phase shift appears for each pair of occupied modes in the eigenstate. It is simplest to demonstrate the procedure for the sector N = 2. The construction for any N is presented in the Appendix. Consider a state with two "1" particles. In free field theory, we could write it as

$$|k_1 k_2>_0 = A^\dagger_{1k_1} A^\dagger_{1k_2}|0> \equiv \int dx_1 dx_2\, \chi(x_i, k_i)\, \Psi^\dagger(x_1, k_1)\Psi^\dagger(x_2, k_2)|0>$$

$$\Psi(x, k) = \cos\theta_k\, \psi_1(x) + \sin\theta_k\, \psi_2(x) \qquad . \qquad (2.8)$$

where $\chi = \exp i(k_1x_1 + k_2x_2)$ for free fermions. We find that by including a factor $(1 + i\lambda\varepsilon(x_1 - x_2))$ in Eq. (2.8) (where $\varepsilon(x)$ is a step function), an eigenstate of H is obtained. λ is determined in the course of the calculation. We proceed by applying H_o to $|k_1k_2>$ and commuting ψ_i's to the right until they annihilate $|0>$. The derivatives of the kinetic Hamiltonian are then integrated by parts to act on $\chi = e^{ik_1x_1} e^{ik_2x_2}(1 + i\lambda\varepsilon(x_1-x_2))$. We have

$$H_o|k_1k_2> = \int dx_1dx_2 \Big\{ \Big[\{(-i\frac{\partial}{\partial x_1}\chi)\cos\theta_1 + m_o\chi\sin\theta_{k_1}\}\psi_1^\dagger(x_1)$$

$$- \{(-i\frac{\partial}{\partial x_1}\chi)\sin\theta_1 - m_o\chi\cos\theta_1\}\psi_2^\dagger(x_1)\Big]\Psi^\dagger(x_2, k_2) + (x_1,k_1 \leftrightarrow x_2,k_2)\Big\}|0> \; . \quad (2.9)$$

If $\lambda = 0$, we have $(-i\frac{\partial}{\partial x_i}\chi) = k_i\chi$. Then, using $\tan 2\theta_k = m_o/k$, it is easy to show

$$k\cos\theta_k + m_o\sin\theta_k = E_k\cos\theta_k$$

$$k\sin\theta_k - m_o\cos\theta_k = -E_k\sin\theta_k \qquad . \qquad (2.10)$$

These substitutions in (2.9) yield (2.7). Of course, $\lambda \neq 0$, and we have an additional contribution. By taking $\delta(x)\varepsilon(x) = 0$, we have

$$(-i\frac{\partial}{\partial x_1}\chi) = \Big(k_1 + 2\lambda\delta(x_1 - x_2)\Big)\chi \qquad . \qquad (2.11)$$

The first term in (2.11) can be substituted into (2.9) to get the free field result again. The second term gives

$$|R> = 4\lambda \int dx \chi(x,x) \sin(\theta_1 + \theta_2) \psi_1^\dagger(x) \psi_2^\dagger(x) |0> \quad . \tag{2.12a}$$

By applying $H_I = H - H_o$ to $|k,k_2>$, we obtain

$$|R'> = -2g_o \int dx \chi(x,x) \sin(\theta_1 - \theta_2) \psi_1^\dagger(x) \psi_2^\dagger(x) |0> \quad . \tag{2.12b}$$

Taking

$$\lambda(k_1, k_2) = \tfrac{1}{2} g_o \frac{\sin\left(\theta_{k_1} - \theta_{k_2}\right)}{\sin\left(\theta_{k_1} + \theta_{k_2}\right)} \tag{2.13}$$

we see that $|R> + |R'> = 0$, and hence

$$H|k_1 k_2> = \left(E_{k_1} + E_{k_2}\right) |k_1, k_2> \tag{2.14}$$

completing the demonstration.

In deriving Eq. (2.14), the symmetric choice $\delta(x)\varepsilon(x) = 0$ has been made. Since H and the number operator N commute, we might regard the diagonalization problem as a many-body Dirac equation for χ. The interaction induces a delta function potential in the eigenvalue problem for χ, and by integrating across the boundary $x_1 = x_2$, we obtain the result that the scattering introduces a phase shift. Eq. (2.13) is that phase shift, since the relative phase of χ between the regions $x_1 > x_2$ and $x_1 < x_2$ is just

$$\frac{1 + i\lambda}{1 - i\lambda} = e^{i\phi(k_1,k_2)}$$

$$\phi(k_1, k_2) = 2 \tan^{-1} \left\{ \tfrac{1}{2} g_o \frac{\sin\left(\theta_{k_1} - \theta_{k_2}\right)}{\sin\left(\theta_{k_1} + \theta_{k_2}\right)} \right\} \quad . \tag{2.15a}$$

It is possible to modify all free Hamiltonian eigenstates in this manner (see Appendix A) obtaining new states $|k_1 \ldots k_n, p_1 \ldots p_m>$ with

$$H\,|k_1 \ldots k_n, p_1 \ldots p_m> \;=\; (\,\Sigma E_{k_i} - \Sigma E_{p_j})\,|k_1 \ldots k_n, p_1 \ldots p_m>$$

$$P\,|k_1 \ldots k_n, p_1 \ldots p_m> \;=\; (\,\Sigma k_i + \Sigma p_j)\,|k_1 \ldots k_n; p_1 \ldots p_m> \qquad (2.16)$$

where P is the momentum operator of the theory. It is important to realize that the k's and p's no longer have the simple meaning they had for free field theory. Including a function of $(x_1 - x_2)$ in Eq. (2.8) destroys the relationship of k in $A_k^\dagger$ to k labeling a state. The k's in the state are now to be regarded only as labels. $|k_1 k_2>$ contains Fock space components with arbitrarily high momentum.

For free field theory, the physical vacuum is the state with all "2" particle modes filled. By quantizing the theory in a box of length L and imposing periodic boundary conditions, these modes become denumerable. We have

$$|\Omega> \;=\; \prod_n A^\dagger_{2k_n}\,|0>\,, \qquad k_n = \frac{2\pi n}{L} \qquad (2.17)$$

where n = integer. The spectrum of states in this vacuum is manifestly positive. It consists of "1" particles with $E_{1k} = \sqrt{k^2 + m_o^2}$, and holes with energy $E_{\text{hole }k} = +\sqrt{k^2 + m_o^2}$, (all energies relative to the vacuum). We shall fill the Dirac sea in the same manner for the interacting theory. All "2" modes are filled in the physical vacuum. It is more difficult to do this for the full theory. Periodic boundary conditions (PBC's), trivial for free field theory, are an infinite set of coupled transcendental equations when $g_o \neq 0$. Fortunately, similar problems occur in other models solved by Bethe's ansatz and have been successfully treated by converting the PBC's to linear integral equations.[5,15,16,17] We will analyze the periodic boundary conditions for the massive Thirring model in Section III. It is the non-trivial nature of the PBC's that prevents us from concluding that Eq. (2.16) implies the spectrum of the interacting theory is identical to the free theory.

It is convenient to take advantage of the Lorentz invariance of the Hamiltonian by introducing rapidity variables. For a "1" mode, let

$$\tanh \beta = k/E \quad . \tag{2.18}$$

In terms of β, we have

$$\phi(\beta_1, \beta_2) = +2 \tan^{-1} \{\cot \mu \tanh \tfrac{1}{2}(\beta_1 - \beta_2)\} \tag{2.15b}$$

where we have set $\cot \mu = -\tfrac{1}{2} g_0$ for reasons which will become apparent. The spectrum is most elegantly presented in terms of the coupling μ. Note that $0 < \mu < \pi$ covers the range $-\infty < g_0 < \infty$. Free field theory is $\mu = \pi/2$. For $\mu < \pi/2$ ($g_0 < 0$), particles and holes repel. For $\mu > \pi/2$ ($g_0 > 0$), and particles and holes attract. Here we find bound states. There is an alternative form for the phase shift, emphasizing the naturalness of the coupling μ. It is

$$\phi(\beta) = -i \ln \left\{ -\frac{\sinh \tfrac{1}{2}(\beta - 2i\mu)}{\sinh \tfrac{1}{2}(\beta + 2i\mu)} \right\} \quad . \tag{2.15c}$$

The precise branch structure of this will be examined in detail later.

Finally, we note that $\beta \to i\pi - \beta$ takes a "1" particle to a "2" particle of the same momentum. We have

$$E = m_o \cosh\beta \rightarrow m_o \cosh(i\pi - \beta) = -m_o \cosh\beta$$

$$P = m_o \sinh\beta \rightarrow m_o \sinh(i\pi - \beta) = +m_o \sinh\beta \qquad .$$

This substitution can be systematically applied to all results, in particular the phase shift. We may obtain phase shifts between "1" and "2" particles, or two "2" particles in this manner. With the introduction of complex rapidity, the filled Dirac sea may be considered as occupying the line $\mathrm{Im}\,\beta = \pi$ in the rapidity plane (see Fig. 1). The notion of complex rapidity will be extremely useful in developing the structure of the theory. In terms of rapidities, we can construct a general eigenstate of the Hamiltonian. It is $|\beta_1 \ldots \beta_n>$ where real β's correspond to "1" particles and $\mathrm{Im}\,\beta = i\pi$ corresponds to "2" particles. The general wavefunction is

$$\chi(x_1, \ldots x_n) = e^{im_o \Sigma x_i \sinh \beta_i} \sum_{1 \leq i < j \leq n} \Big(1 + i\lambda(\beta_i, \beta_j)\, \varepsilon(x_i - x_j)\Big) \tag{2.19}$$

and the eigenvalue equation is

$$H\,|\beta_1 \ldots \beta_n> = \left(\sum_{i=1}^{n} m_o \cosh \beta_i\right) |\,\beta_1 \ldots \beta_n> \qquad . \tag{2.20}$$

III. PERIODIC BOUNDARY CONDITIONS

In order to define the process of filling the Dirac sea, we put the theory in a box of length L and impose periodic boundary conditions (PBC's) on the states. For free field theory ($\mu = \pi/2$), this simply restricts allowed momenta to integral multiples of $(2\pi/L)$. When $\mu \neq \pi/2$, the phase shifts among modes are non-vanishing, and the imposition of PBC's is more involved.

We demand that $\chi(x_1,\ldots,x_n)$ be periodic in each argument x_i. This gives the boundary condition

$$\chi(x_i = 0) = \chi(x_i = L) \quad . \tag{3.1}$$

Comparing χ at the boundaries of the x_i variable and using Eq. (A.3), we find

$$e^{im_0 L \sinh \beta_i} = e^{-i \sum_j \phi(\beta_i - \beta_j)} \quad . \tag{3.2}$$

This set of equations must be satisfied for $|\beta_1,\ldots,\beta_n>$ to be a physically admissible state. Taking the logarithm of Eq. (3.2), we find

$$m_0 \sinh \beta_i = -\frac{2\pi n_i}{L} - \frac{1}{L} \sum_j \phi(\beta_i - \beta_j) \tag{3.3}$$

where $\phi(\beta)$ is given in Eq. (2.15b). The n_i in Eq. (3.3) are integers specifying the branch of the logarithm of Eq. (3.2) to be taken. We always choose $-\pi/2 < \tan^{-1} x < \pi/2$ for the phase shifts in (3.3).

In free field theory, the phase shift $\phi = 0$, so the PBC's require $k_i = 2\pi n_i/L$. The vacuum is the state with no holes, that is to say $n_i - n_{i-1} = 1$. As μ varies from $\pi/2$, the position of each β shifts, but the ground state distribution remains without holes. Figure 1 represents a section of the vacuum distribution of modes for free field theory and for $\mu > \pi/2$ and $\mu < \pi/2$. In Section IV, we shall show how to get exact information about the spectral properties of the theory from the PBC's.

