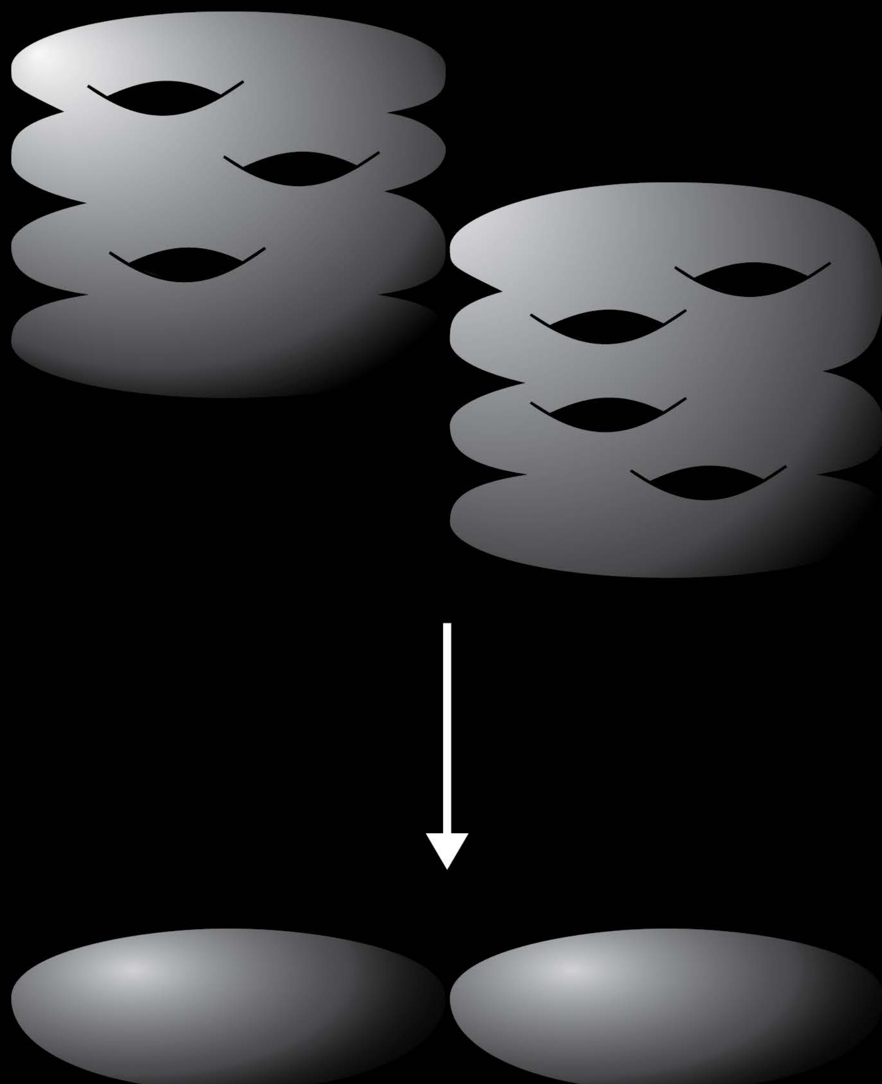
Hurwitz Numbers, moduli of curves, topological recursion, Givental's theory and their relations



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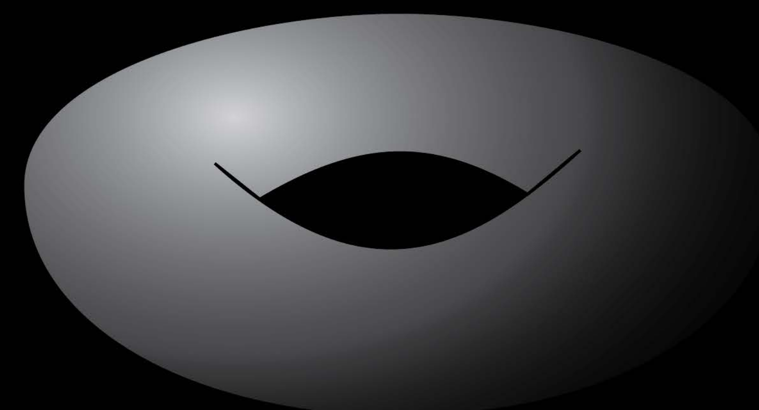

Loek Spitz

Hurwitz numbers, moduli of curves, topological recursion, Givental's theory and their relations.

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Promotor: prof. dr. S. V. Shadrin

Overige leden: prof. dr. A. Chiodo
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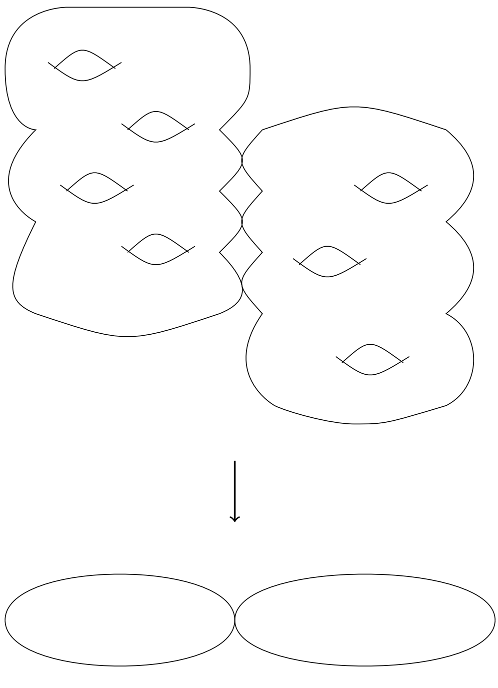

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The past four years have been an amazing experience for me. On the one hand, working on my thesis has been very interesting, and I do not think I have enjoyed myself with mathematics as much as I have during these years. On the other, during this work, I have had the pleasure of interacting with many great people, professionally as well as personally.

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I really think Sergey turned out to be the best possible advisor I could have had, always making time to answer questions and discuss problems, but mainly encouraging me to make as much out of those four years as I could. So, Sergey, thank you; I learned a lot from you, both mathematically and otherwise.

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Bedankt allemaal!
Loek.

Contents

–1– Introduction

In my thesis, I investigate four different concepts that play an important role in mathematical physics, and in particular I discuss some of the relations between these subjects. They are *Hurwitz numbers*, *moduli spaces of curves*, *Chekhov-Eynard-Orantin recursion theory* and *Givental's theory*.

The work presented here is based on the papers [112, 113, 114, 115, 116, 117]. Chapters 2 through 8 are basically a collection of these papers, though reorganized in a way I hope presents the material in a clearer way. The first chapter contains an introduction to provide an overview of the different topics and their relations.

In the section below, intended for non-experts, or even non-mathematicians, I try to give some intuitive idea of what all these things are, leaving out almost all formulas. In the rest of the introduction I then give more precise definitions and explain some of the basic properties, and I state and try to explain the results proved in the main part of the thesis.

1.1 Introduction for non-experts

One of the most basic objects occurring in this thesis is the concept of a *curve*. Curves play an important role in all four of the main subjects, so let me first explain what the word "curve" means for me.

The most important thing about curves is that they are one-dimensional and that they are allowed to be curved; that is, they are not necessarily straight, or as mathematicians would say, linear. Thus, one might think of a curve as the object represented by the picture in Figure 1.1.

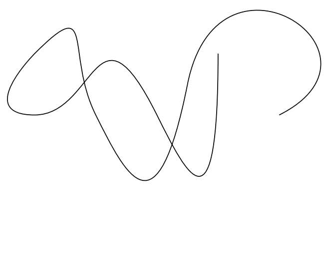


Figure 1.1: Cartoon of a curve.

However, Figure 1.1 is not entirely accurate, because in this case we want to work with *complex curves*. Since the complex numbers are two-dimensional over the real numbers, more accurate pictures are given in Figure 1.2. Mathematicians preferring to work over the real numbers would refer to these as *surfaces*.

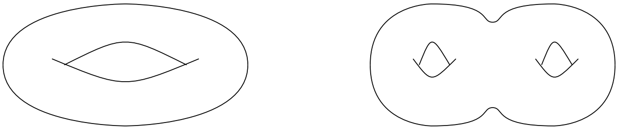


Figure 1.2: A curve of genus 1 and a curve of genus 2.

Looking at the picture in Figure 1.2, one could ask when two of those pictures actually represent the same curve. In different branches of mathematics, different answer would be given to this question. For instance in topology, a curve only depends on its number of 'holes', which is called the *genus*; for each genus, topologically there is precisely one curve of that genus.

Here, we consider curves with some additional structure, called a *complex structure*. Basically, that means that our curves locally resemble the plane of complex numbers. Then, we consider two curves to be the same if they have the same (or, mathematically; *isomorphic*) complex structures.

Once we have defined what we mean when we say that two curves are equivalent, we can:

- Count all the curves with certain properties. Of course, this will only be meaningful if we choose those properties in such a way that the number of curves having those properties is finite.

- Study the set of all curves with some given properties. Examples are moduli spaces of curves, and classes on those moduli spaces.

Now that we know something about curves, it is time to introduce the main subjects of this thesis.

Hurwitz numbers. Hurwitz numbers are an example of the first point above. They count how many *coverings* there are of the sphere by curves of a given genus, and with some specified *ramification profile*. Basically, that means that we look at pictures as in Figure 1.3; there is a curve C of genus g drawn above a sphere S (curve of genus 0). One should imagine that above every point in the sphere, there are some finite number d of points of C, and this number d is the same for all of them (this is what it means to be a covering), except for some special points where it can be smaller (this is specified by the ramification profile).

Moduli spaces of curves. The *moduli spaces of curves* that we are mostly interested in are those that parametrize curves of some specific genus g, where additionally, we have marked some specified number n of their points. When studying such moduli spaces, one of the most common things to do is to try to compute the *intersection* of some classes that are naturally defined on them. That is, we look at the set of all curves in the moduli space that have a certain natural property. In fact, we do this for all kinds of different properties. Then, we are interested in the intersection of those classes; that is, given two or more such properties, we try to find all the curves that have all of those properties, and we try to describe this new class once again by specifying some natural property that is obeyed precisely by the curves in this class. Sometimes, depending on the properties we started with, the number of curves in the intersection will be finite; in that case we are back to counting curves with specific properties, and the number is called an *intersection number*.

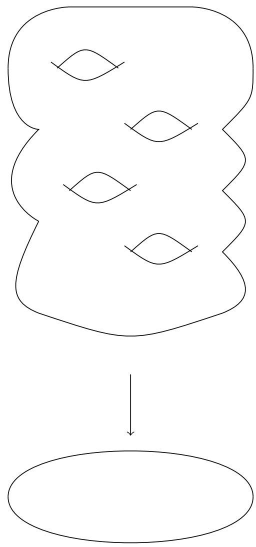


Figure 1.3: A Hurwitz cover.

Chekhov-Eynard-Orantin recursion theory. Whenever we are given a sequence of numbers indexed by some number g, as is the case with the Hurwitz numbers and some of the intersection numbers described above (here, g is always the genus of the curves in question), we could ask whether there is some way to compute the numbers corresponding to higher g directly from the ones corresponding to lower g. A formula describing this type of behaviour is called a *recursion formula*. In fact, in both cases described above there is also another index n, which is the number of points above some special point in S in the case of Hurwitz numbers, and the number of marked points in the case of intersection numbers, and it turns out that Hurwitz numbers as well as some types of intersection numbers obey a recursion of a specific type involving both g and n, called the *Chekhov-Eynard-Orantin recursion*.

One of the most important features of this theory is that the whole formula is encoded by a very simple set of data; basically, by just a curve. Thus, it allows to represent all the data that is encoded in for instance the infinite set of Hurwitz numbers by just giving the equation for one curve. Furthermore, general facts about the CEO recursion can then be used to relate such a set of numbers to other branches of mathematics, such as matrix models and cohomological field theories.

Givental's theory. To say something about Givental's theory, we have to introduce cohomological field theories, which could be described in many different ways. For now, let us think about them as a way to encompass the structure of *Gromov-Witten* theories, which means that they provide some formal way to study curves on a given space. That is, given some space, or *manifold* X (which is like a curve, but may be higher dimensional), we can study all the curves that could be 'drawn' on them. Note that 'drawn' is in parentheses because of the higher dimensional nature of these spaces. However, we can define a mathematical equivalent of drawing that still works in the higher dimensional case (using maps). Once again, we can count the number of curves with specific properties that can be 'drawn' on X, or we can intersect two or more classes of curves on X, and try to understand what class we end up with.

Computation like these are what constitute the Gromov-Witten theory of X, and cohomological field theories are then a way to formalise the properties of such Gromov-Witten theories.

Finally, Giventals theory is a powerful method to compute many things in cohomological field theories by relating them to the trivial cohomological field theory (which in fact is just the moduli space of curves) using operators which have been quantized in a certain way. The precise details of this construction are too involved for this part of the introduction, they are introduced more fully in Section 1.4.4 of this introduction, as well as in Chapter 5.

This concludes the informal introduction to the subjects of my thesis; in the next four sections I give their precise definitions and describe their basic properties.

1.2 Hurwitz numbers

In this section I introduce (various types of) Hurwitz numbers from two different points of view. The interplay between these viewpoints will be used later on, and is one of the main reasons for being interested in Hurwitz nubmers.

1.2.1 Hurwitz numbers

Hurwitz numbers enumerate branched covers of $\mathbb{P}^1$ by smooth curves of some specified genus and with specified ramification profile over some chosen points. I start by stating the most general definition, and then specialize it to the different cases that we encounter in this thesis.

Definition 1.1. Choose some number m, and let $p_1, \cdots, p_m$ be m chosen pair-wise different points on $\mathbb{P}^1$. Given some partitions $\mu^{(1)}, \cdots, \mu^{(m)}$ of a positive integer d and a non-negative integer g, the Hurwitz number $h_{g;\mu^{(1)},\cdots,\mu^{(m)}}$ is defined to be the weighted count of isomorphism classes of degree d covers $f\colon C \to \mathbb{P}^1$, where C is a (possibly disconnected) smooth algebraic curve of genus g, such that the ramification profile of f over p_i is given by $\mu^{(i)}$, and f is unramified on $\mathbb{P}^1 \backslash \{p_1, \cdots, p_m\}$. Two covers $f_1\colon C_1 \to \mathbb{P}^1$ and $f_2\colon C_2 \to \mathbb{P}^1$ are said to be isomorphic if there is an isomorphisms of curves $i\colon C_1 \to C_2$ such that $f_2 \circ i = f_1$, and covers are weighted by the reciprocal of the size of their automorphism group. Often, the inverse images of some of the p_i are considered to be labelled, and isomorphisms are required to respect the labelling. For each such branch point p_i, this has the effect of multiplying the Hurwitz number by $|\mathrm{Aut}(\mu^{(i)})|$.

Note that the Hurwitz number thus defined is finite, and does not depend on the chosen points $p_1, \dots, p_m$, which justifies not including them in the notation.

Here, we do not study Hurwitz numbers in this general form; instead we limit the allowed ramification profiles in different ways. For instance, there are *double Hurwitz numbers* $h_{g;\mu,\nu}$, where the only arbitrary ramification (given by μ and ν) allowed is over two points (which we take to be 0 and ∞ for definiteness), and all other ramification is simple, which means that it has profile of the form $(2, 1, 1, \dots, 1)$. For *single Hurwitz numbers* $h_{g;\mu}$, the only arbitrary ramification allowed is over infinity (it is given by the partition μ), there is no ramification over 0 and all other ramification is again simple.

For us, the most important types of Hurwitz numbers are variants of those called Hurwitz numbers *with completed* $(r+1)$*-cycles* and denoted $h^{(r)}_{g;\mu}$ and $h^{(r)}_{g;\mu,\nu}$. They are just the ordinary single and double Hurwitz numbers, but where any ramification over the points away from 0 and ∞ is not simple, but instead given by the *completed* $(r+1)$*-cycle*. That is, over those points, either $r+1$ sheets come together, corresponding to the partition $(r+1, 1, 1, \dots, 1)$, or some completion effects take place which are more involved to define; their definition is given in Chapter 2.

The branch points where the ramification profile is specified by the partitions that appear in the notation are called *special* branch points (in the case of double Hurwitz numbers they are $0, \infty$; for single Hurwitz numbers it is ∞), whereas the other branch points are called *non-special*. The inverse images of the special branch points are considered to be labelled, the others are not.

It is important to note that in all four cases defined above, the number of non-special branch points is completely determined by the Riemann-Hurwitz formula; for ordinary double Hurwitz numbers it reads:

$$m = 2g - 2 + \ell(\mu) + \ell(\nu) \tag{1.1}$$

while for double Hurwitz numbers with completed $(r+1)$-cycles we have

$$m = \frac{2g - 2 + \ell(\mu) + \ell(\nu)}{r}. \tag{1.2}$$

Here $\ell(\mu)$ is the length of the partition μ (the number of entries). To get the formulas for single Hurwitz numbers, just insert the trivial partition $(1, 1, \ldots, 1)$ for ν, so that $\ell(\nu) = |\mu|$, where $|\mu|$ is the size of μ (the sum of all the entries). These formulas allow us to specify the number of non-special branch points instead of the genus; the notation then becomes $h^{(r),m}_{\mu,\nu}$ (and analogously for the other types).

1.2.2 Combinatorial definition

One of the main reasons that Hurwitz numbers are so interesting, is that they automatically combine geometry and combinatorics. As we have seen in the previous section, the definition of Hurwitz numbers, in terms of covers of curves, is geometric. Here, I will explain why this definition actually leads to combinatorial objects, by giving an equivalent definition in terms of factorizations in the symmetric groups.

For simplicity, we study the single Hurwitz number h^m_μ. We label the points of non-special ramification $p_1, \ldots, p_m$. Choose a base-point p away from $0, \infty, p_1, \ldots, p_m$. Since there is no ramification at p, there are $d = |\mu|$ points in the fibre above p, which we number $1, \ldots, d$ in some way. Now, moving along a simple loop γ_i based at p around one of the points p_i interchanges two of the points of the fibre, which can be described by a transposition σ_i in the symmetric group S_d. On the other hand, moving along a simple loop γ based at p around ∞ interchanges the points in the fibre above p in some way allowed by the ramification profile μ, which means that it can be described by an element σ of S_d of cycle-type μ. If we move around all the loops γ_i and γ consecutively, then the points of the fibre are interchanged according to the product $\sigma_1 \cdots \sigma_m \cdot \sigma$. However, since we are on the sphere, the concatenation of all those loops is equivalent to the trivial loop, so the interchange of the points in the fibre must also be trivial. Thus, any Hurwitz cover induces a factorization in the symmetric group

$$\sigma^{-1} = \sigma_1 \cdots \sigma_m. \tag{1.3}$$

In fact, it turns out that such factorizations are in one-to-one correspondence with Hurwitz covers (one can reconstruct the cover from the data of the factorization, as is shown in Chapter 2 for Hurwitz numbers with completed cycles). Therefore, the Hurwitz number h^m_μ is just the number of factorizations of elements of $S_{|\mu|}$ of cycle-type μ into transpositions.

Computing such numbers of factorizations is a purely combinatorial exercise that could be done on a computer for any individual Hurwitz number. Note that it is not obvious from the original definition of Hurwitz numbers that such a combinatorial way of computing these numbers should exist. In fact, in Chapter 2, I describe an algorithm to compute Hurwitz numbers using their connection to symmetric groups, which will also allow us to deduce more general properties of Hurwitz numbers, such as polynomial behaviour.

1.3 Moduli space of curves

In this section, I introduce the moduli space of curves, some standard classes on this moduli space, and a remarkable relation to Hurwitz numbers that has been used as a powerful tool in the proof of some famous theorems.

1.3.1 The moduli space of curves

A *nodal curve* is a compact connected algebraic curve with finitely many singularities, all of which are nodes. A node is a singularity locally isomorphic to $\{xy = 0\}$ in $\mathbb{C}^2$. Such a curve is called *stable* if it has finitely many automorphisms. For $g \geq 0$ and $n > 0$, we denote by $\overline{\mathcal{M}}_{g,n}$ the moduli space parametrizing stable nodal curves of genus g with n marked points, labelled by $1, \dots, n$. It turns out that a nodal curve is stable if and only if every irreducible component of genus 0 has at least three special points (nodes and marked points). Thus, $\overline{\mathcal{M}}_{g,n}$ is non-empty as long as $(g, n) \neq (0, 1),\ (0, 2)$. It naturally has the structure of a compact Deligne-Mumford stack of dimension $3g - 3 + n$. As an analytic space, it is an orbifold. An excellent introduction to the moduli space of curves is given in [104].

We will also need some natural maps between moduli space of curves that correspond to forgetting marked points and to gluing curves together at marked points. The *forgetful map* $\pi\colon \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the map that forgets the last marked point. The *gluing maps* $\mathrm{gl}\colon \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$ and $\mathrm{gl}\colon \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$ glue two curves together at their last marked points, or glue the last two marked points of a curve together, increasing the genus. Remark that we abuse notation and denote both maps by gl, and also that we do not include g and n in the notation.

1.3.2 Natural classes on $\overline{\mathcal{M}}_{g,n}$.

We introduce some natural classes on $\overline{\mathcal{M}}_{g,n}$. For the *boundary classes*, we associate to any curve in $\overline{\mathcal{M}}_{g,n}$ a graph in the following way;

1. replace each irreducible component by a vertex labelled by the genus of the component,

2. replace each node by an edge connecting the corresponding node(s),

3. attach a leaf to each vertex for each marked point on the corresponding component, labelled by the label of the marked point.

This graph is called the dual graph of the curve, and curves with the same dual graph are said to have the same topological type. Given any graph Γ that appears as the dual graph of some curve, the boundary class D_Γ is the class of all curves in $\overline{\mathcal{M}}_{g,n}$ of topological type Γ.

To introduce the *ψ-classes*, denote by $\mathcal{L}_i$ the line bundle on $\overline{\mathcal{M}}_{g,n}$ whose fibre at a point $(C, p_1, \dots, p_n)$ is the cotangent line to the i^{th} marked point p_i, glued together in some natural way. Define $\psi_i \in H^2(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$ to be the first chern class $c_1(\mathcal{L}_i)$ of this line bundle.

Using the forgetful map $\pi\colon \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$, we introduce *$\kappa$-classes* by the following formula:

$$\kappa_i = \pi_* \psi_{n+1}^{i+1} \tag{1.4}$$

Finally, we introduce so-called *λ-classes.* Let $\mathbb{E}$ be the rank g vector bundle whose fibre at each curve is the space of sections of the canonical line bundle, called the *Hodge bundle.* Then for any $i \in \{1, \dots, g\}$, define λ_i to be the i^{th} chern class of that bundle; $\lambda_i = c_i(\mathbb{E}) \in H^{2i}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$.

Now that we have defined some natural classes on the moduli space, one of the things to understand is their intersection theory. In particular, a natural question to ask is how many points lie in the intersection of a set of classes the sum of whose degrees is equal to the dimension on the space. Such a number of points is denoted by an integral. For instance

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \quad \text{for } d_1, \dots d_n \in \mathbb{Z}_{\geq 0} \tag{1.5}$$

denotes the number of points in the intersection of d_1 copies of ψ_1 with d_2 copies of ψ_2, etc. Note that this intersection is zero-dimensional if and only if $d_1 + \cdots + d_n = 3g - 3 + n$; if this equality does not hold, the integral is defined to be zero. It turns out that those intersections of ψ classes are governed by the Korteweg-de-Vries hierarchy; this is the subject of the Witten-Kontsevich theorem.

1.3.3 The ELSV formula

Another example of natural occurring intersection numbers that are governed by an integrable hierarchy are the so-called *linear Hodge integrals*, which are of the form

$$\int_{\overline{\mathcal{M}}_{g,n}} \lambda_i \psi_1^{d_1} \cdots \psi_n^{d_n} \quad \text{for } d_1, \dots d_n \in \mathbb{Z}_{\geq 0}. \tag{1.6}$$

Again, the integral is defined to be zero when $i + d_1 + \cdots + d_n = 3g - 3 + n$ does not hold.

One of the reasons that these linear Hodge integrals are interesting for us is that they are related to Hurwitz numbers by a remarkable formula due to Ekedahl, Shapiro, Lando and Vainshtein [31];

$$h_{g,\mu} = \prod_{i=1}^{\ell(\mu)} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{1 - \lambda_1 + \lambda_2 - \cdots \pm \lambda_g}{\prod_{i=1}^{\ell(\mu)} (1 - \mu_i \psi_i)} \; . \tag{1.7}$$

Here the right-hand side should be interpreted using the expansion

$$\frac{1}{1 - \mu_i \psi_i} = \sum_{k=0}^{\infty} (\mu_i \psi_i)^k \tag{1.8}$$

and all terms where the sum of the degrees of the classes does not match the dimension of the moduli space are disregarded. The formula is remarkable, because it allows us to compute some a priori geometrical quantities (intersection numbers) in terms of combinatorial ones (Hurwitz numbers). It turns out that this is a very powerful tool, which was for instance used in one of the proofs of the Witten-Kontsevich theorem ([63]).

1.4 Cohomological field theories

In this section, I give the definition of a cohomological field theory, and I try to explain why they are interesting to study. To do that, I introduce the notion of Gromov-Witten theory before continuing to more general cohomological field theories. This finally allows me to introduce Givental's formalism, which will later be related to the CEO recursion of the next section.

1.4.1 Gromov-Witten theory

In the previous section we studied intersection theory on the moduli space of curves. It turns out that it can be even more interesting to look not just at curves, but at curves inside some given space. One of the ways to describe such a situation is called Gromov-Witten theory; basically, given a space X, it studies the moduli space of curves in X, and in particular the intersections of naturally defined classes on that space. One of the reason for the interest in this type of theory comes from physics, where one is interested for instance in the world-sheet of a closed string in space-time, which can indeed be described mathematically as a curve in some manifold.

To be more precise, let $g \geq 0$ and $n \geq 1$ be integers, let X be a smooth projective manifold, and let $\beta \in H^2(X, \mathbb{Z})$ be a class in the second homology of X. Then a genus g, n-pointed stable map to (X, β)

$$f\colon (C, p_1, \ldots, p_n) \to X \tag{1.9}$$

is a map from an n-pointed nodal curve C of genus g to X such that f has only finitely many automorphisms (stability), and $f_*[C] = \beta$. Concretely, the stability means that C should have at least one stable irreducible component, and all irreducible components of C that are mapped to a point should be stable (both in the sense of stable curves). We denote by $\overline{\mathcal{M}}_{g,n}(X, \beta)$ the moduli space of such stable maps to (X, β). Note that when X is a point, this is just the moduli space of curves, so the Gromov-Witten theory of a point reduces to ordinary intersection theory on the moduli space of curves. Also note that there is a natural forgetful map to the moduli space of curves

$$\rho_\beta\colon \overline{\mathcal{M}}_{g,n}(X, \beta) \to \overline{\mathcal{M}}_{g,n} \tag{1.10}$$

that forgets the map and the target, and returns the source curve after an appropriate stabilization.

Once again, we want to integrate some natural classes on the moduli space. However, it is quite singular (it is not even equidimensional), so one has to do some work to be able to do this. The solution turns out to be the construction of a *virtual fundamental class* $[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\mathrm{virt}}$, which behaves just like the fundamental class of a smooth space of dimension

$$\dim = (3 - \dim(X))\,(g - 1) + c_1(TX) \cap \beta + n. \tag{1.11}$$

To define suitable classes to integrate, let $\mathcal{L}_i$ once again be the line bundle on $\overline{\mathcal{M}}_{g,n}(X, \beta)$ with fibre given by the cotangent line to the i^{th} marked point, and let $\psi_i = c_1(\mathcal{L}_i)$. Furthermore, we define some classes associated to the space X. Let $\mathrm{ev}_i\colon \overline{\mathcal{M}}_{g,n}(X, \beta) \to X$ be the map that sends a stable map to the image of the i^{th} marked point. Then, for any even cohomology class α on X, we get a class $\mathrm{ev}_i^*(\alpha)$ on $\overline{\mathcal{M}}_{g,n}(X, \beta)$ by pull-back.

The most common integrals we are interested in are the intersection of these classes, called *correlators*, of the following form:

$$\int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{virt}}} \prod_{i=1}^{n} \psi_i^{d_i} ev_i^*(e_i), \tag{1.12}$$

where e_i are some classes in the even cohomology of X. The analogues of these correlators play an important role in the cohomological field theories introduced in the next subsection, where they are used to build the so-called *descendant potentials*.

1.4.2 Cohomological field theory

Cohomological field theories can be viewed as a way to axiomatize some of the properties of Gromov-Witten theory. The definition is the following.

Let V be a finite dimensional vector space with a scalar product $(\cdot, \cdot)$ and a distinguished vector $e_{\mathbf{1}}$. Denote by $H^*_{\mathrm{even}}(\overline{\mathcal{M}}_{g,n}, \mathbb{C})$ the even cohomology of the moduli space of curves. A cohomological field theory is a collection of linear homomorphisms $c_{g,n}\colon V^{\otimes n} \to H^*_{\mathrm{even}}(\overline{\mathcal{M}}_{g,n}, \mathbb{C})$ for all g and n that behaves nicely with respect to permutation of marked points and tensor factors, and also with respect to the standard morphism between moduli spaces of curves. To be precise, the following should hold:

- For any g and n, the morphism $c_{g,n}$ is $\mathbb{S}_n$ equivariant, where the action is given by permutation of tensor factors in $V^{\otimes n}$ and of marked points in $\overline{\mathcal{M}}_{g,n}$.
- For any $a, b \in V$ we have $(a, b) = c_{0,3}(e_{\mathbf{1}} \otimes a \otimes b) \in H^*(\overline{\mathcal{M}}_{0,3}) = \mathbb{C}$.

Remember that $\pi \colon \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the morphism that forgets the last marked point, and gl denotes the gluing morphism, either gluing two marked points on one curve, increasing the genus, or gluing two curves together at marked points.

- For any $a_1, \ldots, a_n \in V$, we have
$$\pi^* c_{g,n}(a_1 \otimes \cdots \otimes a_n) = c_{g,n+1}(a_1 \otimes \cdots \otimes a_n \otimes \mathbf{1}). \tag{1.13}$$
- Let $\{e_i\}$ be a basis of V and denote $\eta^{ij} = (e_i, e_j)$. Then we have for all $a_1, \ldots, a_{n_1+n_2}$ in V:
$$\begin{aligned} \mathrm{gl}^* c_{g_1+g_2, n_1+n_2}(a_1 \otimes \cdots \otimes a_{n_1+n_2}) = \\ = c_{g_1,n_1+1}(a_1 \otimes \cdots \otimes a_{n_1} \otimes e_i) \cdot \\ \cdot c_{g_2,n_2+1}(a_{n_1+1} \otimes \cdots \otimes a_{n_2} \otimes e_j) \cdot \eta^{ij}. \end{aligned} \tag{1.14}$$
- Finally, for all $a_1, \ldots, a_n$, we have
$$\mathrm{gl}^* c_{g,n}(a_1 \otimes \cdots \otimes a_n) = c_{g-1,n+2}(a_1 \otimes \cdots \otimes a_n \otimes e_i \otimes e_j) \cdot \eta^{ij}. \tag{1.15}$$

Indeed, this definition generalizes Gromov-Witten theories in the following way. Let X be a smooth projective variety, and β a class in the second homology of X. Assume that X does not have odd cohomology. Let $V = H^*(X, \mathbb{C})$ with scalar product $(\cdot, \cdot)$ given by the Poincaré pairing and unit $\mathbf{1}$ by the unit in cohomology. Define

$$c_{g,n}(a_1 \otimes \cdots \otimes a_n) = (\rho_\beta)_* \left([\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\mathrm{virt}} \cap \prod_{i=1}^n \mathrm{ev}_i^*(a_i) \right). \tag{1.16}$$

Then $c_{g,n}$ obeys the axioms of a cohomological field theory, making Gromov-Witten theories into a large class of examples of cohomological field theories.

As in Gromov-Witten theory, it turns out that we are often interested in certain types of intersection numbers of cohomological field theories called correlators. They are denoted by $\langle \tau_{d_1}(a_1) \cdots \tau_{d_n}(a_n) \rangle_{g,n}$ and defined as

$$\langle \tau_{d_1}(a_1) \cdots \tau_{d_n}(a_n) \rangle_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}} c_{g,n}(a_1 \otimes \cdots \otimes a_n) \psi_1^{d_1} \cdots \psi_n^{d_n}. \tag{1.17}$$

It is convenient to collect correlators of a cohomological field theory in a generating function Z, called the *partition function* or *descendant potential*. It is defined as follows. Let $e_1, \ldots, e_N$ be some chosen basis for V. Then:

$$\begin{aligned} Z\left(\hbar; \{t^{d,i}\}\right) := \exp\left(\sum_g \hbar^{g-1} \mathcal{F}_g \right) = \\ = \exp\left(\sum_{g,n} \hbar^{g-1} \sum_{\substack{d_1,\ldots,d_n \geq 0 \\ 1 \leq i_1,\ldots,i_n \leq N}} \frac{\langle \tau_{d_1}(e_{i_1}) \cdots \tau_{d_n}(e_{i_n}) \rangle}{|\mathrm{Aut}((d_j, i_j)_{j=1}^n)|} t^{d_1,i_1} \cdots t^{d_n,i_n} \right). \end{aligned} \tag{1.18}$$

Here, $|\mathrm{Aut}((d_j, i_j)_{j=1}^n)|$ denotes the number of automorphism of the collection of multi-indices (d_j, i_j), and the sum over g, n contains all stable contributions. The variable $\hbar$ keeps track of the genus, and the variables $t^{d,i}$ keep track of the number of ψ-classes and the basis vectors in V.

1.4.3 Frobenius manifolds

It turns out that the genus zero part of a cohomological field theory has an interesting structure of its own; it is a Frobenius manifold. We describe it here.

Definition 1.2. A *Frobenius algebra* is a commutative associative algebra A with unit e together with a bilinear symmetric non-degenerate inner product

$$\langle\cdot,\cdot\rangle\colon A \times A \to \mathbb{C} \tag{1.19}$$

such that

$$\langle ab, c\rangle = \langle a, bc\rangle. \tag{1.20}$$

Definition 1.3. A *Frobenius manifold* is a manifold M together with a stucture of Frobenius algebra on the tangent space at each point, varying smoothly, such that the following hold.

1. The inner product $\langle\cdot,\cdot\rangle$ induces a flat metric η on M.
2. The unit vector field is covariantly constant with respect to the Levi-Cevitá connection ∇ for the metric η: $\nabla e = 0$.
3. Define a symmetric three-tensor by $c(u, v, w) = \langle u \cdot v, w\rangle$. The four-tensor $(\nabla_z c)(u, v, w)$ should be symmetric in the four vector fields z, u, v, w.

Such a Frobenius manifold is called *conformal*, if in addition, it is equiped with a vector field E, called the *Euler vector field*, satisfying

4. $\nabla(\nabla E) = 0$.

For any system of local flat coordinates $t^1, \ldots, t^N$ on a Frobenius manifold M, denote by $c^\alpha_{\beta\gamma}$ the structure constants of the algebra structure on the tangent space. That is

$$\frac{\partial}{\partial t^\beta} \cdot \frac{\partial}{\partial t^\gamma} = c^\alpha_{\alpha\beta} \frac{\partial}{\partial t^\alpha}. \tag{1.21}$$

For any N-dimensional Frobenius manifold M, locally, there exist flat coordinates $t^1, \ldots, t^N$ and a function $G(t)$ called the *potential* such that

$$e = \frac{\partial}{\partial t^1} \tag{1.22}$$

$$\eta_{\alpha\beta} = \frac{\partial^3 G}{\partial t^1 \partial t^\alpha \partial t^\beta} \tag{1.23}$$

$$c^\alpha_{\beta\gamma} = \eta^{\alpha\mu} \frac{\partial^3 G}{\partial t^\mu \partial t^\alpha \partial t^\beta}, \tag{1.24}$$

where the matrix $\eta_{\alpha\beta}$ is just the metric η in the coordinates $\{t^\alpha\}$ and $\eta^{\alpha\beta}$ is its inverse. It turns out that the definition of Frobenius manifolds implies that the potential G satisfies the WDVV equation:

$$\frac{\partial^3 G}{\partial t^\alpha \partial t^\beta \partial t^\mu} \eta^{\mu\nu} \frac{\partial^3 G}{\partial t^\nu \partial t^\gamma \partial t^\delta} = \frac{\partial^3 G}{\partial t^\beta \partial t^\delta \partial t^\mu} \eta^{\mu\nu} \frac{\partial^3 G}{\partial t^\nu \partial t^\gamma \partial t^\alpha} \ . \tag{1.25}$$

Conversely, suppose a holomorphic function $G(t^1, \ldots, t^N)$ on an open subset $U \subset \mathbb{C}^N$ and a constant matrix η satisfying equations (1.23) and (1.25) are given. Then equations (1.22) and (1.24) determine the structure of a Frobenius manifold on U.

Consider an N-dimensional cohomological field theory with vector space $V = \langle e_1, \ldots, e_N \rangle$, unit e_1 and scalar product η. Let G be the genus zero part of the partition function with no descendants (ψ-classes):

$$G(t^{0,1}, \ldots, t^{0,N}) = \mathcal{F}_0(\{t^{d,i}\})\big|_{t^i_{\geq 1}=0} \tag{1.26}$$

Then by the definition of cohomological field theories, G and η fulfil the requirements of Equations (1.23) and (1.25). Since G does not necessarily converge to a function on some open subset of $\mathbb{C}^N$, we say that it defines a *formal Frobenius manifold* structure on V.

It turns out that the whole genus zero potential can be recovered from the series G using the so-called *topological recursion relation* in genus zero (note that the CEO recursion is also sometimes called topological recursion; to avoid confusion here we always call it CEO recursion). Thus, the information of a Frobenius manifold corresponding to a cohomological field theory is enough to recover the whole genus zero partition function.

In the next subsection we describe a construction, due to Givental and proved by Teleman, that allows to reconstruct the full partition function of a cohomological field theory from a local conformal Frobenius manifold structure.

1.4.4 Givental's formalism

Givental's formalism essentially consists of two parts. The first is a group action on the space of cohomological field theories, and the second is a way to reconstruct a group element that takes the trivial cohomological field theory to a given one.

Consider the space of partition functions of cohomological field theories on an N-dimensional vector space $V = \langle e_1, \ldots, e_N \rangle$ with scalar product η:

$$Z = \exp\left(\sum_{g\geq 0} \hbar^{g-1}\mathcal{F}_g\right). \tag{1.27}$$

Let $r_l \in \mathrm{Hom}(V,V)$ for $l \geq 0$ be some operators such that the operators with odd indices are symmetric and those with even indices are skew-symmetric. We define their quantization $(r_l z^l)\hat{}$ by

$$(r_l z^l)\hat{} := -(r_l)^i_{\mathbf{1}} \frac{\partial}{\partial t^{l+1,i}} + \sum_{d=0}^{\infty} v^{d,i} (r_l)^j_i \frac{\partial}{\partial t^{d+l,j}} + \frac{\hbar}{2}\sum_{m=0}^{l-1} (-1)^{m+1} (r_l)^{i,j} \frac{\partial^2}{\partial t^{m,i} \partial t^{l-1-m,j}}. \tag{1.28}$$

Here the indices $i, j \in \{1, \ldots, N\}$ on r_l correspond to the basis $\{e_1, \ldots, e_N\}$ of V, and the index $\mathbf{1}$ corresponds to the unit vector $e_{\mathbf{1}}$. When we write r_l with two upper-indices we mean as usual that we raise one of the indices using the scalar product η.

Given such a sequence of operators r_l, we define an operator series $R(z)$ in the following way

$$R(z) = \sum_{l=0}^{\infty} R_l z^l := \exp\left(\sum_{l=1}^{\infty} r_l z^l\right). \tag{1.29}$$

The quantization $\hat{R}$ of this series is defined by

$$\hat{R} = \exp\left(\sum_{l=1}^{\infty} ((-1)^l r_l z^l)\hat{}\right). \tag{1.30}$$

Givental observed that the action of such operators on tame formal power series (series for which the number of ψ-classes (given by the first index of $t^{d,i}$) at any monomial is not too high) is well defined. The infinitesimal form of the quantization (1.28) was found by Y.P. Lee [68, 69, 70].

It turns out that the natural action of such differential operators $\hat{R}$ on the space of partition functions of cohomological field theories with target space (V, η) is well-defined [44, 62, 103]. This is the first part of Givental's formalism.

For the second part, suppose that we start with some N-dimensional semi-simple conformal cohomological field theory partition function Z. Here semi-simplicity means that the associated Frobenius manifold has a semi-simple Frobenius algebra structure on the tangent space at a generic point. Then, from the data of just the genus zero part without descendants of Z, so from the associated Frobenius manifold, one can construct an operator series $R(z) = \sum_{k\geq 0} R_k z^k$ as above, such that the quantization of this operator takes the trivial N-dimensional cohomological field theory (properly rescaled) to the cohomological field theory we started with [45, 46, 47].

Thus, for semi-simple conformal cohomological field theories, the whole partition function can in principle be reconstructed from just the genus zero data without descendants. Since it is a bit involved, I leave the actual construction of the operator series R to Chapter 5.

For us, one of the most important applications of this theory is that the action of a series R can be represented using graphs, so the partition function of a given cohomological field theory is expressed as a sum of contributions over a specific set of labelled graphs, each of whose building blocks contribute in a specific way.

1.5 CEO recursion

In this section I describe the Chekhov-Eynard-Orantin recursion. It is a way to recursively define a series of invariants $\omega_{g,n}$ (where g is a non-negative integer, and n is a positive integer) from a simple set of data; a curve in $\mathbb{C}^2$ called the spectral curve, together with a meromorphic differential, often called the Bergman kernel. Note that in the literature, it often called topological recursion theory. Here we avoid that name to avoid any confusion with the topological recursion relation mentioned in the last section.

This recursion was first established to deal with expectation values in the theory of matrix models, but it turns out that many series of invariants that arise naturally in different parts of mathematics also fit into this framework. For instance, Hurwitz numbers, some intersection numbers on the moduli space of curves, and cohomological field theories can all be described in some way by CEO recursion.

Let $\mathcal{S}$ be a smooth complex curve in $\mathbb{C}^2$ with coordinates x, y. Let $a_1, \ldots, a_N$ be the critical points of the coordinate function x. Assume that x is a ramified covering with ramification of order 2 at each of the points $a_1, \ldots, a_N$, and no other ramification points. Thus, in the neighbourhood of each point a_i one can define a local coordinate $z^{(i)}$ such that

$$x(z^{(i)}) = (z^{(i)})^2 + x(a_i). \tag{1.31}$$

Furthermore, there is a natural involution σ_i that interchanges the sheets of the covering x; for simplicity, for any function or differential form α we often denote this involution by $\bar{\alpha} = \sigma_i^*(\alpha)$ when the ramification point a_i is clear. Now, let B be a meromorphic symmetric 2-differential on $\mathcal{S} \times \mathcal{S}$ such that near (a_i, a_j) it can be expressed as

$$B(z_1^{(i)}, z_2^{(j)}) = \delta_{ij} \frac{\mathrm{d}z_1^{(i)} \mathrm{d}z_2^{(j)}}{(z_1^{(i)} - z_2^{(j)})^2} + B_{\mathrm{reg}}^{(ij)}(z_1^{(i)}, z_2^{(j)}) \tag{1.32}$$

in the local coordinate $(z^{(i)}, z^{(j)})$, where $B_{\mathrm{reg}}^{(ij)}$ is a 2-differential holomorphic at (a_i, a_j).

To such a set of data $(\mathcal{S}, x, y, B)$ we recursively associate a tower of n-differentials $\omega_{g,n}$ on $\mathcal{S}^{\times n}$, for $g \geq 0$ and $n \geq 1$, in the following way:

$$\omega_{0,1}(z) = 0, \quad \omega_{0,2}(z_1, z_2) = B(z_1, z_2), \tag{1.33}$$

and for $2g - 2 + n > 0$,

$$\omega_{g,n}(z, z_2, \dots, z_n) = -\sum_{i=1}^{N} \operatorname*{res}_{z'=a_i} \left(\frac{\tilde{\omega}_{g,n}(z', \bar{z}', z_2, \dots, z_n)}{2(y(z')\mathrm{d}x - \bar{y}(z')\mathrm{d}x)} \int_{z'}^{\bar{z}'} B(z, \cdot) \right), \tag{1.34}$$

where

$$\tilde{\omega}_{g,n}(z', z'', z_2, \dots, z_n) = \omega_{g-1,n+1}(z', z'', z_2, \dots, z_n) + \\ + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = \{2,\dots,n\}}} \omega_{g_1,|I|+1}(z', z_I)\omega_{g_2,|J|+1}(z'', z_J). \tag{1.35}$$

Here $z_I := (z_{i_1}, \dots, z_{i_{|I|}})$ for any subset $I \subset \{2, \dots, N\}$. The $\omega_{g,n}$ thus defined are called the correlation functions associated to the spectral curve (C, x, y, B).

Furthermore, we also associate to this data a set of so-called symplectic invariants F_g, for $g \geq 2$, as follows:

$$F_g := \frac{1}{2 - 2g} \sum_{i=1}^{N} \operatorname*{res}_{z=a_i} (\omega_{g,1}(z)\Phi(z)), \tag{1.36}$$

where Φ is defined locally near a_i and obeys $\mathrm{d}\Phi = y\mathrm{d}x$.

Remark 1.4. In fact, it turns out that it is often interesting to study the just the local behaviour of the CEO recursion at the branch points of the spectral curve. Thus, one defines local CEO recursion, where the curve is replaced by a set of disks centered at the ramification points, and all differentials are replaced by their local germs. I will use this set-up when discussing the relation with Giventals formalism.

Example 1.5. Consider the spectral curve given by

$$x(z) = z^2 + a, \quad y(z) = z \quad \text{and} \quad B(z, z') = \frac{\mathrm{d}z \otimes \mathrm{d}z'}{(z - z')^2}, \tag{1.37}$$

which is called the Airy curve. It has just one ramification point at $(x, y) = (a, 0)$. The corresponding correlation functions $\omega_{g,n}$ encode the intersections of ψ-classes on the moduli spaces of curves:

$$\omega_{g,n}(z_1, \dots, z_n) = \left(-\frac{1}{2}\right)^{2g-2+n} \sum_{d_1,\dots,d_n \geq 0} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n} \prod_{i=1}^{n} \frac{(2d_i + 1)!!\mathrm{d}z_i}{z_i^{2d_i+2}}. \tag{1.38}$$

This example turns out to be very important, since it constitutes a basic building block for the correlation functions of any spectral curve. That is, for any spectral curve, the correlation functions can be written as a sum over a set of labelled graphs, where each part of a graphs is assigned some contribution. Then, the contribution of the vertices are precisely the correlation functions of the Airy curve described above (1.38).

