

Precision holography for non-conformal branes

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ABSTRACT: We set up precision holography for the non-conformal branes preserving 16 supersymmetries. The near-horizon limit of all such p -brane solutions with $p \leq 4$, including the case of fundamental string solutions, is conformal to $AdS_{p+2} \times S^{8-p}$ with a linear dilaton. We develop holographic renormalization for all these cases. In particular, we obtain the most general asymptotic solutions with appropriate Dirichlet boundary conditions, find the corresponding counterterms and compute the holographic 1-point functions, all in complete generality and at the full non-linear level. The result for the stress energy tensor properly defines the notion of mass for backgrounds with such asymptotics. The analysis is done both in the original formulation of the method and also using a radial Hamiltonian analysis. The latter formulation exhibits most clearly the existence of an underlying generalized conformal structure. In the cases of Dp -branes, the corresponding dual boundary theory, the maximally supersymmetric Yang-Mills theory SYM_{p+1} , indeed exhibits the generalized conformal structure found at strong coupling. We compute the holographic 2-point functions of the stress energy tensor and gluon operator and show they satisfy the expected Ward identities and the constraints of generalized conformal structure. The holographic results are also manifestly compatible with the M-theory uplift, with the asymptotic solutions, counterterms, one and two point functions etc. of the IIA F1 and D4 appropriately descending from those of M2 and M5 branes, respectively. We present a few applications including the computation of condensates in Witten's model of holographic YM_4 theory.

KEYWORDS: Gauge-gravity correspondence, D-branes, AdS-CFT Correspondence.

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1. Introduction

The AdS/CFT correspondence [1] is one of the most far reaching and important ideas to emerge in recent years. On the one hand it opens a window into the strong coupling dynamics of gauge theories, whilst on the other hand it provides a qualitatively new paradigm for gravitational physics: spacetime is emergent, reconstructed from gauge theory data. A key ingredient in using gravity/gauge theory duality in such a way is the holographic dictionary. One needs to know the precise relationship between bulk and boundary physics before one can use the weakly coupled description on one side to compute quantities in the other. In the case of asymptotically $AdS \times X$ backgrounds (with X compact) the underlying principles of the correspondence were laid out in the foundational papers on the subject [2, 3]: for every bulk field Φ there is a corresponding gauge invariant operator \mathcal{O}_Φ in the boundary theory, and the bulk partition function with given boundary conditions for Φ acts as the generating functional for correlation functions of this operator.

To promote the bulk/boundary correspondence from a formal relation to a framework in which one can calculate, one needs to specify how divergences on both sides are treated. In the boundary theory, these are the UV divergences, which are dealt with by standard techniques of renormalization. In the bulk, the divergences are due to the infinite volume, and are thus IR divergences, which need to be dealt with by holographic renormalization, the precise dual of standard QFT renormalization [4–9, 11, 12]; for a review see [10]. The procedure of holographic renormalization in asymptotically AdS spacetimes allows one to extract the renormalized one point functions for local gauge invariant operators from the asymptotics of the spacetime; these can then be functionally differentiated in the standard way to obtain higher correlation functions.

By now there are many other conjectured examples of gravity/gauge theory dualities in string theory, which involve backgrounds with different asymptotics. The case of interest for us is the dualities involving non-conformal branes [13, 14] which follow from decoupling limits, and are thus believed to hold, although rather few quantitative checks of the dualities have been carried out. It is important to develop our understanding of these dualities for a number of reasons. First of all, a primary question in quantum gravity is whether the theory is holographic. Examples such as AdS/CFT indicate that the theory is indeed holographic for certain spacetime asymptotics, but one wants to know whether this holds more generally. Exploring cases where the asymptotics are different but one has a proposal for the dual field theory is a first step to addressing this question.

Secondly, the cases mentioned are interesting in their own right and have many useful applications. For example, one of the major aims of work in gravity/gauge dualities is to find holographic models which capture features of QCD. A simple model which includes confinement and chiral symmetry breaking can be obtained from the decoupling limit of a D4-brane background, with D8-branes added to include flavor, the Witten-Sakai-Sugimoto model [15–17]. This model has been used extensively to extract strong coupling behavior as a model for that in QCD. More generally, non-conformal p -brane backgrounds with $p = 0, 1, 2$ may have interesting unexploited applications to condensed matter physics; the conformal backgrounds have proved useful in modeling strong coupling behavior of transport properties and the non-conformal examples may be equally useful.

The non-conformal brane dualities have not been extensively tested, although some checks of the duality can be found in [18–21] whilst the papers [22–24] discuss the underlying symmetry structure on both sides of the correspondence. Recently, there has been progress in using lattice methods to extract field theory quantities, particularly for the D0-branes [25]. Comparing these results to the holographic predictions serves both to test the duality, and conversely to test lattice techniques (if one assumes the duality holds).

Given the increasing interest in these gravity/gauge theory dualities, one would like to develop precision holography for the non-conformal branes, following the same steps as in AdS: one wants to know exactly how quantum field theory data is encoded in the asymptotics of the spacetime. Precision holography has not previously been extensively developed for non-conformal branes (see however [26–30]), although as we will see the analysis is very close to the analysis of the Asymptotically AdS case. The reason is that the non-conformal branes admit a generalized conformal symmetry [22–24]: there is an underlying conformal symmetry structure of the theory, provided that the string coupling (or in the gauge theory, the Yang-Mills coupling) is transformed as a background field of appropriate dimension under conformal transformations. Whilst this is not a symmetry in the strict sense of the word, the underlying structure can be used to derive Ward identities and perhaps even prove non-renormalization theorems.

In this paper we develop in detail how quantum field theory data can be extracted from the asymptotics of non-conformal brane backgrounds. We begin in section 2 by recalling the correspondence between non-conformal brane backgrounds and quantum field theories. We also introduce the dual frame, in which the near horizon metric is $AdS_{p+2} \times S^{8-p}$. In section 3 we give the field equations in the dual frame for both D-brane and fundamental string solutions.

In the near horizon region of the supergravity solutions conformal symmetry is broken only by the dilaton profile. This means that the background admits a generalized conformal structure: it is invariant under generalized conformal transformations in which the string coupling is also transformed. This generalized conformal structure and its implications are discussed in section 4.

Next we proceed to set up precision holography. The basic idea is to obtain the most general asymptotic solutions of the field equations with appropriate Dirichlet boundary conditions. Given such solutions, one can identify the divergences of the onshell action, find the corresponding counterterms and compute the holographic 1-point functions, in

complete generality and at the non-linear level. This is carried out in section 5. In particular, we give renormalized one point functions for the stress energy tensor and the gluon operator, in the presence of general sources, for all cases.

In section 6 we proceed to develop a radial Hamiltonian formulation for the holographic renormalization. As in the asymptotically AdS case, the Hamiltonian formulation is more elegant and exhibits clearly the underlying generalized conformal structure. In the following sections, 7 and 8, we give a number of applications of the holographic formulae. In particular, in section 7 we compute two point functions and in section 8 we compute condensates in Witten's model of holographic QCD and the renormalized action, mass etc in a non-extremal D1-brane background.

In section 9 we give conclusions and a summary of our results. The appendices A, B, C and D contain a number of useful formulae and technical details. Appendix A summarizes useful formulae for the expansion of the curvature whilst appendix B discusses the holographic computation of the stress energy tensor for asymptotically AdS_{D+1} , with $D = 4, 6$; in the latter the derivation is streamlined, relative to earlier discussions, and the previously unknown traceless, covariantly constant contributions to the stress energy tensor in six dimensions are determined. Appendix C contains the detailed relationship between the M5-brane and D4-brane holographic analysis whilst appendix D gives explicit expressions for the asymptotic expansion of momenta.

The results of this work have been reported at a number of recent conferences [31]. As this paper was finalized we received [32] which contains related results.

2. Non-conformal branes and the dual frame

Let us begin by recalling the brane solutions of supergravity, see for example [33] for a review. The relevant part of the supergravity action in the string frame is

$$S = \frac{1}{(2\pi)^7 \alpha'^4} \int d^{10}x \sqrt{-g} \left[e^{-2\phi} \left(R + 4(\partial\phi)^2 - \frac{1}{12} H_3^2 \right) - \frac{1}{2(p+2)!} F_{p+2}^2 \right]. \quad (2.1)$$

The Dp-brane solutions can be written in the form:

$$\begin{aligned} ds^2 &= (H^{-1/2} ds^2(E^{p,1}) + H^{1/2} ds^2(E^{9-p})); \\ e^\phi &= g_s H^{(3-p)/4}, \\ C_{0\dots p} &= g_s^{-1} (H^{-1} - 1) \quad \text{or} \quad F_{8-p} = g_s^{-1} *_{9-p} dH, \end{aligned} \quad (2.2)$$

where the latter depends on whether the brane couples electrically or magnetically to the field strength. Here g_s is the string coupling constant. We are interested in the simplest supersymmetric solutions, for which the defining function H is harmonic on the flat space E^{9-p} transverse to the brane. Choosing a single-centered harmonic function

$$H = 1 + \frac{Q_p}{r^{7-p}}, \quad (2.3)$$

then the parameter Q_p for the brane solutions of interest is given by $Q_p = d_p N g_s l_s^{7-p}$ with the constant d_p equal to $d_p = (2\sqrt{\pi})^{5-p} \Gamma(\frac{7-p}{2})$, whilst $l_s^2 = \alpha'$ and N denotes the integral quantized charge.

Soon after the AdS/CFT duality was proposed [1], it was suggested that an analogous correspondence exists between the near-horizon limits of non-conformal D-brane backgrounds and (non-conformal) quantum field theories [13]. More precisely, one considers the field theory (or decoupling) limit to be:

$$g_s \rightarrow 0, \quad \alpha' \rightarrow 0, \quad U \equiv \frac{r}{\alpha'} = \text{fixed}, \quad g_d^2 N = \text{fixed}, \quad (2.4)$$

where g_d^2 is the Yang-Mills coupling, related to the string coupling by

$$g_d^2 = g_s (2\pi)^{p-2} (\alpha')^{(p-3)/2}. \quad (2.5)$$

Note that N can be arbitrary for $p < 3$ but (2.4) requires that $N \rightarrow \infty$ when $p > 3$. The decoupling limit implies that the constant part in the harmonic function is negligible:

$$H = 1 + \frac{D_p g_d^2 N}{\alpha'^2 U^{7-p}} \Rightarrow \frac{1}{\alpha'^2} \frac{D_p g_d^2 N}{U^{7-p}}, \quad (2.6)$$

where $D_p \equiv d_p (2\pi)^{2-p}$.

The corresponding dual $(p+1)$ -dimensional quantum field theory is obtained by taking the low energy limit of the $(p+1)$ -dimensional worldvolume theory on N branes. In the case of the Dp -branes this theory is the dimensional reduction of $\mathcal{N} = 1$ SYM in ten dimensions. Recall that the action of ten-dimensional SYM is given by

$$S_{10} = \int d^{10}x \sqrt{-g} \text{Tr} \left(-\frac{1}{4g_{10}^2} F_{mn} F^{mn} + \frac{i}{2} \bar{\psi} \Gamma^m [D_m, \psi] \right), \quad (2.7)$$

with $D_m = \partial_m - iA_m$. The dimensional reduction to d dimensions gives the bosonic terms

$$S_d = \int d^d x \sqrt{-g} \text{Tr} \left(-\frac{1}{4g_d^2} F_{ij} F^{ij} - \frac{1}{2} D_i X D^i X + \frac{g_d^2}{4} [X, X]^2 \right) \quad (2.8)$$

where $i = 0, \dots, (d-1)$ and there are $(9-p)$ scalars X . The fermionic part of the action will not play a role here. Note that the Yang-Mills coupling in $d = (p+1)$ dimensions, g_d^2 , has (length) dimension $(p-3)$, and thus the theory is not renormalizable for $p > 3$. Since the coupling constant is dimensionful, the effective dimensionless coupling constant $g_{\text{eff}}^2(E)$ is

$$g_{\text{eff}}^2(E) = g_d^2 N E^{p-3}. \quad (2.9)$$

at a given energy scale E .

This discussion of the decoupling limit applies to D-branes, but we will also be interested in fundamental strings. The fundamental string solutions can be written in the form:

$$\begin{aligned} ds^2 &= (H^{-1} ds^2(E^{1,1}) + ds^2(E^8)); \\ e^\phi &= g_s H^{-1/2}; \\ B_{01} &= (H^{-1} - 1), \end{aligned} \quad (2.10)$$

where the harmonic function $H = 1 + Q_{F1}/r^6$ with $Q_{F1} = d_1 N g_s^2 l_s^6$. For completeness, let us also mention that the NS5-brane solutions can be written in the form:

$$\begin{aligned} ds^2 &= (ds^2(E^{1,5}) + H ds^2(E^4)); \\ e^\phi &= g_s H^{1/2}; \\ H_3 &= *_4 dH, \end{aligned} \tag{2.11}$$

where the harmonic function $H = 1 + Q_{NS5}/r^2$ with $Q_{NS5} = N l_s^2$.

Whilst the fundamental string solutions have a near string region which is conformal to $AdS_3 \times S^7$ with a linear dilaton, they do not appear to admit a decoupling limit like the one in (2.4) which decouples the asymptotically flat region of the geometry and has a clear meaning from the worldsheet point of view. Nonetheless one can discuss holography for such conformally $AdS_3 \times S^7$ linear dilaton backgrounds, using S duality and the relation to M2-branes: IIB fundamental strings can be included in the discussion by applying S duality to the D1 brane case, and IIA fundamental strings by using the fact they are related to M2 branes wrapped on the M-theory circle.

In the cases of Dp-branes the decoupled region is conformal to $AdS_{p+2} \times S^{8-p}$ and there is a non-vanishing dilaton. The same holds for the near string region of the fundamental string solutions. This implies that there is a Weyl transformation such that the metric is exactly $AdS_{p+2} \times S^{8-p}$. This Weyl transformation brings the string frame metric g_{st} to the so-called *dual frame* metric g_{dual} [14] and is given by

$$ds_{dual}^2 = (N e^\phi)^c ds_{st}^2, \tag{2.12}$$

with

$$c = -\frac{2}{(7-p)} \quad \text{Dp.} \tag{2.13}$$

In this frame the action is

$$S = \frac{N^2}{(2\pi)^7 \alpha'^4} \int d^{10}x \sqrt{-g} (N e^\phi)^\gamma \left(R + 4 \frac{(p-1)(p-4)}{(7-p)^2} (\partial\phi)^2 - \frac{1}{2(p+2)! N^2} F_{p+2}^2 \right). \tag{2.14}$$

with $\gamma = 2(p-3)/(7-p)$. The terminology dual frame has the following origin. Each p -brane couples naturally to a $(p+1)$ potential. The corresponding (Hodge) dual field strength is an $(8-p)$ form. In the dual frame this field strength and the graviton couple to the dilaton in the same way. For example the dual frame of the NS5 branes is the string frame: the dual $(8-p)$ form is H_3 and the metric and H_3 couple the same way to the dilaton in the string frame, as can be seen from (2.1).¹

¹The dual frame was originally introduced in [34] and the rationale behind its introduction was the following. If one has a formulation where the fundamental degrees of freedom are p -branes that couple electrically to a p form, then one expects there to exist non-singular magnetic solitonic solutions. Let us exemplify this for the case of perturbative strings, where the elementary objects are strings. The corresponding magnetic objects, the NS5 branes, indeed appear as solitonic objects. Moreover, the target space metric and the B field couple to the coupling constant of the theory, the dilaton, in the same way, so the low energy effective action is in the string frame. Suppose now that there is a formulation where the

The D5-brane behaves qualitatively differently, as the solution in the dual frame is a linear dilaton background with metric $E^{5,1} \times R \times S^3$:

$$ds_{\text{dual}}^2 = ds^2(E^{5,1}) + Q \left(\frac{dr^2}{r^2} + d\Omega_3^2 \right); \quad (2.15)$$

$$e^\phi = \frac{r}{\sqrt{Q}}; \quad F_3 = Q d\Omega_3.$$

Holography for both D5 and NS5 branes involves such linear dilaton background geometries, and will not be discussed further in this paper.

Here we will be interested in precision holography for the cases where the geometry is conformal to $AdS_{p+2} \times S^{8-p}$; this encompasses Dp-branes with $p = 0, 1, 2, 3, 4, 6$. In all such cases the dual frame solution takes the form

$$ds_{\text{dual}}^2 = \alpha' d_p^{\frac{2}{(7-p)}} \left(D_p^{-1} (g_d^2 N)^{-1} U^{5-p} ds^2(E^{p,1}) + \frac{dU^2}{U^2} + d\Omega_{8-p}^2 \right); \quad (2.16)$$

$$e^\phi = \frac{1}{N} (2\pi)^{2-p} D_p^{(3-p)/4} ((g_d^2 N) U^{p-3})^{(7-p)/4},$$

with the field strength being

$$F_{8-p} = (7-p) d_p N (\alpha')^{(7-p)/2} d\Omega_{8-p}. \quad (2.17)$$

Note that the factors of α' cancel in the effective supergravity action, with only dependence on the dimensionful 't Hooft coupling and N remaining. It is convenient to express the field strength magnetically; for $p < 3$ this should be interpreted as $F_{p+2} = *F_{8-p}$, with the Hodge dual being taken in the dual frame metric.

Changing the variable,

$$u^2 = \mathcal{R}^{-2} (D_p g_d^2 N)^{-1} U^{5-p}, \quad \mathcal{R} = \frac{2}{5-p}, \quad (2.18)$$

brings the AdS metric into the standard form

$$ds_{\text{dual}}^2 = \alpha' d_p^{\frac{2}{7-p}} \left[\mathcal{R}^2 \left(\frac{du^2}{u^2} + u^2 ds^2(E^{p,1}) \right) + d\Omega_{8-p}^2 \right], \quad (2.19)$$

$$e^\phi = \frac{1}{N} (2\pi)^{2-p} (g_d^2 N)^{\frac{(7-p)}{2(5-p)}} D_p^{\frac{(3-p)}{2(p-5)}} (\mathcal{R}^2 u^2)^{\frac{(p-3)(p-7)}{4(p-5)}}.$$

with the field strength being (2.17). Note that by rescaling the metric, dilaton and field strength as

$$ds_{\text{dual}}^2 = \alpha' d_p^{\frac{2}{7-p}} \tilde{d}s^2; \quad N e^\phi = (2\pi)^{2-p} (g_d^2 N)^{\frac{(7-p)}{2(5-p)}} D_p^{\frac{(3-p)}{2(p-5)}} e^{\tilde{\phi}}; \quad F_{8-p} = d_p N (\alpha')^{(7-p)/2} \tilde{F}_{8-p}.$$

elementary degrees of freedom are p -branes. Then one would anticipate that there exist smooth solitonic $(6-p)$ -brane solutions of the theory of the effective action in the p -frame, which is precisely the dual frame. Indeed, the spacetime metric of Dp branes when expressed in the dual frame is non-singular. We should note though that there is currently no formulation of string theory where p branes appear to be the elementary degrees of freedom. Other special properties of the dual frame solutions are discussed in [35, 36].

the factors of D_p , N and the 't Hooft coupling can be absorbed into the overall normalization of the action.

It has been argued in [14] that the dual frame is the holographic frame in the sense that the radial direction u in this frame is identified with the energy scale of the boundary theory,

$$u \sim E. \tag{2.20}$$

More properly, as we will discuss later, the dilatations of the boundary theory are identified with rescaling of the u coordinate. Using (2.20) and (2.9) the dilaton in (2.19) and for the case of D-branes becomes

$$e^\phi = \frac{1}{N} c_d (g_{\text{eff}}^2(u))^{\frac{7-p}{2(5-p)}}, \quad c_d = (2\pi)^{2-p} D_p^{\frac{(p-3)}{2(5-p)}} \mathcal{R}^{\frac{(p-3)(7-p)}{2(5-p)}}. \tag{2.21}$$

The validity of the various approximations was discussed in [13, 37, 14]. In particular, we consider the large N limit, keeping fixed the effective coupling constant g_{eff}^2 , so the dilaton is small in all cases (recall that the decoupling limit when $p > 3$ requires $N \rightarrow \infty$). If $g_{\text{eff}}^2 \ll 1$ then the perturbative SYM description is valid, whereas in the opposite limit $g_{\text{eff}}^2 \gg 1$ the supergravity approximation is valid.

As a consistency check, one can also derive (2.21) using the open string description. The low energy description in the string frame is given by

$$S_{\text{st}} = -\frac{1}{(2\pi)^{p-2} (\alpha')^{(p-3)/2}} \int d^{p+1}x \sqrt{-g_{\text{st}}} e^{-\phi} \frac{1}{4} \text{Tr}(F_{ij} F_{kl}) g_{\text{st}}^{ik} g_{\text{st}}^{jl} + \dots, \tag{2.22}$$

where we indicate explicitly that the metric involved is the string frame metric. In the case of flat target spacetime, g_{st} is the Minkowski metric and $e^\phi = g_s$ and we recover (2.5) by identifying the overall prefactor of $\text{Tr}F^2$ with $1/(4g_d^2)$. In our case, transforming to the dual frame and using the form of the metric in (2.19) we get

$$S_{\text{dual}} = -\frac{\mathcal{R}^{p-3} d_p^{\frac{(p-3)}{(7-p)}}}{(2\pi)^{p-2}} \int d^{p+1}x (N e^\phi)^{\frac{2(p-5)}{(7-p)}} (N u^{p-3}) \frac{1}{4} (\text{Tr}F^2) + \dots \tag{2.23}$$

where now the Lorentz index contractions in $\text{Tr}F^2$ are with the Minkowski metric. Identifying now the overall prefactor of $\text{Tr}F^2$ with $1/(4g_d^2)$ is indeed equivalent to (2.21).

As mentioned above, we will also include fundamental strings in our analysis, exploiting the relation to D1-branes and M2-branes. In this case we focus on the near string geometry, dropping the constant term in the harmonic function, and introduce a dual frame metric $ds_{\text{dual}}^2 = (N e^\phi)^c ds_{\text{st}}^2$ with

$$c = -\frac{2}{3} \quad \text{F1}, \tag{2.24}$$

with the dual frame metric being $AdS_3 \times S^7$. The detailed form of the effective action in the dual frame will be given in the next section.

The aim of this paper will be to consider solutions which asymptote to the decoupled non-conformal brane backgrounds and show how renormalized quantum field theory information can be extracted from the geometry. It may be useful to recall first how the

conformal case of $p = 3$ works. Given the $AdS_5 \times S^5$ background, the spectrum of supergravity fluctuations about this background corresponds to the spectrum of single trace gauge invariant chiral primary operators in the dual $\mathcal{N} = 4$ SYM theory. The spectrum includes stringy modes and D-branes, which correspond to other non primary, high dimension and non-local operators in the dual $\mathcal{N} = 4$ SYM theory. Encoded in the asymptotics of any asymptotically $AdS_5 \times S^5$ supergravity background are one point functions of the chiral primary operators. These allow one to extract the vacuum structure of the dual theory (its vevs and deformation parameters), and if one switches on sources one can also extract higher correlation functions.

The sphere in this background has a radius which is of the same order as the AdS radius, so the higher KK modes are not suppressed relative to the zero modes and one cannot ignore them. It is nevertheless possible to only keep a subset of modes when the equations of motion admit solutions with all modes except the ones kept set equal to zero, i.e. there exist consistent truncations. The existence of such truncations signify the existence of a subset of operators of the dual theory that are closed under OPEs. The resulting theory is a $(d + 1)$ -dimensional gauged supergravity and such gauged supergravity theories have been the starting point for many investigations in AdS/CFT. Gauged supergravity retains only the duals to low dimension chiral primaries in SYM, those in the same multiplet as the stress energy tensor. More recently, the method of Kaluza-Klein holography [46] has been developed to extract systematically one point functions of all other single trace chiral operators.

The goal here is to take the first step in holographic renormalization for non-conformal branes. We will consistently truncate the bulk theory to just the $(p + 2)$ -dimensional graviton and the dilaton, and compute renormalized correlation functions in this sector. Unlike the $p = 3$ case one must retain the dilaton as it is running: the gauge coupling of the dual theory is dimensionful and runs. Such a truncation was considered already in [14] and we will recall the resulting $(p + 2)$ -dimensional action in the next section. Given an understanding of holographic renormalization in this truncated sector, it is straightforward to generalize this setup to include fields dual to other gauge theory operators.

3. Lower dimensional field equations

The supergravity solutions for Dp-branes and fundamental strings in the decoupling limit can be best analyzed by going to the *dual frame* reviewed in the previous section, (2.12) and (2.24). The dual frame is defined as $ds_{\text{dual}}^2 = (Ne^\phi)^c ds^2$, with $c = -2/(7 - p)$ for Dp-branes and $c = -2/3$ for fundamental strings. The Weyl transformation to the dual frame in ten dimensions results in the following action:

$$S = -\frac{N^2}{(2\pi)^7 \alpha'^4} \int d^{10}x \sqrt{g} N^\gamma e^{\gamma\phi} \left[R + \beta(\partial\phi)^2 - \frac{1}{2(8-p)! N^2} |F_{8-p}|^2 \right] \quad (3.1)$$

where the constants (β, γ) are given below in (3.5) for Dp-branes and (3.6) for fundamental strings respectively. Note that it is convenient to express the field strength magnetically; for $p < 3$ this should be interpreted as $F_{p+2} = *F_{8-p}$. From here onwards we will also work in Euclidean signature.

For $p \neq 5$, the field equations in this frame admit $AdS_{p+2} \times S^{8-p}$ solutions with linear dilaton. One can reduce the field equations over the sphere, truncating to the $(p+2)$ -dimensional graviton $\tilde{g}_{\mu\nu}$ and scalar $\tilde{\phi}$. For the Dp-branes the reduction ansatz is

$$\begin{aligned} ds_{\text{dual}}^2 &= \alpha' d_p^{-c} (\mathcal{R}^2 \tilde{g}_{\mu\nu}(x^\rho) dx^\mu dx^\nu + d\Omega_{8-p}^2); \\ F_{8-p} &= (7-p) g_s^{-1} Q_p d\Omega_{8-p}; \\ e^\phi &= g_s (r_o^2 \mathcal{R}^2)^{(p-3)(7-p)/4(5-p)} e^{\tilde{\phi}}, \end{aligned} \tag{3.2}$$

with $r_o^{7-p} \equiv Q_p$ and $\mathcal{R} = 2/(5-p)$. The ten-dimensional metric is in the dual frame and prefactors are chosen to absorb the radius and overall metric and dilaton prefactors of the AdS_{p+2} solution. For the fundamental string one reduces the near horizon geometry as:

$$\begin{aligned} ds_{\text{dual}}^2 &= \alpha' (d_1 N^{-1})^{1/3} (\mathcal{R}^2 \tilde{g}_{\mu\nu}(x^\rho) dx^\mu dx^\nu + d\Omega_7^2); \\ H_7 &= 6Q_{F1} d\Omega_7; \\ e^\phi &= g_s (r_o \mathcal{R})^{3/2} e^{\tilde{\phi}}, \end{aligned} \tag{3.3}$$

where $H_7 = *H_3$, $r_o^6 \equiv Q_{F1}$ and $\mathcal{R} = 2/(5-p)$. It is then straightforward to show that the equations of motion for the lower-dimensional fields for both Dp-branes and fundamental strings follow from an action of the form:

$$S = -L \int d^{d+1}x \sqrt{\tilde{g}} e^{\gamma\tilde{\phi}} [\tilde{R} + \beta(\partial\tilde{\phi})^2 + C]. \tag{3.4}$$

Here $d = p + 1$ and the constants (L, β, γ, C) depend on the case of interest; since from here onwards we are interested only in $(d+1)$ -dimensional fields we suppress their tilde labeling. For Dp-branes the constants are given by

$$\begin{aligned} \gamma &= \frac{2(p-3)}{7-p}, & \beta &= \frac{4(p-1)(p-4)}{(7-p)^2}, \\ \mathcal{R} &= \frac{2}{5-p}, & C &= \frac{1}{2}(9-p)(7-p)\mathcal{R}^2, \\ L &= \frac{\Omega_{8-p} r_o^{(7-p)^2/(5-p)} \mathcal{R}^{(9-p)/(5-p)}}{(2\pi)^7 \alpha'^4} = \frac{(d_p N)^{(7-p)/(5-p)} g_d^{2(p-3)/(5-p)} \mathcal{R}^{(9-p)/(5-p)}}{64\pi^{(5+p)/2} (2\pi)^{(p-3)(p-2)/(5-p)} \Gamma(\frac{9-p}{2})}. \end{aligned} \tag{3.5}$$

For the fundamental string one gets instead:

$$\begin{aligned} \gamma &= \frac{2}{3}, & \beta &= 0, & C &= 6, \\ L &= \frac{\Omega_7 r_o^9}{4(2\pi)^7 g_s^2 (\alpha')^4} = \frac{g_s N^{3/2} (\alpha')^{1/2}}{6\sqrt{2}}, \end{aligned} \tag{3.6}$$

This expression is related to that for the D1-brane background by $g_s \rightarrow 1/g_s$ with $\alpha' \rightarrow \alpha' g_s$, as one would expect from S duality. The truncation is consistent, as one can show that any solution of the lower-dimensional equations of motion also solves the ten-dimensional equations of motion, using the reduction given in (3.2). Note that more general reductions of type II theories on spheres to give gauged supergravity theories were discussed in [39].

These reductions would be relevant if one wants to include additional operators in the boundary theory, beyond the stress energy tensor and scalar operator.

In both cases the equations of motion admit an AdS_{d+1} solution

$$ds^2 = \frac{d\rho^2}{4\rho^2} + \frac{dx_i dx^i}{\rho}; \quad (3.7)$$

$$e^\phi = \rho^\alpha,$$

where $i = 1 \cdots d$. Note that ρ is related to the radial coordinate u used earlier by $\rho = 1/u^2$. The constant α again depends on the case of interest:

$$\alpha = -\frac{(p-7)(p-3)}{4(p-5)}; \quad \text{Dp} \quad (3.8)$$

$$\alpha = -\frac{3}{4}; \quad \text{F1.}$$

Note that for computational convenience the metric and dilaton have been rescaled relative to [14] to set the AdS radius to one and to pull all factors of N and g_s into an overall normalization factor. The radial variable ρ then has length dimension 2 and e^ϕ has length dimension 2α .

