Geometry from Algebra: The Holographic Emergence of Spacetime in String Theory

by

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Abstract

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In the quest for the unification of gravity with quantum mechanics, a new idea has emerged called the holographic principle which states that gravitational physics in D spacetime dimensions can be described by a dual nongravitational theory in $D-1$ dimensions. Thus spacetime geometry is not fundamental but rather emerges holographically from a theory obeying the algebraic laws of quantum mechanics. In this thesis we explore the interplay between geometry and algebra implied by this principle in several new contexts. Using the techniques of general relativity, we extend the holographic principle to Gödel spacetimes, where we discover a possible holographic protection of chronology. Then we study matrix models which provide a simplified toy model of holography and apply these models to find new relations between gauge theories in six dimensions and integrable dynamical systems. Finally we view the geometry of M-theory orbifolds from the algebraic perspective of infinite dimensional Lie algebras and find simple conditions obeyed by the twisted sectors of these orbifolds.

Professor Petr Hořava
Dissertation Committee Chair

I dedicate this work to Baba and Ma.

Contents

List of Figures

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Chapter 1

Introduction

1.1 General Relativity, Quantum Mechanics, and Perturbative String Theory

Our current, experimentally verified, understanding of the fundamental physical laws underlying processes occuring in our universe rests on the twin pillars of general relativity [1]-[4] and quantum mechanics [5, 6]. Each of these theories employs very different mathematical formalisms. While general relativity is written in the language of Lorentzian geometry, quantum mechanics, with its Hilbert spaces and self-adjoint operator observables, is written in the language of algebra. Straightforward attempts to unify these two theories have failed in the past.

More specifically, the usual methods of perturbative quantum field theory, which are adequate for the field theories of the standard model, lead to a nonrenormalizable theory when applied to gravity. The perturbation parameter in quantum gravity is Newton's constant G_N with dimensions of inverse energy squared. Thus the dimensionless coupling for a process of characteristic energy scale E is $G_N E^2$ and diverges quadratically at high energy. Physically this means as one approaches the Plank mass $M_P = G_N^{-1/2} = 1.22 \times 10^{19}$ Gev, or equivalently probes down to the plank length $l_p = M_P^{-1} = 1.6 \times 10^{-33}$ cm, a perturbative quantum field theory of gravity will yield divergent answers.

Historically such ultraviolet divergences have signalled that the current theory is incomplete, and new physics must emerge at small distances to soften these divergences. For example, in the Fermi theory of weak interations, the four fermi vertex also had an inverse energy squared coupling G_F. At energy scales where $G_F E^2 \approx 1$, this divergence was resolved by the appearance of the electroweak gauge bosons, which mediate the contact interaction present in the four-fermi vertex. When these gauge bosons are included in the theory, the full theory becomes renormalizable.

Given our experience with the Fermi theory, it is natural to ask if there is any modification to the perturbation expansions of both the standard model and gravity that can both introduce new physics at short distances to soften the divergences of gravity and unify the two theories. Remarkably, many efforts to answer this question have produced only one consistent modification: replace the point particles of quantum field theory with one-dimensional extended objects, or strings. This is the fundamental idea underlying perturbative string theory [7]-[9], and this simplification almost automatically leads to consistent finite perturbative quantum gravity amplitudes, as well as several other appealing consequences including grand unified gauge groups, extra dimensions, supersymmetry, and chiral gauge couplings.

In perturbative string theory, as the string evolves through spacetime, it traces out a two dimensional surface or world sheet. This evolution has a Lagrangian description and leads to a two dimensional quantum field theory on the world sheet, which itself has a perturbation expansion parameter α' where the tension of the string is $T = \frac{1}{\alpha'}$. [1] In the spacetime picture, α' sets a fundamental string length scale $l_s = \sqrt{\alpha'}$, which is the scale at which effects due to the extended nature of the string are first seen. Furthermore in calculating perturbative amplitudes, one must sum over all two dimensional surfaces, including those of higher genus. Higher genus contributions are the analog of quantum field theory loop corrections, and are suppressed by a factor of $1/g_s^{2N}$ for closed, genus N surfaces. Here g_s is a the dimensionless string coupling constant, which is actually derived from a vacuum expectation value in the theory. Given the fundamental string parameters l_s and g_s, after matching string derived amplitudes to perturbative quantum gravity amplitudes, one finds that they match as long as the Planck scale is $l_p = g_s l_s$.

The fact that the perturbation expansion of string theory recovers that of gravity (and gauge theory for open strings) is at once exciting and unsatisfactory. In principle one would like a nonperturbative and less ad hoc approach in which the perturbative expansion in l_s

[1] Note that α', like G_N, also has dimensions of inverse energy squared but in two dimensions such a coupling constant is marginal as opposed to irrelevant and so the worldsheet theory does not suffer from UV divergences.

and g_s emerges naturally. In section 1.3 we will see how this might be accomplished.

1.2 The Holographic Principle from Gravity

A possible explanation for the failure of most traditional methods to unite the geometry of general relativity with the algebra of quantum mechanics may lie in the recently discovered holographic principle [10] (see [11] for reviews) which points to a radical misunderstanding of the fundamental degrees of freedom underlying spacetime physics. The origin of this holographic principle arises from a careful consideration of black hole thermodynamics [12, 13], which we briefly review.

A non-rotating neutral black hole, as a solution of Einstein's equations, is characterized by its mass M. The black hole also has an event horizon with area A and these two quantities obey the relation

$$dM = \frac{1}{8\pi}\kappa dA \tag{1.1}$$

where κ is the surface gravity, a measure of the strength of the gravitational field at the horizon. Einstein's equations also predict that in any classical gravitational process, the event horizon grows: $dA > 0$. Equation (1.1) bears a resemblance to the first law of thermodynamics $dE = TdS$ while $dA > 0$ is analogous to the second law: $dS > 0$. Hawking put these analogies on firmer footing when he used semiclassical gravity techniques to show that black holes are actually hot and emit thermal radiation at a temperature $T = \frac{\kappa}{2\pi}$ [14]. Equating M with E, we conclude that black holes are thermal objects with an entropy $S = \frac{A}{4}$. One of the great successes of string theory is a microscopic derivation of this black hole entropy for certain supersymmetric black hole solutions [15].

Armed with a knowledge of black hole thermodynamics, the holographic principle follows almost immediately. Consider the question of how much entropy a given region of space of volume V can support. In principle one could keep throwing in matter into the region to increase its entropy. This process however has a limit because the matter will backreact on the geometry of the region. After a critical mass has been thrown in, the region will gravitationally collapse, and a black hole whose event horizon subsumes the region will form. The entropy of the black hole will be $A/4$. Any original matter that was in the region before collapse must, by the second law of thermodynamics, have an entropy less than $A/4$. Based on these arguments we are forced to conclude that no region of space of volume V

can ever support an entropy greater than an amount proportional to its surface area A. This is the statement of the holographic principle. Note that it is in distinct contradiction with quantum field theory which posits the existence of local degrees of freedom assigned to each plank volume of space, and therefore predicts a maximal entropy that scales with the volume V of space.