1.6 Results

In this thesis, I describe and prove various results about the four areas described above, about their internal structure as well as the relations between them. Here I give a description of those results, as well as a plan of how they are presented.

1.6.1 Hurwitz numbers with completed cycles

In Chapter 2, I introduce Hurwitz numbers with completed $(r+1)$-cycles and prove various results about their structure.

After defining the Hurwitz numbers $h^{(r)}_{g,\mu,\nu}$ following [88], I show how to describe them as the vacuum expectation values of certain operators on the so-called semi-infinite wedge space. This allows us to define an algorithm to compute $h^{(r)}_{g,\mu,\nu}$ by commuting those operators, based on the algorithm described by Johnson in [59] for ordinary Hurwitz numbers. In fact, we prove the following theorem:

Theorem 1.6. *The Hurwitz number $h^{(r),m}_{\mu,\nu}$ is given by the following formula:*

$$h^{(r),m}_{\mu,\nu} = \frac{1}{\prod_{i=1}^{l(\mu)} \mu_i \prod_{j=1}^{l(\nu)} \nu_j} [z_1^{r+1} \cdots z_s^{r+1}] \left(\sum_{P \in \mathrm{CP}^s_{\mu,\nu}} \prod_{t \in T(P)} \frac{1}{\zeta(z_{S_t})} \prod_{l \in L(P)} \zeta \begin{pmatrix} P_I^l & P_J^l & P_K^l \\ P_M^l & P_N^l & P_R^l \end{pmatrix} \right) \tag{1.39}$$

Here $[z^k]Q(z)$ denotes the coefficient of z^k in the power series $Q(z)$, the set $\mathrm{CP}^s_{\mu,\nu}$ indexes the possible ways of commuting the operators in the vacuum expectation value describing $h^{(r),m}_{\mu,\nu}$, and the function ζ is given by

$$\zeta(z) = e^{z/2} - e^{-z/2}. \tag{1.40}$$

The matrix argument for ζ in equation (1.39) is just a notation for a certain comibination of the variables $z_1, \dots, z_s$ depending on the commutation pattern P.

The proof of Theorem 1.6 provides us with an algorithm (described in detail in Chapter 2) to compute any Hurwitz numbers $h^{(r)}_{g,\mu,\nu}$ individually. This algorithm can then be used to derive some of their general properties. For instance, it is used to prove the following theorem.

Choose two positive integers s and n and let V be the set of pairs of partitions of length s and n respectively, and of the same size. Define a function $h^{(r)}_g : V \to \mathbb{Q}$ by $h^{(r)}_g(\mu,\nu) = h^{(r)}_{g,\mu,\nu}$. For any subsets $I \subset \{1,\dots,s\}$ and $J \subset \{1,\dots,n\}$ let $W_{I,J}$ be the hyperplane

$$W_{I,J} := \left\{ (\mu,\nu) \in V \;\middle|\; \sum_{i \in I} x_i - \sum_{j \in J} y_J = 0 \right\} \subset V \tag{1.41}$$

Then the following theorem holds.

Theorem 1.7. *The function $h^{(r)}_g$ is piecewise polynomial on V, with walls given by the hyperplanes $W_{I,J}$ for any non-empty proper subsets I and J as above.*

In fact, the algorithm also allows to describe the structure of this piecewise polynomial (Theorem 2.47) and provide wall-crossing formulas for all the hyperplanes $W_{I,J}$ (Theorem 2.49). These theorems are direct generalizations of known theorems for ordinary Hurwitz numbers, which were first conjectured by Goulden, Jackson and Vakil in [51] and later partially proved by Cavalieri, Johnson and Markwig in [14] and fully proved by Johnson in [59]. In fact, the techniques used to prove the theorems described here are direct generalizations of the ones in [59].

The description of the completed Hurwitz number in terms of vacuum expectation values in the infinite wedge space can be used directly to prove two further theorems that are also direct analogues of known theorems for ordinary Hurwitz numbers.

For the first, define a generating series for completed Hurwitz numbers by

$$H_{r+1}(\beta, p_1, p_2, \dots, q_1, q_2, \dots) = \sum_{n,m,s} \sum_{\substack{\mu_1,\dots,\mu_s \\ \nu_1,\dots,\nu_n}} h^{(r+1),m}_{\mu,\nu} \frac{\beta^m}{m!} \frac{p_{\mu_1} \cdots p_{\mu_s}}{s!} \frac{q_{\nu_1} \cdots q_{\nu_n}}{n!}. \tag{1.42}$$

Then this generating series obeys a so-called cut-and-join equation.

Theorem 1.8. *The generating series H_{r+1} obeys the following equation:*

$$\frac{\partial H_{r+1}}{\partial \beta} = Q_{r+1} H_{r+1}, \tag{1.43}$$

where the operators Q_{r+1} are defined as the coefficients of the expansion of the following series

$$Q_1 z + Q_2 z^2 + \cdots = \frac{1}{\zeta(z)} \sum_{n\geq 1} \frac{1}{n!} \sum_{k_1+\cdots+k_n=0} \zeta(k_1 z)\cdots\zeta(k_n z) \frac{:a_{k_1}\cdots a_{k_n}:}{k_1\cdots k_n}. \tag{1.44}$$

Here, for $k \geq 1$ we define $a_{-k} = p_k$ and $a_k = k\frac{\partial}{\partial p_k}$, and the normal ordering $:\cdot:$ means that the differential operators go to the right.

For the second theorem define some 'combinatorial intersection numbers' by the formula

$$h^{(r)}_{g,\mu} = \frac{m!}{d} \int_{X^{(r)}_{g,n}} \frac{1 - \Lambda^{(r)}_2 + \Lambda^{(r)}_4 - \cdots + (-1)^g \Lambda^{(r)}_{2g}}{(1-\mu_1 \Psi^{(r)}_1)\cdots(1-\mu_n\Psi^{(r)}_n)}. \tag{1.45}$$

Here, by combinatorial intersection numbers, we mean that the right-hand side of this formula is defined by the equality with the left-hand side, but one should hope to be able to define some spaces $X^{(r)}_{g,n}$ together with classes $\Lambda^{(r)}_{2l}$ and $\Psi^{(r)}_i$ on them such that the Equality (1.45) holds. Then, Equation (1.45) becomes an ELSV-type formula for one-part double Hurwitz numbers with completed cycles.

As in the case of the usual ELSV formula, it turns out that the generating series for these intersection numbers, after an appropriate change of variables, obeys a particular integrable hierarchy.

Theorem 1.9. *Let*

$$G(u) := \sum_{j,k_1,k_2,\ldots} (-1)^j \langle \Lambda_{2j} \tau_0^{k_0} \tau_1^{k_1} \ldots \rangle_g u^{2j} \frac{T_0^{k_0}}{k_0!} \frac{T_1^{k_1}}{k_1!} \cdots. \tag{1.46}$$

Then, after an appropriate triangular change of variables, we have that for any function $c(u)$ the series $c(u) + G(u, q_1, q_2, \ldots)$ is a solution of the Hirota equations in variables $q_1, q_2, \ldots$.

The first few equations of the change of variables are given by

$$\begin{aligned} T_0 &= q_1, \\ T_1 &= uq_1 + q_2, \\ T_2 &= u^2 q_1 + 3uq_2 + 2q_3, \end{aligned} \tag{1.47}$$

while the complete change is described in Chapter 2.

1.6.2 CEO recursion for Hurwitz numbers with completed cycles

In Chapter 3, I describe a spectral curve whose correlations functions conjecturally generate single Hurwitz numbers with completed cycles. This is a generalization of the Bouchard-Marino conjecture [10], which was proved in various ways [35, 8].

Let $H^{(r)}_{g,n}$ be the generating function for completed single Hurwitz numbers in the following sense:

$$H^{(r)}_{g,n}(x_1,\ldots,x_n) := \sum_{\mu,\ldots,\mu_n} \frac{h^{(r)}_{g,\mu}}{m!} \exp(\mu_1 x_1 + \cdots + \mu_n x_n), \tag{1.48}$$

where m denotes the Euler characteristic (Equation (1.2)) as always. Also, for any function $f(x_1, \ldots, x_n)$, let

$$Df(x_1, \ldots, x_n) = \frac{\partial^n f}{\partial x_1 \cdots \partial x_n} \mathrm{d}x_1 \cdots \mathrm{d}x_n \tag{1.49}$$

Conjecture 1.10. *Let $\omega_{g,n}^{(r)}$ be the correlation functions associated to the spectral curve*

$$\begin{cases} x(z) &= -z^r + \log z \\ y(z) &= z \end{cases} \tag{1.50}$$

$$B(z_1, z_2) = \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2}. \tag{1.51}$$

Then we have

$$DH_{g,n}^{(r)}(x_1, \ldots, x_n) = \omega_{g,n}(x_1, \ldots, x_n). \tag{1.52}$$

The evidence for this conjecture is four-fold. First, there is a proof of the Bouchard-Mariño conjecture by Borot, Eynard, Mulase and Safnuk ([8]) that uses matrix models, which can be fully generalized to the case of completed Hurwitz numbers. Unfortunately, this proof is not completely rigorous, so it should be regarded as strong evidence from mathematical physics for the conjecture.

Second is the general idea [30] that the spectral curve for an enumerative problem should be given by its $(0, 1)$-geometry. In the case of completed Hurwitz numbers it is easy to see that the $(0, 1)$-geometry leads to the spectral curve from Conjecture 1.10.

Third, there is a conjecture by Gukov and Sułkowski [53] which states that if an enumerative problem can be described using the CEO recursion, then there should be a way to 'quantize' the spectral curve to produce an operator that annihilates a certain specialization of the partition function for the enumerative problem. In the case of completed Hurwitz numbers, we prove that the spectral curve from Conjecture 1.10 can be quantized in such a way that the resulting operator annihilates this specialization of the partition function, simultaneously providing some evidence for the r-Bouchard-Mariño conjecture and the Gukov-Sułkowski conjecture.

Let $Z^{(r)}$ be the following generating function for single Hurwitz numbers with completed $(r + 1)$-cycles:

$$Z^{(r)}(\lambda, p_1, p_2, \ldots) := \exp\left(\sum_{g,\mu} h_{g,\mu}^{(r)} \lambda^{2g-2+\ell(\mu)} p_{\mu_1} \cdots p_{\mu_{\ell(\mu)}}\right). \tag{1.53}$$

Remark 1.11. In the generating function above, the parameter λ keeps track of the Euler characteristic of the curve. We will also sometimes use parameters $\hbar$ or g_{s} for this purpose. In that case they will appear with exponents $\hbar^{g-1}$ and g_{s}^{2g-2}. We abuse notation and denote all three generating functions by $Z^{(r)}$.

Theorem 1.12. *Denote by*

$$Z^{(r)}(\lambda, x) = Z^{(r)}(\lambda, p_1, p_2, \ldots)|_{p_i \to x^i} \tag{1.54}$$

the principal specialization of $Z^{(r)}$. Then we have the following Schroedinger type equation:

$$\left(\lambda x \frac{\mathrm{d}}{\mathrm{d}x} - x^{\frac{3}{2}} \exp\left(\frac{\sum_{i=0}^r x^{-1}(\lambda x \frac{\mathrm{d}}{\mathrm{d}x})^i x (\lambda x \frac{\mathrm{d}}{\mathrm{d}x})^{r-i}}{r+1}\right) x^{-\frac{1}{2}}\right) Z^{(r)}(\lambda, x) = 0. \tag{1.55}$$

Furthermore, the operator in front of $Z^{(r)}$ in this equation is a quantization of the spectral curve (1.50) *in some appropriate sense.*

The precise sense in which the operator in Equation (1.55) is a quantization of the spectral curve (1.50) is explained in Chapter 3. One interesting fact is that is one of the first cases of the Gukov-Sułkowski conjecture where one has to use non-normal ordering for the operators in the quantization.

In fact, we prove a similar theorem for the so-called q-double Hurwitz numbers. In that case the spectral curve is known ([9, 26]), and we show that the quantization, once again using non-normal ordering for the operators, annihilates the principal specialization of their generating function.

The final piece of evidence for the r-Bouchard-Mariño conjecture is provided in Chapter 8, where we prove that it is equivalent to a generalization of the ELSV formula relating completed Hurwitz numbers with linear Hodge integrals on the moduli space of r-spin curves. Thus, the evidence for this conjecture (in the form of computer calculation, and proofs in small genus cases) also contributes to the evidence for the r-BM conjecture.

1.6.3 Integrals of ψ-classes over double ramifcation cycles.

In Chapter 4, I define so-called *double-ramification cycles*, which intuitively are classes on $\overline{\mathcal{M}}_{g,n}$ consisting of those curves on which a covering to $\mathbb{P}^1$ can be defined with some specified ramification over 0 and ∞. Thus, for a list of integers $a_1, \ldots, a_n$ satisfying $\sum a_i = 0$, the double ramification cycles $\mathrm{DR}_g(a_1, \ldots, a_n)$ is the class of curves in $\overline{\mathcal{M}}_{g,n}$ for which there exist a covering of $\mathbb{P}^1$ by that curve such that the ramification over 0 is of the type given by all the positive a_i and whose ramification over ∞ is of the type given by the absolute values of all the negative a_i.

Although it is known that these classes can be expressed in terms of the ψ, κ and boundary classes described above, there is no known explicit expression of this type. Here, we give a partial result towards this answer which is interesting in its own right; that is, we compute the intersection of any double-ramification cycles with any monomial in ψ-classes when the result is zero-dimensional.

Theorem 1.13. *Given a list of n integers $a_1, \ldots, a_n$, satisfying $\sum a_i = 0$ and a list of non-negative integers $d_1, \ldots, d_n$ satisfying $\sum d_i = 2g - 3 + n$, the integral*

$$\mathrm{DR}_g(a_1, \cdots a_n)\psi_1^{d_1} \cdots \psi_n^{d_n} \tag{1.56}$$

of a monomial in ψ-classes over a DR-cycle is equal to the coefficient of

$$z_1^{d_1} \cdots z_n^{d_n} \tag{1.57}$$

in the generating function

$$\frac{z_1 \cdots z_n}{\zeta(z_1 + \cdots + z_n)} \sum_{\substack{\sigma \in S_n \\ \sigma(1)=1}} \frac{\zeta\left(\begin{vmatrix} a'_1 & a'_2 \\ z'_1 & z'_2 \end{vmatrix}\right) \zeta\left(\begin{vmatrix} a'_1 + a'_2 & a'_3 \\ z'_1 + z'_2 & z'_3 \end{vmatrix}\right) \cdots \zeta\left(\begin{vmatrix} a'_1 + \cdots + a'_{n-1} & a'_n \\ z'_1 + \cdots + z'_{n-1} & z'_n \end{vmatrix}\right)}{z'_1 \begin{vmatrix} a'_1 & a'_2 \\ z'_1 & z'_2 \end{vmatrix} \begin{vmatrix} a'_2 & a'_3 \\ z'_2 & z'_3 \end{vmatrix} \cdots \begin{vmatrix} a'_{n-1} & a'_n \\ z'_{n-1} & z'_n \end{vmatrix} z'_n}. \tag{1.58}$$

1.6.4 Correspondence between Givental and CEO theories

In Chapter 5, I describe a correspondence between Givental's action on cohomological field theories and the Chekhov-Eynard-Orantin recursion theory. That is, given a semi-simple conformal cohomological field theory, or equivalently, an operator series $R(z)$ as described in Section 1.4.4 of this introduction, together with a rescaling Δ of the trivial cohomological field theory (it will be properly introduced in Chapter 5), one can define a spectral curve such that the associated partition functions are equal.

Theorem 1.14. *Let $R(z)$ be some operators series on an N-dimensional vector space, as in Section 1.4.4, and let Δ be a rescaling of the trivial N-dimensional cohomological field theory. Let Z be the partition function of the corresponding semi-simple cohomological field theory.*

Define a local spectral curve by the following data

$$[z^{2p}w^{2q}]B^{i,j}(z,w) := \frac{1}{(2p-1)!!(2q-1)!!}[z^p w^q]\frac{\delta^{ij} - \sum_{s=1}^N R^i_s(-z)R(-w)^j_s}{z+w} \tag{1.59}$$

and

$$[z^{2k-1}]y^i(z) := \frac{1}{2(2k-1)!!}[z^{k-1}]\left(-R(-z))^i_{\mathbf{1}}\right) \quad (k>1) \tag{1.60}$$

$$[z^1]y^i(z) := -\frac{1}{2\sqrt{\Delta^i}}. \tag{1.61}$$

Then after an appropriate change of variables (that is introduced explicitly in Chapter 5) the partition function of the cohomological field theory and the CEO recursion invariants agree in the following sense:

$$Z = \exp\left(\sum_{g,n}\hbar\omega_{g,n}\right). \tag{1.62}$$

Remark 1.15. Note that in the theorem we only define the even coefficients of the the expansion of B, and only the odd coefficients of the expansion of y. It turns out to be a general property of the CEO invariants $\omega_{g,n}$ that they do not depend on the full data of the spectral curve and two-point differential, but only on those coefficients.

The proof of this theorem is combinatorial; in fact, we express both partition functions as a sum over the same set of decorated graphs, where each component of the a graph (vertex, edge, leaf) contributes in a standard way. Then we show that, using the identification of Theorem 1.14, the contribution of each individual component is equal in both partition functions.

1.6.5 CEO-recusrion for the Gromov-Witten theory of $\mathbb{P}^1$

The first application of Theorem 1.14 is a proof of the Norbury-Scott conjecture [84] in Chapter 6. That is, we describe a spectral curve whose CEO invariants completely describe the stationary Gromov-Witten invariants of $\mathbb{P}^1$. Here stationary means that we always pull-back the class of the point from $\mathbb{P}^1$.

Theorem 1.16. *Let $\omega_{g,n}$ be the CEO invariants associated to the spectral curve*

$$\begin{cases} x &= z + \frac{1}{z}; \\ y &= \log z, \end{cases} \tag{1.63}$$

$$B(z,w) = \frac{\mathrm{d}z \otimes \mathrm{d}w}{(z-w)^2} \tag{1.64}$$

Then, for $2g-2+n>0$ and $a_1,\dots,a_n \geq 0$, we have:

$$\prod_{j=1}^n \left(-\operatorname*{res}_{x_j=\infty}\frac{1}{(a_j+1)!}x_j^{a_j+1}\right)\omega_{g,n}(x_1,\dots,x_n) = \left\langle \prod_{j=1}^n \mathrm{ev}_j^*(\mathrm{pt})\psi_j^{a_j}\right\rangle_g, \tag{1.65}$$

where pt *is the class of the point on $\mathbb{P}^1$, and $\langle\cdot\rangle_g$ denotes the integral on the moduli space of stable genus g maps to $\mathbb{P}^1$.*

1.6.6 Equivalence of ELSV and Bouchard-Mariño; a new proof of the ELSV formula

The second application of Theorem 1.14 is a proof of the equivalence of the Bouchard-Mariño conjecture and the ELSV formula in Chapter 7. Together with a new proof of the Bouchard-Mariño conjecture that does not use the ELSV formula, this is also a new proof of the ELSV formula.

Theorem 1.17. *Let $\omega_{g,n}$ be the correlation functions associated to the spectral curve*

$$\begin{cases} x(z) &= -z + \log z \\ y(z) &= z \end{cases} \tag{1.66}$$

$$B(z_1, z_2) = \frac{\mathrm{d}z_1 \otimes \mathrm{d}z_2}{(z_1 - z_2)^2}. \tag{1.67}$$

Then we have

$$DH_{g,n}(x_1, \ldots, x_n) = \omega(x_1, \ldots, x_n) \tag{1.68}$$

where D is the operator defined in equation (1.49) *and $H_{g,n}$ is the generating function for single Hurwitz numbers:*

$$H_{g,n}(x_1, \ldots, x_n) := \sum_{\mu, \ell(\mu)=n} \frac{h_{g,\mu}}{(2g - 2 + \ell(\mu) + |\mu|)!} \exp(\mu_1 x_1 + \cdots + \mu_n x_n). \tag{1.69}$$

Theorem 1.18. *The Bouchard Mariño conjecture (Theorem 1.17) is equivalent to the ELSV formula* (1.7).

In fact, as an intermediate step in the new proof of the Bouchard-Mariño conjecture, we prove that single Hurwitz numbers have a quasi-polynomial behaviour that is immediately obvious from the ELSV formula, but which had not been proved without the use of that formula.

Theorem 1.19. *Single Hurwitz numbers depend quasi-polynomially on the ramification data in the following sense:*

$$h^{\circ}_{g;\mu_1,\ldots,\mu_n} = m! \left(\prod_{i=1}^{n} \frac{\mu_i^{\mu_i}}{\mu_i!} \right) P_{g,n}(\mu_1, \ldots, \mu_n), \tag{1.70}$$

where $P_{g,n}(\mu_1, \ldots, \mu_n)$ are some polynomials in $\mu_1, \ldots, \mu_n$.

1.6.7 Equivalence of r-ELSV and r-BM conjectures

The final application of Theorem 1.14 in this thesis is a generalization of the previous one, which is the proof (mentioned earlier) of the equivalence of the r-spin variants of the Bouchard-Mariño conjecture and ELSV formula. This is the content of Chapter 8.

Theorem 1.20. *Introduce the following integrals on the space of r-spin structures:*

$$f^{(r)}_{g;k_1,\ldots,k_n} = m!\, r^{m+n+2g-2} \prod_{i=1}^{n} \frac{\left(\frac{k_i}{r}\right)^{p_i}}{p_i!} \times \int_{\overline{\mathcal{M}}^{1/r}_{g,n;a_1,\ldots,a_n}} \frac{\mathcal{S}}{\left(1 - \frac{k_1}{r}\psi_1\right) \ldots \left(1 - \frac{k_n}{r}\psi_n\right)}. \tag{1.71}$$

and let $F_{g,n}^{(r)}$ *be their generating function:*

$$F_{g,n}^{(r)}(x_1,\dots,x_n) := \sum_{\mu,\ell(\mu)=n} \frac{f_{g,\mu}^{(r)}}{m!} \exp(\mu_1 x_1 + \dots + \mu_n x_n) \tag{1.72}$$

Let $\omega_{g,n}^{(r)}$ *be the correlators of the spectral curve*

$$\begin{cases} x(z) &= -z^r + \log z \\ y(z) &= z \end{cases} \tag{1.73}$$

$$B(z_1,z_2) = \frac{\mathrm{d}z_1 \otimes \mathrm{d}z_2}{(z_1-z_2)^2} \tag{1.74}$$

that appears in Conjecture 1.10. Then we have

$$DF_{g,n}^{(r)}(x_1,\dots,x_n) = \omega_{g,n}^{(r)}(x_1,\dots,x_n). \tag{1.75}$$

Note that this theorem implies that Conjecture 1.10 is equivalent to the equation

$$f_{g;k_1,\dots,k_n}^{(r)} = h_{g,k_1,\dots,k_n}^{(r)} \tag{1.76}$$

which is called the r-ELSV formula. Indeed, in the case $r=1$ it reduces to the ordinary ELSV formula (1.7).

Part I

Hurwitz numbers with completed cycles

–2– Hurwitz numbers with completed cycles

2.1 Introduction

The theory of usual double Hurwitz numbers has been considered from several different points of view, usually in a purely combinatorial way. First, Okounkov [85] observed a relation to integrable systems of KP-type that is also nicely explained and heavily used in Johnson's recent paper [59]. Second, double Hurwitz numbers were studied from different points of view in a purely combinatorial way in a foundational paper of Goulden, Jackson, and Vakil [51], where most of the typical contemporary questions were first posed. Third, there is a tropical approach to double Hurwitz numbers developed by Cavalieri, Johnson, and Markwig [13, 14]. Fourth, double Hurwitz numbers were studied in genus 0 through the intersection theory on double ramification cycles in the moduli space of curves by Shapiro, Vainshtein, and Shadrin, see [95, 98].

The theory of Hurwitz numbers with completed cycles has emerged through several relations with geometry of the moduli space of curves. First, it has emerged in [88] as a way to encode higher degrees of ψ-classes in the Gromov-Witten theory of $\mathbb{P}^1$. Second, it is related to the intersection theory of the moduli space of r-spin structures via an analogue of the ELSV formula involving the classes studied by Chiodo in [17] conjectured by Zvonkine [111]. This r-ELSV formula is discussed more thoroughly in Chapter 8.

This motivates us to look attentively into the combinatorial structure of double Hurwitz numbers with completed cycles, since we expect that some kind of intrinsic combinatorial niceness of these numbers might be an explanation of the existing connections of completed cycles to geometry of moduli spaces and predict some further relations. So, we investigate double Hurwitz numbers with completed cycles in a purely combinatorial way generalizing various results in [51, 59, 96, 99].

2.1.1 Organization of the chapter

The chapter starts with an introduction of the necessary theory. We define completed cycles as elements of the class algebra of the symmetric group using an isomorphism with the algebra of shifted symmetric functions in Section 2.2. In Section 2.3, the infinite wedge space is explained and the necessary operators are introduced.

The main part of the chapter consists of four sections. In Section 2.4, we introduce an algorithm to calculate Hurwitz numbers in a practical way, which allows us to prove the theorems in Section 2.6.

In Section 2.5, we use the infinite wedge formalism together with calculations on the representation theory of the symmetric group to prove so called cut-and-join equations for completed Hurwitz numbers. These equations are direct generalizations of the the cut-and-join equation for ordinary Hurwitz numbers.

In Section 2.6, we use the algorithm defined in Section 2.4 to prove a strong piecewise polynomiality property in the sense of [51] for completed Hurwitz numbers. The proof is analogous to the proof in [59] for strong piecewise polynomiality of ordinary Hurwitz numbers. We also derive the corresponding wall crossing formulas.

Finally, in the last section we give a formula for completed one-part double Hurwitz numbers in terms of the intersection theory of some conjectural moduli spaces. We give an explicit formula for the generating series for these conjectural intersection numbers and we prove that it obeys the Hirota equations. The same was done for ordinary one-part double Hurwitz numbers in for instance [96].

2.1.2 Notation

Throughout this chapter, we will use the following notation.

- By $[n]$, $n = 1, 2, \dots$, we denote the set $\{1, \dots, n\}$.
- By $\ell(\mu)$ we denote the length of the partition $\mu = (\mu_1, \dots, \mu_\ell)$, and by $|\mu|$ we denote the size $\sum_{i=1}^{\ell(\mu)} \mu_i$.
- Let $U(z)$ be a formal power series in z. By $[z^a]U(z)$ we denote the coefficient of z^a in U, that is, $U(z) = \sum_{a=1}^{\infty} z^a \cdot [z^a]U(z)$.

2.2 Hurwitz numbers with completed cycles

Following [88] and [111], we give a purely combinatorial definition of Hurwitz numbers with completed cycles. Intuitively, completed Hurwitz numbers count covers of $\mathbb{P}^1$ with given ramification over 0 and ∞, like ordinary Hurwitz numbers, but instead of allowing ramification given by ordinary branch points elsewhere, we allow ramification by completed cycles.

2.2.1 Shifted symmetric functions

Let $\mathbb{Q}[x_1, \dots, x_n]$ be the algebra of polynomials in n variables over $\mathbb{Q}$. We define the shifted action of S_n on this algebra by

$$\sigma(f(x_1 - 1, \dots, x_n - n)) = f(x_{\sigma(1)} - \sigma(1), \dots, x_{\sigma(n)} - \sigma(n)) \tag{2.1}$$

for any element $\sigma \in S_n$ and any polynomial f written in terms of the variables $x_i - i$.

Denote by $\mathbb{Q}[x_1, \dots, x_n]^{*S_d}$ the algebra of polynomials which are invariant with respect to this action. It has a natural filtration by degree.

Definition 2.1. The algebra of shifted symmetric functions is the algebra

$$\Lambda^* = \varprojlim \mathbb{Q}[x_1, \dots, x_n]^{*S_d}, \tag{2.2}$$

where the projective limit is taken in the category of filtered algebras with respect to the homomorphism which sends the last variable x_n to 0.

In other words, the elements of the algebra Λ^* are given as series $f = \{f^{(n)}\}$, $f^{(n)} \in \mathbb{Q}[x_1, \dots, x_n]^{*S(n)}$, such that the polynomials $f^{(n)}$ are of uniformely bounded degree and stable under the restriction $f^{(n+1)}|_{x_{n+1}=0} = f^{(n)}$.

2.2.2 Basis of the algebra of shifted symmetric functions

The shifted analogues of the power sums form a basis of the algebra of shifted symmetric functions.

Definition 2.2. For any positive integer k, the corresponding shifted symmetric power sum p_k is defined as

$$p_k(\lambda) = \sum_{i=1}^{\infty} \left((\lambda_i - i + \frac{1}{2})^k - (-i + \frac{1}{2})^k \right). \tag{2.3}$$

For any partition μ, define $p_\mu = \prod_{i=1}^{\ell(\mu)} p_{\mu_i}$.

The functions p_μ form a basis for the algebra of shifted symmetric funtions. Another basis is obtained in the following way.

Irreducible representations of the symmetric group S_d are in one to one correspondence with partitions of d. On the other hand, partitions of d are also in one to one correspondence with conjugacy classes in S_d. For partitions μ and λ, let $\dim(\lambda)$ be the dimension of the representation given by λ, let χ_μ^λ be the character of this representation evaluated on the conjugacy class C_μ given by μ, and let $|C_\mu|$ be the size of this conjugacy class.

Definition 2.3. We define for any partition μ a function f_μ on the set of partitions of $|\mu|$ by

$$f_\mu(\lambda) = |C_\mu| \frac{\chi_\mu^\lambda}{\dim(\lambda)}. \tag{2.4}$$

When the partition μ is of the form $(r+1, 1, \ldots, 1)$, we denote f_μ by f_r, $r = 0, 1, 2, \ldots$.

Kerov and Olshanski proved that the functions f_μ are shifted symmetric [65]. They also form a basis for the space of shifted symmetric funtions.

2.2.3 Completed cycles

Let $Z\mathbb{C}S_d$ be the class algebra of the symmetric group S_d. We can define a linear isomorphism $\phi\colon \bigoplus_{d=0}^{\infty} Z\mathbb{C}S_d \to \Lambda^*$, $\phi\colon C_\mu \mapsto f_\mu$.

Definition 2.4. For any partition μ, the completed μ-conjugacy class $\overline{C}_\mu$ in the class algebra of the symmetric group is defined as $\overline{C}_\mu := \phi^{-1}(p_\mu)/\prod_{i=1}^{\ell(\mu)} \mu_i!$. A special role is played by the *completed cycles* $\overline{(r)} := \overline{C}_{(r)}$, $r = 1, 2, \ldots$.

Remark 2.5. There are actually two conventions for the definition of the completed $(r+1)$-cycle. The definition that is given here differs by a factor $r!$ from the definitions in [88, 111]. In fact, in later chapters (Chapter 3 and 8) we will use the other definition, which is more convenient for the purposes there.

The first few completed cycles are

$$\begin{aligned}
0! \cdot \overline{(1)} &= (1)\\
1! \cdot \overline{(2)} &= (2)\\
2! \cdot \overline{(3)} &= (3) + (1,1) + \frac{1}{12} \cdot (1)\\
3! \cdot \overline{(4)} &= (4) + 2 \cdot (2,1) + \frac{5}{4} \cdot (2)\\
4! \cdot \overline{(5)} &= (5) + 3 \cdot (3,1) + 4 \cdot (2,2) + \frac{11}{2} \cdot (3)\\
&\quad + 4 \cdot (1,1,1) + \frac{3}{2} \cdot (1,1) + \frac{1}{80} \cdot (1).
\end{aligned}$$

2.2.4 Hurwitz numbers with completed cycles

We fix two non-empty partitions μ and ν such that $|\mu| = |\nu|$ and non-negative integers r and m such that $2g - 2 + \ell(\mu) + \ell(\nu) = rm$.

Definition 2.6. We define *disconnected double Hurwitz numbers with completed* $(r+1)$*-cycles* via the character formula:

$$h^{(r)}_{g,\mu,\nu} := \frac{1}{\prod_{i=1}^{\ell(\mu)} \mu_i \prod_{j=1}^{\ell(\nu)} \nu_j} \sum_{|\lambda|=d} \chi^\lambda_\mu \left(\frac{p_{r+1}(\lambda)}{(r+1)!} \right)^m \chi^\lambda_\nu \tag{2.5}$$

We also denote the same Hurwitz number by $h^{(r),m}_{\mu,\nu}$, and we often omit the superscript (r) when r is fixed in advance.

Remark 2.7. Since the completed 2-cycle is equal to the ordinary 2-cycle, the double Hurwitz numbers $h^{(1),m}_{\mu,\nu}$ are just the ordinary Hurwitz numbers for not necessarily connected curves.

2.2.5 Geometric interpretation

Hurwitz numbers with completed cycles possess a geometric interpretation.

Suppose we are given a partition λ and a nonnegative interger γ. To this data we can assign a singularity of stable maps to $\mathbb{P}^1$ by the following rule.

- If $2 - 2\gamma - l(\lambda) < 0$ then the singularity consists of a contracted curve of genus γ (that is, a curve on which the stable map has degree 0) intersecting the remaining components of the source curve at $l = l(\lambda)$ branches on which the stable map has ramification points with indices $\lambda_1, \dots, \lambda_\ell$. Note that in this case the contracted curve is required to be connected, but not necessarily irreducible.

- If $\gamma = 0$ and $l(\lambda) = 2$, then the singularity consists of a simple self-intersection of the source curve such that the stable maps presents ramification points with indices λ_1, λ_2.

- Finally, if $\gamma = 0$ and $l(\lambda) = 1$, then the singularity is just a ramification point with index λ_1.

This list covers all possible connected singular loci of a stable map to $\mathbb{P}^1$. In all three cases it is natural to consider the image of the singular locus under the stable map as a branch point of multiplicity $2\gamma + |\lambda| - l(\lambda)$, see [41]. For this reason we will say that the singular locus described by the data (λ, γ) has multiplicity $2\gamma + |\lambda| - l(\lambda)$.

Now, a completed $(r+1)$-cycle is a linear combination of conjugacy classes λ. To each conjugacy class it is easy to assign a nonnegative integer γ such that $2\gamma + |\lambda| - l(\lambda) = r$. Thus every term of the completed cycle corresponds to a singular locus of multiplicity r. Moreover, the terms of the completed cycle cover all types of singular loci like that. The numerical coefficient of a partition (λ) in the completed $(r+1)$-cycle is some kind of intersection number on the moduli space $\overline{\mathcal{M}}_{\gamma,l(\lambda)}$ of contracted components; its nature is still not entirely clear.

We can now give the geometric meaning of Hurwitz numbers with completed cycles. Call a stable map $f : C \to \mathbb{P}^1$ an *r-covering* if (i) it has a finite number of preimages of 0 and ∞ and (ii) all its singular loci have multiplicity r except possibly the preimages of 0 and ∞. Call the *weight of a singular locus* described by (λ, γ) the coefficient of (λ) in the completed $(r+1)$-cycle, where $r = 2\gamma + |\lambda| - l(\lambda)$. Call the *weight of an r-covering* the product of weights of its singular loci divided by the number of automorphisms of the covering.

Proposition 2.8. *The Hurwitz number $h^{(r)}_{g,\mu,\nu}$ is equal to the sum of weights of not necessarily connected r-coverings $f : C \to \mathbb{P}^1$ with m fixed branch points, where the Euler characteristic of C equals $2-2g$ and the number of branch points equals $m = (2g-2+\ell(\mu)+\ell(\nu))/r$.*

This proposition leads to a natural definition of connected Hurwitz numbers: just replace "not necessarily connected" by "connected" in the above formulation. Connected Hurwitz numbers can be computed from the disconnected ones by the exclusion-inclusion formula, see also [88, 111].

Remark 2.9. The explicit formula for the coefficients of the completed $(r+1)$-cycle that we use in this geometric description is given in Section 2.4.6.

2.3 Semi-infinite wedge formalism

In this section we sketch the theory of semi-infinite wedge space following [88] and [59].

2.3.1 Infinite wedge

Let V be an infinite dimensional vector space with basis labelled by the half integers. Denote the basis vector labelled by $m/2$ by $\underline{m/2}$, so $V = \bigoplus_{i \in \mathbb{Z}+\frac{1}{2}} \underline{i}$.

Definition 2.10. The semi-infinite wedge space is the span of all wedge products of the form

$$\underline{i_1} \wedge \underline{i_2} \wedge \cdots \tag{2.6}$$

for any decreasing sequence of half integers (i_k) such that there is an integer c (called the charge) with $i_k + k - \frac{1}{2} = c$ for k sufficiently large.

Here, we are mostly concerned with the zero charge subspace of the semi-infinite wedge space, which is the space of all wedge products of the form (2.6) such that

$$i_k + k = \frac{1}{2} \tag{2.7}$$

for k sufficiently large. For brevity, we will call this space the infinite wedge space from now on.

Remark 2.11. An element of the infinite wedge space is of the form $\underline{\lambda_1 - \frac{1}{2}} \wedge \underline{\lambda_2 - \frac{3}{2}} \wedge \cdots$ for some partition λ. This follows immediately from condition (2.7). Thus, we canonically have a basis for the infinite wedge space labelled by all partitions. The inner product associated with this basis will be denoted $(\cdot,\cdot)$.

Notation 2.12. We denote by v_λ the vector labelled by a partition λ. The vector labelled by the empty partition is called the vacuum vector and denoted by $|0\rangle = v_\emptyset = \underline{-\frac{1}{2}} \wedge \underline{-\frac{3}{2}} \wedge \cdots$.

Notation 2.13. If $\mathcal{P}$ is an operator on the infinite wedge space, then we define the vacuum expectation value of $\mathcal{P}$ by $\langle \mathcal{P} \rangle = \langle 0|\mathcal{P}|0\rangle$, where $\langle 0|$ is the dual of the vacuum vector with resprect to the inner product $(\cdot,\cdot)$, and called the covacuum vector.

2.3.2 Operators

We now define some operators on the infinite wedge space.

Definition 2.14. Let k be any half integer. Then the operator ψ_k is defined by

$$\psi_k \colon (\underline{i_1} \wedge \underline{i_2} \wedge \cdots) \;\mapsto\; (\underline{k} \wedge \underline{i_1} \wedge \underline{i_2} \wedge \cdots). \tag{2.8}$$

This operator acts on the whole semi-infinite wedge space (the sum of spaces with all charges). It increases the charge by 1.

The operator ψ_k^* is defined to be the adjoint of the operator ψ_k with respect to the inner product $(\cdot,\cdot)$.

Definition 2.15. The normally ordered products of ψ-operators are defined in the following way

$$:\psi_i\psi_j^*: \;:= \begin{cases} \psi_i\psi_j^*, & \text{if } j > 0 \\ -\psi_j^*\psi_i & \text{if } j < 0\ . \end{cases} \tag{2.9}$$

This operator does not change the charge and can be restricted to the infinite wedge space. Its action on the basis vectors v_λ can be described as follows: $:\psi_i\psi_j^*:$ checks if v_λ contains $\underline{j}$ as a wedge factor and if so replaces it by $\underline{i}$. Otherwise it yields 0. In the case $i = j > 0$, we have $:\psi_i\psi_j^*:(v_\lambda) = v_\lambda$ if v_λ contains $\underline{j}$ and 0 if it does not; in the case $i = j < 0$, we have $:\psi_i\psi_j^*:(v_\lambda) = -v_\lambda$ if v_λ does not contain $\underline{j}$ and 0 if it does. These are the only two cases where the normal ordering is important.

Remark 2.16. Let E_{ij} for $i,j \in \mathbb{Z}+\frac{1}{2}$ denote the standard basis of matrix units of $\mathfrak{gl}(\infty) = \mathfrak{gl}(V)$. Then the assignment $E_{ij} \mapsto \;:\psi_i\psi_j^*:$ defines a projective representation of the Lie algebra $\mathfrak{gl}(V)$ on $\Lambda^{\frac{\infty}{2}}(V)$.

Notation 2.17. We denote by $\zeta(z)$ the function $e^{z/2} - e^{-z/2}$.

Definition 2.18. Let $n \in \mathbb{Z}$ be any integer. We define two operators $\mathcal{E}_n(z)$ and $\tilde{\mathcal{E}}_n(z)$ depending on a formal variable z by

$$\mathcal{E}_n(z) = \sum_{k \in \mathbb{Z}+\frac{1}{2}} e^{z(k-\frac{n}{2})} E_{k-n,k} + \frac{\delta_{n,0}}{\zeta(z)} \tag{2.10}$$

$$\tilde{\mathcal{E}}_n(z) = \sum_{k \in \mathbb{Z}+\frac{1}{2}} e^{z(k-\frac{n}{2})} E_{k-n,k}\ . \tag{2.11}$$

The operator $\mathcal{E}_n(z)$ is called the deregularized $\mathcal{E}$-operator, while $\tilde{\mathcal{E}}_n(z)$ is the regularized $\mathcal{E}$-operator.

Notation 2.19. We denote by α_n, $n \neq 0$, the operator $\mathcal{E}_n(0)$. We denote by $\mathcal{F}_{r+1}$, $r \geq 1$, the operator $[z^{r+1}]\tilde{\mathcal{E}}_0(z)$.

2.3.3 Properties of operators

These operators relate the infinite wedge space to the representation theory of the symmetric group and to double Hurwitz numbers via the following properties.

Proposition 2.20. *Let $\mu = (\mu_1, \ldots, \mu_{\ell(\mu)})$ be some partition. Then we have*

$$\prod_{i=1}^{\ell(\mu)} \alpha_{-\mu_i} |0\rangle = \sum_{|\lambda|=|\mu|} \chi^\lambda_\mu v_\lambda \quad \text{and} \quad \langle 0| \prod_{i=1}^{\ell(\mu)} \alpha_{\mu_i} v_\lambda = \chi^\lambda_\mu, \tag{2.12}$$

where χ^λ_μ is the character of the conjugacy class μ in the representation λ and is set to be 0 if μ and λ are partitions of two different integers.

This proposition is a reformulation of the Murnaghan-Nakayama rule [57]. It is proved, for instance, in [61] or [59].

Proposition 2.21. *We also have*

$$\mathrm{id} = \sum_{k\geq 1} \frac{1}{k!} \sum_{\nu_1,\dots,\nu_k} \frac{1}{\prod \nu_i} \prod \alpha_{-\nu_i} |0\rangle\langle 0| \prod \alpha_{\nu_i}, \tag{2.13}$$

where id *is the identity operator in the infinite wedge space.*

This proposition follows from the orthogonality of characters.

Finally, the algorithm that we will use to compute double Hurwitz numbers is based on the commutation relations of the $\mathcal{E}$-operators.

Proposition 2.22. *The $\mathcal{E}$-operators satisfy the following commutation relation (see Equation (2.17) in [88])*

$$[\mathcal{E}_k(w), \mathcal{E}_l(z)] = \zeta(kz - lw)\mathcal{E}_{k+l}(z+w); \tag{2.14}$$

in particular,

$$[\mathcal{E}_k(0), \mathcal{E}_l(z)] = \zeta(kz)\mathcal{E}_{k+l}(z) \tag{2.15}$$

and, taking a limit as $z \to 0$,

$$[\mathcal{E}_k(0), \mathcal{E}_l(0)] = k\delta_{k+l,0}. \tag{2.16}$$

2.3.4 Hurwitz numbers with completed cycles

Proposition 2.23. *A double Hurwitz number with completed cycles can be expressed as a vacuum expectation value in the infinite wedge space in the following way:*

$$h^{(r),m}_{\mu,\nu} = \frac{\langle \prod_{i=1}^{\ell(\mu)} \alpha_{\mu_i} \mathcal{F}^m_{r+1} \prod_{i=1}^{\ell(\nu)} \alpha_{-\nu_i} \rangle}{\prod_{i=1}^{\ell(\mu)} \mu_i \prod_{j=1}^{\ell(\nu)} \nu_j}. \tag{2.17}$$

Proof. This is just a way to rewrite Equation (2.5) using Proposition 2.20.

Then, the fact that for an arbitrary partition μ the vector v_μ is an eigenvector of $\mathcal{F}_{r+1}$ with eigenvalue $p_{r+1}(\mu)/(r+1)!$, where p_{r+1} is the shifted symmetric power sum defined in Definition 2.2, completes the proof. □

2.4 Computation of Hurwitz numbers

In this section we generalize the algorithm for computation of double Hurwitz numbers in [59] to the case of double Hurwitz numbers with completed $(r+1)$-cycles. This algorithm presents several advantages with respect to direct copmputations with characters. In particular, it will allow us to prove the piecewise polynomiality of Hurwitz numbers. In Sections 2.4.5 and 2.4.6 we also show that in certain cases it leads to quite explicit expressions for Hurwitz numbers, that cannot be directly deduced from the character formulas.

Throughout the section we fix $r \geq 1$ (and therefore omit it in all notations). We also fix two partitions, μ and ν, such that $|\mu| = |\nu|$, and an integer $m \geq 0$. They are the ramification profiles over two special points and the number of completed cycles of a particular Hurwitz number that we consider here.

2.4.1 Notations and properties of operators

For any subsets $I \subset [\ell(\mu)]$, $J \subset [\ell(\nu)]$, and $K \subset [m]$ we introduce the following notation. We denote by μ_I, ν_J, and z_K the sums $\mu_I := \sum_{i\in I} \mu_i$, $\mu_J := \sum_{j\in J} \nu_j$, and $z_K := \sum_{k\in K} z_k$. We denote by $\mathcal{E}(I, J, K)$ the operator $\mathcal{E}_{\mu_I - \nu_J}(z_K)$. We use the same notation also in the case of the operator $\tilde{\mathcal{E}}$. Let $M \subset [\ell(\mu)]$, $N \subset [\ell(\nu)]$, and $L \subset [m]$. The last piece of notation we need is

$$\zeta\left(\begin{smallmatrix} I & J & K \\ M & N & L \end{smallmatrix}\right) := \zeta\left(\det\left(\begin{smallmatrix} \mu_I - \nu_J & z_K \\ \mu_M - \nu_N & z_L \end{smallmatrix}\right)\right), \tag{2.18}$$

where $\zeta(z) = e^{z/2} - e^{-z/2}$ is as defined in Notation 2.17.

Using this notation, we can rewrite a Hurwitz number with completed cycles as

$$h^m_{\mu,\nu} = \frac{1}{\prod_{i=1}^{\ell(\mu)} \mu_i \prod_{j=1}^{\ell(\nu)} \nu_j} \cdot [z_1^{r+1} \cdots z_m^{r+1}] \left\langle \prod_{i=1}^{\ell(\mu)} \mathcal{E}(\{i\}, \emptyset, \emptyset) \prod_{k=1}^{m} \tilde{\mathcal{E}}(\emptyset, \emptyset, \{k\}) \prod_{j=1}^{\ell(\nu)} \mathcal{E}(\emptyset, \{j\}, \emptyset) \right\rangle. \tag{2.19}$$

By the symbol $[z_1^{r+1} \cdots z_m^{r+1}]$ we denote the coefficient of the monomial $z_1^{r+1} \cdots z_m^{r+1}$ in the formal power series which follows it.