For arbitrary d, β and γ , the field equations for the metric and scalar field following from (3.4) are²

$$-R_{\mu\nu} + (\gamma^2 - \beta)\partial_\mu\phi\partial_\nu\phi + \gamma\nabla_\mu\partial_\nu\phi + \frac{1}{2}g_{\mu\nu}[R + (\beta - 2\gamma^2)(\partial\phi)^2 - 2\gamma\nabla^2\phi + C] = 0,$$

$$\gamma R - \beta\gamma(\partial\phi)^2 + C\gamma - 2\beta\nabla^2\phi = 0. \quad (3.9)$$

These equations admit an AdS solution with linear dilaton provided that α and C satisfy

$$\alpha = -\frac{\gamma}{2(\gamma^2 - \beta)}, \quad C = \frac{(d(\gamma^2 - \beta) + \gamma^2)(d(\gamma^2 - \beta) + \beta)}{(\gamma^2 - \beta)^2}. \quad (3.10)$$

We can thus treat both Dp-brane and fundamental string cases simultaneously, by processing the field equations for arbitrary (d, β, γ) and writing (α, C) in terms of these parameters. It might be interesting to consider whether other choices of (d, β, γ) admit interesting physical interpretations.

By taking the trace of the first equation in (3.9) and combining it with the second one can obtain the more convenient three equations

$$-R_{\mu\nu} + (\gamma^2 - \beta)\partial_\mu\phi\partial_\nu\phi + \gamma\nabla_\mu\partial_\nu\phi - \frac{\gamma^2 + d(\gamma^2 - \beta)}{\gamma^2 - \beta}g_{\mu\nu} = 0, \quad (3.11)$$

$$\nabla^2\phi + \gamma(\partial\phi)^2 - \frac{\gamma(d(\gamma^2 - \beta) + \gamma^2)}{(\gamma^2 - \beta)^2} = 0,$$

$$R + \beta(\partial\phi)^2 + \frac{(d(\gamma^2 - \beta) + \gamma^2)(d(\gamma^2 - \beta) - \beta)}{(\gamma^2 - \beta)^2} = 0,$$

where the last line follows from the first two.

²Our conventions for the Riemann and Ricci tensor are $R^\sigma{}_{\mu\nu\rho} = -2\Gamma^\sigma{}_{\mu[\nu,\rho]} - 2\Gamma^\sigma{}_{\mu[\nu}\Gamma^\sigma{}_{\rho]\tau}$, $R_{\mu\nu} = R^\sigma{}_{\mu\sigma\nu}$.

The type IIA fundamental strings and D4-branes are related to the M theory M2-branes and M5-branes respectively under dimensional reduction along a worldvolume direction. The M brane theories fall within the framework of AdS/CFT, with the correspondence being between $AdS_4 \times S^7$ and $AdS_7 \times S^4$ geometries, respectively, and the still poorly understood conformal worldvolume theories. Reducing on the spheres gives four and seven dimensional gauged supergravity, respectively, which can be truncated to Einstein gravity with negative cosmological constant. That is, the effective actions are simply

$$S_M = -L_M \int d^{d+2}x \sqrt{G} (R(G) + d(d+1)), \quad (3.12)$$

where $d = 2$ for the M2-brane and $d = 5$ for the M5-brane. The normalization constant is

$$L_{M2} = \frac{\sqrt{2}N^{3/2}}{24\pi}; \quad L_{M5} = \frac{N^3}{3\pi^3}. \quad (3.13)$$

and the action clearly admits an AdS_{d+2} -dimensional space with unit radius as a solution:

$$ds^2 = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}(dx_i dx^i + dy^2), \quad (3.14)$$

where $i = 1, \dots, d$.

Now consider a diagonal dimensional reduction of the $(d+2)$ -dimensional solution over y , i.e. let the metric be

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu + e^{4\phi(x)/3}dy^2. \quad (3.15)$$

Substituting into the $(d+2)$ -dimensional field equations gives precisely the field equations following from the action (3.4); note that $\gamma = 2/3, \beta = 0$ for both the fundamental string and D4-branes. It may be useful to recall here that the standard dimensional reduction of an M theory metric to a (string frame) type IIA metric g_{MN} is

$$ds_{11}^2 = e^{-2\phi/3}g_{MN}dx^M dx^N + e^{4\phi/3}dy_{11}^2. \quad (3.16)$$

The relation between dual frame and string frame metrics given in (2.12) leads to (3.15). Note that

$$L = L_M(2\pi R_y) = 2\pi g_s l_s L_M, \quad (3.17)$$

where we use the standard relation for the radius of the M theory circle.

The other Dp-branes of type IIA are of course also related to M theory objects: the D0-brane background uplifts to a gravitational wave background, the D6-brane background uplifts to a Kaluza-Klein monopole background whilst the D2-branes are related to the reduction of M2-branes transverse to the worldvolume. These connections will not play a role in this paper. The uplifts reviewed above are useful here as holographic renormalization for the conformal branes is well understood, but holography for gravitational wave backgrounds and Kaluza-Klein monopoles is less well understood than that for the non-conformal branes.

One could use a different reduction and truncation of the theory in the $AdS_4 \times S^7$ background to obtain the action (3.4) for D2-branes. In this case one would embed the M theory circle into the S^7 , and then truncate to only the four-dimensional graviton, along with the scalar field associated with this M theory circle. This reduction will not however be used here.

4. Generalized conformal structure

In this section we will discuss the underlying generalized conformal structure of the non-conformal brane dualities. Recall that the corresponding worldvolume theory is SYM_{p+1} . We will be interesting in computing correlation functions of gauge invariant operators in this theory. Recall that gauge/gravity duality maps bulk fields to boundary operators. In our discussion in the previous section we truncated the bulk theory to gravity coupled to a scalar field in $(d + 1)$ dimensions. The bulk metric corresponds to the stress energy tensor as usual, while as we will see the scalar field corresponds to a scalar operator of dimension four. As usual the fields that parametrize their boundary conditions are identified with sources that couple to gauge invariant operators.

Consider the following $(p + 1)$ -dimensional (Euclidean) action,

$$S_d[g_{(0)ij}(x), \Phi_{(0)}(x)] = - \int d^d x \sqrt{g_{(0)}} \left(-\Phi_{(0)} \frac{1}{4} \text{Tr} F_{ij} F^{ij} + \frac{1}{2} \text{Tr} \left(X \left(D^2 - \frac{(d-2)}{4(d-1)} R \right) X \right) + \frac{1}{4\Phi_{(0)}} \text{Tr}[X, X]^2 \right). \quad (4.1)$$

where $g_{(0)ij}$ is a background metric $\Phi_{(0)}(x)$ is a scalar background field. Setting

$$g_{(0)ij} = \delta_{ij}, \quad \Phi_{(0)} = \frac{1}{g_d^2}, \quad (4.2)$$

the action (4.1) becomes equal to the action of the SYM_{p+1} given in (2.8) (here and it what follows we suppress the fermionic terms). The action (4.1) is invariant under the following Weyl transformations

$$g_{(0)} \rightarrow e^{2\sigma} g_{(0)}, \quad X \rightarrow e^{(1-\frac{d}{2})\sigma} X, \quad A_i \rightarrow A_i, \quad \Phi_{(0)} \rightarrow e^{-(d-4)\sigma} \Phi_{(0)} \quad (4.3)$$

Note that the combination $P_1 = D^2 - \frac{d-2}{4(d-1)} R$, is the conformal Laplacian in d dimensions, which transforms under Weyl transformations as $P_1 \rightarrow e^{-(d/2+1)\sigma} P_1 e^{(d/2-1)\sigma}$.

Let us now define,

$$T_{ij} = \frac{2}{\sqrt{g_{(0)}}} \frac{\delta S_d}{\delta g_{(0)}^{ij}}, \quad \mathcal{O} = \frac{1}{\sqrt{g_{(0)}}} \frac{\delta S_d}{\delta \Phi_{(0)}} \quad (4.4)$$

They are given by

$$T_{ij} = \text{Tr} \left(\Phi_{(0)} F_{ik} F_j^k + D_i X D_j X + \frac{d-2}{4(d-1)} (X^2 R_{ij} - D_i D_j X^2 + g_{(0)ij} D^2 X^2) - g_{(0)ij} \left(\frac{1}{4} \Phi_{(0)} F^2 + \frac{1}{2} (DX)^2 + \frac{(d-2)}{8(d-1)} R X^2 - \frac{1}{4\Phi_{(0)}} [X, X]^2 \right) \right) \quad (4.5)$$

$$\mathcal{O} = \text{Tr} \left(\frac{1}{4} F^2 + \frac{1}{4\Phi_{(0)}} [X, X]^2 \right). \quad (4.6)$$

Using standard manipulations, see for example [8, 9], we obtain the standard diffeomorphism and trace Ward identities,

$$\nabla^j \langle T_{ij} \rangle_J + \langle \mathcal{O} \rangle_J \partial_i \Phi_{(0)} = 0, \quad (4.7)$$

$$\langle T_i^i \rangle_J + (d-4) \Phi_{(0)} \langle \mathcal{O} \rangle_J = 0, \quad (4.8)$$

where $\langle B \rangle_J$ denotes an expectation value of B in the presence of sources J . One can verify that these relations are satisfied at the classical level, i.e. by using (4.5) and the equations of motion that follow from (4.1). Setting $g_{(0)ij} = \delta_{ij}$, $\Phi_{(0)} = g_d^{-2}$ one recovers the conservation of the energy momentum tensor of the SYM $_d$ theory and the fact that conformal invariance is broken by the dimensionful coupling constant. Note that the kinetic part of the scalar field does not contribute to the breaking of conformal invariance because this part of the action is conformally invariant in any dimension (using the conformal Laplacian). This also dictates the position of the coupling constant in (2.8). In a flat background one can change the position of the coupling constant by rescaling the fields. For example, by rescaling $X \rightarrow X/g_d$ the coupling constant becomes an overall constant. This is the normalization one gets from worldvolume D-brane theory in the string frame. This action however does not generalize naturally to a Weyl invariant action. Instead it is (2.8) (with the coupling constant promoted to a background field) that naturally couples to a metric in a Weyl invariant way.

The Ward identities (4.7) lead to an infinite number of relations for correlation functions obtained by differentiating with respect to the sources and setting the sources to $g_{(0)ij} = \eta_{ij}$, where η_{ij} is the Minkowski metric and $\Phi_{(0)} = 1/g_d^2$. The first non-trivial relations are at the level of 2-point functions ($x \neq 0$).

$$\begin{aligned} \partial_x^j \langle T_{ij}(x) T_{kl}(0) \rangle &= 0, \\ \partial_x^j \langle T_{ij}(x) \mathcal{O}(0) \rangle &= 0 \\ \langle T_i^i(x) T_{kl}(0) \rangle + (p-3) \frac{1}{g_d^2} \langle \mathcal{O}(x) T_{kl}(0) \rangle &= 0 \\ \langle T_i^i(x) \mathcal{O}(0) \rangle + (p-3) \frac{1}{g_d^2} \langle \mathcal{O}(x) \mathcal{O}(0) \rangle &= 0. \end{aligned} \tag{4.9}$$

The Ward identities (4.7) were derived by formal path integral manipulations and one should examine whether they really hold at the quantum level. Firstly, for the case of the D4 brane the worldvolume theory is non-renormalizable, so one might question whether the correlators themselves are meaningful. At weak coupling, renormalizing the correlators would require introducing new higher dimension operators in the action, as well as counterterms that depend on the background fields. This process should preserve diffeomorphism and supersymmetry, but it may break the Weyl invariance. Introducing a new source $\Phi_{(0)}^j$ for every new higher dimension operator \mathcal{O}_j added in the process of renormalization would then modify the trace Ward identity as

$$\langle T_i^i \rangle - \sum_{j \geq 0} (d - \Delta_j) \Phi_{(0)}^j \langle \mathcal{O}_j \rangle = \mathcal{A}, \tag{4.10}$$

where Δ_j is the dimension of the operator \mathcal{O}_j (with $\Phi_{(0)}^0 = \Phi_{(0)}$, $\mathcal{O}_0 = \mathcal{O}$, $\Delta_0 = 4$). Due to supersymmetry one would anticipate that Δ_i are protected. One would also anticipate that these operators are dual to the KK modes of the reduction over the sphere S^{8-p} . As discussed in the previous section, one can consistently truncate these modes at strong coupling, so the gravitational computation should lead to Ward identities of the form (4.8),

up to a possible quantum anomaly \mathcal{A} . \mathcal{A} originates from the counterterms that depend on the background fields only $(g_{(0)}, \Phi_{(0)}, \dots)$. In general, \mathcal{A} would be restricted by the Wess-Zumino consistency and therefore should be built from generalized conformal invariants. We will show the extracted holographic Ward identities, (5.78), indeed agree with (4.7)–(4.8) with a quantum anomaly only for $p = 4$.

In a $(p + 1)$ -dimensional conformal field theory, the entropy S at finite temperature T_H necessarily scales as

$$S = c(g_{\text{YM}}^2 N, N, \dots) V_p T_H^p \tag{4.11}$$

where V_p is the spatial volume, g_{YM} is the coupling, N is the rank of the gauge group, $g_{\text{YM}}^2 N$ is the 't Hooft coupling constant and the ellipses denote additional dimensionless parameters. $c(g_{\text{YM}}^2 N, N, \dots)$ denotes an arbitrary function of these dimensionless parameters. In the cases of interest here, scaling indicates that the entropy behaves as

$$S = \tilde{c}((g_{\text{eff}}^2(T_H), N, \dots) V_p T_H^p, \tag{4.12}$$

where $g_{\text{eff}}^2(T_H) = g_d^2 N T_H^{p-3}$ is the effective coupling constant and $\tilde{c}((g_d^2 N T_H^{p-3}), N, \dots)$ denotes a generic function of the dimensionless parameters.

Next let us consider correlation functions, in particular of the gluon operator $\mathcal{O} = -\frac{1}{4} \text{Tr}(F^2 + \dots)$. In a theory which is conformally invariant the two point function of any operator of dimension Δ behaves as

$$\langle \mathcal{O}(x) \mathcal{O}(y) \rangle = f(g_{\text{YM}}^2 N, N, \dots) \frac{1}{|x - y|^{2\Delta}}, \tag{4.13}$$

where $f(g_{\text{YM}}^2 N, N, \dots)$ denotes an arbitrary function of the dimensionless parameters. Now consider the constraints on a two point function in a theory with generalized conformal invariance; these are far less restrictive, with the correlator constrained to be of the form:

$$\langle \mathcal{O}(x) \mathcal{O}(0) \rangle = \tilde{f}(g_{\text{eff}}^2(x), N, \dots) \frac{1}{|x|^{2\Delta}}. \tag{4.14}$$

where $g_{\text{eff}}^2(x) = g_d^2 N |x|^{3-p}$ and $\tilde{f}(g_{\text{eff}}^2(x), N, \dots)$ is an arbitrary function of these (dimensionless) variables. Note that the scaling dimension of the gluon operator as defined above is 4. Both (4.13) and (4.14) are over-simplified as even in a conformal field theory the renormalized correlators can depend on the renormalization group scale μ . For example, for $p = 3$ the renormalized two point function of the dimension four gluon operator is

$$\langle \mathcal{O}(x) \mathcal{O}(0) \rangle = f(g_{\text{YM}}^2 N, N) \square^3 \left(\frac{1}{|x|^2} \log(\mu^2 x^2) \right), \tag{4.15}$$

where note that the renormalized version $\mathcal{R} \frac{1}{|x|^8}$ of $\frac{1}{|x|^8}$ is given by:

$$\mathcal{R} \left(\frac{1}{|x|^8} \right) = -\frac{1}{3 \cdot 2^8} \square^3 \left(\frac{1}{|x|^2} \log(\mu^2 x^2) \right). \tag{4.16}$$

$\mathcal{R}(\frac{1}{|x|^8})$ and $\frac{1}{|x|^8}$ are equal when $x \neq 0$ but they differ by infinite renormalization at $x = 0$. In particular, it is only $\mathcal{R} \frac{1}{|x|^8}$ that has a well defined Fourier transform, given by $p^4 \log(p^2/\mu^2)$,

which may be obtained using the identity

$$\int d^4x e^{ipx} \frac{1}{|x|^2} \log(\mu^2 x^2) = -\frac{4\pi^2}{p^2} \log(p^2/\mu^2). \quad (4.17)$$

(see appendix A, [40]). Thus the correlator in a theory with generalized conformal invariance is

$$\langle \mathcal{O}(x)\mathcal{O}(0) \rangle = \mathcal{R} \left(\tilde{f}(g_{\text{eff}}^2(x), \mu|x|, N, \dots) \frac{1}{|x|^{2\Delta}} \right) \quad (4.18)$$

Note that this is of the same form as a two point function of an operator with definite scaling dimension in any quantum field theory; the generalized conformal structure does not restrict it further, although as discussed above the underlying structure does relate two point functions via Ward identities.

The general form of the two point function (4.18) is compatible with the holographic results discussed later. One can also compute the two point function to leading (one loop) order in perturbation theory, giving:

$$\langle \mathcal{O}(x)\mathcal{O}(0) \rangle = \langle : \text{Tr}(F^2)(x) :: \text{Tr}(F^2)(0) : \rangle \sim \mathcal{R} \left(\frac{g_{\text{eff}}^4(x)}{|x|^8} \right), \quad (4.19)$$

which is also compatible with the general form. (Note that although the complete operator includes in addition other bosonic and fermionic terms the latter do not contribute to the two point function at one loop, whilst the former contribute only to the overall normalization.) One shows this result as follows. The gauge field propagator for $SU(N)$ in Feynman gauge in momentum space is

$$\langle A_{b\mu}^a(k) A_{d\nu}^c(-k) \rangle = ig_d^2 \left(\delta_d^a \delta_b^c - \frac{1}{N} \delta_b^a \delta_d^c \right) \frac{\eta_{\mu\nu}}{|k|^2}, \quad (4.20)$$

where (a, b) are color indices. Then the one loop contribution to the correlation function in momentum space reduces (at large N) to

$$\langle \mathcal{O}(k)\mathcal{O}(-k) \rangle \sim N^2(d-1)|k|^4 \int d^d q \frac{1}{|q|^2|k-q|^2}. \quad (4.21)$$

Using the integral

$$\begin{aligned} I &= \int d^d q \frac{1}{|q|^{2\alpha}|k-q|^{2\beta}} \\ &= \frac{\Gamma(\alpha + \beta - d/2)\Gamma(d/2 - \beta)\Gamma(d/2 - \alpha)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(d - \alpha - \beta)} |k|^{d-2\alpha-2\beta}, \end{aligned} \quad (4.22)$$

one finds that

$$\langle \mathcal{O}(k)\mathcal{O}(-k) \rangle \sim N^2(g_d^2)^2(d-1)|k|^d \frac{\Gamma(2-d/2)\Gamma(d/2-1)^2}{\Gamma(d-2)}. \quad (4.23)$$

This is finite for d odd, as expected given the general result that odd loops are finite in odd dimensions; dimensional regularization when d is even results in a two point function of the form $N^2 g_d^4 |k|^d \log(|k^2|)$. Fourier transforming back to position space results in

$$\langle \mathcal{O}(x)\mathcal{O}(0) \rangle \sim \mathcal{R} \left(\frac{g_{\text{eff}}^4(x)}{|x|^8} \right), \quad (4.24)$$

where again in even dimensions the renormalized expression is of the type given in (4.16). This is manifestly consistent with the form (4.18).

The structure that we find at weak coupling is also visible at strong coupling. The gravitational solution is the linear dilaton AdS_{d+1} solutions in (3.7) and conformal symmetry is broken only by the dilaton profile. Therefore the background is invariant under generalized conformal transformations in which one also transforms the string coupling g_s appropriately. This *generalized conformal structure* was discussed in [22–24], particularly in the context of D0-branes.

5. Holographic renormalization

In this section we will determine how gauge theory data is extracted from the asymptotics of the decoupled non-conformal brane backgrounds, following the same steps as in the asymptotically AdS case. In particular, one first fixes the non-normalizable part of the asymptotics: we will consider solutions which asymptote to a linear dilaton asymptotically locally AdS background. Next one needs to analyze the field equations in the asymptotic region, to understand the asymptotic structure of these backgrounds near the boundary.

Given this analysis, one is ready to proceed with holographic renormalization. Recall that the aim of holographic renormalization is to render well-defined the definition of the correspondence: the onshell bulk action with given boundary values $\Phi_{(0)}$ for the bulk fields acts as the generating functional for the dual quantum field theory in the presence of sources $\Phi_{(0)}$ for operators \mathcal{O} . The asymptotic analysis allows one to isolate the volume divergences of the onshell action, which can then be removed with local covariant counterterms, leading to a renormalized action. The latter allows one to extract renormalized correlators for the quantum field theory.

5.1 Asymptotic expansion

In determining how gauge theory data is encoded in the asymptotics of the non-conformal brane backgrounds the first step is to understand the asymptotic structure of these backgrounds in the asymptotic region near $\rho = 0$ where the solution becomes a linear dilaton locally AdS background. Let us expand the metric and dilaton as:

$$\begin{aligned}
 ds^2 &= \frac{d\rho^2}{4\rho^2} + \frac{g_{ij}(x, \rho)dx^i dx^j}{\rho}, \\
 \phi(x, \rho) &= \alpha \log \rho + \frac{\kappa(x, \rho)}{\gamma},
 \end{aligned}
 \tag{5.1}$$

where we expand $g(x, \rho)$ and $\kappa(x, \rho)$ in powers of ρ :

$$\begin{aligned}
 g(x, \rho) &= g_{(0)}(x) + \rho g_{(2)}(x) + \dots \\
 \kappa(x, \rho) &= \kappa_{(0)}(x) + \rho \kappa_{(2)}(x) + \dots
 \end{aligned}
 \tag{5.2}$$

For $p = 3$ we should instead expand the scalar field as

$$\phi(x, \rho) = \kappa_{(0)}(x) + \rho \kappa_{(2)}(x) + \dots,
 \tag{5.3}$$

since $\alpha = \gamma = 0$. Note that by allowing $(g_{(0)}, \kappa_{(0)})$ to be generic the spacetime is only asymptotically locally AdS.

Consider first the case of $p = 3$, so that the action is Einstein gravity in the presence of a negative cosmological constant, and a massless scalar. The latter couples to the dimension four operator $\text{Tr}(F^2)$. The metric is expanded in the Fefferman-Graham form, with the scalar field expanded accordingly. By the standard rules of AdS/CFT $g_{(0)}$ acts as the source for the stress energy tensor and $\kappa_{(0)}$ acts as the source for the dimension four operator, i.e. it corresponds to the Yang-Mills coupling. The vevs of these operators are captured by subleading terms in the asymptotic expansion.

For general p an analogous relationship should hold: $g_{(0)}$ sources the stress energy tensor and the scalar field determines the (dimensionful) gauge coupling. More precisely, the bulk field that is dual to the operator \mathcal{O} in (4.5) is

$$\Phi(x, \rho) = \exp(\chi\phi(x, \rho)) = \rho^{-\frac{1}{2}(p-3)} (\Phi_{(0)}(x) + \rho\Phi_{(2)}(x) + \dots) \quad (5.4)$$

$$\Phi_{(0)}(x) = \exp\left(-\frac{(p-5)}{(p-3)}\kappa_{(0)}(x)\right) \quad (5.5)$$

The $\Phi_{(0)}$ appearing here is identified with $\Phi_{(0)}$ in (4.1). It will be convenient however to work on the gravitational side with $\phi(x, \rho)$ instead of $\Phi(x, \rho)$.

In the asymptotic expansion we fix the non-normalizable part of the asymptotics, and the vevs should be captured by subleading terms. One now needs to show that such an expansion is consistent with the equations of motion, and what terms occur in the expansion for given (α, β, γ) .

Substituting the scalar and the metric given in (5.1) into the field equations (3.11) gives

$$-\frac{1}{4}\text{Tr}(g^{-1}g')^2 + \frac{1}{2}\text{Tr}g^{-1}g'' + \kappa'' + \left(1 - \frac{\beta}{\gamma^2}\right)(\kappa')^2 = 0, \quad (5.6)$$

$$-\frac{1}{2}\nabla^i g'_{ij} + \frac{1}{2}\nabla_j(\text{Tr}g^{-1}g') + \left(1 - \frac{\beta}{\gamma^2}\right)\partial_j\kappa\kappa' + \partial_j\kappa' - \frac{1}{2}g_j^k\partial_k\kappa = 0, \quad (5.7)$$

$$\begin{aligned} &[-\text{Ric}(g) - (d-2-2\alpha\gamma)g' - \text{Tr}(g^{-1}g')g + \rho(2g'' - 2g'g^{-1}g' + \text{Tr}(g^{-1}g')g')]_{ij} + \\ &+ \nabla_i\partial_j\kappa + \left(1 - \frac{\beta}{\gamma^2}\right)\partial_i\kappa\partial_j\kappa - 2(g_{ij} - \rho g'_{ij})\kappa' = 0, \quad (5.8) \end{aligned}$$

$$4\rho(\kappa'' + (\kappa')^2) + (8\alpha\gamma + 2(2-d))\kappa' + \nabla^2\kappa + (\partial\kappa)^2 + 2\text{Tr}(g^{-1}g')(\alpha\gamma + \rho\kappa') = 0, \quad (5.9)$$

where differentiation with respect to ρ is denoted with a prime, ∇_i is the covariant derivative constructed from the metric g and $d = p + 1$ is the dimension of the space orthogonal to ρ . Note that coefficients in these equations are polynomials in ρ implying that this system of equations admits solutions with $g(x, \rho)$ and $\kappa(x, \rho)$ being regular functions of ρ and this justifies (5.2). To solve these equations one may successively differentiate the equations w.r.t. ρ and then set $\rho = 0$.

Let us first recall how these equations are solved in the pure gravity, asymptotically

locally AdS_{d+1} case, i.e. when the scalar is trivial. Then the equations become

$$\begin{aligned}
 &-\frac{1}{4}\text{Tr}(g^{-1}g')^2 + \frac{1}{2}\text{Tr}g^{-1}g'' = 0; \\
 &-\frac{1}{2}\nabla^i g'_{ij} + \frac{1}{2}\nabla_j(\text{Tr}g^{-1}g') = 0 \quad (5.10) \\
 &[-Ric(g) - (d-2)g' - \text{Tr}(g^{-1}g')g + \rho(2g'' - 2g'g^{-1}g' + \text{Tr}(g^{-1}g')g')]_{ij} = 0,
 \end{aligned}$$

The structure of the expansions depends on whether d is even or odd. For d odd, the expansion is of the form

$$g(x, \rho) = g_{(0)}(x) + \rho g_{(2)}(x) + \dots + \rho^{d/2} g_{(d)}(x) + \dots \quad (5.11)$$

Terms with integral powers of ρ in the expansion are determined locally in terms of $g_{(0)}$ but $g_{(d)}(x)$ is not determined by $g_{(0)}$, except for its trace and divergence, i.e. $g_{(0)}^{ij}g_{(d)ij}$ and $\nabla^i g_{(d)ij}$, which are forced by the field equations to vanish. In this case $g_{(d)}(x)$ determines the vev of the dual stress energy tensor, whose trace must vanish as the theory is conformal and there is no conformal anomaly in odd dimensions. The fact that $g_{(d)}$ is divergenceless leads to the conservation of the stress energy tensor.

For d even, the structure is rather different:

$$g(x, \rho) = g_{(0)}(x) + \rho g_{(2)}(x) + \dots + \rho^{d/2}(g_{(d)}(x) + h_{(d)}(x) \log \rho) + \dots \quad (5.12)$$

In this case one needs to include a logarithmic term to satisfy the field equations; the coefficient of this term is determined by $g_{(0)}$ whilst only the trace and divergence of $g_{(d)}(x)$ are determined by $g_{(0)}$. This structure reflects the fact that the trace of the stress energy tensor of an even-dimensional conformal field theory on a curved background is non-zero and picks up an anomaly determined in terms of $g_{(0)}$; the explicit expression for the stress energy tensor in terms of $(g_{(0)}, g_{(d)})$ is rather more complicated than in the other case but it is such that the divergence of $g_{(d)}$ leads again to conservation of the stress energy tensor.

Let us return now to the cases of interest. As mentioned above, the field equations are solved by successively differentiating the equations w.r.t. ρ and then setting ρ to zero. This procedure leads to equations of the form

$$c(n, d)g_{(2n)ij} = f(g_{(2k)ij}, \kappa_{(2k)}), \quad k < n \quad (5.13)$$

where the right hand side depends on the lower order coefficients and $c(n, d)$ is a numerical coefficient that depends on n and d . If this coefficient is non-zero, one can solve this equation to determine $g_{(n)ij}$. However, in some cases this coefficient is zero and one has to include a logarithmic term at this order for the equations to have a solution. An example of this is the case of pure gravity with d even, where $c(d/2, d) = 0$. Furthermore, note that since in (5.8)-(5.9) only integral powers of ρ enter, likewise only integral powers in (5.2) will depend on $g_{(0)}$ and $\kappa_{(0)}$. In general however non-integral powers can also appear at some order and one must determine these terms separately. An example of this is the case of pure gravity with d odd reviewed above, where a half integral power of ρ appears at order $\rho^{d/2}$.

Let us first consider when one needs to include non-integral powers in the expansion. Let us assume that ρ^σ is the lowest non-integral power that appears in the asymptotic expansion

$$\begin{aligned}\kappa(x, \rho) &= \kappa_{(0)} + \rho\kappa_{(2)} + \cdots + \rho^\sigma\kappa_{(2\sigma)} + \cdots \\ g_{ij}(x, \rho) &= g_{(0)ij} + \rho g_{(2)ij} + \cdots + \rho^\sigma g_{(2\sigma)ij} + \cdots\end{aligned}\tag{5.14}$$

Differentiating the scalar equation (5.9) $[\sigma]$ times, where $[\sigma]$ is the integer part of σ , and taking $\rho \rightarrow 0$ after multiplying with $\rho^{1+[\sigma]-\sigma}$ one obtains

$$(2\sigma + 4\alpha\gamma - d)\kappa_{(2\sigma)} + \alpha\gamma\text{Tr}g_{(2\sigma)} = 0,\tag{5.15}$$

Similarly, equation (5.8) yields,

$$(2\sigma - d + 2\alpha\gamma)g_{(2\sigma)ij} - (\text{Tr}g_{(2\sigma)} + 2\kappa_{(2\sigma)})g_{(0)ij} = 0.\tag{5.16}$$

which upon taking the trace becomes

$$-d\kappa_{(2\sigma)} + (\sigma - d + \alpha\gamma)\text{Tr}g_{(2\sigma)} = 0,\tag{5.17}$$

If the determinant of the coefficients of the system of equation (5.15)–(5.17) is non-zero,

$$D = (2\sigma + 4\alpha\gamma - d)(\sigma - d + \alpha\gamma) + \alpha\gamma d \neq 0\tag{5.18}$$

the only solution of these equations is

$$\text{Tr}g_{(2\sigma)} = \kappa_{(2\sigma)} = 0\tag{5.19}$$

which then using (5.16) implies

$$g_{(2\sigma)ij} = 0\tag{5.20}$$

i.e. in these cases no non-integral power appears in the expansion.