One may object to this argument by saying that is it is not covariant; namely a specific region of space V was singled out, and space alone has no coordinate invariant meaning in general relativity. Nevertheless, the above simplified argument has an extension to a covariant holographic principle discovered by Bousso [38]. Bousso considers light sheets emanating from any initial surface. Using only classical general relativity, he shows that the entropy flux across the light sheets is bounded from above by the initial surface area. The basic idea is that if more entropy flows across the light sheet, the back reaction of the matter contributing to this entropy will cause the light sheet to contract thereby limiting the total amount that can flow across. This effect leads to the required bound.

Ultimately, regardless of the particular technical tools used to derive entropy bounds, all such bounds rest on two eminently reasonable assumptions. The first is that entropy cannot be increased without an energy contribution. The second is that energy contributes to gravitational collapse. These two facts conspire to limit the total amount of entropy in spacetime. The fundamental relations between entropy, energy, and geometry can only be understood within a fully quantum mechanical theory of gravity, but it is fortunate that all we needed to know about these quantites to discover the holographic principle could be found in the classical limit of Einstein's equations.

In the final analysis, a comparison of the holographic principle with local quantum field theory tells us that our current local description of spacetime physics is highly redundant. It is natural to ask if there exists a dual, nonredundant description of spacetime physics. In particular, does the gravitational physics in a region R of spacetime have a nonperturbative description in a holographically dual theory that lives in one lower dimension, possibly involving degrees of freedom related to the boundary of R? Remarkably, in certain special cases, this dual theory has been found as we see next.

1.3 The Holographic Principle from Gauge Theory

One the one hand, In section 1.1 we have seen that quantum mechanics and gravity can be united in a perturbative expansion involving strings moving through spacetime. On the other hand in section 1.2 we have seen that any fundamental nonperturbative theory of quantum gravity should involve degrees of freedom that are far removed from the classical concepts of metric based spacetime. These two observations can be reconciled beautifully by the correspondence between large N gauge theories and string theory proposed by t'Hooft. [17]

Consider for example $U(N)$ gauge theory when N is large. In addition to the Yang-Mills coupling constant g_{YM}, one can do perturbation theory in a new small parameter, namely $1/N$. If one writes down the Feynman diagrams of the theory in the double line notation, where each gluon propagator has two lines corresponding to two indices in the adjoint representation of $U(N)$, then the diagrams which contribute to order N^{2-2g} are exactly those diagrams that can be embedded into a genus g Riemann surface. More precisely, the sum of all Feynman diagrams that contribute to the partition function Z_{YM} of Yang-Mills theory can be grouped together to yield

$$Z_{YM} = \sum_{g=0}^{\infty} N^{2-2g} f_g(\lambda_{YM}) \tag{1.2}$$

where $\lambda_{YM} = g_{YM}^2 N$ is the t'Hooft coupling and $f_g(\lambda_{YM})$ is the sum of all Feynman diagrams that can be drawn on a surface of genus g. This expansion looks remarkably like the perturbation expansion of a string theory. Recall from section 1.1 that each successive surface in the genus expansion of string theory is suppressed by a factor of g_s^2. Thus in (1.2) $\frac{1}{N}$ plays the role of the string coupling g_s. Then the t'Hooft coupling λ_{YM} must be related to l_s, and $f_g(\lambda_{YM})$ can be interpreted not as a sum of Feynman diagrams, but as the partition function of a string theory on a genus g Riemann surface. So far, the above observations are tantalizing but merely suggestive. What would be more exciting is an independent identification of a string theory action which reproduces the amplitudes $f_g(\lambda_{YM})$ without reference to the original gauge theory. In many cases this has now been done.

The most famous concrete illustration of the above ideas is Maldacena's AdS-CFT [18] correspondence. In this case the gauge theory is $N = 4$ $SU(N)$ superconformal Yang-Mills theory living on $S^3 \times \mathbb{R}$ and the string theory is the type IIB superstring living on $AdS_5 \times S^5$.

The radius of curvature of AdS_5 and S^5 are both given by R. The precise mapping between the string theory parameters g_s, l_s, and R and the gauge theory parameters λ_{YM} and N are given by

$$\frac{R^4}{l_s^4} = \lambda_{YM} \tag{1.3}$$

$$4\pi g_s = \frac{\lambda_{YM}}{N}. \tag{1.4}$$

Note that there is no limit of parameter space in which both these theories are weakly coupled. When the physics of the string theory reduces to classical ($g_s \ll 1$) nonstringy ($l_s \ll R$) spacetime physics well described by Einstein's equations, the gauge theory is strongly coupled ($\lambda_{YM} \gg 1$) with many colours ($N \gg 1$). Conversely, when the gauge theory is weakly coupled ($\lambda_{YM} \ll 1$), the $AdS^5 \times S^5$ space is substring scale and all the fluctuations of the string worldsheet must be summed over, destroying the semiclassical spacetime picture. Also, reducing the number of colours of the gauge theory makes g_s larger, in essence making quantum effects stronger on the string theory side. Thus it is hard to prove this duality directly. Nevertheless the AdS-CFT correspondence has passed numerous checks and is generally believed to be correct. Indeed many now view $N = 4$ $SU(N)$ superconformal Yang Mills theory as the nonperturbative definition of quantum supergravity on $AdS^5 \times S^5$.

Assuming the validity of the AdS-CFT conjecture, one might draw several lessons about the structure of spacetime at small distances. It seems that spacetime, general relativity, and even perturbative string theory are not fundamental in any sense, but rather represent emergent properties of the strongly coupled dynamics of a large number of partons which all obey the laws of quantum mechanics. From (1.4) we learn that because there are many partons, our world at low energies is very classical as opposed to quantum mechanical. From (1.3) we learn that because these partons are very strongly coupled, we do not see stringy effects in our classical world. However if we could probe spacetime down to the string scale l_s, we would start seeing new states corresponding to the fluctuations of an effective, spherical string worldsheet. But even these string quanta would not be fundamental. Rather they would be effective quanta derived from the sum of a large number of Feynman diagrams in the theory describing the partons. If furthermore we could probe down to l_p, we would be able to tease apart the now weakly coupled partons into groups with small numbers. In this regime, our original spacetime language would be meaningless and the physics would best be described in terms of the quantum mechanics of this small number of weakly coupled partons. Finally these partons do not even live on the original spacetime, which instead emerges holographically from their dynamics.