We will use a special case of the commutation relation in Proposition 2.22 that in this notation is given by the following lemma.

Lemma 2.24. *For any subsets $I, M \subset [\ell(\mu)]$, $J, N \subset [\ell(\nu)]$ and $K, L \subset [m]$ such that $I \cap M = J \cap N = K \cap L = \emptyset$, we have*

$$[\mathcal{E}(I, J, K), \mathcal{E}(M, N, L)] = \zeta\left(\begin{smallmatrix} I & J & K \\ M & N & L \end{smallmatrix}\right) \mathcal{E}\left(I \cup M, J \cup N, K \cup L\right). \tag{2.20}$$

Remark 2.25. As before, Lemma 2.24 is also true if one of the $\mathcal{E}$-operators on the left hand side of the Equation (2.20) is replaced by $\tilde{\mathcal{E}}$.

2.4.2 An algorithm for computation

We say that the operator $\mathcal{E}_n(z)$ has positive (respectively, negative, zero) energy if the integer n is positive (respectively, negative, zero). We see immediately that $\mathcal{E}(I, J, K)|0\rangle$ (respectively, $\langle 0|\mathcal{E}(I, J, K)$) is zero when $\mathcal{E}(I, J, K)$ has positive (respectively, negative) energy. The vacuum expectation value in Equation (2.19) has operators of negative energy on the right and operators of positive energy on the left; by commuting them, we will be able to make use of this observation.

Remark 2.26. Below we present an algorithm that computes the vacuum expectation value on the right hand side of Equation (2.19). It consists of commuting operators of negative energy to the left. Since $\tilde{\mathcal{E}}(\emptyset, \emptyset, k)|0\rangle = 0$, each of the $\tilde{\mathcal{E}}$-operators must be involved in a commutator somewhere in our computations. Then Remark 2.25 implies that we might as well have started with the vacuum expectation value

$$\left\langle \prod_{i=1}^{\ell(\mu)} \mathcal{E}(\{i\}, \emptyset, \emptyset) \prod_{k=1}^{m} \mathcal{E}(\emptyset, \emptyset, \{k\}) \prod_{j=1}^{\ell(\nu)} \mathcal{E}(\emptyset, \{j\}, \emptyset) \right\rangle. \tag{2.21}$$

if we additionally demand that each of the zero energy operators will be involved in a commutator at a certain step of the algorithm.

Now we describe the algorithm. Note that the vacuum expectation value (2.21) is of the form

$$\prod_{q\in Q}\zeta\left(\begin{smallmatrix} F_q & G_q & H_q \\ M_q & N_q & L_q \end{smallmatrix}\right)\left\langle\prod_{t\in T}\mathcal{E}(I_t,J_t,K_t)\right\rangle \tag{2.22}$$

for some finite index sets T and Q and subsets

$$I_t, F_q, M_q \subset [\ell(\mu)], \quad J_t, G_q, N_q \subset [\ell(\nu)] \quad \text{and} \quad K_t, H_q, L_q \subset [m]. \tag{2.23}$$

At the beginning of any step in the algorithm we have vacuum expectation value of this form. Suppose that the operators in the vacuum expectation value of some step do not all have zero energy. Then the step will consist of the following actions.

Let t_0 be the index of the leftmost operator of negative energy. If it is the leftmost operator, then this vacuum expectation value is zero by the remarks above and the algorithm terminates. If it is not, commute it to the left, that is, apply the equality

$$\begin{aligned}&\mathcal{E}(I_{t_0-1},J_{t_0-1},K_{t_0-1})\mathcal{E}(I_{t_0},J_{t_0},K_{t_0})\\&=\mathcal{E}(I_{t_0},J_{t_0},K_{t_0})\mathcal{E}(I_{t_0-1},J_{t_0-1},K_{t_0-1})\\&+[\mathcal{E}(I_{t_0-1},J_{t_0-1},K_{t_0-1}),\mathcal{E}(I_{t_0},J_{t_0},K_{t_0})].\end{aligned} \tag{2.24}$$

The vacuum expectation resulting from the first (respectively, second) term on the right-hand side is called the passing (respectively, commutator) term. By Lemma 2.24, the commutator in the commutator term is equal to

$$\zeta\left(\begin{smallmatrix} I_{t_0-1} & J_{t_0-1} & K_{t_0-1} \\ I_{t_0} & J_{t_0} & K_{t_0} \end{smallmatrix}\right)\mathcal{E}(I_{t_0}\cup I_{t_0-1}, J_{t_0}\cup J_{t_0-1}, K_{t_0}\cup K_{t_0-1}). \tag{2.25}$$

We now choose either the passing term or the commutator term and continue the algorithm with it.

In the end we will sum over the contributions from all possible choices. Because both the passing term and the commutator term are again of the form (2.22), we can iterate this procedure. The algorithm terminates when the result is zero because an operator of negative energy is on the far left, or one of the $\mathcal{E}(\emptyset,\emptyset,\{k\})$ wasn't commuted with any negative energy operators and moved to the far right, or an operator of positive energy is on the far right. It also terminates when all operators in the vacuum expectation have zero energy.

Since taking the passing term results in having an operator of negative energy further to the left and taking the commutator term results in having one less operator in the vacuum expectation value, the algorithm will terminate for any values of the partitions μ and ν and any non-negative integer m.

Remark 2.27. Since we demand $|\mu|=|\nu|$ (otherwise the Hurwitz number $h^m_{\mu,\nu}$ makes no sense), a vacuum expectation value with only one operator can only appear in the algorithm if this operator has zero energy.

Remark 2.28. For definiteness, we have chosen here to always move the left-most operator with negative energy to the left. It is clear, however, that moving any negative energy operator to the left, or any positive energy operator to right, will bring the algorithm closer to termination. In practice, we will often choose to compute Hurwitz numbers in such a different way, since it might simplify the computations. The rest of the discussion in this section still goes through even for those different orderings of taking commutators (see Remark 2.35 for the only subtlety).

2.4.3 Commutation pattern

Using the algorithm in the previous section, we can give an expression for a Hurwitz number in terms of ζ-functions.

Definition 2.29. A commutation pattern P is a set of six-tuples of sets

$$\{(P_I^l, P_J^l, P_K^l, P_M^l, P_N^l, P_R^l)\}_{l\in L(P)} \tag{2.26}$$

(where $L(P) := [|L|]$ is some index set) such that we get a non-vanishing contribution to the vacuum expection value (2.19) when we go through the algorithm in such a way that the l-th commutator computed is $[\mathcal{E}(P_I^l, P_J^l, P_K^l)\ ,\ \mathcal{E}(P_M^l, P_N^l, P_R^l)]$.

The set of all commutation patterns for given values of μ, ν, and m is denoted by $\mathrm{CP}^m_{\mu,\nu}$.

Note that the final vacuum expectation value in any commutation pattern $P \in \mathrm{CP}^m_{\mu,\nu}$ will always be the vacuum expectation of a product of zero energy operators, that is, it will always be of the form $\langle \prod_{t\in T(P)} \mathcal{E}_0(z_{S_t})\rangle$ for some index set $T(P)$ and some non-intersecting subsets $S_t \subset [m]$, $t \in T(P)$ whose union is equal to $[m]$.

Theorem 2.30. *The Hurwitz number $h^m_{\mu,\nu}$ is given by the following formula:*

$$h^m_{\mu,\nu} = \frac{1}{\prod_{i=1}^{\ell(\mu)} \mu_i \prod_{j=1}^{\ell(\nu)} \nu_j}[z_1^{r+1}\cdots z_m^{r+1}]\left(\sum_{P\in\mathrm{CP}^m_{\mu,\nu}} \prod_{t\in T(P)} \frac{1}{\zeta(z_{S_t})} \prod_{l\in L(P)} \zeta\begin{pmatrix} P_I^l & P_J^l & P_K^l \\ P_M^l & P_N^l & P_R^l \end{pmatrix}\right). \tag{2.27}$$

Proof. This follows immediately from Equation (2.19) and the description of the algorithm in Section 2.4.2. We use that $\langle \prod_{t\in T(P)} \mathcal{E}_0(z_{S_t})\rangle = 1/\prod_{t\in T(P)} \zeta(z_{S_t})$ which follows directly from the definition of the operators $\mathcal{E}_0(z)$. □

Remark 2.31. In Theorem 2.30, a factor $\zeta(0)\mathcal{E}_0(0)$ that arises through the contribution of the commutator $[\mathcal{E}_n(0), \mathcal{E}_{-n}(0)]$ in the contribution of a commutation pattern to the Hurwitz number should be interpreted in the following way. When the factor is produced, replace the zero in the argument of the ζ-function by n times a formal variable t, and replace the zero in the argument of the $\mathcal{E}$-operator by t. Then when the whole commutation pattern is completed, let t go to zero. We see immediately that this is the same as replacing the commutator $[\mathcal{E}_n(0), \mathcal{E}_{-n}(0)]$ by the scalar operator n instead of by $\zeta(0)\mathcal{E}_0(0)$. This also agrees with the analysis in [88] (Equation 2.19).

2.4.4 Connected Hurwitz numbers

The algorithm also allows for the computation of the connected Hurwitz number. For this, we need one more definition.

Definition 2.32. A commutation pattern $P \in \mathrm{CP}^m_{\mu,\nu}$ is called *connected* if the set $T(P)$ consists of exactly one element. The set of all connected commutation patterns is denoted by $\mathrm{CP}^{m,\circ}_{\mu,\nu}$.

Theorem 2.33. *The connected Hurwitz number $h^{m,\circ}_{\mu,\nu}$ is equal to*

$$h^{m,\circ}_{\mu,\nu} = \frac{1}{\prod_{i=1}^{\ell(\mu)} \mu_i \prod_{j=1}^{\ell(\nu)} \nu_j}[z_1^{r+1}\cdots z_m^{r+1}]\left(\frac{1}{\zeta(z_{[m]})}\sum_{P\in\mathrm{CP}^{m,\circ}_{\mu,\nu}} \prod_{l\in L(P)} \zeta\begin{pmatrix} P_I^l & P_J^l & P_K^l \\ P_M^l & P_N^l & P_R^l \end{pmatrix}\right). \tag{2.28}$$

Proof. Since $\mathcal{E}_0(z)$ is a scalar operator, for any commutation pattern $P \in \mathrm{CP}^m_{\mu,\nu}$ we have $\langle \prod_{t\in T(P)} \mathcal{E}_0(z_{S_t})\rangle = \prod_{t\in T(P)} \langle \mathcal{E}_0(z_{S_t})\rangle$. Furthermore, when $|T(P)| \geq 2$, the operators from the start of the algorithm contributing to $\mathcal{E}_0(z_{S_t})$ for different $t \in T(P)$ do not interact with each other at all. That is, a commutator term involving operators eventually contributing to $\mathcal{E}_0(z_{S_t})$ for different t is never taken. Therefore, for any integer $n \geq 1$, the contribution to $h^m_{\mu,\nu}$ by covers with at least n connected components is given exactly by the contribution to Equation (2.27) of commutation patterns P for which $|T(P)| \geq n$.

That means that the exclusion-inclusion formula for the connected Hurwitz number in terms of ordinary Hurwitz numbers coincides precisely with the exclusion-inclusion formula for the contribution to Equation (2.27) by commutation patterns P with $|T(P)| = 1$ in terms of contributions by arbitrary commutation paterns. This completes the proof. □

Definition 2.34. More generally, the contribution of a connected commutation pattern to the vacuum expectation value of a product of $\mathcal{E}$-operators is called a *connected contribution*. The sum of all connected contributions of any such product is called a *connected vacuum expectation value* and denoted by $\langle\cdot\rangle^\circ$ (adding a super-script circle to the full vacuum expectation value).

Remark 2.35. Even if we take a different ordering of the commutators in the algorithm (see Remark 2.28), the connected vacuum expectation value will still be the sum of the connected contributions, that is, the contributions from commutation patterns that have only one $\mathcal{E}$-operator in the final expectation value. This can be easily seen through the method of the proof of Theorem 2.33.

2.4.5 Example: one-part double Hurwitz numbers

In the case of one-part double Hurwitz numbers, that is, $\ell(\nu) = 1$, we have just one commutation pattern. Therefore, applying Equation (2.27) (or, equivalently, Equation (2.28)), we obtain

$$h^m_{\mu,|\mu|} = \frac{1}{|\mu|\prod_{i=1}^{\ell(\mu)}\mu_i}[z_1^{r+1}\cdots z_m^{r+1}]\frac{\prod_{k=1}^m \zeta(|\mu|z_k)\prod_{i=1}^{\ell(\mu)}\zeta(\mu_i z_{[m]})}{\zeta(z_{[m]})} \tag{2.29}$$

2.4.6 Example: coefficients of the completed cycles

The coefficients of the completed cycles are obtained from Theorem 2.33 in the case $m = 1$, $\nu = (1,\dots,1)$. In that case there is only one connected commutation pattern, and we get for the coefficient of the partition μ in the completed $r+1$-cycle.

$$\frac{h^{1,\circ}_{\mu,(1,\dots,1)}}{|C_\mu||\mathrm{Aut}(\mu)|} = \frac{1}{|\mu|!}[z^{r+1}]\zeta(z)^{|\mu|-1}\prod_{i=1}^{\ell(\mu)}\zeta(\mu_i z), \tag{2.30}$$

(where $|\mathrm{Aut}(\mu)|$ is the number of automorphisms of the partition μ and $|C_\mu|$ is the size of the conjugacy class C_μ). This formula agrees with the one given in [88, Equation (0.22)] modulo the diffences in conventions and notation. As an example, we compute the coefficients of the different terms in the completed 4-cycle. Since the power series expansion of ζ is given by

$$\zeta(z) = \sum_{k=1}^{\infty}\frac{z^k}{2^{k-1}k!} \tag{2.31}$$

we get

$$\frac{h^{1,\circ}_{(4),(1,1,1,1)}}{|C_{(4)}||\mathrm{Aut}((4))|} = \frac{1}{4!}[z^4]\zeta(z)^3\zeta(4z) = \frac{1}{4!}4 = \frac{1}{3!}, \tag{2.32}$$

$$\frac{h^{1,\circ}_{(2,1),(1,1,1)}}{|C_{(2,1)}||\mathrm{Aut}((2,1))|} = \frac{1}{3!}[z^4]\zeta(z)^3\zeta(2z) = \frac{1}{3!}2 = \frac{2}{3!}, \tag{2.33}$$

$$\frac{h^{1,\circ}_{(2),(1,1)}}{|C_{(2)}||\mathrm{Aut}((2))|} = \frac{1}{2!}[z^4]\zeta(z)^1\zeta(2z) = \frac{1}{2}(\frac{1}{24}2 + \frac{2^3}{24}) = \frac{1}{3!}\frac{5}{4} \tag{2.34}$$

which agrees with the coefficients of the completed 4-cycle given just below Definition 2.4. All the coefficients there can be computed in this way.

2.5 The cut-and-join operators

2.5.1 Three vector spaces

Let $p_1, p_2, \dots$ be an infinite sequence of formal variables. Given a permutation $\sigma \in S_d$ with cycle lengths $n_1, \dots, n_s$ denote by $p(\sigma)$ the product $p_{n_1} \cdots p_{n_s}$. If we assign the weight n to the variable p_n, then the monomial $p(\sigma)$ is of total weight d. The map p is extended by linearity to an isomorphism

$$p\colon Z\mathbb{C}S_d \mapsto \mathbb{C}_n[p_1, \dots, p_d], \tag{2.35}$$

where $Z\mathbb{C}S_d$ is the center of the group algebra of the symmetric group S_d and $\mathbb{C}_d[p_1, \dots, p_d]$ is the space of quasi-homogeneous polynomials of weight d. This is an isomorphism of vector spaces, but not of algebras, since the target space of p does not have a natural algebra structure. However the action of elements of $Z\mathbb{C}S_d$ by multiplication gives rise to interesting operators in the space of homogeneous polynomials. In particular, the map p transforms the operator of multiplication by the sum of all transpositions into the well-known cut-and-join operator:

$$\frac{1}{2}\sum_{i,j\geq 1}\left(ijp_{i+j}\frac{\partial^2}{\partial p_i \partial p_j} + (i+j)p_i p_j \frac{\partial}{\partial p_{i+j}}\right). \tag{2.36}$$

The first term corresponds to the case where two cycles of lenghts i and j are merged together by the transposition; the second term corresponds to the case where a cycle of length $i + j$ is cut into two cycles of lengths i and j.

The operator corresponding to the mutiplication by the sum of all $(r + 1)$-cycles was determined for every r by Goulden and Jackson in [49] and by Mironov, Morozov, and Natanzon in [79] and [80].

The space $\mathbb{C}[[p_1, p_2, \dots]]$ of formal power series in variables $p_1, p_2, \dots$ is a completion of the direct sum of the spaces of quasi-homogeneous polynomials. It is natually isomorphic to a completion of the direct sum of spaces $Z\mathbb{C}S_d$. Moreover, both vector spaces are naturally identified with the infinite wedge space via the isomorphism $v_\lambda \leftrightarrow s_\lambda(p)$, where s_λ is the Schur polynomial:

$$s_\lambda(p) = \frac{1}{d!}\sum_{\sigma\in S_d}\chi_\lambda(\sigma)p(\sigma). \tag{2.37}$$

Under this identification, the multiplication by the completed $(r + 1)$-cycle corresponds to the operator

$$\mathcal{F}_{r+1} = \frac{1}{(r+1)!}\sum_{k\in\mathbb{Z}+1/2} k^{r+1}E_{k,k} \tag{2.38}$$

in the infinite wegde space. Our goal is to construct the corresponding cut-and-join operator in the space $\mathbb{C}[[p_1, p_2 \dots,]]$.

2.5.2 The construction of operators

For $n \geq 1$, let $a_{-n} = p_n$ be the operator of multiplication by p_n and $a_n = n\,\partial/\partial p_n$. We let $a_0 = 0$. In the infinite wedge space the operator a_n becomes $\alpha_n = \tilde{\mathcal{E}}_n(0) = \sum_{k\in\mathbb{Z}+1/2} E_{k-n,k}$ (see Notation 2.19). For $n < 0$ it transforms v_λ into $\sum \epsilon(\mu)v_\mu$, where the Young diagrams μ are all diagrams that can be obtained from λ by adding a ribbon of length n and the sign $\epsilon(\mu)$ is the number of horizontal steps in the ribbon, as in the Murnaghan-Nakayama rule [57]. Similarly, for $n > 0$ it transforms v_λ into $\sum \epsilon(\mu)v_\mu$, where the Young diagrams μ are all diagrams that can be obtained from λ by removing a ribbon of length n and the sign $\epsilon(\mu)$ is the number of horizontal steps in the ribbon.

The *normal ordering* $:a_{n_1}\cdots a_{n_s}:$ of a monomial $a_{n_1}\cdots a_{n_s}$ is the non-decreasing order of indices; in other words the derivations go to the right and the multiplication operators to the left. Recall (Notation 2.17) that $\zeta(z) = e^{z/2} - e^{-z/2}$.

Definition 2.36. The coefficients $Q_1, Q_2, \dots$ of the series

$$Q_1 z + Q_2 z^2 + \cdots = \frac{1}{\zeta(z)} \sum_{s\geq 1} \frac{1}{s!} \sum_{n_1+\cdots+n_s=0} \zeta(n_1 z)\cdots\zeta(n_s z)\, \frac{:a_{n_1}\cdots a_{n_s}:}{n_1\cdots n_s} \tag{2.39}$$

are called the *completed cut-and-join operators.*

Theorem 2.37. *The map $p : Z\mathbb{C}S_d \to \mathbb{C}_d[p_1, \dots, p_d]$ transforms the operator of multiplication by the completed $(r+1)$-cycle into the $(r+1)^{\text{st}}$ completed cut-and-join operator.*

For instance, we have

$$Q_1 = \sum_{i\geq 1} i p_i \frac{\partial}{\partial p_i}, \tag{2.40}$$

$$Q_2 = \frac{1}{2} \sum_{i,j\geq 1} \left(ij p_{i+j} \frac{\partial^2}{\partial p_i \partial p_j} + (i+j) p_i p_j \frac{\partial}{\partial p_{i+j}} \right), \tag{2.41}$$

$$Q_3 = \frac{1}{6} \sum_{i,j,k\geq 1} \left(ijk p_{i+j+k} \frac{\partial^3}{\partial p_i \partial p_j \partial p_k} + (i+j+k) p_i p_j p_k \frac{\partial}{\partial p_{i+j+k}} \right) \tag{2.42}$$

$$+ \frac{1}{4} \sum_{i+j=k+l} ij p_k p_l \frac{\partial^2}{\partial p_i \partial p_j} + \frac{1}{24} \sum_{i\geq 1} (2i^3 - i) p_i \frac{\partial}{\partial p_i}. \tag{2.43}$$

The multiplication of $\sigma \in S_d$ by the completed cycle $\overline{(1)} = (1)$ corresponds to picking an element of σ, which just mutiplies the permutation by d. Hence the operator Q_1 multiplies a homogeneous polynomial by its total weight.

The operator Q_2 is the standard cut-and-join operator.

The operator Q_3 is the more complicated cut-and-join operator whose action corresponds to multiplying a permutation by the completed 3-cycle. Let us briefly explain how its terms are related to the expression of the completed 3-cycle $\frac{1}{2}(3) + \frac{1}{2}(1,1) + \frac{1}{24}(1)$.

The last term (1) is the operator of picking a sheet of the ramified covering or an element of the permutation. So $\frac{1}{24}(1)$ corresponds to

$$\frac{1}{24} \sum i p_i \frac{\partial}{\partial p_i}. \tag{2.44}$$

Here i is the length of the cycle of the permutation that contains the picked element.

The second term $(1,1)$ is the operator of picking two sheets of the covering or two elements of the permutation. So $\frac{1}{2}(1,1)$ corresponds to

$$\frac{1}{4}\sum_{i,j\geq 1} ijp_ip_j\frac{\partial^2}{\partial p_i \partial p_j} + \frac{1}{4}\sum_{i\geq 1} i(i-1)p_i\frac{\partial}{\partial p_i}. \tag{2.45}$$

The first sum describes the case when the chosen elements belong to two different cycles of lengths i and j, while the second sum describes the case where they lie in the same cycle of length i.

Finally, the term (3) is the operator of multiplication by a 3-cycle. It's action is more complicated. If the elements of the 3-cycle belong to three different cycles of the permutations of lenghts i, j, k, these cycles are merged into one. This gives us the term

$$\frac{1}{6}\sum_{i,j,k\geq 1} ijkp_{i+j+k}\frac{\partial^3}{\partial p_i \partial p_j \partial p_k}. \tag{2.46}$$

If one element lies in one cycle and two other elements lie in another cycle, then a piece of the second cycle is cut off and attached to the first one. But there is a subtlety: there are two ways to go from cycles of lengths, say, 2 and 19 to cycles of lengths 9 and 12: one can either take a 7-elements piece of the 19-cycle and attach it to the 2-cycle, or one can take a 10-elements piece. On the other hand, to go from cycles of lengths 2 and 19 to cycles of lengths 2 and 19 again there is only one way: one should take a 17-element piece from the 19-cycle and attach it to the 2-cycle. As a result, we get the following sum:

$$\frac{1}{4}\sum_{\substack{i+j=k+l\\ \{i,j\}\neq\{k,l\}}} ijp_kp_l\frac{\partial^2}{\partial p_i \partial p_j} + \frac{1}{4}ijp_ip_j\frac{\partial^2}{\partial p_i \partial p_j}. \tag{2.47}$$

Finally, all three elements of the 3-cycle can lie in the same cycle of the permutation. If the cycles "turn" in two opposite ways, the cycle of the permutation is split into three parts and we get the term

$$\frac{1}{6}\sum_{i,j,k\geq 1} (i+j+k)p_ip_jp_k\frac{\partial}{\partial p_{i+j+k}}. \tag{2.48}$$

If both cycles "turn" in the same direction, then the cycle of the permutation remains in one piece, though the order of elements changes. This corresponds to the operator

$$\frac{1}{12}\sum_{i\geq 1} i(i-1)(i-2)p_i\frac{\partial}{\partial p_i}. \tag{2.49}$$

The reader can check that if we add all these terms we recover the operator Q_3.

Remark 2.38. As we see from this example, the operators Q_k, $k \geq 3$, can also be obtained as linear combinations of the operators $W(\lambda)$ that correspond to the partitions λ in the expressions for the completed cycles. These operators are discussed in detail in [79] and [80]. This approach to the cut-and-join operators for completed cycles is discussed in a very recent preprint [7], and it seems that so far it doesn't allow to reproduce the explicit formulas that we have here for an arbitrary k.

2.5.3 The generating series for Hurwitz numbers

Introduce the following generating series for the disconnected Hurwitz numbers with completed cycles:

$$H_{r+1}(\beta, p_1, p_2, \dots, q_1, q_2, \dots) = \sum_{\ell_1, \ell_2, m} \sum_{\substack{\mu_1,\dots,\mu_{\ell_1} \\ \nu_1,\dots,\nu_{\ell_2}}} h^{(r+1)}_{g,\mu,\nu} \frac{\beta^m}{m!} \frac{p_{\mu_1}\cdots p_{\mu_{\ell_1}}}{\ell_1!} \frac{q_{\nu_1}\cdots q_{\nu_{\ell_2}}}{\ell_2!}. \tag{2.50}$$

Here, as before, $m = (2g-2+\ell_1+\ell_2)/r$ and, by convention, the summands with $\sum \mu_i \neq \sum \nu_i$ are set to 0.

It follows from the discussion in Section 2.3 and [88] that H_{r+1} is a tau-function of the KP hierarchy in both sets of variables p_i and q_i, where $i \geq 1$.

Theorem 2.39. *The series H_{r+1} satisfies the partial differential equation*

$$\frac{\partial H_{r+1}}{\partial \beta} = Q_{r+1} H_{r+1}. \tag{2.51}$$

This theorem is actually an equivalent formulation of Theorem 2.37.

2.5.4 Proofs

Now we are going to prove Theorem 2.39 and hence the equivalent Theorem 2.37.

According to Proposition 2.23, we have

$$H_{r+1} = \left\langle \exp\left(\sum_{n\geq 1} p_n \alpha_n/n\right) \exp(\beta F_{r+1}) \exp\left(\sum_{n\geq 1} q_n \alpha_{-n}/n\right)\right\rangle. \tag{2.52}$$

Hence,

$$\frac{\partial H_{r+1}}{\partial \beta} = \left\langle \exp\left(\sum_{n\geq 1} p_n \alpha_n/n\right) F_{r+1} \exp(\beta F_{r+1}) \exp\left(\sum_{n\geq 1} q_n \alpha_{-n}/n\right)\right\rangle. \tag{2.53}$$

We will prove several lemmas that will allow us to simplify the above expression and to relate it to the cut-and-join operator Q_{r+1}. First of all recall (Lemma 2.24) that

$$[\tilde{\mathcal{E}}_a(z), \tilde{\mathcal{E}}_b(w)] = \zeta(aw - bz)\tilde{\mathcal{E}}_{a+b}(z+w) \tag{2.54}$$

and (Notation 2.19) that

$$\alpha_n = \tilde{\mathcal{E}}_n(0), \qquad \tilde{\mathcal{E}}_0(z) = \sum F_{r+1} z^{r+1}. \tag{2.55}$$

Lemma 2.40. *We have*

$$\exp\left(\sum_{n\geq 1} p_n \alpha_n/n\right) \tilde{\mathcal{E}}_0(z) \exp\left(-\sum_{n\geq 1} p_n \alpha_n/n\right) \tag{2.56}$$

$$= \tilde{\mathcal{E}}_0(z) + \frac{1}{1!}\sum_{i=1}^{\infty} \zeta(iz)\frac{p_i}{i}\tilde{\mathcal{E}}_i(z) + \frac{1}{2!}\sum_{i,j=1}^{\infty} \zeta(iz)\zeta(jz)\frac{p_i}{i}\frac{p_j}{j}\tilde{\mathcal{E}}_{i+j}(z)$$

$$+ \frac{1}{3!}\sum_{i,j,k=1}^{\infty} \zeta(iz)\zeta(jz)\zeta(kz)\frac{p_i}{i}\frac{p_j}{j}\frac{p_k}{k}\tilde{\mathcal{E}}_{i+j+k}(z) + \dots$$

Proof. The map $u \mapsto e^x u e^{-x}$ is the exponent of the map $u \mapsto [x, u]$. Substituing $u = \tilde{\mathcal{E}}_0(z)$, $x = \sum_{n\geq 1} p_n \alpha_n / n$ and using the commutation relations for the operators $\tilde{\mathcal{E}}$ we obtain the above formula. □

Lemma 2.41. *Let*

$$O_K(z) = \sum_{s=1}^{\infty} \frac{1}{s!} \sum_{n_1+\cdots+n_s=K} \frac{\zeta(n_1 z)\cdots\zeta(n_s z)}{\zeta(z)} \frac{\partial}{\partial p_{n_1}} \cdots \frac{\partial}{\partial p_{n_s}}. \tag{2.57}$$

Then we have

$$\sum_{r\geq 0} Q_{r+1} z^{r+1} = \sum_{s=1}^{\infty} \frac{1}{s!} \sum_{n_1,\ldots,n_s>0} \zeta(n_1 z)\cdots\zeta(n_s z) \frac{p_{n_1}}{n_1} \cdots \frac{p_{n_s}}{n_s} O_{n_1+\cdots+n_s}(z). \tag{2.58}$$

Proof. This is obtained by a simple computation. □

Lemma 2.42. *Let X be any operator in the infinite wedge space independent of p_1, p_2, ... Then*

$$\left\langle \tilde{\mathcal{E}}_K(z) \exp\left(\sum_{n\geq 1} p_n \alpha_n / n\right) X \right\rangle = O_K(z) \left\langle \exp\left(\sum_{n\geq 1} p_n \alpha_n / n\right) X \right\rangle. \tag{2.59}$$

Proof. It is enough to prove the lemma under the assumption that $X(v_\emptyset) = v_\lambda$. The general case is obtained by taking a linear combination of operators X like that. Let $|\lambda| = N$.

We are going to evaluate the right-hand side of the equality and simplify it finally obtaining the left-hand side. Note that the vacuum expectation value $\langle \exp\left(\sum_{n\geq 1} p_n \alpha_n / n\right) v_\lambda \rangle$ is equal to the Schur polynomial s_λ which is conveniently written as

$$\frac{1}{N!} p \left(\sum_{\sigma\in S_N} \chi_\lambda(\sigma)\sigma\right). \tag{2.60}$$

The action of the operator $O_K(z)$ has a natural interpretation in terms of permutations: the operator picks (in all possible ways) a set of cycles of σ with total length K. These cycles will be called distinguished. A distinguished cycle of length n is assigned a factor of $\zeta(nz)$. A non-distinguished cycle of length n is assigned a factor p_n as before. To σ is assigned the product of these factors. And the result of the action of $O_K(z)$ is the sum of the contributions of all permutations $\sigma \in S_N$ divided by $N!$ and by $\zeta(z)$.

The summation over all permutations $\sigma \in S_N$ with a set of distinguished cycles of total length K can be replaced by a summation over the permutations whose distinguished cycles cover the elements from 1 to K. A permutation like that actually lies in $S_K \times S_{N-K}$. The contribution of permutations like that is $N!/K!(N-K)!$ times smaller than the contribution of all permutations, so the new sum should be divided by $K!(N-K)!$ instead of $N!$.

We can decompose the representation λ of S_N into a direct sum of representations of $S_K \times S_{N-K}$ as follows:

$$\bigoplus_{\mu\subset\lambda} \mu \otimes (\lambda \setminus \mu). \tag{2.61}$$

Here μ denotes an irreducible representation of S_K corresponding to a Young diagram included in λ and $\lambda \setminus \mu$ is the (possibly reducible) representation $\mathrm{Hom}_{S_K}(\mu, \lambda)$ of S_{N-K}. Using this decomposition we obtain:

$$\mathrm{RHS} = \frac{1}{\zeta(z)} \sum_{\mu\subset\lambda} s_\mu(\zeta(z), \zeta(2z), \ldots) \cdot s_{\lambda\setminus\mu}(p_1, p_2, \ldots). \tag{2.62}$$

This expression can be further simplified using the following lemma.

A partition is called a *hook* if it has the form $\mu_{a,b} = (a\,1^b)$ for $a, b \geq 0$. This name is due to the shape of the corresponding Young diagram.

Lemma 2.43. *We have*

$$\frac{1}{\zeta(z)} s_\mu(\zeta(z), \zeta(2z), \dots) = (-1)^b e^{(a-b-1)z/2} \tag{2.63}$$

if $\mu = \mu_{a,b}$ *and* $s_\mu(\zeta(z), \zeta(2z), \dots) = 0$ *otherwise.*

We will prove this lemma later; at present we continue to simplify the right-hand side of the equality of Lemma 2.42. Using Lemma 2.43 we get

$$\mathrm{RHS} = \sum_{a+b=K} (-1)^b e^{(a-b-1)z/2} s_{\lambda\setminus\mu}(p_1, p_2, \dots). \tag{2.64}$$

Now let us explain why this coincides with the left-hand part. The vector $\tilde{\mathcal{E}}_K(z)(\mu)$ has a non-zero $v_\emptyset$ component only when μ is a hook partition. In that case, we have

$$\tilde{\mathcal{E}}_K(z)(\mu_{a,b}) = (-1)^b e^{(a-b-1)z/2} v_\emptyset. \tag{2.65}$$

We also need to recall (see, for instance, [106]) that the Schur polynomial of the representation $\lambda \setminus \mu$ of S_{N-K} is obtained by the Murnaghan-Nakayama rule, that is, it is equal to

$$s_{\lambda\setminus\mu}(p_1, p_2, \dots) = \left\langle v_\mu \exp\left(\sum_{k\geq 1} \alpha_k p_k / k\right) v_\lambda \right\rangle. \tag{2.66}$$

Therefore we have

$$\begin{aligned} \mathrm{LHS} &= \sum_\mu \langle v_\emptyset \tilde{\mathcal{E}}_K(z) v_\mu \rangle \left\langle v_\mu \exp\left(\sum_{k\geq 1} \alpha_k p_k / k\right) v_\lambda \right\rangle \\ &= \sum_{a+b=K} (-1)^b e^{(a-b-1)z/2} s_{\lambda\setminus\mu_{a,b}}(p_1, p_2, \dots). \end{aligned} \tag{2.67}$$

□

Now we prove Lemma 2.43.

Proof. We use the well-known identity for Schur polynomials

$$p_n s_\mu = \sum \pm s_\lambda, \tag{2.68}$$

where the sum is taken over the Young diagrams λ obtained by adding a ribbon of length n to μ and the sign $\pm$ is the parity of the number of downward steps in the ribbon. (This is a reformulation of the action of the operator α_{-n}.) First let us check that this formula is compatible with the claim of the theorem.

If μ is not a hook, then neither of the λ's will be a hook. So after the substitution $p_n = \zeta(nz)$ we get the correct equality

$$\zeta(nz) \cdot 0 = \sum \pm 0. \tag{2.69}$$

If $\mu = \mu_{a,b}$ then there are exactly two ways to add an n-ribbon to μ in such a way that it remains a hook: we can increase either a or b by n. In the first case the sign of the ribbon is $+1$, in the second case it is $(-1)^{n-1}$. Thus we get the correct equality

$$\zeta(nz) \cdot (-1)^b e^{(a-b-1)z/2} = (-1)^b e^{(a+n-b-1)z/2} + (-1)^{n-1}(-1)^{b+n} e^{(a-b-n-1)z/2}. \tag{2.70}$$

Now, every Schur polynomial can be obtained as a linear combination of the form

$$\sum_i c_i p_{n_i} s_{\lambda_i}. \tag{2.71}$$

For instance, we can use the formula

$$s_\mu = \frac{1}{|\mu|} \sum p_n \frac{n\, \partial s_\mu}{\partial p_n} \tag{2.72}$$

and decompose every $n\, \partial s_\mu / \partial p_n$ into a linear combination of Schur polynomials. These expressions allow us to find the Schur polynomials before or after the substitution $p_n = \zeta(nz)$ by induction on the degree. The equality

$$\frac{1}{\zeta(z)} s_1(\zeta(z), \zeta(2z), \dots) = \frac{\zeta(z)}{\zeta(z)} = 1 = (-1)^0 e^{1-0-1} \tag{2.73}$$

provides the base of induction. Since we know these inductive relations are compatible with the formula given in the lemma, we conclude that the lemma is true. $\square$

Finally, Theorem 2.39 follows immediately from Lemmas 2.40, 2.41, 2.42. Indeed, in the expression (2.53) for $\partial H_{r+1}/\partial \beta$, the operator F_{r+1} is the coefficient of z^{r+1} in $\tilde{\mathcal{E}}_0(z)$. Using Lemma 2.40 we obtain that $\partial H_{r+1}/\partial \beta$ is the coefficient of z^{r+1} in

$$\begin{aligned} &\sum_{s \geq 0} \frac{1}{s!} \sum_{n_1, \dots, n_s} \prod_{i=1}^{s} \frac{\zeta(n_i z) p_{n_i}}{n_i} \\ &\times \left\langle \tilde{\mathcal{E}}_{\sum n_i}(z) \exp\left(\sum_{n \geq 1} p_n \alpha_n / n \right) \exp(\beta F_{r+1}) \exp\left(\sum_{n \geq 1} q_n \alpha_{-n}/n \right) \right\rangle. \end{aligned} \tag{2.74}$$

According to Lemma 2.42 this is equal to

$$\begin{aligned} &\sum_{s \geq 0} \frac{1}{s!} \sum_{n_1, \dots, n_s} \prod_{i=1}^{s} \frac{\zeta(n_i z) p_{n_i}}{n_i} O_{\sum n_i}(z) \\ &\times \left\langle \exp\left(\sum_{n \geq 1} p_n \alpha_n / n \right) \exp(\beta F_{r+1}) \exp\left(\sum_{n \geq 1} q_n \alpha_{-n}/n \right) \right\rangle \\ &= \sum Q_{r+1} z^{r+1} H_{r+1}, \end{aligned} \tag{2.75}$$

where in the last equality we have used Lemma 2.41 for the expression of Q_{r+1} and Equation (2.52) for H_{r+1}. Extracting the coefficient of z^{r+1}, we get

$$\frac{\partial H_{r+1}}{\partial \beta} = Q_{r+1} H_{r+1} \tag{2.76}$$

as claimed.

2.6 Strong piecewise polynomiality

In this section, we prove an analogue of strong piecewise polynomiality for Hurwitz numbers with completed $(r+1)$-cycles and derive the wall crossing formulas for this piecewise polynomial. It is a generalization of Johnson's results in [59].

2.6.1 Notation

Throughout this section we fix two positive integers ℓ_1 and ℓ_2. Let V be the subset of $(\mathbb{Z}_{\geq 0})^{\ell_1} \oplus (\mathbb{Z}_{\geq 0})^{\ell_2}$ defined by

$$V := \left\{ (x_1, \dots, x_{\ell_1}, y_1, \dots, y_{\ell_2}) \;\middle|\; \sum_{i=1}^{\ell_1} x_i = \sum_{j=1}^{\ell_2} y_j \right\} \tag{2.77}$$

We consider double Hurwitz number with m completed $(r+1)$-cycles as a function $h^m \colon V \to \mathbb{Q}$ such that $h^m(\mu,\nu) = h^m_{\mu,\nu}$

Definition 2.44. Let $I \subset [\ell_1]$ and $J \subset [\ell_2]$ be any non-empty proper subsets. Then the hyperplane

$$\{(x,y) \in V \mid x_I - y_J = 0\} \subset V \tag{2.78}$$

is called the hyperplane given by I and J and denoted $W_{I,J}$.

Remark 2.45. Consider a pair $(\mu,\nu) \in V$ such that it does not lie on any of the hyperplanes $W_{I,J}$. Then $h^m_{\mu,\nu} = h^{m,\circ}_{\mu,\nu}$, since there are no covers of $\mathbb{P}^1$ with ramification over 0 and ∞ given by μ and ν with more than one connected component. Thus, if we interpret the $W_{I,J}$ as the walls of a hyperplane arrangement, then at the internal points of the chambers of this arrangement the disconnected and connected Hurwitz numbers are equal.

2.6.2 Polynomiality in a chamber

Theorem 2.46. *The function $h^m \colon V \to \mathbb{Q}$ is a piecewise polynomial function with the walls given by the hyperplanes $W_{I,J}$.*

Proof. The proof is analogous to the one in [59] of piecewise polynomiality for ordinary double Hurwitz numbers. Let $\mathfrak{c}$ be some chamber of the hyperplane arrangement mentioned in Remark 2.45. We have to prove that $h^m|_{\mathfrak{c}}$ is polynomial.

The crucial point is that the set of commutation patterns $\mathrm{CP}^m_{\mu,\nu}$ does not depend on $(\mu,\nu) \in \mathfrak{c}$, but only on the chamber $\mathfrak{c}$ itself. Note that for any subsets $I \subset [\ell_1]$ and $J \subset [\ell_2]$ the sign of the number $\mu_I - \nu_J$ is determined by the chamber $\mathfrak{c}$ containing (μ,ν), and it is these signs which determine the set of commutation patterns for (μ,ν). Thus, $\mathrm{CP}^m_{\mu,\nu}$ depends only on the chamber $\mathfrak{c}$ containing (μ,ν). From now on, we will denote it by $\mathrm{CP}(\mathfrak{c})$.

Furthermore, since we are in a chamber of the hyperplane arrangement, by Remark 2.45, the connected and disconnected Hurwitz numbers are equal, so $|T(P)| = 1$ for any commutation pattern in $\mathrm{CP}(\mathfrak{c})$. Thus, the Hurwitz number is determined by Theorem 2.33 instead of Theorem 2.30. Let us prove that the factor

$$\frac{1}{\zeta(z_{[m]})} \sum_{P \in \mathrm{CP}(\mathfrak{c})} \prod_{l \in L(P)} \zeta \begin{pmatrix} P^l_I & P^l_J & P^l_K \\ P^l_M & P^l_N & P^l_R \end{pmatrix} \tag{2.79}$$

in Equation (2.28) restricted to $\mathfrak{c}$ is a power series in $z_1, \dots z_m$ with coefficients depending polynomially on (μ,ν). Indeed, the only problem is the factor $1/\zeta(z_{[m]})$. Meanwhile, in any commutation pattern in $\mathrm{CP}(\mathfrak{c})$ the last commutator taken will produce a factor $\zeta(nz_{[m]})$ for some integer n. Since $\zeta(nz)/\zeta(z)$ is holomorphic at $z=0$, the whole expression is indeed a power series in $z_1,\dots,z_m$. Clearly, the coefficients depend polynomially on (μ,ν).

Therefore, the coefficient of $z_1^{r+1}\cdots z_m^{r+1}$ will depend polynomially on μ and ν and it only remains to show that it contains a factor $\prod_{i=1}^{\ell_1}\mu_i\prod_{j=1}^{\ell_2}\nu_j$. Indeed, in the vacuum expectation value at the start of the algorithm (equation (2.19)) we have the operator $\mathcal{E}(\{i\},\emptyset,\emptyset)$ for all $i=1,\dots,\ell_1$. In any commutation pattern $P\in\mathrm{CP}(\mathfrak{c})$ it will eventually be commuted with some operator, which will provide a factor $\zeta(\mu_i z_L)$ for a certain subset $L\subset[m]$ that is divisible by μ_i. The same argument also works for ν_j, for all $j=1,\dots,\ell_2$, and we see from this argument that the product $\prod_{i=1}^{\ell_1}\mu_i\prod_{j=1}^{\ell_2}\nu_j$ also divides the product of ζ-functions on the right side of Equation (2.28). Thus, h_g is polynomial on any chamber $\mathfrak{c}$, which proves the theorem. □

2.6.3 The structure of the polynomial

Theorem 2.47. *Let $\mathfrak{c}$ be a chamber of the hyperplane arrangement of Remark 2.45. Then $h^m|_{\mathfrak{c}}$ has the following form:*

$$h^m|_{\mathfrak{c}}(\mu,\nu)=\sum_{k=0}^{g}(-1)^k P^m_{\mathfrak{c},k}(\mu,\nu), \tag{2.80}$$

where $P^m_{\mathfrak{c},k}\colon V\to\mathbb{Q}$ is a homogeneous polynomial of degree $(r+1)m+1-\ell(\mu)-\ell(\nu)-2k$ with $P^m_{\mathfrak{c},k}(\mu,\nu)>0$ for all $(\mu,\nu)\in\mathfrak{c}$, and $g=(rm-\ell(\mu)-\ell(\nu)+2)/2$ is the genus of the covering.

Proof. Since we know that $h^m|_{\mathfrak{c}}$ is a polynomial, we only need to prove that it is either even or odd, that it is of degree $(r+1)m+1-\ell(\mu)-\ell(\nu)$, that the lowest occurring term is of degree $2g$ less than the highest, and the alternating nature of the homogeneous terms. It is clear that $h^m|_{\mathfrak{c}}$ is either even or odd from Theorem 2.33 and the fact that $\zeta(z)$ is an odd function.

Let $P\in\mathrm{CP}(\mathfrak{c})$ be a commutation pattern. Then for any $l\in L(P)$ we will always have that either P^l_K or P^l_R is non-empty (it is easy to see the vacuum expectation will be zero otherwise, contradicting that P is a commutation pattern). Therefore $\zeta\left(\begin{smallmatrix}P^l_I & P^l_J & P^l_K\\ P^l_M & P^l_N & P^l_R\end{smallmatrix}\right)$ has equal total order in (μ,ν) and $(z_1,\dots z_m)$. So the highest total order in (μ,ν) in $h^m|_{\mathfrak{c}}$ occurs when we take the lowest possible total order in $(z_1,\dots,z_m)$ in the factor $1/\zeta(z_1+\cdots+z_m)$ in Equation (2.28); which is equal to -1. Thus, the highest occuring order in (μ,ν) is the total degree in $(z_1,\dots,z_s)$ (which is $m(r+1)$), plus 1, minus the degree in (μ,ν) of $\prod_{i=1}^{\ell(\mu)}\mu_i\prod_{j=1}^{\ell(\nu)}\nu_j$, for a total degree of $(r+1)m+1-\ell(\mu)-\ell(\nu)$.