We can now discuss excitations built upon the vacuum. Since a term in H proportional to N, the number of filled modes, was dropped in Eq. (2.1), we will always remain in the neutral sector. Thus, an excitation will be constructed by removing a certain number of modes from the filled vacuum line $\mathrm{Im}\,\beta = \pi$ and placing them in configurations which satisfy the PBC's.

The simplest excitation possible is obtained by removing a mode from $\beta = i\pi + \alpha_o$ and placing it at $\beta = \alpha_o'$, where α_o and α_o' are real. In free field theory, this particle-hole pair is a state with a fermion and an antifermion. The antifermion (hole) has energy $-m_o \cosh(i\pi + \alpha_o) = +m_o \cosh\alpha_o$, and the fermion has energy $+m_o \cosh\alpha_o'$. For $\mu < \pi/2$, such a state persists, but for $\mu > \pi/2$, only a particle hole pair with $\alpha_o = \alpha_o'$ is allowed. To see this, we consider first a simpler problem. Given a filled mode at $\beta = 0$, we can ask which modes along the line $\mathrm{Im}\,\beta = \pi$ can be filled while satisfying the PBC's. For a state with a mode at $\beta = i\pi + \alpha_o$ (α_o real) Eq. (3.3) becomes

$$m_o L \sinh\alpha_o = 2\pi n + 2\tan^{-1}\cot\mu \coth \tfrac{1}{2}\alpha_o \quad . \tag{3.4}$$

When $\mu < \pi/2$, $\cot\mu > 0$, and for any n, we have a solution to Eq. (3.4), as shown in Fig. 2a. For $n = 0$, we have, in fact, 2 solutions. One is positive and one is negative. The discontinuity introduced in the PBC's by the presence of a particle at $\alpha = 0$ has provided an extra place on the $\mathrm{Im}\ \alpha = i\pi$ line for a mode.

For $\mu > \pi/2$, $\cot\mu < 0$, and, for $n = 0$, Eq. (3.4) has no solutions. This is depicted in Fig. 2b. Of course, solutions exist for $|n| > 0$. As μ increases past $\pi/2$, we go from having two solutions to the PBC's with $n = 0$ to having none. In this discussion, we have ignored the effects of other particles along the line $\mathrm{Im}\,\alpha = i\pi$. These will be discussed shortly.

There are other possible excitations in the interacting theory. The modes can be restricted, without loss of generality, to the strip $-\pi < \mathrm{Im}\,\beta \leq \pi$. In general, we must be careful to place modes along the lines $\mathrm{Im}\,\beta = 0$ or π to insure that $k = m_0 \sinh \beta$ is real. An imaginary k introduces the possibility of an exponentially growing wavefunction in some direction. In the interacting theory, it turns out that very special configurations of modes off the lines $\mathrm{Im}\,\beta = 0$ or π are permitted. They are called n-strings.[5] Although they have imaginary momentum labels, the coefficient of the exponentially growing term in the wavefunction vanishes. They will be found to represent bound states for a certain range of the coupling μ, and unbound pairs for another range.

For simplicity, we will first construct an n-string in the absence of the Dirac sea. Consider a state with n modes $\{\beta_i\}$ filled. All β_i have the same real part, α_s. The wavefunction χ contains plane wave factors $e^{i m_0 x_I \sinh \beta_i}$. If $0 < \mathrm{Im}\,\beta_i \leq \pi$, then $\mathrm{Im} \sinh \beta_i > 0$, so as $x_i \to -\infty$, the wavefunction will blow up unless we can arrange for the relevant coefficients to vanish. In the limit $x_i \to -\infty$, we can replace $\varepsilon(x_i - x_j)$ by (-1), and require

$$\prod_j \Big(1 - i\lambda(\beta_i, \beta_j)\Big) = 0 \qquad . \tag{3.5}$$

From this it is found that the β's must be distributed symmetrically about the real axis (so that the total momentum of the n-string is real), and that the factors in (3.5) connecting adjacent β's must vanish. This leads to the requirement $1 + \cot \mu \tan \frac{1}{2} \Delta = 0$, or

$$\Delta = 2(\pi - \mu) \tag{3.6}$$

where Δ is the spacing between adjacent β's. Thus, an n-string consists of n β's spaced $2(\pi - \mu)$ apart in the imaginary direction, all with the same real part. This is shown in Fig. 3. If n such points cannot be fit within $-\pi < \text{Im}\,\beta \le \pi$, the n-string is not allowed by the PBC's. An n-string may exist for $\left(\frac{n-2}{n-1}\right)\pi < \mu < \pi$.

In order to discuss the excitations available for a given value of the coupling parameter μ, we divide the domain $(0, \pi)$ into intervals I_r where r is an integer. $\mu \in I_r$ will denote a value in the range $\left(\frac{r}{r+1}\right)\pi < \mu < \left(\frac{r+1}{r+2}\right)\pi$. Thus, for $\mu \in I_r$, n strings for $n \le r + 2$ are permitted. We shall see, however, that in the presence of the Dirac sea, the character of the r + 2 and r + 1 string is different from the character of the $n \le r$ string. For $\mu \in I_r$, the largest two n-strings (namely $n = r + 2$, $r + 1$) require $(n - 2)$ holes at $\alpha_s + i\pi$ while all other n-strings (namely $n \le r$) require n holes at $\alpha_s + i\pi$. An r + 2 (or r + 1) string state has 3 free parameters. They are α_s, the location of the string, and the location of the 2 holes in the Dirac sea which were not forced to be at $\alpha_s + i\pi$. An n-string for $n \le r$ has only 1 free parameter, namely α_s, since all n holes are in this case required to be at $\alpha_s + i\pi$. We will find that the two longest string states correspond to an unbound fermion-antifermion pair, while the smaller strings correspond to bound states.

The restriction on the placement of holes in n-string states described above will now be shown to follow directly from the PBC's. The first step in this demonstration is to compute the phase shift suffered by a vacuum particle labeled by $\beta = i\pi + \alpha$ (α real) due to an n-string at α_s. This will be called $\Phi_n(\alpha, \alpha_s)$. We define

$$F_n(\alpha) = 2\tan^{-1}\left\{\cot\frac{n\mu}{2}\tanh\frac{1}{2}\beta\right\} \tag{3.6'}$$

so that $F_2(\beta)$ is the phase shift. As always, we choose the principle branch for $\tan^{-1}$. The sum of phase shifts is computed using the form of $\phi(\beta)$ given in Eq. (2.15c). When we add the phase shifts due to each mode in the n-string, the sum telescopes,with the result (for $n > 1$)

$$\Phi_n(\alpha, \alpha_s) = F_{n+1}(i\pi + \alpha - \alpha_s) + F_{n-1}(i\pi + \alpha - \alpha_s) - A\varepsilon(\alpha - \alpha_s) \quad (n \text{ odd})$$

$$\Phi_n(\alpha, \alpha_s) = F_{n+1}(\alpha - \alpha_s) + F_{n-1}(\alpha - \alpha_s) - A\varepsilon(\alpha - \alpha_s) \quad (n \text{ even}) \quad . \quad (3.7)$$

Here the A's are the constant multiples of π needed to match the branches of the left and right-hand sides of the equations. They can be determined by examining the equations in the limit $\mathrm{Re}\,\alpha \to \pm\infty$. From Eq. (2.15b), we see $\Phi_n(\alpha, \alpha_s) \to n(\pi - 2\mu)$ in this limit. By computing the limits of the F_n, we find for $\mu \in I_r$, $A = (n-2)\pi$ for the $r+2$ and $r+1$ strings, $A = n\pi$ for even n strings with $n \leq r$, and $A = (n-2)\pi$ for odd n-strings with $n \leq r$. This specifies the phase shifts to be used in the PBC's.

As we saw from the analysis of the 1-string (Eq. 3.4), the discontinuity of $\Phi_n(\alpha_0, \alpha_s)$ at $\alpha_0 = \alpha_s$ is of central importance in determining the proper placement of modes along the $i\pi$ line. This discontinuity is readily computed from Eq. (3.7) and the knowledge of A. For even strings, the F's are continuous, and the sole source of the discontinuity is the $\varepsilon(\alpha)$ in Eq. (3.7). Thus, the discontinuity is $-2\pi(n-2)$ for the longest even n-string. For the smaller even n-strings, the jump is $-2\pi n$. Odd n-strings have an additional discontinuity due to the fact that the F's are discontinuous (containing $\coth(\alpha - \alpha_s)$) with total discontinuity 2π. For the longest odd n-string, the discontinuities of F_{n+1} and F_{n-1} cancel, leaving a total discontinuity of $-2\pi(n-2)$. For the others, the discontinuities add, leaving a total discontinuity $-2\pi n$. Thus, for $\mu \in I_r$, the $n = r + 2$ and $r + 1$ strings have a discontinuity in Φ_n of $-2\pi(n - 2)$, and the others have a discontinuity of $-2\pi n$. This is also valid for the 1 string.

We can now analyze the PBC's for vacuum modes in the presence of an n-string. First, consider the filled Dirac sea. This has no holes ($n_i - n_{i-1} = 1$), and the rapidities of the vacuum modes at $\beta_i = i\pi + \alpha_i$ satisfy

$$m_o \sinh \alpha_i = \frac{2\pi n_i}{L} + \frac{1}{L} \sum_j \phi(\alpha_i - \alpha_j) \quad . \tag{3.8a}$$

Suppose we put an n-string between α_m and α_{m+1} (e.g. $\alpha_m < \alpha_s < \alpha_{m+1}$, see Fig. 4). The rapidity gap between these modes is O(1/L). The new PBC's are

$$m_o \sinh \bar{\alpha}_i = \frac{2\pi \bar{n}_i}{L} + \frac{1}{L} \sum_j \phi(\bar{\alpha}_i - \bar{\alpha}_j) + \frac{1}{L} \Phi_n(\bar{\alpha}_i, \bar{\alpha}_s) \quad . \tag{3.8b}$$

We can construct an approximate solution to these equations by setting $\bar{\alpha}_i = \alpha_i$, because the extra term Φ_n is a sum of a finite number of terms each O(1/L). We must, however, change $\bar{n}_i$ by introducing an appropriate number of holes, i.e., $n_{m+1} - n_m > 1$ (where $\alpha_m < \alpha_s < \alpha_{m+1}$). This can be seen by subtracting the $i = m$ Eq. (3.8b) from the $i = m + 1$ equation, and using Eq. (3.8a) this gives

$$m_o \sinh \alpha_{n+1} - m_o \sinh \alpha_n = \frac{2\pi}{L}(\tilde{n}_{m+1} - \tilde{n}_m) + \left[\frac{1}{L} \sum_j \phi(\alpha_{m+1} - \alpha_j) - \frac{1}{L} \sum \phi(\alpha_m - \alpha_j)\right]$$
$$+ \frac{1}{L}\Big(\Phi_n(\alpha_{m+1}, \alpha_s) - \Phi_n(\alpha_m, \alpha_s)\Big)$$

$$\frac{2\pi}{L}(n_{m+1} - n_m) = \frac{2\pi}{L}(\tilde{n}_{m+1} - \tilde{n}_m) + \frac{1}{L}\,\mathrm{Disc}\,(\Phi_n) \quad .$$

Since the vacuum distribution has $n_{m+1} - n_m = 1$, we conclude

$$\tilde{n}_{m+1} - \tilde{n}_m = -\frac{1}{2\pi}\,\mathrm{Disc}\,(\Phi_n) + 1 \qquad . \tag{3.9}$$

Eq. (3.9) leads to the hole counting described above. For $\mu \in I_r$, the (r + 2)- and (r + 1)-strings have discontinuities $-2\pi(n-2)$, or, according to Eq. (3.9), (n - 2) holes ($n_{n+1} - n_n = 1$ means no holes). The n strings with $n \leq r$ have discontinuities of $-2\pi n$, or, according to Eq. (3.9), require n holes at $i\pi + \alpha_s$ for their construction.