On the other hand, when $D = 0$ equations (5.17)–(5.15) admit a non-trivial solution. The two solution of $D = 0$ are $\sigma_1 = d/2 - \alpha\gamma$ and $\sigma_2 = 2(d/2 - \alpha\gamma)$. Clearly, $\sigma_2 > \sigma_1$ and when σ_2 in non-integer so is σ_1 , so a non-integer power first appears at:

$$\sigma = \frac{d}{2} - \alpha\gamma\tag{5.21}$$

When this holds equations (5.17)–(5.15) reduce to

$$\text{Tr}g_{(2\sigma)} + 2\kappa_{(2\sigma)} = 0.\tag{5.22}$$

and the coefficient of $g_{(2\sigma)ij}$ in (5.16) vanishes, so apart from its trace, these equations leave $g_{(2\sigma)ij}$ undetermined. The remaining Einstein equation (5.7) also imposes a constraint on the divergence of the terms occurring at this order, as will be discussed later. To summarize, the expansion contains a non-integer power of ρ^σ in the following cases

$$\sigma = \frac{p-7}{p-5} \Rightarrow \quad D0 : \sigma = 7/5; \quad D1, F1 : \sigma = 3/2; \quad D2 : \sigma = 5/3,\tag{5.23}$$

and the coefficient multiplying this power is only partly constrained. As we will see, this category is the analogue of even dimensional asymptotically AdS backgrounds, which are dual to odd dimensional boundary theories.

The second case to discuss is the case of only integral powers. In this case the undetermined term occurs at an integral power ρ^σ with

$$\sigma = \frac{p-7}{p-5} \Rightarrow \quad D3 : \sigma = 2; \quad D4 : \sigma = 3, \quad (5.24)$$

and logarithmic terms need to be included in the expansions. In these cases the combination $(\text{Tr}g_{(2\sigma)} + 2\kappa_{(2\sigma)})$ is determined by $g_{(0)}$ and $\kappa_{(0)}$. This category is analogous to odd-dimensional asymptotically AdS backgrounds, which are dual to even-dimensional boundary theories. The remaining Einstein equation (5.7) also imposes a constraint on the divergence of the terms occurring at this order.

Actually one can see on rather general grounds why the undetermined terms occur at these powers: the undetermined terms will relate to the vev of the stress energy tensor, which is of dimension $(p+1)$ for a $(p+1)$ -dimensional field theory. However, the overall normalization of the action behaves as $l_s^{(p-3)^2/(5-p)}$, and therefore on dimensional grounds the vev should sit in the $g_{(2\sigma)}\rho^\sigma$ term where

$$\sigma = (p+1) + \frac{(p-3)^2}{(5-p)} = \frac{(p-7)}{(p-5)}, \quad (5.25)$$

which agrees with the discussion above. Put differently we can compare the power of the first undetermined term to pure AdS and notice that it is shifted by $-\alpha\gamma = -\frac{(p-3)^2}{2(p-5)}$ (for both Dp-branes and the fundamental string). This is just what is needed to offset the background value of the $e^{\gamma\phi}$ term multiplying the Einstein-Hilbert action in (3.4), in order to ensure that all divergent terms in the action are still determined by the asymptotic field equations.

One should note here that the case of $p=6$ is outside the computational framework discussed above. In this case the prefactor in the action is of positive mass dimension nine, whilst the stress energy tensor in the dual seven-dimensional theory must be of dimension seven. Therefore one finds a (meaningless) negative value for σ , indicating that one is not making the correct asymptotic expansion. In other words, one finds that the “subleading terms” are more singular than the leading term.

5.2 Explicit expressions for expansion coefficients

In all cases of interest $2\sigma > 2$ and thus there are $g_{(2)}$ and $\kappa_{(2)}$ terms. Evaluating (5.9) and (5.8) at $\rho = 0$ gives in the case of $\beta = 0$ and $2\alpha\gamma = -1$ (relevant for D1-branes, fundamental strings and D4-branes):

$$\begin{aligned} \kappa_{(2)} &= \frac{1}{2d} \left(\nabla^2 \kappa_{(0)} + g_{(0)}^{ij} \partial_i \kappa_{(0)} \partial_j \kappa_{(0)} + \frac{1}{2(d-1)} R_{(0)} \right), \\ g_{(2)ij} &= \frac{1}{d-1} \left(-R_{(0)ij} + \frac{1}{2d} R_{(0)} g_{(0)ij} + (\nabla_{\{i} \partial_{j\}} \kappa)_{(0)} + \partial_{\{i} \kappa_{(0)} \partial_{j\}} \kappa_{(0)} \right) \end{aligned} \quad (5.26)$$

Here the parentheses in a quantity $A_{\{ab\}}$ denote the traceless symmetric tensor and ∇_i is the covariant derivative in the metric $g_{(0)ij}$.

If $\beta \neq 0$, as for $p = 0, 2$, the expressions are slightly more involved:

$$\begin{aligned} \kappa_{(2)} &= -\frac{1}{M} \left(2\alpha\gamma R_{(0)} - 2(d-1)\nabla^2\kappa_{(0)} + \left(\frac{2\alpha\beta}{\gamma} - 2d + 2 \right) (g_{(0)}^{ij} \partial_i \kappa_{(0)} \partial_j \kappa_{(0)}) \right), \\ g_{(2)ij} &= \frac{1}{d - 2\alpha\gamma - 2} \left(-R_{(0)ij} + \nabla_i \partial_j \kappa_{(0)} + \left(1 - \frac{\beta}{\gamma^2} \right) \partial_i \kappa_{(0)} \partial_j \kappa_{(0)} \right. \\ &\quad \left. + \frac{\gamma^2 - \beta}{2(\gamma^2 d - \beta d + \beta)} g_{(0)ij} \left(R_{(0)} - 2\nabla^2 \kappa_{(0)} - 2 \left(1 - \frac{\beta}{2\gamma^2} \right) (g_{(0)}^{ij} \partial_i \kappa_{(0)} \partial_j \kappa_{(0)}) \right) \right), \\ M &\equiv 16\alpha^2\beta - 2(d-1)(8\alpha\gamma + 4 - 2d) = \frac{16(9-p)}{(5-p)^2}. \end{aligned} \tag{5.27}$$

The final equality, expressing the coefficient M in terms of p , holds for the Dp-branes of interest here.

5.2.1 Category 1: undetermined terms at non-integral order

Let us first consider the case where the undetermined terms occur at non-integral order.

In the cases of $p = 0, 1, 2$ the terms given above in (5.27) are the only determined terms. The underdetermined terms appear at order $\rho^{(p-7)/(p-5)}$ and satisfy the constraints

$$2\kappa_{(2\sigma)} + \text{Tr}g_{(2\sigma)} = 0, \quad \sigma = \frac{p-7}{p-5} \tag{5.28}$$

$$\nabla^i g_{(2\sigma)ij} - 2 \left(1 - \frac{\beta}{\gamma^2} \right) \partial_j \kappa_{(0)} \kappa_{(2\sigma)} + g_{(2\sigma)ij} \partial^i \kappa_{(0)} = 0. \tag{5.29}$$

We will see that the trace and divergent constraints translate into conformal and diffeomorphism Ward identities respectively.

5.2.2 Category 2: undetermined terms at integral order

Let us next consider the case where the undetermined terms occur at integral order: this includes the D3 and D4 branes. Explicit expressions for the conformal cases, including the case of D3-branes, are given in [6]. For the D4-branes, the equations at next order can be solved to determine $\kappa_{(4)}$ and $g_{(4)ij}$:

$$\begin{aligned} \kappa_{(4)} &= \frac{1}{8} \left((\nabla^2 \kappa)_{(2)} + 6\kappa_{(2)}^2 + (\partial\kappa)_{(2)}^2 + \frac{1}{2} \text{Tr}g_{(2)}^2 + 2\kappa_{(2)} \text{Tr}g_{(2)} \right), \\ g_{(4)ij} &= \frac{1}{4} \left[\left(2\kappa_{(2)}^2 + \frac{1}{2} \text{Tr}g_{(2)}^2 \right) g_{(0)ij} - R_{(2)ij} - 2(g_{(2)}^2)_{ij} + (\nabla_i \partial_j \kappa)_{(2)} + 2\partial_i \kappa_{(2)} \partial_j \kappa_{(0)} \right]. \end{aligned} \tag{5.30}$$

where we introduce the notation

$$A[g(x, \rho), \kappa(x, \rho)] = A_{(0)}(x) + \rho A_{(2)}(x) + \rho^2 A_{(4)}(x) + \dots \tag{5.31}$$

for composite quantities $A[g, \kappa]$ of $g(x, \rho)$ and $\kappa(x, \rho)$. For (5.30) we need the coefficients of $A = \{\nabla^2 \kappa, (\partial\kappa)^2, R_{ij}\}$. The explicit expression for these coefficients can be worked out

straightforwardly using the asymptotic expansion of $g(x, \rho)$ and $\kappa(x, \rho)$ and we give these expressions for the Christoffel connections and curvature coefficients in appendix A. Note also that we use the compact notation

$$(g_{(2)}^2)_{ij} \equiv (g_{(2)}g_{(0)}^{-1}g_{(2)})_{ij}, \quad \text{Tr}(g_{(2n)}) \equiv \text{Tr}(g_{(0)}^{-1}g_{(2n)}). \quad (5.32)$$

Proceeding to the next order, one finds that the expansion coefficients $\kappa_{(6)}$ and $g_{(6)ij}$ cannot be determined independently in terms of lower order coefficients because after further differentiating the highest derivative terms in (5.8) and (5.9) both vanish. Only the combination $(2\kappa_{(6)} + \text{Tr}g_{(6)})$ is fixed, along with a constraint on the divergence. Furthermore one has to introduce logarithmic terms in (5.2) for the equations to be satisfied, namely

$$\begin{aligned} g(x, \rho) &= g_{(0)}(x) + \rho g_{(2)}(x) + \rho^2 g_{(4)}(x) + \rho^3 g_{(6)}(x) + \rho^3 \log(\rho) h_{(6)}(x) + \dots \\ \kappa(x, \rho) &= \kappa_{(0)}(x) + \rho \kappa_{(2)}(x) + \rho^2 \kappa_{(4)}(x) + \rho^3 \kappa_{(6)}(x) + \rho^3 \log(\rho) \tilde{\kappa}_{(6)}(x) + \dots \end{aligned} \quad (5.33)$$

For the logarithmic terms one finds

$$\begin{aligned} \tilde{\kappa}_{(6)} &= -\frac{1}{12} \left[(\nabla^2 \kappa)_{(4)} + (\partial \kappa)_{(4)}^2 + 20\kappa_{(2)}\kappa_{(4)} - \frac{1}{2} \text{Tr}g_{(2)}^3 + \text{Tr}g_{(2)}g_{(4)} \right. \\ &\quad \left. + 2\kappa_{(2)}(-\text{Tr}g_{(2)}^2 + 2\text{Tr}g_{(4)}) + 4\kappa_{(4)}\text{Tr}g_{(2)} \right], \\ h_{(6)ij} &= -\frac{1}{12} \left[-2R_{(4)ij} + (-\text{Tr}g_{(2)}^3 + 2\text{Tr}g_{(2)}g_{(4)} + 8\kappa_{(2)}\kappa_{(4)})g_{(0)ij} + 2\text{Tr}g_{(2)}g_{(4)ij} \right. \\ &\quad \left. - 8(g_{(4)}g_{(2)})_{ij} - 8(g_{(2)}g_{(4)})_{ij} + 4g_{(2)}^3_{ij} + 2(\nabla_i \partial_j \kappa)_{(4)} + 2(\partial_i \kappa \partial_j \kappa)_{(4)} + 4\kappa_{(2)}g_{(4)ij} \right], \end{aligned} \quad (5.34)$$

Note that these coefficients satisfy the following identities

$$\text{Tr}h_{(6)} + 2\tilde{\kappa}_{(6)} = 0, \quad (5.35)$$

$$g_{(0)}^{ki} (\nabla_k h_{(6)ij} + h_{(6)ij} \partial_k \kappa_{(0)}) - 2\partial_j \kappa_{(0)} \tilde{\kappa}_{(6)} = 0. \quad (5.36)$$

Furthermore, $\kappa_{(6)}$, $\text{Tr}g_{(6)}$ and $\nabla^i g_{(6)ij}$ are constrained by the following equations,

$$2\kappa_{(6)} + \text{Tr}g_{(6)} = -\frac{1}{6}(-4\text{Tr}g_{(2)}g_{(4)} + \text{Tr}g_{(2)}^3 + 8\kappa_{(2)}\kappa_{(4)}), \quad (5.37)$$

$$\nabla^i g_{(6)ij} - 2\partial_j \kappa_{(0)} \kappa_{(6)} + g_{(6)ij} \partial^i \kappa_{(0)} = T_j, \quad (5.38)$$

where T_j is locally determined in terms of $(g_{(2n)}, \kappa_{(2n)})$ with $n \leq 2$,

$$\begin{aligned} T_j &= \nabla^i A_{ij} - 2\partial_j \kappa_{(0)} \left(A - \frac{2}{3} \kappa_{(2)}^3 - 2\kappa_{(2)}\kappa_{(4)} \right) + A_{ij} \partial^i \kappa_{(0)} \\ &\quad + \frac{1}{6} \text{Tr}(g_{(4)} \nabla_j g_{(2)}) + \frac{2}{3} (\kappa_{(4)} + \kappa_{(2)}^2) \partial_j \kappa_{(2)}, \end{aligned} \quad (5.39)$$

with

$$\begin{aligned}
 A_{ij} &= \frac{1}{3} \left((2g_{(2)}g_{(4)} + g_{(4)}g_{(2)})_{ij} - (g_{(2)}^3)_{ij} \right. \\
 &\quad + \frac{1}{8} (\text{Tr}(g_{(2)}^2) - \text{Tr}g_{(2)}(\text{Tr}g_{(2)} + 4\kappa_{(2)}))g_{(2)ij} \\
 &\quad - (\text{Tr}g_{(2)} + 2\kappa_{(2)})(g_{(4)ij} - \frac{1}{2}(g_{(2)}^2)_{ij}) \\
 &\quad - \left(\frac{1}{8}\text{Tr}g_{(2)}\text{Tr}g_{(2)}^2 - \frac{1}{24}(\text{Tr}g_{(2)})^3 - \frac{1}{6}\text{Tr}g_{(2)}^3 + \frac{1}{2}\text{Tr}g_{(2)}g_{(4)} \right) g_{(0)ij} \\
 &\quad \left. + \left(\frac{1}{4}\kappa_{(2)}((\text{Tr}g_{(2)})^2 - \text{Tr}g_{(2)}^2) - \frac{4}{3}\kappa_{(2)}^3 - 2\kappa_{(2)}\kappa_{(4)} \right) g_{(0)ij} \right) \\
 A &= \frac{1}{6} \left(- \left(\frac{1}{8}\text{Tr}g_{(2)}\text{Tr}g_{(2)}^2 - \frac{1}{24}(\text{Tr}g_{(2)})^3 - \frac{1}{6}\text{Tr}g_{(2)}^3 + \frac{1}{2}\text{Tr}g_{(2)}g_{(4)} \right) \right. \\
 &\quad \left. - \frac{32}{3}\kappa_{(2)}^3 - 6\kappa_{(2)}\kappa_{(4)} - \kappa_{(2)}^2\text{Tr}g_{(2)} - 2\kappa_{(4)}\text{Tr}g_{(2)} \right).
 \end{aligned} \tag{5.40}$$

We would now like to integrate the equations (5.37) and (5.38). Following the steps in [6], it is convenient to express $g_{(6)ij}$ and $\kappa_{(6)}$ as

$$\begin{aligned}
 g_{(6)ij} &= A_{ij} - \frac{1}{24}S_{ij} + t_{ij}; \\
 \kappa_{(6)} &= A - \frac{1}{24}S - 2\kappa_{(2)}\kappa_{(4)} - \frac{2}{3}\kappa_{(2)}^3 + \varphi,
 \end{aligned} \tag{5.41}$$

where (S_{ij}, S) are local functions of $g_{(0)}, \kappa_{(0)}$,

$$\begin{aligned}
 S_{ij} &= (\nabla^2 + \partial^m \kappa_{(0)} \nabla_m) I_{ij} - 2\partial^m \kappa_{(0)} \partial_{(i} \kappa_{(0)} I_{j)m} + 4\partial_i \kappa_{(0)} \partial_j \kappa_{(0)} I \\
 &\quad + 2R_{kilj} I^{kl} - 4I(\nabla_i \partial_j \kappa_{(0)} + \partial_i \kappa_{(0)} \partial_j \kappa_{(0)}) + 4(g_{(2)}g_{(4)} - g_{(4)}g_{(2)})_{ij} \\
 &\quad + \frac{1}{10}(\nabla_i \partial_j B - g_{(0)ij}(\nabla^2 + \partial^m \kappa_{(0)} \partial_m)B) \\
 &\quad + \frac{2}{5}B + g_{(0)ij} \left(-\frac{2}{3}\text{Tr}g_{(2)}^3 - \frac{4}{15}(\text{Tr}g_{(2)})^3 + \frac{3}{5}\text{Tr}g_{(2)}\text{Tr}g_{(2)}^2 \right. \\
 &\quad \left. - \frac{8}{3}\kappa_{(2)}^3 - \frac{8}{5}\kappa_{(2)}(\text{Tr}g_{(2)})^2 - \frac{4}{5}\kappa_{(2)}^2\text{Tr}g_{(2)} + \frac{6}{5}\kappa_{(2)}\text{Tr}g_{(2)}^2 \right),
 \end{aligned} \tag{5.42}$$

$$\begin{aligned}
 S &= (\nabla^2 + \partial^m \kappa_{(0)} \partial_m) I + \partial_i \kappa_{(0)} \partial_j \kappa_{(0)} I^{ij} - 2(\partial \kappa)_{(0)}^2 I \\
 &\quad - (\nabla_k \partial_l \kappa_{(0)} + \partial_k \kappa_{(0)} \partial_l \kappa_{(0)}) I^{kl} - \frac{1}{20}(\nabla^2 + \partial^m \kappa_{(0)} \partial_m) B \\
 &\quad + \frac{2}{5}B\kappa_{(2)} - \frac{4}{3}\kappa_{(2)}^3 - \frac{4}{5}\kappa_{(2)}(\text{Tr}g_{(2)})^2 - \frac{2}{5}\kappa_{(2)}^2\text{Tr}g_{(2)} + \frac{3}{5}\kappa_{(2)}\text{Tr}g_{(2)}^2, \\
 I_{ij} &= \left(g_{(4)} - \frac{1}{2}g_{(2)}^2 + \frac{1}{4}g_{(2)}(\text{Tr}g_{(2)} + 2\kappa_{(2)}) \right)_{ij} + \frac{1}{8}g_{(0)ij}B, \\
 I &= \kappa_{(4)} + \frac{1}{2}\kappa_{(2)}^2 + \frac{1}{4}\kappa_{(2)}\text{Tr}g_{(2)} + \frac{B}{16}, \\
 B &= \text{Tr}g_{(2)}^2 - \text{Tr}g_{(2)}(\text{Tr}g_{(2)} + 4\kappa_{(2)}).
 \end{aligned} \tag{5.43}$$

Note that these definitions imply the following identities

$$\begin{aligned} \nabla^i S_{ij} - 2\partial_j \kappa_{(0)} S + S_{ij} \partial^i \kappa_{(0)} &= -4(\text{Tr}(g_{(4)} \nabla_j g_{(2)}) + 4(\kappa_{(4)} + \kappa_{(2)}^2) \partial_j \kappa_{(2)}); \\ \text{Tr}(S_{ij}) + 2S &= -8\text{Tr}(g_{(2)} g_{(4)}) - 32\kappa_{(2)}(\kappa_{(2)}^2 + \kappa_{(4)}). \end{aligned} \quad (5.44)$$

Now, these definitions imply that t_{ij} defined in (5.41) is a symmetric tensor: A_{ij} contains an antisymmetric part but this is canceled by a corresponding antisymmetric part in S_{ij} . Inserting (5.41) in (5.37) one finds that the quantities (t_{ij}, φ) satisfy the following divergence and trace constraints:

$$\begin{aligned} \nabla^i t_{ij} &= 2\partial_j \kappa_{(0)} \varphi - t_{ij} \partial^i \kappa_{(0)}; \\ \text{Tr}t + 2\varphi &= -\frac{1}{3} \left(\frac{1}{8} (\text{Tr}g_{(2)})^3 - \frac{3}{8} \text{Tr}g_{(2)} \text{Tr}g_{(2)}^2 + \frac{1}{2} \text{Tr}g_{(2)}^3 - \text{Tr}g_{(2)} g_{(4)} \right. \\ &\quad \left. - \frac{3}{4} \kappa_{(2)} (\text{Tr}g_{(2)}^2 - (\text{Tr}g_{(2)})^2) - 4\kappa_{(2)} \kappa_{(4)} + 2\kappa_{(2)}^3 \right). \end{aligned} \quad (5.45)$$

We will find that the one point functions are expressed in terms of (t_{ij}, φ) and these constraints translate into the conformal and diffeomorphism Ward identities.

5.3 Reduction of M-branes

The D4-brane and type IIA fundamental string solutions are obtained from the reduction along a worldvolume direction of the M5 and M2 brane solutions respectively. The boundary conditions for the supergravity solutions also descend directly from dimensional reduction: diagonal reduction on a circle of an asymptotically (locally) AdS_{d+2} spacetime results in an asymptotically (locally) AdS_{d+1} spacetime with linear dilaton. Therefore the rather complicated results for the asymptotic expansions in the D4 and fundamental string cases should follow directly from the previously derived results for AdS_7 and AdS_4 given in [6], and we show that this is indeed the case in this subsection.

As discussed in section 3, solutions of the field equations of (3.12) are related to solutions of the field equations of the action (3.4) via the reduction formula (3.15). In the cases of F1 and D4 brane this means in particular

$$e^{4\phi/3} = \frac{1}{\rho} e^{2\kappa}, \quad (5.46)$$

where in comparing with (5.1) one should note that $\alpha = -3/4, \gamma = 2/3$ for both F1 and D4. This implies that the $(d+2)$ solution is automatically in the Fefferman-Graham gauge:

$$ds_{d+2}^2 = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} (g_{ij} dx^i dx^j + e^{2\kappa} dy^2). \quad (5.47)$$

Recall that for an asymptotically AdS_{d+2} Einstein manifold, the asymptotic expansion in the Fefferman-Graham gauge is

$$ds_{d+2}^2 = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} G_{ab} dx^a dx^b \quad (5.48)$$

where $a = 1, \dots, (d+1)$ and

$$G = G_{(0)}(x) + \rho G_{(2)}(x) + \dots + \rho^{(d+1)/2} G_{(d+1)/2}(x) + \rho^{(d+1)/2} \log(\rho) H_{(d+1)/2}(x) + \dots, \quad (5.49)$$

with the logarithmic term present only when $(d+1)$ is even. The explicit expression for $G_{(2)}(x)$ in terms of $G_{(0)}(x)$ is³

$$G_{(2)ab} = \frac{1}{d-1} \left(-R_{ab} + \frac{1}{2d} R G_{(0)ab} \right). \quad (5.50)$$

where the R_{ab} is the Ricci tensor of $G_{(0)}$, etc.

Comparing (5.47) with (5.48) one obtains

$$G_{ij} = g_{ij}; \quad G_{yy} = e^{2\kappa}. \quad (5.51)$$

In particular $G_{(0)ij} = g_{(0)ij}$ and $G_{(0)yy} = e^{2\kappa(0)}$, so

$$\begin{aligned} R[G_{(0)}]_{ij} &= R_{(0)ij} - \nabla_i \partial_j \kappa_{(0)} - \partial_i \kappa_{(0)} \partial_j \kappa_{(0)}; \\ R[G_{(0)}]_{yy} &= e^{2\kappa(0)} (-\nabla^i \partial_i \kappa_{(0)} - \partial_i \kappa_{(0)} \partial^i \kappa_{(0)}), \end{aligned} \quad (5.52)$$

with $R[G_{(0)}]_{yi} = 0$. Substituting into (5.50) gives

$$\begin{aligned} G_{(2)ij} &= \frac{1}{d-1} \left(-R_{(0)ij} + \frac{1}{2d} R_{(0)} g_{(0)ij} + (\nabla_{\{i} \partial_{j\}} \kappa)_{(0)} + \partial_{\{i} \kappa_{(0)} \partial_{j\}} \kappa_{(0)} \right); \\ G_{(2)yy} &= e^{2\kappa(0)} \left(\frac{1}{2d(d-1)} R_{(0)} + \frac{1}{d} (\nabla^2 \kappa_{(0)} + (\partial \kappa_{(0)})^2) \right), \end{aligned} \quad (5.53)$$

with $G_{(2)yi} = 0$. We thus find exact agreement between $G_{(2)ij}$ and $g_{(2)ij}$ in (5.26). Now using

$$G_{yy} = e^{2\kappa} = e^{(2\kappa(0) + 2\rho\kappa(2) + \dots)} = e^{2\kappa(0)} (1 + 2\rho\kappa(2) + \dots) \quad (5.54)$$

one determines $\kappa(2)$ to be exactly the expression given in (5.26).

Now restrict to the asymptotically AdS_4 case; the next coefficient in the asymptotic expansion occurs at order $\rho^{3/2}$, in $G_{(3)ab}$, and is undetermined except for the vanishing of its trace and divergence:

$$G_{(0)}^{ab} G_{(3)ab} = 0; \quad D^a G_{(3)ab} = 0. \quad (5.55)$$

Reducing these constraints leads immediately to

$$\begin{aligned} g_{(0)}^{ij} g_{(3)ij} + 2\kappa_{(3)} &= 0; \\ \nabla^i g_{(3)ij} - 2\partial_j \kappa_{(0)} \kappa_{(3)} + g_{(3)ij} \partial^i \kappa_{(0)} &= 0, \end{aligned} \quad (5.56)$$

in agreement with (5.28) and (5.29).

Similarly if one considers the asymptotically AdS_7 case, the determined coefficients $G_{(4)}$ and $H_{(6)}$ reduce to give $(g_{(4)}, \kappa_{(4)})$ and $(h_{(6)}, \tilde{\kappa}_{(6)})$ respectively. Furthermore, the trace of $G_{(6)}$ fixes the combination $(2\kappa_{(6)} + \text{Tr}g_{(6)})$. One can show that all explicit formulae agree precisely with the dimensional reduction of the formulae in [6]; the details are discussed in appendix C.

³Note that the conventions for the curvature used here differ by an overall sign from those in [6].

5.4 Renormalization of the action

Having derived the general form of the asymptotic expansion one can now proceed to holographic renormalization, following the discussion in [6]. In this method one substitutes the asymptotic expansions back into the regulated action and then introduces local covariant counterterms to cancel the divergences and renormalise the action. Whilst this method is conceptually very simple, in practice it is rather cumbersome for explicit computations. A more efficient method based on a radial Hamiltonian formalism [11, 12] will be discussed in the next section.

Let us choose an illustrative yet simple example to demonstrate this method of holographic renormalization: we will work out the renormalised on-shell action and compute the one-point function of the energy-momentum tensor and the operator \mathcal{O} for the case $p = 1$, both fundamental strings and D1-branes.

Since in this case $\beta = 0$, $\hat{\Phi} \equiv e^{\gamma\phi}$ behaves like a Lagrange multiplier and the bulk part of the action vanishes on-shell. The only non-trivial contribution comes then from the Gibbons-Hawking boundary term:

$$S_{\text{boundary}} = -L \int_{\rho=\epsilon} d^2x \sqrt{h} 2\hat{\Phi} K, \tag{5.57}$$

where h_{ij} is the induced metric on the boundary and K is the trace of the extrinsic curvature. Since (5.57) is divergent we regularise the action by evaluating it at $\rho = \epsilon$.

We would like now to find counterterms to remove the divergences in (5.57). From the discussion in section 5.1 we know the asymptotic expansion for Φ and $h_{ij}(x, \rho) = g_{ij}(x, \rho)/\rho$:

$$\begin{aligned} \hat{\Phi} &= \frac{e^{\kappa(0)}}{\sqrt{\rho}} (1 + \rho\kappa(2) + \rho^{3/2}\kappa(3) + \dots), \\ h &= \frac{1}{\rho} (g(0) + \rho g(2) + \rho^{3/2}g(3) + \dots), \end{aligned} \tag{5.58}$$

where $\kappa(3)$ and $g(3)$ are the lowest undetermined coefficients. Note that the expansions are the same for both fundamental strings and D1-branes, since in both cases $\alpha\gamma = -1/2$. Inserting the expansion (5.58) in (5.57) we find for the divergent part

$$S_{\text{div}} = -4L \int_{\rho=\epsilon} d^2x e^{\kappa(0)} \sqrt{g(0)} (\epsilon^{-3/2} + \epsilon^{-1/2}\kappa(2)), \tag{5.59}$$

using the formula

$$K = d - \rho \text{Tr}(g^{-1}g') \tag{5.60}$$

for the trace of the extrinsic curvature in the asymptotically AdS_{d+1} background. The trace term here cancels against the one in the expansion of the determinant.