The lowest degree term in (μ,ν) of $h^m|_{\mathfrak{c}}$ occurs when we take the lowest possible degree in z in $\zeta\left(\begin{smallmatrix}P^l_I & P^l_J & P^l_K\\ P^l_M & P^l_N & P^l_R\end{smallmatrix}\right)$ for all $l\in L(P)$; which is equal to 1. Therefore, the lowest occurring degree in (μ,ν) is equal to the number of commutator terms taken in P. Taking a commutator term reduces the number of $\mathcal{E}$-operators by one. We start with $\ell(\mu)+\ell(\nu)+m$ of these operators and we end up with one. Therefore, the number of commutator terms taken is equal to $\ell(\mu)+\ell(\nu)+m-1$. Thus, the lowest degree in (μ,ν) occuring is $\ell(\mu)+\ell(\nu)+m-1-\ell(\mu)-\ell(\nu)=m-1$.

By the Riemann-Hurwitz formula, the difference between the highest and the lowest degree in (μ,ν) is then equal to $(r+1)m+1-\ell(\mu)-\ell(\nu)-m+1=2g$.

The coefficients of the homogeneous summands of the expansion of $1/\zeta(z_{[m]})$ have alternating signs. Therefore, to prove positivity of $P^m_{\mathfrak{c},k}$ it is enough to prove that all coefficients of odd-degree terms in the expansion of the ζ-functions in the product in Equation (2.28) are positive. By definition of the algorithm, we can only get a factor

$$\zeta\left(\begin{smallmatrix}P^l_I & P^l_J & P^l_K\\ P^l_M & P^l_N & P^l_R\end{smallmatrix}\right)=\zeta\left(\det\begin{pmatrix}|\mu_{P^l_I}|-|\nu_{P^l_J}| & z_{P^l_K}\\ |\mu_{P^l_M}|-|\nu_{P^l_N}| & z_{P^l_R}\end{pmatrix}\right) \tag{2.81}$$

when $\mathcal{E}(P_I^l, P_J^l, P_K^l)$ has positive energy and $\mathcal{E}(P_M^l, P_N^l, P_R^l)$ has negative energy. But then

$$\det \begin{pmatrix} |\mu_{P_I^l} - |\nu_{P_J^l}| & z_{P_K^l} \\ |\mu_{P_M^l} - |\nu_{P_N^l}| & z_{P_R^l} \end{pmatrix} = a z_{P_K^l} + b z_{P_R^l}, \tag{2.82}$$

for some $a, b > 0$, and since the odd-degree coefficients in the expansion of $\zeta(z)$ are all positive, this shows that all coefficients of odd-degree terms in the expansion in $z_1, \ldots, z_m$ of the ζ-functions in the product in Theorem 2.33 are positive. This completes the proof of the theorem. □

2.6.4 Wall-crossing formula

In this section we obtain the wall crossing formula for double Hurwitz numbers with completed cycles with respect to the walls $W_{I,J}$ described in Remark 2.45.

Fix $I \subset [\ell_1]$ and $J \subset [\ell_2]$. Let $\mathfrak{c}_1$ and $\mathfrak{c}_2$ be the two chambers bordering along the wall $W_{I,J}$. The wall crossing formula is a formula for the difference between the polynomials describing the Hurwitz numbers on the different chambers: $\mathrm{WC}_{I,J}^{(r)} = h_g^{(r)}|_{\mathfrak{c}_1} - h_g^{(r)}|_{\mathfrak{c}_2}$.

In order to compute it, we define a series which captures the information about double Hurwitz numbers with completed $(r+1)$-cycles for any value of r with given ramification over 0 and ∞:

$$H_{\mu,\nu}^m(z_1, \ldots, z_s) := \frac{1}{\prod_{i=1}^{\ell(\mu)} \mu_i \prod_{j=1}^{\ell(\nu)} \nu_j} \langle \prod_{i=1}^{\ell(\mu)} \mathcal{E}_{\mu_i}(0) \prod_{k=1}^{m} \mathcal{E}_0(z_k) \prod_{i=1}^{\ell(\nu)} \mathcal{E}_{-\nu_j}(0) \rangle. \tag{2.83}$$

Remark 2.48. The information of the double Hurwitz number with completed $(r+1)$-cycles and ramification given by (μ, ν) is encoded in $H_{\mu,\nu}^m$ for any value of r, that is, $h_{\mu,\nu}^{(r),m} = [z_1^{r+1} \cdots z_m^{r+1}] H_{\mu,\nu}^m$. However, the coefficients of other monomials in $z_1, \ldots, z_m$ also have an interpretation as some Hurwitz numbers. The coefficient of the monomial $z_1^{r_1+1} \cdots z_m^{r_m+1}$ for any non-negative integers $r_1, \cdots r_m$ is the number of covers of $\mathbb{P}^1$ with ramification over 0 and ∞ given μ and ν, and ramification over m more points given by $\overline{(r_1)}, \ldots, \overline{(r_m)}$.

It is clear from the proof of Theorem 2.46 that all coefficients of the power series $H_{\mu,\nu}^m$ are piecewise polynomial with respect to the described hyperplane arrangement. Thus, if we take completed cycles of different values for different ramification points, the corresponding Hurwitz number will still be piecewise polynomial.

So, let $W_{I,J}$ be a given wall in the hyperplane arrangement of Remark 2.45. Let μ, ν also be given. Let δ denote the difference $\delta := |\mu_I| - |\nu_J|$.

Theorem 2.49. *The wall crossing formula is given by*

$$\mathrm{WC}_{I,J}^{(r)}(\mu, \nu) = [z_1^{r+1} \cdots z_m^{r+1}] \sum_{K \subset [m]} \delta^2 \frac{\zeta(z_K)\zeta(z_{K^c})\zeta(\delta z_{[m]})}{\zeta(\delta z_k)\zeta(\delta z_{K^c})\zeta(z_{[m]})} \cdot \tag{2.84}$$
$$\cdot H_{\mu_I, \nu_J + \delta}^{|K|}(\{z_k\}_{k \in K}) H_{\mu_{I^c}+\delta, \nu_{J^c}}^{|K^c|}(\{z_k\}_{k \in K^c}).$$

Proof. Let P be a commutation pattern in $\mathrm{CP}(\mathfrak{c}_1)$. If P does not produce the operator $\mathcal{E}(I, J, K)$ for some $K \subset [m]$ at some point, it will also be a commutation pattern in $\mathrm{CP}(\mathfrak{c}_2)$. Thus, a commutation pattern P only contributes to the wall crossing formula if at some point it produces $\mathcal{E}(I, J, K)$ for some $K \subset [m]$. Let P be such a pattern. Using that the operators in any of the three products in Equation (2.19) commute amongst themselves, we may start the algorithm with the vacuum expectation value

$$\left\langle \prod_{i\notin I} (i,\emptyset,\emptyset) \prod_{i\in I} \mathcal{E}(i,\emptyset,\emptyset) \prod_{k=1}^{m} \mathcal{E}(\emptyset,\emptyset,k) \prod_{j\in J} \mathcal{E}(\emptyset,j,\emptyset) \prod_{j\notin J} (\emptyset,j,\emptyset) \right\rangle . \tag{2.85}$$

If a pattern produces $\mathcal{E}(I,J,K)$, the first vacuum expectation value where it occurs must be

$$\left\langle \prod_{i\notin I} \mathcal{E}(i,\emptyset,\emptyset) \prod_{k\notin K} \mathcal{E}(\emptyset,\emptyset,k)\ \mathcal{E}(I,J,K) \prod_{j\notin J} \mathcal{E}(\emptyset,j,\emptyset) \right\rangle . \tag{2.86}$$

Let T_1^K be the product of ζ-functions produced by the algorithm up until this point. Note that it does not depend on whether we run the algorithm on $\mathfrak{c}_1$ or $\mathfrak{c}_2$. It is easy to see that it is given by

$$T_1^K(\{z_k\}_{k\in K}) := \left\langle \prod_{i\in I} \mathcal{E}(i,\emptyset,\emptyset) \prod_{k\in K} \mathcal{E}(\emptyset,\emptyset,k) \prod_{j\in J} \mathcal{E}(\emptyset,j,\emptyset) \mathcal{E}_{-\delta}(0) \right\rangle \frac{\zeta(z_K)}{\zeta(\delta z_K)} . \tag{2.87}$$

On the other hand, by defintion of the function H we have:

$$\begin{aligned} H_{\mu_I,\nu_J+\delta}(\{z_k\}_{k\in K}) &= \frac{1}{\delta \prod_{i\in I}\mu_i \prod_{j\in J}\nu_j} \cdot \\ &\cdot \left\langle \prod_{i\in I} \mathcal{E}(i,\emptyset,\emptyset) \prod_{k\in K} \mathcal{E}(\emptyset,\emptyset,k) \prod_{j\in J} \mathcal{E}(\emptyset,j,\emptyset) \mathcal{E}_{-\delta}(0) \right\rangle . \end{aligned} \tag{2.88}$$

Therefore,

$$T_1^K(\{z_k\}_{k\in K}) = \delta \prod_{i\in I}\mu_i \prod_{j\in J}\nu_j \frac{\zeta(z_K)}{\zeta(\delta z_K)} H_{\mu_I,\nu_J+\delta}(\{z_k\}_{k\in K}). \tag{2.89}$$

Let T_2^K denote the difference between the polynomials computing the vacuum expectation value (2.86) on $\mathfrak{c}_1$ and $\mathfrak{c}_2$. To compute this, it will be better to let the algorithm run according to the rules of the chamber $\mathfrak{c}_1$ on both sides (i.e.; we use the set of commumation patterns $\mathrm{CP}(\mathfrak{c}_1)$). This means we will move the operator $\mathcal{E}(I,J,K)$ to the left on both chambers, even though on $\mathfrak{c}_2$ it has positive energy.

If the operator $\mathcal{E}(I,J,K)$ is involved in a commutator term at any point, the algorithm will run as normal afterwards on both chambers, since the chambers differed by only one wall. Therefore, the only contribution to T_2^K comes from commutation patterns where $\mathcal{E}(I,J,K)$ is moved entirely to the left. The result will then be zero on $\mathfrak{c}_1$, since there is an operator of negative energy on the far left, but it will be non-zero on $\mathfrak{c}_2$. The last step in the algorithm on $\mathfrak{c}_2$ for Equation (2.86) will be

$$\langle \mathcal{E}(I,J,K)\ \mathcal{E}(I^{\mathrm{c}},J^{\mathrm{c}},K^{\mathrm{c}})\rangle = \frac{\zeta(\delta z_{[m]})}{\zeta(z_{[m]})}, \tag{2.90}$$

where I^{c} denotes the complement of $I \subset [\ell_1]$, and the same for J^{c} and K^{c}. Also using that

$$\langle \mathcal{E}_\delta(0)\ \mathcal{E}(I^{\mathrm{c}},J^{\mathrm{c}},K^{\mathrm{c}})\rangle = \frac{\zeta(\delta z_{K^{\mathrm{c}}})}{\zeta(z_{K^{\mathrm{c}}})}, \tag{2.91}$$

we see that

$$T_2^K(\{z_k\}_{k\notin K}) = \left\langle \mathcal{E}_\delta(0) \prod_{i\notin I} \mathcal{E}(i,\emptyset,\emptyset) \prod_{k\notin K} \mathcal{E}(\emptyset,\emptyset,k) \prod_{j\notin J} \mathcal{E}(\emptyset,j,\emptyset) \right\rangle \frac{\zeta(z_{K^c})\zeta(\delta z_{[m]})}{\zeta(\delta z_{K^c})\zeta(z_{[m]})}. \tag{2.92}$$

(by $\prod_{k\notin K}$ we denote $\prod_{k\in K^c}$; the same for I and J). On the other hand

$$H_{\mu_{I^c}+\delta,\nu_{J^c}}(\{z_k\}_{k\in K^c}) = \frac{1}{\delta \prod_{i\notin I}\mu_i \prod_{j\notin J}\nu_j} \cdot \tag{2.93}$$
$$\cdot \left\langle \mathcal{E}_\delta(0) \prod_{i\notin I} \mathcal{E}(i,\emptyset,\emptyset) \prod_{k\notin K} \mathcal{E}(\emptyset,\emptyset,k) \prod_{j\notin J} \mathcal{E}(\emptyset,j,\emptyset)\mathcal{E}_{-\delta}(0) \right\rangle,$$

therefore

$$T_2^K(\{z_k\}_{k\notin K}) = \delta \prod_{i\notin I}\mu_i \prod_{j\notin J}\nu_j \zeta(\mathfrak{z}_{\mathfrak{K}^c})\zeta(\delta\mathfrak{z}_{[\mathfrak{s}]})\zeta(\delta z_{K^c})\zeta(z_{[s]})H_{\mu_{I^c}+\delta,\nu_{J^c}}(\{z_k\}_{k\in K^c}). \tag{2.94}$$

It is clear that

$$\mathrm{WC}^{(r)}_{I,J}(\mu,\nu) = \frac{1}{\prod_{i\in[\ell_1]}\mu_i \prod_{j\in[\ell_2]}\nu_j}[z_1^{r+1}\cdots z_m^{r+1}] \sum_{K\subset[m]} T_1^K T_2^K. \tag{2.95}$$

Substituting Equations (2.89) and (2.94) into this formula, we obtain the wall crosing formula (2.84). □

2.7 An analogue of GJV-formula

In this section, we discuss an analogue of the Goulden-Jackson-Vakil formula for one-part double Hurwitz numbers that might relate them to the intersection theory of some moduli spaces. These conjectural "intersection numbers" have very nice properties as some explicit solutions of the KP hierarchy. The number $r \geq 1$ is fixed throughout the section, so we omit the superscript (r) in all notations.

2.7.1 The formula

Let μ be an arbitrary partition $(\mu_1,\dots,\mu_\ell)$ and $g \geq 0$ be an arbitrary non-negative integer. We consider the one-part double Hurwitz number with completed $(r+1)$-cycles $h_{g,|\mu|,\mu}$. We propose the following formula:

$$h_{g,|\mu|,\mu} = \frac{m!}{d} \int_{X_{g,\ell(\mu)}} \frac{1-\Lambda_2+\Lambda_4-\cdots+(-1)^g\Lambda_{2g}}{(1-\mu_1\Psi_1)\cdots(1-\mu_{\ell(\mu)}\Psi_{\ell(\mu)})}, \tag{2.96}$$

where $X_{g,n}$ is a sequence of spaces of complex dimension $2g(r+1)+n-1$, and we fix the degrees of the rational cohomology classes $\Lambda_{2k} \in H^{4rk}(X_{g,n})$ and $\Psi_1,\dots,\Psi_n \in H^{2r}(X_{g,n})$. Existence of these geometric objects is a pure speculation, so a way to understand this formula is the following.

One-part double Hurwitz numbers with completed cycles $h_{g,|\mu|,\mu}$ are expressed in terms of some new combinatorially significant numbers that we denote by

$$\langle \Lambda_{2k} \prod_{i=1}^n \tau_{d_i} \rangle_g := \int_{X_{g,n}} \Lambda_{2rk} \prod_{i=1}^n \Psi_i^{d_i} \tag{2.97}$$

that are symmetric in $d_1,\dots,d_n$, non-zero only if $2g(r+1)+n-1 = (2k+\sum_{i=1}^n d_i)r$, and have interesting properties together with a hope to be related to geometry in future.

2.7.2 Generating function for intersection numbers

We consider a generating function for the numbers $\langle\Lambda_{2k}\prod_{i=1}^{n}\tau_{d_i}\rangle_g$. Let

$$G(u) := \sum_{j,k_1,k_2,\ldots}(-1)^j\langle\Lambda_{2j}\tau_0^{k_0}\tau_1^{k_1}\ldots\rangle_g u^{2j}\frac{T_0^{k_0}}{k_0!}\frac{T_1^{k_1}}{k_1!}\ldots. \tag{2.98}$$

Here we take the sum of all non-negative integer indices $j,k_1,\ldots,k_n$, $n\geq 0$, such that there exists a non-negative integer g such that $2g(r+1)+n-1=(2k+\sum_{i=1}^{n}d_i)r$.

Notation 2.50. In this section, we use the isomorphism described in Section 2.5 between the infinite wedge space and the space of formal power series $\mathbb{C}[[q_1,q_2,\ldots]]$ to interpret the $\mathcal{E}$-operators as operators on $\mathbb{C}[[q_1,q_2,\ldots]]$. By abuse of notation, we denote these operators in the same way. We denote by $\mathcal{E}_{k,a}$ the operator $[z^a]\mathcal{E}_k(z)$.

We would like to consider the formal variables T_k, $k=0,1,\ldots$, as linear functions in formal variables q_i, $i=1,2,\ldots$. We set $T_0=q_1$, and $T_{k+1}=(u\mathcal{E}_{0,1}+\mathcal{E}_{-1,1})T_k$. We list the first few expressions:

$$\begin{aligned} T_0 &= q_1, \\ T_1 &= uq_1+q_2, \\ T_2 &= u^2q_1+3uq_2+2q_3, \end{aligned} \tag{2.99}$$

and so on.

Theorem 2.51. *For any function $c(u)$, the series $c(u)+G(u,q_1,q_2,\ldots)$ is a solution of the Hirota equations in variables q_i, $i=1,2,\ldots$ (u is just a parameter).*

In particular, we consider the series $F:=G|_{u=0}$, that is, the generation function for the intersection numbers without Λ-classes:

$$F(q_0,q_1,\ldots)=\sum_{k_0,k_1,\ldots}\langle\tau_0^{k_0}\tau_1^{k_1}\ldots\rangle_g\frac{(0!q_1)^{k_0}}{k_0!}\frac{(1!q_2)^{k_1}}{k_1!}\ldots. \tag{2.100}$$

Theorem 2.51 implies the following property of F.

Corollary 2.52. *The series $F(q_1,q_2,\ldots)$ is a solution of the Hirota equations and linearized Hirota equations.*

2.7.3 An explicit formula for G and F

In this section, we give explicit formulas for the series G and F. For that we need to introduce some operators Y_i, $i\geq 0$. We denote by Y_0 the operator $\tilde{\mathcal{E}}_0$. We denote by Y_{i+1}, $i\geq 0$, the operator

$$Y_{i+1}(w) := \zeta(w)^{i+1}\left(\prod_{k=0}^{i}\left(\frac{\partial}{\partial w}-\frac{i}{2}+k\right)\right)\mathcal{E}_{-(i+1)}(w) \tag{2.101}$$

Observe that $Y_{i+1}(w)=O(w^{i+1})$.

Theorem 2.53. *We have:*

$$G(u,q_1,q_2,\ldots)=\exp\left([w^{r+1}]\sum_{k=0}^{r+1}u^k\frac{Y_{r+1-k}(w)}{(r+1-k)!}\right)q_1. \tag{2.102}$$

Corollary 2.54. *We have:*

$$F(q_1, q_2, \dots) = \exp\left(Y_{r+1}(0)\right) q_1. \tag{2.103}$$

Remark 2.55. The constant term of Y_{r+1} (used in the Equation (2.103)) is a linear combination of $\mathcal{E}_{-(r+1),i}$, where $i = 1, 3, 5, \dots, r+1$ ($i = 0, 2, 4, \dots, r+1$) for even (respectively, odd) r, and the coefficients are *central factorial numbers* [1].

2.7.4 Rearranging the generating series

Consider the following version of a generating series for Hurwitz numbers,

$$H(\beta, p_1, p_2, \dots) := \sum_{g,n} \frac{1}{n!} \sum_{\mu_1,\dots,\mu_n} |\mu| \cdot h_{g,|\mu|,\mu} p_{b_1} \cdots p_{b_n} \frac{\beta^m}{m!}. \tag{2.104}$$

Here $m = (2g + n - 1)/r$. In fact, it is the way we rather present a generating series for the integrals in the formula (2.96). Since

$$H + c(\beta) = \exp\left(\beta \cdot [w^{r+1}]\tilde{\mathcal{E}}_0(w)\right) \left(c(\beta) + \sum_{i=1}^{\infty} p_i\right), \tag{2.105}$$

we conclude that $H + c(\beta)$ satisfies the Hirota equations, for an arbitrary function $c(\beta)$. Note that Equation (2.105) is just the cut-and-join equation and that the operator $[w^{r+1}]\tilde{\mathcal{E}}_0(w)$ is actually explicitly given by the operator Q_{r+1} discussed Section 2.5 (see Notation 2.50).

Now, using that $(r+1)m = \dim X_{g,n}/r + n - 1$, we obtain:

$$\begin{aligned} H &= \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{g,\mu_1,\dots,mu_n} \int_{X_{g,n}} \frac{1 - \Lambda_2 + \dots \pm \Lambda_{2g}}{(1-\mu_1\Psi_1)\cdots(1-\mu_n\Psi_n)} p_{\mu_1} \cdots p_{\mu_n} \beta^m \\ &= \frac{1}{u} \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{g,\mu_1,\dots,b\mu_n} \int_{X_{g,n}} (1 - u^2\Lambda_2 + \dots \pm u^{2g}\Lambda_{2g}) \prod_{i=1}^{n} \frac{up_{\mu_i}}{(1-u\mu_i\Psi_i)} \\ &= \frac{1}{u} \sum_{g,n} \frac{1}{n!} \left\langle (1 - u^2\Lambda_2 + u^4\Lambda_4 - \dots) \prod_{i=1}^{n} \left(\sum_{d \geq 0} \tau_d T_d \right) \right\rangle_g, \end{aligned} \tag{2.106}$$

where $u^{r+1} = \beta$ and $T_d = \sum_{b\geq 1} up_b \cdot (ub)^d = u^{d+1} \sum_{b \geq 1} b^d p_b$. Observe that $T_{d+1} = u\tilde{\mathcal{E}}_{0,1} T_d$.

2.7.5 Change of variables

We use the same change of variables as in [99, 96]. We rescale the variables by setting $p_b = q_b/u^b$, and then we replace q_i with $\exp(-\mathcal{E}_{-1,1}/u)q_i$, $i = 1, 2, \dots$. An explicit formula for this linear triangular change of variables is given by

$$p_b = \sum_{i=b}^{\infty} \frac{1}{u^i} (-1)^{i-b} \binom{i-1}{b-1} q_i. \tag{2.107}$$

Under this change of variable a series $f(u, p_1, p_2, \dots)$ transforms into

$$g(u, q_1, q_2, \dots) := \exp(-\mathcal{E}_{-1,1}/u) f(u, q_1/u, q_2/u^2, \dots). \tag{2.108}$$

This change of variable is a symmetry of the Hirota equations.

A straightforward computation shows that T_0 turns into q_1 and $u\tilde{\mathcal{E}}_{0,1}$ turns into

$$\exp(-\mathcal{E}_{-1,1}/u)u\tilde{\mathcal{E}}_{0,1}\exp(\mathcal{E}_{-1,1}/u) = u\tilde{\mathcal{E}}_{0,1} + \mathcal{E}_{-1,1}. \tag{2.109}$$

This means that $T_d = (u\tilde{\mathcal{E}}_{0,1} + \mathcal{E}_{-1,1})^d q_1$.

In order to apply the change of variables to the operator $\tilde{\mathcal{E}}_0(w)$, we need the following lemma.

Lemma 2.56. *We have:* $[Y_i, \mathcal{E}_{-1,1}] = Y_{i+1}$, $i = 0, 1, 2, \dots$.

Proof. Indeed,

$$[Y_i, \mathcal{E}_{-1,1}] = \left[\zeta(w)^i \left(\prod_{k=0}^{i-1}\left(\frac{\partial}{\partial w} - \frac{i-1}{2} + k\right)\right)\mathcal{E}_{-(i)}(w), [z]\mathcal{E}_{-1}(z)\right]. \tag{2.110}$$

Observe that

$$\begin{aligned}[z]\left[\mathcal{E}_{-(i)}(w), [z]\mathcal{E}_{-1}(z)\right] = [z]\left(\zeta(w-iz)\mathcal{E}_{-(i+1)}(z+w)\right) = \\ = \left(\zeta(w)\frac{\partial}{\partial w} - i\zeta'(w)\right)\mathcal{E}_{-(i+1)}(w).\end{aligned} \tag{2.111}$$

A straighforward computation implies that

$$\begin{aligned}&\left(\frac{\partial}{\partial w} + \frac{i-1-2k}{2}\right)\left(\zeta(w)\frac{\partial}{\partial w} - (i-k)\zeta'(w) - \frac{k}{2}\zeta(w)\right) \\ &= \left(\zeta(w)\frac{\partial}{\partial w} - (i-1-k)\zeta'(w) - \frac{k+1}{2}\zeta(w)\right)\left(\frac{\partial}{\partial w} + \frac{i-2k}{2}\right),\end{aligned} \tag{2.112}$$

$k = 0, 1, \dots, i-1$. Therefore,

$$\left(\prod_{k=0}^{i-1}\left(\frac{\partial}{\partial w} - \frac{i-1}{2} + k\right)\right)\left(\zeta(w)\frac{\partial}{\partial w} - i\zeta'(w)\right) = \zeta(w)\prod_{k=0}^{i}\left(\frac{\partial}{\partial w} - \frac{i}{2} + k\right). \tag{2.113}$$

Thus we see that the right hand side of Equation (2.110) is equal to

$$\zeta(w)^i \cdot \left(\zeta(w)\prod_{k=0}^{i}\left(\frac{\partial}{\partial w} - \frac{i}{2} + k\right)\right)\mathcal{E}_{-(i+1)}(w) = Y_{i+1}. \tag{2.114}$$

□

Corollary 2.57. *Under the change of variable the operator* $\tilde{\mathcal{E}}_0(w)$ *turns into* $\sum_{i=0}^{\infty} u^{-i}Y_i/i!$

2.7.6 Proof of Theorems 2.51 and 2.53

The generating series given in Equation (2.104), $H + c(u)$, is a solution to Hirota equations. We apply the change of variables (2.107). Using Corollary 2.57, we see that under this change of variables Equation (2.105) turns into

$$c(u) + \exp\left(u^{r+1}[w^{r+1}]\sum_{i=0}^{\infty} u^{-i}Y_i/i!\right)\frac{q_1}{u}. \tag{2.115}$$

Using that $Y_i(w) = O(w^i)$, we see that this formula multiplied by u is equal to the right hand side of the Equation (2.102) (if we choose $c(u) = 0$). On the other hand, from Equation (2.106) we know that H multiplied by u is equal to G in coordinates $u, q_1, q_2, \dots$. This completes the proof of Theorem 2.53. Theorem 2.51 is then obvious since the change of variables (2.107) is a symmetry of the KP-hierarchy.

–3– CEO recursion for Hurwitz numbers with completed cycles

3.1 Introduction

In this chapter we introduce the r-spin variant of the Bouchard-Mariño conjecture (the r-BM conjecture), stating that the spectral curve for Hurwitz numbers with completed $(r+1)$ cycles is the r-Lambert curve $x = \log(y) - y^r$, and we provide three pieces of evidence for this conjecture. A fourth piece of evidence is discussed in Chapter 8.

First is an attempt at a direct proof using matrix models, generalizing the direct proof of the Bouchard-Mariño conjecture in [8]. Unfortunately, the proof in [8] turned out not to be completely rigorous; thus, our generalization should be viewed as a strong indication in favour of the r-BM conjecture. More precisely, we prove Theorem 3.3, which states that Hurwitz numbers with completed $(r+1)$-cycles are given by a matrix model. Then we argue that the spectral curve corresponding to this matrix model should be the one described by the r-BM conjecture, and we show where there are still gaps in the proof of this second statement.

Second is the general idea that the spectral curve for an enumerative problem should be given by its $(0,1)$-geometry, as discussed in [30]. In the case of completed Hurwitz numbers, this leads us immediately to the r-BM conjecture.

For the third piece of evidence, we turn to a conjecture from physics [3, 23, 24, 25, 53]. That is, when we have some invariants $\omega_{g,n}$ coming from the CEO-recursion for a genus zero spectral curve, it is conjectured that the following holds.

- There exists a unique procedure to calculate the canonical primitive functions of the symmetric differential forms that are obtained by the CEO-recursion.

- The *partition function* of the theory, which is the exponential generating function of the *principal specialization* of these primitive functions, satisfies a holonomic system generated by a single stationary Schrödinger operator.

- Moreover, the total symbol of the holonomic system defines a Lagrangian subvariety immersed into the cotangent bundle of $\mathbb{C}^*$, which is exactly the same as the realization of the spectral curve as a plane curve.

- In other words, the spectral curve and its immersion as a Lagrangian into the cotangent bundle are recovered from the semi-classical limit of the Schrödinger equation.

In the physics literature cited above, this Schrödinger operator is called a *quantum curve*. It is the Weyl quantization of the defining equation of the spectral curve in the cotangent bungle. Mathematical proofs of this conjecture for a few simple cases have been established in [81].

In the case of simple Hurwitz numbers, whose spectral curve is the Lambert curve $x = \log(y) - y$ ([10, 8, 35, 33]), Zhou has shown [110] the existence of the quantum curve, quantizing in a proper way the equation of the Lambert curve (see also [81]).

Here, we show that a proper quantization of the r-Lambert curve indeed annihilates the principal specialization of the partition function for Hurwitz numbers with completed cycles, simultaneously providing evidence for the conjecture described above and the r-BM conjecture.

Furthermore, we show that the conjecture also holds for so-called q-double Hurwitz numbers (described below), as well as for the mixed case of q-double Hurwitz numbers with completed $(r+1)$-cycles.

Together, these already provide a good indication that the r-BM conjecture should be true. In Chapter 8, we provide a final piece of evidence, showing that the r-BM conjecture is equivalent to the r-ELSV conjecture, which has independent evidence of its own.

3.1.1 Plan of the chapter.

We start by stating once again the r-BM conjecture in Section 3.2. For the first part of the evidence, we state and prove Theorem 3.3 in Section 3.3, providing a matrix model for completed Hurwitz numbers. In Section 3.4, we indicate why it should be expected that the spectral curve for this matrix model is the one specified by the r-BM conjecture.

In Section 3.5, we collect the data of the spectral curves for the different types of Hurwitz numbers, as well as their quantizations.

For the second and third parts of the evidence, in each of the following three sections we (a) introduce a particular generalization of Hurwitz numbers; (b) derive the formula for the principal specialization of their partition function; (c) identify the formula for the spectral curve using the $(0,1)$-geometry; and (d) prove the existence of the quantum curve, or the stationary Schrödinger equation. The q-double Hurwitz numbers are studied in Section 3.6, the r-spin Hurwitz numbers in Section 3.7, and finally in Section 3.8 we prove the results for the mixed case.

3.2 The r-Bouchard-Mariño conjecture

We repeat the statement of the r-Bouchard-Mariño conjecture.

Let $H^{(r)}_{g,n}$ be n-point generating function of genus g for Hurwitz numbers with completed $(r+1)$-cycles as defined in Chapter 1:

$$H^{(r)}_{g,n}(x_1,\dots,x_n) = \sum_{k_1,\dots,k_n} \frac{h^{(r)}_{g;k_1,\dots,k_n}}{m!} \exp(k_1x_1+\dots+k_nx_n). \tag{3.1}$$

Remark 3.1. In this chapter, we use a different convention for the definition of the completed $(r+1)$-cycle from what we did in Chapter 2; the completed $(r+1)$-cycle here is $r!$ times the completed $(r+1)$ cycle from that chapter. To avoid too cluttered notation, we still denote Hurwitz numbers with completed cycles in the same way. See also Remark 2.5. If we would use the definition from Chapter 2, Equation (3.3) would have $x(z) = -\frac{z^r}{r!} + \log(z)$.

Also, for any function $f(x_1,\dots,x_n)$, let

$$Df = \frac{\partial^n f}{\partial x_1 \cdots \partial x_n} \mathrm{d}x_1 \cdots \mathrm{d}x_n. \tag{3.2}$$

Then we conjecture the following generalization of the Bouchard-Mariño conjecture.

Conjecture 3.2 (r-BM)**.** *Let $\omega_{g,n}$ be the n-point correlation forms associated by the CEO-recursion to the plane curve*

$$\begin{cases} x(z) &= -z^r + \log z \\ y(z) &= z \end{cases} \tag{3.3}$$

$$B(z_1, z_2) = \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2}. \tag{3.4}$$

Then we have

$$DH_{g,r}(x_1, \dots, x_n) = \omega_{g,n}. \tag{3.5}$$

3.3 Completed Hurwitz numbers as a matrix model

In this section we state and prove a theorem, representing the partition function for completed Hurwitz numbers as a matrix integral, that is shown in Section 3.4 to lead to strong evidence for the r-BM conjecture. Both the theorem presented in this section and the resulting evidence in the next section are based on a direct generalization of one of the proofs of the original Bouchard-Mariño conjecture in [8]. Unfortunately, it seems that this proof is not completely rigorous, meaning that we can only present the generalization to completed Hurwitz numbers as evidence for the r-BM conjecture.

3.3.1 The statement

Let Z be the partition function for Hurwitz numbers with completed $(r+1)$-cycles:

$$Z := \exp\left(\sum_{g=0}^{\infty} g_s^{2g-2} \sum_{\mu} \frac{h_{g;\mu}^{(r)}}{m!} p_\mu \right). \tag{3.6}$$

The character formula for not necessarily connected Hurwitz numbers allows us to write

$$Z(\mathbf{p}, g_s; t) = \sum_{K=0}^{\infty} t^K \sum_{|\mu|=K} \sum_{m=0}^{\infty} \frac{g_s^{rm-K-l(\mu)}}{m!} p_\mu \times \\ \sum_{|\lambda|=K} \left(\frac{\dim(\lambda)}{K!} \right)^2 \frac{|C_\mu| \chi_\lambda(\mu)}{\dim(\lambda)} \left(\frac{\mathbf{p}_{r+1}(\lambda)}{(r+1)} \right)^m. \tag{3.7}$$

Here $rm - K - |\mu|$ is equal to the Euler characteristic of the curve (by the Riemann-Hurwitz formula), λ and μ are partitions of K encoding an irreducible representation and a conjugacy class C_μ respectively, and $p_\mu = p_{\mu_1} \cdots p_{\mu_n}$ for $\mu = (\mu_1, \dots, \mu_n)$. For every μ the coefficient of p_μ in this expression is a formal Laurent series in g_s with a finite number of negative degree terms.

Note that we have inserted an extra formal variable t to encode the degree of the covering. It is redundant, since the degree can also be recovered from the total degree in $\mathbf{p}$, but turns out to be convenient later on.

Fix a positive integer N. We use the following substitution for the variables p_k, $k = 1, 2, \dots,$ as symmetric functions:

$$p_k = g_s \sum_{i=1}^{N} v_i^k. \tag{3.8}$$

Moreover, introduce the N-tuple of variables $\mathbf{v} := (v_1, \dots, v_N)$ and the diagonal matrix of their formal logarithms $\mathbf{R} := \mathrm{diag}(\log v_1, \dots \log v_N)$. We use Δ to denote the Vandermonde determinant:

$$\Delta(\mathbf{v}) := \prod_{1\le j<i\le N} (v_i - v_j), \tag{3.9}$$

and similarly for $\Delta(\mathbf{R})$, where we take the Vandermonde determinant of its diagonal entries.

Let B_k be the Bernoulli numbers, and introduce the following functions depending also on the parameter N:

$$\begin{aligned} A_{r+1}(x) = \sum_{k=0}^{r+1} \Bigg(& r! \frac{(-N+\frac{1}{2})^k}{k!} \frac{x^{r+1-k}}{(r+1-k)!} \\ & + (-1)^{r+1} r! \frac{(-1)^k B_k}{k!} \frac{(2^{k-1}-1)}{2^{k-1}} \frac{N^{r+1-k}}{(r+2-k)!} \Bigg); \end{aligned} \tag{3.10}$$

$$\begin{aligned} V(x) = & - g_s^{r+1} A_{r+1}(x/g_s) + g_s \log(g_s/t) A_1(x/g_s) \\ & + \mathrm{I}\pi x - g_s \log(\Gamma(-x/g_s)) + \mathrm{I}\pi g_s. \end{aligned} \tag{3.11}$$

The function A_{r+1} is just a function of x, whereas V is a function of x that also depends on the variables g_s and t that live on $\mathbb{C}\backslash(-\infty, 0)$. These functions originate from the combinatorics of Young diagrams, their meaning will be explained later in this section.

Let $\mathcal{C}_D$ be a fixed contour in the complex plane that goes around the integers h with $0 \le h \le D$. Let $\mathcal{H}_N(\mathcal{C}_D)$ be the space of N by N *normal matrices* M with eigenvalues in $\mathcal{C}_D$. In other words, $M \in \mathcal{H}_N(\mathcal{C}_D)$ if and only if M can be diagonalized by conjugation with a unitary matrix, and its eigenvalues belong to $\mathcal{C}_D$:

$$M = U^\dagger X U, \quad U \in U(N), \quad X = \mathrm{diag}(x_1, \ldots, x_N), \quad x_i \in \mathcal{C}_D. \tag{3.12}$$

We use the following measure on $\mathcal{H}_N(\mathcal{C}_D)$:

$$\mathrm{d}M = \Delta(X)^2 \mathrm{d}X \mathrm{d}U, \tag{3.13}$$

where $\mathrm{d}U$ is the Haar measure on $U(N)$ and $\mathrm{d}X$ is the product of Lebesgue curvilinear measures along $\mathcal{C}_D$.

Now we formulate the main theorem of this section.

Theorem 3.3. *We have:*

$$Z(\mathbf{p}, g_s; t) \sim \frac{g_s^{-N^2}}{N!} \frac{\Delta(\mathbf{R})}{\Delta(\mathbf{v})} \int_{\mathcal{H}_N(\mathcal{C}_D)} \mathrm{d}M e^{-\frac{1}{g_s}\mathrm{Tr}(V(M) - M\mathbf{R})}. \tag{3.14}$$

Notice that the left-hand side is a formal Laurent series in $\mathbf{p}, g_s, t$, whereas the right-hand side is a meromorphic function of those variables, defined for t and g_s on the domain $\mathbb{C}\backslash(-\infty, 0)$, that also depends on two parameters N and D. The symbol $\sim$ means that for any K, the coefficient of t^K on the left-hand side is given by the coefficient of t^K in the series expansion around $t = 0$ of the function on the right-hand side for any choice of the parameters such that $N > K$ and $D > K + \frac{N-1}{2}$.

Remark 3.4. The form of Theorem 3.3 differs from that of the analogous statement in [8]; here the contour is around a finite set of integers, whereas in [8] it goes around all non-negative integers. According to our understanding, the contour should be finite in both cases, since the integral over the infinite contour does not converge to a meromorphic function, which makes it impossible to have an expansion for it in powers of t. See Remark 3.7 for a more precise discussion of the origin of this problem. Note that this is one of the reasons we are not able to convert the evidence in the next section into a formal theorem (see Section 3.4.1).

The proof of this theorem occupies the rest of this section.

3.3.2 Schur polynomials

We recollect some facts about the Schur polynomials $s_\lambda(\mathbf{v})$ that can be defined, for a sufficiently large N, by the following formula:

$$s_\lambda(\mathbf{v}) := \frac{\det(v_i^{\lambda_j - j + N})}{\Delta(\mathbf{v})} \ . \tag{3.15}$$

The Schur polynomials are related to representations of the symmetric group (and thus to Hurwitz numbers) by the Frobenius formula

$$s_\lambda(\mathbf{v}) = \frac{1}{n!}\sum_{|\mu|=n} |C_\mu|\chi_\lambda(C_\mu)\tilde{p}_\mu \quad (\text{where } \tilde{p}_m = \sum_{i=1}^{\ell(\mu)} v_i^m\ ) \ . \tag{3.16}$$

There is an expression for s_λ in terms of the Itzykson-Zuber integral (see [56, 8])

$$I(X,Y) := \int_{U(N)} \mathrm{d}U e^{\mathrm{Tr}(XUYU^\dagger)} = \frac{\det(e^{x_i y_j})}{\Delta(X)\Delta(Y)}, \tag{3.17}$$

where $\mathrm{d}U$ is the Haar measure on $U(N)$, normalized according to the second equality. Denote by $\mathbf{h}_\lambda$ the diagonal matrix $\mathrm{diag}(h_1 \dots h_N)$, where $h_i = \lambda_i - i + N$, and by $\Delta(\mathbf{h}_\lambda)$ the Vandermonde determinant of its diagonal entries. Then we have:

$$s_\lambda(\mathbf{v}) = \Delta(\mathbf{h}_\lambda)\frac{\Delta(\mathbf{R})}{\Delta(\mathbf{v})} I(\mathbf{h}_\lambda, \mathbf{R}). \tag{3.18}$$

3.3.3 Partition function in terms of the Itzykson-Zuber integral

First, we rearrange the partition function for completed Hurwitz numbers in the following way:

$$\begin{aligned} Z(\mathbf{p}, g_s; t) &= \sum_{K=0}^{\infty} t^K \sum_{m=0}^{\infty} \frac{g_s^{mr-K}}{m!} \sum_{\substack{|\lambda|=K \\ \ell(\lambda)\le N}} s_\lambda(\mathbf{v}) \frac{\dim(\lambda)}{|\lambda|!}\left(\frac{\mathbf{p}_{r+1}(\lambda)}{r+1}\right)^m \\ &= \sum_{\ell(\lambda)\le N} \left(\frac{t}{g_s}\right)^{|\lambda|} \frac{\dim(\lambda)}{|\lambda|!} s_\lambda(\mathbf{v}) e^{g_s^r \frac{\mathbf{p}_{r+1}(\lambda)}{r+1}}. \end{aligned} \tag{3.19}$$

Here we use the interpretation of p_k, $k = 1, 2, \dots,$ as symmetric functions in v_i, $1 \le i \le N$. Furthermore, the formula above should be interpreted order by order in powers of t; for any given power K of t, the formula is true for N larger than K. We should keep this interpretation in mind throughout the rest of the computations.

Suppose that we have found functions $A_{r+1}(x)$ such that

$$\sum_{i=1}^{N} A_{r+1}(h_i) = \frac{\mathbf{p}_{r+1}(\lambda)}{r+1}, \tag{3.20}$$

so that in particular

$$\sum_{i=1}^{N} A_1(h_i) = |\lambda|. \tag{3.21}$$

Then, applying Equation (3.18) and the equality

$$\frac{\dim(\lambda)}{|\lambda|!} = \frac{\Delta(\mathbf{h})}{\prod_{i=1}^{N} h_i!} \quad \text{for} \quad N \geq \ell(\lambda), \tag{3.22}$$

we get the following:

$$\begin{aligned} Z(\mathbf{p}, g_s; t) &= \frac{\Delta(\mathbf{R})}{\Delta(\mathbf{v})} \sum_{\lambda} \left(\frac{t}{g_s}\right)^{|\lambda|} I(\mathbf{h}_\lambda, \mathbf{R}) \frac{(\Delta(\mathbf{h}_\lambda))^2}{\prod_{i=1}^{N} h_i!} \prod_{i=1}^{N} e^{g_s^r A_{r+1}(h_i)} \\ &= \frac{\Delta(\mathbf{R})}{\Delta(\mathbf{v})} \sum_{h_1 > \cdots > h_N \geq 0} I(\mathbf{h}, \mathbf{R})(\Delta(\mathbf{h}))^2 \prod_{i=1}^{N} \frac{e^{g_s^r A_{r+1}(h_i)} (g_s/t)^{-A_1(h_i)}}{\Gamma(h_i+1)} \\ &= \frac{1}{N!} \frac{\Delta(\mathbf{R})}{\Delta(\mathbf{v})} \sum_{h_1, \ldots, h_N \geq 0} I(\mathbf{h}, \mathbf{R})(\Delta(\mathbf{h}))^2 \prod_{i=1}^{N} \frac{e^{g_s^r A_{r+1}(h_i)} (g_s/t)^{-A_1(h_i)}}{\Gamma(h_i+1)}. \end{aligned} \tag{3.23}$$

Remark 3.5. Note that we should be careful when writing $(g_s/t)^{A_1(h)}$, since this might introduce non-integer powers of the formal variables g_s and t. In fact, by Equation (3.27), we obtain half-integer powers. However, it is clear that in Equation (3.23) they will cancel in the full product over i.

3.3.4 Computation of A_{r+1}

In this section we compute explicitly the polynomials A_{r+1} using Equation (3.20) as a definition. The result will coincide with Equation (3.10).

We have

$$\begin{aligned} \mathbf{p}_{r+1}(\lambda) &= \sum_{i=1}^{N} \left((\lambda_i - i + \frac{1}{2})^{r+1} - (-i + \frac{1}{2})^{r+1} \right) \\ &= \sum_{i=1}^{N} \left((h_i - N + \frac{1}{2})^{r+1} - (-i + \frac{1}{2})^{r+1} \right) \\ &= \sum_{i=1}^{N} \sum_{k=0}^{r+1} \binom{r+1}{k} (-N + \frac{1}{2})^k h_i^{r+1-k} - \sum_{j=1}^{N} \left(\frac{-2j+1}{2} \right)^{r+1}. \end{aligned} \tag{3.24}$$

The second term can be represented in the following form:

$$\begin{aligned} \sum_{j=1}^{N} \left(\frac{-2j+1}{2} \right)^{r+1} &= \frac{1}{(-2)^{r+1}} \left(\sum_{j=1}^{2N-1} j^{r+1} - \sum_{k=1}^{N-1} (2k)^{r+1} \right) \\ &= \frac{(-1)^{r+1}}{r+2} \sum_{k=0}^{r+1} \binom{r+2}{k} (-1)^k B_k \left(\frac{1}{2^{r+1}} (2N)^{r+2-k} - N^{r+2-k} \right) \\ &= \frac{(-1)^r}{r+2} \sum_{k=0}^{r+1} \binom{r+2}{k} (-1)^k B_k \left(\frac{(2^{k-1} - 1) N^{r+2-k}}{2^{k-1}} \right) \end{aligned} \tag{3.25}$$

(here $B_k := B_k(0)$, $k = 0, 1, \ldots$, are the Bernoulli numbers).