1.4 Overview

The moral of the story we have seen so far is that in the battle for fundamental physics at small distances, algebra wins over geometry; the geometry of spacetime is a derived concept and emerges holographically from a dual theory obeying the algebraic laws of quantum mechanics. The fact that spacetime is not fundamental can be seen from both within the formalism of general relativity as well as through a microscopic formulation involving large N gauge theory. In this thesis we explore the above interplay between geometry and algebra in the context of three concerete systems.

In chapter 2 we explore the physics of holography in the Gödel universe. The main motivation for doing so is that while the AdS-CFT correspondence illustrates how holography could work in spaces with negative cosmological constant, little is known about holography in flat space, or deSitter spaces. With so few examples of holography at work, every additional example is precious and the geometry of the Gödel universe presents some interesting considerations for holography. At the face of it, the Gödel universe appears inconsistent, with closed-timelike curves going through every point, and no asymptotic infinity with which to define S-matrices. However, taking the view point that strict geometry is misleading, we conduct a phenomenological analysis of holography in the Gödel universe. By employing the same light sheet technology used by Buosso to derive the holographic bound, we calculate the location of holographic screens in the Gödel universe. We find that similar to deSitter space, the holographic screen is observer dependent. Furthermore the screen surrounds the observer and any closed timelike curve that starts within the vicinity of the observer must pierce the screen before returning to the starting point. Thus holography suggests that the spacetime geometry of the Gödel universe is not fundamental and perhaps there is a holographic mechanism for chronology protection at work. Furthermore we show that the Gödel universe is T-dual to a pp-wave which is in turn holographically dual to a large R charge limit of a CFT, thus placing for the first time Gödel universes within the duality web of string theory.

Having shown that holography can radically reinterpret the geometry of these Gödel spacetimes, we move on in chapter 3 to studying extremely simplified models for the holographic emergence of geometry from algebraic objects, within the context of matrix models. These matrix models, or integrals, can be thought of as 0+0 dimensional quantum mechanics, and are specified by an integral over the space of NxN matrices. Whereas in the AdS-CFT correspondence it is very hard to evaluate the partition function of the gauge theory, and therefore to understand the emergence of geometry, the matrix model has the advantage that it can be solved exactly in the large N limit using a saddle point method. In this solution one can explicitly understand how the geometry arises in the large N limit. Basically in the saddle point approximation, the large number of eigenvalues of the matrix fall along an interval in the complex plane in the vicinity of a critical point of the matrix energy functional. This interval on which the eigenvalues condense can be interpreted as a branch cut of a Riemann surface Σ, called the spectral curve. Indeed the effective force on a probe eigenvalue in the presence of the condensed eigenvalues is a multivalued function on the complex plane, branched over the cut and Σ is the Riemann surface for this function. If one allows fluctuations in the condensed eigenvalues away from the saddlepoint, the Riemann surface will see this as fluctuations of effective fields living on its surface.

In particular in chapter 3 we use matrix models after the Dijkgraaf-Vafa technique [19] to study massive vacua of $6D$ $U(N)$ super Yang-Mills theories with R-symmetry twists. In this case the geometry emerging from the matrix model is not that of spacetime, but rather that of a Seiberg-Witten curve which encodes the physics of the 6D gauge theory. The particular matrix models we study live on tori and as a consequence have a genus two spectral curve. The Jacobian of this curve is closely related to a twisted four torus T in which the Seiberg-Witten curves of the theory are embedded. We also analyze R-symmetry twists in a bundle with nontrivial first Chern class which yields intrinsically 6D SUSY breaking and a novel matrix integral whose eigenvalues float in a sea of background charge. Next we analyze the underlying integrable system of the theory, whose phase space we show to be a system of N-1 points on T. We write down an explicit set of Poisson commuting Hamiltonians for this system for arbitrary N and use them to prove that equilibrium configurations with respect to all Hamiltonians correspond to points in moduli space where the Seiberg-Witten curve maximally degenerates to genus 2, thereby recovering the matrix model spectral curve. We also write down a conjecture for a dual set of Poisson commuting variables which could shed light on a particle-like interpretation of the system.

After considering this simple algebraic toy model for the emergence of geometry (albeit Seiberg-Witten geometry and not spacetime geometry), we return to the full problem of trying to understand a possible fundamental algebraic structure underlying toroidal compactifications of M-theory. Again, the hope is to understand the algebraic, holographic description of toroidial compactifications of M-theory. M-theory [20] unifies the 5 string theories and has eleven dimensional supergravity as its low energy limit, but beyond that we know very little about the theory. In the absence of a detailed dynamical understanding of M-theory, we use the next most powerful tool we have at our disposal, namely its symmetry properties. In the same way that isometries of $AdS_5 \times S^5$ were a clue to identifying the gauge theory dual, a detailed manifestly symmetric formulation of M theory would be instrumental in identifying its holographic dual.

In contrast to the relatively simple case of $AdS_5 \times S^5$, toroidial compactifications of M-theory have a host of perturbative as well as nonperturbative symmetries called the U-duality group which includes various T-dualities and S-dualities. These groups act on the nonperturbative objects of the theory, permuting various branes and fluxes. In the fully compactified theory on T^{10}, it would be useful to have a single algebraic structure which represented all the branes and fluxes at once. Then all the duality transformations could be understood simply as automorphisms of this structure, and various brane actions could be written in a manifestly duality invariant form. In a recent publication [21] it was posited that E_{10} provides exactly this structure. Zero modes of supergravity fields correspond to the cartan subalgebra of E_{10}. Fluxes correspond to real roots of E_{10}. Branes correspond to certain imaginary roots of E_{10}. The U-duality group corresponds to the Weyl group of the root lattice of E_{10}. The masses of the branes are simply related to the norm of the corresponding root and interactions between branes and fluxes can be classified in a duality invariant manner by the inner product of the associated roots.

In chapter 4 we build upon this work by analyzing orbifolds of M-theory on T^{10} in the algebraic formulation. Traditionally orbifolds of string and M-theories by a symmetry group involve truncating the spectrum of the theory to invariant states. This truncated spectrum is often inconsistent, and through a thorough analysis of all possible quantum anomalies, one finds that one must add a twisted sector to complete the theory. Although there is no general principle to discover the twisted sector, we find that when these orbifolds are reanalyzed from the algebraic perspective of E_{10} they can be understood simply. Basically, in every case a simple relation exists between the action of the orbifold group on the E_{10} root lattice and the root corresponding to the branes that need to be added in the twisted sector. Indeed this relation can be used to predict the twisted sector. Furthermore the famous $\mathbb{Z}_2$ orbifold of Horava-Witten [22, 23] is dual to heterotic string theory, which has been conjectured to be related to DE_{18}. The subalgebra of E_{10} invariant under this $\mathbb{Z}_2$ orbifold is the untwisted sector and contains, but is larger than, the algebra DE_{10}, which is also a subalgebra of DE_{18}. We elucidate the relationships between the untwisted sector of E_{10}, DE_{10} and DE_{18}. Furthermore by decomposing roots of DE_{18} under its $DE_{10} \times SO(16)$ subalgebra we arrive at a physical interpretation of some of the roots of DE_{18} in terms of states of the heterotic string. Roots that fall into nontrivial reps of $SO(16)$ correspond to heterotic states that are in the twisted sector from the point of view of the M-theory orbifold.