From (5.58) and (5.26) we find

$$\sqrt{g(0)} = \rho\sqrt{h} \left(1 + \frac{1}{4(d-1)} R[h] \right), \tag{5.61}$$

which allows us to write the counterterms in a gauge-invariant form:

$$S_{\text{ct}} = -S_{\text{div}} = 4L \int_{\rho=\epsilon} d^2x \sqrt{h} \hat{\Phi} \left(1 + \frac{1}{4} R[h] \right). \tag{5.62}$$

The renormalised action is then

$$S_{\text{ren}}[g_{(0)}, \kappa_{(0)}] = \lim_{\epsilon \rightarrow 0} S_{\text{sub}}[h(x, \epsilon), \hat{\Phi}(x, \epsilon); \epsilon] \quad (5.63)$$

where

$$\begin{aligned} S_{\text{sub}} &= S_{\text{bulk}} + S_{\text{boundary}} + S_{\text{ct}} \\ &= -L \left[\int_{\rho \geq \epsilon} d^3x \sqrt{g} \hat{\Phi} (R + C) + \int_{\rho = \epsilon} d^2x \sqrt{h} \hat{\Phi} (2K - 4 - R[h]) \right]. \end{aligned} \quad (5.64)$$

This allows us to compute the renormalised vevs of the operator dual to $\hat{\Phi}$ and the stress-energy tensor. For the former, only the boundary part contributes, since $R + C = 0$ from the equation of motion for $\hat{\Phi}$. It can be easily checked that the divergent parts cancel and we obtain the finite result

$$\langle \mathcal{O} \rangle = \frac{1}{\sqrt{g_{(0)}}} \frac{\delta S_{\text{sub}}}{\delta \hat{\Phi}_{(0)}} = -\frac{1}{2} e^{3\kappa_{(0)}} \lim_{\epsilon \rightarrow 0} \left(\frac{1}{\epsilon^{3/2} \sqrt{h}} \frac{\delta S_{\text{sub}}}{\delta \hat{\Phi}} \right) = \frac{3}{2} e^{3\kappa_{(0)}} L \text{Tr} g_{(3)} = -3e^{3\kappa_0} L \kappa_{(3)}. \quad (5.65)$$

where we used (5.4) and the definition of $\hat{\Phi}$. The vev of the stress-energy tensor $\langle T_{ij} \rangle = \lim_{\epsilon \rightarrow 0} T_{ij}[h]$ gets a contribution from the bulk term as well. We can split it into the contribution of the regularised action and the counterterms

$$T_{ij}[h] = T_{ij}^{\text{reg}} + T_{ij}^{\text{ct}}, \quad (5.66)$$

where

$$\begin{aligned} T_{ij}^{\text{reg}}[h] &= 2L[\hat{\Phi}(Kh_{ij} - K_{ij}) - 2\rho\partial_\rho\hat{\Phi}h_{ij}], \\ T_{ij}^{\text{ct}}[h] &= 2L\left[\hat{\Phi}\left(R_{ij} - \frac{1}{2}Rh_{ij} - 2h_{ij}\right) + \nabla^2\hat{\Phi}h_{ij} - \nabla_i\partial_j\hat{\Phi}\right]. \end{aligned} \quad (5.67)$$

One can again check that the divergent terms cancel and obtain the finite contribution

$$\langle T_{ij} \rangle = \lim_{\epsilon \rightarrow 0} \left(\frac{2}{\sqrt{h}} \frac{\delta S_{\text{ren}}}{\delta h^{ij}} \right) = 3Le^{\kappa_{(0)}} g_{(3)ij}. \quad (5.68)$$

Note that the expressions for the vevs take the same form for both D1-brane and fundamental string cases. The one point functions satisfy the following Ward identities:

$$\begin{aligned} \langle T_i^i \rangle - 2\Phi_{(0)}\langle \mathcal{O} \rangle &= 0, \\ \nabla^i \langle T_{ij} \rangle + \partial_j \Phi_{(0)} \langle \mathcal{O} \rangle &= 0. \end{aligned} \quad (5.69)$$

To derive these one needs the trace and divergence identities given in (5.28) and (5.29) and the relation $\Phi_{(0)} = e^{-2\kappa_{(0)}}$ (see (5.4)). These Ward identities indeed agree exactly with what we derived on the QFT side, (4.7)–(4.8).

The first variation of the renormalized action yields the relation between the 1-point functions and non-linear combinations of the asymptotic coefficients. The one point functions are obtained in the presence of sources, so higher point functions can be obtained by further functional differentiation with respect to sources.

One should note here that the local boundary counterterms are required, irrespectively of the issue of finiteness, by the more fundamental requirement of the well-posedness of the appropriate variational problem [41]. The conformal boundary of asymptotically AdS spacetimes has a well-defined conformal class of metric rather than an induced metric. This means that the appropriate variational problem involves keeping fixed a conformal class and not an induced metric as in the usual Dirichlet problem for gravity in a spacetime with a boundary. The new variational problem requires the addition of further boundary terms, on top of the Gibbons-Hawking term. In the context of asymptotically AdS spacetimes (with no linear dilaton) these turn out to be precisely the boundary counterterms, see [41] for the details and a discussion of the subtleties related to conformal anomalies.

5.5 Relation to M2 theory

In the case of fundamental strings these formulae again follow directly from dimensional reduction of the AdS_4 case, since for the latter the renormalized stress energy tensor is [6]

$$\langle T_{ab} \rangle = 3L_M G_{(3)ab}. \tag{5.70}$$

Recalling the dimensional reduction formula (5.51), and noting that

$$L_M = L e^{\kappa_0}, \tag{5.71}$$

one finds immediately that

$$\langle T_{ij} \rangle = 3L e^{\kappa_0} g_{(3)ij}, \tag{5.72}$$

in agreement with (5.68). Noting that $G_{yy} = e^{4\phi/3} \rho = \hat{\Phi}^2 \rho$ one finds

$$\langle T_{yy} \rangle = 6L e^{3\kappa_0} \kappa_{(3)} = -2\langle \mathcal{O} \rangle, \tag{5.73}$$

in agreement with (5.65). The first Ward identity in (5.69) is thus an immediate consequence of the conformal Ward identity of the M2 brane theory, i.e. the tracelessness of the stress energy tensor. The second Ward identity in (5.69) similarly follows from the vanishing divergence of the stress energy tensor in the M2-brane theory.

5.6 Formulae for other Dp-branes

It is straightforward to derive analogous formulae for the other Dp-branes. Note that in general there is also a bulk contribution to the on-shell action

$$S_{on-shell} = L \frac{4\alpha\beta(d-2\alpha\gamma)}{h} \int_{\rho \geq \epsilon} d^{d+1}x \sqrt{g} e^{\gamma\phi} + L \int_{\rho=\epsilon} d^d x \sqrt{h} e^{\gamma\phi} 2K \tag{5.74}$$

where h_{ij} is the induced metric on the boundary, K is the trace of the extrinsic curvature and the action is regularised at $\rho = \epsilon$. Focusing first on the cases $p < 3$ the divergent terms are:

$$S_{div} = -L \int_{\rho=\epsilon} d^d x \sqrt{g_{(0)}} e^{\kappa_{(0)}} \epsilon^{-d/2+\alpha\gamma} \left(2d - \frac{4\alpha\beta}{\gamma} + \left(-\frac{4\alpha\beta(d-2\alpha\gamma)}{\gamma(d-2\alpha\gamma-2)} + 2d \right) \rho \kappa_{(2)} \right) + \left(-\frac{2\alpha\beta(d-2\alpha\gamma)}{\gamma(d-2\alpha\gamma-2)} + d - 2 \right) \rho \text{Tr} g_{(2)}, \tag{5.75}$$

which can be removed with the counterterm action

$$\begin{aligned}
 S_{ct} &= L \int_{\rho=\epsilon} d^d x \sqrt{\bar{h}} e^{\gamma\phi} \left(2d - \frac{4\alpha\beta}{\gamma} + C_R(\hat{R}[h] + \beta(\partial_i\phi)^2) \right) \\
 &= L \int_{\rho=\epsilon} d^d x \sqrt{\bar{h}} e^{\gamma\phi} \left(\frac{2(9-p)}{5-p} + \frac{5-p}{4}(\hat{R}[h] + \beta(\partial_i\phi)^2) \right) \\
 C_R &\equiv \frac{\gamma^2 - \beta}{d\gamma^2 - d\beta - \gamma^2 + 2\beta} = \frac{5-p}{4}.
 \end{aligned} \tag{5.76}$$

Again for convenience we give the formulae both in terms of (α, β, γ) and for the specific cases of interest here, the Dp-branes. The renormalised vevs of the operator⁴ \mathcal{O}_ϕ dual to ϕ and the stress-energy tensor can now be computed giving:

$$\begin{aligned}
 \langle \mathcal{O}_\phi \rangle &= 2\sigma L e^{\kappa(0)} \frac{1}{\alpha} \kappa_{(2\sigma)}, \\
 \langle T_{ij} \rangle &= 2\sigma L e^{\kappa(0)} g_{(2\sigma)ij}.
 \end{aligned} \tag{5.77}$$

Using (5.28) and (5.29) one obtains

$$0 = \langle T_i^i \rangle + 2\alpha \langle \mathcal{O}_\phi \rangle = \langle T_i^i \rangle + (p-3)\Phi_{(0)} \langle \mathcal{O} \rangle \tag{5.78}$$

$$0 = \nabla^i \langle T_{ij} \rangle - \frac{1}{\gamma} \partial_j \kappa_{(0)} \langle \mathcal{O}_\phi \rangle = \nabla^i \langle T_{ij} \rangle + \partial_j \Phi_{(0)} \langle \mathcal{O} \rangle, \tag{5.79}$$

where in the second equality we use the relation between $\kappa_{(0)}$ and $\Phi_{(0)}$ in (5.4) which implies in particular that $\langle \mathcal{O}_\phi \rangle = \chi \Phi_{(0)} \langle \mathcal{O} \rangle$. These are the anticipated dilatation and diffeomorphism Ward identities.

Next let us consider the case of D4-branes, for which one needs more counterterms:

$$\begin{aligned}
 S_{ct} &= L \int_{\rho=\epsilon} d^5 x \sqrt{\bar{h}} e^{\gamma\phi} \left(10 + \frac{1}{4} \hat{R}[h] + \frac{1}{32} (\hat{R}[h]_{ij} - \gamma(\hat{\nabla}_i \partial_j \phi + \partial_i \phi \partial_j \phi))^2 \right. \\
 &\quad \left. + \frac{1}{32} \gamma^2 (\hat{\nabla}^2 \phi + (\partial_i \phi)^2)^2 - \frac{3}{320} (\hat{R}[h] - 2\gamma(\hat{\nabla}^2 \phi + (\partial_i \phi)^2))^2 + a_{(6)} \log \epsilon \right),
 \end{aligned} \tag{5.80}$$

where the coefficient of the logarithmic term $a_{(6)}$ is given by

$$\begin{aligned}
 a_{(6)} &= 6 \text{Tr} h_{(6)}; \\
 &= \frac{1}{8} (\text{Tr} g_{(2)})^3 - \frac{3}{8} \text{Tr} g_{(2)} \text{Tr} g_{(2)}^2 + \frac{1}{2} \text{Tr} g_{(2)}^3 - \text{Tr} g_{(2)} g_{(4)} \\
 &\quad - \frac{3}{4} \kappa_{(2)} \text{Tr} g_{(2)}^2 + \frac{3}{4} \kappa_{(2)} (\text{Tr} g_{(2)})^2 - 4 \kappa_{(2)} \kappa_{(4)} - 2 \kappa_{(2)}^3.
 \end{aligned} \tag{5.81}$$

Note that in cases such as the D4-brane, where one needs to compute many counterterms, it is rather more convenient to use the Hamiltonian formalism, which will be discussed in the next section. We will also discuss the structure of this anomaly further in the following section.

⁴Note that $\langle \mathcal{O}_\phi \rangle = \chi \Phi_{(0)} \langle \mathcal{O} \rangle$. This is obtained using (5.4) and the chain rule.

The renormalised vevs of the operator dual to ϕ and the stress-energy tensor can now be computed giving:

$$\begin{aligned}\langle \mathcal{O}_\phi \rangle &= -Le^{\kappa(0)} \left(8\varphi + \frac{44}{3} \tilde{\kappa}_{(6)} \right), \\ \langle T_{ij} \rangle &= Le^{\kappa(0)} (6t_{ij} + 11h_{(6)ij}),\end{aligned}\tag{5.82}$$

where (t_{ij}, φ) are defined in (5.41). Note that the contributions proportional to $\tilde{\kappa}_{(6)}, h_{(6)ij}$ are scheme dependent; one can remove these contributions by adding finite local boundary terms.

The dilatation Ward identity is

$$\langle T_i^i \rangle + \Phi_{(0)} \langle \mathcal{O} \rangle = -2Le^{\kappa(0)} a_{(6)},\tag{5.83}$$

whilst the diffeomorphism Ward identity is

$$\nabla^i \langle T_{ij} \rangle + \partial_j \Phi_{(0)} \langle \mathcal{O} \rangle = 0.\tag{5.84}$$

The terms involving $(h_{(6)ij}, \tilde{\kappa}_{(6)})$ drop out of the Ward identities because of the trace and divergence identities given in (5.35).

These formulae are as expected consistent with the reduction of the M5 brane formulae given in [6]. This computation of the renormalized stress energy tensor for the M5-brane case is reviewed in appendix B. In fact in [6] the renormalized stress energy tensor for the AdS_7 case was given only up to scheme dependent traceless, covariantly constant terms, proportional to the coefficient $H_{(6)ab}$ of the logarithmic term in the asymptotic expansion. In appendix B we determine these contributions to the stress energy tensor, with the resulting stress energy tensor being (B.9):

$$\langle T_{ab} \rangle = \frac{N^3}{3\pi^3} (6t_{ab} + 11H_{(6)ab}).\tag{5.85}$$

The streamlined method of derivation of the renormalized stress energy tensor given in appendix B is also useful in the explicit derivation of the D4-brane formulae given in (5.82). Dimensional reduction of the t_{ab} term in the stress energy tensor results in the (t_{ij}, φ) terms in the D4-brane operator vevs, whilst reduction of the $H_{(6)ab}$ term gives the terms involving $(h_{(6)ij}, \tilde{\kappa}_{(6)})$. The details of this dimensional reduction are discussed in appendix C.

6. Hamiltonian formulation

In the previous section we showed how correlation functions can be computed using the basic holographic dictionary that relates the on-shell gravitational action to the generating functional of correlators, and we renormalized the action with counterterms to obtain finite expressions. This method of holographic renormalization is conceptually very simple but does not exploit all the structure of the theory.

The underlying structure of the correlators is best exhibited in the radial Hamiltonian formalism, which is a Hamiltonian formulation with the radius playing the role of time.

The Hamilton-Jacobi theory, introduced in this context in [42], relates the variation of the on-shell action w.r.t. boundary conditions, thus the holographic 1-point functions, to radial canonical momenta. It follows that one can bypass the on-shell action and directly compute renormalized correlators using radial canonical momenta π , as was developed for asymptotically AdS spacetimes in [11, 12].

A fundamental property of asymptotically (locally) AdS spacetimes is that dilatations are part of their asymptotic symmetries. This implies that all covariant quantities can be decomposed into a sum of terms each of which has definite scaling. These coefficients are in 1-1 correspondence with the asymptotic coefficients in (5.1) with the exact relation being in general non-linear. The advantage of working with dilatation eigenvalues rather than with asymptotic coefficients is that the former are manifestly covariant while the latter in general are not: the asymptotic expansion (5.1) singles out one coordinate so it is not covariant. Holographic 1-point functions can be expressed most compactly in terms of eigenfunctions of the dilatation operator, and this explains the non-linearities found in explicit computations of 1-point functions.

6.1 Hamiltonian method for non-conformal branes

We now develop a Hamiltonian version of the holographic renormalization of these backgrounds following closely the steps of [11, 12]. We consider the action (3.4) with the Gibbons-Hawking boundary term added to ensure that the action depends only on first radial derivatives (as we will see shortly), so a radial Hamiltonian formalism can be set up:

$$S = -L \int_{AdS_{d+1}} d^{d+1}x \sqrt{g} e^{\gamma\phi} [R + \beta(\partial\phi)^2 + C] - 2L \int_{\partial AdS_{d+1}} d^d x \sqrt{h} e^{\gamma\phi} K. \quad (6.1)$$

Note that we are again working in Euclidean signature. Next we introduce a radial Hamiltonian formulation. In the usual Hamiltonian formulation of gravity in the ADM formalism one foliates spacetime by hypersurfaces of constant time. Here analogously we introduce a family of hypersurfaces Σ_r of constant radius r near the boundary and denote by n^μ their unit normal. For asymptotically locally AdS manifolds there always exists a radial function normal to the boundary which can be used to foliate the space in such radial slices, at least in a neighborhood of the boundary.

In order to give a Hamiltonian description of the dynamics, one needs to express the action (3.4) in terms of quantities on Σ_r . In particular, this means that the Ricci scalar in the action (3.4) should be expressed in terms of expressions which only contain first derivatives in the radial variable. The induced metric on the hypersurface Σ_r can be expressed as $h_{\sigma\mu} = g_{\sigma\mu} - n_\sigma n_\mu$, with $h_\mu^\rho \equiv g^{\rho\sigma} h_{\sigma\mu}$. Now let us define the radial flow vector field r^μ by the relation $r^\mu \partial_\mu r = 1$, such that the components of r^μ tangent and normal to Σ_r define shift and lapse functions respectively:

$$r_{\parallel}^\mu = h_\rho^\mu r^\rho \equiv N^\mu; \quad r_{\perp}^\mu = N n^\mu. \quad (6.2)$$

Thus the metric is decomposed as

$$ds^2 = (N^2 + N_\mu N^\mu) dr^2 + 2N_\mu dx^\mu dr + h_{\mu\nu} dx^\mu dx^\nu, \quad (6.3)$$

analogously to the usual ADM decomposition.

A useful tool in our analysis is the extrinsic curvature $K_{\mu\nu}$ of the hypersurface given by the covariant derivative of the unit normal

$$K_{\mu\nu} = h_{\sigma(\mu} \nabla^\sigma n_{\nu)}. \quad (6.4)$$

The geometric *Gauss-Codazzi* equations (in the contracted form of [11, 12]) can be used to express the curvature of the embedding space in terms of extrinsic and intrinsic curvatures on the hypersurface:⁵

$$\begin{aligned} K^2 - K_{\mu\nu} K^{\mu\nu} &= \hat{R} + 2G_{\mu\nu} n^\mu n^\nu, \\ \hat{\nabla}_\mu K_\nu^\mu - \partial_\nu K &= G_{\rho\sigma} h_\nu^\rho n^\sigma, \\ \mathcal{L}_n K_{\mu\nu} + K K_{\mu\nu} - 2K_\mu^\rho K_{\rho\nu} &= \hat{R}_{\mu\nu} - h_\mu^\rho h_\nu^\sigma R_{\rho\sigma}, \end{aligned} \quad (6.5)$$

where $G_{\mu\nu}$ is the Einstein tensor in the embedding spacetime, K is the trace of the extrinsic curvature, $\hat{R}_{\mu\nu}$ is the intrinsic curvature and $\hat{\nabla}$ is the covariant derivative on the hypersurface. Note that these equations become completely equivalent to Einstein's equations, on gauge fixing the lapse and the shift.

Combining the first equation in (6.5) with the Ricci identity $R_{\mu\nu} n^\mu n^\nu = n^\nu (\nabla_\sigma \nabla_\nu - \nabla_\nu \nabla_\sigma) n^\sigma$ the Ricci scalar can be expressed as

$$R = K^2 - K_{\mu\nu} K^{\mu\nu} + \hat{R} - 2\nabla_\mu (n^\mu \nabla_\nu n^\nu) + 2\nabla_\nu (n^\mu \nabla_\mu n^\nu), \quad (6.6)$$

Inserting this expression into the action (3.4), the last two terms cancel the Gibbons-Hawking boundary term in (3.4) after partial integration and the remaining term is

$$\begin{aligned} S = -L \int d^{d+1}x \sqrt{g} e^{\gamma\phi} &[\hat{R} + K^2 - K_{\mu\nu} K^{\mu\nu} + \beta(\partial\phi)^2 + C \\ &+ 2\gamma\partial_\mu\phi n^\mu \nabla_\nu n^\nu - 2\gamma\partial_\nu\phi n^\mu \nabla_\mu n^\nu]. \end{aligned} \quad (6.7)$$

Note that the extrinsic curvature can be expressed as

$$K_{\mu\nu} = \frac{1}{2N} (\partial_r h_{\mu\nu} - \hat{\nabla}_\mu N_\nu - \hat{\nabla}_\nu N_\mu), \quad (6.8)$$

and thus the action can be expressed entirely in terms of the fields $(h_{\mu\nu}, N^\mu, N)$ and the scalar field ϕ , and their derivatives. The canonical momenta conjugate to these fields are given by

$$\pi^{\mu\nu} \equiv \frac{\delta L}{\delta \dot{h}_{\mu\nu}}, \quad \pi_\phi \equiv \frac{\delta L}{\delta \dot{\phi}}, \quad (6.9)$$

where $\dot{f} \equiv \partial_r f$ and the momenta conjugate to the lapse and shift functions vanish identically. The corresponding equations of motion in the canonical formalism become constraints, which are precisely those obtained from the first two equations in (6.5) and are the Hamiltonian and momentum constraints respectively.

⁵The Lie derivative in our conventions is defined as $\mathcal{L}_n K_{\mu\nu} = n^\sigma K_{\mu\nu,\sigma} - 2n^\sigma_{,(\mu} K_{\nu)\sigma}$.

The diffeomorphism gauge is most naturally fixed by choosing Gaussian normal coordinates ($N^\mu = 0$ and $N = 1$), such that

$$\begin{aligned} ds^2 &= dr^2 + h_{ij}(r, x)dx^i dx^j, & K_{ij} &= \frac{1}{2}\dot{h}_{ij} \\ n^\mu &= \delta_r^\mu, & \nabla_\mu n^\mu &= K, & n^\mu \nabla_\mu n^\nu &= 0, \end{aligned} \quad (6.10)$$

where the dot denotes differentiation with respect to r . The action becomes

$$S = -L \int d^{d+1}x \sqrt{h} e^{\gamma\phi} [\hat{R} + K^2 - K_{ij}K^{ij} + \beta(\dot{\phi}^2 + (\partial_i\phi)^2) + C + 2\gamma\dot{\phi}K]. \quad (6.11)$$

and the canonical momenta are given by

$$\begin{aligned} \pi_\phi &= 2B(\beta\dot{\phi} + \gamma K), & B &\equiv -Le^{\gamma\phi}\sqrt{h}. \\ \pi^{ij} &= B(Kh^{ij} - K^{ij} + \gamma\dot{\phi}h^{ij}), \end{aligned} \quad (6.12)$$

The Gauss-Codazzi identities in this gauge become:

$$\begin{aligned} K^2 - K_{ij}K^{ij} &= \hat{R} + 2G_{rr}, \\ D_i K_j^i - D_j K &= G_{jr}, \\ \dot{K}_j^i + K K_j^i &= \hat{R}_j^i - R_j^i, \end{aligned} \quad (6.13)$$

Now consider the regulated manifold \mathcal{M}_{r_0} defined as the submanifold of \mathcal{M} bounded by the hypersurface Σ_{r_0} . The values of the induced fields on Σ_{r_0} become boundary conditions for the action, and therefore the momenta on the regulating surface can be obtained from variations of the on-shell action with respect to the boundary values of the induced fields. The Hamilton-Jacobi identities thus allow the momenta (6.12) on the regulating surface to be expressed in terms of the on-shell action by

$$\pi^{ij}(r_0, x) = \frac{\delta S_{on-shell}}{\delta h_{ij}(r_0, x)}, \quad \pi_\phi(r_0, x) = \frac{\delta S_{on-shell}}{\delta \phi(r_0, x)}. \quad (6.14)$$

Since the choice of the regulator r_0 is arbitrary, the equations (6.17) and (6.14) can be used not just to compute the on-shell action and momentum on the regulating surface Σ_{r_0} but on any radial surface Σ_r .

Now to calculate the regulated on-shell action one uses the first of the Gauss-Codazzi identities, together with the field equations (3.11):

$$S_{on-shell} = -2L \int_{\mathcal{M}_{r_0}} d^{d+1}x \sqrt{h} e^{\gamma\phi} [\hat{R} + \beta(\partial_i\phi)^2 + C]. \quad (6.15)$$

However, since the field equations follow from the variation of the bulk part of the action, the radial derivative of the on-shell action can be expressed as a purely as a boundary term,

$$\dot{S}_{on-shell} = -2L \int_{\Sigma_{r_0}} d^d x \sqrt{h} e^{\gamma\phi} [\hat{R} + \beta(\partial_i\phi)^2 + C]. \quad (6.16)$$

From this expression follows that the regulated on-shell action can itself also be written as a d -dimensional integral by introducing a covariant variable λ ,

$$S_{on-shell} = -2L \int_{\Sigma_{r_0}} d^d x \sqrt{h} e^{\gamma\phi} [K - \lambda], \quad (6.17)$$

where λ satisfies the equation

$$\begin{aligned} \dot{\lambda} + \lambda(K + \gamma\dot{\phi}) + E &= 0, \\ E &= \frac{(\gamma^2 + d(\gamma^2 - \beta))\beta}{(\gamma^2 - \beta)^2} = -\frac{2(p-1)(p-4)(p-7)}{(p-5)^2}, \end{aligned} \quad (6.18)$$

and the trace of the third equation in (6.13) is used, along with the field equations (3.11). Note that since Σ_{r_0} is compact λ is defined only up to a total divergence.

The Hamilton-Jacobi identities then imply that:

$$\pi^{ij} \delta h_{ij} + \pi_\phi \delta \phi = -2L \delta[\sqrt{h} e^{\gamma\phi} (K - \lambda)], \quad (6.19)$$

up to a total derivative. One can always use the total divergence ambiguity in λ to ensure that this expression holds without integrating it over Σ_r . First one chooses any λ satisfying (6.18), and then one calculates the variation $\delta[\sqrt{h} e^{\gamma\phi} (K - \lambda)]$. This variation necessarily gives the left hand side of (6.19), up to total derivative terms, which can be absorbed into the definition of λ . (Strictly speaking, this argument applies only to the local terms in λ ; the finite part of λ as $r \rightarrow \infty$ is actually non-local in the sources, and only the integrated identity holds for this part.)

6.2 Holographic renormalization

We next turn to the formulation of a Hamiltonian method of holographic renormalization. In the earlier sections, we discussed holographic renormalization by solving asymptotically the field equations, as a function of sources. Here we will instead use the equations of motion to determine the asymptotic form of the momenta as functionals of induced fields. Such a procedure is manifestly covariant at all stages, with the Ward identities being manifest and the one point functions of dual operators being naturally expressed in terms of the momenta.

An important tool in the Hamiltonian method is the dilatation operator, whose eigenfunctions are covariant expressions on the hypersurface Σ_r , and which asymptotically behaves like the radial derivative. The radial derivative acting on the on-shell action and on the momenta can be represented as a functional derivative, since by means of the field equations the on-shell action and the momenta are given as functionals of h_{ij} and ϕ :

$$\partial_r = \int d^d x \left(2K_{ij}[h, \phi] \frac{\delta}{\delta h_{ij}} + \dot{\phi}[h, \phi] \frac{\delta}{\delta \phi} \right) \quad (6.20)$$

where we used (6.10). Now, recall that the dilatation operator for a d -dimensional theory on a curved background containing sources for operators of dimension Δ is given by

$$\delta_D \equiv \int d^d x \left(2h_{ij} \frac{\delta}{\delta h_{ij}} + (\Delta - d) \Phi \frac{\delta}{\delta \Phi} \right) \quad (6.21)$$

In our case, the field $\Phi = \exp \frac{2(p-5)}{(7-p)}\phi$ couples to \mathcal{O} which has dimension 4. Using the chain rule we obtain

$$\delta_D \equiv \int d^d x \left(2h_{ij} \frac{\delta}{\delta h_{ij}} - 2\alpha \frac{\delta}{\delta \phi} \right) = \partial_r + \mathcal{O}(e^{-2r}), \quad (6.22)$$

so indeed the radial derivative can be asymptotically identified with the dilatation operator since asymptotically $\dot{\phi} \rightarrow -2\alpha$ and $\dot{h}_{ij} \rightarrow 2h_{ij}$.

The next key observation is that the momenta and on-shell action can be expanded asymptotically in terms of eigenfunctions of the dilatation operator δ_D . The structure one expects in these expansions of K_j^i , λ and $\dot{\phi}$ in terms of weights of the dilatation operator is similar to the radial expansions (5.2), except that every term in the expansion also contains terms subleading in e^{-2r} :

$$\begin{aligned} K_j^i[h, \phi] &= K_{(0)j}^i + K_{(2)j}^i + \cdots + K_{(d-2\alpha\gamma)j}^i + \tilde{K}_{(d-2\alpha\gamma)j}^i \log e^{-2r}, \\ \lambda[h, \phi] &= \lambda_{(0)} + \lambda_{(2)} + \cdots + \lambda_{(d-2\alpha\gamma)} + \tilde{\lambda}_{(d-2\alpha\gamma)} \log e^{-2r}, \\ \dot{\phi}[h, \phi] &= p_{(0)}^\phi + p_{(2)}^\phi + \cdots + p_{(d-2\alpha\gamma)}^\phi + \tilde{p}_{(d-2\alpha\gamma)}^\phi \log e^{-2r}. \end{aligned} \quad (6.23)$$

(We will see that the logarithmic terms appear only if $(d - 2\alpha\gamma)$ is an even integer, i.e. for $p = 3, 4$.) The transformation properties of these terms under the dilatation operator are:

$$\begin{aligned} \delta_D K_{(n)j}^i &= -n K_{(n)j}^i, \\ \delta_D \tilde{K}_{(d-2\alpha\gamma)j}^i &= -(d - 2\alpha\gamma) \tilde{K}_{(d-2\alpha\gamma)j}^i, \\ \delta_D K_{(d-2\alpha\gamma)j}^i &= -(d - 2\alpha\gamma) K_{(d-2\alpha\gamma)j}^i - 2\tilde{K}_{(d-2\alpha\gamma)j}^i, \end{aligned} \quad (6.24)$$

and similarly for λ_k and p_k^ϕ . Thus terms with weight $n < (d - 2\alpha\gamma)$ transform homogeneously, whilst terms with weight $n = (d - 2\alpha\gamma)$ transform inhomogeneously, indicating that these terms depend non-locally on the induced fields. As we will see below, the terms with weight $n < d - 2\alpha\gamma$ are algebraically (locally) determined in terms of the asymptotics, while the weight $(d - 2\alpha\gamma)$ terms are undetermined up to some constraints. The latter will be identified with the renormalized one point functions and the on-shell action, which are non-local functionals of the sources. Given a solution from which one wishes to extract the 1-point function dual to a given field, one simply subtracts the lower weight terms in the dilatation expansion of the corresponding momentum. We will show below how these lower weight terms can be determined recursively in terms of the asymptotic data.