Thus we have the following formula for A_{r+1}:

$$\begin{aligned} A_{r+1}(x) = & \sum_{k=0}^{r+1} \left(\binom{r+1}{k} (-N + \frac{1}{2})^k \frac{x^{r+1-k}}{r+1} \right. \\ & \left. + \frac{(-1)^{r+1}}{(r+2)(r+1)} \binom{r+2}{k} (-1)^k B_k \frac{(2^{k-1}-1)N^{r+1-k}}{2^{k-1}} \right) \\ = & \sum_{k=0}^{r+1} \left(r! \frac{(-N+\frac{1}{2})^k}{k!} \frac{x^{r+1-k}}{(r+1-k)!} \right. \\ & \left. + (-1)^{r+1} r! \frac{(-1)^k B_k}{k!} \frac{(2^{k-1}-1)}{2^{k-1}} \frac{N^{r+1-k}}{(r+2-k)!} \right) \end{aligned} \tag{3.26}$$

in agreement with (3.10). In particular, we have

$$A_1(x) = x - \frac{N-1}{2}. \tag{3.27}$$

3.3.5 Contour integral

We now replace the N sums in Equation (3.23) for the partition function by integrals over a contour $\mathcal{C}_D$ enclosing the non-negative integers less than or equal to D. For that we use a function which has simple poles with residue 1 at all integers:

$$f(\xi) := \frac{\pi e^{-\mathrm{I}\pi\xi}}{\sin(\pi\xi)} = -\Gamma(\xi+1)\Gamma(-\xi)e^{-\mathrm{I}\pi\xi}. \tag{3.28}$$

Note that for any K, only finitely many terms of the sum in (3.23) contribute to the coefficient of t^K. Thus, when we want to compute any such coefficient, we can replace the sum by a finite one:

$$\begin{aligned} & [t^K]Z(\mathbf{p}, g_s; t) = [t^K]Z_D(\mathbf{p}, g_s; t) := \\ & \qquad [t^K] \frac{1}{N!} \frac{\Delta(\mathbf{R})}{\Delta(\mathbf{v})} \sum_{h_1,\dots,h_N=0}^{D} I(\mathbf{h}, \mathbf{R})(\Delta(\mathbf{h}))^2 \prod_{i=1}^{N} \frac{e^{g_s^r A_{r+1}(h_i)} (g_s/t)^{-A_1(h_i)}}{\Gamma(h_i+1)}, \end{aligned}$$

which is true as long as $D \geq K + \frac{N-1}{2}$.

Remark 3.6. While Z is a Laurent series that does not converge to a function, the truncated series Z_D obviously does converge to a meromorphic function with domain $\mathbb{C}$ for all the variables, since it is a finite sum of such functions.

Using the function f defined in equation (3.28) we can rewrite the function Z_D in terms of residues, if we restrict the domain of g_s and t to $\mathbb{C}\backslash(-\infty, 0)$:

$$\begin{aligned} Z_D(\mathbf{p}, g_s; t) = & \frac{1}{N!} \frac{\Delta(\mathbf{R})}{\Delta(\mathbf{v})} \sum_{h_1,\dots,h_N=0}^{D} \operatorname*{res}_{z_1 \to h_1} \cdots \operatorname*{res}_{z_N \to h_N} \\ & I(\mathbf{z}, \mathbf{R})(\Delta(\mathbf{z}))^2 \prod_{i=1}^{N} \frac{f(z_i) e^{g_s^r A_{r+1}(z_i)} (g_s/t)^{-A_1(z_i)}}{\Gamma(z_i+1)} \end{aligned} \tag{3.29}$$

On the right-hand side, $(g_s/t)^{-A_1(z)}$ is defined as $\exp(-A_1(x)\log(g_s/t))$, which requires a choice of branch of the logarithm (one can see that the end result does not depend on this choice), and explains the change in domain. Here, it is important that g_s and t are no longer just formal variables, but the arguments of a function.

Finally, the sum over residues can be replaced by a contour integral:

$$Z_D(\mathbf{p}, g_s; t) = \frac{1}{N!}\frac{\Delta(\mathbf{R})}{\Delta(\mathbf{v})}\oint_{\mathcal{C}_D^N} \mathrm{d}h_1 \cdots \mathrm{d}h_N (\Delta(\mathbf{h}))^2 I(\mathbf{h}, \mathbf{R}) \prod_{i=1}^{N} \frac{f(h_i) e^{g_s^r A_{r+1}(h_i)} (g_s/t)^{-A_1(h_i)}}{\Gamma(h_i+1)}. \tag{3.30}$$

Remark 3.7. Equation (3.30) is an equality of functions, and the function on the left-hand side is defined as a (converging) series in t, implying that the function on the right has the same series expansion at $t = 0$. Note that we have to work with Z_D because it is not possible to write a formula like (3.29) for Z, since it is only a formal series, and does not converge to any function. Another way to see this is that the integral around all non-negative integers does not converge, so that it is meaningless to take the coefficient of t^K in that integral.

Rescaling the integration variables $h_i \to h_i/g_s$, we get:

$$Z_D(\mathbf{p}, g_s; t) = \frac{g_s^{-N^2}}{N!}\frac{\Delta(\mathbf{R})}{\Delta(\mathbf{v})}\oint_{\mathcal{C}_D^N} \mathrm{d}h_1 \cdots \mathrm{d}h_N (\Delta(\mathbf{h}))^2 \times I\left(\frac{\mathbf{h}}{g_s}, \mathbf{R}\right) \prod_{i=1}^{N} -\Gamma\left(-\frac{h_i}{g_s}\right) e^{g_s^r A_{r+1}(\frac{h_i}{g_s}) - \frac{I\pi h_i}{g_s}} \left(\frac{g_s}{t}\right)^{-A_1(\frac{h_i}{g_s})}. \tag{3.31}$$

3.3.6 Normal matrices and final formula

As it is done in [8], we now replace the integration along the N copies of the contour $\mathcal{C}_D$ by integration over the space $\mathcal{H}_N(\mathcal{C}_D)$ of N by N normal matrices with eigenvalues in $\mathcal{C}_D$. We get

$$Z_D(\mathbf{p}, g_s; t)) = \lim_{N\to\infty} \frac{1}{N!}\frac{\Delta(\mathbf{R})}{\Delta(\mathbf{v})}\int_{\mathcal{H}_N(\mathcal{C}_\mathcal{D})} \mathrm{d}M e^{-\mathrm{Tr}V(M) + \mathrm{Tr}(M\mathbf{R})}, \tag{3.32}$$

where V is as in formula(3.11):

$$V(\xi) = -g_s^r A_{r+1}(\frac{\xi}{g_s}) + g_s \log(\frac{g_s}{t}) A_1(\frac{\xi}{g_s}) + \mathrm{I}\pi\xi - g_s \log(\Gamma(-\frac{\xi}{g_s})) + \mathrm{I}\pi g_s.$$

In particular, it implies

$$Z(\mathbf{p}, g_s; t) \sim \frac{g_s^{-N^2}}{N!}\frac{\Delta(\mathbf{R})}{\Delta(\mathbf{v})}\int_{\mathcal{H}_N(\mathcal{C}_\mathcal{D})} \mathrm{d}M e^{-\frac{1}{g_s}\mathrm{Tr}(V(M) - M\mathbf{R})}, \tag{3.33}$$

concluding the proof of Theorem 3.3.

3.4 Spectral curve for the r-spin Hurwitz matrix model

3.4.1 Spectral curve associated to a matrix model

In [8], a "physics proof" of the Bouchard-Mariño conjecture is given by first representing the generating function for Hurwitz numbers as a matrix model, and then showing that the spectral curve for this matrix model is equal to the one predicted by the Bouchard-Mariño conjecture.

We generalized the first part of this proof in the previous section, where we showed that the generating function for completed Hurwitz numbers is given by the matrix model (3.14).

Unfortunately, it seems that the reasoning in [8] does not constitute a precise mathematical proof, nor can it easily be made into one. On the other hand, all the reasoning in [8] generalizes directly to the case of completed Hurwitz numbers. Thus, if a way could be found to transform this into a rigorous mathematical proof, it would immediately prove the r-BM conjecture.

Remark 3.8. There is another proof of the Bouchard-Mariño conjecture in [35], but it is based on the ELSV formula, so it is not useful for our purposes.

In the rest of this section, we briefly describe the steps taken in [8] and how they generalize to completed Hurwitz numbers, and we note the places were we believe the reasoning is not mathematically rigorous. Since all the steps in [8] generalize directly to our case, we do not repeat the detailed steps of that paper, and just give an overview of the reasoning.

3.4.2 Loop equations and topological expansion

It is a general theme in the theory of matrix models that they can be related to a spectral curve by way of so-called *loop equations*. That is, to any matrix model one can associate a free energy $F = \log Z$ and a tower of n-point correlation functions $W_n(x_1, \dots, x_n)$, $n \geq 1$. Then, one can ask whether there exists a curve such that those invariants coincide with the symplectic invariants and n-point correlation forms associated to this curve by CEO-recursion.

In general, the answer to this question is given by varying the integration variable in the matrix integral in a specific way. The resulting equations are called the loop equations, and in good situations they imply that the free energy and correlation functions of the matrix model are given by a specific spectral curve.

To derive such loop equations, we need F and W_n to have a Laurent series expansion in powers of g_s with finite tail (this is called the topological expansion property of the matrix model). In particular, this allows us to define invariants F_g and $W_{g,n}$ as the coefficients of powers of g_s in the expansions of F and W_n, which in turn makes it possible to compare them to the symplectic invariants and correlation forms of a spectral curve. In [8], it is shown that the matrix model discussed there has this topological expansion property, and the proof goes through in exactly the same way for our matrix model.

Remark 3.9. In [8], the correlators $W_{g,n}$ are themselves power series in g_s. Because of the triangular nature of the relation between $W_{g,n}$ in that paper and the coefficients of powers of g_s in W_n, it is immediate that those coefficients of powers of g_s are well-defined and they are non-zero only for finitely many negative powers of g_s. This does mean that the powers of g_s in the expansion of W_n do not necessarily increase in steps of 2, but that does not present any problems in the rest of the reasoning.

Loop equations

Given that our matrix model has the topological expansion property, the loop equations are derived as the invariance of the integral under a certain change of variables. In [8], the change

$$M \to M + \epsilon \frac{1}{x - M} \frac{1}{y - R} \tag{3.34}$$

for ϵ small is used, but this does not preserve the property of being a normal matrix, so we prefer the change

$$M \to M + \frac{\epsilon}{2} \frac{1}{x - M} \frac{1}{y - R} + \frac{\epsilon}{2} \frac{1}{y - R} \frac{1}{x - M} \tag{3.35}$$

which does preserve that property. In Appendix D of [36] the spectral curve equation and the CEO-recursion for the correlators are derived from such a change of variables for a matrix model of the form (3.14). However, their derivation depends on the potential $V(x)$ being a rational function of x, independent of g_s, neither of which holds for the matrix model in [8] or the generalization described here. This does not affect their reasoning when deriving the spectral curve, but they really use those properties of V to show that the $W_{g,n}$ indeed obey the CEO-recursive relations associated to that spectral curve.

Furthermore, the invariance of the integral under the change of variables depends on the fact that the domain of integration does not change under this change of variables. When using the infinite contour of [8], this in fact holds, but for the finite contour $\mathcal{C}_D$ (see Remark 3.4), it is not the case. That is, one easily sees that the change (3.35) does not affect the property of a matrix being Hermitian (normal matrix with real eigenvalues), but it sends the space of normal matrices with eigenvalues in $\mathcal{C}_D$ to the space of normal matrices with eigenvalues on some different contour $\widetilde{\mathcal{C}}_D$.

Remark 3.10. If we were integrating over the space of diagonal matrices with eigenvalues in $\mathcal{C}_D$ instead of those that are diagonalizable using unitary matrices, the space would also effectively be invariant under the change of variables, since the integral would only depend on the homotopy type of the contour with respect to the non-negative integers. However, the unitary matrices spoil this symmetry.

Spectral curve for completed Hurwitz numbers

Suppose that we would overcome the problems described above in some way. Then, the loop equations would lead to a spectral curve (depending on g_s) and corresponding topological recursion for the $W_{g,n}$. The proof of Conjecture 3.2 could then be completed as in [8], using the relation between the n-point genus g correlation functions for Hurwitz numbers and the free energy of the matrix model

$$\frac{\partial^n H^{(r)}_{g,n}(R_1,\dots,R_n)}{\partial R_1\cdots\partial R_n} = \frac{1}{g_s^n}\frac{\partial^n F_g}{\partial R_1\cdots\partial R_n}\bigg|_{g_s=0} \tag{3.36}$$

and some properties of the topological recursion theory and its relation to matrix models. Together, those show that the spectral curve for completed Hurwitz numbers is given by $\mathcal{S}^{(r)}_{\mathrm{s}}: x = -y^r + \log y$, concluding the evidence for the spectral curve of the matrix model (3.14).

3.5 Spectral curve and quantization

In the remainder of this chapter we consider two different generalizations of the usual simple Hurwitz numbers. The first are the Hurwitz numbers with completed $(r+1)$-cycles discussed in the previous sections, and one of the main subjects in this thesis. In the remainder of this chapter we denote them by $h^{r,1}_{g,\mu}$; the reason for the extra index 1 in the notation will become clear when we introduce the mixed case.

The other type of Hurwitz numbers we study here are a certain kind of double Hurwitz numbers. They count ramified coverings of $\mathbb{P}^1$ with two special fibers as follows. One of the special fibers has an arbitrary fixed cyclic type of monodromy $\mu = (\mu_1,\dots,\mu_\ell)$, and the other has the cyclic type of monodromy equal to $(q,q,\dots,q)$. All other critical points are assumed to be simple. This type of Hurwitz numbers we call *q-double Hurwitz numbers* and denote by $h^{1,q}_{g,\mu}$. There is a closed formula for these numbers in terms of the so-called Hurwitz-Hodge integrals, see [60]. For $q=1$ we recover the usual simple Hurwitz numbers.

Finally we also consider the mixed case of the above two generalizations. Geometrically, this is the case of two special fibers, where one has an arbitrary fixed monodromy, the other has the cyclic type of $(q, q, \ldots, q)$, and all other ramifications are the completed $(r+1)$-cycles. We call these numbers *q-double r-spin Hurwitz numbers*, and denote them by $h^{r,q}_{g,\mu}$.

3.5.1 Spectral curves

If we have a partition function Z that is the exponential generating function of the *free energies*, i.e., if Z has an expansion of the form

$$Z = \exp\left(\sum_{g=0}^{\infty} \lambda^{2g-2} \sum_{\ell=1}^{\infty} F_{g,\ell}\right), \tag{3.37}$$

then a natural question is whether we can produce a spectral curve and the other input data of the CEO-recursion procedure so that the ℓ-point differential forms $\omega_{g,\ell}$ determined by the recursion would coincide with the exterior derivatives $\mathrm{d}_1 \cdots \mathrm{d}_\ell F_{g,\ell}$ of the free energies.

We do not have a general answer to this question. If we can find the holonomic system satisfied by Z, then its semi-classical limit gives a spectral curve as a holomorphic Lagrangian subvariety. Another mechanism was proposed in [30]. The idea is that the spectral curve can be obtained via the analysis of the $(0,1)$-geometry, that is, the spectral curve is the Riemann surface (the maximal domain of holomorphy) of the one variable function $F_{0,1}$. This mechanim works for many examples, including simple Hurwitz numbers [30].

Note that in both cases, it is not à priori clear that the ℓ-point differential forms produced from the resulting spectral curve will coincide with the exterior derivatives of the free energies; it appears to be the case in many known examples, but has to be proved in each individual case.

We first examine the latter idea in the case of various generalizations of simple Hurwitz numbers described above. This way we obtain the spectral curves in Table 3.1.

q-Double Hurwitz Numbers	$x = y^{1/q} e^{-y}$
r-Spin Hurwitz Numbers	$x = y e^{-y^r}$
Mixed q-Double r-Spin Hurwitz Numbers	$x = y^{1/q} e^{-y^r}$

Table 3.1: Spectral Curves.

Remark 3.11. In the table above, a change of coordinates $x \to \exp(x)$ occurred compared to the previous section. This does not change the spectral curve itself, but it does change the functions x and y defined on it, so it also changes the correlators $\omega_{g,n}$. Thus, when computing correlators, one should always use the parametrization of the previous section. In this section we use the new coordinates since they make for slightly more attractive formulas. Note that one could also do all the computations in this section in the old coordinates, as long as one uses the quantization $\hat{y} = \lambda \mathrm{d}/\mathrm{d}x$ instead of $\hat{y} = \lambda x \mathrm{d}/\mathrm{d}x$. See also Remark 3.13 and Equation (3.59).

We note that the spectral curve for the case of q-double Hurwitz numbers was recently proved in [9, 26]. The formula for the spectral curve for r-spin Hurwitz numbers is the subject of the r-BM conjecture. The mixed case is so far still conjectural.

3.5.2 Schrödinger equations

The formulas for the spectral curves, even still conjectural for the most general case, give enough input to test the conjecture of the existence of the quantum curves, or the Schrödinger equation for the principal specialization of the partition function. We prove it in all three cases mentioned above, generalizing in this way the result of [110] for simple Hurwitz numbers.

It is worth mentioning that when we apply Weyl quantization, we need to find the correct ordering of the operators. Our guiding principle is the straightforward application of the semi-infinite wedge product formalism of the various Hurwitz numbers, as described in Chapter 2.

The main result of the quantum curves we establish are summarized in the following table.

q-Double Hurwitz Numbers	$\hat{y} - \left(e^{\frac{q-1}{2}\hat{y}}\hat{x}e^{-\frac{q-1}{2}\hat{y}}\right)^q e^{q\hat{y}}$
r-Spin Hurwitz Numbers	$\hat{y} - \hat{x}^{\frac{3}{2}} \exp\left(\frac{\sum_{i=0}^{r} \hat{x}^{-1}\hat{y}^i\hat{x}\hat{y}^{r-i}}{r+1}\right)\hat{x}^{-\frac{1}{2}}$
Mixed Hurwitz Numbers	$\hat{y} - \hat{x}^{q+1/2}e^{\frac{q}{r+1}\sum_{i=0}^{r}\hat{x}^{-q}\hat{y}^i\hat{x}^q\hat{y}^{r-1}}\hat{x}^{-1/2}$

Table 3.2: Quantum Curves.

Here the canonical quantization of the coordinate functions x and y are defined by

$$\begin{cases} \hat{x} = x \\ \hat{y} = \lambda x \frac{d}{dx}, \end{cases} \tag{3.38}$$

reflecting the nature of the cotangent bundle $T^*(\mathbb{C}^*)$ and the holomorphic tautological 1-form $y\, \mathrm{d}(\log x)$ on it.

3.6 q-Double Hurwitz numbers

In this section, we study *q-double Hurwitz numbers*. Their geometric definition, mentioned in the previous section, is equivalent ([85, 59]) to the following one in terms of connected (see Definition 2.34) vacuum expectation values in the infinite wedge space.

Definition 3.12. We define the (connected) q-double Hurwitz numbers as

$$h^{1,q}_{g;\mu} := [w_1^2 \cdots w_m^2] \left\langle \prod_{i=1}^{\ell(\mu)} \frac{\alpha_{\mu_i}}{\mu_i} \cdot \prod_{j=1}^{m} \tilde{\mathcal{E}}_0(w_j) \cdot \frac{(\alpha_{-q})^s}{q^s \cdot s!} \right\rangle^{\circ}, \tag{3.39}$$

where $[w_1^{d_1} \cdots w_n^{d_n}]$ denotes the coefficient of the monomial $w_1^{d_1} \cdots w_n^{d_n}$ in the power series that follows it. Note that $s = |\mu|/q$ is an integer since $|\mu|$ is the degree of the covering, and m is the number of simple ramification points away from 0 and ∞; it is given by the Riemann-Hurwitz formula:

$$m = 2g - 2 + \ell(\mu) + s\ . \tag{3.40}$$

Note that the Hurwitz numbers defined here differ slightly from those in [59] in that we do not remember the ordering of the branch points over ∞, reflected in the factor $1/s!$. Write

$$F^{1,q}_{g,\ell}(p_1, p_2, \ldots) := \sum_{\mu\colon\, \ell(\mu)=\ell} \frac{h^{1,q}_{g;\mu}}{m!}\, p_{\mu_1} \cdots p_{\mu_n} \tag{3.41}$$

for the generating series of genus g, q-double Hurwitz numbers whose partition μ has ℓ parts.

The full generating series is given by

$$\log Z^{1,q}(p_1, p_2, \ldots; \lambda) := \sum_{g,\ell} F^{1,q}_{g,\ell}(p_1, p_2, \ldots)\lambda^{2g-2+\ell} \tag{3.42}$$

$$= \sum_{g,\mu} \frac{h^{1,q}_{g;\mu}}{m!} \lambda^{2g-2+\ell(\mu)} p_{\mu_1} \cdots p_{\mu_{\ell(\mu)}}$$

$$= \left\langle \exp\left(\sum_{i=1}^{\infty} \frac{\alpha_i p_i}{i\lambda^{i/q}}\right) \exp\left([w^2]\tilde{\mathcal{E}}_0(w)\lambda\right) \exp\left(\frac{\alpha_{-q}}{q}\right)\right\rangle^{\circ}.$$

3.6.1 Spectral curve from $(0,1)$ geometry

To find an equation for the spectral curve, we compute the $(g,n) = (0,1)$ part of the generating function

$$F^{1,q}_{0,1}(\mathbf{p}) = [w_1^2 \cdots w_{n-1}^2] \sum_{n=1}^{\infty} p_{nq} \left\langle \frac{\alpha_{nq}}{nq} \cdot \prod_{i=1}^{n-1} \frac{\tilde{\mathcal{E}}_0(w_i)}{(n-1)!} \cdot \frac{\alpha_{-q}^n}{q^n n!} \right\rangle^{\circ}. \tag{3.43}$$

Using the commutation relation (2.20) to commute the operator α_{nq} to the right, we obtain:

$$F^{1,q}_{0,1}(\mathbf{p}) = \sum_{n=1}^{\infty} \frac{(nq)^{n-2}}{n!} p_{nq}. \tag{3.44}$$

We will abuse notation and write $F^{1,q}_{0,1}(x) = F^{1,q}_{0,1}(\mathbf{p})|_{p_i \mapsto x^i}$ for the principal specialization of $F^{1,q}_{0,1}$.

Remark 3.13. Suppose the generating function for these Hurwitz numbers comes from a spectral curve in $\mathbb{C}^2$. Denote by x and y the coordinates on the two copies of $\mathbb{C}$. Then by the topological recursion theory, the one-form $\omega_{0,1}(x) = \mathrm{d}F^{1,q}_{0,1}(x)$ should be equal to $y(x)\mathrm{d}x$. Sometimes, it will be more natural to think of the spectral curve as living in $\mathbb{C}^* \times \mathbb{C}$ or in $(\mathbb{C}^*)^2$. In that case $\omega_{0,1}(x)$ should be equal to $y(x)\frac{\mathrm{d}x}{x}$ or $\log(y)\frac{\mathrm{d}x}{x}$ respectively.

We define an auxiliary function. Let W be the main branch of the Lambert function [21]. It has a power-series expansion around zero with radius of convergence of $1/e$ given by

$$W(z) = -\sum_{n=1}^{\infty} \frac{n^{n-1}}{n!}(-z)^n. \tag{3.45}$$

and has the property that

$$W(z)e^{W(z)} = z. \tag{3.46}$$

Using this definition, we have

$$\omega_{0,1}(x) = \mathrm{d}F^{1,q}_{0,1}(x) = \frac{1}{q}\sum_{n=1}^{\infty} \frac{n^{n-1}}{n!}(qx^q)^n \frac{\mathrm{d}x}{x} = -\frac{1}{q}W(-qx^q)\frac{\mathrm{d}x}{x}, \tag{3.47}$$

where the last equality is true as long as $|x| \leq (qe)^{-1/q}$.

Therefore, Remark 3.13 leads us to think of the spectral curve $S^{1,q}$ as living in $\mathbb{C}^* \times \mathbb{C}$, given by the equation

$$S^{1,q}\colon\ y = -\frac{1}{q}W(-qx^q). \tag{3.48}$$

which can be rewritten to get

$$-qx^q = -qye^{-qy} \quad \Leftrightarrow \quad x = y^{1/q}e^{-y}. \tag{3.49}$$

3.6.2 Principal specialization

Here we once again abuse notation and write

$$Z^{1,q}(x;\lambda) := Z^{1,q}(\mathbf{p};\lambda)|_{p_i \mapsto x^i} \tag{3.50}$$

for the principal specialization of $Z^{1,q}$.

Let $s_\sigma(\mathbf{p})$ be the Schur function corresponding to a partition σ, which is given as the following vacuum expectation value in the infinite wedge space

$$s_\sigma(\mathbf{p}) := \left\langle 0 \middle| \exp\left(\sum_{i=0}^{\infty} \frac{\alpha_i p_i}{i}\right) \middle| v_\sigma \right\rangle. \tag{3.51}$$

It is a standard fact in the theory of Schur functions that its principal specialization is given by

$$s_\sigma(\mathbf{p})|_{p_i \mapsto x^i} = \begin{cases} x^l & \text{if } \sigma = (l, 0, \ldots) \text{ for some } l \\ 0 & \text{otherwise .} \end{cases} \tag{3.52}$$

Using this it is easy to see that the principal specialization of $Z^{1,q}$ is given by

$$Z^{1,q}(x;\lambda) = \sum_{i=0}^{\infty} \frac{x^{iq}}{i!(\lambda q)^i} \exp\left(\lambda \frac{(iq - \frac{1}{2})^2 - (-\frac{1}{2})^2}{2}\right). \tag{3.53}$$

To find an operator that annihilates this power-series, we proceed as follows. Denote the i^{th} summand in $Z^{1,q}(x;\lambda)$ by a_i:

$$a_i := \frac{x^{iq}}{i!(\lambda q)^i} \exp\left(\lambda \frac{(iq - \frac{1}{2})^2 - (-\frac{1}{2})^2}{2}\right). \tag{3.54}$$

Then

$$\frac{a_{i+1}}{a_i} = \frac{x^q}{(i+1)\lambda q} e^{\lambda\left(iq^2 + \frac{q(q-1)}{2}\right)}, \tag{3.55}$$

which implies that the coefficients of $Z^{1,q}(x;\lambda)$ are related by

$$\lambda q(i+1)a_{i+1} = \left(x e^{\lambda \frac{q-1}{2}}\right)^q e^{\lambda i q^2} a_i. \tag{3.56}$$

In terms of operators, this can be rewritten as

$$\lambda x \frac{\mathrm{d}}{\mathrm{d}x} a_{i+1} - \left(x e^{\lambda \frac{q-1}{2}}\right)^q e^{q\lambda x \frac{\mathrm{d}}{\mathrm{d}x}} a_i = 0, \tag{3.57}$$

which implies that the operator

$$\lambda x \frac{\mathrm{d}}{\mathrm{d}x} - \left(x e^{\lambda \frac{q-1}{2}}\right)^q e^{q\lambda x \frac{\mathrm{d}}{\mathrm{d}x}} \tag{3.58}$$

annihilates $Z^{1,q}(x;\lambda)$.

3.6.3 Quantization

We show that the operator that annihilates the principal specialization of $Z^{1,q}$ can be obtained as a quantization of the equation of the spectral curve $S^{1,q}$.

The spectral curve $S^{1,q}$ is defined in $\mathbb{C}^* \times \mathbb{C}$, where the symplectic form is $\lambda \mathrm{d}(\log(x)) \wedge \mathrm{d}y$, so we have the following rules of quantization:

$$\begin{cases} \hat{x} = x \cdot \\ \hat{y} = \lambda \frac{\mathrm{d}}{\mathrm{d}(\log(x))} = \lambda x \frac{\mathrm{d}}{\mathrm{d}x} \end{cases} . \tag{3.59}$$

In order to have the right ordering, we rewrite the equation for $S^{1,q}$ as follows:

$$S^{1,q}\colon\ y - \left(e^{\frac{q-1}{2}y} x e^{-\frac{q-1}{2}y}\right)^q e^{qy} = 0. \tag{3.60}$$

Theorem 3.14. *Quantization of the equation of $S^{1,q}$ in this form annihilates $Z^{1,q}(x,\lambda)$.*

Proof. Indeed, direct computation implies that

$$\hat{y} - \left(e^{\frac{q-1}{2}\hat{y}} \hat{x} e^{-\frac{q-1}{2}\hat{y}}\right)^q e^{q\hat{y}} = \lambda x \frac{\mathrm{d}}{\mathrm{d}x} - \left(x e^{\lambda \frac{q-1}{2}}\right)^q e^{q\lambda x \frac{\mathrm{d}}{\mathrm{d}x}}, \tag{3.61}$$

and we have seen in the previous section that this operator annihilates $Z^{1,q}(x,\lambda)$. □

We see that q-double Hurwitz numbers are an example of a theory obeying a Schrödinger-like equation with respect to the quantization of the spectral curve as expected by [53], but contrary to the previous known cases [81, 109, 110] we have to take a non-trivial ordering of the operators to obtain this result.

3.7 r-Spin Hurwitz numbers

In this section, we look at the r-spin single Hurwitz numbers. They were described as vacuum expectation values in the infinite wedge space in Chapter 2; we repeat that description here as a definition.

Definition 3.15. We define the (connected) r-spin Hurwitz numbers as

$$h^{r,1}_{g,\mu} := \left\langle \prod_{i=1}^{\ell(\mu)} \frac{\alpha_{\mu_i}}{\mu_i} \cdot \left(r![w^{r+1}]\tilde{\mathcal{E}}_0(w)\right)^m \cdot \frac{(\alpha_{-1})^{|\mu|}}{|\mu|!} \right\rangle^\circ, \tag{3.62}$$

where m is the number of ramification points other than 0, which is given by the Riemann-Hurwitz formula:

$$m = \frac{2g - 2 + \ell(\mu) + |\mu|}{r}. \tag{3.63}$$

Note the extra factor of $\frac{1}{|\mu|!}$ compared to Proposition 2.23; it appears because for single Hurwitz numbers, we only label the the inverse images of the special ramification point.

For $r = 1$, this definition reduces to the definition of ordinary single Hurwitz numbers. We remind the reader that there are different conventions on the coefficient of $[w^{r+1}]\tilde{\mathcal{E}}_0(w)$ in different sources; in particular, a different convention is used in [88, 112].

Similar to to previous section, we denote by $F^{r,1}_{g,n}(\mathbf{p})$ the generating function for genus g, r-spin Hurwitz numbers $h^{r,1}_{g,\mu}$ whose partition μ has ℓ parts. That is,

$$F^{r,1}_{g,\ell}(\mathbf{p}) := \sum_{\mu\colon \ell(\mu)=\ell} h^{r,1}_{g;\mu} p_{\mu_1} \cdots p_{\mu_\ell}. \tag{3.64}$$

For the full generating function $Z^{r,1}$ we then have

$$\begin{aligned}\log Z^{r,1}(\mathbf{p},\lambda) &:= \sum_{g,\ell} F^{r,1}_{g,\ell}(\mathbf{p})\lambda^{2g-2+\ell} \\ &= \left\langle \exp\left(\sum_{i=1}^{\infty}\frac{\alpha_i p_i}{i\lambda^i}\right)\exp\left(r![w^{r+1}]\tilde{\mathcal{E}}_0(w)\lambda^r\right)\exp(\alpha_{-1})\right\rangle^{\circ}.\end{aligned} \tag{3.65}$$

3.7.1 Spectral curve from $(0,1)$-geometry

To find an equation for the spectral curve, we compute the $(g,n)=(0,1)$ part of the generating function. Commuting the operator α_d responsible for the total ramification over 0 in (3.65) to the right, we obtain

$$F^{r,1}_{0,1}(\mathbf{p}) = \sum_{n=0}^{\infty}\frac{(rn+1)^{n-2}}{n!}p_{rn+1}. \tag{3.66}$$

Applying the principal specialization, this means that

$$F^{r,1}_{0,1}(x) = \sum_{n=0}^{\infty}\frac{(rn+1)^{n-2}}{n!}x^{rn+1}, \tag{3.67}$$

which leads to

$$\omega_{0,1}(x) = \mathrm{d}F^{r,1}_{0,1} = \sum_{n=0}^{\infty}\frac{(rn+1)^{n-1}}{n!}x^{rn+1}\frac{\mathrm{d}x}{x}. \tag{3.68}$$

We use the following formula from [21]:

$$\left(\frac{W(x)}{x}\right)^{\alpha} = \sum_{n=0}^{\infty}\frac{\alpha(n+\alpha)^{n-1}}{n!}(-x)^n \tag{3.69}$$

to express the right hand side of Equation (3.68) in a more convenient way. That is

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(rn+1)^{n-1}}{n!}x^{rn+1} &= x\sum_{n=0}^{\infty}\frac{\frac{1}{r}(n+\frac{1}{r})^{n-1}}{n!}(rx^r)^n \\ &= x\left(\frac{W(-rx^r)}{-rx^r}\right)^{1/r} = \frac{W(-rx^r)^{1/r}}{(-r)^{1/r}}\end{aligned} \tag{3.70}$$

Thus, by Remark 3.13 we arrive at the following equation for the spectral curve $S^{r,1}$ in $\mathbb{C}^*\times\mathbb{C}$:

$$S^{r,1}\colon\; y = \left(\frac{W(-rx^r)}{-r}\right)^{1/r} \quad\Leftrightarrow\quad x = ye^{-y^r}. \tag{3.71}$$

3.7.2 Principal specialization

Once again we look at the principal specialization of the full generating function

$$Z^{r,1}(x;\lambda) = Z^{r,1}(x;\lambda)|_{p_i\mapsto x^i} = \sum_{d=0}\frac{x^d}{\lambda^d d!}\exp\left(\lambda^r\frac{(d-\frac{1}{2})^{r+1}-(-\frac{1}{2})^{r+1}}{r+1}\right). \tag{3.72}$$

We define a_d to be the d^{th} summand this expression. The quotient of a_{d+1} and a_d is given by

$$\frac{a_{d+1}}{a_d} = \frac{x}{\lambda(d+1)}\exp\left(\lambda^r\frac{(d+\frac{1}{2})^{r+1}-(d-\frac{1}{2})^{r+1}}{r+1}\right), \tag{3.73}$$

which is equivalent to

$$(d+1)\lambda a_{d+1} = x \exp\left(\lambda^r \frac{(d+\frac{1}{2})^{r+1} - (d-\frac{1}{2})^{r+1}}{r+1}\right) a_d. \tag{3.74}$$

To get this into a more convenient form to compare later on with quantization, we define an operator

$$\mathcal{A} := x^{\frac{3}{2}} \exp\left(\frac{x^{-1}\sum_{i=0}^{r}\left(\lambda x \frac{\mathrm{d}}{\mathrm{d}x}\right)^i x \left(\lambda x\frac{\mathrm{d}}{\mathrm{d}x}\right)^{r-i}}{r+1}\right) x^{-\frac{1}{2}}. \tag{3.75}$$

Observe that

$$\begin{aligned} \mathcal{A}x^n &= \exp\left(\frac{\lambda^r}{r+1}\sum_{i=0}^{r}(n+\frac{1}{2})^i(n-\frac{1}{2})^{r-i}\right)x^{n+1} \\ &= \exp\left(\frac{\lambda^r}{r+1}\left((n+\frac{1}{2})^{r+1}-(n-\frac{1}{2})^{r+1}\right)\right)x^{n+1}. \end{aligned} \tag{3.76}$$

Thus, equation (3.74) implies that

$$(\lambda x\frac{\mathrm{d}}{\mathrm{d}x} - \mathcal{A})Z^{r,1}(x;\lambda) = 0. \tag{3.77}$$

3.7.3 Quantization

We show that the operator that annihilates the principal specialization of $Z^{r,1}$ can be obtained as a quantization of the equation of the spectral curve $S^{r,1}$.

We can rewrite the equation of the spectral curve (3.71) as

$$S^{r,1}:\; y - x^{\frac{3}{2}}\exp\left(\frac{\sum_{i=0}^{r} x^{-1}y^ixy^{r-i}}{r+1}\right)x^{-\frac{1}{2}} = 0. \tag{3.78}$$

Theorem 3.16. *Quantization of the equation of $S^{r,1}$ in this form annihilates $Z^{r,1}(x,\lambda)$.*

Proof. Indeed, apllying the standard quantization (3.59) to Equation (3.78) we obtain the operator $\lambda x\frac{\mathrm{d}}{\mathrm{d}x} - \mathcal{A}$. □

3.8 Mixed case

In this section we provide a slight generalization of the previous two sections, were we look at (connected) r-spin q-double Hurwitz numbers $h^{r,q}_{g,\mu}$. Since the computations are basically the same as in the previous two sections, we just give the main formulas. Note that for $r=1$ this reduces to the computations of Section 3.6, and for $q=1$ this reduces to those of Section 3.7.

These Hurwitz numbers are given as vacuum expectation values by:

$$h^{r,q}_{g,\mu} = \left\langle \prod_{i=1}^{\ell}\frac{\alpha_{\mu_i}}{\mu_i}\cdot\left(r![z^{r+1}]\tilde{\mathcal{E}}_0(z)\right)^m\cdot\frac{(\alpha_{-q})^s}{q^s s!}\right\rangle^{\circ}. \tag{3.79}$$

Here the degree of the covering is given by $d=\sum_{i=1}^{\ell(\mu)}\mu_i = qs$, and the Riemann-Hurwitz formula reads $2g-2+\ell(\mu) = mr - s$.

The full generating function is given by

$$\log Z^{r,q}(\mathbf{p};\lambda) = \sum_{g,\mu} \frac{h^{r,q}_{g;\mu}}{m!} \lambda^{2g-2+\ell(\mu)} p_{\mu_1}\cdots p_{\mu_{\ell(\mu)}} \tag{3.80}$$
$$= \left\langle \exp\left(\sum_{i=1}^{n} \frac{\alpha_i p_i}{i\lambda^{i/q}}\right) \exp\left(r![w^{r+1}]\tilde{\mathcal{E}}_0(w)\lambda^r\right) \exp\left(\frac{\alpha_{-q}}{q}\right)\right\rangle^{\circ},$$

and the $(0,1)$-function is given by

$$F_{0,1}(x) = q\sum_{n=0}^{\infty} \frac{((nr+1)q)^{n-2}}{n!} x^{(nr+1)q} \tag{3.81}$$

This leads to the following spectral curve:

$$S\colon\ x = y^{1/q} e^{-y^r}, \tag{3.82}$$

which means that

$$y = -\left(\frac{1}{rq}\right)^{\frac{1}{r}} W(-rqx^{rq})^{\frac{1}{r}}, \tag{3.83}$$

where W is the standard Lambert function.

The principal specialization $(p_i \mapsto x^i)$ of $Z^{r,q}$ is given by

$$Z^{r,q}(x,\lambda) = \sum_{n=0}^{\infty} \frac{x^{qn}}{\lambda^n q^n n!} e^{\frac{\lambda^r}{r+1}\left((qn-\frac{1}{2})^{r+1}-(-\frac{1}{2})^{r+1}\right)}, \tag{3.84}$$

which is annihilated by the operator

$$\lambda x\frac{d}{dx} - x^{q+1/2} e^{\frac{q}{r+1}\sum_{i=0}^{r} x^{-q}\left(\lambda x\frac{d}{dx}\right)^{i} x^{q}\left(\lambda x\frac{d}{dx}\right)^{r-i}} x^{-1/2} \tag{3.85}$$

This operator dequantizes to $y - x^q\exp(qy^r)$, which is equivalent to the equation (3.82) of the spectral curve $S^{r,q}$ computed from the $(0,1)$-geometry.

Furthermore, one sees immediately that under the specializations $(r,q) = (1,q)$ and $(r,1)$ we recover all the formulas we had in Sections 3.6 and 3.7.

–4– Integrals of ψ-classes over double ramication cycles

4.1 Introduction

In this chapter we introduce the so-called double ramification cycles on the the moduli space of curves, and prove a direct formula for their intersection with monomials in ψ-classes of complementary codimension.

4.1.1 Relative stable maps and double ramification cycles

Let $a_1, \ldots, a_n$ be a list of integers satisfying $\sum a_i = 0$. To a list like that we assign a space of "rubber" stable maps to $\mathbb{P}^1$ relative to 0 and ∞ in the following way.

Denote by n_+ the number of positive integers among the a_i. They form a partition $\mu = (\mu_1, \ldots, \mu_{n_+})$. Similarly, denote by n_- the number of negative integers among the a_i. After a change of sign they form another partition $\nu = (\nu_1, \ldots, \nu_{n_-})$. Both μ and ν are partitions of the same integer

$$d = \frac{1}{2} \sum_{i=1}^{n} |a_i|. \tag{4.1}$$

Finally, let n_0 be the number of vanishing a_i.

To the list $a_1, \ldots, a_n$ we assign the space

$$\overline{\mathcal{M}}_{g;a_1,\ldots,a_n} := \overline{\mathcal{M}}^{\sim}_{g,n_0;\mu,\nu}(\mathbb{P}^1, 0, \infty) \tag{4.2}$$

of degree d "rubber" stable maps to $\mathbb{P}^1$ relative to 0 and ∞ with ramification profiles μ and ν, respectively. Here "rubber" means that we factor the space by the $\mathbb{C}^*$ action in the target $\mathbb{P}^1$. We consider the pre-images of 0 and ∞ as marked points and there are n_0 more additional marked points.

Thus in the source curve there are n numbered marked points with labels $a_1, \ldots, a_n$. The relative stable map sends the points with positive labels to 0, those with negative labels to ∞, while those with zero labels do not have a fixed image.

We have a forgetful map

$$p \colon \overline{\mathcal{M}}_{g;a_1,\ldots,a_n} \to \overline{\mathcal{M}}_{g,n}. \tag{4.3}$$

Definition 4.1. The push-forward

$$p_*[\overline{\mathcal{M}}_{g;a_1,\ldots,a_n}]^{\text{virt}} \tag{4.4}$$

of the virtual fundamental class under the forgetful map p is called a *double ramification cycle* or a *DR-cycle* and is denoted by $\mathrm{DR}_g(a_1, \ldots, a_n)$.

It is known (see [43]) that the Poincaré dual cohomology class of $\mathrm{DR}_g(a_1,\ldots,a_n)$ lies in the tautological Chow ring of $\overline{\mathcal{M}}_{g,n}$. The virtual dimension of $\overline{\mathcal{M}}_{g;a_1,\ldots,a_n}$ and hence the dimension of $\mathrm{DR}_g(a_1,\ldots,a_n)$ equals $2g-3+n$.

A well-known problem, publicized in particular by Y. Eliashberg in view of applications to Symplectic Field Theory, is to find an explicit expression for the class $\mathrm{DR}_g(a_1,\ldots,a_n)$ in terms of the standard tautological classes. Recently R. Hain [54] found the restriction of $\mathrm{DR}_g(a_1,\ldots,a_n)$ to the locus $\overline{\mathcal{M}}^{\mathrm{c}}_{g,n}$ of curves with compact Jacobians. His expression is a homogeneous polynomial of degree $2g$ in $a_1,\ldots,a_n$ with coefficients in $H^g(\overline{\mathcal{M}}^{\mathrm{c}}_{g,n})$. In this paper we find the intersection numbers of $\mathrm{DR}_g(a_1,\ldots,a_n)$ with monomials in ψ-classes. Note that these numbers involve more than the knowledge of $\mathrm{DR}_g(a_1,\ldots,a_n)$ on $\overline{\mathcal{M}}^{\mathrm{c}}_{g,n}$. Thus our results are in some sense complementary with Hain's, even though they are still insufficient to deduce the complete expression for the double ramification cycles. For a given monomial in $\psi_1,\ldots,\psi_n$ the intersection number we find is a non-homogeneous polynomial of degree $2g$ in variables $a_1,\ldots,a_n$. This gives additional evidence to the following folklore conjecture.

Conjecture 4.2. $\mathrm{DR}_g(a_1,\ldots,a_n)$ *is a polynomial in* $a_1,\ldots,a_n$ *with coefficients in* $H^g(\overline{\mathcal{M}}_{g,n})$.

4.1.2 Plan of the chapter

In Section 4.2, we give a general formula for the intersection number of a double ramification cycle with any monomial in ψ-classes. We also give a particular case of this formula where the monomial consists of just some power of one ψ-class. The reason is that this formula is a lot simpler, interesting in its own right, and is used as a base case for an inductive proof of the more general formula later in this chapter.

In Section 4.3 we provide formulas for the intersection of a DR-cycle with a ψ-class in terms of other DR-cycles. In Section 4.4 we use those formulas to inductively proof Theorem 4.3. Theorem 4.4 is then proved in Section 4.5 using Theorem 4.3 as a base case and the splitting formulas from section 4.3 for the induction step.

4.2 Integral of ψ-classes over a DR-cycle: Theorem

We first give a formula for the intersection number of a double ramification cycle with a power of just one ψ-class, then we generalize this formula to the intersection number with any monomial in ψ-classes.

4.2.1 Intersection with one ψ-class

For the first formula, denote by $S(z)$ the power series

$$S(z)=\frac{\sinh(z/2)}{z/2}=\sum_{k\geq 0}\frac{z^{2k}}{2^{2k}\,(2k+1)!}=1+\frac{z^2}{24}+\frac{z^4}{1920}+\frac{z^6}{322560}+\ldots. \tag{4.5}$$

Theorem 4.3. *We have*

$$\psi_s^{2g-3+n}\mathrm{DR}_g(a_1,\ldots,a_n)=[z^{2g}]\frac{\prod_{i\neq s}S(a_iz)}{S(z)}, \tag{4.6}$$

where $[z^{2g}]$ *denotes the coefficient of* z^{2g}.

4.2.2 Intersection with several ψ-classes

Our next goal is to express the integral over a DR-cycle of a monomial in ψ-classes at different marked points. We will use the following notation.

- We let $\zeta(z) = e^{z/2} - e^{-z/2}$. (In the previous section we were using $S(z) = \zeta(z)/z$, but here $\zeta(z)$ is much more convenient.)
- For a permutation $\sigma \in S_n$ denote $a'_i = a_{\sigma(i)}$ and $z'_i = z_{\sigma(i)}$.
- Finally,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \tag{4.7}$$

Theorem 4.4. *Given a list of n integers $a_1, \ldots, a_n$, satisfying $\sum a_i = 0$ and a list of non-negative integers $d_1, \ldots, d_n$ satisfying $\sum d_i = 2g - 3 + n$, the integral*

$$\mathrm{DR}_g(a_1, \cdots a_n)\psi_1^{d_1} \cdots \psi_n^{d_n} \tag{4.8}$$

of a monomial in ψ-classes over a DR-cycle is equal to the coefficient of

$$z_1^{d_1} \cdots z_n^{d_n} \tag{4.9}$$

in the generating function

$$\frac{z_1 \cdots z_n}{\zeta(z_1 + \cdots + z_n)} \sum_{\substack{\sigma \in S_n \\ \sigma(1)=1}} \frac{\zeta\left(\begin{vmatrix} a'_1 & a'_2 \\ z'_1 & z'_2 \end{vmatrix}\right) \zeta\left(\begin{vmatrix} a'_1 + a'_2 & a'_3 \\ z'_1 + z'_2 & z'_3 \end{vmatrix}\right) \cdots \zeta\left(\begin{vmatrix} a'_1 + \cdots + a'_{n-1} & a'_n \\ z'_1 + \cdots + z'_{n-1} & z'_n \end{vmatrix}\right)}{z'_1 \begin{vmatrix} a'_1 & a'_2 \\ z'_1 & z'_2 \end{vmatrix} \begin{vmatrix} a'_2 & a'_3 \\ z'_2 & z'_3 \end{vmatrix} \cdots \begin{vmatrix} a'_{n-1} & a'_n \\ z'_{n-1} & z'_n \end{vmatrix} z'_n}. \tag{4.10}$$

Remark 4.5. The expression for the generating function is not written in a symmetrical form: the first marked point is singled out, since we only sum over the permutations that fix the element 1. However the generating function turns out to be symmetric in all n variables. The expression can be symmetrized by extending the summation to all permutations and dividing by n.