Again, the simplification of M-theory orbifolds when viewed from an algebraic perspective provides yet another example of the supremacy of algebra over geometry that is a recurring theme in this thesis. Although a complete understanding of a nonperturbative holographic dual of M-theory is not yet at hand (although see [24]), it is likely that a duality invariant formulation of the theory and its orbifolds will be important preliminary work towards such a goal. The further elucidation of this algebraic structure, as well as the search for other examples of gauge/string correspondences present exciting short-term avenues for research on the way toward the long term goal of a detailed understanding of the emergent properties of the spacetime in which we live. It would also be interesting to study the emergent properties of the brains that study branes.

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Appendix A

The Algebraic Geometry of Elliptic Curves

A.1 Meromorphic functions on T^2

Consider a torus T^2 with complex structure τ (Im $\tau > 0$). A theta function ϑ on the torus is a quasi-periodic function, with the following periodicity conditions:

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z+1|\tau) = e^{2\pi i a} \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z|\tau) \tag{A.1.1}$$

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z+\tau|\tau) = e^{-\pi i \tau - 2\pi i (z+b)} \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z|\tau) \tag{A.1.2}$$

An explicit formula for ϑ is

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z|\tau) = \sum_{n=-\infty}^{\infty} \exp[\pi i (n+a)^2 \tau + 2\pi i (n+a)(z+b)]. \tag{A.1.3}$$

We can use three methods to construct meromorphic functions on T^2 from theta functions. The first is to form ratios.

$$\frac{\prod_{i=1}^{n} \vartheta \left[\begin{smallmatrix} a_i \\ b_i \end{smallmatrix}\right] (z|\tau)}{\prod_{i=1}^{n} \vartheta \left[\begin{smallmatrix} a'_i \\ b'_i \end{smallmatrix}\right] (z|\tau)}$$

is a meromorphic function on T^2 provided $\sum a_i \equiv \sum a'_i, \sum b_i \equiv \sum b'_i \mathrm{mod}\, \mathbb{Z}$. Another method is the derivative of the logarithm of a ratio of theta function.

$$\partial_{z_i} \ln \frac{\vartheta \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] (z|\tau)}{\vartheta \left[\begin{smallmatrix} a' \\ b' \end{smallmatrix}\right] (z|\tau)}$$

The last is the second derivative of the logarithm of a theta function.

$$\partial_{z_i}\partial_{z_j} \ln \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z|\tau)$$

Using the transformation properties above, these can be shown to be periodic in both directions.

Another method of constructing meromorphic functions on a T^2 involves the Weierstrass $\wp$-function, and its derivative $\wp'$. We define it as

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda\in\Lambda,\lambda\neq 0} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right), \qquad \text{(A.1.4)}$$

where $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ is the lattice defining the torus. This is even, (doubly) periodic in Λ, analytic on $\mathbb{C}\backslash\Lambda$, and has a pole of order two at the points on Λ. $\wp$ and $\wp'$ satisfy the differential equation

$$\wp^2 = 4\wp^3 - g_2\wp - g_3, \qquad \text{(A.1.5)}$$

where g_2 and g_3 are constants determined by the lattice Λ (and therefore τ). Note that as $\wp$ is even and doubly periodic, $\wp'$ is odd and doubly periodic. It turns out that any doubly periodic function F can be written as

$$F(z) = R_1(\wp) + \wp' R_2(\wp), \qquad \text{(A.1.6)}$$

with R_1 and R_2 rational functions. Morally one decomposes F into odd and even parts.

Two other Weierstrass functions deserve mention: the Weierstrass σ function and the Weierstrass ζ function, the latter not to be confused with the Riemann ζ function. We define them as

$$\sigma(z) = z \prod_{\lambda\in\Lambda,\lambda\neq 0} \left(1 - \frac{z}{\lambda}\right) \exp\left[\frac{z}{w} + \frac{1}{2}\left(\frac{z}{\lambda}\right)^2\right] \qquad \text{(A.1.7)}$$

$$\zeta(z) = \frac{1}{z} + \sum_{\lambda\in\Lambda,\lambda\neq 0} \left(\frac{1}{z-\lambda} + \frac{1}{\lambda} + \frac{z}{\lambda^2} \right). \qquad \text{(A.1.8)}$$

ζ has a simple pole with residue 1 at every point in Λ, and is analytic on $\mathbb{C}\backslash\Lambda$. Lastly, we note the relations between these various functions and their periodicity properties.

$$\zeta(z) = \frac{d}{dz}\log\sigma(z) \tag{A.1.9}$$

$$\wp(z) = -\frac{d}{dz}\zeta(z) \tag{A.1.10}$$

$$\zeta(z+n+m\tau) = \zeta(z)+n\eta_1+m\eta_2 \tag{A.1.11}$$

$$\sigma(z+n+m\tau) = (-1)^{nm+n+m}\sigma(z)\exp\left[(n\eta_1+m\eta_2)(z+\frac{1}{2}(n+m\tau))\right] \tag{A.1.12}$$

$$\eta_1\tau-\eta_2 = 2\pi i. \tag{A.1.13}$$

A.2 Higher dimensional Theta functions

For a higher dimensional complex torus $\mathbb{C}^g/\Lambda$, where Λ is a lattice of rank $2g$, the analogy of the complex structure τ is a $g\times g$ complex matrix Ω. Ω must be symmetric, and $\mathrm{Im}\,\Omega$ must be positive definite. Then the higher dimensional ϑ functions are defined on $\mathbb{C}^g$ as

$$\vartheta\begin{bmatrix}\vec{a}\\ \vec{b}\end{bmatrix}(\vec{z}|\Omega) = \sum_{\vec{n}\in\Lambda}\exp[\pi i(\vec{n}+\vec{a})\cdot\Omega\cdot(\vec{n}+\vec{a})+2\pi i(\vec{n}+\vec{a})\cdot(\vec{z}+\vec{b})]. \tag{A.2.1}$$

Similar to the T^2 case, where we could holomorphically transform the lattice to $\mathbb{Z}+\tau\mathbb{Z}$, in the g-complex dimensional case, we can view the lattice as $\Lambda=\mathbb{Z}^g+\Omega\mathbb{Z}^g$. The periodicity properties are also analogous. Let $\vec{m}\in\mathbb{Z}^g$.