Although it is as mentioned above not necessary to compute the renormalised action to obtain renormalised 1-point functions, the Hamiltonian method is more efficient at determining the counterterms. The divergences in the on-shell action can be expressed in terms of the terms in the expansions which are divergent as $r_0 \rightarrow \infty$. These divergences can be removed by a counterterm action which consists of these divergent terms in the expansions, namely:

$$I_{ct} = 2L \int_{\Sigma_{r_0}} \sqrt{h} e^{\gamma\phi} \left(\sum_{0 \leq n < d-2\alpha\gamma} (K_{(n)} - \lambda_{(n)}) + (\tilde{K}_{(n)} - \tilde{\lambda}_{(n)}) \log e^{-2r_0} \right). \quad (6.25)$$

This counterterm action also leads through the Hamilton-Jacobi relations to the covariant counterterms of the momenta. The renormalised action is then given by the terms of appropriate weight in the on-shell action (6.17):

$$I_{\text{ren}} = -2L \int_{\Sigma_{r_0}} d^d x \sqrt{h} e^{\gamma\phi} [K_{(d-2\alpha\gamma)} - \lambda_{(d-2\alpha\gamma)}]. \quad (6.26)$$

The gravity/gauge theory prescription identifies this with the generating functional in the dual field theory, and so, in particular, the first derivatives of this actions with respect to the sources correspond to the one point functions of the dual operators. Since the Hamilton-Jacobi relations identify these first derivatives with the non-local terms in the expansions of the momenta one obtains immediately the relations:

$$\langle T_{ij} \rangle = \pi_{(d-2\alpha\gamma)ij}; \quad \langle \mathcal{O}_\phi \rangle = (\pi_\phi)_{(d-2\alpha\gamma)}. \quad (6.27)$$

From (6.12) one sees that the one-point functions are given by:

$$\begin{aligned} \langle \mathcal{O}_\phi \rangle &= -2L e^{\gamma\phi} (\beta p^\phi_{(d-2\alpha\gamma)} + \gamma K_{(d-2\alpha\gamma)}), \\ \langle T_{ij} \rangle &= 2L e^{\gamma\phi} ((K_{(d-2\alpha\gamma)} + \gamma p^\phi_{(d-2\alpha\gamma)}) h_{ij} - K_{(d-2\alpha\gamma)ij}). \end{aligned} \quad (6.28)$$

Thus to obtain both the counterterms and the one-point functions one needs to solve for the terms in the dilatation expansions.

6.3 Ward identities

The diffeomorphism Ward identity can be derived from the momentum constraint, the second Gauss-Codazzi equation in (6.13):

$$\hat{\nabla}_i K_j^i - \hat{\nabla}_j K = G_{jr} = (\gamma^2 - \beta) \partial_j \phi \dot{\phi} + \gamma \partial_j \dot{\phi} - \gamma K_j^i \partial_i \phi. \quad (6.29)$$

Using (6.12) this can easily be expressed in terms of momenta:

$$\hat{\nabla}_i \left(\frac{\pi^{ij}}{\sqrt{h}} \right) = \frac{1}{2\sqrt{h}} \partial^j \phi \pi_\phi. \quad (6.30)$$

Expressing this identity at weight $(d-2\alpha\gamma)$ in terms of one-point functions yields the Ward identity

$$\hat{\nabla}_i \langle T^{ij} \rangle - \gamma^{-1} \langle \mathcal{O}_\phi \rangle \partial^j \kappa_{(0)} = 0. \quad (6.31)$$

which becomes of the standard QFT form (4.7) upon expressing this identity in terms of $\langle O \rangle$ and $\Phi_{(0)}$. To determine the dilatation Ward identity one computes the infinitesimal Weyl transformation of the renormalised action (6.26)

$$\delta_\sigma I_{\text{ren}} = 4L \int_{\Sigma_r} d^d x \sqrt{h} (N e^\phi)^\gamma [\tilde{K}_{(d-2\alpha\gamma)} - \tilde{\lambda}_{(d-2\alpha\gamma)}] \delta\sigma, \quad (6.32)$$

where one uses the non-diagonal behaviour of $K_{(d-2\alpha\gamma)}$ and $\lambda_{(d-2\alpha\gamma)}$ under the dilatation operator exhibited in (6.24). However, this infinitesimal Weyl transformation is also given by the renormalised version of the Hamilton-Jacobi relations (6.14) given by⁶

$$\delta_\sigma I_{\text{ren}} = - \int_{\Sigma_r} d^d x \sqrt{h} [2\pi_{(d-2\alpha\gamma)i}^i - 2\alpha\pi_{\phi(d-2\alpha\gamma)}] \delta\sigma. \quad (6.33)$$

⁶We define e.g. $\pi_{\phi(d-2\alpha\gamma)}$ to be the weight $(d-2\alpha\gamma)$ part of π_ϕ/\sqrt{h} .

Since these identities hold for arbitrary $\delta\sigma$ we can infer the conformal Ward identity

$$\langle T_i^i \rangle + 2\alpha \langle \mathcal{O}_\phi \rangle = \mathcal{A}, \quad (6.34)$$

where the anomaly is given by

$$\mathcal{A} = -4L[\tilde{K}_{(d-2\alpha\gamma)} - \tilde{\lambda}_{(d-2\alpha\gamma)}]. \quad (6.35)$$

The anomaly for the D4-brane will be computed below. Again this becomes the standard Ward identity (4.8) (with an anomaly) upon replacing $\langle \mathcal{O}_\phi \rangle$ by $\chi \Phi_{(0)} \langle \mathcal{O} \rangle$ (see footnote 4).

6.4 Evaluation of terms in the dilatation expansion

Let us now discuss how to evaluate the local terms in the dilatation expansion. In the previous section we have derived a number of identities which can be solved recursively to determine terms in the expansions. In particular, applying the Hamilton-Jacobi identity (6.19) to dilatations gives

$$(1 + \delta_D)K - (d - 2\alpha\gamma + \delta_D)\lambda - (d\gamma - 2\alpha\beta)\dot{\phi} = 0. \quad (6.36)$$

The Hamilton-Jacobi relations (6.14) and (6.12) also imply expressions for the extrinsic curvature and scalar field momenta:

$$\begin{aligned} (Kh^{ij} - K^{ij} + \gamma\dot{\phi}h^{ij}) &= \frac{2}{e^{\gamma\phi}\sqrt{h}} \frac{\delta}{\delta h_{ij}} \int_{\Sigma_{r_0}} d^d x \sqrt{h} e^{\gamma\phi} (K - \lambda); \\ (\beta\dot{\phi} + \gamma K) &= \frac{1}{e^{\gamma\phi}\sqrt{h}} \frac{\delta}{\delta \phi} \int_{\Sigma_{r_0}} d^d x \sqrt{h} e^{\gamma\phi} (K - \lambda). \end{aligned} \quad (6.37)$$

Next one has the Einstein equations, rewritten as the Gauss-Codazzi equations (6.13). Note that the Hamiltonian constraint in (6.13) can be written as

$$K^2 - K_{ij}K^{ij} = \hat{R} - \beta\dot{\phi}^2 + (\beta - 2\gamma^2)(\partial_i\phi)^2 - 2\gamma\hat{\nabla}^2\phi - 2\gamma K\dot{\phi} + C, \quad (6.38)$$

where the field equations (3.11) are used on the right hand side, and the double radial derivative terms $\ddot{\phi}$ are eliminated using the scalar equation of motion. One can also use the scalar equation of motion (the second equation in (3.11)), which in Gaussian normal coordinates reads

$$\ddot{\phi} + \hat{\nabla}^2\phi + K\dot{\phi} + \gamma\dot{\phi}^2 + \gamma(\partial_i\phi)^2 - \frac{\gamma(d(\gamma^2 - \beta) + \gamma^2)}{(\gamma^2 - \beta)^2} = 0. \quad (6.39)$$

as well as the differential equation for λ (6.18). Not all of these identities are necessary in order to recursively determine the lower terms in the dilatation expansion.

In practice it is convenient to first use the Hamilton-Jacobi identity (6.36) to express the local coefficients of λ in terms of those in K and $\dot{\phi}$:

$$\lambda_{(2n)} = \frac{(1 - 2n)K_{(2n)} - (d\gamma - 2\alpha\beta)p_{(2n)}^\phi}{d - 2\alpha\gamma - 2n}. \quad (6.40)$$

Thus this identity ensures that all counterterms are expressed in terms of the momenta.

Next one needs to solve for the momenta, using the Hamilton-Jacobi relations, Gauss-Codazzi relations and the scalar equation of motion. Consider first the Hamilton constraint (6.38); this equation can be expanded into terms of given dilatation weight, and solving at each weight yields a recursion relation for terms in the expansion of K_{ij} and $\dot{\phi}$. At dilatation weight zero this constraint yields merely a check of the background solution. Noting that $K_{(0)} = K_{(0)ij}K_{(0)}^{ij} = d$ the zero weight constraint is

$$d(d-1) = -\beta(p_{(0)}^\phi)^2 - 2\gamma dp_{(0)}^\phi + C, \quad (6.41)$$

which is satisfied given that $p_{(0)}^\phi = -2\alpha$ and the definition of α in terms of (β, d, C) .

At higher dilatation weight one obtains a recursion relation for a linear combination for $K_{(2n)}$ and $p_{(2n)}^\phi$ at a given weight $2n$:

$$\begin{aligned} K_{(2)} + \gamma p_{(2)}^\phi &= \frac{1}{2(d-2\alpha\gamma-1)} [\hat{R} + (\beta - 2\gamma^2)(\partial_i\phi)^2 - 2\gamma\hat{\nabla}^2\phi], \\ K_{(2n)} + \gamma p_{(2n)}^\phi &= \frac{1}{2(d-2\alpha\gamma-1)} \left[\sum_{m=1}^{n-1} (K_{(2m)j}^i K_{(2n-2m)i}^j - K_{(2m)}K_{(2n-2m)}) \right. \\ &\quad \left. - \sum_{m=1}^{n-1} (\beta p_{(2m)}^\phi p_{(2n-2m)}^\phi + 2\gamma K_{(2m)} p_{(2n-2m)}^\phi) \right]. \end{aligned} \quad (6.42)$$

Note that if $(d-2\alpha\gamma)$ is not an even integer one immediately finds the relation

$$K_{(d-2\alpha\gamma)} + \gamma p_{(d-2\alpha\gamma)}^\phi = 0, \quad (6.43)$$

since no terms on the right hand side can contribute at this weight. This relation precisely corresponds to (5.22) in the old formalism, in the case where the undetermined term appears at a non-integral power of ρ .

Consider next the scalar equation of motion; to express this in terms of terms of given dilatation weight, it is necessary to expand $\ddot{\phi}$ in terms of eigenfunctions of the dilatation operator. (Note that eliminating $\ddot{\phi}$ using the other field equations does not give an identity which is independent of (6.38).) The additional radial derivative in $\ddot{\phi}$ can be expressed in terms of the dilatation operator by keeping higher terms in the expansion of the radial derivative:

$$\begin{aligned} \partial_r &= \int d^d x \left(2K_{ij} \frac{\delta}{\delta h_{ij}} + \dot{\phi} \frac{\delta}{\delta \phi} \right) \\ &= \delta_D + \sum_{n \geq 1} \int d^d x \left(2K_{(2n)ij} \frac{\delta}{\delta h_{ij}} + p_{(2n)}^\phi \frac{\delta}{\delta \phi} \right) \equiv \delta_D + \sum_{n \geq 1} \delta_{(2n)}. \end{aligned} \quad (6.44)$$

Given the transformation properties (6.24) of the expansion coefficients of the momenta, the subleading terms in the expansion of ∂_r must satisfy the commutation relation $[\delta_D, \delta_{(2n)}] = -2n\delta_{(2n)}$.

Solving the scalar field equation at weight zero, (6.39) is automatically satisfied given the leading asymptotic behavior. At higher weights $2n$ with $n > 1$ a recursion relation for a distinct linear combination of $K_{(2n)}$ and $p^\phi_{(2n)}$ is obtained:

$$\begin{aligned} (d-2-4\alpha\gamma)p^\phi_{(2)} - 2\alpha K_{(2)} &= -\hat{\nabla}^2\phi - \gamma(\partial_i\phi)^2, \\ (d-2n-4\alpha\gamma)p^\phi_{(2n)} - 2\alpha K_{(2n)} &= -\sum_{m=1}^{n-1} (\delta_{(2m)}p^\phi_{(2n-2m)} + K_{(2m)}p^\phi_{(2n-2m)}). \end{aligned} \quad (6.45)$$

In the case that $(d-2\alpha\gamma)$ is not an even integer, the relevant term in the recursion relation becomes

$$-2\alpha(K_{(d-2\alpha\gamma)} + \gamma p^\phi_{(d-2\alpha\gamma)}) = 0, \quad (6.46)$$

since no terms on the right hand side can contribute at this weight, and thus reproduces the trace constraint (6.43).

The Hamiltonian constraint (6.42) together with the scalar equation (6.45) thus constitutes a linear system of equations which allows one to express $K_{(2n)}$ and $p^\phi_{(2n)}$ in terms of lower order coefficients. One can then determine $\lambda_{(2n)}$ from (6.40), and use the Hamilton-Jacobi relations (6.37) to determine the extrinsic curvature $K_{(2n)}^i_j$. This is all information needed to proceed in the recursion.

It is useful to recall here the equation (6.18) for the variable λ , which determines the on-shell action. Here again the radial derivative can be expressed in terms of the dilatation operator, giving:

$$\left(\delta_D + \sum_{n=1}^{d/2-\alpha\gamma} \delta_{(2n)} \right) \lambda + \lambda(K + \gamma\dot{\phi}) + E = 0. \quad (6.47)$$

Note that in the case of $E = 0$, i.e. for F1,D1 and D4 branes $\lambda = 0$ solves the differential equation, and thus the coefficients $\lambda_{(2n)}$ consist only of total derivative terms which are determined by (6.40).

6.4.1 Category 1: Undetermined terms at non-integral order

Let us consider first the case where the undetermined terms occur at non-integral order, namely $p < 3$, and obtain the counterterms and one point functions.

The Hamiltonian constraint (6.42) together with the scalar equation (6.45) can be solved at first order to give:

$$\begin{aligned} K_{(2)} &= \frac{1}{2(d-2\alpha\gamma-1)(d-2\alpha\gamma-2)} \\ &\quad \times \left((d-2-4\alpha\gamma)(\hat{R} + \beta(\partial\phi)^2) + 2(1+2\alpha\gamma)e^{-\gamma\phi}\nabla^2(e^{\gamma\phi}) \right); \\ p^\phi_{(2)} &= \frac{1}{\gamma(d-2\alpha\gamma-1)(d-2\alpha\gamma-2)} \\ &\quad \times \left(\gamma\alpha(\hat{R} + \beta(\partial\phi)^2) - (d-1)e^{-\gamma\phi}\nabla^2(e^{\gamma\phi}) \right); \end{aligned} \quad (6.48)$$

Next note that the counterterms $\lambda_{(2n)}$ follow from (6.40), and are given by

$$\begin{aligned}\lambda_{(0)} &= -\frac{2\alpha\beta}{\gamma}; \\ \lambda_{(2)} &= -\frac{K_{(2)} + (d\gamma - 2\alpha\beta)p^\phi_{(2)}}{(d - 2\alpha\gamma - 2)}.\end{aligned}\tag{6.49}$$

For the cases $p < 3$ one only needs to solve up to this order to obtain all counterterms, with the counterterm action being:

$$\begin{aligned}I_{ct} &= L \int_{\Sigma_{r_0}} \sqrt{h} e^{\gamma\phi} \left(2d - \frac{4\alpha\beta}{\gamma} + \frac{\gamma^2 - \beta}{(d-1)\gamma^2 + \beta(2-d)} (\hat{R} + \beta(\partial\phi)^2) \right) \\ &\quad - L \int_{\Sigma_{r_0}} \sqrt{h} \frac{d}{(d - 2\alpha\gamma - 2)} \nabla^2(e^{\gamma\phi}).\end{aligned}\tag{6.50}$$

This coincides with the counterterm action found earlier in (5.76), up to the (irrelevant) total derivative term in the second line.

Next consider the one point functions. To apply the general formula (6.28), one needs to relate the momentum coefficients with terms in the asymptotic expansion of the metric and the scalar field. In the case that $(d-2\alpha\gamma)$ is not an even integer, this identification turns out to be very simple. Recall that in the original method of holographic renormalization one expanded the induced metric asymptotically in the radial coordinate $\rho = e^{-2r}$ as

$$h_{ij} = \frac{1}{\rho} \left(g_{(0)ij} + \rho g_{(2)ij} + \dots + \rho^{\frac{1}{2}(d-2\alpha\gamma)} g_{(d-2\alpha\gamma)ij} + \rho^{\frac{1}{2}(d-2\alpha\gamma)} \ln \rho h_{(d-2\alpha\gamma)ij} + \dots \right),\tag{6.51}$$

where the logarithmic term is included when $(d-2\alpha\gamma)$ is an even integer. Differentiating with respect to r gives

$$\begin{aligned}K_{ij} &= \frac{1}{2} \dot{h}_{ij} \\ &= \frac{1}{\rho} g_{(0)ij} - \rho g_{(4)ij} + \dots + \rho^{\frac{1}{2}(d-2\alpha\gamma-2)} \left(\left(1 - \frac{1}{2}(d-2\alpha\gamma) \right) g_{(d-2\alpha\gamma)ij} - h_{(d-2\alpha\gamma)ij} \right) \\ &\quad + \rho^{\frac{1}{2}(d-2\alpha\gamma-2)} \ln \rho \left(1 - \frac{1}{2}(d-2\alpha\gamma) \right) h_{(d-2\alpha\gamma)ij} + \dots\end{aligned}\tag{6.52}$$

However, each term in the covariant expansion of the extrinsic curvature is a functional of h_{ij} and can be expanded as:

$$\begin{aligned}K_{(0)ij}[h] &= h_{ij} = \frac{1}{\rho} \left(g_{(0)ij} + \rho g_{(2)ij} + \dots + \rho^{\frac{1}{2}(d-2\alpha\gamma)} g_{(d-2\alpha\gamma)ij} \right. \\ &\quad \left. + \rho^{\frac{1}{2}(d-2\alpha\gamma)} \ln \rho h_{(d-2\alpha\gamma)ij} + \dots \right); \\ K_{(2)ij}[h] &= K_{(2)ij}[g_{(0)}] + \rho \int d^d x g_{(2)kl} \frac{\delta K_{(2)ij}}{\delta g_{(0)kl}} + \dots; \\ K_{(d-2\alpha\gamma)ij}[h] &= \rho^{\frac{1}{2}(d-2\alpha\gamma-2)} K_{(d-2\alpha\gamma)ij}[g_{(0)}] + \dots; \\ \tilde{K}_{(d-2\alpha\gamma)ij}[h] &= \rho^{\frac{1}{2}(d-2\alpha\gamma-2)} \tilde{K}_{(d-2\alpha\gamma)ij}[g_{(0)}] + \dots.\end{aligned}\tag{6.53}$$

Inserting these expressions into the expansion and comparing with (6.52) implies:

$$\begin{aligned}
 K_{(0)ij}[g_{(0)}] &= g_{(0)ij}; & (6.54) \\
 K_{(2)ij}[g_{(0)}] &= -g_{(2)ij}; \\
 K_{(d-2\alpha\gamma)ij}[g_{(0)}] &= -\frac{1}{2}(d-2\alpha\gamma)g_{(d-2\alpha\gamma)ij} - h_{(d-2\alpha\gamma)ij} + \dots; \\
 \tilde{K}_{(d-2\alpha\gamma)ij}[g_{(0)}] &= -\frac{1}{2}(d-2\alpha\gamma)h_{(d-2\alpha\gamma)ij}.
 \end{aligned}$$

Here the ellipses denote terms involving functional derivatives with respect to $g_{(0)ij}$ of lower order coefficients $g_{(2n)ij}[g_{(0)}]$.

The formulae are thus simplified in the case where $(d-2\alpha\gamma)$ is not an even integer, since no lower weight terms can contribute and we obtain $K_{(d-2\alpha\gamma)ij} = -(\frac{d}{2}-\alpha\gamma)g_{(d-2\alpha\gamma)ij}$. Similarly treating the scalar field expansion, one finds that

$$\gamma p^{\phi}_{(d-2\alpha\gamma)} = -(d-2\alpha\gamma)\kappa_{(d-2\alpha\gamma)}, \tag{6.55}$$

which yields for the one point functions:

$$\begin{aligned}
 \langle \mathcal{O}_{\phi} \rangle &= (d-2\alpha\gamma) \left(\gamma - \frac{\beta}{\gamma} \right) L e^{\kappa(0)} \text{Tr} g_{(d-2\alpha\gamma)}, & (6.56) \\
 \langle T_{ij} \rangle &= (d-2\alpha\gamma) L e^{\kappa(0)} g_{(d-2\alpha\gamma)ij},
 \end{aligned}$$

where we used the constraint (6.43) in the last equation. Note that the mixing of K and $\dot{\phi}$ in the momenta conspires to ensure that the expectation value of the energy-momentum tensor is proportional to just $g_{(d-2\alpha\gamma)ij}$, without involving $\text{Tr} g_{(d-2\alpha\gamma)}$. These formulas exactly agree with the ones in (5.77) we derived earlier (upon use of (5.21) and (3.10)).

The D4-branes are the only case under consideration where $(d-2\alpha\gamma)$ is an even integer; here the lower weight terms do contribute and the expressions for the vevs are considerably more complicated. We thus turn next to the evaluation of the momentum coefficients in this case.

6.4.2 Category 2: The D4-brane

In this section we will consider the case of the D4-branes, where $(d-2\alpha\gamma)$ is an even integer, and derive the counterterms; the anomaly term \mathcal{A} in the dilatation Ward identity (6.34) and the one point functions. Note that the anomaly appears only if $(d-2\alpha\gamma)$ is an even integer, since only then do we need nonzero coefficients $\tilde{K}_{(d-2\alpha\gamma)}$ and $\tilde{p}^{\phi}_{(d-2\alpha\gamma)}$ of the logarithmic terms in (6.23) to fulfill the field equations. For the branes of interest, only the cases of $p=3$ and $p=4$ have anomalies, and the coefficients can be calculated from the counterterms. The case $p=3$ was discussed already in [4, 6] and will not be discussed further here.

The counterterms and the anomaly are found by recursively computing the momentum coefficients. The Hamiltonian constraint (6.42) along with the scalar equation (6.45) provides a system of equations to determine $K_{(2n)}$ and $p^{\phi}_{(2n)}$, whilst the uncontracted Hamilton-Jacobi identity (6.37) can be used to obtain $K_{(2n)j}^i$. Recall that in this case

$E = 0$, and thus λ is zero, up to total derivatives. This means in particular that the dilatation equation (6.40) can always be written as

$$(1 - 2n)K_{(2n)} - d\gamma\dot{\phi}_{(2n)} = (d - 2\alpha\gamma - 2n)\lambda_{(2n)} \equiv \hat{\Phi}^{-1}\hat{\nabla}_l Y_{(2n)}^l, \quad (6.57)$$

where $\hat{\Phi} \equiv e^{\gamma\phi}$. As λ is zero, up to these total derivatives, the only counterterms needed are the $K_{(2n)}$, along with the logarithmic counterterm $\tilde{K}_{(6)}$. Explicit expressions for the momenta found by solving the recursion relations are given in appendix D, with the terms $K_{(2)}$ and $K_{(4)}$ agreeing with the (non-logarithmic) counterterms found previously, see (5.80).

At weight $(d - 2\alpha\gamma) = 6$ the dilatation equation (6.40) breaks down and only a linear combination of $K_{(6)}$ and $p^{\phi}_{(6)}$ can thus be determined. This however is sufficient to determine the anomaly

$$\langle T_i^i \rangle - \frac{3}{2}\langle \mathcal{O}_\phi \rangle = \mathcal{A}, \quad (6.58)$$

which is given by

$$\mathcal{A} = -4L\tilde{K}_{(6)} = 2Ld(K_{(6)} + \gamma p^{\phi}_{(6)}), \quad (6.59)$$

where the right hand side is the combination of $K_{(6)}$ and $p^{\phi}_{(6)}$ which is determined by (6.42) in terms of lower counterterms. The anomaly in terms of the momentum coefficients is therefore:

$$\begin{aligned} \mathcal{A} &= 10L(K_{(6)} + \gamma p^{\phi}_{(6)}) \\ &= 2L(K_{(2)}^i{}_j K_{(4)}^j{}_i - K_{(2)}K_{(4)} - K_{(2)}\gamma p^{\phi}_{(4)} - K_{(4)}\gamma p^{\phi}_{(2)}). \end{aligned} \quad (6.60)$$

Explicit expressions for each of these terms are given in appendix D; the total anomaly can then be written as

$$\begin{aligned} \mathcal{A} &= -\frac{N^3}{48\pi^2}g_s(\alpha')^{1/2} \left[-R^{ljk i}\mathcal{R}_{lk}\mathcal{R}_{ij} - 2\hat{\Phi}^{-2}\nabla^2\hat{\Phi}\nabla_i\partial_j\hat{\Phi}\mathcal{R}^{ij} \right. \\ &\quad + \frac{1}{2}\mathcal{R}(\mathcal{R}_{ij}\mathcal{R}^{ij} + \hat{\Phi}^{-2}(\nabla^2\hat{\Phi})^2) - \frac{3}{50}\mathcal{R}^3 + \frac{1}{5}\mathcal{R}_{ij}\nabla^i\partial^j\mathcal{R} + \frac{1}{20}\mathcal{R}(\nabla^2 + \hat{\Phi}^{-1}\partial^i\hat{\Phi}\partial_i)\mathcal{R} \\ &\quad - \frac{1}{2}\mathcal{R}_{ij}[(\nabla^2 + \hat{\Phi}^{-1}\partial^l\hat{\Phi}\nabla_l)\mathcal{R}^{ij} - 2\hat{\Phi}^{-2}\partial_l\hat{\Phi}\partial^{(i}\hat{\Phi}\mathcal{R}^{j)l} - 2\hat{\Phi}^{-3}\partial^i\hat{\Phi}\partial^j\hat{\Phi}\nabla^2\hat{\Phi}] \\ &\quad \left. + \frac{1}{2}\hat{\Phi}^{-1}\nabla^2\hat{\Phi}[-(\nabla^2 + \hat{\Phi}^{-1}\partial^i\hat{\Phi}\partial_i)(\hat{\Phi}^{-1}\nabla^2\hat{\Phi}) + 2\hat{\Phi}^{-2}\partial_i\hat{\Phi}\partial_j\hat{\Phi}\mathcal{R}^{ij} + 2\hat{\Phi}^{-3}\partial_i\hat{\Phi}\partial^i\hat{\Phi}\nabla^2\hat{\Phi}] \right], \end{aligned} \quad (6.61)$$

where ∇ is the covariant derivative in the five-dimensional metric and

$$\begin{aligned} \mathcal{R} &\equiv R - 2\hat{\Phi}^{-1}\nabla^2\hat{\Phi}, \\ \mathcal{R}_{ij} &\equiv R_{ij} - \hat{\Phi}^{-1}\nabla_i\partial_j\hat{\Phi}. \end{aligned} \quad (6.62)$$

Here the anomaly has been expressed in such a way to demonstrate that it agrees with the dimensional reduction of the anomaly of the M5-brane theory found in [4, 6]. The latter is given in terms of the six-dimensional curvature $R_{abcd}(G)$ of the six-dimensional metric G_{ab} by

$$\begin{aligned} \langle T_a^a \rangle &= \frac{N^3}{96\pi^3} \left(R^{ab}R^{cd}R_{abcd} - \frac{1}{2}RR^{ab}R_{ab} + \frac{3}{50}R^3 \right. \\ &\quad \left. + \frac{1}{5}R^{ab}D_aD_bR - \frac{1}{2}R^{ab}\square R_{ab} + \frac{1}{20}R\square R \right). \end{aligned} \quad (6.63)$$

In particular, the anomaly vanishes for a Ricci flat manifold (more generally it vanishes for conformally Einstein manifolds). Now recall that on diagonal reduction the six-dimensional Ricci tensor $R(G)_{ab}$ can be written as:

$$R(G)_{ij} = R_{ij} - \hat{\Phi}^{-1} \nabla_i \partial_j \hat{\Phi}; \quad R(G)_{yy} = -\hat{\Phi}^{-1} \nabla^2 \hat{\Phi}. \quad (6.64)$$

Clearly,

$$R_{ij} = \nabla_i \partial_j \hat{\Phi} = 0, \quad (6.65)$$

in the reduced theory implies that the six dimensional manifold is Ricci flat. Comparing with (6.61) one sees that indeed the anomaly vanishes under these conditions.

The anomaly of the six-dimensional theory can be expressed in terms of conformal invariants, such that it is of the form

$$\mathcal{A} = aN^3(E_{(6)} + I_{(6)} + D_a J_{(5)}^a), \quad (6.66)$$

where a is an appropriate constant, $E_{(6)}$ is proportional to the six-dimensional Euler density (type A anomaly), $I_{(6)}$ is a conformal invariant (type B anomaly) and the $D_a J_{(5)}^a$ terms are scheme dependent, as they can always be canceled by adding finite counterterms.

The D4 anomaly can necessarily be expressed in terms of invariants of the generalized conformal structure: dimensional reduction of each of the six-dimensional conformal invariants gives a generalized conformal invariant. Note however that the reduction of the six-dimensional Euler density will give an invariant which is not topological with respect to the five-dimensional background. It is also not clear that the basis of generalized conformal invariants obtained by dimensional reduction would be irreducible; it would be interesting to explore this issue further.