Remark 4.6. At first sight it appears that the generating function has simple poles along the hyperplanes $a_i z_j - a_j z_i$ (because of the determinants in the denominator) and $z_1 + \cdots + z_n = 0$ (because of the $\zeta(z_1 + \cdots + z_n)$ in the denominator). It is easy to see, however, that these denominators actually simplify.

Indeed, in each summand the factor $a'_1 z'_2 - a'_2 z'_1$ simplifies with $\zeta(a'_1 z'_2 - a'_2 z'_1)$. But this was the only factor of the form $a_1 z_i - a_i z_1$, thus no factor like that remains in the denominator of any summand and hence of the total sum. Since the first marked point was singled out arbitrarily, this implies that no factor of the form $a_i z_j - a_j z_i$ remains in the denominator.

As for the factor $z_1 + \cdots + z_n$, it simplifies with

$$\zeta\left(\begin{vmatrix} a'_1 + \cdots + a'_{n-1} & a'_n \\ z'_1 + \cdots + z'_{n-1} & z'_n \end{vmatrix}\right), \tag{4.11}$$

if we take into account that $a'_1 + \cdots + a'_{n-1} = -a'_n$.

The only case where this reasoning breaks down is when $n = 2$. Indeed, in this case $z_1 + z_2$ and $a_1 z_2 - a_2 z_1 = a_1(z_1 + z_2)$ are twice the same factor, but this factor is only compensated for once in the numerator. In this case the generating function does contain a singularity of the form $1/(z_1 + z_2)$ (see Example 4.7). This singular term should be ignored when we extract the coefficients.

Example 4.7. For $n = 2$ we let $a_1 = a$, $a_2 = -a$. There is only one permutation in S_2 that fixes the first element. Thus we get the generating function

$$\frac{z_1 z_2}{\zeta(z_1+z_2)} \frac{\zeta(a(z_1+z_2))}{z_1\, a(z_1+z_2)\, z_2} = \frac{\zeta(a(z_1+z_2))}{a(z_1+z_2)\zeta(z_1+z_2)} \tag{4.12}$$

$$= \frac{1}{z_1+z_2} + \frac{a^2-1}{24}(z_1+z_2) + \frac{(a^2-1)(3a^2-7)}{5760}(z_1+z_2)^3 + \cdots \tag{4.13}$$

It follows that

$$\mathrm{DR}_1(a,-a)\psi_1 = \mathrm{DR}_1(a,-a)\psi_2 = \frac{a^2-1}{24}, \tag{4.14}$$

$$\mathrm{DR}_2(a,-a)\psi_1^3 = \mathrm{DR}_2(a,-a)\psi_2^3 = \frac{(a^2-1)(3a^2-7)}{5760}, \tag{4.15}$$

$$\mathrm{DR}_2(a,-a)\psi_1^2\psi_2 = \mathrm{DR}_2(a,-a)\psi_1\psi_2^2 \tag{4.16}$$

$$= \frac{3(a^2-1)(3a^2-7)}{5760} = \frac{(a^2-1)(3a^2-7)}{1920}. \tag{4.17}$$

Example 4.8. For $n = 3$ we have $a_3 = -(a_1 + a_2)$. There are two summands in the formula corresponding to the permutations $(1, 2, 3)$ and $(1, 3, 2)$. We get

$$\frac{1}{\zeta(z_1+z_2+z_3)} \left\{ \frac{\zeta(a_1z_2 - a_2z_1)}{a_1z_2 - a_2z_1} \frac{z_2\,\zeta\big((a_1+a_2)(z_1+z_2+z_3)\big)}{a_2z_3 + (a_1+a_2)z_2} + \right. \tag{4.18}$$

$$\left. \frac{\zeta(a_1z_3 + (a_1+a_2)z_1)}{a_1z_3 + (a_1+a_2)z_1} \frac{z_3\,\zeta\big(a_2(z_1+z_2+z_3)\big)}{a_2z_3 + (a_1+a_2)z_2} \right\}. \tag{4.19}$$

Expanding this expression we get, in particular,

$$\mathrm{DR}_1(a_1,a_2,a_3)\psi_1^2 = \frac{a_2^2 + a_3^2 - 1}{24}, \tag{4.20}$$

$$\mathrm{DR}_1(a_1,a_2,a_3)\psi_1\psi_2 = \frac{a_1^2 + a_2^2 + a_3^2 - 2}{24}, \tag{4.21}$$

where we have re-introduced a_3 for more symmetry.

4.2.3 Completed cycles as a particular case of Theorem 4.3

Let $\overline{\mathcal{M}}^0_{g,1,K;\kappa}(\mathrm{CP}^1,\infty)$ be the space of degree K relative stable maps $f\colon C \to \mathrm{CP}^1$ with branching profile $\kappa = (k_1, \ldots, k_n)$ over ∞ and with one marked point $x \in C$ satisfying the condition that $f(x) = 0$. It is a natural problem to find an effective cycle representing the homology class

$$[\overline{\mathcal{M}}^0_{g,1,K;\kappa}(\mathrm{CP}^1,\infty)]^{\mathrm{virt}}\ \psi_x^m. \tag{4.22}$$

Okounkov and Pandharipande gave an answer to this question when m is equal to the virtual dimension $K + n + 2g - 2$ of $\overline{\mathcal{M}}^0_{g,1,K;\kappa}(\mathrm{CP}^1,\infty)$ and thus the answer is just a number. To simplify the formula we assume that the n pre-images of ∞ in our space of relative stable maps are numbered. Then we have the following equality.

Theorem. *(Okounkov, Pandharipande [88]). For $m = K + n + 2g - 2$ we have*

$$[\overline{\mathcal{M}}^0_{g,1,K;\kappa}(\mathrm{CP}^1,\infty)]^{\mathrm{virt}}\ \psi_x^m = m!\frac{\prod_{i=1}^n k_i}{K!}[z^{2g}]S(z)^{K-1}\prod_{i=1}^n S(k_iz). \tag{4.23}$$

Using the degeneration of the target it is not hard to generalize this formula to several relative points with ramification types $\mu_1, \ldots, \mu_s$. In particular, for the case of two relative points, the following expression is given in [88], Eq. (3.11) or [94], Eq. (10). Let $a_1, \ldots, a_n$ be the list of elements of μ_1 merged with the list of elements of μ_2 with reversed signs. Thus $\sum a_i = 0$. Denote by ψ_x the ψ-class at the marked point x.

Theorem (Okounkov-Pandharipande, Rossi). *We have*

$$[\overline{\mathcal{M}}^0_{g,1,d;\mu_1,\mu_2}(\mathrm{CP}^1, p_1, p_2)]^{\mathrm{virt}}\ \psi_x^{n+2g-2} = [z^{2g}] \frac{\prod_{i=1}^n S(a_i z)}{S(z)}. \tag{4.24}$$

It is easy to see that this formula is a particular case of Theorem 4.3, namely, the case when $a_s = 0$ while all other a_i's do not vanish. The case where $a_s = 0$ and some other a_i's may also vanish is covered by a more general computation in Proposition 2.5 of [89].

Actually, we don't have an independent proof for the case $a_s = 0$; we just invoke the above result. Our proof for the case $a_s \neq 0$ is quite different and does not generalize to $a_s = 0$. Thus we get the same answer for $a_s = 0$ and $a_s \neq 0$, even though we do not know any proof that would work in both situations.

4.3 DR-cycle times a ψ-class: splitting formulas

In this section we express the intersection of a ψ-class with a double-ramification cycle in terms of "splittings" of the DR-cycle. These formulas can then be used in the next sections to give inductive proofs of Theorems 4.3 and 4.4.

4.3.1 The splitting formulas - formulations

In this section we express the product of a double ramification cycle $\mathrm{DR}_g(a_1, \ldots, a_n)$ and the class ψ_s for some $s \in \{1, \ldots, n\}$ in terms of other DR-cycles. This will make it possible to evaluate monomials in ψ-classes on a DR-cycle by induction. Note that we can only do that if $a_s \neq 0$.

The picture below shows a cycle in $\overline{\mathcal{M}}_{g,n}$ obtained from two DR-cycles via a gluing map.

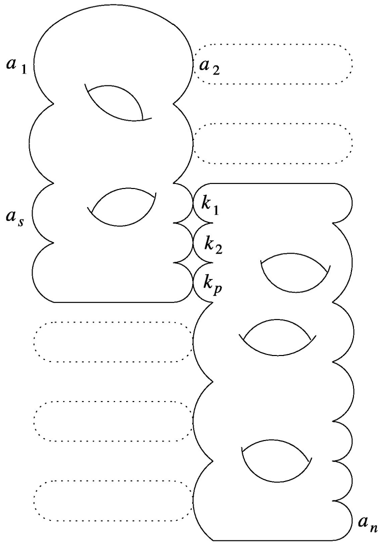
The two DR-cycles are constructed in the following way. The list $a_1, \ldots, a_n$ is divided into two disjoint parts: $I \sqcup J = \{1, \ldots, n\}$ in such a way that $\sum_{i \in I} a_i > 0$ or, equivalently, $\sum_{i \in J} a_i < 0$. In the figure, for instance, we have $1, 2, s \in I$ and $n \in J$. Then a new list of positive integers $k_1, \ldots, k_p$ is chosen in such a way that

$$\sum_{i \in I} a_i - \sum_{i=1}^p k_i = \sum_{i \in J} a_i + \sum_{i=1}^p k_i = 0. \tag{4.25}$$

Now two DR-cycles of genera g_1 and g_2 are formed as shown in the figure and glued together at the "new" marked points labelled $k_1, \ldots, k_p$. Since we want to get a genus g in the end we impose the condition $g_1 + g_2 + p - 1 = g$. We denote by

$$\mathrm{DR}_{g_1}(a_I, -k_1, \ldots, -k_p) \boxtimes \mathrm{DR}_{g_2}(a_J, k_1, \ldots, k_p) \tag{4.26}$$

the resulting cycle in $\overline{\mathcal{M}}_{g,n}$.



Let $r = 2g - 2 + n$ be the number of branch points of our initial DR-cycle $\mathrm{DR}(a_1, \dots, a_n)$. Let $r' = 2g_1 - 2 + |I| + p$ and $r'' = 2g_2 - 2 + |J| + p$ be the numbers of branch points in the two components of the target curve. (In both cases we do not count 0 and ∞.)

Theorem 4.9. *Let $a_1, \dots, a_n$ be a list of integers with vanishing sum. Assume that $a_s \neq 0$. Then we have*

$$a_s \psi_s \mathrm{DR}_g(a_1, \dots, a_n) = \tag{4.27}$$

$$\sum_{I,J} \sum_{p \geq 1} \sum_{g_1, g_2} \sum_{k_1, \dots, k_p} \frac{\rho}{r} \frac{\prod_{i=1}^{p} k_i}{p!} \mathrm{DR}_{g_1}(a_I, -k_1, \dots, -k_p) \boxtimes \mathrm{DR}_{g_2}(a_J, k_1, \dots, k_p). \tag{4.28}$$

Here the first sum is taken over all $I \sqcup J = \{1, \dots, n\}$ such that $\sum_{i \in I} a_i > 0$; the third sum is over all non-negative genera g_1, g_2 satisfying $g_1 + g_2 + p - 1 = g$; the fourth sum is over the p-uplets of positive integers with total sum $\sum_{i \in I} a_i = -\sum_{i \in J} a_i$. The number ρ is defined by

$$\rho = \begin{cases} r'' & \text{if} \quad s \in I, \\ -r' & \text{if} \quad s \in J. \end{cases} \tag{4.29}$$

Theorem 4.10. *Let $a_1, \dots, a_n$ be a list of integers with vanishing sum. Assume that $a_s \neq 0$ and $a_l = 0$. Then we have*

$$a_s \psi_s \mathrm{DR}_g(a_1, \dots, a_n) = \tag{4.30}$$

$$\sum_{I,J} \sum_{p \geq 1} \sum_{g_1, g_2} \sum_{k_1, \dots, k_p} \varepsilon \, \frac{\prod_{i=1}^{p} k_i}{p!} \mathrm{DR}_{g_1}(a_I, -k_1, \dots, -k_p) \boxtimes \mathrm{DR}_{g_2}(a_J, k_1, \dots, k_p). \tag{4.31}$$

Here the first sum is taken over all $I \sqcup J = \{1, \dots, n\}$ such that $\sum_{i \in I} a_i > 0$; the third sum is over all non-negative genera g_1, g_2 satisfying $g_1 + g_2 + p - 1 = g$; the fourth sum is over the p-uplets of positive integers with total sum $\sum_{i \in I} a_i = -\sum_{i \in J} a_i$. The number ε is defined by

$$\varepsilon = \begin{cases} 1 & \text{if} \quad s \in I, l \in J, \\ -1 & \text{if} \quad s \in J, l \in I, \\ 0 & \text{otherwise.} \end{cases} \tag{4.32}$$

Theorem 4.9 is called the *splitting formula with respect to branching points*, while Theorem 4.10 is the *splitting formula with respect to a marked point.* Before proving the theorems let us formulate some corollaries that we will use in our computations.

Corollary 4.11. *Assume that $a_s \neq 0$. We have*

$$\begin{aligned} &r a_s\, \psi_s^{r-1} \mathrm{DR}_g(a_1, \ldots, a_n) = \\ &-\frac{1}{2} \sum_{i,j \neq s} (a_i + a_j)\, \psi_s^{r-2} \mathrm{DR}_g(a_1, \ldots, \widehat{a_i}, \ldots, \widehat{a_j}, \ldots, a_n, a_i + a_j) \\ &-\frac{1}{2} \sum_{i \neq s} \mathrm{sign}(a_i) \sum_{\substack{b+c=a_i \\ b \cdot c > 0}} bc\, \psi_s^{r-2} \mathrm{DR}_{g-1}(a_1, \ldots, \widehat{a_i}, \ldots, a_n, b, c). \end{aligned} \tag{4.33}$$

Here, as before, $r = 2g - 2 + n$ and a hat means that the element is skipped.

Proof. We will use the splitting formula with respect to the branch points. Since we are interested in the intersection number of our DR-cycle with ψ_s^{r-1} we only need to keep those terms of the splitting formula for which the s^{th} marked point stays on a DR-cycle of dimension $r-2$. This implies that the remaining DR-cycle is of dimension 0, that is, it is of the form $\mathrm{DR}_0(a, b, c)$. The expression in the corollary is a sum over all splittings of this form. □

This corollary gives a recursive relation for intersection numbers of DR-cycles with powers of *one* ψ-class. We will use it to prove Theorem 4.3.

Corollary 4.12. *Let t and s be two different elements in $\{1, \ldots, n\}$. Assume that both a_s and a_t are non-zero. Then we have*

$$(a_s \psi_s - a_t \psi_t) \mathrm{DR}_g(a_1, \ldots, a_n) \tag{4.34}$$

$$= \sum_{s \in I, t \in J} \sum_{p \geq 1} \sum_{g_1, g_2} \sum_{k_1, \ldots, k_p} \frac{\prod_{i=1}^p k_i}{p!} \mathrm{DR}_{g_1}(a_I, -k_1, \ldots, -k_p) \boxtimes \mathrm{DR}_{g_2}(a_J, k_1, \ldots, k_p) \tag{4.35}$$

$$- \sum_{t \in I, s \in J} \sum_{p \geq 1} \sum_{g_1, g_2} \sum_{k_1, \ldots, k_p} \frac{\prod_{i=1}^p k_i}{p!} \mathrm{DR}_{g_1}(a_I, -k_1, \ldots, -k_p) \boxtimes \mathrm{DR}_{g_2}(a_J, k_1, \ldots, k_p). \tag{4.36}$$

Here, as before, the first sum is taken over all $I \sqcup J = \{1, \ldots, n\}$ such that $\sum_{i \in I} a_i > 0$; the third sum is over all non-negative genera g_1, g_2 satisfying $g_1 + g_2 + p - 1 = g$; the fourth sum is over the p-uplets of positive integers with total sum $\sum_{i \in I} a_i = -\sum_{i \in J} a_i$.

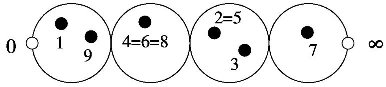
Proof. This follows directly from the splitting formula with respect to the branch points. It suffices to notice that the expressions it provides for $a_s \psi_s$ and $a_t \psi_t$ only differ in the definition of r'. □

Multiplying the identity in this corollary by any monomial in ψ-classes of degree $2g-4+n$ we obtain a simple way to "move" a ψ-class from one marked point to another.

4.3.2 The splitting formulas - proofs

Plan of proof

Our proof uses the Losev-Manin compactification LM_r of $\mathcal{M}_{0,r+2}$. It is the moduli space of chains of spheres with two special "white" marked points 0 and ∞ at the extremities of the chain and r more "black" marked points on the other spheres. The black points are allowed to coincide with each other and there should be at least one black point per sphere. For more details see [75].



We have two forgetful maps from the DR-space $\overline{\mathcal{M}}_{g;a_1,\dots,a_n}$:

$$\mathrm{LM}_{r+n_0}/S_r \xleftarrow{q} \overline{\mathcal{M}}_{g;a_1,\dots,a_n} \xrightarrow{p} \overline{\mathcal{M}}_{g,n}, \tag{4.37}$$

where n_0 is the number of indices i such that $a_i = 0$ and $r = 2g-2+n$ is the number of branch points.

The map q assigns to a relative stable map its target rational curve. The marked points are the r branch points and the images of the marked points in the source curve. The map p assigns to a relative stable map its stabilized source curve. (This is the map that we used to define the DR-cycle $\mathrm{DR}_g(a_1,\dots,a_n)$.)

The proof of the splitting formulas proceeds as follows.

1. Identify the ψ-class on the DR-space with the ψ-class on the Losev-Manin space.
2. Express the ψ-class on the Losev-Manin space as a sum of boundary divisors (we will do that in two ways, whence two splitting formulas).
3. Lift these divisors to the DR-space
4. Subtract the difference between the ψ-class on the DR-space and the ψ-class on $\overline{\mathcal{M}}_{g,n}$.

We start with two lemmas that will be needed in the course of the proof.

DR-cycles with disconnected domains

Consider the space of stable maps to CP^1 relative to 0 and ∞, but with disconnected domains

$$\overline{\mathcal{M}}_{g_1;a^1_1,\dots,a^1_{n_1}} \times \dots \times \overline{\mathcal{M}}_{g_k;a^k_1,\dots,a^k_{n_k}}. \tag{4.38}$$

We assume that $2g_i - 2 + n_i > 0$ for each i. From the corresponding rubber space

$$\left(\overline{\mathcal{M}}_{g_1;a^1_1,\dots,a^1_{n_1}} \times \dots \times \overline{\mathcal{M}}_{g_k;a^k_1,\dots,a^k_{n_k}}\right)^\sim \tag{4.39}$$

there is a natural forgetful map p to the product of moduli spaces $\overline{\mathcal{M}}_{g_1,n_1} \times \dots \times \overline{\mathcal{M}}_{g_k,n_k}$.

Lemma 4.13. *The image of the virtual fundamental class of*

$$\left(\overline{\mathcal{M}}_{g_1;a^1_1,\dots,a^1_{n_1}} \times \dots \times \overline{\mathcal{M}}_{g_k;a^k_1,\dots,a^k_{n_k}}\right)^\sim \tag{4.40}$$

in $\overline{\mathcal{M}}_{g_1,n_1} \times \dots \times \overline{\mathcal{M}}_{g_k,n_k}$ under the forgetful map p vanishes.

Even though the computations of this section take place in the DR-space, the goal of the paper is to study the DR-cycles, that is, the images of the virtual fundamental classes of DR-spaces by the map p. Therefore in the sequel of this section we will perform all our computations "modulo terms with disconnected domains". In other words, we will disregard all the terms that, according to the lemma, vanish after the push-forward by p.

Proof. We will call *parts* the k connected components of the curves.

Adding a new marked point. Consider the space

$$\left(\overline{\mathcal{M}}_{g_1;a_1^1,\dots,a_{n_1}^1,0}\times\cdots\times\overline{\mathcal{M}}_{g_k;a_1^k,\dots,a_{n_k}^k}\right)^{\sim}. \tag{4.41}$$

If π is the forgetful map that forgets the new point, we have, by the dilaton relation,

$$\pi_*\left\{\left[\left(\overline{\mathcal{M}}_{g_1;a_1^1,\dots,a_{n_1}^1,0}\times\cdots\times\overline{\mathcal{M}}_{g_k;a_1^k,\dots,a_{n_k}^k}\right)^{\sim}\right]^{\mathrm{virt}}\psi_{n_1+1}\right\} \tag{4.42}$$

$$=(2g_1-2+n_1)\left[\left(\overline{\mathcal{M}}_{g_1;a_1^1,\dots,a_{n_1}^1}\times\cdots\times\overline{\mathcal{M}}_{g_k;a_1^k,\dots,a_{n_k}^k}\right)^{\sim}\right]^{\mathrm{virt}}. \tag{4.43}$$

Thus it suffices to prove that the image of

$$\left[\left(\overline{\mathcal{M}}_{g_1;a_1^1,\dots,a_{n_1}^1,0}\times\cdots\times\overline{\mathcal{M}}_{g_k;a_1^k,\dots,a_{n_k}^k}\right)^{\sim}\right]^{\mathrm{virt}}\psi_{n_1+1} \tag{4.44}$$

vanishes in $\overline{\mathcal{M}}_{g_1,n_1+1}\times\cdots\times\overline{\mathcal{M}}_{g_k,n_k}$.

Introducing a $\mathbb{C}^*$-action On the space

$$\left(\overline{\mathcal{M}}_{g_1;a_1^1,\dots,a_{n_1}^1,0}\times\cdots\times\overline{\mathcal{M}}_{g_k;a_1^k,\dots,a_{n_k}^k}\right)^{\sim} \tag{4.45}$$

we can introduce a $\mathbb{C}^*$-action in the following way. Let $f: C\to S$ be a rubber map, where S is a genus 0 curve from the Losev-Manin space. Let $S_\bullet$ be the irreducible component of S that contains the image of the new marked point, that is, the $(n_1+1)^{\mathrm{st}}$ marked point in the first part of C. (The purpose of adding a new marked point was precisely to be able to single out a component of S in this way.) Now, for $\lambda\in\mathbb{C}^*$, we let $\lambda.f$ be equal to f on every component of C that does not map to $S_\bullet$ or is in the first part (that is, the part that contains the new marked point). On the components of the other parts that map to $S_\bullet$ we let $\lambda.f=\lambda f$.

The pull-back of any differential form from $\overline{\mathcal{M}}_{g_1,n_1+1}\times\cdots\times\overline{\mathcal{M}}_{g_k,n_k}$ to our DR-space is $\mathbb{C}^*$-invariant, because the action of $\mathbb{C}^*$ does not change the complex structure of the source curve. We are going to prove by localization that the integral against

$$\left[\left(\overline{\mathcal{M}}_{g_1;a_1^1,\dots,a_{n_1}^1,0}\times\cdots\times\overline{\mathcal{M}}_{g_k;a_1^k,\dots,a_{n_k}^k}\right)^{\sim}\right]^{\mathrm{virt}}\psi_{n_1+1} \tag{4.46}$$

of any $\mathbb{C}^*$-invariant form vanishes.

Localization. The invariant locus of the $\mathbb{C}^*$-action is composed of maps that have no marked or ramification points over $S_\bullet$ on parts $2,\dots,k$. Thus the invariant locus has three types of components, classified by the topological type of the target genus 0 curve S (at the generic point of the component of the locus):

1. the curve S has the form $S'\cup S_\bullet$;
2. the curve S has the form $S_\bullet\cup S''$;
3. the curve S has the form $S'\cup S_\bullet\cup S''$.

Each component of the invariant locus is the product of two (in the first two cases) or three (in the last case) disconnected DR-spaces and has the same virtual fundamental class. A simple dimension count shows that the virtual dimension of each component of the invariant locus is less than the virtual dimension of the original DR-space. (Indeed, the dimension is equal to the number of marked and branch points minus the number of components of S.) Therefore each term in the localization formula vanishes, completing the proof of the lemma. □

Pull-backs of divisors from the Losev-Manin space

Consider a DR-space $\overline{\mathcal{M}}_{g;a_1,\dots,a_n}$ and consider the forgetful map

$$q : \overline{\mathcal{M}}_{g;a_1,\dots,a_n} \to \mathrm{LM}_{r+n_0}/S_r. \tag{4.47}$$

Let $\alpha \sqcup \beta$ be a partition of the set of indices i such that $a_i = 0$. Let $r' + r'' = r$. Denote by $D_{(r',\alpha|r'',\beta)}$ the boundary divisor in the space LM_{r+n_0}/S_r with self-explanatory notation.

Lemma 4.14. *Modulo terms with disconnected domains, we have*

$$q^* D_{(r',\alpha|r'',\beta)} \left[\overline{\mathcal{M}}_{g;a_1,\dots,a_n}\right]^{\mathrm{virt}} = \tag{4.48}$$

$$\sum_{I,J} \sum_{p\geq 1} \sum_{g_1,g_2} \sum_{k_1,\dots,k_p} \frac{\prod_{i=1}^p k_i}{p!} \left[\overline{\mathcal{M}}_{g_1;a_I,-k_1,\dots,-k_p}\right]^{\mathrm{virt}} \boxtimes \left[\overline{\mathcal{M}}_{g_2;a_J,k_1,\dots,k_p}\right]^{\mathrm{virt}}. \tag{4.49}$$

Here the first sum is taken over all $I \sqcup J = \{1,\dots,n\}$ such that $\alpha \subset I$, $\beta \subset J$ and $\sum_{i\in I} a_i > 0$; the third sum is over all non-negative genera g_1, g_2 satisfying $g_1 + g_2 + p - 1 = g$; the fourth sum is over the p-uplets of positive integers with total sum $\sum_{i\in I} a_i = -\sum_{i\in J} a_i$.

Proof. This lemma is a version of Jun Li's degeneration formula [72, 73]. It should be applied to the target rational curve where all branch points have been marked and numbered. We then take the sum of contributions from all possible ways to put r' marked point on one component of the degeneration and r'' on the other component. After this we can forget the numbering of the branch points once again.

In the degeneration formula we see DR-spaces with both connected and disconnected domains over each component of the target. However, since we are working modulo terms with disconnected domains, only the terms indicated in the lemma survive. □

Comparing the ψ-classes on different spaces

Recall the two forgetful maps from the DR-space

$$p : \overline{\mathcal{M}}_{g;a_1,\dots,a_n} \to \overline{\mathcal{M}}_{g,n} \tag{4.50}$$

and

$$q : \overline{\mathcal{M}}_{g;a_1,\dots,a_n} \to \mathrm{LM}_{r+n_0}/S_r. \tag{4.51}$$

Denote by Ψ_s and ψ_s the ψ-classes at the sth marked point on $\overline{\mathcal{M}}_{g;a_1,\dots,a_n}$ and on $\overline{\mathcal{M}}_{g,n}$ respectively. Denote by ψ_0 and ψ_∞ the ψ-classes on the Losev-Manin space.

Proposition 4.15. *Assume that $a_s \neq 0$. We have*

$$a_s \Psi_s = q^*\psi_0 \quad \textit{if} \quad a_s > 0, \tag{4.52}$$

$$-a_s \Psi_s = q^*\psi_\infty \quad \textit{if} \quad a_s < 0. \tag{4.53}$$

Proof. This simple but very useful statement first appeared in Ionel's paper [55]. Assume for definiteness that $a_s > 0$. Then q obviously identifies the tangent line to 0 in the target with the a_sth power of the tangent line to the sth marked point in the source, which proves the proposition. □

Lemma 4.16. *Assume that* $a_s \neq 0$. *Modulo terms with disconnected domains we have*

$$\Psi_s - p^*\psi_s = \tag{4.54}$$

$$\frac{1}{|a_s|}\sum_{I,J}\sum_{p\geq 1}\sum_{g_1,g_2}\sum_{k_1,\dots,k_p}\frac{\prod_{i=1}^p k_i}{p!}\left[\overline{\mathcal{M}}_{g_1;a_I,-k_1,\dots,-k_p}\right]^{\mathrm{virt}}\boxtimes\left[\overline{\mathcal{M}}_{g_2;a_J,k_1,\dots,k_p}\right]^{\mathrm{virt}}. \tag{4.55}$$

Here the first sum is taken over all $I \sqcup J = \{1,\dots,n\}$ *such that* $\sum_{i\in I} a_i > 0$ *and* $s \in J$ *if* $a_s > 0$ *or* $s \in I$ *if* $a_s < 0$; *the third sum is over all non-negative genera* g_1, g_2 *satisfying* $g_1 + g_2 + p - 1 = g$; *the fourth sum is over the p-uplets of positive integers with total sum* $\sum_{i\in I} a_i = -\sum_{i\in J} a_i$.

Proof. The sum in the lemma enumerates all the boundary divisors, modulo the ones with disconnected domains, on which the sth marked point lies on a bubble (that is, on a rational component that gets contracted by the forgetful map p). It is precisely those divisors that contribute to the difference between the two ψ-classes. It remains to determine the coefficients.

Assume, for definiteness, that $a_s > 0$. A divisor enumerated in the sum splits the marked and branch points in the target into two groups: those that lie on the component of 0 and those that lie on the component of ∞. Consider the map $\tilde{q}$ that forgets all the marked and branch points from the component of 0. It is a forgetful map between two Losev-Manin spaces. Denote by ψ_0' the ψ-class at 0 on the smaller Losev-Manin space, that is, on the image of the forgetful map. It is easy to see that in the neighbourhood of our divisor and outside of the other divisors enumerated in the lemma we have $a_s\psi_s = q^*\tilde{q}^*\psi_0'$ and therefore

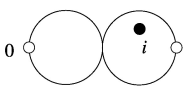
$$a_s(\Psi_s - p^*\psi_s) = q^*(\psi_0 - \tilde{q}^*\psi_0'). \tag{4.56}$$

The difference $\psi_0 - \tilde{q}^*\psi_0'$ is exactly given by the divisor where the target curve degenerates. Therefore the coefficient of our divisor in $\Psi_s - p^*\psi_s$ is equal to the coefficient of the same divisor in Lemma 4.14 divided by a_s. ☐

Expressing ψ_0 and ψ_∞ as boundary divisors

The class ψ_0 on the Losev-Manin space LM_{r+n_0} is easily expressed as a sum of boundary divisors: namely, for any $i \in \{1,\dots,r+n_0\}$, we have

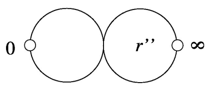
$$\psi_0 = \sum \left[\quad\right], \tag{4.57}$$



where the sum is over all boundary divisors such that the ith marked point lies on the same component as ∞. If i is the image of a marked point we leave this expression as it is.

If i is a branch point, it makes sense to symmetrize the expression with respect to the S_r action, since we are working with the quotient LM_{r+n_0}/S_r. We get

$$\psi_0 = \sum \frac{r''}{r}\left[\quad\right], \tag{4.58}$$



where the sum is over all boundary divisors and r'' is the number of branch points on the component of ∞.

Computing $p^*\psi_s$

Now we prove Theorems 4.9 and 4.10 using the preceding lemmas to express $p^*\psi_s$ in terms of boundary divisors. Assume for definiteness that $a_s > 0$. Then we have

$$a_s p^*\psi_s = a_s\Psi_s - a_s(\Psi_s - p^*\psi_s) = q^*\psi_0 - a_s(\Psi_s - p^*\psi_s). \tag{4.59}$$

Equations (4.58) and (4.57) give two alternative expressions for $q^*\psi_0$ while Lemma 4.16 gives an expression for $a_s(\Psi_s - p^*\psi_s)$. All three expressions involve very similar summations over the same set of divisors, but with different coefficients.

Proof of Theorem 4.9. We use Equation (4.58) for $q^*\psi_0$. The coefficient of

$$\frac{\prod_{i=1}^p k_i}{p!}\left[\overline{\mathcal{M}}_{g_1;a_I,-k_1,\dots,-k_p}\right]^{\mathrm{virt}} \boxtimes \left[\overline{\mathcal{M}}_{g_2;a_J,k_1,\dots,k_p}\right]^{\mathrm{virt}} \tag{4.60}$$

in Eq. (4.58) equals r''/r. Its coefficient in Lemma 4.16 multiplied by a_s equals 1 if $s \in J$ or 0 if $s \in I$. Subtracting the second coefficient from the first one and using $r' + r'' = r$ we get

$$\frac{r''}{r} \quad \text{if} \quad s \in I, \tag{4.61}$$

$$-\frac{r'}{r} \quad \text{if} \quad s \in J. \tag{4.62}$$

These are exactly the coefficients from Theorem 4.9. □

Proof of Theorem 4.10. We use Equation (4.57) for $q^*\psi_0$. Denote by l the index of the marked point with $a_l = 0$ that appears in this equation. The coefficient of

$$\frac{\prod_{i=1}^p k_i}{p!}\left[\overline{\mathcal{M}}_{g_1;a_I,-k_1,\dots,-k_p}\right]^{\mathrm{virt}} \boxtimes \left[\overline{\mathcal{M}}_{g_2;a_J,k_1,\dots,k_p}\right]^{\mathrm{virt}} \tag{4.63}$$

in Eq. (4.57) equals 1 if $l \in J$ and 0 otherwise. Its coefficient in Lemma 4.16 multiplied by a_s equals 1 if $s \in J$ and otherwise. Subtracting the second coefficient from the first we get

$$1 \quad \text{if} \quad s \in I, l \in J, \tag{4.64}$$

$$-1 \quad \text{if} \quad s \in J, l \in I, \tag{4.65}$$

and 0 otherwise. These are exactly the coefficients from Theorem 4.10. □

Both theorems are proved.

4.3.3 A digression on admissible coverings

Double ramification cycles have an alternative definition, using admissible coverings rather than relative stable maps (see, for instance, [55]). To distinguish the two notions, just for the length of this section, we will write $\mathrm{DR}^{\mathrm{adm}}$ and $\mathrm{DR}^{\mathrm{stab}}$. The goal of this section is to explain what would change in our results if we replaced $\mathrm{DR}^{\mathrm{stab}}$ by $\mathrm{DR}^{\mathrm{adm}}$. This section is not self-contained, since we don't introduce the admissible coverings here; it can be skipped in first reading.

Example 4.17. We have

$$\mathrm{DR}_1^{\mathrm{adm}}(a, \widetilde{-a}) = a^2 - 1 \in H^0(\overline{\mathcal{M}}_{1,1}), \tag{4.66}$$

$$\mathrm{DR}_1^{\mathrm{stab}}(a, \widetilde{-a}) = a^2 \in H^0(\overline{\mathcal{M}}_{1,1}), \tag{4.67}$$

where the tilde means that the the corresponding marked point is forgotten.

Indeed, given an elliptic curve (C, x) with one marked point, there exists a^2 points y such that $x - y$ is an a-torsion point in the Jacobian of C. The space of admissible coverings contains one point per $y \neq x$, that is, $a^2 - 1$ points. The space of rubber maps contains one additional point corresponding to $y = x$: it represents the map with a contracted elliptic component.

Theorem 4.18. *The intersection numbers of a monomial* $\psi_1^{d_1} \cdots \psi_n^{d_n}$ *with* $\mathrm{DR}_g^{\mathrm{adm}}(a_1, \dots, a_n)$ *and with* $\mathrm{DR}_g^{\mathrm{stab}}(a_1, \dots, a_n)$ *coincide if none of the* a_i*'s vanishes. These intersection numbers may differ in presence of an* $a_i = 0$*.*

Corollary 4.19. *At least for some* g *and* n *the class* $\mathrm{DR}_g^{\mathrm{adm}}(a_1, \dots, a_n)$ *does not have a polynomial dependence on* $a_1, \dots, a_n$*.*

Proof. Our formulas show that the intersection number of a given monomial in ψ-classes with $\mathrm{DR}_g^{\mathrm{stab}}(a_1, \dots, a_n)$ depends polynomially on $a_1, \dots, a_n$. The intersection number of the same monomial with $\mathrm{DR}_g^{\mathrm{adm}}(a_1, \dots, a_n)$ has the same values for non-zero a_i's, but different values if some of the a_i's vanish. Therefore this intersection number cannot depend polynomially on $a_1, \dots, a_n$. Hence the class itself cannot depend polynomially on $a_1, \dots, a_n$. □

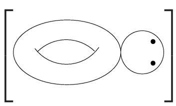
Remark 4.20. Ultimately it's Corollary 4.19 that convinced us that $\mathrm{DR}^{\mathrm{stab}}$-cycles must be preferred to $\mathrm{DR}^{\mathrm{adm}}$-cycles.

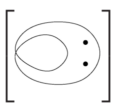
Proof of Theorem 4.18. The first claim of the theorem is proved by checking that all the steps of our computation of $\mathrm{DR}_g(a_1, \dots, a_n)\psi_1^{d_1} \cdots \psi_n^{d_n}$ go through in the same way for $\mathrm{DR}^{\mathrm{adm}}$ and $\mathrm{DR}^{\mathrm{stab}}$ as long as $a_1, \dots, a_n$ do not vanish.

Theorem 4.9 is the base of our computations. Its analogue for $\mathrm{DR}^{\mathrm{adm}}$-cycles is well-known (see [55], Lemma 2.4 for a proof modulo some omitted terms; see [100] Lemmas 3.2, 3.3, 3.6, 3.7 for a detailed proof is genus 1 that easily generalizes to higher genus). The proof is actually even simpler for $\mathrm{DR}^{\mathrm{adm}}$-cycles because the space of admissible coverings has the expected dimension, so there is no virtual fundamental class involved. The combinatorial part of the computation only uses Theorem 4.9 as long as there are no vanishing a_i's, therefore it works in the same way for both $\mathrm{DR}^{\mathrm{adm}}$ and $\mathrm{DR}^{\mathrm{stab}}$.

The only part of the computation that does not generalize is the use of Okounkov and Pandharipande's computation in the case where there are ψ-classes only at the marked point with vanishing a_i's. If there are no vanishing a_i's this part is not needed.

To prove the second claim of the theorem we will use the following example. Let

$\beta =$  , (4.68)

$\gamma =$  (4.69)

be two cohomology classes in $H^2(\overline{\mathcal{M}}_{1,2})$.

Proposition 4.21. *We have*

$$\mathrm{DR}_1^{\mathrm{adm}}(a,-a) = (a^2-1)\beta + \frac{a^2-1}{12}\gamma, \tag{4.70}$$

$$\mathrm{DR}_1^{\mathrm{stab}}(a,-a) = a^2\beta + \frac{a^2-1}{12}\gamma. \tag{4.71}$$

Proof. In this case the space of rubber maps has two irreducible components of the same dimension equal to the expected dimension. Its virtual fundamental class is the sum of the fundamental classes of the two components. The first component coincides with the space of admissible coverings. The second component is composed of maps with a contracted torus in the source curve. Thus the difference between $\mathrm{DR}_1^{\mathrm{adm}}(a,-a)$ and $\mathrm{DR}_1^{\mathrm{stab}}(a,-a)$ comes from the component with contracted tori, whose fundamental class projects to β. In other words, $\mathrm{DR}^{\mathrm{stab}}(a,-a) - \mathrm{DR}^{\mathrm{adm}}(a,-a) = \beta$, which is actually the only thing that is needed for the proof of the theorem.

The expression for $\mathrm{DR}_1^{\mathrm{stab}}(a_1,\dots,a_n)$ is given by Hain [54] in full generality, so our expression can be found as a particular case. Both formulas can also be proved using a lifting of the WDVV relation in the Losev-Manin space. □

Corollary 4.22. *We have*

$$\mathrm{DR}_1^{\mathrm{adm}}(a,-a,0)\psi_3^2 = (a^2-1)/12, \tag{4.72}$$

$$\mathrm{DR}_1^{\mathrm{stab}}(a,-a,0)\psi_3^2 = (2a^2-1)/24. \tag{4.73}$$

Proof. The classes $\mathrm{DR}_1(a,-a,0)$ are obtained from $\mathrm{DR}_1(a,-a)$ by pull-backs under the forgetful map $\pi^*\colon \overline{\mathcal{M}}_{1,3} \to \overline{\mathcal{M}}_{1,2}$. It is straightforward to compute the intersection of the classes thus obtained with ψ_3^2. Note that the second equality is a particular case of Example 4.8. What matters for us is that

$$\psi_3^2(\mathrm{DR}^{\mathrm{stab}}(a,-a,0) - \mathrm{DR}^{\mathrm{adm}}(a,-a,0) = \psi_3^2\pi^*\beta = \frac{1}{24} \neq 0. \tag{4.74}$$

□

Thus we have found an example where the intersection numbers of the same monomial in ψ-classes with a $\mathrm{DR}^{\mathrm{adm}}$-cycles and with a $\mathrm{DR}^{\mathrm{stab}}$-cycle differ. This proves the second claim of the theorem. □

4.4 Generating functions for one ψ-class

In this section we prove Theorem 4.3 that evaluates the power of a ψ-class on a DR-cycle.

4.4.1 Proof of Theorem 4.3

The proof of Theorem 4.3 splits into two very different cases: $a_s = 0$ and $a_s \neq 0$.

Proof for $a_s = 0$. We use two lemmas.

Lemma 4.23. *Let $p\colon \overline{\mathcal{M}}_{g;a_1,\dots,a_n} \to \overline{\mathcal{M}}_{g,n}$ be the forgetful map from the rubber space to the moduli space. Assume that $a_s = 0$. Then we have $p^*\psi_s = \psi_s$.*

Proof. Let $f\colon C \to \mathrm{CP}^1$ be a point of $\overline{\mathcal{M}}_{g;a_1,\dots,a_n}$. If the sth marked point lies on a component of the source curve contracted by f then this component is stable, because f is stable. If it lies on a component that is not contracted by f then this component contains at least two more marked points: a pre-image of 0 and a pre-image of ∞. Therefore it is also stable. Thus the sth marked point never lies on a component of the source curve contracted by the forgetful map. This allows us to identify the cotangent lines to the curve at the sth marked point before and after the forgetful map. □

Note that the statement of the lemma is completely wrong if $a_s \neq 0$.

Lemma 4.24. *Let μ, ν be two partitions of the same integer d. Consider the rubber space*

$$\overline{\mathcal{M}}^{\sim}_{g,p,\mu,\nu}(\mathrm{CP}^1, 0, \infty) \tag{4.75}$$

of relative stable maps to CP^1 with p marked points $x_1, \dots, x_p$. Also consider the moduli space

$$\overline{\mathcal{M}}^{1}_{g,p,\mu,\nu}(\mathrm{CP}^1, 0, \infty) \tag{4.76}$$

of relative stable maps to CP^1 with p marked points $x_1, \dots, x_p$ such that the image of x_1 is fixed to be $1 \in \mathrm{CP}^1$. These two spaces are isomorphic to each other; their perfect obstruction theories and virtual fundamental classes coincide.

This is a well-known fact and the proof is a simple check.

The consequence of these two lemmas is that the intersection number

$$\psi_s^{2g-3+n}\mathrm{DR}_g(a_1, \dots, a_n) \tag{4.77}$$

is actually a Gromov-Witten invariant of CP^1 relative to two points. Indeed, by Lemma 4.23, instead of evaluating ψ_s^{2g-3+n} we can evaluate it directly on the rubber space $\overline{\mathcal{M}}_{g;a_1,\dots,a_n}$. And, according to Lemma 4.24, this is equivalent to finding a Gromov-Witten invariant of CP^1 relative to two points.

The Gromov-Witten invariants that we need were computed in [88], Eq. (3.11) and [94], Eq. (10) if a_s is the only vanishing marking; while the general case is covered by Proposition 2.5 of [89]. We do not have any new contribution to this computation.

This proves the theorem for $a_s = 0$. □

Proof for $a_s \neq 0$. We proceed by induction on the number of branch points $r = 2g-2+n_+ + n_-$. The base case is $r = 1$, that is, $g = 0$, $n_+ + n_- = 3$. (Genus 1 is impossible, because $n_+ + n_- \geq 2$.) We have

$$\psi_s^{n_0}\mathrm{DR}_0(a_1, a_2, a_3, \underbrace{0, \dots, 0}_{n_0}) = \int_{\overline{\mathcal{M}}_{0,n_0+3}} \psi_s^{n_0} = 1, \tag{4.78}$$

which coincides with the constant term of the generating function in the theorem.

Recall the recursion from Corollary 4.11:

$$ra_s\, \psi_s^{r-1}\mathrm{DR}_g(a_1, \dots, a_n) = \tag{4.79}$$

$$-\frac{1}{2}\sum_{i,j\neq s}(a_i + a_j)\, \psi_s^{r-2}\mathrm{DR}_g(a_1, \dots, \widehat{a_i}, \dots, \widehat{a_j}, \dots, a_n, a_i + a_j) \tag{4.80}$$

$$-\frac{1}{2}\sum_{i\neq s}\mathrm{sign}(a_i)\sum_{\substack{b+c=a_i\\ b\cdot c>0}} bc\, \psi_s^{r-2}\mathrm{DR}_{g-1}(a_1, \dots, \widehat{a_i}, \dots, a_n, b, c). \tag{4.81}$$

By the induction assumption, the sum (4.80) is equal to

$$\begin{aligned}
&-[z^{2g}]\frac{1}{ra_s}\sum_{i,j\neq s}\frac{\sinh\left(\frac{a_iz}{2}\right)\cosh\left(\frac{a_jz}{2}\right)}{\frac{z}{2}}\frac{\prod_{l\neq i,j,s}S(a_lz)}{S(z)}= \\
&=[z^{2g}]\frac{1}{ra_s}\sum_{i\neq s}(a_i+a_s)\cosh\left(\frac{a_iz}{2}\right)\frac{\prod_{j\neq i,s}S(a_jz)}{S(z)}.
\end{aligned} \tag{4.82}$$

By the induction assumption, the sum (4.81) is equal to

$$\begin{aligned}
&-[z^{2g-2}]\frac{1}{ra_s}\sum_{i\neq s}\mathrm{sign}(a_i)\sum_{\substack{b+c=a_i\\ bc>0}}\frac{bc}{2}S(bz)S(cz)\frac{\prod_{l\neq i,s}S(a_lz)}{S(z)}= \\
&=-[z^{2g}]\frac{1}{ra_s}\sum_{i\neq s}\left(a_i\cosh\left(\frac{a_iz}{2}\right)-\frac{\sinh\left(\frac{a_iz}{2}\right)\cosh\left(\frac{z}{2}\right)}{\sinh\left(\frac{z}{2}\right)}\right)\frac{\prod_{j\neq i,s}S(a_jz)}{S(z)}.
\end{aligned} \tag{4.83}$$

Thus, (4.79) is equal to

$$\begin{aligned}
&[z^{2g}]\frac{1}{ra_s}\sum_{i\neq s}\left(a_s\cosh\left(\frac{a_iz}{2}\right)+\frac{\sinh\left(\frac{a_iz}{2}\right)\cosh\left(\frac{z}{2}\right)}{\sinh\left(\frac{z}{2}\right)}\right)\frac{\prod_{j\neq i,s}S(a_jz)}{S(z)}= \\
&=[z^{2g}]\frac{1}{r}\left(z\frac{d}{dz}+n-2\right)\frac{\prod_{i\neq s}S(a_iz)}{S(z)}= \\
&=[z^{2g}]\frac{\prod_{i\neq s}S(a_iz)}{S(z)}.
\end{aligned} \tag{4.84}$$

The theorem is proved. □

Remark 4.25. Our formula for $\psi_s^{r-1}\mathrm{DR}(a_1,\dots,a_n)$ coincides, up to a simple factor, with the formula for one-part double Hurwitz numbers found in [51] (first equality of Theorem 3.1). This is due to the fact that the recursion relation of Corollary 4.11 coincides with the cut-and-join equation for Hurwitz numbers. Note, however, that in the Hurwitz numbers theory the formula only holds for *one-part* numbers; in other words, all the numbers $a_1,\dots,a_n$ must be of the same sign except for a_s that is of the opposite sign. If this condition is not satisfied then the cut-and-join equation fails. In our situation, however, the signs of the numbers a_i do not matter. Thus we get the same generating function and the same cut-and-join equation, but their interpretations and their ranges of applicability are different.