In particular, for $\mu \in I_o$ (i.e. $0 < \mu < \pi/2$), a 1 string has (-1) holes. This is precisely the situation found in Eq. (3.4), namely, there were two solutions to the periodic boundary condition equation for the vacuum distribution over the 1 string. As μ increases towards $\pi/2$, these solutions approach one another. When μ is slightly greater than $\pi/2$ ($\mu \in I_1$), we know a 3-string is allowed. It has modes at $\mathrm{Im}\,\alpha = 0$ and $\mathrm{Im}\,\alpha = \pm i(\pi - 2\varepsilon)$ where $\mu = (\pi/2) + \varepsilon$. We can add $2\pi i$ to the mode at $-i(\pi - 2\varepsilon)$ bringing it to $i(\pi + 2\varepsilon)$. In this way, the two new modes might be thought of as the two solutions to the vacuum PBC's which were approaching each other for $\mu < \pi/2$. Instead of colliding (and causing $|\alpha_1 \ldots \alpha_{n_1} \ldots \rangle$ to vanish) they move aside (see Fig. 5) causing the 1 string with -1 holes (part of an unbound fermion-antifermion state) to evolve into a 3-string with 1 hole (part of the unbound fermion-antifermion state in the region I_1). A similar picture may be constructed for the evolution of this fermion-antifermion state into the region I_2, and so on. One can also be constructed for even strings.

IV. THE SPECTRUM

Now that we understand the character of the states allowed by the periodic boundary conditions, we can use the PBC's to compute energies and momenta. First, consider the ground state. As $L \to \infty$, the modes along $\mathrm{Im}\,\beta = \pi$ approach a continuous distribution. Expecting $\Delta\alpha_i \equiv \alpha_i - \alpha_{i-1}$ to be $O(1/L)$, we define

$$\rho(\alpha) = \frac{1}{L\Delta\alpha} \quad . \tag{4.1}$$

If the modes filled for the vacuum are $\{i\pi + \alpha_i\}$, they satisfy

$$m_o \sinh \alpha_i = \frac{2\pi n_i}{L} + \frac{1}{L} \sum_j \phi(\alpha_i - \alpha_j) \quad . \tag{3.8a}$$

Subtracting the i^{th} equation (3.8a) from the (i - 1)st, we get

$$(m_o \cosh \alpha_i)(\Delta\alpha_i) = \frac{2\pi}{L} + \frac{1}{L} \sum_j [\phi(\alpha_i - \alpha_j) - \phi(\alpha_{i-1} - \alpha_j)]$$

$$m_o \cosh \alpha = 2\pi\rho(\alpha) + \sum_j \Delta\alpha_j \left(\frac{1}{L\Delta\alpha_j}\right) \frac{\partial}{\partial\alpha_i} \phi(\alpha_i - \alpha_j)$$

$$m_o \cosh \alpha = 2\pi\rho(\alpha) + \int_{-\Lambda}^{\Lambda} d\alpha' \rho(\alpha') \frac{\partial}{\partial\alpha} \phi(\alpha - \alpha') \quad . \tag{4.2}$$

Here a rapidity cutoff Λ has been introduced. The Dirac sea is filled up to rapidities of $\pm\Lambda$. This equation can be solved in the limit $\Lambda \to \infty$, yielding

$$\rho(\alpha) = m_o \left(\frac{\gamma - 1}{2 \sin 2\mu}\right) e^{(1-\gamma)\Lambda} \cosh \gamma\alpha \tag{4.3}$$

where γ satisfies

$$\frac{\sin(\pi - 2\mu)\gamma}{\sin \pi\gamma} = -1$$

Possible values of γ are $\frac{\pi}{2\mu}$, $\frac{\pi}{\pi - \mu}$, etc. In free field theory, we know $\rho_0(\alpha) = m_0 \cosh \alpha$. The only choice for γ yielding this in the limit $\mu \to \pi/2$ is $\gamma = \frac{\pi}{2\mu}$. This choice does not cross any other choice for $\mu > \pi/3$. In particular, it is the only value of γ that develops continuously from the free field value for $\mu > \pi/2$, the region of interest for the study of bound states. At $\mu = \pi/3$, the solution $\gamma = \pi/2\mu$ crosses the choice $\frac{\pi}{\pi - \mu}$. For $\mu \leq \pi/3$, the simple cutoff used in (4.2) must be modified. This problem is best understood by comparison with the eight-vertex model (see Sec. V). For the remainder of this Section we will assume $\mu > \pi/3$.

The bare mass, m_0 appears multiplied by a cutoff dependent factor $e^{(1-\gamma)\Lambda}$. In the remainder of this section we will study the excitation spectrum and find that the combination $m_0 e^{(1-\gamma)\Lambda}$ is finite and proportional to the physical mass in the limit $\Lambda \to \infty$. Thus, $\rho(\alpha)$ is completely cutoff independent, and, in fact, of the same form as $\rho_0(\alpha)$ ($\mu = \pi/2$). The only differences are the appearance of a renormalized mass and a rapidity rescaling ($\alpha \to \gamma\alpha$).

We now turn to computing the spectral properties of n-strings. The vacuum distribution will be affected by the presence of an n string. First of all, we must include the holes an n string requires. Recall that in the neutral charge sector with $\mu \in I_r$, n strings with $n = r + 2$ or $r + 1$ need $n - 2$ holes at $i\pi + \alpha_s$, and two others somewhere in the vacuum distribution. The n-strings with $n \leq r$ need n holes at $i\pi + \alpha_s$. In addition to introducing holes along the $i\pi$ line, the presence of an n string shifts the remaining α_i of the vacuum distribution. This results from the fact that the new Dirac sea satisfies PBC's with extra phase shifts from the n-string. The addition of a finite number of terms each $O(1/L)$ is expected to shift the α_i's of the undisturbed Dirac sea by O(1/L). Since the number of modes in the sea is O(L), this gives a contribution to the energy which is finite as $L \to \infty$. We will refer to this effect as the "backflow" of the Dirac sea. It must be taken into account if we are to compute the energies of states properly.

Consder an n-string whose real coordinate is α_s. The perturbed Dirac sea has modes at $\{i\pi + \bar{\alpha}_i\}$, where $\bar{\alpha}_i$ are real. There are n holes at $\beta_1^{(h)} = i\pi + \alpha_1$, $\beta_2^{(h)} = i\pi + \alpha_2$, and $\beta_3^{(h)} = \ldots = \beta_n^{(h)} = i\pi + \alpha_s$. For $n \leq r$, $\alpha_1 = \alpha_2 = \alpha_s$. The undisturbed vacuum has particles of rapidity $\{i\pi + \alpha_i\}$. The PBC's for the sea modes in an n string state are

$$m_o \sinh \bar{\alpha}_i = \frac{2\pi \bar{n}_i}{L} + \frac{1}{L} \sum_j \phi(\bar{\alpha}_i - \bar{\alpha}_j) + \frac{1}{L} \Phi_n^{(s)}(\bar{\alpha}_i, \alpha_s) \tag{4.3}$$

where $\Phi_n^{(s)}$ is given by Eq. (3.7). An integral equation describing the backflow is obtained by subtracting the PBC's for the undisturbed Dirac sea from those of the excited state, Eq. (4.3). If $\bar{n}_i' = n_i$, we subtract the $(i')^{th}$ Eq. (4.3) from the $(i)^{th}$ Eq. (3.8a). This is shown in Fig. 6. Let $\omega(\alpha_i) = (\bar{\alpha}_i' - \alpha_i)L$. $(\bar{\alpha}_i' - \alpha_i)$ is expected to be O(1/L), so $\omega(\alpha)$ should have a finite limit. Then,

$$\left(m_o \cosh \alpha_i\right) \frac{\omega(\alpha_i)}{L} = \frac{1}{L} \sum_j \left[\frac{\partial}{\partial \alpha_i} \phi(\alpha_i - \alpha_j)\right] \left[\frac{\omega(\alpha_i)}{L} - \frac{\omega(\alpha_j)}{L}\right]$$

$$+ \frac{1}{L} \Phi_n(\alpha_i, \alpha_s) - \frac{1}{L}(n-2)\phi(\alpha_i - \alpha_s) - \frac{1}{L}\phi(\alpha_i - \alpha_1) - \frac{1}{L}\phi(\alpha_i - \alpha_2)$$

$$\left(m_o \cosh \alpha\right) \omega(\alpha) = \int d\alpha' \rho(\alpha') \left[\omega(\alpha) - \omega(\alpha')\right] \frac{\partial}{\partial \alpha} \phi(\alpha - \alpha') + \Phi_n(\alpha, \alpha_s)$$

$$- (n-2)\phi(\alpha - \alpha_s) - \phi(\alpha - \alpha_1) - \phi(\alpha - \alpha_s) \quad .$$

Using Eq. (4.2), we finally obtain

$$2\pi F(\alpha) = \Phi_n(\alpha, \alpha_s) - (n-2)\phi(\alpha - \alpha_s) - \phi(\alpha - \alpha_1) - \phi(\alpha - \alpha_2)$$

$$- \int d\alpha' F(\alpha') \frac{\partial}{\partial \alpha} \phi(\alpha - \alpha') \quad . \tag{4.4}$$

Here, the limits of integration can be extended to infinity since the integral is found to be convergent. Now we have a simple convolution equation which can be solved via Fourier transformation. Letting

$$\tilde{F}(y) = \int d\alpha \, e^{-i\alpha y} F(\alpha) \tag{4.5}$$

we find that

$$\tilde{F}(y) = \frac{\tilde{I}(y; \alpha_s, \alpha_1, \alpha_2)}{2\pi(1 + \tilde{K}(y))} \tag{4.6}$$

where

$$I(\alpha; \alpha_s, \alpha_1, \alpha_2) = \Phi_n(\alpha, \alpha_s) - (n-2)\phi(\alpha - \alpha_s) - \phi(\alpha - \alpha_1) - \phi(\alpha - \alpha_2) \tag{4.7}$$

and $\tilde{I}(y; \alpha_s, \alpha_1, \alpha_2)$ is its Fourier transform. Also,

$$\tilde{K}(y) = \frac{1}{2\pi} \int d\alpha \, e^{-i\alpha y} \frac{\partial}{\partial \alpha} \phi(\alpha) = \frac{\sinh(\pi - 2\mu)y}{\sinh \pi y} \quad . \tag{4.8}$$

$F(\alpha)$ can be found by inverting the Fourier transform (4.6). Using this result, we can compute the energy and momentum E_n and P_n of an n-string state relative to the ground state. The energy is obtained by subtracting eigenvalues of the form (2.20), giving

$$E_n = \sum_{\text{string}} m_o \cosh \beta_i^{(s)} - \sum_{\text{holes}} m_o \cosh \beta_i^{(h)} + \sum_{\text{sea}} [m_o \cosh \overline{\beta}_i - m_o \cosh \beta_i]$$

$$= m_o \sum_{\text{string}} \cosh \beta_i^{(s)} + m_o [(n-2)\cosh \alpha_s + \cosh \alpha_1 + \cosh \alpha_2] + B_n \qquad (4.9a)$$

where $\beta_i^{(s)}$ and $\beta_i^{(h)}$ are the positions of the n-string modes and holes respectively, i.e. $\beta_\ell^{(s)} = \alpha_s + i\ell(\pi - \mu)$ with $\ell = (n-1), (n-3),\ldots,-(n-1)$, $\beta_1^{(h)} = i\pi + \alpha_1$, $\beta_2^{(h)} = i\pi + \alpha_2$, $\beta_3^{(h)} = \ldots = \beta_n^{(h)} = i\pi + \alpha_s$ with $\alpha_1 = \alpha_2 = \alpha_2$, if $n \leq r$. Similarly, the momentum of an excitation is

$$P_n = \sum_{\text{string}} m_o \sinh \beta_i^{(s)} - \sum_{\text{holes}} m_o \sinh \beta_i^{(h)}$$

$$+ \sum_{\text{sea}} [m_o \sinh \overline{\beta}_i - m_o \sinh \beta_i] \quad . \qquad (4.9b)$$

The first two terms in (4.9a) will be called the "bare" energy of the n-string and holes. The last term constitutes the backflow energy of the Dirac sea. For $L \to \infty$ it can be written

$$B_n = m_o \sum_{\text{sea}} [\cosh \overline{\beta}_i - \cosh \beta_i]$$

$$= - m_o \int_{-\Lambda}^{\Lambda} d\alpha F(\alpha) \sinh \alpha \qquad (4.10)$$

where we have used $\beta_i = i\pi + \alpha_i$, α_i real. The backflow momentum may also be written as an integral over $F(\alpha)$. Note that these integrals require a cutoff, which will ultimately be absorbed into a renormalized mass. As in the ground state integral equation, a sharp cutoff in rapidity is suggested by comparison with the 8-vertex model (see Section V).