$$\vartheta\begin{bmatrix}\vec{a}\\ \vec{b}\end{bmatrix}(\vec{z}+\vec{m}|\Omega) = e^{2\pi i\vec{a}\cdot\vec{m}}\vartheta\begin{bmatrix}\vec{a}\\ \vec{b}\end{bmatrix}(\vec{z}|\Omega), \tag{A.2.2}$$

$$\vartheta\begin{bmatrix}\vec{a}\\ \vec{b}\end{bmatrix}(\vec{z}+\Omega\vec{m}|\Omega) = e^{-\pi i\vec{m}\cdot\Omega\vec{m}-2\pi i\vec{m}\cdot(\vec{z}+\vec{b})}\vartheta\begin{bmatrix}\vec{a}\\ \vec{b}\end{bmatrix}(\vec{z}|\Omega). \tag{A.2.3}$$

We can use the same three methods as before to construct meromorphic functions on $T^{2g}=\mathbb{C}^g/\Lambda$ using theta functions. We repeat them here in the multi-dimensional notation for appendectical completeness.

$$\frac{\prod_{i=1}^n\vartheta\begin{bmatrix}\vec{a}_i\\ \vec{b}_i\end{bmatrix}(\vec{z}|\Omega)}{\prod_{i=1}^n\vartheta\begin{bmatrix}\vec{a}_i\prime\\ \vec{b}_i\prime\end{bmatrix}(\vec{z}|\Omega)} \tag{A.2.4}$$

is a meromorphic function on T^{2g} provided $\sum a_i\equiv\sum a_i', \sum b_i\equiv\sum b_i'\mathrm{mod}\,\mathbb{Z}^g$.

$$\text{So are}\quad \partial_{z_i}\ln\frac{\vartheta\begin{bmatrix}\vec{a}\\ \vec{b}\end{bmatrix}(\vec{z}|\Omega)}{\vartheta\begin{bmatrix}\vec{a}'\\ \vec{b}'\end{bmatrix}(\vec{z}|\Omega)}\quad\text{and}\quad\partial_{z_i}\partial_{z_j}\ln\vartheta\begin{bmatrix}\vec{a}\\ \vec{b}\end{bmatrix}(\vec{z}|\Omega), \tag{A.2.5}$$

for any choice of characters. Again, using the above transformation properties, these can be shown to be periodic in all $2g$ directions.

These functions can be used to define functions on genus g Riemannian surfaces. Consider such a Σ_g. There are g holomorphic 1-forms on Σ_g, call them ω_i. Denote the canonical basis of $H_1(\Sigma_g, \mathbb{Z})$ by the $2g$ cycles A_i and B_i, where $A_i \cap A_j = B_i \cap B_j = 0$, and $A_i \cap B_j = \delta_{ij}$. Then we can define the $g \times 2g$ *period matrix* as

$$\begin{bmatrix} \int_{A_1} \omega_1 & \cdots & \int_{A_g} \omega_1 & \int_{B_1} \omega_1 & \cdots & \int_{B_g} \omega_1 \\ \vdots & & \vdots & \vdots & & \vdots \\ \int_{A_1} \omega_g & \cdots & \int_{A_g} \omega_g & \int_{B_1} \omega_g & \cdots & \int_{B_g} \omega_g \end{bmatrix}. \tag{A.2.6}$$

We can choose the ω_i such that $\int_{A_j} \omega_i = \delta_{ij}$; then the period matrix is in the form $[I, \Omega]$ for the $g \times g$ identity matrix I and a $g \times g$ symmetric matrix Ω, where $\mathrm{Im}\,\Omega$ is positive definite. This similarity with the complex structure matrix in the beginning of this appendix, which was cunningly also named Ω, is not coincidental, as we now show.

Now the columns of the period matrix are $2g$ vectors in $\mathbb{C}^g$; these naturally form a lattice Λ and thus induce a torus $T^{2g} = \mathbb{C}^g/\Lambda$. This torus is called the *Jacobian* of Σ_g, often denoted $\mathcal{J}(\Sigma_g)$. What is the relation between these two objects? The answer is given by the Abel-Jacobi map.

Choose a $p_0 \in \Sigma_g$, and consider the function $\mu : \Sigma_g \to \mathbb{C}^g/\Lambda$, under which

$$p \mapsto \left(\int_{p_0}^{p} \omega_1, \ldots, \int_{p_0}^{p} \omega_g \right). \tag{A.2.7}$$

Note this is only defined up to Λ, since in choosing a contour from p_0 to p we could go around any combination of the cycles of the torus. In fact, we can generalize this as a map from any degree 0 divisor to $\mathcal{J}$. This is the Abel-Jacobi map, $\mu : \mathrm{Div}^0 \Sigma_g \to \mathcal{J}(\Sigma_g)$, where

$$\sum_i (p_i - q_i) \mapsto \left(\sum_i \int_{q_i}^{p_i} \omega_1, \ldots, \sum_i \int_{q_i}^{p_i} \omega_g \right). \tag{A.2.8}$$

As Σ_g is one-complex dimensional, and $\mathcal{J}$ is g-complex dimensional, in order to get a surjective map we need to pick g points on Σ_g, say p_i, $i = 1, \ldots, g$. The Jacobi Inversion Theorem states this explicitly: Given any $\lambda \in \mathcal{J}(\Sigma_g)$, there exist g points $p_i \in \Sigma_g$ such that $\mu\left(\sum_i (p_i - p_0)\right) = \lambda$. Moreover, these p_i are generically unique. Finally, Abel's theorem states that if the divisor $\sum (p_i - q_i)$ is a divisor of some meromorphic function, then $\mu\left(\sum (p_i - q_i)\right) = 0$. These two results mean the Abel-Jacobi map is an isomorphism between the moduli space of line bundles of degree 0, $\mathrm{Pic}^0(\Sigma_g)$, and the Jacobian $\mathcal{J}(\Sigma_g)$.

As promised, this allows us to find meromorphic functions on Σ_g. The Abel-Jacobi map gives us an embedding of Σ_g into $\mathcal{J}(\Sigma_g)$. By constructing meromorphic functions on $T^{2g} \cong \mathcal{J}(\Sigma_g)$, we can simply pull them back under the Abel-Jacobi map to get meromorphic functions on Σ_g.

More complete expositions can be found in [94] and [82].

Appendix B

Lie Algebraic Proofs

B.1 Proof that $DE_{10} \subset \mathfrak{g}^{(inv)}$

We will now complete the details of the proof from **S**4.4.3. Using the map υ, defined in (4.4.8), we can construct an injective homomorphism of Lie algebras,

$$\tilde{\upsilon} : DE_{10} \to \mathfrak{g}^{(inv)} \subset E_{10}$$

such that

a. For any $\alpha \in \Delta(DE_{10})$ we have $\tilde{\upsilon}(\mathfrak{g}(DE_{10})_\alpha) \subset \mathfrak{g}_{\psi(\alpha)}$, and the restriction of $\tilde{\upsilon}$ to $\mathfrak{g}(DE_{10})_\alpha$ is an injection. [Here $\mathfrak{g}(E_{10})_\alpha$ and $\mathfrak{g}(DE_{10})_\alpha$ are the root spaces of DE_{10} and E_{10}, as defined in (4.2.1).]

b. $\tilde{\upsilon}$ is an isomorphism between the Cartan subalgebras $\mathfrak{h}(DE_{10})$ and $\mathfrak{h}(E_{10})$.

c. If $\alpha \in \Delta_{Re}(DE_{10})$ (a real root) then $\tilde{\upsilon}$ is an isomorphism between the root spaces $\mathfrak{g}(DE_{10})_\alpha$ and $\mathfrak{g}_{\psi(\alpha)}$.