4.5 Integrals of ψ-classes over a DR-cycle: Proof

In this section we use the infinite wedge formalism to prove Theorem 4.4. The proof is by induction, and the base case is Theorem 4.3. For that we first need to restate both theorems in terms of the infinite wedge formalism.

In the rest of this section g will always be used to denote the genus of some DR-cycle which is intersected with some ψ-classes. It is determined by the dimension constraint that this intersection should be a number.

Proposition 4.26. *Let $a_1,\dots,a_n$ be a list of real numbers such that $\sum a_i=0$. Denote $J=\{1,\dots n\}\backslash\{s\}$ and define*

$$J_+=\{\ j\in J\ :\ a_j\geq 0\ \} \tag{4.85}$$

and $J_- = J\backslash J_+$. Then

$$[z^{2g}]\frac{\prod_{j\in J} S(a_j z)}{S(z)} = [x^{2g-2+n}]\left\langle \prod_{j\in J_+} \frac{\mathcal{E}_{a_j}(0)}{a_j}\mathcal{E}_{a_s}(x)\prod_{j\in J_-}\frac{\mathcal{E}_{a_j}(0)}{-a_j}\right\rangle^{\circ}. \tag{4.86}$$

Remark 4.27. When $a_i = 0$ for some i, we interpret the right-hand side of the formula as follows. We first compute the vacuum expectation value using the commutation relations of the $\mathcal{E}$-operators (Proposition 2.22) where we keep a_i as a variable. It is easy to see that the result can be continued analytically to a neighbourhood of $a_i = 0$, so we take the limit $a_i \to 0$. Note that way the apparent singularity coming from the factor $\frac{1}{a_i}$ will be cancelled with the zero coming from $\zeta(a_i x)$.

Lemma 4.28. *Let $a_1, \ldots, a_n$ be a list of non-zero real numbers with vanishing sum. Assume that it is split into a disjoint union of three sets*

$$\{1, \ldots, n\} = \{s\} \sqcup J_+ \sqcup J_-. \tag{4.87}$$

Then the vacuum expectation value

$$\left\langle \prod_{j\in J_+} \mathcal{E}_{a_j}(0)\mathcal{E}_{a_s}(x)\prod_{j\in J_-}\mathcal{E}_{a_j}(0)\right\rangle^{\circ} \tag{4.88}$$

vanishes unless all the elements of J_+ are positive and all the elements of J_- are negative.

Proof. Assume, for instance, that an element $a_j = a$ of J_- is positive. We apply our algorithm by moving this operator to the right. Since

$$[\mathcal{E}_a(0), \mathcal{E}_b(0)] = \lim_{t\to 0}\zeta(at)\mathcal{E}_{a+b}(t) = a\delta_{0,a+b} \tag{4.89}$$

by Proposition 2.22, we see that the commutator term either vanishes or contains an $\mathcal{E}_0$ factor that is prohibited in a connected expectation value. Thus to compute the connected expectation value we only need to take the passing term. In other words we can just move the operator $\mathcal{E}_a(0)$ to the right-most position and we see that the expectation value vanishes, because $\mathcal{E}_a(0)v_\emptyset = 0$. □

Proof of Proposition 4.26. We first assume that the $a_i \neq 0$ for all $i \in \{1, \ldots, s\}$.

To compute the expectation value we apply our algorithm by commuting $\mathcal{E}_{a_s}(x)$ with its right and left neighbours in an arbitrary order. At every step the contribution of the passing term vanishes by the lemma. Using the commutation formulas we get

$$[x^{2g-2+n}]\langle\mathcal{E}_0(x)\rangle\prod_{j\in J_+}\frac{\zeta(a_j x)}{a_j}\prod_{j\in J_-}\frac{\zeta(-a_j x)}{-a_j} = [x^{2g-2+n}]\frac{x^{n-1}\prod_{j\in J} S(a_j x)}{\zeta(x)} = [x^{2g}]\frac{\prod_{i\neq s} S(a_i x)}{S(x)}, \tag{4.90}$$

where we used $S(0) = 1$ and $S(-a_j x) = S(a_j x)$. Now note that when $a_i = 0$ for i in some index set I, we can view this case as the previous one, where now we have to add those a_i by hand. The left-hand side remains unchanged, since $S(0) = 1$, while the right-hand side is multiplied by

$$\lim_{a_i\to 0}\prod_{i\in I}\frac{\zeta(a_i x)}{a_i} = x \tag{4.91}$$

while simultaneously n is increased by $|I|$. Because of the $[x^{2g-2+n}]$ in front, these two changes exactly cancel. This completes the proof of the proposition. □

Proposition 4.26 allows us to restate Theorem 4.3 in terms of the infinite wedge formalism:

Corollary 4.29. *We have*

$$\psi_s^{2g-3+n}\mathrm{DR}_g(a_1,\ldots,a_n) = [x^{2g-2+n}]\left\langle \prod_{j\in J_+} \frac{\mathcal{E}_{a_j}(0)}{a_j}\mathcal{E}_{a_s}(x)\prod_{j\in J_-}\frac{\mathcal{E}_{a_j}(0)}{-a_j}\right\rangle^\circ. \tag{4.92}$$

On the other hand, Theorem 4.4 is equivalent to the following:

Theorem 4.30. *Let n be a positive integer, and let $a_1,\ldots,a_n$ be integers such that $\sum a_i = 0$ and $d_1,\ldots,d_n$ non-negative integers such that $\sum d_i = 2g-3+n$. Then we have*

$$\psi_1^{d_1}\cdots\psi_n^{d_n}\mathrm{DR}_g(a_1,\ldots,a_n) = [x_1^{d_1}\cdots x_n^{d_n}]\sum_{\sigma\in S'_n}\frac{\left\langle\left[\cdots\left[\mathcal{E}_{a'_1}(x'_1),\mathcal{E}_{a'_2}(x'_2)\right],\ldots\right],\mathcal{E}_{a'_n}(x'_n)\right]\right\rangle^\circ}{x'_1(a'_2-\frac{a'_1x'_2}{x'_1})\cdots(a'_n-\frac{a'_{n-1}x'_n}{x'_{n-1}})}, \tag{4.93}$$

where S'_n is the group of permutations σ of the set $\{1,\ldots,n\}$ with $\sigma(1)=1$. As before, we define $x'_i := x_{\sigma(i)}$ and $a'_i := a_{\sigma(i)}$.

The fact that Theorem 4.30 is equivalent to Theorem 4.4 follows immediately by repeated application of the commutation relation of Proposition 2.22.

Notation 4.31. In the following, if a sequence of integers $a_1,\ldots,a_n$ and a corresponding sequence of formal variables $x_1,\ldots,x_n$ have been defined, we will often use the following abbreviation for the sake of clarity:

$$\mathcal{E}_j := \mathcal{E}_{a_j}(x_j). \tag{4.94}$$

Definition 4.32. Let $I\subset\{1,\ldots,n\}$. Given a list of integers $a_1,\ldots,a_n$ and a list of variables $x_1,\ldots,x_n$, let P be a polynomial in operators $\mathcal{E}_i$, $i\in I$, whose coefficients are rational functions in x_i and a_i. Let $t\notin I$.

For any $i\in I$ we define $\mathcal{O}_i^tP$ to be the result of the substitution

$$\mathcal{E}_i \mapsto \frac{1}{\frac{a_ix_t}{x_i}-a_t}[\mathcal{E}_i,\mathcal{E}_t]. \tag{4.95}$$

Furthermore, we define $\mathcal{O}^tP = \sum_{i\in I}\mathcal{O}_i^tP$.

Definition 4.33. Let n be some positive integer, let $a_1,\ldots,a_n$ be some sequence of integers, and let $x_1,\ldots,x_n$ be a sequence of formal variables. Then we define

$$G^t(a_1,\ldots,a_n;x_1,\ldots,x_n) = \sum_{\sigma\in S'_t}\frac{\left[\cdots\left[\mathcal{E}_{\sigma_1},\mathcal{E}_{\sigma_2}\right],\ldots\right],\mathcal{E}_{\sigma_t}\right]}{x_{\sigma_1}(\frac{a_{\sigma_1}x_{\sigma_2}}{x_{\sigma_1}}-a_{\sigma_2})\cdots(\frac{a_{\sigma_{t-1}}x_{\sigma_t}}{x_{\sigma_{t-1}}}-a_{\sigma_t})}. \tag{4.96}$$

To prove Theorem 4.30, we will need the following lemmas.

Lemma 4.34. *For any positive integers $t\le n$, and for all $a_1,\ldots,a_n\in\mathbb{Z}$, we have*

$$G^t(a_1,\ldots,a_n;x_1,\ldots,x_n) = \mathcal{O}^t\cdots\mathcal{O}^2\frac{1}{x_1}\mathcal{E}_1. \tag{4.97}$$

Note that the empty product of operators that appears on the right-hand side in the case $t=1$ should be interpreted as the identity operator, as usual.

Proof. We prove that the coefficients of $x_1^{d_1}\cdots x_n^{d_n}$ are equal on both sides of the equation for any non-negative integers $d_1,\ldots,d_n$. The lemma is clearly true when $t=1$. We proceed by induction on t.

Denote by F^t the right-hand side of equation (4.97):

$$F^t := \mathcal{O}^t\cdots\mathcal{O}^2\frac{1}{x_1}\mathcal{E}_1 \tag{4.98}$$

Now assume that F^t and G^t are equal for some t, and are related by just a series of applications of the Jacobi identity. Defining

$$\tilde{G}_i^{t+1} = \mathcal{O}_i^{t+1}G^t \tag{4.99}$$

it follows that $\mathcal{O}_i^{t+1}F^t = \tilde{G}_i^{t+1}$, again related by a series of applications of the Jacobi identity. We complete the proof by showing that G^{t+1} is equal to $\tilde{G}^{t+1} := \sum_{i=0}^t \tilde{G}_i^{t+1}$, and this equality can be given just by application of the Jacobi identity.

The terms of G^{t+1} are of the form

$$\frac{[\cdots[\mathcal{E}_{\sigma_1},\mathcal{E}_{\sigma_2}],\ldots],\mathcal{E}_{\sigma_i}],\mathcal{E}_{t+1}],\ldots],\mathcal{E}_{\sigma_t}]}{x_{\sigma_1}(\frac{a_{\sigma_i}x_{t+1}}{x_{\sigma_i}}-a_{t+1})(\frac{a_{t+1}x_{\sigma_{i+1}}}{x_{t+1}}-a_{\sigma_{i+1}})\prod_j(\frac{a_{\sigma_j}x_{\sigma_{j+1}}}{x_{\sigma_j}}-a_{\sigma_{j+1}})}, \tag{4.100}$$

where σ is some permutation appearing in the sum in G^t, and where j runs from 1 to $t-1$, skipping i.

First we look at the case where $0<i<t$. A term with the iterated commutator appearing in (4.100) arises in $\tilde{G}^{t+1}$ in precisely two ways:

1. In $\tilde{G}_i^{t+1}$, as the first term of the Jacobi identity applied to

$$[\cdots[\mathcal{E}_{\sigma_1},\mathcal{E}_{\sigma_2}],\ldots],\mathcal{E}_{\sigma_{i-1}}],[\mathcal{E}_{\sigma_i},\mathcal{E}_{\sigma_{t+1}}],\mathcal{E}_{\sigma_{i+1}}],\ldots],\mathcal{E}_{\sigma_t}] \tag{4.101}$$

2. In $\tilde{G}_{i+1}^{t+1}$, as the second term of the Jacobi identity applied to

$$[\cdots[\mathcal{E}_{\sigma_1},\mathcal{E}_{\sigma_2}],\ldots],\mathcal{E}_{\sigma_i}],[\mathcal{E}_{\sigma_i+1},\mathcal{E}_{\sigma_{t+1}}],\mathcal{E}_{\sigma_{i+2}}],\ldots],\mathcal{E}_{\sigma_t}]. \tag{4.102}$$

Taking into account the coefficients of these two contributions, we get

$$\begin{aligned}&\frac{1}{\prod_j(\frac{a_{\sigma_j}x_{\sigma_{j+1}}}{x_{\sigma_j}}-a_{\sigma_{j+1}})}\left(\frac{1}{(\frac{a_{\sigma_i}x_{\sigma_{i+1}}}{x_{\sigma_i}}-a_{\sigma_{i+1}})(\frac{a_{\sigma_i}x_{t+1}}{x_{\sigma_i}}-a_{t+1})}\right.\\&\qquad\left.-\frac{1}{(\frac{a_{\sigma_i}x_{\sigma_{i+1}}}{x_{\sigma_i}}-a_{\sigma_{i+1}})(\frac{a_{\sigma_{i+1}}x_{t+1}}{x_{\sigma_{i+1}}}-a_{t+1})}\right)\\&\qquad=\frac{1}{\prod_j(\frac{a_{\sigma_j}x_{\sigma_{j+1}}}{x_{\sigma_j}}-a_{\sigma_{j+1}})}\frac{1}{\frac{a_{\sigma_i}x_{t+1}}{x_{\sigma_i}}-a_{t+1}}\frac{1}{\frac{a_{t+1}x_{\sigma_{i+1}}}{x_{t+1}}-a_{\sigma_{i+1}}},\end{aligned} \tag{4.103}$$

(where j runs over the same set as above) which is precisely the coefficient of the iterated commutator appearing in (4.100).

In this way, we use all the terms of $\tilde{G}^{t+1}$ except three to get the terms of the form (4.100) in G^{t+1} with $0<i<t$. The three terms we did not yet use are $\tilde{G}_0^{t+1}$, the second term of the Jacobi identity applied to $\tilde{G}_1^{t+1}$ and the first term of the Jacobi identity applied to $\tilde{G}_t^{t+1}$. The only remaining terms in G^{t+1} are those of the form (4.100) with $i=0$ and with $i=t$. By a similar argument as the one above, the first is easily seen to be equal to the sum of $\tilde{G}_0^{t+1}$ and the second term of the Jacobi identity applied to $\tilde{G}_1^{t+1}$, whereas it is immediate that the second is equal to the first term of the Jacobi identity applied to $\tilde{G}_t^{t+1}$. This completes the induction. ☐

Corollary 4.35. *Let $\bar{a} = (a_1, \dots, a_n)$ be an ordered set of integers. Denote $\bar{x} = (x_1, \dots, x_n)$. Then the expression*

$$F(\bar{x}; \bar{a}) := \mathcal{O}^n \cdots \mathcal{O}^2 \frac{1}{x_1} \mathcal{E}_{a_1}(x_1) \tag{4.104}$$

is symmetric with respect to the action of S_n on $\{1, \dots, n\}$.

Proof. The group S_n is generated by S'_n and the transposition $(1,2)$. The fact that $F(\bar{x}, \bar{a})$ is symmetric with respect to the action of S'_n follows immediately because the quantity on the left-hand side of (4.97) is symmetric with respect to this action. The invariance under the transposition $(1,2)$ is shown as follows:

$$\mathcal{O}^2 \frac{1}{x_1} \mathcal{E}_1 = \frac{[\mathcal{E}_1, \mathcal{E}_2]}{a_1 x_2 - a_2 x_1} = \frac{[\mathcal{E}_2, \mathcal{E}_1]}{x_2(\frac{a_2 x_1}{x_2} - a_1)} = \mathcal{O}^1 \frac{1}{x_2} \mathcal{E}_2. \tag{4.105}$$

□

Lemma 4.36. *Let n be any positive integer, and let $a_1, \dots, a_n$ be any integers. For any subset $I \subset \{2, \dots, n\}$ we have*

$$[\prod_{i \in I^c} x_i^0] \left\langle \mathcal{O}^n \cdots \mathcal{O}^2 \frac{1}{x_1} \mathcal{E}_1 \right\rangle^\circ = \left\langle \prod_{i \in I^c, a_i \geq 0} \frac{\mathcal{E}_i}{a_i} \left(\prod_{t \in I} \mathcal{O}^t \frac{1}{x_1} \mathcal{E}_1 \right) \prod_{j \in I^c, a_j < 0} \frac{\mathcal{E}_j}{-a_j} \right\rangle^\circ, \tag{4.106}$$

where I^c denotes the complement of $I \subset \{2, \dots, n\}$.

Proof. Let $k = |I| + 1$. By Corollary 4.35, we can assume that $I = \{2, \cdots, k\}$.

First note that

$$[\prod_{i \in I^c} x_i^0] \mathcal{O}^n \cdots \mathcal{O}^2 \frac{1}{x_1} \mathcal{E}_1 = \frac{\tilde{\mathcal{O}}^n}{a_n} \cdots \frac{\tilde{\mathcal{O}}^{k+1}}{a_{k+1}} \mathcal{O}^k \cdots \mathcal{O}^2 \frac{1}{x_1} \mathcal{E}_1 \tag{4.107}$$

where $\tilde{\mathcal{O}}^t$ is defined by $\tilde{\mathcal{O}}^t := \mathrm{ad}_{\mathcal{E}_{a_t}(0)}$. This follows immediately from the fact that we take the coefficient of $x_{k+1}^0 \cdots x_n^0$ and the expansion of the factor $\frac{1}{\frac{a_j x_t}{x_j} - a_t}$ appearing in the operator $\mathcal{O}_j^t$.

Whenever $a_t = 0$ for some $t > k$, the operator $\frac{\tilde{\mathcal{O}}_t}{a_t}$ just acts as multiplication by $\frac{\mathcal{E}_t}{a_t}$ from the left; since we take the connected vacuum expectation value, it will be forced to involved in a commutator with a negative energy operator from the left at some point. It is easy to see that the effect of this commutator will be the same as the effect from the one coming from the definition of $\tilde{\mathcal{O}}_t$.

For $t > k$ with $a_t \neq 0$, we use the standard Lie-theory fact that for any $t > k$

$$\begin{aligned} &\tilde{\mathcal{O}}^t \cdots \tilde{\mathcal{O}}^{k+1} \mathcal{O}^k \cdots \mathcal{O}^2 \frac{1}{x_1} \mathcal{E}_1 = \\ &\quad = \mathcal{E}_{a_t}(0) \tilde{\mathcal{O}}^{t-1} \cdots \tilde{\mathcal{O}}^{k+1} \mathcal{O}^k \cdots \mathcal{O}^2 \frac{1}{x_1} \mathcal{E}_1 - \tilde{\mathcal{O}}^{t-1} \cdots \tilde{\mathcal{O}}^{k+1} \mathcal{O}^k \cdots \mathcal{O}^2 (\frac{1}{x_1} \mathcal{E}_1) \mathcal{E}_{a_t}(0). \end{aligned} \tag{4.108}$$

Depending on the sign of a_t, only one of these two terms will contribute when we take the vacuum expectation value. Iterating this procedure from $t = k+1$ up to $t = n$ completes the proof of the lemma. □

Remark 4.37. Using Lemmas 4.34 and 4.36, it is easy to see that Corollary 4.29 is a special case of Theorem 4.30, i.e. that Theorem 4.3 is a special case of Theorem 4.4.

Lemma 4.38. *Let n be a positive integer, let $a_1, \ldots, a_n$ be a sequence of integers and let $x_1, \ldots, x_n$ be a sequence of formal variables. Denote $\bar{a} := (a_1, \ldots, a_n)$ and $\bar{x} = (x_1, \ldots, x_n)$. Then we have, for any p, q with $1 \leq p < q \leq n$:*

$$G^n(\bar{a};\bar{x}) = \frac{x_p x_q}{a_p x_q - a_q x_p} \sum_{I,J} [G^{|I|}(a_I; x_I), G^{|J|}(a_J; x_J)] \tag{4.109}$$

where the sum is over all disjoint sets I and J such that $I \cup J = \{1, \ldots, n\}$ and $p \in I$, $q \in J$, and G is as defined in definition 4.33.

Proof. Let us introduce the following notation. Suppose $h_1, h_2, \ldots, h_n$ are operators. Let

$$Q_n(h_1, \ldots, h_n; x_1, \ldots, x_n) = \sum_{\sigma \in S'_n} \frac{x_{\sigma_2} x_{\sigma_3} \cdots x_{\sigma_{n-1}} \left[\left[\cdots\left[h_{\sigma_1}, h_{\sigma_2}\right], \ldots\right], h_{\sigma_n}\right]}{(x_{\sigma_1} - x_{\sigma_2}) \cdots (x_{\sigma_{n-1}} - x_{\sigma_n})}. \tag{4.110}$$

For any $\sigma \in S_n$, we have the symmetry

$$Q_n(h_{\sigma_1}, \ldots, h_{\sigma_n}; x_{\sigma_1}, \ldots, x_{\sigma_n}) = Q_n(h_1, \ldots, h_n; x_1, \ldots, x_n). \tag{4.111}$$

It can be proved in the same way as the symmetry of $G^n(\bar{a};\bar{x})$. We have

$$G^n(\bar{a};\bar{x}) = \frac{(-1)^{n-1}}{a_1 a_2 \ldots a_n} Q_n(\mathcal{E}_1, \ldots, \mathcal{E}_n; \frac{x_1}{a_1}, \ldots, \frac{x_n}{a_n}). \tag{4.112}$$

The lemma obviously follows from the formula

$$Q_n(h; z) = \frac{x_p x_q}{x_p - x_q} \sum_{\substack{I \coprod J = \{1,\ldots,n\} \\ p \in I,\ q \in J}} [Q_{|I|}(h_I; x_I), Q_{|J|}(h_J; x_J)]. \tag{4.113}$$

We prove (4.113) by induction on n. The case $n = 2$ is obvious. Suppose $n \geq 3$. Let us denote the set $\{1, 2, \ldots, n\}$ by $[n]$. We have

$$\frac{x_p x_q}{x_p - x_q} \sum_{\substack{I \coprod J = [n] \\ p \in I \\ q \in J}} [Q_{|I|}(h_I; x_I), Q_{|J|}(h_J; x_J)] =$$

$$= \sum_{i \neq p,q} \frac{x_p (x_i - x_q)}{(x_p - x_q)(x_p - x_i)} \times$$

$$\times \frac{x_i x_q}{x_i - x_q} \sum_{\substack{I \coprod J = [n] \\ i,p \in I \\ q \in J}} [Q_{|I|-1}([h_p, h_i], h_{I \setminus \{i,p\}}; x_i, x_{I \setminus \{i,p\}}), Q_{|J|}(h_J; x_J)] + \tag{4.114}$$

$$+ \frac{x_q}{x_p - x_q} [h_p, Q_{n-1}(h_{[n] \setminus \{p\}}; x_{[n] \setminus \{p\}})]. \tag{4.115}$$

By the induction assumption the sum (4.114) is equal to

$$\sum_{i \neq p,q} \frac{x_p (x_i - x_q)}{(x_p - x_q)(x_p - x_i)} Q_{n-1}([h_p, h_i], h_{[n] \setminus \{i,p\}}; x_i, x_{[n] \setminus \{i,p\}}) =$$

$$= \sum_{i \neq p,q} \frac{x_i}{x_p - x_i} Q_{n-1}([h_p, h_i], h_{[n] \setminus \{i,p\}}; x_i, x_{[n] \setminus \{i,p\}}) + \tag{4.116}$$

$$+ \sum_{i \neq p,q} \frac{x_q}{x_q - x_p} Q_{n-1}([h_p, h_i], h_{[n] \setminus \{i,p\}}; x_i, x_{[n] \setminus \{i,p\}}) \tag{4.117}$$

It is easy to see that (4.117) is equal to

$$\frac{x_q}{x_q - x_p}[h_p, Q_{n-1}(h_{[n]\setminus\{p\}}; x_{[n]\setminus\{p\}})]+ \tag{4.118}$$

$$+\frac{x_q}{x_p - x_q}Q_{n-1}([h_p, h_q], h_{[n]\setminus\{p,q\}}; x_q, x_{[n]\setminus\{p,q\}}). \tag{4.119}$$

Clearly, the sum of (4.116) and (4.119) is equal to $Q_n(h; x)$ and the sum of (4.115) and (4.118) is zero. The formula (4.113) is proved. □

The proof of Theorem 4.4 is by induction, starting from two base cases. In the first case all ψ-classes are at points on the DR-cycle which are not mapped to zero or infinity. In the second case there is one point in the inverse image of zero or infinity were some non-zero power of a ψ-class appears, and all other ψ-classes again are at points which are not mapped to zero or infinity. The induction will then be completed using Corollary 4.12, which allows us to move ψ-classes between points on the boundary. We now prove the two base cases.

Proposition 4.39. *Let n be some positive integers, and let $a_1, \ldots, a_n$ be integers. Let $d_1, \ldots, d_n$ be a set of non-negative integers such that d_i is zero whenever $a_i \neq 0$. Then Theorem 4.30 holds. That is, under the conditions described above, we have:*

$$\mathrm{DR}_g(a_1, \ldots, a_n) \prod_{i=1}^n \psi_i^{d_i} = [\prod_{i=1}^n x_i^{d_i}] \left\langle \mathcal{O}^n \cdots \mathcal{O}^2 \frac{1}{x_1}\mathcal{E}_1 \right\rangle. \tag{4.120}$$

Proof. First note that by Lemma 4.34, equation (4.120) is indeed equivalent to the described special case of Theorem 4.30.

Denote by I the set $I = \{1 \le i \le n : d_i > 0\}$. It is clear that the in the left-hand side of (4.120), we can replace the product over i from 1 to n by a product over the set I.

By Corollary 4.35 we can assume that $1 \in I$ and that $i < j$ whenever $a_i = 0$ and $a_j \neq 0$. Let t be the number of i for which $a_i = 0$. By Lemma 4.36, the right-hand side of equation (4.120) is equal to

$$[\prod_{i=1}^t x_i^{d_i}] \left\langle \prod_{i, a_i \ge 0} \frac{\mathcal{E}_i}{a_i} \left(\mathcal{O}^t \cdots \mathcal{O}^2 \frac{1}{x_1}\mathcal{E}_1\right) \prod_{j, a_j < 0} \frac{\mathcal{E}_j}{-a_j} \right\rangle^\circ, \tag{4.121}$$

which shows that when I contains only one element, the statement of the proposition is a direct consequence of Corollary 4.29.

Furthermore, by [89], Proposition 2.5;

$$\mathrm{DR}_g(a_1, \ldots, a_n) \prod_{i \in I} \psi_i^{d_i} = \binom{d_1 + \cdots + d_n}{d_1, \ldots, d_n} \mathrm{DR}_g(a_1, \ldots, a_n) \psi_1^{d_1 + \cdots + d_n - n + 1}. \tag{4.122}$$

Using this and the expression above for the right-hand side of (4.120), it only remains to show that

$$[x_1^{d_1} \cdots x_t^{d_t}] \mathcal{O}^t \cdots \mathcal{O}^2 \frac{1}{x_1}\mathcal{E}_1 = \binom{d_1 + \cdots + d_t}{d_1, \ldots, d_t} [x^{d_1 + \cdots d_t - t + 1}] \frac{1}{x}\mathcal{E}_0(x). \tag{4.123}$$

It is a direct computation that this equation is equivalent to

$$\mathcal{O}^t \cdots \mathcal{O}^2 \frac{1}{x_1}\mathcal{E}_1 = (x_1 + \cdots + x_t)^{t-2} \mathcal{E}_0(x_1 + \cdots + x_t), \tag{4.124}$$

which is clearly true when $t = 1$, so we proceed by induction. Suppose that the equation above is true for all $t \le l$ for some $l \ge 1$.

Note that the action of $\mathcal{O}_j^l$ on $\prod_i \mathcal{O}^i \mathcal{E}_1$ is given by the following actions.

- replace x_j by $x_j + x_l$
- replace a_j by $a_j + a_l$
- multiply the result by $\frac{x_j\zeta(a_jx_l - a_lx_j)}{a_jx_l - a_lx_j} = x_jS(a_jx_l - a_lx_j)$, which is equal to x_j when a_j and a_l tend to 0 (which is the case for us since $i \in I$ implies $a_i = 0$).

Thus, we have, by the induction hypothesis

$$\begin{aligned}\mathcal{O}^{l+1}\cdots\mathcal{O}^2\mathcal{E}_{a_1}(x_1) &= \mathcal{O}^{l+1}x_1(x_1+\cdots+x_l)^{l-2}\mathcal{E}_0(x_1+\cdots+x_l)\\ &= (x_1+x_{l+1})(x_1+\cdots+x_{l+1})^{l-2}\mathcal{E}_0(x_1+\cdots+x_{l+1})x_1\\ &\quad+\sum_{j=2}^{l} x_1(x_1+\cdots+x_{l+1})^{l-2}\mathcal{E}_0(x_1+\cdots x_{l+1})x_j\\ &= x_1(x_1+\cdots+x_{l+1})^{l-1}\mathcal{E}_0(x_1+\cdots+x_{l+1}). \end{aligned} \tag{4.125}$$

□

Proposition 4.40. *Let n be some positive integer, and let $a_1,\dots,a_n$ be some sequence of integers with $a_1 \neq 0$. Let K be a subset of $\{2,\dots,n\}$, and suppose that a_i is zero for all $i \in K$. Then for all sequences of integers $d_1,\dots,d_n$ with $d_i \neq 0$ if and only if $i \in K \cup \{1\}$ we have*

$$\mathrm{DR}_g(a_1,\dots,a_n)\prod_{i=1}^{n}\psi_i^{d_i} = [\prod_{i=1}^{n}x_i^{d_i}]\left\langle \mathcal{O}^n\cdots\mathcal{O}^2\frac{1}{x_1}\mathcal{E}_{a_1}(x_1)\right\rangle^{\circ}. \tag{4.126}$$

where g is determined by the usual formula.

Proof. Let $k := |K|$. By Corollary 4.35 we can reorder the set $\{1,\dots,n\}$ in such a way that the elements of K correspond to $2,\dots,k$. By Lemma 4.36, the statement of the proposition is equivalent to

$$\begin{aligned}&\sum_{d_1,\dots,d_k}\mathrm{DR}_g(a_1,\dots,a_n)\prod_{i=1}^{k}(\psi_ix_i)^{d_i}\\ &\qquad= \left\langle\left(\prod_{i>k,a_i\geq 0}\frac{\mathcal{E}_i}{a_i}\right)\left(\mathcal{O}^k\cdots\mathcal{O}^2\frac{1}{x_1}\mathcal{E}_1\right)\left(\prod_{j>k,a_j<0}\frac{\mathcal{E}_j}{-a_j}\right)\right\rangle^{\circ}. \end{aligned}\tag{4.127}$$

We prove this statement using induction on k. If $k = 0$, the statement is a direct consequence of Corollary 4.29. On the other hand, the right-hand side of (4.127) is equal to

$$\begin{aligned}&\frac{1}{x_1(\frac{a_1x_2}{x_1}-a_2)}\left\langle\left(\prod_{i>k,a_i\geq 0}\frac{\mathcal{E}_i}{a_i}\right)(\mathcal{O}^k\cdots\mathcal{O}^3[\mathcal{E}_1,\mathcal{E}_2])\left(\prod_{j>k,a_j<0}\frac{\mathcal{E}_j}{-a_j}\right)\right\rangle^{\circ}\\ &= \frac{1}{a_1x_2-a_2x_1}\sum_{S\subset\{2,\dots,k\}}\left\langle\left(\prod_{i>k,a_i\geq 0}\frac{\mathcal{E}_i}{a_i}\right)\left([\prod_{s\in S}\mathcal{O}^s\mathcal{E}_1,\prod_{t\in S^c}\mathcal{O}^t\mathcal{E}_2]\right)\left(\prod_{j>k,a_j<0}\frac{\mathcal{E}_j}{-a_j}\right)\right\rangle^{\circ}\end{aligned}$$

$$= \frac{1}{a_1x_2 - a_2x_1} \sum \Bigg\{ \frac{k_1 \cdots k_p}{k!} \left\langle \prod_{i\in I} \frac{\mathcal{E}_i}{a_i} \left(\prod_{s\in S} \mathcal{O}^s \mathcal{E}_1 \right) \prod_{j\in J} \frac{\mathcal{E}_j}{-a_j} \prod_{q=1}^{p} \frac{\mathcal{E}_{-k_q}(0)}{k_q} \right\rangle^{\circ} \cdot$$
$$\cdot \left\langle \prod_{q=1}^{p} \frac{\mathcal{E}_{k_q}(0)}{k_q} \prod_{i\in I^{\mathrm{c}}} \frac{\mathcal{E}_i}{a_i} \left(\prod_{s\in S^{\mathrm{c}}} \mathcal{O}^s \mathcal{E}_2 \right) \prod_{j\in J^{\mathrm{c}}} \frac{\mathcal{E}_j}{-a_j} \right\rangle^{\circ}$$
$$- \left\langle \prod_{i\in I} \frac{\mathcal{E}_i}{a_i} \left(\prod_{s\in S} \mathcal{O}^s \mathcal{E}_2 \right) \prod_{j\in J} \frac{\mathcal{E}_j}{-a_j} \prod_{q=1}^{p} \frac{\mathcal{E}_{-k_q}(0)}{k_q} \right\rangle^{\circ} \cdot$$
$$\cdot \left\langle \prod_{q=1}^{p} \frac{\mathcal{E}_{k_q}(0)}{k_q} \prod_{i\in I^{\mathrm{c}}} \frac{\mathcal{E}_i}{a_i} \left(\prod_{s\in S^{\mathrm{c}}} \mathcal{O}^s \mathcal{E}_1 \right) \prod_{j\in J^{\mathrm{c}}} \frac{\mathcal{E}_j}{-a_j} \right\rangle^{\circ} \Bigg\}, \quad (4.128)$$

where the last sum is over all subsets S as in the line above, all subsets $I \subset \{i; i > k, a_i \geq 0\}$ and $J \subset \{j; j > k, a_j < 0\}$, all positive integers t and all sequences of integers $k_1, \ldots, k_p$. If we take the coefficient of $x_1^{d_1} \cdots x_n^{d_n}$ in this expression, using the induction hypothesis and Proposition 4.39, we get

$$\sum \frac{1}{a_1} \frac{k_1 \cdots k_p}{p!} \big(\mathrm{DR}_{g_1}(a_1, \bar{a}_1, k_1, \ldots, k_p) \boxtimes \mathrm{DR}_{g_2}(a_2, \bar{a}_2, k_1, \ldots, k_p)$$
$$- \mathrm{DR}_{g_1}(a_2, \bar{a}_1, k_1, \ldots, k_p) \boxtimes \mathrm{DR}_{g_2}(a_1, \bar{a}_2, k_1, \ldots, k_p) \big) \psi_1^{d_1 - 1} \psi_2^{d_2} \cdots \psi_n^{d_n} \quad (4.129)$$

where we use $\bar{a}_1$ as shorthand for the set of variables $\{a_i\}_{i \in I \cup J \cup S}$ and $\bar{a}_2$ as shorthand for $\{a_i\}_{i \in I^{\mathrm{c}} \cup J^{\mathrm{c}} \cup S^{\mathrm{c}}}$, and where the sum is over the same range as in the previous equation (note that the genera g_1 and g_2 are determined by dimensional constraints). By Theorem 4.10, this combination is precisely equal to the intersection of ψ-classes and a DR-cycle as in the proposition. □

Proof of Theorem 4.30. The left-hand side of equation (4.93) is completely determined by the case where a_i is zero whenever d_i is not zero, the case there is precisely one i with $a_i \neq 0$ and $d_i \neq 0$, and Corollary 4.12.

The right-hand side of the equation is completely determined by the same two cases, and Lemma 4.38. By Proposition 4.39 the left- and right-hand side are equal in the first case, and by Proposition 4.40 they are equal in the second case.

Furthermore, intersecting the equation of Corollary 4.12 with a monomial in ψ-classes of total degree $2g - 4 + n$, we see that the application of Corollary 4.12 and Corollary 4.38 lead to equivalent operations on the left- and right-hand sides of equation (4.93). □

Part II

Identification of Givental and CEO theories

–5– Givental's action and CEO-recursion

5.1 Introduction

5.1.1 Givental theory

Givental theory [46, 45, 47] is one of the most important tools in the study of Gromov-Witten invariants of target varieties and general cohomological field theories that allows, in particular, to obtain explicit relations between the partition functions of different theories, to reconstruct higher genera correlators from the genus 0 data, and to establish general properties of semi-simple theories.

The core of the theory is Givental's formula that gives a formal Gromov-Witten potential associated to a calibrated semi-simple Frobenius structure. Teleman proves [103] that the formal Gromov-Witten potential associated to the calibrated Frobenius structure of a target variety with semi-simple quantum cohomology coincides with the actual Gromov-Witten potential in all genera.

Roughly speaking, to a calibrated Frobenius structure of dimension N with a chosen semi-simple point t one can associate two $N \times N$ matrix series, $S_t(\zeta^{-1})$ and $R_t(\zeta)$, and $N \times N$ matrices Ψ_t and Δ_t (the latter one is diagonal), such that for a certain quantization of these matrices we have the following formula for the corresponding Gromov-Witten potential

$$\hat{S}_t^{-1}\hat{\Psi}_t\hat{R}_t\hat{\Delta}_t Z_{\mathrm{KdV}}^{\otimes r}, \tag{5.1}$$

where by Z_{KdV} we denote the Kontsevich-Witten tau-function of the KdV hierarchy; that is, the function parametrizing the intersection indices of ψ-classes on the moduli space of curves.

5.1.2 CEO-recursion theory

The Chekhov-Eynard-Orantin-recursion, or CEO-recursion (see [36, 38]), is a procedure that takes the following objects as input. First, a particular Riemann surface, which is usually called the *spectral curve*. Second, two functions x and y on this surface, and third, a choice of a bi-differential on this surface, which we will call the *two-point function* (it has often been referred to as the Bergman kernel, but since this term has other uses as well, we refrain from using it here). Occasionally, a particular extra choice of a coordinate on an open part of the Riemann surface is also made. The output of the CEO-recursion is a set of n-forms $\omega_{g,n}$, whose expansion in this additional coordinate generates interesting numbers.

In some cases these numbers are correlators of a matrix model (that was the original motivation for introducing the CEO-recursion; it is a natural generalization of the reconstruction procedure for the correlators of a certain class of matrix models, see, e.g. [5]), in some other cases they appear to be related to Gromov-Witten theory and to various intersection numbers on the moduli space of curves.

One of the ways to think about the input data of the CEO-recursion theory is to say that the $(g, n) = (0, 1)$ part of a partition function in some geometrically motivated theory determines the spectral curve; the $(g,n)=(0,2)$ part of a partition function determines the two-point function, and the rest of the correlators can be reconstructed from these two via CEO-recursion, in terms of a proper expansion of $\omega_{g,n}$ (see [30]).

The CEO-recursion theory is often used to reproduce known partition functions, to extract from them some higher genus correlators which were until then unreachable and to give new non-trivial relations for the correlators, see e. g. [39].

5.1.3 Identification

As we see, there is a lot of similarities in both theories (which was first noted by Alexandrov, Mironov and Morozov in [4, 5, 6]). In both cases we have to start with a small amount of data fixed in genus zero, and in both cases the intersection indices of ψ-classes on the moduli space of curves are some kind of structure constants of the reconstruction procedure (in the case of Givental this is just a part of Givental's formula for the formal Gromov-Witten potential, and in the case of CEO-recursion it is recovered locally in an expansion near a simple critical point of the spectral curve, see [33]).

Moreover, in both cases we have an expansion of the correlators in terms of Feynman graphs, see [118] on the Givental side and [34, 39, 67] on the spectral curve side. So, the natural question is whether we can precisely identify both theories in some setup.

On the Givental side we restrict ourselves to a part of the Givental formula, namely, $\hat{R}\hat{\Delta}Z_{\mathrm{KdV}}^{\otimes r}$ (this expression gives the so-called *total ancestor potential*, written in the normalized canonical basis). In some sense, it is the most important part of the Givental formula since it determines the underlying Frobenius structure, while the rest of the formula is a linear change of variables (action of the matrix $\hat{\Psi}$) and a change of calibration rather than of the Frobenius structure itself (action of the matrix series $\hat{S}^{-1}$). Note that for a cohomological field theory which does not have quadratic terms in the potential, the S-action becomes trivial when one takes the origin as the chosen point on the Frobenius manifold. For Gromov-Witten applications, where quadratic terms do appear, the S-action is nontrivial, but, together with Ψ-action, it amounts to a linear change of variables. This still allows for the correspondence below to be established, as long as one makes a specific choice of coordinates on the CEO-recursion side. We describe this in detail in the case of the particular example of $\mathbb{P}^1$, see below.

On the CEO-recursion side we consider a collection of local germs of a spectral curve at a finite number of points, with fixed expansions of the coordinate functions x and y and the two-point function near these points. The result of the CEO-recursion are local germs of n-forms $\omega_{g,n}$ defined on the products of the given germs of the curve, which we expand in a particular basis of forms that also depends on the expansions of the two-point function.

The resulting systems of correlators coincide for consistent choices of the input data in both theories. We prove this fact, essentially using the graphical interpretation of the formulas given in [118, 34], and provide a dictionary to translate Givental data into local spectral curve data and vice versa.

Thus, we solve the problem about the mysterious relation between CEO-recursion and enumerative geometry.

5.1.4 Organization of the chapter

The chapter assumes some pre-knowledge of both Givental and CEO-recursion theory; we refer to [38, 71, 97] as possible sources. In Section 5.2 we recall the Givental theory, and we present the Givental formula as a sum over graphs. In Section 5.3 we do the same for CEO-recursion theory. In Section 5.4 we prove the theorem on identification of the two theories and provide a corresponding dictionary.

5.2 Givental group action as a sum over graphs

In this section we review the Givental group action and we remind the reader how it can be used to write the partition function of an N-dimensional semi-simple cohomological field theory as an operator acting on the product of N KdV τ-functions. Using this, we write the partition function for such a cohomological field theory as a sum over decorated graphs. This is essentially the same as what was done in [118]; in the present chapter the contributions are distributed in a slightly different way over the components of the graph to make the comparison with the CEO-recursion.

5.2.1 Givental group action

We remind the reader of the original formulation, due to Y.-P. Lee, of the infinitesimal Givental group action in terms of differential operators [68, 69, 70].

Consider the space of partition functions for N-dimensional cohomological field theories

$$Z = \exp\left(\sum_{g\geq 0} \hbar^{g-1}\mathcal{F}_g\right) \tag{5.2}$$

in variables $v^{d,i}$, $d \geq 0$, $i = 1, \ldots, N$. There is a fixed scalar product $\eta_{ij} = \delta_{ij}$ on the vector space $V := \langle e_1, \ldots, e_N\rangle$ of primary fields corresponding to the indices $i = 1, \ldots, N$. Furthermore, we will denote by $e_{\mathbf{1}}$ the vector in V that plays the role of the unit.

Later on we will also use the so-called *correlators*

$$\langle \tau_{d_1}(e_{i_1})\tau_{d_2}(e_{i_2})\cdots\tau_{d_k}(e_{i_k})\rangle_g \tag{5.3}$$

which correspond to the coefficients of formal power series $\mathcal{F}_g$ in the following way:

$$\mathcal{F}_g = \sum \frac{\langle \tau_{d_1}(e_{i_1})\tau_{d_2}(e_{i_2})\cdots\tau_{d_k}(e_{i_k})\rangle_g}{|\mathrm{Aut}((i_m, d_m)_{m=1}^k)|} v^{d_1,i_1}\cdots v^{d_k,i_k}, \tag{5.4}$$

where $|\mathrm{Aut}((i_m, d_m)_{m=1}^k)|$ denotes the number of automorphisms of the collection of multi-indices (i_m, d_m) and where the sum is such that it includes each monomial $v^{d_1,i_1}\cdots v^{d_k,i_k}$ exactly once. Note that in the special case of a Gromov-Witten theory for some manifold X, these correlators carry the following meaning:

$$\langle \tau_{d_1}(e_{i_1})\tau_{d_2}(e_{i_2})\cdots\tau_{d_k}(e_{i_k})\rangle_g = \sum_{\deg}\int_{[X_{g,k,\deg}]} ev_1^*(e_{i_1})\psi_1^{d_1}\, ev_2^*(e_{i_2})\psi_2^{d_2}\cdots ev_k^*(e_{i_k})\psi_k^{d_k}, \tag{5.5}$$

where $[X_{g,k,\deg}]$ is the moduli space of degree deg stable maps to X of genus-g curves with k marked points, ev_i is the evaluation map at the i^{th} point and the ψ correspond to ψ-classes.

Consider a sequence of operators $r_l \in \mathrm{Hom}(V,V)$ for $l \geq 1$, such that the operators with odd (resp., even) indices are symmetric (resp., skew-symmetric). Then we denote by $(r_l z^l)\hat{}$ the following differential operator:

$$\begin{aligned}(r_l z^l)\hat{} :=& -(r_l)^i_{\mathbf{1}}\frac{\partial}{\partial v^{l+1,i}} + \sum_{d=0}^{\infty} v^{d,i}(r_l)^j_i\frac{\partial}{\partial v^{d+l,j}} \\ &+\frac{\hbar}{2}\sum_{m=0}^{l-1}(-1)^{m+1}(r_l)^{i,j}\frac{\partial^2}{\partial v^{m,i}\partial v^{l-1-m,j}}.\end{aligned} \tag{5.6}$$

Here the indices $i, j \in \{1, \dots, N\}$ on r_l correspond to the basis $\{e_1, \dots, e_N\}$ of V, and the index $\mathbf{1}$ corresponds to the unit vector $e_{\mathbf{1}}$. When we write r_l with two upper-indices we mean as usual that we raise one of the indices using the scalar product η.