To compute the backflow integrals, we will use Eq. (4.6) directly by introducing the Fourier transform $\tilde{F}(y)$ into (4.10) and doing the α integral first. The Fourier transforms of Φ_n and ϕ which appear in Eq. (4.6) are given by

$$\tilde{\Phi}_n(y, \alpha_s) = \frac{2\pi}{iy} e^{i\alpha_s y} \frac{1}{\sinh \pi y} \Big[\sinh [(n-1)\pi - (n+1)\mu] y + \sinh (n-1)(\pi - \mu)y - (n-2)\sinh \pi y \Big] \qquad n = r+1,\ r+2 \tag{4.11a}$$

$$= \frac{2\pi}{iy} e^{i\alpha_s y} \frac{1}{\sinh \pi y} [\sinh (n+1)(\pi - \mu)y + \sinh (n-1)(\pi - \mu)y - n \sinh \pi y] \qquad n \leq r \tag{4.11b}$$

$$\tilde{\phi}(y, \alpha_1) = \frac{2\pi}{iy} e^{i\alpha_1 y} \frac{1}{\sinh \pi y} \sinh (\pi - 2\mu)y \qquad . \tag{4.11c}$$

Now the backflow energy can be written as the difference of two terms

$$B_\pm = -\frac{m_o}{2} \int_{-\Lambda}^{\Lambda} d\alpha\, e^{\pm\alpha} F(\alpha) \qquad .$$

Carrying out the α integration, this gives

$$B_+ = -\frac{m_o}{2} \int_{-\Lambda}^{\Lambda} d\alpha\, e^{\alpha} \int_{-\infty}^{\infty} \frac{dy}{2\pi} e^{i\alpha y} \frac{\tilde{I}(y; \alpha_s, \alpha_1, \alpha_2)}{2\pi(1 + \tilde{K}(y))}$$

$$= -\frac{m_o}{2} e^{\Lambda} \int \frac{dy}{2\pi} e^{i\Lambda y} \frac{1}{1+iy} \frac{\sinh \pi y}{2 \sinh (\pi - \mu)y \cosh \mu y} \frac{I(y)}{2\pi} \qquad . \tag{4.12}$$

There is also a term proportional to $e^{-\Lambda}$ which vanishes in the limit $\Lambda \to \infty$. The y integral may now be done by contour integration, closing the contour in the upper half plane. For $\mu > \pi/3$, the leading poles are at $y = i$ and $y = \frac{i\pi}{2\mu}$. Other poles give contributions which are down by positive powers of $e^{-\Lambda}$. It is found that the contribution of the $y = i$ pole (and $y = -i$ pole from B_-) exactly cancels the bare energy. The exact expression for the energy comes entirely from the $y = \pm\frac{i\pi}{2\mu}$ poles.

The $y = \pm i$ poles contribute to the total backflow energy

$$B_n(y = \pm i) = -m_o\left[\cosh\alpha_s \frac{\sin n(\pi - \mu)}{\sin\mu} + (n-2)\cosh\alpha_s + \cosh\alpha_1 + \cosh\alpha_2\right]. \quad (4.13)$$

Eq. (4.13) holds for all n-strings. It is not hard to add the energies of the modes of an n string. The bare energy thus obtained is

$$m_o \sum_{\text{string}} \cosh\beta_i^{(s)} = m_o \frac{\sin n(\pi - \mu)}{\sin\mu}\cosh\alpha_s \quad . \quad (4.14)$$

From (4.13), (4.14), and (4.9a) it is seen that the $y = \pm i$ poles exactly cancel the bare energy of the n-string and holes. One may obtain similar results for the momentum. Now we compute the contribution to the backflow energy from the poles at $y = \pm\frac{i\pi}{2\mu} \equiv \pm i\gamma$. A straightforward evaluation yields, for $n = r + 1$ or $r + 2$,

$$B_n(y = \pm i\gamma) = +\frac{m_o e^{\Lambda(1-\gamma)}}{\pi(1-\gamma)} \tan\pi\gamma\ [\cosh\gamma\alpha_1 + \cosh\gamma\alpha_2] \quad .$$

Now define the renormalized mass

$$m = m_o \frac{\tan\pi\gamma}{\pi(\gamma - 1)} e^{\Lambda(1-\gamma)} \quad . \quad (4.15)$$

(Note that $m > 0$ since we have assumed $\mu > \pi/3$.) The energy of an n-string with $n = r + 2$ or $r + 1$, and holes at $i\pi + \alpha_1$ and $i\pi + \alpha_2$ is then given by

$$E = m(\cosh\gamma\alpha_1 + \cosh\gamma\alpha_2) \quad . \quad (4.16a)$$

It is worth remarking that this expression is independent of α_s. We interpret (4.16a) as the energy of an unbound fermion-antifermion pair. A similar computation shows

$$P = m(\sinh \gamma\alpha_1 + \sinh \gamma\alpha_2) \tag{4.16b}$$

For $n \leq r$, we know $\alpha_1 = \alpha_2 = \alpha_s$. From (4.11b) we obtain

$$E = m \cosh \gamma\alpha_s \left\{ \frac{\sin[(r+1)(\pi-\mu)\gamma]}{\sin \pi\gamma} + \frac{\sin[(n-1)(\pi-\mu)\gamma]}{\sin \pi\gamma} - n + n \right\}$$

$$= m \cosh \gamma\alpha_s \left\{ \frac{\sin[n(\pi-\mu)\gamma] \cos[(\pi-\mu)\gamma]}{\sin \pi\gamma} \right\}$$

which gives

$$E = m \sin\left[\frac{n\pi}{2}\left(\frac{\pi}{\mu} - 1\right)\right] \cosh \gamma\alpha_s \tag{4.17a}$$

$$P = m \sin\left[\frac{n\pi}{2}\left(\frac{\pi}{\mu} - 1\right)\right] \sinh \gamma\alpha_s \quad . \tag{4.17b}$$

This is precisely the bound state spectrum of the sine-Gordon theory computed in the WKB approximation by Dashen, Hasslacher and Neveu.[12] The parameter μ is related to the g used in Ref. 3 by

$$\mu = \frac{\pi}{2}\left(\frac{2g+\pi}{g+\pi}\right) \quad . \tag{4.18}$$

In general terms, the effect of the Dirac sea has again been to renormalize the mass and rescale the rapidity. The renormalized mass in Eq. (4.15) has precisely the cutoff dependence found in the coefficient of the vacuum density of states Eq. (4.3). Furthermore, the rapidity for these excitations is rescaled by the same quantity as the rapidity for the distribution of the Dirac sea.

V. THE MASSIVE THIRRING MODEL AND THE EIGHT-VERTEX LATTICE NEAR THE CRITICAL POINT

In the preceding sections we have described an exact field theoretic treatment of the massive Thirring model. The method begins with a diagonalization of the Hamiltonian (2.1) in an unphysical Hilbert space built upon the state $|0\rangle$ defined by $\psi_1(x)|0\rangle = \psi_2(x)|0\rangle = 0$. The physical vacuum is then described by a Bethe wave function with all negative energy modes filled. A consistent formulation of this step requires the introduction of a rapidity cutoff Λ. The structure of the excitation spectrum is delineated by the periodic boundary conditions, which also serve as the essential tool for computing physical quantities such as the energy and momentum of an excitation. To gain a broader perspective on these results, it is of great interest to explore the connection originally discussed by Luther[6] which relates the massive Thirring model to a continuum limit of the XYZ spin chain and thus to the critical behavior of the Baxter 8-vertex lattice. In fact, Luther's analysis and the lattice techniques of Baxter[4] and of Johnson, Krinsky and McCoy[5] provided much of the inspiration for the methods described in this paper. A comparison of these models provides an instructive example of the relationship between lattice statistics and quantum field theory and also allays possible suspicions about the legitimacy of the rapidity cutoff procedure used in section IV to regularize the divergence associated with mass renormalization. The rapidity cutoff parameter Λ is found to be closely related to the elliptic modulus which measures the temperature interval $(T - T_c)$ in Baxter's parametrization of the 8-vertex model. The limit $\Lambda \to \infty$ in the field theory can then be associated with the approach to the critical point in the lattice problem.

The main intent of this section is to relate the periodic boundary conditions for the massive Thirring model, eq. (3.3), and the subsequent spectral calculations to the developments which follow from Baxter's fundamental equation for the eigenvalues of the 8-vertex model transfer matrix. We begin with a brief discussion of methods and results for the 8-vertex model and XYZ spin chain, emphasizing only those aspects which relate directly to the massive Thirring model. The 8-vertex model is formulated on a square lattice with toroidal boundary conditions and horizontal and vertical bonds connecting adjacent lattice sites (vertices). An arrow is placed on each bond, with the only allowed configurations having an even number of arrows pointing into each vertex. This limits the allowed vertex configurations to the eight shown in Fig. 7. Associating energies E_j, $j = 1,\dots,8$ with these vertices, the symmetric 8-vertex model is defined by the four vertex weights

$$\omega_1 = \omega_2 = a \qquad \omega_3 = \omega_4 = b$$

$$\omega_5 = \omega_6 = c \qquad \omega_7 = \omega_8 = d \tag{5.1}$$

where

$$\omega_j = e^{-\beta E_j} \quad . \tag{5.2}$$

Particularly useful combinations of these weights are

$$w_1 = \tfrac{1}{2}(c + d) \qquad w_2 = \tfrac{1}{2}(c - d)$$

$$w_3 = \tfrac{1}{2}(a - b) \qquad w_4 = \tfrac{1}{2}(a + b) \quad . \tag{5.3}$$

By standard manipulations, the problem of calculating the partition function can be reduced to that of finding the largest eigenvalue of the transfer matrix $\underline{T}$.