Proof. $\tilde{\upsilon}$ can be defined naturally on the Cartan subalgebra $\mathfrak{h}(DE_{10})$. Pick Chevalley generators $e'_i \in \mathfrak{g}(DE_{10})_{\gamma_i}$ $(i = -1 \dots 8)$, and pick nonzero elements $x_i \in \mathfrak{g}(E_{10})_{\upsilon(\gamma_i)}$. Define $\tilde{\upsilon}(e'_i) = x_i$. The Serre relations among the e'_i's are satisfied by the x_i's. We can see this by using the Weyl-group $W(E_{10})$ to turn pairs of x_i's into simple roots. Using the invariant bilinear forms on DE_{10} and on E_{10} we can find $y_i \in \mathfrak{g}(E_{10})_{-\upsilon(\gamma_i)}$ such that $(x_i|y_j) = \delta_{ij}$. Pick Chevalley generators $f'_i \in \mathfrak{g}(DE_{10})_{-\gamma_i}$ and set $\tilde{\upsilon}(f'_i) = y_i$.

The map $\tilde{v}$ is well-defined. To see that it is an injection, note that the kernel $\operatorname{Ker}\tilde{v}$ is an ideal of DE_{10} that intersects $\mathfrak{h}(DE_{10})$ trivially. Therefore, according to [119], $\operatorname{Ker}\tilde{v} = 0$.

Parts (b) and (c) follow immediately from (a). □

B.2 Proof that $DE_{10} \subset \mathfrak{g}^{(com)}$

We now complete the missing details of Proposition 2. Set $x^{\pm}$ to be nonzero generators of the root spaces $\mathfrak{g}(DE_{18})_{\pm\chi}$, and let $\mathfrak{g}' \subset DE_{18}$ be the smallest subalgebra that contains the set

$$\{e_{-1}, f_{-1}, \cdots, e_6, f_6, e_8, f_8, x^+, x^-\} \subset DE_{18(10)}. \quad \text{(B.2-1)}$$

We will now show that $\mathfrak{g}' \simeq DE_{10}$. Consider the set of positive real roots $\beta_{-1}, \ldots, \beta_6, \beta_8, \chi$. These roots all square to 2, and their intersection matrix is encoded in the Dynkin diagram of Figure 4.8, which, as we saw, is the Dynkin diagram of DE_{10}. We can therefore construct a surjective map $\phi : DE_{10} \to \mathfrak{g}'$ that maps the Chevalley generators of DE_{10} to the corresponding elements (B.2-1). We need to prove that ϕ is injective, i.e. that there are no extra relations among the elements of (B.2-1) in addition to those of the Kac-Moody algebra DE_{10}. To see this note that $\{\phi^{-1}(h_{-1}), \ldots, \phi^{-1}(h_8)\}$ generate the Cartan subalgebra $\mathfrak{h}(DE_{10})$, and therefore the kernel of ϕ intersects $\mathfrak{h}(DE_{10})$ trivially. But the kernel of ϕ is an ideal of DE_{10} and a Kac-Moody algebra has no nontrivial ideals that intersect the Cartan subalgebra trivially. (This follows from the construction in **S**1 of [119].) It follows that ϕ is an isomorphism of algebras and $DE_{10} \simeq \mathfrak{g}'$.

B.3 Proof of Proposition 3

Proposition 3 states that for

$$0 \neq x \in \mathfrak{g}_\gamma, \qquad \gamma = \sum_{i=-1}^{8} k_i \beta_i \in \Delta(DE_{18}),$$

$U(\mathfrak{so}(16))x \simeq L_{\mathfrak{so}(16)}(|k_7|\tilde{\tilde{\Lambda}}_9)$.

Proof. Let e_i, f_i, h_i ($i = -1, \ldots, 16$) be Chevalley generators for DE_{18}. Suppose, without loss of generality, that $k_7 < 0$. The element x is a linear combination of multiple commutators of $f_{-1},\dots,f_8$ and f_7 appears $|k_7|$ times. Consider a particular commutator

$$z := \underbrace{[\cdots[f_7,\cdots[f_7,\cdots\cdots]]]}_{|k_7|\ times}, \tag{B.3-1}$$

where the other generators that appear in $\cdots$ are from the list $f_{-1},\dots,f_6,f_8$. $\mathfrak{so}(16)$ commutes with all these generators, and

$$V := U(\mathfrak{so}(16))f_7 \simeq L(\tilde{\tilde{\Lambda}}_9)$$

is isomorphic to the fundamental representation of $\mathfrak{so}(16)$ [since $e_7, f_7, h_7, e_9, \dots, h_{16}$ generate a finite $\mathfrak{so}(18)$ Lie algebra]. Note that $f_7 \in V$ is a lowest-weight vector (a generator of the weight-space of $\tilde{\tilde{\Lambda}}_9$). Let $V^{\otimes|k_7|}$ be the tensor product of $|k_7|$ copies of the fundamental representation **16**. There is a surjective map $g: V^{\otimes|k_7|} \to U(\mathfrak{so}(16))z$ generated by

$$v_1 \otimes v_2 \otimes \cdots \otimes v_{|k_7|} \mapsto [\cdots[v_1,\cdots[v_2,\cdots\cdots]]],$$

where the various $(\cdots)$'s are the same series of commutators as appear in the corresponding expression (B.3-1). This map sends $f_7 \otimes \cdots \otimes f_7$ to z. Now set

$$W := U(\mathfrak{so}(16))(\underbrace{f_7 \otimes \cdots \otimes f_7}_{|k_7|\ times}) \simeq L(|k_7|\tilde{\tilde{\Lambda}}_9).$$

W is isomorphic to the irreducible representation of rank-$|k_7|$ traceless symmetric tensors. g induces a map $g': W \to U(\mathfrak{so}(16))z$, since $W \subset V^{\otimes|k_7|}$. Also, $f_7 \otimes \cdots \otimes f_7 \in W$, so g' is surjective. Since W is irreducible, g' is an isomorphism. This proves that $U(\mathfrak{so}(16))z$ is isomorphic to $L_{\mathfrak{so}(16)}(|k_7|\tilde{\tilde{\Lambda}}_9)$. It is easy to extend this proof to x, which is a linear combination of expressions like (B.3-1). □