The general one point functions in this case are given by evaluating the expressions:

$$\begin{aligned} \langle \mathcal{O}_\phi \rangle &= -2Le^{\gamma\phi}(\gamma K_{(d-2\alpha\gamma)}), \\ \langle T_{ij} \rangle &= 2Le^{\gamma\phi} \left((K_{(d-2\alpha\gamma)} + \gamma p_{(d-2\alpha\gamma)}^\phi) h_{ij} - K_{(d-2\alpha\gamma)ij} \right). \end{aligned} \quad (6.67)$$

The resulting expressions are as found before, see (5.82):

$$\langle \mathcal{O}_\phi \rangle = -Le^{\kappa(0)} \left(8\varphi + \frac{44}{3} \tilde{\kappa}_{(6)} \right); \quad \langle T_{ij} \rangle = Le^{\kappa(0)} (6t_{ij} + 11h_{(6)ij}), \quad (6.68)$$

where (φ, t_{ij}) are given in (5.41).

7. Two-point functions

In this section we will discuss the computation of 2-point functions for backgrounds with the asymptotics of the non-conformal branes. Transforming to the dual frame, these become Asymptotically locally AdS backgrounds with a linear dilaton and this implies that their analysis is essentially the same as the analysis of the more familiar holographic RG flows with conformal asymptotics [8, 9, 12]. In the next subsection we briefly review the basic principles of the computation of 2-point functions, mostly following the discussion in [8]. Then we compute the 2-point functions for the D-branes in subsection 7.2 and finally we will discuss the computation for the general case in subsection 7.3.

7.1 Generalities

Let us start by recalling the basic formula relating bulk and boundary quantities:

$$\langle \exp(-S_{\text{QFT}}[g_{(0)}, \Phi_{(0)}]) \rangle = \exp(-S_{\text{SG}}[g_{(0)}, \Phi_{(0)}]). \quad (7.1)$$

The left hand side denotes the functional integration involving the field theory action S_{QFT} coupled to background metric $g_{(0)}$ and sources $\Phi_{(0)}$ that couple to composite operators. For the case of Dp-branes the action S_{QFT} is given in (4.1). On the right hand side $S_{\text{SG}}[g_{(0)}, \Phi_{(0)}]$ is the bulk supergravity action evaluated on classical solutions with boundary data $g_{(0)}, \Phi_{(0)}$. For the cases at hand this action is given in (3.4). As discussed extensively in previous sections, this relation needs to be renormalized and we have determined the renormalized action S_{ren} for all cases. By definition the variation of the renormalized action is given by

$$\delta S_{\text{ren}}[g_{(0)}, \Phi_{(0)}] = \int d^{d+1}x \sqrt{g_{(0)}} \left(\frac{1}{2} \langle T_{ij} \rangle \delta g_{(0)}^{ij} + \langle \mathcal{O} \rangle \delta \Phi_{(0)} \right). \quad (7.2)$$

Higher point functions are determined by further differentiation of the 1-point functions, e.g. for the case of Dp-branes

$$\langle \mathcal{O}(x) \mathcal{O}(y) \rangle = - \frac{1}{\sqrt{g_{(0)}}} \frac{\delta \langle \mathcal{O}(x) \rangle}{\delta \Phi_{(0)}(y)} \Big|_{g_{(0)ij}=\delta_{ij}, \Phi_{(0)}=g_d^{-2}}. \quad (7.3)$$

As we have shown in earlier sections, the 1-point functions in the presence of sources are expressed in terms of the asymptotic coefficients in the near-boundary expansion of the bulk solution. In particular, they depend on the coefficients that the asymptotic analysis does not determine. To obtain those we need exact regular solutions with prescribed boundary conditions. On general grounds, regularity in the interior should fix the relation between the asymptotically undetermined coefficients and the boundary data. Having obtained such relations one can then proceed to compute the holographic n -point functions. To date, this program has only been possible to carry out perturbatively around given solutions. In particular, linearized solutions determine 2-point functions, second order perturbations determine 3-point functions etc. Here we will discuss the 2-point functions involving the stress energy tensor T_{ij} and the scalar operator \mathcal{O} .

Let us decompose the metric perturbation as,

$$\delta g_{(0)ij}(x) = \delta h_{(0)ij}^T + \nabla_{(i} \delta h_{(0)j)}^L + g_{(0)ij} \frac{1}{d-1} \delta f_{(0)} - \nabla_i \nabla_j \delta H_{(0)} \quad (7.4)$$

where

$$\nabla^i h_{(0)ij}^T = 0, \quad h_{(0)i}^T{}^i = 0, \quad \nabla^i h_{(0)i}^L = 0. \quad (7.5)$$

All covariant derivatives are that of $g_{(0)}$. Then the different components source different irreducible components of the stress energy tensor,

$$\begin{aligned} \delta S_{\text{ren}}[g_{(0)}, \Phi_{(0)}] = \int d^{d+1}x \sqrt{g_{(0)}} \left(\langle \mathcal{O} \rangle \delta \Phi_{(0)} - \frac{1}{2} \langle T_{ij} \rangle \delta h_{(0)}^{Tij} - \frac{1}{2(d-1)} \langle T_i^i \rangle \delta f_{(0)} \right. \\ \left. + \nabla^i \langle T_{ij} \rangle \delta h_{(0)}^{Lj} + \nabla^i \nabla^j \langle T_{ij} \rangle \delta H_{(0)} \right) \end{aligned} \quad (7.6)$$

Now, recall that in the cases we discuss here we have already established that the holographic Ward identities,

$$\nabla^j \langle T_{ij} \rangle_J + \langle \mathcal{O} \rangle_J \partial_i \Phi_{(0)} = 0, \quad (7.7)$$

$$\langle T_i^i \rangle_J + (d-4) \Phi_{(0)} \langle \mathcal{O} \rangle_J = \mathcal{A}, \quad (7.8)$$

where there is an anomaly only for $p = 4$. These and the fact that $\Phi_{(0)}$ in the background solution is a constant imply that the second line in (7.6) does not contribute to 2-point functions. Note also that the source for the trace of stress energy tensor is $-f_{(0)}/(2(d-1))$.

We will be interested in cases with $g_{(0)ij} = \delta_{ij}$ (or somewhat more generally the cases with $g_{(0)}$ being conformally flat). The two-point functions of T_{ij} and \mathcal{O} have the following standard representation in momentum space,

$$\begin{aligned} \langle T_{ij}(q) T_{kl}(-q) \rangle &= \Pi_{ijkl}^{TT} A(q^2) + \pi_{ij} \pi_{kl} B(q^2) \\ \langle T_{ij}(q) \mathcal{O}(-q) \rangle &= \pi_{ij} C(q^2) \\ \langle \mathcal{O}(q) \mathcal{O}(-q) \rangle &= D(q^2) \end{aligned} \quad (7.9)$$

where A, B, C, D are functions of q^2 and

$$\begin{aligned} \pi_{ij} &= \delta_{ij} - \frac{q_i q_j}{q^2} \\ \Pi_{ijkl}^{TT} &= -\frac{\delta h_{(0)ij}^{TT}}{\delta h_{(0)kl}^{TT}} = \frac{1}{2} (\pi_{ik} \pi_{jl} + \pi_{il} \pi_{jk}) - \frac{1}{d-1} \pi_{ij} \pi_{kl} \end{aligned} \quad (7.10)$$

are transverse and transverse traceless projectors, respectively. The trace Ward identity implies

$$\begin{aligned} \langle T_{ij}(q) T_k^k(-q) \rangle &= -\frac{1}{g_d^2} (d-4) \langle T_{ij}(q) \mathcal{O}(-q) \rangle \\ \langle T_i^i(q) \mathcal{O}(-q) \rangle &= -\frac{1}{g_d^2} (d-4) \langle \mathcal{O}(q) \mathcal{O}(-q) \rangle \end{aligned} \quad (7.11)$$

which then leads to the relations,

$$B(q^2) = -\frac{1}{g_d^2} \frac{(d-4)}{(d-1)} C(q^2) = \left(\frac{1}{g_d^2} \frac{(d-4)}{(d-1)} \right)^2 D(q^2) \quad (7.12)$$

Furthermore, the coefficient $D(q^2)$ is also constrained by the generalized conformal invariance as discussed in section 4.

7.2 Holographic 2-point functions for the brane backgrounds

We next discuss the computation of the 2-point functions in the backgrounds of the non-conformal branes. Earlier discussions of the 2-point functions in the D0-brane background can be found in [27] and for Dp-brane backgrounds they were discussed in [18, 19, 28].

We need to solve for small fluctuations around the background solution given in (3.7). We thus consider a solution of the form

$$ds^2 = \frac{d\rho^2}{4\rho^2} + \frac{g_{ij}(x, \rho) dx^i dx^j}{\rho}, \quad (7.13)$$

$$\phi(x, \rho) = \alpha \log \rho + \varphi(x, \rho), \quad \varphi(x, \rho) \equiv \frac{\kappa(x, \rho)}{\gamma},$$

with

$$g_{ij}(x, \rho) = \delta_{ij} + \gamma_{ij}(x, \rho). \quad (7.14)$$

and φ, γ_{ij} considered infinitesimal. The background metric is translationally invariant, so it is convenient to Fourier transform. The fluctuation $\gamma_{ij}(q, \rho)$ can be decomposed into irreducible pieces as

$$\gamma_{ij}(q, \rho) = e_{ij}(q, \rho) + \frac{d}{d-1} \left(\frac{1}{d} \delta_{ij} - \frac{q_i q_j}{q^2} \right) f(q, \rho) + \frac{q_i q_j}{q^2} S(q, \rho), \quad (7.15)$$

Let us also express the transverse traceless part as $e_{ij}(q, \rho) \equiv h_{(0)ij}^T(q) h(q, \rho)$, where $h(q, \rho)$ is normalized to go to 1 as $\rho \rightarrow 0$. The field theory sources $h_{(0)ij}^T(q), f_{(0)}(q), S_{(0)}(q)$ are the leading ρ independent parts of $e_{ij}(q, \rho), f(q, \rho), S(q, \rho)$. Relative to the discussion in the previous subsection, we have gauged away the longitudinal vector perturbation h_i^L and traded H for $S = \frac{d}{d-1} f + p^2 H$.

The linearized equations are now obtained by inserting (7.14)–(7.15) into (5.6)–(5.9) and treating κ, h, f, S as infinitesimal variables. This leads to the following equations:

$$\frac{1}{2} S'' + \kappa'' = 0; \quad (7.16)$$

$$\frac{1}{2} f' + \kappa' = 0; \quad (7.17)$$

$$2\rho h'' - (d-2-2\alpha\gamma)h' - \frac{1}{2}q^2 h = 0; \quad (7.18)$$

$$2\rho S'' + (2\alpha\gamma + 2 - 2d)S' - 2d\kappa' - q^2(\kappa + f) = 0; \quad (7.19)$$

$$4\rho\kappa'' + (8\alpha\gamma + 4 - 2d)\kappa' + 2\alpha\gamma S' - q^2\kappa = 0, \quad (7.20)$$

where the equations are listed in the same order as in (5.6)–(5.9) with (7.18) and (7.19) being the transverse traceless and trace part of (5.8). Equation (7.18) is already diagonal. The remaining equations can be diagonalized by elementary manipulations leading to the following expressions,

$$\kappa(q, \rho) = 2\alpha\gamma v_0(q) + v_1(q)\chi(q, \rho) \quad (7.21)$$

$$f(q, \rho) = -2(d-1)v_0(q) - 2v_1(q)\chi(q, \rho),$$

$$S(q, \rho) = v_2(q) + \rho q^2 v_0(q) - 2v_1(q)\chi(q, \rho)$$

where v_0, v_1, v_2 are integration constants, which can be expressed in terms of the sources as

$$v_0 = \frac{2\gamma\phi_{(0)} + f_{(0)}}{2(1-2\sigma)}, \quad v_1 = \frac{(d-1)\gamma\phi_{(0)} + \alpha\gamma f_{(0)}}{2\sigma-1}, \quad v_2 = S_{(0)} + 2v_1, \quad (7.22)$$

where $\sigma = d/2 - \alpha\gamma = (p-7)/(p-5)$ and $\phi_{(0)} = \kappa_{(0)}/\gamma$ with $\kappa_{(0)}$ the ρ independent part of $\kappa(q, \rho)$. $\chi(q, r)$ is normalized to go to 1 as $\rho \rightarrow 0$ and satisfies the same differential equation as the transverse traceless mode, namely

$$2\rho\chi'' - 2(\sigma - 1)\chi' - \frac{1}{2}q^2\chi = 0 \tag{7.23}$$

The solution of this equation that is regular in the interior is given in terms of the modified Bessel function of the second kind,

$$\chi_\sigma(q, \rho) = c(\sigma)x^\sigma K_\sigma(x), \quad x = \sqrt{q^2\rho}, \quad \sigma = \frac{p-7}{p-5}, \tag{7.24}$$

where the normalization coefficient $c(\sigma)$ is chosen such that $\chi(q, \rho)$ approaches 1 as $\rho \rightarrow 0$. In our case, $\sigma = \{7/5, 3/2, 5/3, 3\}$ for $p = \{0, 1, 2, 4\}$.

7.2.1 Non-integral cases

The asymptotic expansion for non-integer values of σ is

$$\chi_\sigma(q, \rho) = 1 + \frac{1}{4(1-\sigma)}q^2\rho + \dots + \tilde{\chi}_{(2\sigma)}(q)\rho^\sigma + \dots \quad (\nu \text{ non-integer}) \tag{7.25}$$

where

$$\tilde{\chi}_{(2\sigma)}(q) = -\frac{\Gamma(1-\sigma)}{2^{2\sigma}\Gamma(1+\sigma)}(q^2)^\sigma. \tag{7.26}$$

One can verify that the leading order terms in the exact linearized solution indeed agree with the linearization of the asymptotic coefficients derived earlier and furthermore one can obtain the coefficient that the asymptotic analysis left undetermined. Combining the previous formulas we obtain,

$$\begin{aligned} \kappa_{(2\sigma)} &= v_1(q)\tilde{\chi}_{(2\sigma)}(q^2) \\ g_{(2\sigma)ij} &= \left(h_{(0)ij}^T(q) - \frac{2}{(d-1)}v_1(q)\pi_{ij} \right) \tilde{\chi}_{(2\sigma)}(q^2) \end{aligned} \tag{7.27}$$

which indeed satisfy the linearization of (5.28)–(5.29). Thus the 1-point functions (5.77) to linear order in the sources are then given by

$$\langle \mathcal{O}_\phi \rangle = \frac{2\sigma L\gamma(d-1)}{\alpha(2\sigma-1)} \left(\phi_{(0)} - 2\alpha \left(-\frac{f_{(0)}}{2(d-1)} \right) \right) \tilde{\chi}_{(2\sigma)}(q^2), \tag{7.28}$$

$$\langle T_{ij} \rangle = 2\sigma L \left(h_{(0)ij}^T - \frac{2\gamma}{(2\sigma-1)} \left(\phi_{(0)} - 2\alpha \left(-\frac{f_{(0)}}{2(d-1)} \right) \right) \pi_{ij} \right) \tilde{\chi}_{(2\sigma)}(q^2). \tag{7.29}$$

It follows that the 2-point functions are given by

$$\begin{aligned} \langle T_{ij}(q)T_{kl}(-q) \rangle &= \Pi_{ijkl}^{TT} (4\sigma L\tilde{\chi}_{(2\sigma)}(q^2)) + \pi_{ij}\pi_{kl} \left(-\frac{2\alpha}{(d-1)} \right)^2 \langle \mathcal{O}_\phi(q)\mathcal{O}_\phi(-q) \rangle \\ \langle T_{ij}(q)\mathcal{O}_\phi(-q) \rangle &= \pi_{ij} \left(-\frac{2\alpha}{(d-1)} \right) \langle \mathcal{O}_\phi(q)\mathcal{O}_\phi(-q) \rangle \\ \langle \mathcal{O}_\phi(q)\mathcal{O}_\phi(-q) \rangle &= -\frac{2\sigma L\gamma(d-1)}{\alpha(2\sigma-1)}\tilde{\chi}_{(2\sigma)}(q^2) \end{aligned} \tag{7.30}$$

These relations are of the form (7.9) with the coefficients B, C related to the D coefficient as dictated by the trace Ward identity (with the relation becoming (7.12) once we pass from \mathcal{O}_ϕ to \mathcal{O}). Thus we only need discuss the transverse traceless part of the 2-point function of T_{ij} and the scalar 2-point function.

We now Fourier transform to position space using

$$\int d^d q e^{-iqx} (q^2)^\sigma = \pi^{d/2} 2^{d+2\sigma} \frac{\Gamma(d/2 - \sigma)}{\Gamma(-\sigma)} \frac{1}{|x|^{d+2\sigma}}, \quad (7.31)$$

which is valid when $\sigma \neq -(d/2 + k)$, where k is an integer. Let us first discuss the case of Dp -branes. The scalar two function becomes

$$\begin{aligned} \langle \mathcal{O}_\phi(x) \mathcal{O}_\phi(0) \rangle &= C_\phi N^{(7-p)/(5-p)} (g_d^2)^{(p-3)/(5-p)} |x|^{\frac{p^2-19-2p}{5-p}}, \\ &= C_\phi N^2 \frac{(g_{\text{eff}}^2(x))^{\frac{(p-3)}{(5-p)}}}{|x|^{2d}}, \end{aligned} \quad (7.32)$$

where C_ϕ is a positive numerical constant (obtained by collecting all numerical constants in previous formulas). Note that the characteristic scale in this case is x and therefore the effective coupling constant is $g_{\text{eff}}^2(x) = g_d^2 N |x|^{3-p}$. The g_d and x dependence is consistent with the constraints of generalized conformal invariance discussed in section 4. Recall also that the operator \mathcal{O}_ϕ at weak coupling has dimension d (and \mathcal{O} has dimension 4). So going from weak to strong coupling we find that the dimension is protected but the 2-point function itself gets corrections. The overall factor of N^2 reflects the fact that this is a tree level computation. Similarly, the transverse traceless part of the 2-point function of the stress energy tensor is given by

$$\langle T_{ij}(x) T_{kl}(0) \rangle_{TT} = C_T \Pi_{ijkl}^{TT} \frac{N^2 (g_{\text{eff}}^2(x))^{\frac{(p-3)}{(5-p)}}}{|x|^{2d}}, \quad (7.33)$$

with C_T a positive constant. In this case the dimension is protected because T_{ij} is conserved. We can trust these results provided

$$g_{\text{eff}}^2(x) \gg 1 \quad \Rightarrow \quad |x| \gg (g_d^2 N)^{-1/(3-p)} \quad (7.34)$$

For the fundamental string background we obtain

$$\langle \mathcal{O}_\phi(x) \mathcal{O}_\phi(0) \rangle \sim N^{3/2} g_s (\alpha')^{1/2} \frac{1}{|x|^5}, \quad (7.35)$$

$$\langle T_{ij}(x) T_{kl}(0) \rangle_{TT} \sim N^{3/2} g_s (\alpha')^{1/2} \Pi_{ijkl}^{TT} \frac{1}{|x|^5} \quad (7.36)$$

In the IIB case S-duality relates the fundamental string solution to the D1 brane solution. Indeed, the 2-point function (7.35) becomes equal the $p = 1$ case in (7.32) under S-duality, $g_s \rightarrow 1/g_s, \alpha' \rightarrow \alpha' g_s$.

In the IIA case the fundamental string lifts to the M2 brane. As discussed in section 5.3, the source for the stress energy tensor of the M2 theory is simply related to the sources for

the stress energy tensor of the string and the operator \mathcal{O}_ϕ , see (5.51). Taking into account that the worldvolume theories are related by reduction over the M-theory circle and so their actions are related by the factor of R_{11} , the radius of the M-theory circle, we find (up to numerical constants)

$$T_{ij}^{M2} \sim R_{11}^{-1} T_{ij}, \quad T_{yy}^{M2} \sim R_{11}^{-1} \mathcal{O}_\phi \quad (7.37)$$

Using $R_{11} = g_s l_s$ we get

$$\langle T_{yy}^{M2}(x) T_{yy}^{M2}(0) \rangle = \frac{1}{R_{11}^2} \langle \mathcal{O}_\phi(x) \mathcal{O}_\phi(0) \rangle \sim \frac{N^{3/2}}{R_{11} |x|^5} \quad (7.38)$$

with similar results for the other correlators. The stress energy tensor of the M2 theory has dimension 3, so one expects the correlator to scale as $|x|^{-6}$. However, one of the worldvolume directions is compactified with radius R_{11} . Smearing out over the compactified direction indeed results in the fall off in (7.38). Finally the N scaling is the well-known $N^{3/2}$ scaling of the M2 theory.

7.2.2 The D4 case

For the $\sigma = 3$ case corresponding to D4 branes we have

$$\chi_3(q, \rho) = 1 - \frac{1}{8} q^2 \rho + \dots + \rho^3 \left(\tilde{\chi}_{(6)}(q) + \frac{1}{768} q^6 \log \rho \right) + \dots \quad (7.39)$$

where

$$\tilde{\chi}_{(6)}(q) = \frac{1}{384} q^6 \left(\frac{1}{2} \log q^2 - \log 2 + \gamma - \frac{11}{12} \right) \quad (7.40)$$

and γ is the Euler constant (not to be confused with the γ used in other parts of this paper). The terms without $\log q^2$ are scheme dependent and will be omitted in what follows. The one point functions and two point functions are then given by (7.28), (7.29) and (7.30) respectively. In particular,

$$\langle \mathcal{O}_\phi(q) \mathcal{O}_\phi(-q) \rangle = \frac{L}{180} q^6 \ln q^2. \quad (7.41)$$

Fourier transforming back to position space, the scalar two function becomes

$$\langle \mathcal{O}_\phi(x) \mathcal{O}_\phi(0) \rangle = C_\phi N^2 \mathcal{R} \left(\frac{g_{\text{eff}}^2(x)}{|x|^{10}} \right), \quad (7.42)$$

where C_ϕ is a positive numerical constant (obtained by collecting all numerical constants) and as in section 4 $\mathcal{R}(1/|x|^a)$ denotes the renormalised version of $(1/|x|^a)$. The effective coupling constant is $g_{\text{eff}}^2(x) = g_d^2 N / |x|$, and the g_d and x dependence is consistent with the constraints of generalized conformal invariance discussed in section 4.

This result is also consistent with the uplift to the M5-brane results. The source for the stress energy tensor of the M5 theory is simply related to the sources for the stress energy tensor of the D4-brane and the operator \mathcal{O}_ϕ . Taking into account that the worldvolume

theories are related by reduction over M-theory circle and so their actions are related by the factor of R_{11} , the radius of the M-theory circle, we find (up to numerical constants)

$$T_{ij}^{M5} \sim R_{11}^{-1} T_{ij}, \quad T_{yy}^{M5} \sim R_{11}^{-1} \mathcal{O}_\phi \quad (7.43)$$

Using $R_{11} = g_s l_s$ we then get

$$\langle T_{yy}^{M5}(x) T_{yy}^{M5}(0) \rangle = \frac{1}{R_{11}^2} \langle \mathcal{O}_\phi(x) \mathcal{O}_\phi(0) \rangle \sim \frac{N^3}{R_{11}} \mathcal{R} \left(\frac{1}{|x|^{11}} \right) \quad (7.44)$$

with similar results for the other correlators. The stress energy tensor of the M5 theory has dimension six, and the correlator of the six-dimensional theory behaves as $\mathcal{R}|x|^{-12}$. Here one of the worldvolume directions is compactified with radius R_{11} and smearing out over the compactified direction indeed results in the fall off in (7.44). Note that the N scaling is the well-known N^3 scaling of the M5-brane theory.

7.3 General case

In the simple case discussed above, it was straightforward to solve the equations for linear perturbations, but in more general backgrounds the diagonalisation of the fluctuation equations is more involved. To treat the general case, it is convenient to use the analysis [43, 8, 12] of linear fluctuations around background solutions of a single scalar field coupled to gravity; in these paper the fluctuation equations were diagonalised for a general domain wall scalar system.

In this section we will explain a general method for computing the two point functions which exploits this analysis. As discussed in section 7.1 we need to determine the one point functions to linear order in the sources and in the Hamiltonian method this corresponds to determining the momenta to linear order in the sources. So, as in the previous section, let us begin by considering linear fluctuations around the background of interest in the dual frame:

$$h_{ij} = h_{ij}^B(r) + \gamma_{ij}(r, x) = e^{2A(r)} \delta_{ij} + \gamma_{ij}(r, x), \quad (7.45)$$

$$\phi = \phi_B(r) + \varphi(r, x).$$

Note that the metric fluctuation has already been put into axial gauge. Next we will express the canonical momenta in terms of these fluctuations. To do this, first note that the extrinsic curvature of constant r hypersurfaces can be expressed as:

$$K_j^i = \dot{A} \delta_j^i + \frac{1}{2} \dot{S}_j^i, \quad (7.46)$$

where $S_j^i \equiv h_B^{ik} \gamma_{kj}$. S_j^i can be decomposed into irreducible components as

$$S_j^i = e_j^i + \frac{d}{d-1} \left(\frac{1}{d} \delta_j^i - \frac{\partial^i \partial_j}{\nabla_B^2} \right) f + \frac{\partial^i \partial_j}{\nabla_B^2} S, \quad (7.47)$$

where $\partial_i e_j^i = e_j^i = 0$, $S = S_i^i$, indices are raised with the inverse background metric $e^{-2A} \delta^{ij}$ and $\nabla_B^2 = e^{-2A} \nabla^2 = e^{-2A} \delta^{ij} \partial_i \partial_j$. Here the diffeomorphism invariance of the transverse space can be used to set the vector component to zero.

The momenta (6.12) up to linear order in the fluctuations are then given by

$$\begin{aligned} \pi_\phi &= 2B(\beta\partial_r\phi + \gamma K) = \pi_\phi^B + B(2\beta\partial_r\varphi + \gamma\partial_r S), \\ \pi_i^i &= \pi_i^{i,B} - \frac{1}{2}B(d-1)\partial_r S + Bd\gamma\partial_r\varphi, \quad \pi_{j,TT}^i = \pi_{j,TT}^{i,B} - \frac{1}{2}B\partial_r e_j^i, \end{aligned} \tag{7.48}$$

where π_ϕ^B , $\pi_i^{i,B}$ and $\pi_{j,TT}^{i,B}$ are the background values, in the absence of fluctuations, and TT stands for transverse and traceless. The one point functions are obtained by extracting the components of appropriate dilatation weight from these momenta. So we need to determine $\partial_r\varphi$, $\partial_r S$, $\partial_r e_j^i$.

To obtain these momenta, however, we would need to diagonalise the equations of motion for the linear fluctuations, and solve for $\partial_r\varphi$ etc. Diagonalising such fluctuation equations is in general rather difficult, and thus it is convenient to exploit the analysis of [8, 12], where the fluctuation equations were diagonalised for a generic domain wall dilaton background. In the latter work, however, an Einstein frame bulk action was used, so we will first need to transform our backgrounds to the Einstein frame, and then map our fluctuation equations to the set of equations which were diagonalised in full generality in [8, 12].

The analysis of [8, 12] begins with an Einstein frame bulk action:

$$S = - \int d^{d+1}x \sqrt{G_E} \left(\frac{1}{2\kappa^2} R_E - \frac{1}{2} (\partial\tilde{\phi})^2 - V(\tilde{\phi}) \right). \tag{7.49}$$

and then one considers domain wall solutions of the form

$$ds_B^2 = d\tilde{r}^2 + e^{2A(\tilde{r})} dx_i dx^i, \quad \tilde{\phi} = \tilde{\phi}_B(\tilde{r}), \tag{7.50}$$

which preserve Poincaré symmetry in the transverse directions. Here the subscript B denotes that this is the background solution around which linear fluctuation equations will be solved.

Substituting the ansatz (7.50) into the field equations gives:

$$\begin{aligned} \dot{A}^2 - \frac{\kappa^2}{d(d-1)} (\dot{\tilde{\phi}}_B^2 - 2V(\tilde{\phi}_B)) &= 0, \\ \ddot{A} + d\dot{A}^2 + \frac{2\kappa^2}{d-1} V(\tilde{\phi}_B) &= 0, \\ \ddot{\tilde{\phi}}_B + d\dot{A}\dot{\tilde{\phi}}_B - V'(\tilde{\phi}_B) &= 0, \end{aligned} \tag{7.51}$$

where the dot denotes differentiation with respect to \tilde{r} and the prime denotes differentiation with respect to $\tilde{\phi}$. In explicitly solving these equations one can use the fact that these second order equations are solved by any solution of the first order flow equations [44, 45]:

$$\begin{aligned} \dot{A} &= -\frac{\kappa^2}{d-1} W(\tilde{\phi}_B), \\ \dot{\tilde{\phi}}_B &= W'(\tilde{\phi}_B), \end{aligned} \tag{7.52}$$

with the potential expressed in terms of a superpotential W as:

$$V(\tilde{\phi}_B) = \frac{1}{2} \left[W'^2 - \frac{d\kappa^2}{d-1} W^2 \right]. \quad (7.53)$$

Conversely, given an explicit solution of (7.51), which may not be asymptotically AdS but $\tilde{\phi}_B$ should have at most isolated zeros, one can use (7.52) to define a superpotential $W(\tilde{\phi}_B)$ [46].

Now let us consider the backgrounds of interest here, which are asymptotic to Dp-brane backgrounds. In these cases, the action (3.4) in the dual frame can be transformed to the Einstein frame using the transformation $g_E = \exp(2\gamma\phi/(d-1))g_{\text{dual}}$, giving

$$S = -L \int d^{d+1}x \sqrt{G_E} \left[R_E - \frac{1}{2}(\partial\tilde{\phi})^2 + C e^{-2\gamma\tilde{\phi}/\nu(d-1)} \right]. \quad (7.54)$$

Here the scalar has been rescaled as

$$\tilde{\phi} \equiv \nu\phi, \quad \nu \equiv \sqrt{2 \left(\frac{d\gamma^2}{d-1} - \beta \right)}, \quad (7.55)$$

so that $\tilde{\phi}$ is canonically normalized. The metric and dilaton for the decoupled Dp-brane background can then be written in Einstein frame as

$$\begin{aligned} ds_E^2 &= d\tilde{r}^2 + (\mu\tilde{r})^{2(\mu+1)/\mu} dx_i dx^i, \\ \tilde{\phi} &= -\frac{2\alpha\nu}{\mu} \log(\mu\tilde{r}), \\ \tilde{r} &= \frac{\rho^{-\mu/2}}{\mu} = \frac{e^{\mu r}}{\mu}, \quad \mu = -\frac{2\alpha\gamma}{d-1} = \frac{(p-3)^2}{p(5-p)}. \end{aligned} \quad (7.56)$$

From this solution one can extract the parameters and functions abstractly defined in (7.50), (7.52) and (7.53):

$$\begin{aligned} \kappa^2 &= \frac{1}{2}, \quad A(\tilde{r}) = \frac{\mu+1}{\mu} \log(\mu\tilde{r}), \quad \tilde{\phi}_B = \sqrt{\frac{2(\mu+1)(d-1)}{\mu}} \log(\mu\tilde{r}) \\ V(\tilde{\phi}_B) &= -C \exp \left(-\sqrt{\frac{2\mu}{(\mu+1)(d-1)}} \tilde{\phi}_B \right), \\ W(\tilde{\phi}_B) &= -2(d-1)(\mu+1) \exp \left(-\sqrt{\frac{\mu}{2(\mu+1)(d-1)}} \tilde{\phi}_B \right). \end{aligned}$$

Given a more general solution in the dual frame, which asymptotes to an AdS linear dilaton background, one can similarly transform it into Einstein frame and extract the corresponding superpotential etc.