For a lattice with N sites per row, $\underline{T}$ is a $2^N \times 2^N$ matrix labeled by the configurations of two adjacent rows of vertical arrows. It is given by the product of vertex weights in a row summed over horizontal arrow configurations, specifically

$$T_{\alpha|\alpha'} = \mathrm{Tr}\left\{ \underline{R}(\alpha_1, \alpha_1')\underline{R}(\alpha_2, \alpha_2')\ldots\underline{R}(\alpha_N, \alpha_N') \right\} \quad . \tag{5.4}$$

Here $\underline{R}(\alpha, \alpha')$ is a 2×2 matrix representing an elementary vertex. It is conveniently written by introducing the Pauli matrices $\sigma^1, \sigma^2, \sigma^3$, and the unit matrix σ^4,

$$R(\alpha, \alpha')_{\lambda\lambda'} = \sum_{j=1}^{4} w_j \; \sigma^j_{\alpha\alpha'} \; \sigma^j_{\lambda\lambda'} \quad . \tag{5.5}$$

The partition function is then given by

$$Z = \mathrm{Tr}\, \underline{T}^M \quad , \tag{5.6}$$

where M is the number of rows. The partition function has important symmetry properties which may be derived from the expressions (5.4)-(5.6). Considering Z as a function of the four w's, one finds the following symmetries:

$$Z(w_1, w_2, w_3, w_4) = Z(\pm w_i, \; \pm w_j, \; \pm w_k, \; \pm w_\ell) \tag{5.7}$$

where i,j,k,ℓ is any permutation of 1,2,3,4. Included in (5.7) is the self-duality property which relates eight-vertex models below T_c to eight-vertex models above T_c. This is the generalization of the famous Kramers-Wannier symmetry of the two-dimensional Ising model.

The method devised by Baxter for obtaining the eigenvalues of $\underline{T}$ is facilitated by parametrizing the vertex weights in terms of elliptic functions. This introduces parameters η, v, and k which determine the relative size of the vertex weights by

$$a : b : c : d \; = \; \mathrm{sn}(v+\eta,k) : \mathrm{sn}(v-\eta,k) : \mathrm{sn}(2\eta,k) : k\,\mathrm{sn}(2\eta,k)\mathrm{sn}(v-\eta,k)\mathrm{sn}(v+\eta,k) \quad . \tag{5.8}$$

For discussing phase transitions it is also convenient to use a related set of parameters ζ, V, and ℓ, given by

$$V \; = \; - i(1 + k)\, v \tag{5.9a}$$

$$\zeta \; = \; - i(1 + k)\eta \tag{5.9b}$$

$$\ell \; = \; \frac{1 - k}{1 + k} \quad . \tag{5.9c}$$

Using the properties of elliptic functions under the change of modulus (5.9c),[18] the ratios of w's may be obtained from (5.3) and (5.8), giving

$$w_1 : w_2 : w_3 : w_4 \; = \frac{\mathrm{cn}(V, \ell)}{\mathrm{cn}(\zeta, \ell)} : \frac{\mathrm{dn}(V, \ell)}{\mathrm{dn}(\zeta, \ell)} : 1 : \frac{\mathrm{sn}(V, \ell)}{\mathrm{sn}\,(\zeta, \ell)} \quad . \tag{5.10}$$

Baxter showed that the parameter V plays a special role. Any two transfer matrices $\underline{T}(V)$ and $\underline{T}(V')$ with the same ζ and ℓ will commute for arbitrary values of V and V'. This means that for fixed ζ and ℓ, the matrix $\underline{T}(V)$ is diagonalized by a set of eigenvectors which are independent of V. In a related development, Baxter obtained a precise relationship between the eight-vertex model transfer matrix and the Hamiltonian of the XYZ spin chain. The latter is given by

$$\mathscr{H} = -\tfrac{1}{2}\sum_{n=1}^{N}\{J_x\sigma_n^1\sigma_{n+1}^1 + J_y\sigma_n^2\sigma_{n+1}^2 + J_z\sigma_n^3\sigma_{n+1}^3\} \tag{5.11}$$

where $\sigma_{N+1}^j \equiv \sigma_1^j$. If the constants J_x, J_y, and J_z are parametrized in terms of ζ and ℓ by

$$J_x : J_y : J_z = \mathrm{cn}(2\zeta,\ell) : \mathrm{dn}(2\zeta:\ell) : 1 \quad , \tag{5.12}$$

then the Hamiltonian (5.11) is obtained from the transfer matrix (5.4) by the formula

$$\mathscr{H} = -J_z\,\mathrm{sn}(2\zeta,\ell)\left\{\frac{d}{dV}\ln \underline{T}(V)\Big|_{V=\zeta} - \tfrac{1}{2}N[\mathrm{cn}(2\zeta,\ell) + \mathrm{dn}(2\zeta,\ell) - 1]/\mathrm{sn}(2\zeta,\ell)\right\}. \tag{5.13}$$

In order to restrict the range of parameters we wish to consider, let us recall the relationship between the Hamiltonian (5.11) and that of the massive Thirring model.[6] The essential connection is established by converting spin operators $\sigma^\pm \equiv \tfrac{1}{2}(\sigma^1 \pm i\sigma^2)$ into fermion creation and annihilation operators via a Jordan-Wigner transformation. Under this transformation, the J_x and J_y terms in (5.11) produce the kinetic energy and mass terms for the fermion Hamiltonian, while the J_z term becomes a four-fermion interaction. In order for this particular identification of spin and fermion operators to lead to a sensible quantum field theory, J_z must be smaller in magnitude than J_x and J_y, with J_x and J_y becoming equal in the continuum (scaling) limit. This can be accomplished in Eq. (5.12) by choosing ζ to be pure imaginary. Defining

$$\zeta' = -i\zeta \tag{5.14a}$$

$$\ell' = (1-\ell^2)^{1/2} \tag{5.14b}$$

and using identities involving elliptic functions of imaginary argument,[19] (5.12) can be rewritten

$$J_x : J_y : J_z = 1 : \mathrm{dn}(2\zeta', \ell') : \mathrm{cn}(2\zeta', \ell') \quad . \tag{5.12'}$$

If we choose ζ' real and $0 < \ell' < 1$, we find

$$J_x > J_y > |J_z| \tag{5.15}$$

with $J_x/J_y \to 1$ as $\ell' \to 0$.

By similar manipulations, the ratio of w's in the corresponding eight vertex model becomes

$$w_1 : w_2 : w_3 : w_4 = 1 : \frac{\mathrm{dn}(V', \ell')}{\mathrm{dn}(\zeta', \ell')} : \frac{\mathrm{cn}(V', \ell')}{\mathrm{cn}(\zeta', \ell')} : \frac{\mathrm{sn}(V', \ell')}{\mathrm{sn}(\zeta', \ell')} \tag{5.10'}$$

where $V' = -iV$. The massive Thirring model is associated with an eight-vertex model near the critical point. If we restrict our consideration to V', ζ' real and in the region

$$|V'| < \zeta' < K_{\ell}' \tag{5.16}$$

with

$$0 < \ell' < 1 \quad ,$$

then we find

$$w_3 > w_2 > w_1 > |w_4| \quad . \tag{5.17}$$

For any arrangement of w's, a phase transition takes place when, and only when, the middle two w's cross.[4] Thus, in the region (5.16), the critical point is approached when $w_2 \to w_1$, i.e. $\ell' \to 0$. The region (5.17) differs from the "fundamental region" considered in Refs. 4 and 5. They are related by a dual transformation which interchanges w_1 and w_3.

The fundamental equation upon which eight-vertex model calculations are based is Baxter's[4] relation for the eigenvalues T(v) of the transfer matrix. These eigenvalues are found to satisfy the equation

$$T(v)Q(v) = \phi(v+\eta)Q(v-2\eta) + \phi(v-\eta)Q(v+2\eta) \tag{5.18}$$

where[20]

$$\phi(v) = [\tilde{\rho}\, H(v)]^N \tag{5.19a}$$

$$Q(v) = \prod_{j=1}^{n} H(v-v_j) \tag{5.19b}$$

and H(v) is a Jacobi eta function of modulus ℓ'. Here the v_j's may be associated with the set of occupied modes in a particular eigenstate. $\tilde{\rho}$ is a normalization constant which depends on the elliptic modulus but not on η or v. It need not be specified further for the purposes of our discussion.

The calculation of an eigenvalue T(v) from Eq. (5.18) proceeds in two stages. The first step is to determine the allowed values of v_j. To do this note that H(0) = 0 and hence $Q(v_i) = 0$, $i = 1,\ldots,n$. Evaluating (5.18) at each v_i, the left-hand side vanishes and we obtain a set of n equations for the v_j's,

$$\left[\frac{H(v_i+\eta)}{H(v_i-\eta)}\right]^N = -\prod_{j=1}^{n}\left[\frac{H(v_i-v_j+2\eta)}{H(v_i-v_j-2\eta)}\right] \quad . \tag{5.20}$$

It will be shown that, in the appropriate limit, Eq. (5.20) reduces to the periodic boundary conditions for the massive Thirring model, Eq. (3.2). The second step in computing $T(v)$ is to evaluate (5.19b) using the v_j's determined from (5.20) and then obtain $T(v)$ from (5.18). The corresponding eigenvalue of the XYZ Hamiltonian is given by (5.13). This will reduce to a mode sum identical to those which arise as eigenvalues of the massive Thirring model Hamiltonian.

To understand the connection between (5.20) and the periodic boundary conditions of the Thirring model, we must study the function

$$F_m(v) = \ln\left[\frac{H(v+m\eta)}{H(v-m\eta)}\right] \tag{5.21}$$

where the H's are of modulus ℓ'. Since we are interested in the behavior of $F_m(v)$ as $\ell' \to 0$, it is convenient to consider its expansion in powers of the nome q, where

$$q = \exp\{-\pi K_\ell / K_\ell'\} \quad . \tag{5.22}$$

Note that as $\ell' \to 0$, $K_\ell' \to \pi/2$, $K_\ell \to \ln(4/\ell') \to \infty$, and hence

$$q \xrightarrow[\ell' \to 0]{} \frac{\ell'^2}{16} \to 0 \quad . \tag{5.23}$$

Equation (5.21) can be expanded in powers of q,[19]

$$F_m(v) = \ln\left[\frac{\sin(\tilde{v}+m\tilde{\eta})}{\sin(\tilde{v}-m\tilde{\eta})}\right] + 4\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\sin 2n\tilde{v}\sin 2nm\tilde{\eta} \tag{5.24a}$$

where

$$\tilde{v} = \frac{\pi v}{2K_\ell'} \tag{5.24b}$$

$$\tilde{\eta} = \frac{\pi\eta}{2K_\ell'} \quad . \tag{5.24c}$$

The right-hand side of Eq. (5.20) reduces straightforwardly to the exponentiated sum of phase shifts on the right-hand side of Eq. (3.2). The eight-vertex model parameters v and η are related to rapidity and coupling constant (respectively) in the massive Thirring model. Making the replacement

$$(v_i - v_j) \rightarrow -\frac{i}{2}(\beta_i - \beta_j) \tag{5.25a}$$

$$\eta \rightarrow \tfrac{1}{2}(\pi - \mu) \tag{5.25b}$$

and using (5.23) we find, as $\ell' \rightarrow 0$,

$$F_2(v_i - v_j) \rightarrow \ln\left\{-\frac{\sinh \tfrac{1}{2}(\beta_i - \beta_j - 2i\mu)}{\sinh \tfrac{1}{2}(\beta_i - \beta_j + 2i\mu)}\right\} \tag{5.26}$$

which is precisely the two-body phase shift of the massive Thirring model, Eq. (2.15c).

The reduction of the left-hand side of (5.20) to the corresponding Thirring model expression is somewhat more delicate. This results from the fact that the appropriate limit not only takes $\ell' \rightarrow 0$, but simultaneously $v_i \rightarrow i\infty$. We will find that the proper replacement is

$$v_i \rightarrow -\frac{i}{2}\beta_i + iK_\ell \tag{5.27}$$

with β_i = rapidity remaining finite in the continuum limit. (Note that $K_\ell \to \infty$ as $\ell' \to 0$.) Using (5.27), the desired result follows by keeping both the logarithmic term and the n = 1 term in the sum in Eq. (5.24a), which reduces to

$$F_1(v_i) \to -i(\pi-\mu) - 2ie^{2iv}\sin 2\eta + 2iq^2e^{-2iv}\sin 2\eta$$

$$\to -i(\pi-\mu) - \frac{i\ell'^2}{16}\sin\mu\,\sinh\beta_i \quad . \tag{5.28}$$

The first term in (5.28) represents an overall shift in the origin of momentum space associated with the fact that the "Fermi surface" of the occupied modes in the ground state is at $(\pi - \mu)$. The second term reproduces the Thirring model exponent on the left-hand side of Eq. (3.2), specifically,

$$NF_1(v_i) \to -iL(k_F + m_o \sinh\beta_i) \quad . \tag{5.29}$$

Here,

$$k_F = (\pi-\mu)/a \tag{5.30a}$$

$$m_o = \frac{\ell'^2}{16a}\sin\mu \quad , \tag{5.30b}$$

where a = L/N is the lattice spacing. This dependence of the bare mass on the vanishing elliptic modulus and on the lattice spacing may also be obtained directly from the XYZ Hamiltonian.[6] The constant term k_F in (5.29) has no effect on the integral equations, which are derived by considering differences of PBC's.