B.4 Denominator Formula for $\mathfrak{g}^{(inv)}$

In this section we will present a "denominator formula" that captures the multiplicities of $\mathfrak{g}^{(inv)}$ roots. Recall the *denominator identity* (formula (10.4.4) of [119]),

$$\prod_{\alpha\in\Delta^+}(1-e^{-\alpha})^{\operatorname{mult}\alpha} = \sum_{w\in W}(\operatorname{sgn} w)e^{w(\rho)-\rho},$$

Here, as usual in character formulas, we expand each side in a formal power series in the formal variables $e^{-\alpha_i}$. The multiplication is according to the rule $e^{-\alpha}e^{-\beta} = e^{-\alpha-\beta}$. The sum on the right-hand side is over all Weyl-group elements w, and $\operatorname{sgn} w$ is the "signature" of w – $(+1)$ if w is a product of an even number of simple reflections and (-1) otherwise. The weight ρ is chosen such that $(\rho|\alpha) = 1$ for all simple roots α. For E_{10}, with the assignment of simple roots as in the Figure 4.1, we have

$$\rho = -30\alpha_{-1} - 61\alpha_0 - 93\alpha_1 - 126\alpha_2 - 160\alpha_3 - 195\alpha_4 - 231\alpha_5 - 153\alpha_6 - 76\alpha_7 - 115\alpha_8.$$

We now calculate

$$\prod_{\alpha\in\Delta^+}(1-e^{-t\alpha})^{\operatorname{mult}\alpha} = \sum_{w\in W}(\operatorname{sgn} w)e^{tw(\rho)-t\rho}$$

and

$$\prod_{\alpha\in\Delta^+}(1-(-1)^{(\alpha|\tau)}e^{-\alpha})^{\operatorname{mult}\alpha} = \sum_{w\in W}(\operatorname{sgn} w)(-1)^{(\tau|w(\rho)-\rho)}e^{w(\rho)-\rho}$$

It follows that

$$\prod_{\alpha\in\Delta_+^{(inv)}}(1-e^{-\alpha})^{\operatorname{mult}\alpha}\prod_{\alpha\in\Delta^+\setminus\Delta_+^{(inv)}}(1+e^{-\alpha})^{\operatorname{mult}\alpha} = \sum_{w\in W}(\operatorname{sgn} w)(-1)^{(\tau|w(\rho)-\rho)}e^{w(\rho)-\rho}$$

Multiplying by the original denominator formula, we get

$$\prod_{\alpha\in\Delta_+^{(inv)}}(1-e^{-\alpha})^{2\operatorname{mult}\alpha}\prod_{\alpha\in\Delta^+\setminus\Delta_+^{(inv)}}(1-e^{-2\alpha})^{\operatorname{mult}\alpha}$$
$$= \left(\sum_{w\in W}(\operatorname{sgn} w)(-1)^{(\tau|w(\rho)-\rho)}e^{w(\rho)-\rho}\right)\left(\sum_{w\in W}(\operatorname{sgn} w)e^{w(\rho)-\rho}\right).$$

Finally, dividing by

$$\prod_{\alpha\in\Delta^+}(1-e^{-2\alpha})^{\operatorname{mult}\alpha} = \sum_{w\in W}(\operatorname{sgn} w)e^{2w(\rho)-2\rho}$$

we obtain the requisite formula

$$\prod_{\alpha\in\Delta_+^{(inv)}}(\tanh\frac{\alpha}{2})^{\operatorname{mult}\alpha} = \frac{\left(\sum_{w\in W}(\operatorname{sgn} w)(-1)^{(\tau|w(\rho)-\rho)}e^{w(\rho)-\rho}\right)\left(\sum_{w\in W}(\operatorname{sgn} w)e^{w(\rho)-\rho}\right)}{\sum_{w\in W}(\operatorname{sgn} w)e^{2w(\rho)-2\rho}} \tag{B.4-1}$$

where we used

$$\prod_{\alpha\in\Delta_+^{(inv)}}(\tanh\frac{\alpha}{2})^{\operatorname{mult}\alpha} = \prod_{\alpha\in\Delta_+^{(inv)}}\left(\frac{1-e^{-\alpha}}{1+e^{-\alpha}}\right)^{\operatorname{mult}\alpha} = \prod_{\alpha\in\Delta_+^{(inv)}}\frac{(1-e^{-\alpha})^{2\operatorname{mult}\alpha}}{(1-e^{-2\alpha})^{\operatorname{mult}\alpha}}$$

Now we can compare (B.4-1) to a similar expression for DE_{10}. Using similar manipulations we find

$$\prod_{\alpha\in\Delta_+^{(inv)}}(\tanh\frac{\alpha}{2})^{\text{mult}'\alpha} = \frac{\left(\sum_{w'\in W(DE_{10})}(\operatorname{sgn} w')e^{w'(\rho')-\rho'}\right)^2}{\sum_{w'\in W(DE_{10})}(\operatorname{sgn} w')e^{2w'(\rho')-2\rho'}}, \qquad \rho'\equiv\rho(DE_{10}), \quad \text{(B.4-2)}$$

where mult′ denotes DE_{10} multiplicities, and we used the identification $\Delta_+^{(inv)} = \Delta^+(DE_{10})$, where $\mathfrak{h}(DE_{10})^*$ is implicitly identified with $\mathfrak{h}(E_{10})^*$ using (4.4.8).

Using (4.2.5) and (4.4.5) we can write

$$\rho' = \rho + 8\tau',$$

where τ' is some root such that

$$\tau - \tau' \in 2Q(E_{10})$$

so that

$$(-1)^{(\alpha|\tau)} = (-1)^{(\alpha|\tau')} \qquad \text{for all } \alpha\in Q(E_{10}).$$

Using (B.4-2) we get

$$\prod_{\alpha\in\Delta_+^{(inv)}}(\tanh\frac{\alpha}{2})^{\text{mult}'\alpha} = \frac{\left(\sum_{w'\in W(DE_{10})}(\operatorname{sgn} w')e^{w'(\rho)-\rho+8w'(\tau')-8\tau'}\right)^2}{\sum_{w'\in W(DE_{10})}(\operatorname{sgn} w')e^{2w'(\rho)-2\rho+16w'(\tau')-16\tau'}}$$

$$\prod_{\alpha\in\Delta_+^{(inv)}}(\tanh\frac{\alpha}{2})^{\text{mult}\,\alpha} = \frac{\left(\sum_{w\in W(E_{10})}(\operatorname{sgn} w)(-1)^{\langle\tau',w(\rho)-\rho\rangle}e^{w(\rho)-\rho}\right)\left(\sum_{w\in W(E_{10})}(\operatorname{sgn} w)e^{w(\rho)-\rho}\right)}{\sum_{w\in W(E_{10})}(\operatorname{sgn} w)e^{2w(\rho)-2\rho}}$$

Let us now discuss the relation between the Weyl groups $W(DE_{10})$ and $W(E_{10})$. We can identify both $W(DE_{10})$ and $W(E_{10})$ as subgroups of the isometry group of the dual of the Cartan subalgebra $\mathfrak{h}(DE_{10})^*\simeq\mathfrak{h}(E_{10})^*$.