Suppose the fluctuations in the Einstein frame are given by:

$$g_{E\mu\nu} = g_{E\mu\nu}^B + \tilde{\gamma}_{\mu\nu}; \quad \tilde{\phi} = \tilde{\phi}^B + \tilde{\varphi}, \quad (7.57)$$

where $\tilde{S}_j^i \equiv h_B^{ik} \tilde{\gamma}_{kj}$ is:

$$\tilde{S}_j^i = \tilde{e}_j^i + \frac{d}{d-1} \left(\frac{1}{d} \delta_j^i - \frac{\partial^i \partial_j}{\nabla_B^2} \right) \tilde{f} + \frac{\partial^i \partial_j}{\nabla_B^2} \tilde{S}, \quad (7.58)$$

Then these fluctuations in Einstein frame are related to those in the dual frame defined in (7.45) via:

$$\begin{aligned} \tilde{e}_j^i &= e_j^i, & \tilde{f} &= 2\gamma\varphi + f, \\ \tilde{S} &= \frac{2\gamma d}{(d-1)}\varphi + S, & \nu\tilde{\varphi} &= \varphi, & \tilde{\gamma}_{rr} &= \frac{2\gamma d}{(d-1)}\varphi. \end{aligned} \quad (7.59)$$

Note in particular that the Weyl transformation to the Einstein frame takes the fluctuations outside axial gauge: $\tilde{\gamma}_{rr} \neq 0$.

Using [8, 12], one can write down the diagonalised equations of motion for the linear fluctuations in Einstein frame:

$$\begin{aligned} (\partial_{\tilde{r}}^2 + d\dot{A}\partial_{\tilde{r}} - e^{-2A}q^2)\tilde{e}_j^i &= 0, \\ (\partial_{\tilde{r}}^2 + [d\dot{A} + 2W\partial_{\tilde{\phi}}^2 \log W]\partial_{\tilde{r}} - e^{-2A}q^2)\omega &= 0, \end{aligned} \quad (7.60)$$

$$\partial_{\tilde{r}}\tilde{S} = \frac{1}{(d-1)\dot{A}} \left(e^{-2A}q^2\tilde{f} + 2\kappa^2(\partial_{\tilde{r}}\tilde{\phi}_B\partial_{\tilde{r}}\varphi - V'(\tilde{\phi}_B)\tilde{\varphi} - V(\tilde{\phi}_B)\tilde{\gamma}_{rr}) \right),$$

where

$$\omega \equiv \frac{W}{W'}\tilde{\varphi} + \frac{1}{2\kappa^2}\tilde{f}, \quad (7.61)$$

and we have Fourier transformed to momentum space, with q being the momentum.

To derive the two point functions we will need to obtain the functional dependence of the one-point functions on the sources. The one-point functions are given in terms of the canonical momenta, with the parts dependent on the fluctuations being given by linear combinations of radial derivatives of fluctuations. Hence we write the radial derivatives of the fluctuations \tilde{e}_j^i and ω as functionals of the background fields A and $\tilde{\phi}_B$:

$$\partial_{\tilde{r}}\tilde{e}_j^i = E(A, \tilde{\phi}_B)\tilde{e}_j^i, \quad \partial_{\tilde{r}}\omega = \Omega(A, \tilde{\phi}_B)\omega. \quad (7.62)$$

The first two equations in (7.60) then become first order equations for E and Ω :

$$\begin{aligned} \dot{E} + E^2 + d\dot{A}E - e^{-2A}q^2 &= 0, \\ \dot{\Omega} + \Omega^2 + [d\dot{A} + 2W\partial_{\tilde{\phi}}^2 \log W]\Omega - e^{-2A}q^2 &= 0. \end{aligned} \quad (7.63)$$

Note that in the case of the Dp-brane backgrounds these equations actually coincide since $\partial_{\tilde{\phi}}^2 \log W = 0$. Given the solutions for E and Ω and omitting terms that contribute to contact terms one can obtain the required expressions for the radial derivatives of other fluctuations:

$$\begin{aligned} \partial_{\tilde{r}}\tilde{e}_j^i &= E\tilde{e}_j^i, \\ \partial_{\tilde{r}}\tilde{\varphi} &= \Omega\tilde{\varphi} + \frac{1}{2\kappa^2} \frac{W'}{W} \Omega \tilde{f}, \\ \partial_{\tilde{r}}\tilde{S} &= -\frac{1}{\kappa^2} \left[\left(\frac{W'}{W} \right)^2 \Omega + \frac{e^{-2A}}{W} q^2 \right] \tilde{f} - 2 \frac{W'}{W} \Omega \tilde{\varphi}. \end{aligned} \quad (7.64)$$

This completes the diagonalisation of the fluctuation equations in the Einstein frame. Next one can rewrite these relations in terms of the fluctuations and radial derivative in the dual frame as:

$$\begin{aligned}
 \partial_r \tilde{e}_j^i &= e^{\gamma\phi_B/(d-1)} E e_j^i, & (7.65) \\
 \nu \partial_r \tilde{\varphi} &= e^{\gamma\phi_B/(d-1)} \left(\nu^2 \Omega \left(1 + \frac{\gamma}{\nu\kappa^2} \frac{W'}{W} \right) \varphi + \frac{\nu}{2\kappa^2} \frac{W'}{W} \Omega f \right), \\
 \partial_r \tilde{S} &= e^{\gamma\phi_B/(d-1)} \left(-\frac{1}{\kappa^2} \left[\left(\frac{W'}{W} \right)^2 \Omega + \frac{e^{-2A}}{W} q^2 \right] f \right. \\
 &\quad \left. - 2\nu \left[\left(\frac{W'}{W} + \frac{\gamma}{\nu\kappa^2} \left(\frac{W'}{W} \right)^2 \right) \Omega + \frac{\gamma}{\nu\kappa^2} \frac{e^{-2A}}{W} q^2 \right] \varphi \right).
 \end{aligned}$$

Using (7.59) in (7.48), and applying (6.28) one finds that the expressions for the one point functions to linear order in the fluctuations are:

$$\begin{aligned}
 \langle \mathcal{O}_\phi \rangle &= \langle \mathcal{O}_\phi \rangle_B - B(\nu \partial_r \tilde{\varphi} - \gamma \partial_r \tilde{S})_{(2\sigma)}, & (7.66) \\
 \langle T_i^i \rangle &= \langle T_i^i \rangle_B - B(d-1)(\partial_r \tilde{S})_{(2\sigma)}, \\
 \langle T_{j,TT}^i \rangle &= \langle T_{j,TT}^i \rangle_B + B(\partial_r \tilde{e}_j^i)_{(2\sigma)},
 \end{aligned}$$

where $X_{(2\sigma)}$ denotes the term of dilatation weight $2\sigma \equiv (d - 2\alpha\gamma)$ in X .

To explicitly evaluate these one point functions with linear sources we now need to determine exact regular solutions for E and Ω . Up to this point, we have given completely general expressions, applicable for all solutions asymptotic to the Dp-brane backgrounds. The actual background determines the defining differential equations for E and Ω . Next we will solve these equations for the specific case of the decoupled Dp-brane background; as mentioned before, the equations for E and Ω become identical in this case since $\partial_\phi^2 \log W = 0$. The only equation to be solved is thus:

$$\left(\partial_{\tilde{r}}^2 + \frac{d(\mu+1)}{\mu\tilde{r}} \partial_{\tilde{r}} - (\mu\tilde{r})^{-2(\mu+1)/\mu} q^2 \right) \omega = 0. \quad (7.67)$$

The solution which is regular in the interior, $\tilde{r} \rightarrow 0$, is given by

$$\begin{aligned}
 \omega(\tilde{r}) &= (\mu\tilde{r})^{-c} K_{\mu c} \left(\frac{q}{(\mu\tilde{r})^{1/\mu}} \right) \equiv e^{-\sigma r} K_\sigma(qe^{-r}), & (7.68) \\
 \mu c &= \frac{1}{2}(d - 2\alpha\gamma) \equiv \sigma,
 \end{aligned}$$

where K is the modified Bessel function of the second kind; these are exactly the same functions found in the previous section. The solution for Ω is then

$$\Omega = \partial_{\tilde{r}} \ln \left((\mu\tilde{r})^{-c} K_{\mu c} \left(\frac{q}{(\mu\tilde{r})^{1/\mu}} \right) \right) \equiv e^{-\mu r} \partial_r \ln(\chi_\sigma(q, e^{-2r})), \quad (7.69)$$

where $\chi_\sigma(q, \rho)$ was given in (7.24), and is normalized to approach one as $\rho \equiv e^{-2r} \rightarrow 0$. The terms appearing in the one point functions (7.66) follow from taking the projections onto appropriate dilatation weight:

$$(e^{\gamma\phi_B/(d-1)} \Omega)_{(2\sigma)} \equiv (e^{\mu r} \Omega)_{(2\sigma)} = -2\sigma \tilde{\chi}_{(2\sigma)}(q). \quad (7.70)$$

where we have used the expansions of $\chi_\sigma(q, \rho)$ given in (7.24) and the terms of appropriate dilatation weight, $\tilde{\chi}_{(2\sigma)}(q)$, in these asymptotic expansions, see (7.26) and (7.40).

Using (7.66) one obtains the renormalised one point functions to linear order in the sources:

$$\begin{aligned} \langle \mathcal{O}_\phi(q) \rangle &= L\tilde{\chi}_{(2\sigma)}(q)\nu(d-2\alpha\gamma) \left(-\nu\varphi(q) \left[1 - \frac{2\gamma^2}{(d-1)\nu^2} \right]^2 \right. \\ &\quad \left. + f(q) \left[\frac{\gamma}{\nu(d-1)} - \frac{2\gamma^3}{\nu^3(d-1)^2} \right] \right) \\ \langle T_i^i(q) \rangle &= 2L(d-2\alpha\gamma)\tilde{\chi}_{(2\sigma)}(q) \left(\left[-\frac{\gamma}{\nu} + \frac{2\gamma^3}{\nu^3(d-1)} \right] \nu\varphi(q) + \frac{\gamma^2}{\nu^2(d-1)} f(q) \right), \\ \langle T_j^i(q) \rangle_{TT} &= L(d-2\alpha\gamma)\tilde{\chi}_{(2\sigma)}(q)e_j^i(q), \end{aligned} \tag{7.71}$$

where we have used $W'/W = -\gamma/\nu(d-1)$ and $\kappa^2 = 1/2$. The first two expressions can be rewritten as:

$$\begin{aligned} \langle \mathcal{O}_\phi(q) \rangle &= L\gamma \frac{(d-2\alpha\gamma)}{\alpha(d-1-2\alpha\gamma)} \tilde{\chi}_{(2\sigma)}(q) ((d-1)\phi_{(0)}(q) + \alpha f_{(0)}(q)) \\ \langle T_i^i(q) \rangle &= -2L\gamma \frac{(d-2\alpha\gamma)}{(d-1-2\alpha\gamma)} \tilde{\chi}_{(2\sigma)}(q) ((d-1)\phi_{(0)}(q) + \alpha f_{(0)}(q)), \end{aligned} \tag{7.72}$$

where we have renamed the sources as $\varphi(q) \equiv \phi_{(0)}(q)$ and $f(q) \equiv f_{(0)}(q)$ to demonstrate agreement with the expressions obtained previously in (7.28) and (7.29). The two point functions are given as before by (7.30).

8. Applications

In this section we will present a number of applications of the holographic methods.

8.1 Non-extremal D1 branes

Let us first consider non-extremal D1-branes, and derive the renormalized vevs and onshell action. The ten-dimensional solution for non-extremal D1-branes is:

$$\begin{aligned} ds^2 &= H^{-1/2}(-f dt^2 + dx^2) + H^{1/2} \left(\frac{dr^2}{f} + r^2 d\Omega_7^2 \right); \\ e^\phi &= g_s H^{1/2}; \quad F_{01r} = g_s^{-1} \partial_r \left(1 - \frac{Q}{r^6} H^{-1} \right), \end{aligned} \tag{8.1}$$

with

$$H = 1 + \frac{\mu^6 \sinh^2 \alpha}{r^6}; \quad f = \left(1 - \frac{\mu^6}{r^6} \right); \quad Q \equiv r_o^6 = \mu^6 \sinh \alpha \cosh \alpha. \tag{8.2}$$

The extremal limit is reached by taking $\mu \rightarrow 0$ and $\alpha \rightarrow \infty$ with $\mu^3 \sinh \alpha$ fixed. In the near extremal limit, for which $\mu \ll 1$, the decoupled dual frame metric is

$$ds_{\text{dual}}^2 = (g_s N)^{-1/3} \left(\left(\frac{r}{r_o} \right)^4 (-f dt^2 + dx^2) + r_o^2 \left(\frac{dr^2}{r^2 f} + d\Omega_7^2 \right) \right). \tag{8.3}$$

Applying the reduction formulae (3.2) gives an asymptotically AdS_3 solution of the three-dimensional action:

$$\begin{aligned}
 ds^2 &= \frac{d\rho^2}{4\rho^2 f} + \frac{1}{\rho}(-f dt^2 + dx^2); \\
 e^{-4\phi/3} &= \frac{1}{\rho}, \qquad f = \left(1 - \frac{8\mu^6}{r_o^9} \rho^{3/2}\right).
 \end{aligned}
 \tag{8.4}$$

The inverse Hawking temperature β_H and the area of the horizon A are respectively given by

$$\beta_H = \frac{2\pi r_o^3}{3\mu^2}; \qquad A = \frac{8\pi R_x \mu^4}{r_o^6},
 \tag{8.5}$$

where the x direction is taken to be periodic with period $2\pi R_x$.

Next one can read off the vevs for the stress energy tensor and scalar operator by bringing the metric into Fefferman-Graham form:

$$\begin{aligned}
 ds^2 &= \frac{dz^2}{4z^2} + \frac{1}{z} \left(-dt^2 \left(1 - \frac{16\mu^6}{3r_o^9} z^{3/2}\right) + dx^2 \left(1 + \frac{8\mu^6}{3r_o^9} z^{3/2}\right) \right); \\
 e^{-2\phi/3} &\equiv \frac{1}{\sqrt{z}} e^\kappa = \frac{1}{\sqrt{z}} \left(1 + \frac{4\mu^6}{3r_o^9} z^{3/2}\right).
 \end{aligned}
 \tag{8.6}$$

Then applying (5.65) and (5.68) (analytically continued back to the Lorentzian) the vevs of the stress energy tensor are:

$$\langle T_{tt} \rangle = 16L \frac{\mu^6}{r_o^9}; \qquad \langle T_{yy} \rangle = 8L \frac{\mu^6}{r_o^9}; \qquad \langle \mathcal{O} \rangle = -4L \frac{\mu^6}{r_o^9},
 \tag{8.7}$$

with the conformal Ward identity (5.69) manifestly satisfied. Note that the mass is given by

$$M = \int dx \langle T_{tt} \rangle = LR_x \frac{32\pi\mu^6}{r_o^9}.
 \tag{8.8}$$

The renormalized onshell (Euclidean) action I_E is given by

$$I_E = -L \left[\int_{\rho \geq \epsilon} d^3x \sqrt{g} \Phi(R + C) - \int_{\rho = \epsilon} d^2x \sqrt{h} \Phi(2K - 4 - R[h]) \right].
 \tag{8.9}$$

Evaluating this action on the solution gives

$$I_E = -2\pi\beta_H R_x L \frac{8\mu^6}{r_o^9},
 \tag{8.10}$$

whilst the entropy is

$$S = 4\pi L A = L \frac{32\pi^2 R_x \mu^4}{r_o^6},
 \tag{8.11}$$

and thus the expected relation

$$I_E = \beta_H M - S
 \tag{8.12}$$

is satisfied. Note that $M/T_H S = 2/3$. This result is in agreement with the results found in [47] for the entropy of non-extremal Dp-branes. The entropy can be rewritten as

$$S = \frac{2^4 \pi^{5/2}}{3^3} \frac{N^2}{g_{\text{eff}}(T_H)} (V_1 T_H), \tag{8.13}$$

where $V_1 = 2\pi R_H$ is the spatial volume of the D1 brane and $g_{\text{eff}}^2 = g_2^2 N T_H^{-2}$ is the dimensionless effective coupling (with $g_2^2 = g_s / (2\pi\alpha')$ the dimensionful Yang-Mills coupling constant). This is indeed of the form (4.12) dictated from the generalized conformal structure. The overall N^2 is due to the fact that the bulk computation is a tree-level computation.

8.2 The Witten model of holographic YM_4 theory

As the next application of the formalism let us discuss Witten’s holographic model for four dimensional Yang-Mills theory [15]. In this model one considers D4 branes wrapping a circle of size L_τ with anti-periodic boundary conditions for the fermions, which breaks the supersymmetry. This system at low energies looks like a four-dimensional $SU(N)$ gauge theory, with Yang-Mills coupling $g_4^2 = g_5^2 / L_\tau$. In the limit that $\lambda_4 = g_4^2 N \gg 1$ there is an effective supergravity description given by the D4 brane soliton solution, which (in the string frame) is [15, 48]:

$$\begin{aligned} ds_{\text{st}}^2 &= \left(\frac{r}{r_o}\right)^{3/2} [\eta_{\alpha\beta} dx^\alpha dx^\beta + f(r) d\tau^2] + \left(\frac{r_o}{r}\right)^{3/2} \left(\frac{dr^2}{f(r)} + r^2 d\Omega_4^2\right), \\ e^\phi &= g_s \left(\frac{r}{r_o}\right)^{3/4}, \\ F_4 &= 3g_s^{-1} r_o^3 d\Omega_4, \\ f(r) &= 1 - \frac{r_{\text{KK}}^3}{r^3}, \end{aligned} \tag{8.14}$$

where $d\Omega_4$ is the volume form of the S^4 and r_o was defined below (3.2). Then r_{KK} is the minimum value of the radial coordinate and the circle direction τ must have periodicity $L_\tau = 4\pi r_o^{3/2} / (3r_{\text{KK}}^{1/2})$ to prevent a conical singularity.

By wrapping D8-branes around the S^4 , and along the four flat directions, one can model chiral flavors in the gauge theory [16, 17] and the resulting Witten-Sakai-Sugimoto model has attracted considerable attention as a simple holographic model for a non-supersymmetric four-dimensional gauge theory. The methods developed in this paper immediately allow one to extract holographic data from this background, and to quantify the features of QCD which are well or poorly modeled.

Starting from the ten-dimensional string frame solution, one can move to the dual frame $ds_{\text{dual}}^2 = (N e^\phi)^{-2/3} ds^2$ in which the metric becomes asymptotically $AdS_6 \times S^4$:

$$\begin{aligned} ds_{\text{dual}}^2 &= (N e^\phi)^{-2/3} ds_{\text{st}}^2 = \pi^{2/3} \alpha' \left(4 \left[\frac{d\rho^2}{4\rho^2 f(\rho)} + \frac{\eta_{\alpha\beta} dx^\alpha dx^\beta + f(\rho) d\tau^2}{\rho} \right] + d\Omega_4^2 \right), \\ f(\rho) &\equiv 1 - \frac{\rho^3}{\rho_{\text{KK}}^3} = f(r), \end{aligned} \tag{8.15}$$

with changed variable $\rho = 4r_o^3/r$. Comparing with the reduction given in (3.2), one obtains the following six-dimensional background:

$$ds^2 = \frac{d\rho^2}{4\rho^2 f(\rho)} + \frac{\eta_{\alpha\beta} dx^\alpha dx^\beta + f(\rho) d\tau^2}{\rho}; \quad (8.16)$$

$$e^\phi = \frac{1}{\rho^{3/4}},$$

which is asymptotically AdS_6 with a linear dilaton.

The gauge theory operators dual to the metric and the scalar field are the five-dimensional stress energy tensor T_{ij} and the gluon operator \mathcal{O} respectively, which satisfy the dilatation Ward identity (see (5.83) or (6.34)):

$$\langle T_i^i \rangle + \frac{1}{g_5^2} \langle \mathcal{O} \rangle = 0. \quad (8.17)$$

(There is no anomaly in this case, as both $g_{(0)}$ and $\kappa_{(0)}$ are constant.) This Ward identity can be rewritten in terms of operators in the four-dimensional theory obtained via reduction over the circle: the four-dimensional stress energy tensor $T_{ab}^{(4)} = L_\tau T_{ab}$ and the scalar operator $\mathcal{O}_\tau = L_\tau T_{\tau\tau}$. This gives

$$\langle T_a^a \rangle + \langle \mathcal{O}_\tau \rangle + \frac{1}{g_4^2} \langle \mathcal{O} \rangle = 0. \quad (8.18)$$

Consider the dimensional reduction of the stress energy tensor and gluon operator defined in (4.5) from five to four dimensions. When the reduction over the circle preserves supersymmetry, the operator \mathcal{O}_τ coincides with $-\frac{1}{g_4^2} \mathcal{O}$ and the four-dimensional stress energy tensor is traceless. With non-supersymmetric boundary conditions, this is not the case anymore, since as we will see shortly the vacuum expectation value of the trace of the stress energy tensor is not zero and the vevs of the two operators are different. With the proper identification of the relation between \mathcal{O}_τ and \mathcal{O} , the trace Ward identity would lead to the identification of the beta function.

Next one can extract the one point functions for the stress energy tensor and gluon operators from the coefficients in the asymptotic expansion of this solution near the boundary. To apply the formulae for the holographic vevs, the metric should first be brought into Fefferman-Graham form by changing the radial variable:

$$\tilde{\rho} = \left(1 + \frac{\rho^3}{6\rho_{\text{KK}}^3}\right)\rho + \mathcal{O}(\rho^5), \quad (8.19)$$

$$ds^2 = \frac{d\tilde{\rho}^2}{4\tilde{\rho}^2} + \tilde{\rho}^{-1} \left(1 + \frac{\tilde{\rho}^3}{6\rho_{\text{KK}}^3}\right) \eta_{\alpha\beta} dx^\alpha dx^\beta + \tilde{\rho}^{-1} \left(1 - \frac{5\tilde{\rho}^3}{6\rho_{\text{KK}}^3}\right) d\tau^2 + \dots$$

Using (6.28) the one-point function of the scalar operator is thus:

$$\langle \mathcal{O}_\phi \rangle = -12L\gamma\kappa_{(6)} = -\frac{2L}{3\rho_{\text{KK}}^3}, \quad (8.20)$$

with the vev of the stress energy tensor being:

$$\langle T_{\alpha\beta} \rangle = \frac{L}{\rho_{\text{KK}}^3} \eta_{\alpha\beta}; \quad \langle T_{\tau\tau} \rangle = -5\frac{L}{\rho_{\text{KK}}^3}. \quad (8.21)$$

The gluon condensate can be reexpressed as:

$$\langle \mathcal{O}_\phi \rangle = -\frac{2^6 \pi^2}{3^8} N^2 \frac{\lambda_4}{L_\tau^5}, \tag{8.22}$$

where recall that $\lambda_4 = g_4^2 N$ is the four-dimensional 't Hooft coupling and L_τ is the radius of the circle. In terms of the dimension four operator \mathcal{O} the condensate is

$$\langle \mathcal{O} \rangle = \frac{2^5 \pi^2}{3^7} N \frac{\lambda_4^2}{L_\tau^4}. \tag{8.23}$$

In comparing results for this holographic model with those of QCD, it would be natural to match the condensate values, and thus fix L_τ .

9. Discussion

In this paper we have developed precision holography for the non-conformal branes. We found that all holographic results that were developed earlier in the context of holography for the conformal branes can be extended to this more general setup. All branes under consideration have a near-horizon limit with non-vanishing dilaton and a metric that (in the string frame) is conformal to $AdS_{p+2} \times S^{8-p}$. This implies that there is a frame, the dual frame, where the metric is exactly $AdS_{p+2} \times S^{8-p}$ (one can cancel the overall conformal factor by multiplying the metric with the appropriate power of the dilaton).

There are a number of reasons why this frame is distinguished. Firstly, it is manifest in this frame that there is an effective $(p+1)$ -dimensional gravitational description, obtained by reducing over S^{8-p} , as required by holography. Secondly, the setup becomes the same as that of holographic RG flows studied earlier. Actually the bulk solutions do describe an RG flow, albeit a trivial one driven by the dimension of the coupling constant. Recall that in the holographic RG flows studied in the past the bulk solution asymptotically becomes AdS, corresponding to the fact that the dual QFT approaches a fixed point in the UV. The scalar fields vanish asymptotically, and from the asymptotic fall off one can infer whether the bulk solution corresponds to a deformation of the UV Lagrangian by the addition of the operator dual to the corresponding field or the conformal theory in a non-trivial state characterized (in part) by the vev of the dual operator. The coefficients in the asymptotic expansion of the solution determine the coupling constant multiplying the dual operator in the case of deformations, or the vev of the dual operator in the case of non-trivial states.

The non-conformal branes are analogous to the case of deformations: the asymptotic value of the dilaton determines the value of the coupling constant, which is the (dimensionful) Yang-Mills coupling constant in the case of D_p branes. The main difference is that in the current context the theory does not flow in the UV to a $(p+1)$ -dimensional fixed point. Rather in the regime where the various approximations are valid, the theory runs trivially due to the dimensionality of the coupling constant.

In some cases however we know that a new dimension, the M-theory dimension, opens up at strong coupling and the theory flows to a $(p+2)$ -dimensional fixed point. This is the case for the IIA fundamental string and the D4 brane which uplift to the M2 and M5

brane theories, respectively. Here is another instance that illustrates the preferred status of the dual frame: the general solution in the dual frame

$$ds_d^2 = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} g_{ij} dx^i dx^j \tag{9.1}$$

$$e^{4\phi/3} = \frac{1}{\rho} e^{2\kappa}, \tag{9.2}$$

lifts to

$$ds_{d+1}^2 = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} (g_{ij} dx^i dx^j + e^{2\kappa} dy^2). \tag{9.3}$$

In other words, the dual frame metric in the Fefferman-Graham gauge in d -dimensions is equal to the d -dimensional part of the metric in $(d+1)$ dimensions in the Fefferman-Graham gauge, with the dilaton providing the additional dimension. It was already observed in [14] that the radial coordinate in the dual frame is identified with the energy of the dual theory via the UV-IR connection and here we see a more precise formulation of this statement. The radial direction of the M5 and M2 branes is also the radial direction in the dual frame of the D4 and F1 branes, respectively. In more covariant language, the dilatation operator of the boundary theory is to leading order equal to the radial derivative of the dual frame metric.

Working in the dual frame, we have systematically developed holographic renormalization for all non-conformal branes. In particular, we obtained the general solutions of the field equations with the appropriate Dirichlet boundary conditions. This allowed us to identify the volume divergences of the action, and then remove these divergences with local covariant counterterms. Having defined the renormalized action, we then proceeded to calculate the holographic one-point functions which, by further functional differentiation wrt sources, yield the higher point functions. The counterterm actions can be found in (5.76) and (5.80), whilst the holographic one point functions are given in (5.77) and (5.82). Note that the result for the stress energy tensor properly defines the notion of mass for back-grounds with these asymptotics.

We developed holographic renormalization both in the original formulation, described in the previous paragraph, and in the radial Hamiltonian formalism (in section 6). In the latter, Hamilton-Jacobi theory relates the variation of the on-shell action w.r.t. boundary conditions, thus the holographic 1-point functions, to radial canonical momenta. It follows that one can bypass the onshell action and directly compute renormalized correlators using radial canonical momenta π , as was developed for asymptotically AdS spacetimes in [11, 12]. For explicit calculations, the Hamiltonian method is more efficient and powerful, as it exploits to the full the underlying symmetry structure.

Throughout the existence of an underlying generalized conformal structure plays a crucial role. As we discussed in section 4 SYM in d dimensions admits a generalized conformal structure, in which the action is invariant under Weyl transformations provided that the coupling constant is also promoted to a background field $\Phi_{(0)}$ which transforms appropriately. This background field can be thought of as a source for a gauge invariant operator \mathcal{O} . Then diffeomorphism and Weyl invariance imply Ward identities for the

correlators of the stress energy tensor and the operator \mathcal{O} . This generalized conformal structure is preserved at strong coupling, and governs the holographic Ward identities. In particular, the Dirichlet boundary conditions for the dilaton are determined by the field theory source $\Phi_{(0)}$.

In the cases of the type IIA fundamental string and D4-branes, all the holographic results we find are manifestly compatible with the M theory uplift. In particular, we showed in detail how the asymptotic solutions, counterterms, one point functions and anomalies descend from those of M2 and M5 branes. The generalized conformal structure is also inherited from the higher dimensional conformal symmetry in these cases. This is exactly analogous to the case of the more familiar holographic RG flows, which also have a similar generalized conformal structure inherited from the UV fixed point.

Having set up the formalism in full generality, we then proceeded to discuss a number of examples and applications. In section 7 we calculated two point functions of the stress energy tensor and gluon operator. We computed these two point functions for the supersymmetric backgrounds, and showed that the results were consistent with the underlying generalized conformal structure. In section 7.3 we developed a general method for computing two point functions in any background which asymptotes to the non-conformal brane background.

In section 8 we gave several more applications. One was the explicit evaluation of the mass and action in a non-extremal brane background. The second was Witten's model for a non-supersymmetric four-dimensional gauge theory: we computed the dimension four condensates in this model. One would anticipate that there are many further interesting applications of the formalism developed here, to be explored in future work.