Using the prescription (5.27) we can also consider the contribution of an occupied mode to the energy of a spin chain in the continuum limit. It can be shown that,[4,5] when $0 < \mathrm{Im}\, v < \frac{1}{2}K'_k$, the first term on the right-hand side of (5.18) dominates the second term exponentially for large N. Thus, after determining the v_j's, we may compute the eigenvalue of the transfer matrix from

$$T(v) \approx \phi(v+\eta)\frac{Q(v-2\eta)}{Q(v)} \tag{5.31}$$

in the limit $N \to \infty$. The energy E of the corresponding spin chain state is found from (5.13). Changing to real parameters, this gives

$$E = \tfrac{1}{2}(1+\ell)J_x \mathrm{sn}(2\zeta', \ell') \left\{ \frac{d}{dv} \ln \left[\frac{Q(v-2\eta)}{Q(v)}\right]\right\}\Bigg|_{v=\eta} + (\text{const.}) \tag{5.32}$$

where (const.) includes those terms which are independent of the eigenstate being considered. From (5.32) and (5.19b) it is seen that the energy can be written as a sum over occupied modes, where the contribution of mode j is given by

$$\varepsilon_j = -\tfrac{1}{2}(1+\ell)J_x \mathrm{sn}(2\zeta', \ell')\left[\frac{d}{dv} F_1(v-v_j-\eta)\right]_{v=\eta} \quad . \tag{5.33}$$

Using (5.29) and letting

$$J_x = \frac{1}{2a \sin\mu} \quad , \tag{5.34}$$

Eq. (5.33) becomes, in the limit $\ell' \to 0$,

$$\varepsilon_j = m_o \cosh\beta_j \quad . \tag{5.35}$$

The choice (5.34), which fixes the overall normalization of the spin chain Hamiltonian, may be better understood by considering two special cases, the XY chain ($J_z = 0$, free fermions) and the XXZ($J_x = J_y$, massless Thirring model). For the XY case, the energy of a single mode is[21]

$$\varepsilon_{XY} = \frac{2J_x}{1+\gamma} [\gamma^2 + (1-\gamma^2)\cos^2(ka + \pi/2)]^{1/2} \tag{5.36}$$

where $\gamma = (J_y - J_x)$ and the momentum k is measured from the Fermi surface at $\pi/2a$. In the continuum limit $\gamma \to 0$, $a \to 0$, $\gamma/a = m_o$ fixed, this becomes

$$\varepsilon_{XY} \to 2aJ_x[k^2 + m_o^2]^{1/2} \quad . \tag{5.37}$$

The proper relation between energy and momentum leads to the choice $J_x = 1/(2a)$. This is just (5.34) for $\mu = \pi/2$. In the XXZ chain, the mode energy is[17]

$$\varepsilon_{XXZ} = -2J_x[\cos\mu + \cos(ka + \pi - \mu)]$$

(again measuring momentum from the Fermi surface at $(\pi - \mu)/a$). The continuum limit in this case is just $a \to 0$, whereupon

$$\varepsilon_{XXZ} \to (2aJ_x \sin\mu) \times k \quad . \tag{5.38}$$

Again we are led to the choice (5.34) by requiring the proper energy-momentum relationship for a massless particle. Finally, we note that, as $\ell' \to 0$,

$$J_z = \mathrm{cn}(2\zeta', \ell')J_x \to \frac{1}{2a}\cot\mu \tag{5.39}$$

which accounts for our choice $\frac{1}{2}g_o = -\cot\mu$ in the continuum theory, Eq. (2.15b).

Having established the connection between the eight-vertex model formalism near the critical point and the corresponding results for the massive Thirring model, we can now gain a better perspective on the rapidity cutoff procedure used in Section IV. The limits $\pm\Lambda$ on rapidity integrations (mode sums) which were imposed somewhat artificially in the massive Thirring model arise naturally in the lattice theory as Brillouin zone boundaries. To see this, we first note that the quasiperiodicity of the Jacobi eta function of modulus ℓ',

$$H(v + 2iK_\ell) = -q^{-1}\exp(-i\pi v/K_\ell)H(v) \quad , \tag{5.40}$$

implies that the functions defined by (5.21) have the property

$$F_m(v + 2iK_\ell) = -(2im\pi\eta/K_\ell') + F_m(v) \quad . \tag{5.41}$$

If we now consider the mode energy (5.33) as a function of the rapidity β which is related to v by (5.27), it is seen to be periodic

$$\varepsilon(\beta - 2K_\ell) = \varepsilon(\beta + 2K_\ell) \quad . \tag{5.42}$$

The ground state of the XYZ chain is obtained by filling all modes within a single period along the line $\beta = \alpha + i\pi$, which we can choose to be

$$-2K_\ell < \alpha < 2K_\ell \quad . \tag{5.43}$$

Thus, the rapidity cutoff is related to the elliptic modulus by $\Lambda \leftrightarrow 2K_\ell$.

The difficulties which were encountered in the continuum theory for $\mu < \pi/3$ can be understood in the lattice formalism as resulting from an illegitimate interchange between the $\ell' \to 0$ limit and a mode summation. For example, consider the integrand in the expressions for the backflow energy, eq. (4.12). In addition to $\tilde{I}(y)$, there is a factor $[1 + \tilde{K}(y)]^{-1}$ and a factor coming from the cutoff Fourier transform of $\sinh\alpha$. The difficulty for $\mu < \pi/3$ stems from the fact that the poles in $1/\sinh[(\pi - \mu)y]$ cross the $y = i\pi/2\mu$ pole of $[\cosh\mu y]^{-1}$ and become the dominant singularities. In the lattice theory this does not happen. Instead of a Fourier transform of $\sinh\alpha$, the integrand in this case contains a Fourier transform of the full F_1 function (c.f. eq. (5.29)). This introduces a factor $\sinh(\pi - \mu)y/\sinh \pi y$ which cancels the $[\sinh(\pi - \mu)y]^{-1}$. As a result, the pole at $y = i\pi/2\mu$ always dominates. It seems that the continuum rapidity cutoff procedure used in Sec. IV is inadequate for the range of coupling $0 < \mu < \pi/3$ (which corresponds to $-\pi/2 < g < -\pi/4$ in the notation of Ref. 3). For this range a more sophisitcated cutoff (such as putting the theory on a lattice) is required. It may be that, by paying closer attention to the infinite set of conserved currents in the massive Thirring model, a consistent cutoff scheme could be devised for $\mu < \pi/3$ without resort to a lattice. Since for $\mu < \pi/3$, certain anomalous dimensions become large (for instance, $\gamma = (3/2)$ at $\mu = \pi/3$), there may be other operators which must be included in the Hamiltonian in order to render the $\Lambda \to \infty$ limit physically acceptable. It is likely that a careful analysis of the criticial limit of the <u>operators</u> of the lattice theory would show the presence of such effects. Indeed, such effects are known to be present in this model.[22] We believe that a thorough resolution of this problem requires the calculation of Green's functions. For this, an operator formulation of Bethe's ansatz would be desirable. In any case it is worth re-emphasizing that, for a broad range of coupling $\pi/3 < \mu < \pi$

$(-\pi/4 < g < +\infty)$, the continuum methods presented in Secs. II-IV provide a consistent and exact treatment of the theory which agrees with the appropriate limit of the lattice theory and with other known results for the massive Thirring model. This encourages the application of such methods to other theories which are known or suspected to have an infinite number of conservation laws but for which a soluble lattice theory is not available.

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APPENDIX

In section II, we demonstrated that a certain Bethe ansatz state in the N = 2 sector was an eigenstate of H. In this appendix, we will carry out the demonstration for a general state labeled by rapidities. Dealing with rapidity rather than momentum allows us to consolidate all possible cases of "1" and "2" particles. In terms of rapidity, we have

$$\Psi(x, \alpha) = \left(e^{\frac{1}{2}\alpha}\psi_1(x) + e^{-\frac{1}{2}\alpha}\psi_2(x) \right) \quad . \tag{A.1}$$

This is obtained from Eq. (2.8), $\tan 2\Theta_k = (m_o/k)$, and the definition of rapidity. The eigenstate is then

$$|\alpha_1 \ldots \alpha_n\rangle = \int dx_1 \ldots dx_n \, \chi(x_1 \ldots x_n) \prod_{i=1}^{n} \Psi^\dagger(x_i, \alpha_i)|0\rangle \tag{A.2}$$

with

$$\chi(x_1 \ldots x_n) = e^{im_o \sum_i x_i \sinh \alpha_i} \prod_{1 \le i < j \le n} \left(1 + i\lambda(\alpha_i, \alpha_j)\varepsilon(x_i - x_j) \right) \tag{A.3}$$

and

$$\lambda(\alpha_i, \alpha_j) = -\tfrac{1}{2}g_o \tanh \tfrac{1}{2}(\alpha_i - \alpha_j) \quad . \tag{A.4}$$

First, we apply the kinetic term in the Hamiltonian, commuting $\psi_i^\dagger \nabla \psi_i$ through the product of $\Psi^\dagger$'s to $|0\rangle$. The result after integrating by parts is

$$H_k|\alpha_1\ldots\alpha_n\rangle = \int dx_1\ldots dx_n \sum_{i=1}^{n}(-i\nabla_i\chi)\left[\psi^\dagger(x_1\alpha_1)\ldots\left(e^{\frac{1}{2}\alpha_i}\psi_1(x_i) - e^{-\frac{1}{2}\alpha_i}\psi_2^\dagger(x_i)\right)\ldots\psi^\dagger(x_n\alpha_n)\right]|0\rangle \quad . \tag{A.5}$$

The mass term gives

$$H_m|\alpha_1\ldots\alpha_n\rangle = \int dx_1\ldots dx_n\chi(x)\sum_{i=1}^{n}\left[\psi^\dagger(x_1\alpha_1)\ldots\left(e^{-\frac{1}{2}\alpha_i}\psi_1^\dagger(x_i) + e^{\frac{1}{2}\alpha_i}\psi_2^\dagger(x_i)\right)\ldots\psi^\dagger(x_n\alpha_n)\right]|0\rangle \quad . \tag{A.6}$$

At $\lambda = 0$, we have $-i\nabla_i\chi = (m_o\sinh\alpha_i)\chi$, so we combine appropriate terms in (A.5) and (A.6) via

$$m_o\left[\sinh\alpha\left(e^{\frac{1}{2}\alpha}\psi_1^\dagger - e^{-\frac{1}{2}\alpha}\psi_2^\dagger\right) + \left(e^{-\frac{1}{2}\alpha}\psi_1^\dagger + e^{\frac{1}{2}\alpha}\psi_2^\dagger\right)\right]$$

$$= m_o\cosh\alpha\,\psi^\dagger(x,\alpha) \quad . \tag{A.7}$$

As in the two-body case, we take $\epsilon(x)\delta(x) = 0$ to compute the derivative of χ. Then, we obtain

$$-i\nabla_{i_o}\chi = \left[m_o\sinh\alpha_{i_o} + 2\sum_{\substack{i=1\\ i\neq i_o}}^{n}\lambda(\alpha_{i_o},\alpha_i)\delta(x_{i_o} - x_i)\right]\chi \quad .$$