Lemma 4. *We have:*

a. *$W(DE_{10})\subset W(E_{10})$;*

b. *$W(DE_{10})\simeq\{w\in W(E_{10})\colon\quad w(\tau)-\tau\in 2Q(E_{10})\}$;*

c. *The coset $W(DE_{10})/W(E_{10})$ is finite and has 527 elements.*

Proof. $W(DE_{10})$ is generated by reflections around $\gamma_{-1}, \ldots, \gamma_8$. Using (4.4.8), we can thus identify $W(DE_{10})$ with the subset of $W(E_{10})$ that is generated by reflections around the real roots $\upsilon(\gamma_{-1}), \ldots, \upsilon(\gamma_8)$. This proves part (a).

To prove (b), consider a fundamental Weyl reflection $r_i \in W(DE_{10})$ around the simple root γ_i. This maps to a reflection $r_{\upsilon(\gamma_i)} \in W(E_{10})$. We now calculate

$$r_{\upsilon(\gamma_i)}(\tau) - \tau = (\tau|\upsilon(\gamma_i))\upsilon(\gamma_i) \in 2Q(E_{10}),$$

since $(\tau|\upsilon(\gamma_i))$ is even. $W(E_{10})$ is generated by the fundamental reflections r_i, and it therefore follows that $W(DE_{10}) \subset \{w \in W(E_{10})\colon \quad w(\tau) - \tau \in 2Q(E_{10})\}$.

To prove $\supset$, take $w \in W(E_{10})$ such that

$$w(\tau) - \tau = 2\beta \qquad \text{for some } \beta \in Q(E_{10}).$$

note first that

$$(\tau|\tau) = \tau^2 = 0 \equiv 0 \pmod 2$$

and therefore $\tau \in Q^{(inv)} \simeq Q(DE_{10})$. According to Proposition 5.10b of [119], since both E_{10} and DE_{10} are hyperbolic, the Weyl groups $W(DE_{10})$ and $W(E_{10})$ are equivalent to a $\mathbb{Z}/2\mathbb{Z}$[1] quotient of the group of automorphisms of the respective root lattices $Q(DE_{10})$ and $Q(E_{10})$. The $\mathbb{Z}/2\mathbb{Z}$ quotient is the identification of the automorphisms $\phi, -\phi \in \mathrm{Aut}(Q)$. From part (a) it follows that every automorphism of $Q(DE_{10})$ can be extended to an automorphism of $Q(E_{10})$. Thus

$$q(w(\alpha)) \equiv (w(\alpha)|\tau) = (\alpha|w^{-1}(\tau)) = (\alpha|\tau - 2w^{-1}(\beta)) \equiv (\alpha|\tau) \equiv q(\alpha)$$
$$\text{for any } \alpha \in Q(DE_{10}) \simeq Q^{(inv)} \subset Q(E_{10}).$$

This proves that w preserves $Q^{(inv)} \simeq Q(DE_{10})$ and therefore

$$W(DE_{10}) \supset \{w \in W(E_{10})\colon \quad w(\tau) - \tau \in 2Q(E_{10})\}.$$

To prove (c), note that if $w_1, w_2 \in W(E_{10})$ then $w_1(\tau) - w_2(\tau) \in 2Q(E_{10})$, if and only if $w_1^{-1}w_2(\tau) - \tau \in 2Q(E_{10})$, and so, by (b), $w_1^{-1}w_2 \in W(DE_{10})$. Therefore, the map $t : W(E_{10})/W(DE_{10}) \to Q(E_{10})/2Q(E_{10})$ that sends the equivalence class of $w \in W(E_{10})$ to the equivalence class of $w(\tau) - \tau$ is an injection. This proves the finiteness of $W(E_{10})/W(DE_{10})$.

[1] Here we use the notation $\mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}_2$ to avoid any confusion with the $\mathbb{Z}_2$ orbifold group.

To count the size of $W(E_{10})/W(DE_{10})$ we need to calculate the size of the image of t. This is most conveniently done in the basis (4.2.4). Define the set

$$S{:=}\{(n_1,\dots,n_{10}):\quad n_i\in\mathbb{Z},\quad \sum_{i=1}^{10} n_i\in 3\mathbb{Z},\quad \sum_{i=1}^{10}(-1)^{n_i}\in\{-8,-6,0,2,8\}\}$$

Using (4.2.2), we see that S can be identified with a certain a subset of $Q(E_{10})$. Using (4.2.4), it is easy to check that the fundamental Weyl reflections preserve $S\subset Q(E_{10})$. S is therefore $W(E_{10})$-invariant. Using the explicit expressions found in **S**4.3, we see that $\tau\in S$. The coset $S/2Q(E_{10})$ [where we use the explicit representation of $Q(E_{10})$ as in (4.2.4)] contains

$$\binom{10}{9}+\binom{10}{8}+\binom{10}{5}+\binom{10}{4}+\binom{10}{1}=527\,\text{elements}.$$

Therefore, the image in $Q(E_{10})/2Q(E_{10})$ of the map t contains at most 527 elements. To see that it contains exactly 527 elements, recall that by Proposition 5.7 of [119] all positive prime isotropic roots of E_{10} are $W(E_{10})$-equivalent to τ. Below is a list of such roots $\alpha=w(\tau)$ [in the basis (4.2.4)] such that $t(w)=[\alpha-\tau]$ (the equivalence class of $\alpha-\tau$ in $Q(E_{10})/2Q(E_{10})$) exhaust all 527 possibilities.

$$\binom{10}{9}=10\text{ distinct permutations of}\quad(0,1,1,1,1,1,1,1,1,1),$$

$$\binom{10}{8}=45\text{ distinct permutations of}\quad(2,2,1,1,1,1,1,1,1,1)\equiv(0,0,1,1,1,1,1,1,1,1),$$

$$\binom{10}{5}=252\text{ distinct permutations of}\quad(2,2,2,2,2,1,1,1,1,1)\equiv(0,0,0,0,0,1,1,1,1,1),$$

$$\binom{10}{4}=210\text{ distinct permutations of}\quad(2,2,2,2,2,2,1,1,1,3)\equiv(0,0,0,0,0,0,1,1,1,1),$$

$$\binom{10}{1}=10\text{ distinct permutations of}\quad(2,2,2,2,2,2,2,2,4,1)\equiv(0,0,0,0,0,0,0,0,0,1).$$

The $\equiv$'s are $\pmod 2$. $\square$