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A. Useful formulae

In this appendix we collect some useful formulae for the asymptotic expansions. Given the expansion of the d -dimensional metric g_{ij} as

$$g_{ij} = g_{(0)ij} + \rho g_{(2)ij} + \rho^2 g_{(4)ij} + \dots \tag{A.1}$$

the inverse metric is given by

$$g^{-1} = g_{(0)}^{-1} - \rho g_{(0)}^{-1} g_{(2)} g_{(0)}^{-1} + \rho^2 (g_{(0)}^{-1} g_{(2)} g_{(0)}^{-1} g_{(2)} g_{(0)}^{-1} - g_{(0)}^{-1} g_{(4)} g_{(0)}^{-1}) + \dots \tag{A.2}$$

Next we compute the expansion of the Christoffel connection,

$$\Gamma_{ij}^i = \Gamma_{(0)ij}^i + \rho \Gamma_{(2)ij}^i + \rho^2 \Gamma_{(4)ij}^i + \dots \tag{A.3}$$

Here $\Gamma_{(0)ij}^i$ is the Christoffel connection of the metric $g_{(0)}$ and

$$\Gamma_{(2)ij}^i = \frac{1}{2}g_{(0)}^{il}(\nabla_j g_{(2)kl} + \nabla_k g_{(2)jl} - \nabla_l g_{(2)jk}) \quad (\text{A.4})$$

$$\Gamma_{(4)ij}^i = \frac{1}{2}\left(g_{(0)}^{il}(\nabla_j g_{(4)kl} + \nabla_k g_{(4)jl} - \nabla_l g_{(4)jk}) - g_{(2)}^{il}(\nabla_j g_{(2)kl} + \nabla_k g_{(2)jl} - \nabla_l g_{(2)jk})\right),$$

where ∇ is the covariant derivative in the metric $g_{(0)}$.

From here we then compute the expansion of the associated curvature

$$R_{ij} = R_{(0)ij} + \rho R_{(2)ij} + \rho^2 R_{(4)ij} + \dots \quad (\text{A.5})$$

with $R_{(0)ij}$ the Ricci tensor of $g_{(0)}$ and

$$R_{(2)ij} = \frac{1}{2}\left(\nabla^k \nabla_j g_{(2)ik} - \nabla^i \nabla_j \text{Tr}(g_{(2)}) + R_{(0)kijl}g_{(2)}^{kl} + R_{(0)im}g_{(2)}^m{}_j - \nabla^2 g_{(2)ij} + \nabla_i \nabla^k g_{(2)jk}\right), \quad (\text{A.6})$$

$$\begin{aligned} R_{(4)ij} = \frac{1}{2}\left(\frac{1}{2}\nabla^l \text{Tr}g_{(2)} \nabla_l g_{(2)ij} + g_{(2)}^{kl}(R_{kimj}g_{(2)}^m{}_l + R_{kiml}g_{(2)}^m{}_j) \right. \\ + g_{(2)}^{kl}(R_{kjni}g_{(2)}^m{}_l + R_{kjmli}g_{(2)}^m{}_j) + g_{(2)}^{kl}\nabla_k \nabla_l g_{(2)ij} + \frac{1}{2}\nabla_j g_{(2)lm} \nabla^l g_{(2)}^m{}_i \\ - \frac{1}{2}g_{(2)}^{kl}(\nabla_i \nabla_k g_{(2)jl} + \nabla_j \nabla_k g_{(2)il}) - 2R_{limj}g_{(4)}^{lm} + R_{im}g_{(4)j}^m + R_{jm}g_{(4)i}^m \\ + \frac{1}{4}g_{(2)j}^l \nabla_i \nabla_l \text{Tr}g_{(2)} + \frac{1}{4}g_{(2)i}^l \nabla_j \nabla_l \text{Tr}(g_{(2)}) + \frac{1}{2}\nabla_i g_{(2)lm} \nabla^l g_{(2)}^m{}_j \\ \left. - \nabla^2 g_{(4)ij} - \frac{1}{2}\nabla_i g_{(2)lm} \nabla_j g_{(2)}^{lm} - \nabla_m g_{(2)il} \nabla^l g_{(2)}^m{}_j - \nabla_m g_{(2)il} \nabla^m g_{(2)j}^l\right). \end{aligned} \quad (\text{A.7})$$

B. The energy momentum tensor in the conformal cases

In this section we streamline the derivation of the vev of the energy-momentum tensor in terms of the asymptotic coefficients for the conformal cases $D = 4$ and $D = 6$ given in [6]. The starting point is the expression of the stress energy tensor as sum of two contributions, one originating from the bulk action and the other from the counterterms, eqn (3.5)-(3.6)-(3.7) of [6]:

$$\langle T_{ab} \rangle = 2L_{D+1} \lim_{\rho \rightarrow 0} \left(\frac{1}{\rho^{D/2-1}} T_{ab}[G] \right), \quad (\text{B.1})$$

$$T_{ab}[G] = T_{ab}^{\text{reg}} + T_{ab}^{\text{ct}},$$

$$T_{ab}^{\text{reg}} = G'_{ab} - G_{ab} \text{Tr}[G^{-1}G'] - \frac{1-D}{\rho} G_{ab},$$

$$\begin{aligned} T_{ab}^{\text{ct}} = & -\frac{D-1}{\rho} G_{ab} + \frac{1}{(D-2)} \left(R(G)_{ab} - \frac{1}{2} R(G) G_{ab} \right) \\ & + \frac{\rho}{(D-4)(D-2)^2} \left[\square R(G)_{ab} + 2R(G)_{acbd} R(G)^{cd} - \frac{D-2}{2(D-1)} D_a D_b R(G) \right. \\ & - \frac{D}{2(D-1)} R(G) R(G)_{ab} - \frac{1}{2} G_{ab} (R(G)_{cd} R(G)^{cd}) - \frac{D}{4(D-1)} R(G)^2 \\ & \left. + \frac{1}{D-1} \square R(G) \right] + \frac{1}{2} T_{ab}^{\text{log}} \log \rho, \end{aligned}$$

where $L_{D+1} = \frac{1}{16\pi G_{D+1}}$ with G_{D+1} the Newton constant and T_{ab}^{log} is the stress energy tensor of the action given by the conformal anomaly.⁷ Note that for $D = 2$ only the first term in T_{ab}^{ct} applies; for $D = 4$ only the first line applies plus the logarithmic terms, whilst for $D = 6$ all terms listed are needed and for $D > 6$ one would need to include additional terms.

For $D = 2$ one immediately obtains the answer

$$\langle T_{ab} \rangle = 2L_{D+1} (G_{(2)ab} - G_{(0)ab} \text{Tr}G_{(2)}) \quad (\text{B.2})$$

For $D > 2$ one can simplify the evaluation of (B.1) by using the equation of motions (5.10) to obtain

$$\begin{aligned} R_{ab} - \frac{1}{2}R G_{ab} = & -(D-2)G'_{ab} + (D-2)\text{Tr}(G^{-1}G')G_{ab} \\ & + \rho \left[2G'' - 2G'G^{-1}G' + \text{Tr}(G^{-1}G')G' \right. \\ & \left. + \left(-\text{Tr}(G^{-1}G'') + \text{Tr}(G^{-1}G')^2 - \frac{1}{2}(\text{Tr}G^{-1}G')^2 \right) G \right]_{ab}. \end{aligned} \quad (\text{B.3})$$

Using this identity in (B.1) we see that T_{ab}^{reg} cancels the first line of T_{ab}^{ct} up to the terms proportional to ρ in (B.3), so $T_{ab}[G]$ is manifestly linear in ρ . It follows that we only need to set $\rho = 0$ in the remaining terms to obtain the vev for $D = 4$:

$$\begin{aligned} \langle T_{ab} \rangle = 2L_{D+1} \left(2G_{(4)ab} - G_{(2)ab}^2 + \frac{1}{2}\text{Tr}G_{(2)}G_{(2)ab} \right. \\ \left. + \frac{1}{4}G_{(0)ab}(\text{Tr}(G^{-1}G_{(2)})^2 - (\text{Tr}G^{-1}G_{(2)})^2) + 3H_{(4)ab} \right). \end{aligned} \quad (\text{B.4})$$

For $D = 6$ one can check straightforwardly that order ρ terms in $T_{ab}[G]$ cancel, so there is indeed a finite limit. To obtain the vev one needs to extract the order ρ^2 terms. To simplify this computation we differentiate the field equations (5.10) to obtain a formula for the radial derivative of the Ricci tensor,

$$\begin{aligned} R'_{ab} = & R_{(a}^c G'_{b)c} - R_{abcd}G'^{cd} + D_{(a}D^b G'_{b)c} - \frac{1}{2}\square G'_{ab} + D_a \partial_b \text{Tr}G' \\ = & \frac{1}{D-2} \left[-R_{abcd}R^{cd} + \frac{D-2}{4(D-1)}D_a D_b R + \frac{1}{2}\square R_{ab} + \frac{1}{4(D-1)}\square R G_{ab} \right. \\ & + R_a^c R_{bc} - \rho \left[4R_{(0)c}^c \tilde{C}_{b)c} - 4R_{(0)abcd}\tilde{C}^{cd} - \frac{D-2}{4(D-1)}D_a \partial_b B \right. \\ & \left. \left. - 2\square \tilde{C}_{ab} - \frac{1}{4(D-1)}G_{(0)ab} \right] \right] + \mathcal{O}(\rho^2), \\ \tilde{C}_{ab} = & \left(G_{(4)} - \frac{1}{2}G_{(2)}^2 + \frac{1}{4}G_{(2)}\text{Tr}G_{(2)} \right)_{ab}, \\ B = & \text{Tr}G_{(2)}^2 - (\text{Tr}G_{(2)})^2. \end{aligned} \quad (\text{B.5})$$

Then we note that the terms involving the Riemann tensor and covariant derivatives enter with the same relative factors as in T_{ab}^{ct} , so we can use (B.5) to express T_{ab}^{ct} in terms of

⁷The factor of 1/2 in front of T_{ab}^{log} corrects a typo in [6].

$R'_{ab} = R_{(2)ab} + 2\rho R_{(4)ab} + \dots$, which is easier to relate to higher expansion coefficients. Indeed, as is discussed in the next appendix, the coefficient $R_{(2)ab}, R_{(4)ab}$ can be expressed in terms of $G_{(2)ab}, G_{(4)ab}$ and $H_{(6)ab}$.

Combining these results and setting $D = 6$ we obtain

$$\langle T_{ab} \rangle = 2L_7 \left(3G_{(6)ab} - 3A_{(6)ab} + \frac{1}{8}S_{ab} + \frac{11}{2}H_{(6)ab} \right), \quad (\text{B.6})$$

where $A_{(6)ab}$ and S_{ab} are given by [6]

$$\begin{aligned} S_{ab} &= \square C_{ab} + 2R_{abcd}C^{cd} + 4(G_{(2)}G_{(4)} - G_{(4)}G_{(2)})_{ab} \\ &\quad + \frac{1}{10}(D_a D_b B - G_{(0)ab}\square B) + \frac{2}{5}G_{(2)ab}B \\ &\quad + G_{(0)ab} \left(-\frac{2}{3}\text{Tr}G_{(2)}^3 - \frac{4}{15}(\text{Tr}G_{(2)})^3 + \frac{3}{5}\text{Tr}G_{(2)}\text{Tr}G_{(2)}^2 \right), \\ A_{(6)ab} &= \frac{1}{3} \left((2G_{(2)}G_{(4)} + G_{(4)}G_{(2)})_{ab} - G_{(2)ab}^3 + \frac{1}{8}[\text{Tr}G_{(2)}^2 - (\text{Tr}G_{(2)})^2]G_{(2)ab} \right. \\ &\quad \left. - \text{Tr}G_{(2)} \left[G_{(4)ab} - \frac{1}{2}G_{(2)ab}^2 \right] - \left[\frac{1}{8}\text{Tr}G_{(2)}^2 \text{Tr}G_{(2)} - \frac{1}{24}(\text{Tr}G_{(2)})^3 \right. \right. \\ &\quad \left. \left. - \frac{1}{6}\text{Tr}G_{(2)}^3 + \frac{1}{2}\text{Tr}(G_{(2)}G_{(4)}) \right] G_{(0)ab} \right), \\ C_{ab} &= \left(G_{(4)} - \frac{1}{2}G_{(2)}^2 + \frac{1}{4}G_{(2)}\text{Tr}G_{(2)} \right)_{ab} + \frac{1}{8}G_{(0)ab}B, \\ B &= \text{Tr}G_{(2)}^2 - (\text{Tr}G_{(2)})^2. \end{aligned} \quad (\text{B.7})$$

Noting that $L_7 = N^3/(3\pi^3)$ and introducing the combination

$$t_{ab} = G_{(6)ab} - A_{(6)ab} + \frac{1}{24}S_{ab} \quad (\text{B.8})$$

the stress energy tensor may be expressed as

$$\langle T_{ab} \rangle = \frac{N^3}{3\pi^3} (6t_{ab} + 11H_{(6)ab}). \quad (\text{B.9})$$

This result includes the term in $H_{(6)ab}$ which was not given in [6].

C. Reduction of M5 to D4

The expansion coefficients for an asymptotically local AdS_{D+1} metric were given in [6]. We will be interested in the case where $D = d + 1$, for which the first expansion coefficients are:

$$\begin{aligned} G_{(2)ab} &= \frac{1}{d-1} \left(-R_{(0)ab} + \frac{1}{2d}R_{(0)}G_{(0)ab} \right); \\ G_{(4)ab} &= \frac{1}{2(d-3)} \left(-R_{(2)ab} - 2(G_{(2)}^2)_{ab} + \frac{1}{2}\text{Tr}(G_{(2)}^2)G_{(0)ab} \right). \end{aligned} \quad (\text{C.1})$$

Using the explicit form of $G_{(2)ab}$ and the D -dimensional analogue of (A.6) we obtain:

$$\begin{aligned}
 R_{(2)ab} &= -\frac{1}{2(d-1)} \left(2R_{(0)ac}R_{(0)b}^c - 2R_{(0)cadb}R_{(0)}^{cd} - \frac{d-1}{2d}D_aD_bR_{(0)} \right. \\
 &\quad \left. + D^2R_{(0)ab} - \frac{1}{2d}D^2R_{(0)}G_{(0)ab} \right), \\
 G_{(4)ab} &= -\frac{1}{d-3} \left(-\frac{1}{8d}D_aD_bR + \frac{1}{4(d-1)}D_cD^cR_{ab} \right. \\
 &\quad - \frac{1}{8d(d-1)}D_cD^cRG_{(0)ab} + \frac{1}{2(d-1)}R^{cd}R_{acbd} \\
 &\quad - \frac{d-3}{2(d-1)^2}R_a^cR_{cb} - \frac{1}{d(d-1)^2}RR_{ab} \\
 &\quad \left. - \frac{1}{4(d-1)^2}R^{cd}R_{cd}G_{(0)ab} + 3\frac{(d+1)}{16d^2(d-1)^2}R^2G_{(0)ab} \right),
 \end{aligned} \tag{C.2}$$

where D_a is the covariant derivative in the metric $G_{(0)}$. Note that $R_{(2)} = 0$, and thus

$$\text{Tr}G_{(4)} = \frac{1}{4}\text{Tr}(G_{(2)}^2). \tag{C.3}$$

At next order one finds that the trace and the divergence of $G_{(6)}$ are determined via

$$\begin{aligned}
 \text{Tr}(G_{(6)}) &= \frac{2}{3}\text{Tr}(G_{(2)}G_{(4)}) - \frac{1}{6}\text{Tr}(G_{(2)}^3); \\
 D^aG_{(6)ab} &= D^aA_{(6)ab} + \frac{1}{6}\text{Tr}(G_{(4)}D_bG_{(2)}); \\
 A_{(6)ab} &= \frac{1}{3} \left((2G_{(2)}G_{(4)} + G_{(4)}G_{(2)})_{ab} - (G_{(2)}^3)_{ab} + \frac{1}{8}[\text{Tr}G_{(2)}^2 - (\text{Tr}G_{(2)})^2]G_{(2)ab} \right. \\
 &\quad - \text{Tr}G_{(2)} \left[G_{(4)ab} - \frac{1}{2}(G_{(2)}^2)_{ab} \right] - \left[\frac{1}{8}\text{Tr}G_{(2)}^2 \text{Tr}G_{(2)} - \frac{1}{24}(\text{Tr}G_{(2)})^3 \right. \\
 &\quad \left. \left. - \frac{1}{6}\text{Tr}G_{(2)}^3 + \frac{1}{2}\text{Tr}(G_{(2)}G_{(4)}) \right] G_{(0)ab} \right).
 \end{aligned} \tag{C.5}$$

The logarithmic term in the expansion $H_{(6)}$ is given by

$$\begin{aligned}
 H_{(6)ab} &= \frac{1}{6}(R_{(4)ab} + \left(-\text{Tr}(G_{(2)}G_{(4)}) + \frac{1}{2}\text{Tr}(G_{(2)}^3) \right) G_{(0)ab}) \\
 &\quad - \frac{1}{6}\text{Tr}(G_{(2)}G_{(4)ab}) - \frac{1}{3}(G_{(2)}^3)_{ab} + \frac{2}{3}(G_{(2)}G_{(4)} + G_{(4)}G_{(2)})_{ab}.
 \end{aligned} \tag{C.6}$$

Note that $H_{(6)}$ is traceless and divergence free.

For the dimensional reduction it is useful to note that the non-vanishing components of the Riemann tensor can be expressed as

$$\begin{aligned}
 R(G)_{ijkl} &= R_{ijkl}; \\
 R(G)_{yiyj} &= -e^{2\kappa}(\nabla_i\partial_j\kappa + (\partial_i\kappa)(\partial_j\kappa)),
 \end{aligned} \tag{C.7}$$

and similarly the non-vanishing components of the Ricci tensor are

$$\begin{aligned}
 R(G)_{ij} &= R_{ij} - \nabla_i\partial_j\kappa - \partial_i\kappa\partial_j\kappa; \\
 R(G)_{yy} &= e^{2\kappa}(-\nabla^i\partial_i\kappa - \partial_i\kappa\partial^i\kappa).
 \end{aligned} \tag{C.8}$$

Let furthermore S be a scalar and C_{ab} a symmetric tensor with $C_{iy} = 0$. Then the Laplacian reduces as

$$\begin{aligned} D^2 S &= (\nabla^2 + \partial^i \kappa \partial_i) S, \\ D^2 C_{ij} &= (\nabla^2 + \partial^l \kappa \nabla_l) C_{ij} - 2\partial_l \kappa \partial_{(i} \kappa C_{j)}^l + 2\partial_i \kappa \partial_j \kappa C_y^y, \\ D^2 C_y^y &= (\nabla^2 + \partial^i \kappa \partial_i) C_y^y + 2\partial_i \kappa \partial_j \kappa C^{ij} - 2\partial_i \kappa \partial^i \kappa C_y^y. \end{aligned} \tag{C.9}$$

Letting $G_{(0)ij} = g_{(0)ij}$ and $G_{(0)yy} = e^{2\kappa(0)}$ one finds that

$$\begin{aligned} R(G)_{(0)ij} &= R_{(0)ij} - \nabla_i \partial_j \kappa_{(0)} - \partial_i \kappa_{(0)} \partial_j \kappa_{(0)}; \\ R(G)_{(0)yy} &= e^{2\kappa(0)} (-\nabla^i \partial_i \kappa_{(0)} - \partial_i \kappa_{(0)} \partial^i \kappa_{(0)}), \end{aligned} \tag{C.10}$$

with $R(G)_{(0)yi} = 0$. Substituting into (5.50) gives⁸

$$\begin{aligned} G_{(2)ij} &= \frac{1}{d-1} \left(-R_{(0)ij} + \frac{1}{2d} R_{(0)} g_{(0)ij} + (\nabla_{\{i} \partial_{j\}} \kappa)_{(0)} + \partial_{\{i} \kappa_{(0)} \partial_{j\}} \kappa_{(0)} \right); \\ G_{(2)yy} &= e^{2\kappa(0)} \left(\frac{1}{2d(d-1)} R_{(0)} + \frac{1}{d} (\nabla^2 \kappa_{(0)} + (\partial \kappa_{(0)})^2) \right), \end{aligned} \tag{C.11}$$

with $G_{(2)yi} = 0$. Now using

$$G_{yy} = e^{2\kappa} = e^{(2\kappa(0) + 2\rho\kappa(2) + \dots)} = e^{2\kappa(0)} (1 + 2\rho\kappa(2) + \dots) \tag{C.12}$$

one determines $\kappa(2)$ to be exactly the expression given in (5.26).

One next shows that $G_{(4)ab}$ in (C.1) reduces as

$$G_{(4)ij} = g_{(4)ij}; \quad G_{(4)yy} = e^{2\kappa(0)} (2\kappa_{(2)}^2 + 2\kappa_{(4)}), \tag{C.13}$$

with $g_{(4)ij}$ and $\kappa_{(4)}$ given in (5.30). This follows from the expansion of the six-dimensional curvatures at second order:

$$\begin{aligned} R(G)_{(2)ij} &= R_{(2)ij} - (\nabla_i \partial_j \kappa)_{(2)} - (\partial_i \kappa \partial_j \kappa)_{(2)}; \\ R(G)_{(2)yy} &= -e^{2\kappa(0)} (\nabla^i \partial_i \kappa + \partial_i \kappa \partial^i \kappa)_{(2)} - e^{2\kappa(0)} 2\kappa_{(2)} (\nabla^i \partial_i \kappa + \partial_i \kappa \partial^i \kappa)_{(0)}. \end{aligned} \tag{C.14}$$

Reducing (C.4) gives

$$\begin{aligned} \text{Tr}(G_{(6)}) &= \text{Tr}(g_{(6)}) + 2\kappa_{(6)} + \frac{4}{3}\kappa_{(2)}^3 + 4\kappa_{(2)}\kappa_{(4)}; \\ &= \frac{2}{3}\text{Tr}(g_{(2)}g_{(4)}) + \frac{4}{3}\kappa_{(2)}(\kappa_{(2)}^2 + 2\kappa_{(4)}) - \frac{1}{6}\text{Tr}(g_{(2)}^3), \end{aligned} \tag{C.15}$$

and thus gives

$$\lambda_6 = \text{Tr}(g_{(6)}) + 2\kappa_{(6)} = \frac{2}{3}\text{Tr}(g_{(2)}g_{(4)}) - \frac{4}{3}\kappa_{(2)}\kappa_{(4)} - \frac{1}{6}\text{Tr}(g_{(2)}^3). \tag{C.16}$$

The reduction of (C.6) gives

$$H_{(6)ij} = h_{(6)ij}; \quad H_{(6)yy} = e^{2\kappa_0} 2\tilde{\kappa}_{(6)}, \tag{C.17}$$

⁸Round brackets (ij) denote symmetrisation and curly brackets $\{ij\}$ traceless symmetrisation of indices.

with

$$\begin{aligned}
h_{(6)ij} &= -\frac{1}{12} \left[-2R_{(4)ij} + (-\text{Tr}g_{(2)}^3 + 2\text{Tr}g_{(2)}g_{(4)} + 8\kappa_2\kappa_4)g_{(0)ij} + 2(\text{Tr}g_{(2)})g_{(4)ij} \right. \\
&\quad \left. -8(g_{(4)}g_{(2)})_{ij} - 8(g_{(2)}g_{(4)})_{ij} + 4g_{(2)ij}^3 + 2(\nabla_i\partial_j\kappa)_{(4)} + 2(\partial_i\kappa\partial_j\kappa)_{(4)} + 4\kappa_{(2)}g_{(4)ij} \right] \\
\tilde{\kappa}_{(6)} &= -\frac{1}{12} \left[(\nabla^2\kappa)_{(4)} + (\partial\kappa)_{(4)}^2 + \text{Tr}g_{(2)}g_{(4)} - \frac{1}{2}\text{Tr}g_{(2)}^3 \right. \\
&\quad \left. -\kappa_{(2)}\text{Tr}g_{(2)}^2 + 4\kappa_{(4)}\text{Tr}g_{(2)} - 4\kappa_{(2)}^3 + 12\kappa_{(2)}\kappa_{(4)} \right], \tag{C.18}
\end{aligned}$$

which agree with the expressions (5.34). In reducing the curvature term $R(G)_{(4)yy}$ one should use the identities:

$$\begin{aligned}
-((\nabla^2\kappa) + (\partial\kappa)^2)_{(0)} &= -10\kappa_{(2)} - \text{Tr}g_{(2)}; \tag{C.19} \\
-((\nabla^2\kappa) + (\partial\kappa)^2)_{(2)} &= -8\kappa_{(4)} + 6\kappa_{(2)}^2 + 2\kappa_{(2)}\text{Tr}g_{(2)} + \frac{1}{2}\text{Tr}g_{(2)}^2.
\end{aligned}$$

D. Explicit expressions for momentum coefficients

In the following we give explicit expressions for the terms in the expansions of the momenta in eigenfunctions of the dilatation operator. The expressions given below are applicable for $\beta = 0$ in (3.4) and $d \geq 3$, although in this paper we will use only the case of $d = 5$ (the D4-branes). Here we give $K_{(2n)ij}$ and $p^\phi_{(2n)}$ up to $n = 2$; note that $\hat{\Phi} = e^{\gamma\phi}$. These expressions are needed to compute the anomaly and one point functions for the D4-brane in the Hamiltonian formalism in section 6.4.2:⁹

$$\begin{aligned}
\gamma p^\phi_{(2)} &= -\frac{1}{d} \left[\frac{1}{2(d-1)} R + \hat{\Phi}^{-1} \hat{\nabla}^2 \hat{\Phi} \right], \\
K_{(2)} &= \frac{1}{2(d-1)} R, \\
K_{(2)ij} &= \frac{1}{d-1} \left[R_{ij} - \frac{1}{2d} R h_{ij} - \hat{\Phi}^{-1} \hat{\nabla}_{\{i} \partial_{j\}} \hat{\Phi} \right]; \tag{D.1} \\
\gamma p^\phi_{(4)} &= \frac{1}{2d(d-1)^2(d-3)} \left[-3R_{ij}R^{ij} + \frac{3(d+1)}{4d} R^2 - \frac{3}{d} \hat{\nabla}^2 R - 3(\hat{\Phi}^{-1} \hat{\nabla}_{\{i} \partial_{j\}} \hat{\Phi})^2 \right. \\
&\quad \left. -2(d-3)(\hat{\Phi}^{-1} \hat{\nabla}_i (R^{ij} \partial_j \hat{\Phi}) - \frac{d+2}{2d} \hat{\Phi}^{-1} \hat{\nabla}^j (R \partial_j \hat{\Phi}) + \frac{1}{2d} \hat{\Phi}^{-1} \hat{\nabla}^2 (\hat{\Phi} R)) \right. \\
&\quad \left. -2d(\hat{\Phi}^{-1} \hat{\nabla}^i \hat{\nabla}^j \hat{\nabla}_{\{i} \hat{\nabla}_{j\}} \hat{\Phi} - 2\hat{\Phi}^{-1} \nabla^i (\hat{\Phi}^{-1} \partial^j \hat{\Phi} \hat{\nabla}_{\{i} \hat{\nabla}_{j\}} \hat{\Phi})) \right], \\
K_{(4)} &= -\frac{1}{2(d-3)(d-1)^2} \left[-R_{ij}R^{ij} + \frac{d+1}{4d} R^2 - \frac{1}{d} \hat{\nabla}^2 R - (\hat{\Phi}^{-1} \hat{\nabla}_{\{i} \partial_{j\}} \hat{\Phi})^2 \right. \\
&\quad \left. -2\hat{\Phi}^{-1} \hat{\nabla}^i \hat{\nabla}^j \hat{\nabla}_{\{i} \hat{\nabla}_{j\}} \hat{\Phi} + 4\hat{\Phi}^{-1} \nabla^i (\hat{\Phi}^{-1} \partial^j \hat{\Phi} \hat{\nabla}_{\{i} \hat{\nabla}_{j\}} \hat{\Phi}) \right],
\end{aligned}$$

⁹Round brackets (ij) denote symmetrisation and curly brackets $\{ij\}$ traceless symmetrisation of indices.

$$\begin{aligned}
 K_{(4)}^{ij} &= \gamma p^{\phi}_{(4)} h^{ij} - \frac{1}{(d-1)^2(d-3)} \left[-2R^{ik}R^j{}_k + \frac{d+1}{2d}RR^{ij} - 2\hat{\Phi}^{-2}\hat{\nabla}^i\partial_k\hat{\Phi}\hat{\nabla}^j\partial^k\hat{\Phi} \right. \\
 &\quad \left. - \frac{1}{d}(\hat{\nabla}^i\partial^j R + \hat{\nabla}^2R^{ij}) + \hat{\Phi}^{-1}\hat{\nabla}_l X^{ijl} \right], \\
 X^{ijl} &= -2\hat{\nabla}_k(\hat{\Phi}R^{kl})h^{ij} + 2\hat{\nabla}^{(i}(\hat{\Phi}R^{j)l}) - \hat{\nabla}^l(\hat{\Phi}R^{ij}) \\
 &\quad + \frac{d+1}{2d}[\hat{\nabla}^l(\hat{\Phi}R)h^{ij} - h^{l(i}\hat{\nabla}^{j)}(\hat{\Phi}R)] + 2\hat{\Phi}^{-1}\hat{\nabla}^l\partial^{(i}\hat{\Phi}\partial^{j)}\hat{\Phi} - \hat{\Phi}^{-1}\hat{\nabla}^i\partial^j\hat{\Phi}\partial^l\hat{\Phi} \\
 &\quad - \frac{2}{d}\hat{\Phi}^{-1}h^{l(i}\hat{\nabla}^2\hat{\Phi}\partial^{j)}\hat{\Phi} + \frac{1}{d}\left[h^{l(i}\hat{\Phi}\partial^{j)}R + \frac{d-1}{2}\hat{\Phi}\partial^l R h^{ij} - \hat{\nabla}^l(\hat{\Phi}R^{ij}) \right. \\
 &\quad \left. + 2\hat{\Phi}\hat{\nabla}^l R^{ij} - d\hat{\nabla}^l\hat{\nabla}^2\hat{\Phi}h^{ij} + h^{l(i}\hat{\nabla}^{j)}\hat{\nabla}^2\hat{\Phi} \right].
 \end{aligned}$$

Note that the terms $K_{(2)}$ and $K_{(4)}$ correspond to the (non-logarithmic) counterterms in the action.

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