Putting this in (A.5) and (A.6) and using (A.7), we obtain

$$(H_K + H_M)|\alpha_1\ldots\alpha_n\rangle = \sum_i m_o\cosh\alpha_i\;|\alpha_1\ldots\alpha_n\rangle + |R\rangle \tag{A.8}$$

where

$$|R> = 2\int dx_1 \ldots dx_n \chi(x) \sum_i \sum_{j\neq i} \lambda(\alpha_i, \alpha_j)\delta(x_i - x_j)\Big[\Psi^\dagger(x_1\alpha_1)\ldots$$

$$\left(e^{\frac{1}{2}\alpha_i}\psi_1^\dagger(x_i) - e^{-\frac{1}{2}\alpha_i}\psi_2^\dagger(x_i)\right)\ldots\Psi^\dagger(x_n\alpha_n)\Big]|0>$$

$$= 4\int dx_1 \ldots dx_n \chi(x) \sum_i \sum_{j\neq i} \lambda(\alpha_i, \alpha_j)\delta(x_i - x_j)(-)^{i-j+1}\varepsilon(j-i)\cosh \tfrac{1}{2}(\alpha_i - \alpha_j)$$

$$\psi_1^\dagger(x_i)\psi_2^\dagger(x_i)\;[\Psi\text{'s except for } i,j]\;|0> \quad .$$

The interaction term gives

$$H_g|\alpha_1\ldots\alpha_2> = 2g_o\int dx_1\ldots dx_n\chi(x)\sum_i\sum_{j\neq i}\delta(x_i - x_j)\varepsilon(j-i)(-)^{i-j+1}\sinh \tfrac{1}{2}(\alpha_i - \alpha_j)$$

$$\psi_1^\dagger(x_i)\psi_2^\dagger(x_i)\quad[\Psi\text{'s except } i,j]\;|0> \quad .$$

Since λ is given by Eq. (A.4), this cancels $|R>$, and thus

$$H|\alpha_1\ldots\alpha_n> = \left(\sum_i m_o\cosh\alpha_i\right)|\alpha_1\ldots\alpha_n> \quad . \tag{A.9}$$

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FIGURE CAPTIONS

Fig. 1: Filled modes in the Dirac sea for a range of couplings.

Fig. 2: Graphical solution of periodic boundary conditions in the presence of a mode at $\alpha = 0$.

Fig. 3: Allowed n-strings for $\frac{2}{3}\pi < \mu < \frac{3}{4}\pi$.

Fig. 4: Dirac sea in the presence of a 7-string.

Fig. 5: (a) Dirac sea with a 1-string for $\mu < \pi/2$; note that two α's are labeled by n_o.

(b) Dirac sea with a 1-string for $\mu > \pi/2$; note that the former 1-string is now a 3-string with one hole.

Fig. 6: Scheme for subtracting ground state from excited state mode sums. Dashed lines indicate which ground state modes are to be subtracted from sea modes in the excited state. Far from the string, the x's and dots line up.

Fig. 7: Allowed vertices in the eight-vertex model.

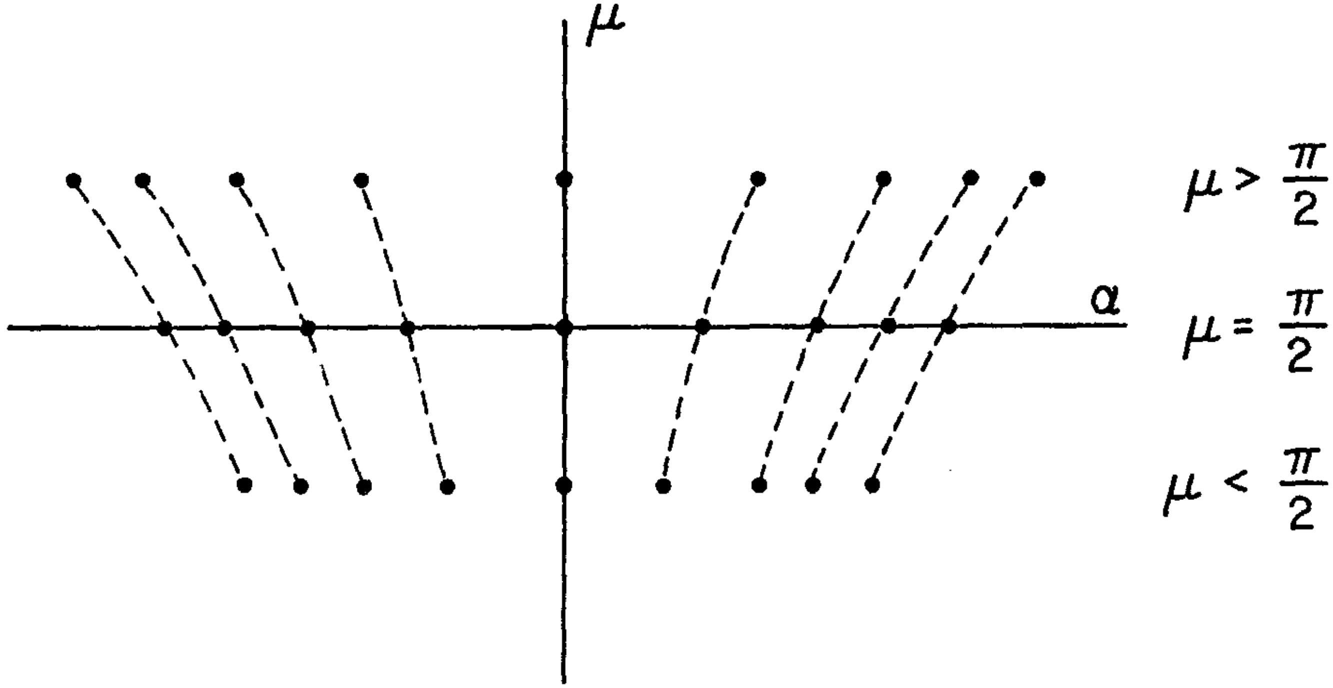

Fig. 1

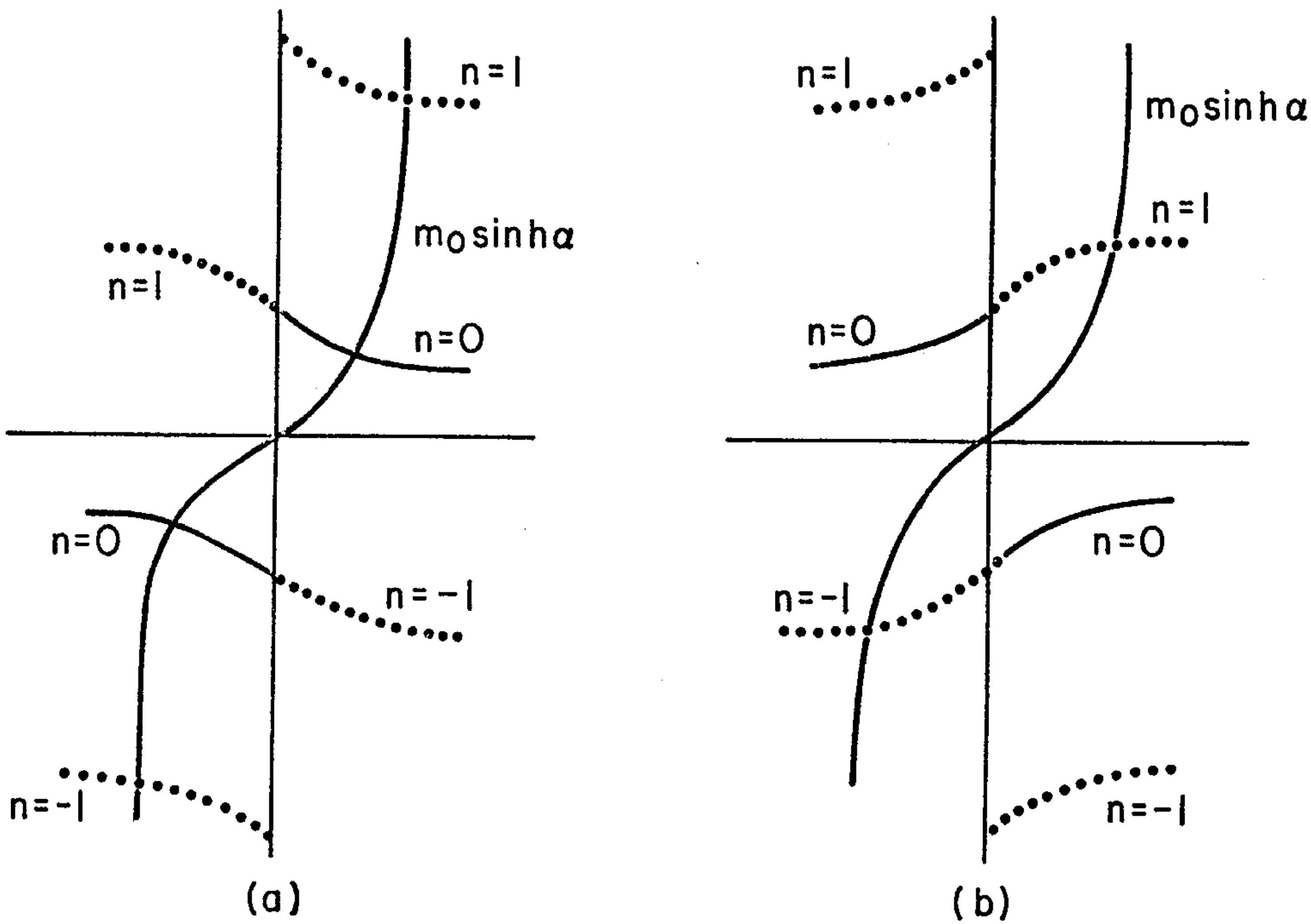

Fig. 2

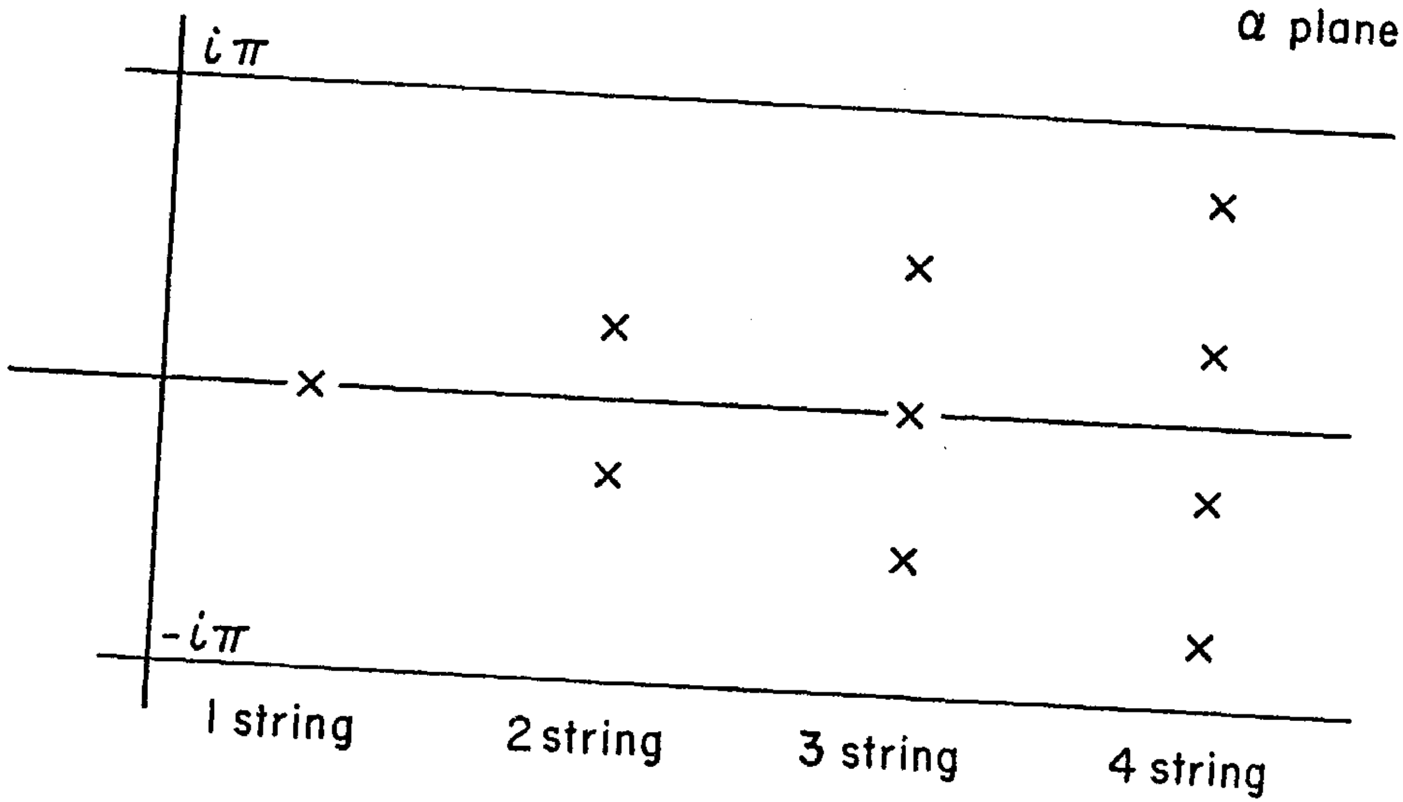

Fig. 3

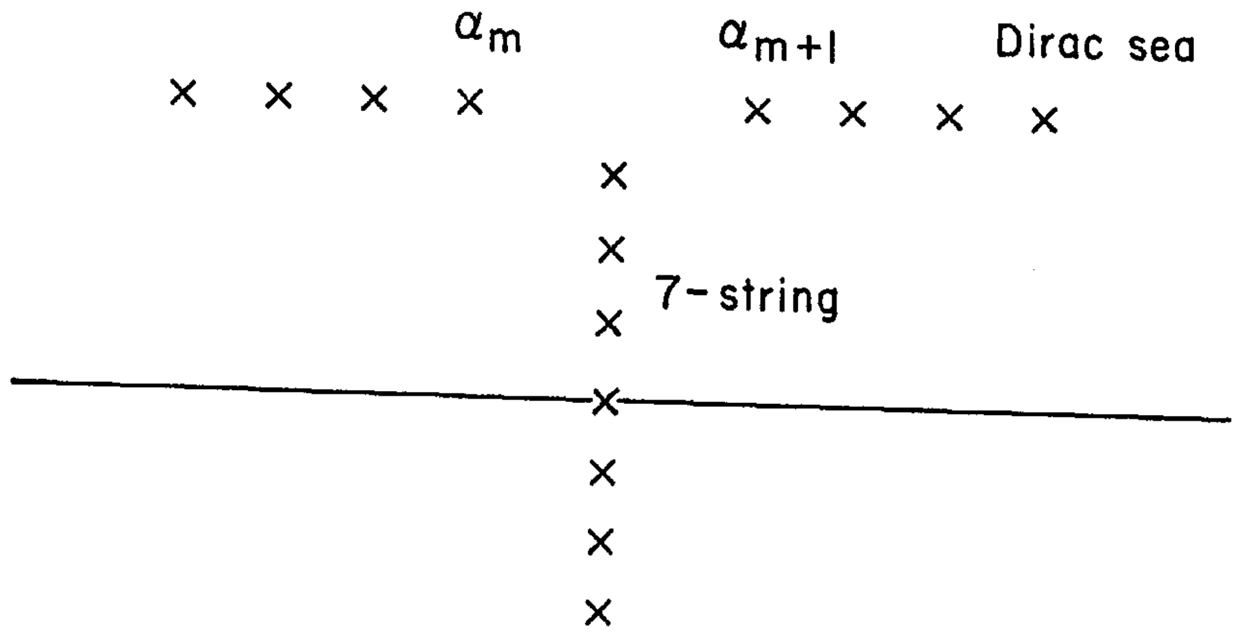

Fig. 4

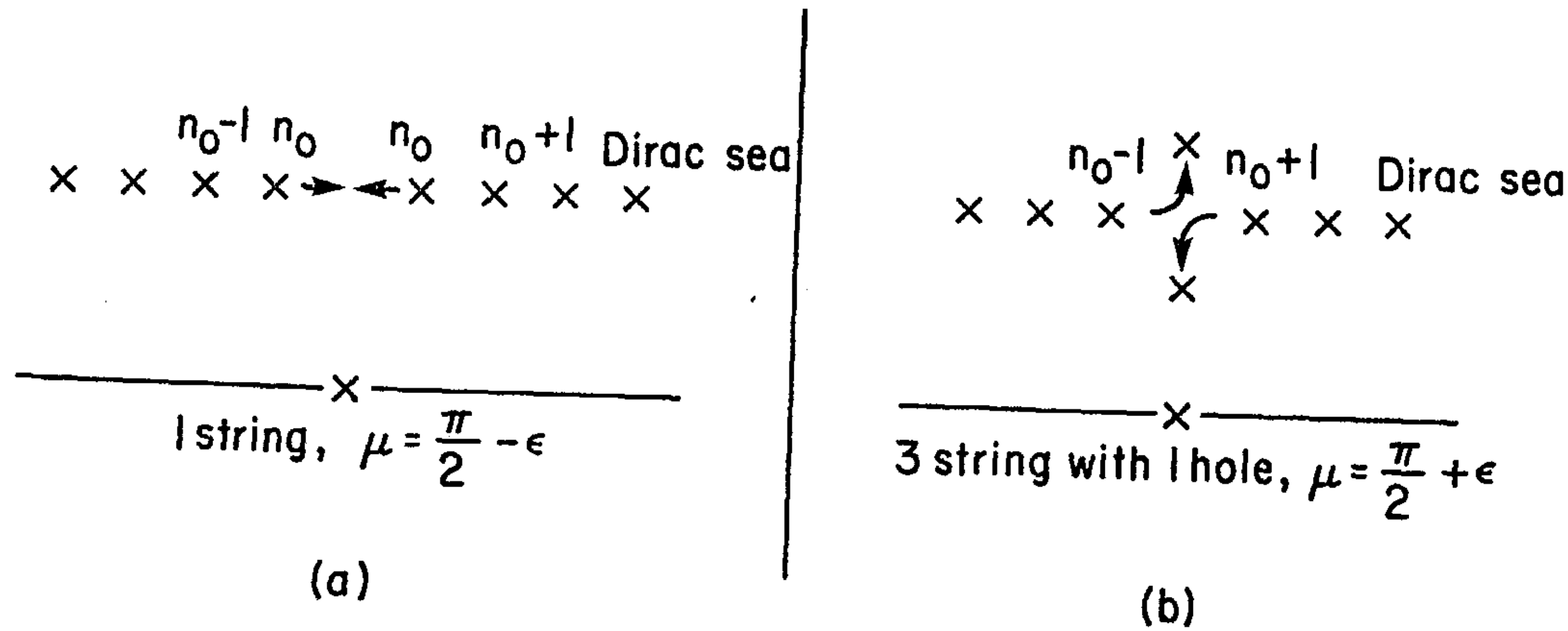
n₀-1 n₀ n₀ n₀+1 Dirac sea
1 string, μ = π/2 − ε
(a)
n₀-1 n₀+1 Dirac sea
3 string with 1 hole, μ = π/2 + ε
(b)

Fig. 5

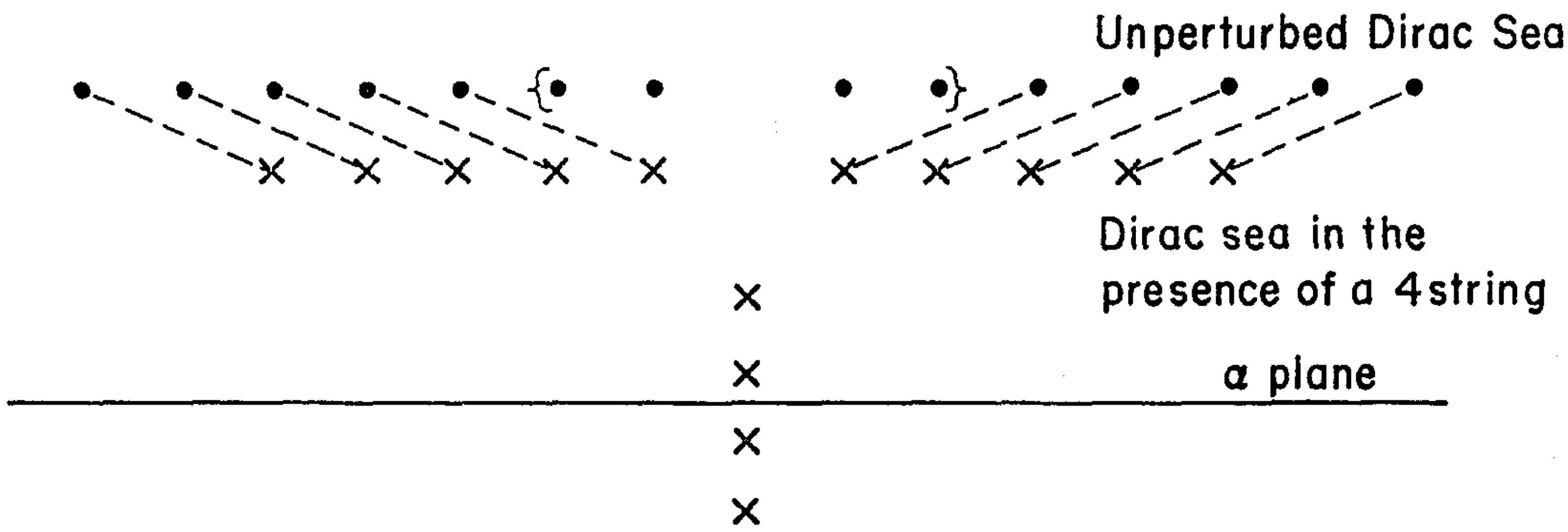
Unperturbed Dirac Sea
Dirac sea in the
presence of a 4string
a plane

Fig. 6

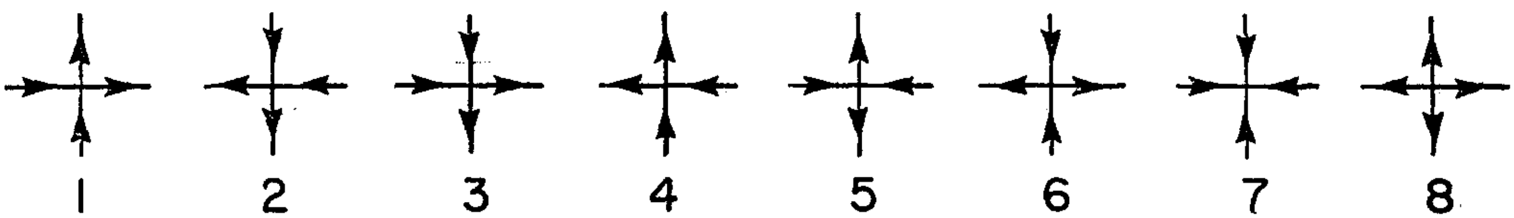

Fig. 7