Given such a sequence of operators r_l, we define an operator series $R(z)$ in the following way

$$R(z) = \sum_{l=0}^{\infty} R_l z^l := \exp\left(\sum_{l=1}^{\infty} r_l z^l\right). \tag{5.7}$$

The quantization $\hat{R}$ of this series is defined by

$$\hat{R} = \exp\left(\sum_{l=1}^{\infty} \left((-1)^l r_l z^l\right)\hat{}\,\right). \tag{5.8}$$

Givental observed that the action of such operators $\hat{R}$ on formal power series Z for which the number of ψ-classes (given by the first index of $v^{d,\mu}$) at any monomial of degree n is no more than $3g-3+n$, is well-defined. The main theorem of [44] states that this action preserves the property that Z is a generating function of the correlators of a cohomological field theory with target space (V, η) (see also [62, 103]).

Remark 5.1. Note that this definition of $\hat{R}$ differs from the one in [118] by the sign $(-1)^l$. It is needed here to agree with Givental's notation in Proposition 5.3, cf. [46, Proposition 7.3]. For the same reason, in order to agree with the conventions of Givental, we label in a matrix by the upper index the column and by the lower index the row.

5.2.2 Givental operator for a Frobenius manifold

Let $Z(\{t^{d,\mu}\})$ be the partition function of some N-dimensional semi-simple conformal cohomological field theory. We recall the construction (due to Givental [46, 45, 47], see also Dubrovin [28]) of an operator series $R(z)$ as in the previous section whose quantization takes the product of N KdV τ-functions to Z.

Let F be the restriction of $\log(Z)$ to the genus zero part without descendants. Denote $t^\mu := t^{0,\mu}$. Then F can be interpreted as a formal Frobenius manifold with metric

$$\eta_{\alpha\beta} = \frac{\partial^3 F}{\partial t^{\mathbf{1}} \partial t^\alpha \partial t^\beta} \tag{5.9}$$

and Frobenius algebra structure $c_{\alpha\beta}^\gamma$

$$c_{\alpha\beta\gamma} = \frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\gamma}. \tag{5.10}$$

We can assume that $\eta_{\alpha\beta} = \delta_{\alpha+\beta,n+1}$ and $e_{\mathbf{1}} = e_1$. According to [27] it is always possible by an appropriate choice of these flat coordinates t^μ.

Canonical coordinates

Another set of coordinates is given by the *canonical coordinates* $\{u^i\}$ which can be found as solutions to Equation (3.54) from [27], and have the property that $\{\partial_i := \partial/\partial u^i\}$ forms a basis of canonical idempotents of the Frobenius algebra product. In these coordinates the metric is diagonal and the unit vector field is given by $e_{\mathbf{1}} = \partial_1 + \cdots + \partial_N$.

Define $\Delta_i := 1/(\partial_i, \partial_i)$ to be the inverse of the square of the length of the i^{th} canonical basis element, and call $\{\partial/\partial v^i := \Delta_i^{1/2} \partial/\partial u^i\}$ the normalized canonical basis in the tangent space. We denote the coordinates corresponding to this basis by v^i, and the formal variables corresponding to these coordinates by $v^{d,i}$. They are precisely the formal variables $v^{d,i}$ appearing in the previous section.

Let U be the matrix of canonical coordinates $U = \mathrm{diag}(u^1, \dots, u^N)$ and denote by Ψ the transition matrix from the flat to the normalized canonical bases. That is, denoting $\mathrm{d}t = (\mathrm{d}t^1, \dots, \mathrm{d}t^N)^{\mathrm{T}}$ and $\mathrm{d}u = (\mathrm{d}u^1, \dots, \mathrm{d}u^N)^{\mathrm{T}}$, one has

$$\Delta^{-1/2}\mathrm{d}u = \Psi \mathrm{d}t, \tag{5.11}$$

where $\Delta = \mathrm{diag}(\Delta_1, \dots, \Delta_N)$.

Remark 5.2. Note that Ψ obtained by the definition above depends on the point p of the Frobenius manifold.

Recursion

Construct an operator series $R(z) = \sum_{k\geq 0} R_k z^k$ as in the previous section in the following way.

Recursively define the off-diagonal entries of R_k in normalized canonical coordinates by solving the equation

$$\Psi^{-1}\mathrm{d}(\Psi R_{k-1}) = [\mathrm{d}U, R_k]. \tag{5.12}$$

using $R_0 = \mathbf{I}$ as a base case. Construct the diagonal entries of R_k by integrating the next equation

$$\Psi^{-1}\mathrm{d}(\Psi R_k) = [\mathrm{d}U, R_{k+1}] \tag{5.13}$$

using the fact that the diagonal entries of $[\mathrm{d}U, R_{k+1}]$ are equal to zero. To fix the integration constant, use the Euler equation

$$R_k = -(i_E \mathrm{d}R_k)/k, \tag{5.14}$$

where $E = \sum u^i \partial_i$ is the Euler field (here we use the fact that we started with a conformal cohomological field theory).

This procedure recursively defines R_k for all k. The following proposition is essentially proved in Givental's papers [46, 45].

Proposition 5.3. *Let F be a local N-dimensional Frobenius manifold structure, semisimple at the origin, and let (R_k) be the series of operators constructed from this F by the recursive procedure described above, at the origin. Let Ψ and Δ be as above, taken at the origin as well. Then we have the following formula:*

$$\mathcal{F}_0 = \operatorname*{Res}_{\hbar=0} \mathrm{d}\hbar \cdot \log \hat{\Psi}\hat{R}\hat{\Delta}\mathcal{T}. \tag{5.15}$$

Here $\mathcal{F}_0 = \mathcal{F}_0(\{t^{d,\mu}\})$ is the genus 0 descendant potential of cohomological field theory associated to F; $\mathcal{T}$ is the product of N KdV tau-functions,

$$\mathcal{T} := Z_{\mathrm{KdV}}(\{u^{d,1}\}) \cdots Z_{\mathrm{KdV}}(\{u^{d,N}\});$$

$\hat{\Delta}$ replaces the variables of i^{th} KdV τ-function according to $u^{d,i} = \Delta_i^{1/2} v^{d,i}$ and replaces $\hbar$ with $\Delta_i \hbar$, while $\hat{\Psi}$ is the change of variables $v^{d,i} = \Psi^i_\nu t^{d,\nu}$. The unit for the R-action is given by $(\Psi^1_1, \dots, \Psi^N_1)$.

Remark 5.4. In fact, using Teleman's result in [103], one has a refined version of Equation (5.15):

$$Z = \hat{\Psi}\hat{R}\hat{\Delta}\mathcal{T}. \tag{5.16}$$

Note that it holds for cohomological field theories. In the Gromov-Witten case, when quadratic terms in the potential cannot be neglected, there appears an additional complication, see the next remark below.

Remark 5.5. Givental's formula [46] for a Gromov-Witten total descendant potential (without the $(g=1, n=0)$-term),

$$Z = \hat{S}^{-1}\hat{\Psi}\hat{R}\hat{\Delta}\mathcal{T}, \tag{5.17}$$

also includes the operator $\hat{S}$, given by

$$\hat{S} = \exp\left(\sum_{l=1}^{\infty}(s_l z^{-l})^\wedge\right), \tag{5.18}$$

where the operators $(s_l z^{-l})^\wedge$ are defined in the following way (see, e. g., [44, Section 4.2]):

$$\sum_{l=1}^{\infty}(s_l z^{-l})^\wedge = -\,(s_1)_1^\mu \frac{\partial}{\partial t^{0,\mu}} + \frac{1}{\hbar}\sum_{d=0}^{\infty}(s_{d+2})_{1,\mu}\, t^{d,\mu} \tag{5.19}$$

$$+\sum_{\substack{d=0\\ l=1}}^{\infty}(s_l)_\nu^\mu\, t^{d+l,\nu}\frac{\partial}{\partial t^{d,\mu}} + \frac{1}{2\hbar}\sum_{\substack{d_1,d_2\\ \mu_2,\mu_2}}(-1)^{d_1}(s_{d_1+d_2+1})_{\mu_1,\mu_2}\, t^{d_1,\mu_1}t^{d_2,\mu_2}.$$

Note that formula (5.18) for the quantization of S differs from the analogous formula (5.8) for R by a factor of $(-1)^l$ in the exponent, which agrees with the definition in Givental's papers [45, 46].

The matrices s_k are defined through the following relation:

$$S(z) = \sum_{k=0}^{\infty} S_k z^{-k} = \exp\left(\sum_{l=0}^{\infty} s_l z^{-l}\right), \tag{5.20}$$

where for $S(z)$, taken at a point p of the Frobenius manifold, we have (see [46]), for any points a and b of the Frobenius manifold,

$$(a, b\, S_p) := (a,b) + \sum_{k=0}^{\infty}\langle\tau_0(a)\exp\left(\tau_0(p)\right)\tau_k(b)\rangle_0\, z^{-1-k}. \tag{5.21}$$

Here on the left hand side the brackets stand for the scalar product on the tangent space to the Frobenius manifold at p, and we used an identification of the tangent space with the whole Frobenius manifold, since in this case the Frobenius manifold is itself a vector space. If p is the origin, we have just

$$(a, b\, S) := (a,b) + \sum_{k=0}^{\infty}\langle\tau_0(a)\tau_k(b)\rangle\;\, z^{-1-k}. \tag{5.22}$$

Note that this S action is defined in the general case when the total descendant genus 0 potential is known. For the case when only a Frobenius potential is specified, the choice of S is then called a *calibration* of the Frobenius manifold, see [45, 28] for related details. In the case of cohomological field theory when we disregard quadratic terms, the S action is trivial if p is taken to be the origin.

It turns out that in most of the relevant cases, e.g. for the Gromov-Witten theory of CP1 (see section 6.1 below), the only relevant term in equation (5.19) is

$$\sum_{\substack{d=0\\ l=1}}^{\infty}(s_l)_\nu^\mu\, t^{d+l,\nu}\frac{\partial}{\partial t^{d,\mu}}, \tag{5.23}$$

since $(s_1)_1^\mu$ vanishes and all other terms just change the unstable terms in the potential.

This means, that in these cases $\hat{S}^{-1}$ just performs a linear change of formal variables $t^{d,\mu}$ in the following way:

$$t^{d,\mu} \mapsto \sum_{m=d}^{\infty} (S_{m-d})^\mu_\nu t^{m,\nu}. \tag{5.24}$$

5.2.3 Expressions in terms of graphs

In [118] the action of an operator series as in equation (5.8) is written as a sum over graphs. By Remark 5.4, this allows us to construct the potential of any semi-simple conformal cohomological field theory as a sum over graphs. Here we repeat the construction of [118] in a slightly different way that will be more convenient for the comparison with the CEO-recursion formalism. Furthermore, we also include the action of $\hat{\Delta}$. It is easy to see that the construction is equivalent to that of [118].

Notation 5.6. Let γ be any graph. By a half-edge we mean either a leaf or an edge together with a choice of one of the two vertices it is attached to. By $V(\gamma)$, $E(\gamma)$, $H(\gamma)$ and $L(\gamma)$ we denote the sets of vertices, edges, half-edges and leaves of γ. For any vertex v of γ, denote by $H(v)$ the set of half-edges connected to v.

Let $\widetilde{\Gamma}$ be the set of all connected graphs γ together with a choice of disjoint splitting $L(\gamma) = L^*(\gamma) \coprod L^\bullet(\gamma)$, a labelling of the vertices by pairs $(g,i) \in \mathbb{Z}_{\geq 0} \times \{1,\dots,N\}$ and a labelling of the elements of $H(\gamma)$ by non-negative integers, such that the label of a leaf in $L^\bullet$ is always greater than one. The elements of $L^*(\gamma)$ are called *ordinary leaves*, the elements of $L^\bullet$ are called *dilaton leaves*. We denote by Γ the subset of all graphs in $\widetilde{\Gamma}$ that are *stable*; that is, any vertex labelled $(0,i)$ for some i is of valence at least three.

For any graph γ denote by $\mathfrak{g}\colon V(\gamma) \to \mathbb{Z}_{\geq 0}$ and $\mathfrak{i}\colon V(\gamma) \to \{1,\dots,N\}$ the maps that associate to any vertex its first and second label respectively, and by $\mathfrak{k}\colon H(\gamma) \to \mathbb{Z}_{\geq 0}$ the map that associates to any half-edge its label. Denote by $\mathfrak{v}\colon L(\gamma) \to V(\gamma)$ the map that associates to each leaf the corresponding vertex, and by $\mathfrak{v}_1, \mathfrak{v}_2\colon E(\gamma) \to V(\gamma)$ and by $\mathfrak{h}_1, \mathfrak{h}_2\colon E(\gamma) \to H(\gamma)$ the maps that associate to an edge the first and second vertex, and the corresponding half-edges respectively.

Remark 5.7. The labels introduced above are used to keep track of different data for the trivial cohomological field theory; g is for the genus, i for the primary field in canonical coordinates and the labelling of the marked half-edges is for the power of ψ-class.

Remark 5.8. As in [118], edges of a graph in Γ are considered to be oriented (this allows to define the maps $\mathfrak{v}_1$ and $\mathfrak{v}_2$ unambiguously); the final result does not depend on the orientation.

Let $R(z)^i_j$ be the components of the operator series $R(z)$ in normalized canonical basis as computed in Section 5.2.2. To each part of a graph $\gamma \in \Gamma$ we assign some polynomial in formal variables $\hbar$ and $v^{d,i}$. Here $\hbar$ is used to keep track of the genus, while the first index of $v^{d,i}$ keeps track of the number of ψ-classes and the second index keeps track of the normalized canonical coordinate.

Leaves

To each ordinary leaf $l \in L^*$ marked by k attached to a vertex marked by the pair (g,i), we assign

$$(\mathcal{L}^*)^i_k(l) := [z^k] \left(\sum_{d \geq 0} \left((R(-z))^i_j v^{d,j} z^d \right) \right), \tag{5.25}$$

which corresponds to the second term in (5.6).

To a dilaton leaf $\lambda \in L^\bullet(\gamma)$ marked by k attached to a vertex marked by (g,i) we assign

$$(\mathcal{L}^\bullet)^i_k(\lambda) := [z^{k-1}]\left(-(R(-z))^i_{\mathbf{1}}\right), \tag{5.26}$$

which corresponds to the first term in (5.6), which is called the *dilaton shift*.

Edges

To an edge e connecting a vertex v_1 marked by (g_1,i_1) to a vertex v_2 marked by (g_2,i_2) and with markings k_1 and k_2 at the corresponding half-edges, we assign

$$\mathcal{E}^{i_1,i_2}_{k_1,k_2}(e) := [z^{k_1}w^{k_2}]\left(\hbar \cdot \frac{\delta^{i_1 i_2} - \sum_s (R(-z))^{i_1}_s (R(-w))^{i_2}_s}{z+w}\right). \tag{5.27}$$

Note that this does not depend on the choice of ordering of the vertices and that it follows from the fact that $R(z)$ can be written as $R(z) = \exp(\sum r_l z^l)$ that the numerator on the right-hand side is equal to the product of $(z+w)$ with some power series in z and w, so this definition makes sense.

Vertices

Let v be a vertex marked by (g,i) with n half-edges attached to it (this includes all ordinary and dilaton leaves and also half-edges that are parts of internal edges) labelled by $k_1,\dots,k_n$. Then we assign to v the following expression:

$$\mathcal{V}^{(g,i)}_{\{k_1,\dots,k_n\}}(v) := \hbar^{g-1}(\Delta_i)^{\frac{1}{2}(2g-2+n)} \int_{\bar{\mathcal{M}}_{g,n}} \psi_1^{k_1}\cdots\psi_n^{k_n}. \tag{5.28}$$

Z as a sum over graphs

It is easy to see that the sum over all graphs in Γ of the product of the contributions described above, weighted by the inverse order of the automorphism group of the graph, is equal to the graph-sum described in [118] (the only difference is that now we have specialized to the action on the trivial cohomological field theory, leading to ψ-class integrals (5.28) as vertex contributions). Thus, we recover the partition function Z of the cohomological field theory we started with as a sum over Γ:

$$(\hat{R}\hat{\Delta}\mathcal{T})(\{v^{d,j}\}) = \sum_{\gamma\in\Gamma}\frac{1}{|\mathrm{Aut}(\gamma)|}\prod_{v\in V(\Gamma)}\hbar^{\mathfrak{g}(v)-1}(\Delta_{\mathfrak{i}(v)})^{\frac{1}{2}(2\mathfrak{g}(v)-2+\mathrm{val}(v))}\left\langle\prod_{h\in H(v)}\tau_{\mathfrak{k}(h)}\right\rangle_{\mathfrak{g}}$$
$$\prod_{e\in E(\gamma)}\mathcal{E}^{\mathfrak{i}(\mathfrak{v}_1(e)),\mathfrak{i}(\mathfrak{v}_2(e))}_{\mathfrak{k}(\mathfrak{h}_1(e)),\mathfrak{k}(\mathfrak{h}_2(e))}(e)\prod_{l\in L^*(\gamma)}(\mathcal{L}^*)^{\mathfrak{i}(\mathfrak{v}(l))}_{\mathfrak{k}(l)}(l)\prod_{\lambda\in L^\bullet(\gamma)}(\mathcal{L}^\bullet)^{\mathfrak{i}(\mathfrak{v}(l))}_{\mathfrak{k}(l)}(\lambda). \tag{5.29}$$

5.3 CEO-recursion as a sum over graphs

In this section, we define a local version of the CEO-recursion and write the corresponding invariants as a sum over graphs, which allows us to compare it to the Givental action in the next section.

5.3.1 Local CEO-recursion

We define a local version of the CEO-recursion in the following way. The term local refers to the fact that the data are all defined locally around the canonical coordinates without any reference to the possible existence of a global manifold where these functions can be defined.

Definition 5.9. For $N \in \mathbb{N}^*$, we call times a set of N families of complex numbers $\{h^i_k\}_{k\in\mathbb{N}}$ for $i = 1, \dots, N$ and jumps another set of $N \times N$ infinite families of complex numbers $\left\{B^{i,j}_{k,l}\right\}_{(k,l)\in\mathbb{N}^2}$ for $i, j = 1, \dots, N$. We finally define a set of canonical coordinates $\{a_i\}_{i=1}^N \in \mathbb{C}^N$ subject to $a_i \neq a_j$ for $i \neq j$.

For all $i, j \in \{1, \dots, N\}$, we define the following set of analytic functions and differential forms in a neighborhood of $0 \in \mathbb{C}$:

$$x^i(z) := z^2 + a_i\,, \quad y^i(z) := \sum_{k=0}^{\infty} h^i_k z^k \tag{5.30}$$

and

$$B^{i,j}(z,z') = \delta_{i,j}\frac{\mathrm{d}z \otimes \mathrm{d}z'}{(z-z')^2} + \sum_{k,l=0}^{\infty} B^{i,j}_{k,l} z^k z'^l \mathrm{d}z \otimes \mathrm{d}z'. \tag{5.31}$$

For $2g - 2 + n > 0$, we define the genus g, n-point correlation functions $\omega^{i_1,\dots,i_n}_{g,n}(z_1, \dots z_n)$ recursively by

$$\omega^{i_0,i_1,\dots,i_n}_{g,n+1}(z_0, z_1, \dots, z_n) := \sum_{j=1}^{N} \operatorname*{res}_{z\to 0} \frac{\int_{-z}^{z} B^{i_0,j}(z_0,\cdot)}{2\left(y^j(z) - y^j(-z)\right)\mathrm{d}x^j(z)}$$
$$\left(\omega^{j,j,i_1,\dots,i_n}_{g-1,n+2}(z,-z,z_1,\dots,z_n) \;+ \sum_{A\cup B=\{1,\dots,n\}} \sum_{h=0}^{g} \omega^{j,\mathbf{i}_A}_{h,|A|+1}(z,\mathbf{z}_A)\,\omega^{j,\mathbf{i}_B}_{g-h,|B|+1}(-z,\mathbf{z}_B)\right), \tag{5.32}$$

where for any set A, we denote by $\mathbf{z}_A$ (resp., $\mathbf{i}_A$) the set $\{z_k\}_{k\in A}$ (resp., $\{i_k\}_{k\in A}$), and where the base of the recursion is given by

$$\omega^i_{0,1}(z) := 0; \qquad \omega^{i,j}_{0,2}(z,z') := B^{i,j}(z,z'). \tag{5.33}$$

For convenience, in the sequel we denote

$$K^{i,j}(z,z') = \frac{\int_{-z}^{z} B^{i,j}(z',\cdot)}{2(y^j(z) - y^j(-z))\mathrm{d}x^j(z)} \tag{5.34}$$

and

$$\omega_{g,n}(\vec{z}) = \sum_{\vec{i}} \omega^{\vec{i}}_{g,n}(\vec{z})\;, \tag{5.35}$$

where the length of $\vec{z}$ and $\vec{i}$ is n.

5.3.2 Correlation functions and intersection numbers

The correlation functions built by this CEO-recursion can actually be written in terms of intersections of ψ classes on the moduli space of Riemann surfaces. This result is a slight generalization of [33, 34] to the local CEO-recursion.

5.3.3 One-branch point case

The link between the CEO-recursion formalism and intersection numbers on the moduli space of Riemann surfaces comes from the application of this formalism to the Airy curve. This case corresponds to $N = 1$ and:

$$x(z) = z^2 + a\,, \quad y(z) = z \quad \text{and} \quad B(z,z') = \frac{\mathrm{d}z \otimes \mathrm{d}z'}{(z-z')^2}. \tag{5.36}$$

Remark 5.10. Since there is only one branch point in this case, i.e. $N = 1$, we omit the superscript indicating which branch point we consider in the notations of this section.

For further convenience, we introduce two additional parameters by considering the curve

$$x(z) = z^2 + a\,, \quad y(z) = \alpha z \quad \text{and} \quad B(z,z') = \beta \frac{\mathrm{d}z \otimes \mathrm{d}z'}{(z-z')^2}, \tag{5.37}$$

the usual Airy curve being $\alpha = \beta = 1$. In this case, the CEO-recursion reads

$$\omega_{g,n+1}(z_0, z_1, \ldots, z_n) := \operatorname*{res}_{z\to 0} \frac{\beta}{2\alpha} \frac{\mathrm{d}z_0}{2z\,\mathrm{d}z} \frac{1}{(z_0^2 - z^2)} \left(\omega_{g-1,n+2}(z,-z,z_1,\ldots,z_n) + \sum_{A\cup B=\{1,\ldots,n\}} \sum_{h=0}^{g} \omega_{h,|A|+1}(z,\mathbf{z}_A)\omega_{g-h,|B|+1}(-z,\mathbf{z}_B) \right) \tag{5.38}$$

and one has

Lemma 5.11. *The correlation functions of the Airy curve can be expressed in terms of intersection numbers:*

$$\omega_{g,n}(z_1,\ldots,z_n) = \left(-\frac{\beta}{2\alpha}\right)^{2g+n-2} \beta^{g+n-1} \sum_{\alpha_1,\ldots,\alpha_n \geq 0} \langle \tau_{a_1} \ldots \tau_{a_n} \rangle_{g,n} \prod_{i=1}^{n} \frac{(2\alpha_i+1)!!\,\mathrm{d}z_i}{z_i^{2\alpha_i+2}}. \tag{5.39}$$

This lemma was proved many times by direct computation [32, 33, 38, 109], matching the CEO-recursion with the recursive definition of the intersection numbers.

As a side note, the first few correlation functions are

$$\omega_{0.3}(z_1,z_2,z_3) = -\frac{\beta^3}{2\alpha} \prod_{i=1}^{3} \frac{\mathrm{d}z_i}{z_i^2}, \tag{5.40}$$

$$\omega_{0,4}(z_1,z_2,z_3,z_4) = \frac{\beta^5}{4\alpha^2} \prod_{i=1}^{4} \frac{\mathrm{d}z_i}{z_i^2} \sum_{i=1}^{4} \frac{3}{z_i^2}, \tag{5.41}$$

$$\omega_{1,1}(z) = \frac{-\beta^2}{2\alpha} \frac{\mathrm{d}z}{8z^4} \tag{5.42}$$

and

$$\omega_{1,2}(z_1,z_2) = \frac{\beta^4}{4\alpha^2} \frac{\mathrm{d}z_1\,\mathrm{d}z_2}{8z_1^2 z_2^2} \left(\frac{5}{z_1^4} + \frac{5}{z_2^4} + \frac{3}{z_1^2 z_2^2} \right). \tag{5.43}$$

Remark 5.12. It is important to remark that there exist different conventions in the literature for defining the CEO-recursion, mainly differing by a change of sign of the recursion kernel. The latter can be recovered by a change of sign $\alpha \to -\alpha$.

Let us now consider a deformation of the Airy curve which we will refer to as the KdV curve in the following. It has only one branch point, $N = 1$, and reads

$$\begin{cases} x(z) = z^2 + a_i \\ y(z) = \alpha \sum_{k=1}^{\infty} h_k z^k \\ B(z,z') = \beta B_{\mathrm{KdV}}(z,z') = \beta \frac{\mathrm{d}z \otimes \mathrm{d}z'}{(z-z')^2} \end{cases}. \tag{5.44}$$

The corresponding correlation functions can also be expressed in terms intersection numbers as follows:

Lemma 5.13. *The correlation functions of the KdV curve read:*

$$\omega_{g,n}(z_1,\dots,z_n) = \left(-\frac{\beta}{2\alpha h_1}\right)^{2g+n-2} \beta^{g+n-1}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}$$
$$\sum_{\vec{\alpha}\in\mathbb{N}^{*m}}\prod_{k=1}^{m}(2\alpha_k+1)!!\frac{h_{2\alpha_k+1}}{h_1}\prod_{i=1}^{n}\frac{(2d_i+1)!!\,\mathrm{d}z_i}{z_i^{2d_i+2}}\left\langle\prod_{j=1}^{n}\tau_{d_j}\prod_{k=1}^{m}\tau_{\alpha_k+1}\right\rangle_{g,n+m}. \tag{5.45}$$

Proof. Once again the proof can be found in the literature [32, 37, 33]. However, let us study a graphical interpretation of this result when considering an arbitrary convention for the CEO-recursion. For $f(z)$ an analytic function around $z\to 0$ and $\{T_k\}_{k\in\mathbb{Z}}$ a set of parameters, one can compute

$$\operatorname*{res}_{Z_1\to 0}\operatorname*{res}_{Z_2\to 0} K(Z_1,z)\left\{\left(\sum_{k\geq 1}T_kZ_1^k\right)\mathrm{d}Z_1\,K(Z_2,-Z_1)f(Z_2)\,[\mathrm{d}Z_2]^2 - \left(\sum_{k\geq 1}T_k(-Z_1)^k\right)\mathrm{d}Z_1\,K(Z_2,Z_1)f(Z_2)\,[\mathrm{d}Z_2]^2\right\} \tag{5.46}$$

where the recursion kernel is the one of the Airy curve, i.e. the one for which $h_k=0$ for $k\geq 2$:

$$K(z,z_0)=\frac{\beta}{2\alpha h_1}\frac{\mathrm{d}z_0}{2z\,\mathrm{d}z}\frac{1}{(z_0^2-z^2)}. \tag{5.47}$$

One can move the integration contours to get

$$\operatorname*{res}_{Z_1\to 0}\operatorname*{res}_{Z_2\to 0} = \operatorname*{res}_{Z_2\to 0}\operatorname*{res}_{Z_1\to 0} + \operatorname*{res}_{Z_2\to 0}\operatorname*{res}_{Z_1\to Z_2} + \operatorname*{res}_{Z_2\to 0}\operatorname*{res}_{Z_1\to -Z_2}. \tag{5.48}$$

The first term of the right hand side vanishes since the integrand does not have any pole at $Z_1\to 0$. Let us now compute one of the other two terms:

$$\begin{aligned}\operatorname*{res}_{Z_2\to 0}\operatorname*{res}_{Z_1\to Z_2} & K(Z_1,z)\left(\sum_{k\geq 1}T_kZ_1^k\mathrm{d}Z_1\right)K(Z_2,-Z_1)f(Z_2)\,[\mathrm{d}Z_2]^2 \\ &= -\operatorname*{res}_{Z_2\to 0}\frac{\beta}{2\alpha h_1}\frac{\mathrm{d}Z_2}{2Z_2}f(Z_2)\operatorname*{res}_{Z_1\to Z_2}\frac{\mathrm{d}z}{2Z_1}\frac{1}{(z^2-Z_1^2)}\frac{\beta}{2\alpha h_1}\frac{1}{(Z_1^2-Z_2^2)}\left(\sum_{k\geq 1}T_kZ_1^k\mathrm{d}Z_1\right) \\ &= -\operatorname*{res}_{Z_2\to 0}\frac{\beta}{2\alpha h_1}\frac{\mathrm{d}Z_2dz}{2Z_2}f(Z_2)\frac{1}{(z^2-Z_2^2)}\frac{\beta}{2\alpha h_1}\sum_{k\geq 1}\frac{T_k}{4}Z_2^{k-2}.\end{aligned} \tag{5.49}$$

In the same way,

$$\begin{aligned}\operatorname*{res}_{Z_2\to 0}\operatorname*{res}_{Z_1\to -Z_2} & K(Z_1,z)\left(\sum_{k\geq 1}T_kZ_1^k\mathrm{d}Z_1\right)K(Z_2,-Z_1)f(Z_2)\,[\mathrm{d}Z_2]^2 = \\ &= -\operatorname*{res}_{Z_2\to 0}\frac{\beta}{2\alpha h_1}\frac{\mathrm{d}Z_2\mathrm{d}z}{2Z_2}f(Z_2)\frac{1}{(z^2-Z_2^2)}\frac{\beta}{2\alpha h_1}\sum_{k\geq 1}\frac{T_k}{4}(-Z_2)^{k-2}.\end{aligned} \tag{5.50}$$

The sum of these two terms reads

$$\operatorname*{res}_{Z_2\to 0}\operatorname*{res}_{Z_1\to\pm Z_2} K(Z_1,z)\left(\sum_{k\geq 1} T_k Z_1^k dZ_1\right) K(Z_2,-Z_1)f(Z_2)\,[\mathrm{d}Z_2]^2 =$$
$$= -\operatorname*{res}_{Z_2\to 0}\frac{\beta}{2\alpha h_1}\frac{\mathrm{d}Z_2\mathrm{d}z}{2Z_2}f(Z_2)\frac{1}{(z^2-Z_2^2)}\frac{\beta}{2\alpha h_1}\sum_{k\geq 1}\frac{T_{2k}}{2}(Z_2)^{2k-2} \quad (5.51)$$

and finally:

$$\operatorname*{res}_{Z_1\to 0}\operatorname*{res}_{Z_2\to 0} K(Z_1,z)\left\{\left(\sum_{k\geq 1} T_k Z_1^k\right)\mathrm{d}Z_1\,K(Z_2,-Z_1)f(Z_2)\,[\mathrm{d}Z_2]^2\right.$$
$$\left. -\left(\sum_{k\geq 1} T_k (-Z_1)^k\right)\mathrm{d}Z_1\,K(Z_2,Z_1)f(Z_2)\,[\mathrm{d}Z_2]^2\right\} =$$
$$= \operatorname*{res}_{Z_2\to 0}\frac{\beta}{2\alpha h_1}\frac{\mathrm{d}Z_2\mathrm{d}z}{2Z_2}f(Z_2)\frac{1}{(z^2-Z_2^2)}\left(-\frac{\beta}{2\alpha h_1}\right)\sum_{k\geq 1} T_{2k}(Z_2)^{2k-2}. \quad (5.52)$$

On the other hand, plugging in the times h_k amounts to computing similar quantities:

$$\operatorname*{res}_{z\to 0}\frac{\beta}{2\alpha h_1}\frac{\mathrm{d}z_0}{2z\,\mathrm{d}z}\frac{1}{(z_0^2-z^2)}\frac{1}{\left(1+\sum_{k=1}^{\infty}\frac{h_{2k+1}}{h_1}z^{2k}\right)}f(z)\,[\mathrm{d}z]^2 =$$
$$= \operatorname*{res}_{z\to 0}\frac{\beta}{2\alpha h_1}\frac{\mathrm{d}z_0}{2z\,\mathrm{d}z}\frac{1}{(z_0^2-z^2)}f(z)\,[\mathrm{d}z]^2$$
$$\cdot\left(1-\sum_{k=1}^{\infty}\frac{h_{2k+1}}{h_1}z^{2k}+\left[\sum_{k=1}^{\infty}\frac{h_{2k+1}}{h_1}z^{2k}\right]^2+\cdots\right) \quad (5.53)$$

The first term of this sum is the Airy recursion kernel. The second one is of the shape of the preceding one with $T_{2k+2}=\frac{2\alpha h_{2k+1}}{h_1}$ for $k\geq 1$ so that:

$$-\operatorname*{res}_{z\to 0}\frac{\beta}{2\alpha}\frac{\mathrm{d}z_0}{2z\,\mathrm{d}z}\frac{1}{(z_0^2-z^2)}f(z)\,[\mathrm{d}z]^2\sum_{k=1}^{\infty}\frac{h_{2k+1}}{h_1}z^{2k} =$$
$$= \operatorname*{res}_{Z_1\to 0}\operatorname*{res}_{Z_2\to 0} K(Z_1,z_0)\left\{g(Z_1)\mathrm{d}Z_1\,K(Z_2,-Z_1)f(Z_2)\,[\mathrm{d}Z_2]^2\right.$$
$$\left. -g(-Z_1)\mathrm{d}Z_1\,K(Z_2,Z_1)f(Z_2)\,[\mathrm{d}Z_2]^2\right\} \quad (5.54)$$

where

$$g(z) := \sum_{k\geq 1}\frac{2\alpha h_{2k+1}}{\beta h_1}z^{2k+2}. \quad (5.55)$$

This same procedure can be applied to the other terms of the sum. The k^{th} order term can be written as a sequence of $k+1$ residues computed with the Airy recursion kernel with $g(z)dz$ on one of the outgoing legs. This computation shows that introducing non-vanishing times amounts to introducing a non-vanishing $\omega_{0,1}(z) := g(z)dz$ in the CEO-recursion.

It is often useful to represent the CEO-recursion in a graphical form by representing the interaction kernel $K(z,z_0)$ by an edge oriented from z_0 towards a trivalent vertex labelled by z

and the function $\omega_{0,2}(z_1, z_2)$ by a non-oriented edge (see [36] for more details about this set of graphs). In this form, $\omega_{g,n}(z_1, \ldots, z_n)$ is a sum over trivalent graphs of genus g with n leaves labelled by the arguments $z_1, \ldots, z_n$. The preceding computation shows that the correlation functions of the KdV curve can be obtained from the correlation functions of the Airy curve by introducing a set of new leaves, called dilation leaves, in the definition of the graphs used. A dilation leaf decorated by a label k is weighted by

$$(2d-1)!! \operatorname*{res}_{z\to 0} g(z)\frac{\mathrm{d}z}{z^{2d+1}} = (2d-1)!!\,\frac{2\alpha h_{2d-1}}{\beta}. \tag{5.56}$$

Plugging this expression into the formula for the Airy correlation functions proves the result. □

General case

In this section we give a formula for the correlation function of the local CEO-recursion.

Definition 5.14. Let $\Gamma_{g,n}$ be the subset of Γ (see Notation 5.6) consisting of graphs of genus g' such that $g' + \sum_{v\in V(\Gamma)} \mathfrak{g}(v) = g$ and with n ordinary leaves. Let us also introduce orderings on the ordinary leaves and denote by $\check{\Gamma}_{g,n}$ the set of all graphs from $\Gamma_{g,n}$ with all possible orderings on the ordinary leaves. For a given graph with a fixed ordering $\check{\gamma} \in \check{\Gamma}_{g,n}$ and for an ordinary leaf of that graph $l \in L^*(\check{\gamma})$ we denote by $\mathfrak{m}(l)$ the index of this particular leaf (then $\mathfrak{m}(l)$ is an integer from 1 to n such that different leaves have different values $\mathfrak{m}(l)$ assigned to them).

Theorem 5.15. *The correlation functions can be written as a sum over decorated graphs whose vertices are weighted by intersection of ψ-classes on $\overline{\mathcal{M}}_{g,n}$, edges by the jumps, ordinary leaves by primitives of B and dilaton leaves by the times.*

For $2-2g-n<0$, one has

$$\omega_{g,n}(\vec{z}) = \frac{1}{n!}\sum_{\check{\gamma}\in\check{\Gamma}_{g,n}} \prod_{v\in V(\check{\gamma})} \left(-2h_1^{\mathfrak{i}(v)}\right)^{2-2\mathfrak{g}(v)-\mathrm{val}(v)} \left\langle \prod_{h\in H(v)} \tau_{\mathfrak{k}(h)} \right\rangle_{\mathfrak{g}(v),\mathrm{val}(v)}$$
$$\prod_{e\in E(\check{\gamma})} \check{B}^{\mathfrak{i}(\mathfrak{v}_1(e)),\mathfrak{i}(\mathfrak{v}_2(e))}_{\mathfrak{k}(\mathfrak{h}_1(e)),\mathfrak{k}(\mathfrak{h}_2(e))} \prod_{l\in L^*(\check{\gamma})} \sum_{j=1}^{N} \mathrm{d}\xi^{\mathfrak{i}(\mathfrak{v}(l))}_{\mathfrak{k}(l)}(z_{\mathfrak{m}(l)}, j) \prod_{\lambda\in L^\bullet(\check{\gamma})} \check{h}^{\mathfrak{i}(\mathfrak{v}(\lambda))}_{\mathfrak{k}(\lambda)} \tag{5.57}$$

with

$$\check{h}^i_k := 2(2k-1)!!h^i_{2k-1}, \tag{5.58}$$

$$\mathrm{d}\xi^i_d(z_\alpha, j) := \operatorname*{res}_{z\to 0} \frac{(2d+1)!!\mathrm{d}z}{z^{2d+2}} \int^z B^{i,j}(z, z_\alpha), \tag{5.59}$$

$$\check{B}^{i,j}_{d_1,d_2} := B^{i,j}_{2d_1,2d_2}\,(2d_1-1)!!\,(2d_2-1)!! \tag{5.60}$$

and

$$\left\langle \prod_{i=1}^{n} \tau_{k_i} \right\rangle_{g,n} := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1}\psi_2^{k_2}\ldots\psi_n^{k_n}. \tag{5.61}$$

Proof. The proof is very similar to the one presented in [34, 67]. However, we prefer to present a completely graphical proof so that the link with the next sections becomes clear.

We follow the proof of [34]. From the definition, one can write the correlation functions as a sum over graphs with oriented and non-oriented arrows linking trivalent vertices resulting in the following expression:

$$\omega^{\vec{i}}_{g,n}(\vec{z}) = \sum_{G\in\widehat{G}_{g,n}} \omega(G) \tag{5.62}$$

with $\widehat{G}_{g,n}$ the set of genus g trivalent graphs with one root and $n-1$ leaves labelled by the arguments z_i and a skeleton tree of oriented edges pointing from the root towards the leaves weighted by

$$\begin{aligned}\omega(G) \quad = \quad & \vec{\prod_{v\in V(G)}} \operatorname*{res}_{Z_v\to 0} \prod_{e\in E_{\mathrm{oriented}}(G)} K^{i(v_1(e)),i(v_2(e))}(Z_{v_1(e)},Z_{v_2(e)}) \\ & \prod_{e\in E_{\mathrm{unoriented}}(G)} B^{i(v_1(e)),i(v_2(e))}(Z_{v_1(e)},Z_{v_2(e)})\end{aligned} \tag{5.63}$$

where each leaf is considered as a one-valent vertex v and one denotes Z_v the variable z_i associated to this leaf in the correlation function, $E_{\mathrm{oriented}}(G)$ is the set of oriented leaves of G and $E_{\mathrm{unoriented}}(G)$ is the set of unoriented leaves of G (see [36] for further details). The product of residues $\vec{\prod}_{v\in V(G)} \operatorname{res}_{Z_v\to 0}$ is oriented following the arrows, i.e. one first computes the residue corresponding to the end of an arrow before the one associated to its root.

It is useful to remark that, for any edge, oriented or not, one has two types of contributions. Indeed, the functions $B^{i,j}(z,z')$ have a singular part

$$B^{i,j}_{\mathrm{KdV}}(z,z') := \delta_{i,j}\frac{\mathrm{d}z\otimes\mathrm{d}z'}{(z-z')^2} \tag{5.64}$$

and a regular part

$$B^{i,j}_{\mathrm{reg}}(z,z') := \sum_{k,l=0}^{\infty} B^{i,j}_{k,l} z^k z'^l \mathrm{d}z\otimes\mathrm{d}z' \tag{5.65}$$

when $z\to z'$:

$$B^{i,j}(z,z') = B^{i,j}_{\mathrm{KdV}}(z,z') + B^{i,j}_{\mathrm{reg}}(z,z'). \tag{5.66}$$

In the same way, one has

$$K^{i,j}(z,z') = K^{i,j}_{\mathrm{KdV}}(z,z') + K^{i,j}_{\mathrm{reg}}(z,z'). \tag{5.67}$$

One can translate this by representing $B^{i,j}_{\mathrm{KdV}}(z,z')$ and $B^{i,j}_{\mathrm{reg}}(z,z'))$ by dashed and dotted unoriented edges respectively, while representing $K^{i,j}_{\mathrm{KdV}}(z,z')$ and $K^{i,j}_{\mathrm{reg}}(z,z'))$ by dashed and dotted oriented edges from z to z'. The preceding sum is thus transformed into a sum over graphs where the edges are dotted or dashed and weighted accordingly.

The dashed edges can be expressed in a slightly different way. Indeed, one has

$$B^{i,j}_{\mathrm{reg}}(z,z') = \operatorname*{res}_{z_1\to z}\operatorname*{res}_{z_2\to z'} B^{i,i}_{\mathrm{KdV}}(z,z_1)\left[\int^{z_1}\int^{z_2} B^{i,j}_{\mathrm{reg}}(z_1,z_2)\right] B^{j,j}_{\mathrm{KdV}}(z_2,z') \tag{5.68}$$

and

$$K^{i,j}_{\mathrm{reg}}(z,z') = \operatorname*{res}_{z_1\to z}\operatorname*{res}_{z_2\to \pm z'} B^{i,i}_{\mathrm{KdV}}(z,z_1)\left[\int^{z_1}\int^{z_2} B^{i,j}_{\mathrm{reg}}(z_1,z_2)\right] K^{j,j}_{\mathrm{KdV}}(z_2,z') \tag{5.69}$$

by a simple application of the Cauchy formula.

Remember that such an edge comes with integration of its boundary variables, thus, one typically has to compute

$$\operatorname*{res}_{z\to 0}\operatorname*{res}_{z'\to 0} g(z)K^{i,j}_{\mathrm{reg}}(z,z')f(z') \tag{5.70}$$

which reads

$$\operatorname*{res}_{z\to 0}\operatorname*{res}_{z_1\to z}\operatorname*{res}_{z'\to 0}\operatorname*{res}_{z_2\to\pm z'}\left(g(z)B^{i,i}_{\mathrm{KdV}}(z,z_1)\left[\int^{z_1}\int^{z_2}B^{i,j}_{\mathrm{reg}}(z_1,z_2)\right]K^{j,j}_{\mathrm{KdV}}(z_2,z')f(z')\right). \tag{5.71}$$

One can move the integration contours around 0 thanks to:

$$\operatorname*{res}_{z\to 0}\operatorname*{res}_{z_1\to z} = \operatorname*{res}_{z_1\to 0}\operatorname*{res}_{z\to 0} - \operatorname*{res}_{z\to 0}\operatorname*{res}_{z_1\to 0} \tag{5.72}$$

and

$$\operatorname*{res}_{z'\to 0}\operatorname*{res}_{z_2\to\pm z'} = \operatorname*{res}_{z_2\to 0}\operatorname*{res}_{z'\to 0} - \operatorname*{res}_{z'\to 0}\operatorname*{res}_{z_2\to 0}. \tag{5.73}$$

Since, the integrand does not have any pole as $z_1\to 0$ nor $z_2\to 0$, this shows that (5.70) is equal to

$$\operatorname*{res}_{z_1\to 0}\operatorname*{res}_{z\to 0}\operatorname*{res}_{z_2\to 0}\left(g(z)\,B^{i,i}_{\mathrm{KdV}}(z,z_1)\left[\int^{z_1}\int^{z_2}B^{i,j}_{\mathrm{reg}}(z_1,z_2)\right]\operatorname*{res}_{z'\to 0}\,K^{j,j}_{\mathrm{KdV}}(z_2,z')\,f(z')\right). \tag{5.74}$$

In the same way, one gets that

$$\operatorname*{res}_{z\to 0}\operatorname*{res}_{z'\to 0}g(z)B^{i,j}_{\mathrm{reg}}(z,z')f(z') \tag{5.75}$$

is equal to

$$\operatorname*{res}_{z_1\to 0}\operatorname*{res}_{z\to 0}\operatorname*{res}_{z_2\to 0}\left(g(z)\,B^{i,i}_{\mathrm{KdV}}(z,z_1)\left[\int^{z_1}\int^{z_2}B^{i,j}_{\mathrm{reg}}(z_1,z_2)\right]\operatorname*{res}_{z'\to 0}\,B^{j,j}_{\mathrm{KdV}}(z_2,z')\,f(z')\right). \tag{5.76}$$

One can finally proceed in a similar way for re-expressing the weights of the root and the leaves by writing

$$\operatorname*{res}_{z'\to 0}K^{i,j}(z,z')f(z') = \operatorname*{res}_{z_2\to 0}\left[\int^{z_2}B^{i,j}(z,z_2)\right]\operatorname*{res}_{z'\to 0}\,K^{j,j}_{\mathrm{KdV}}(z_2,z')\,f(z') \tag{5.77}$$

and

$$\operatorname*{res}_{z\to 0}g(z)B^{i,j}(z,z') = \operatorname*{res}_{z_1\to 0}\operatorname*{res}_{z\to 0}g(z)\,B^{i,i}_{\mathrm{KdV}}(z,z_1)\left[\int^{z_1}B^{i,j}(z_1,z')\right]. \tag{5.78}$$

Remark that, for the roots and leaves, in opposition to the inner edges, the functions are the full ones, not just the regular part.

As a result, by applying this transformation to each dotted line, any graph is composed of a set of dotted subgraphs whose vertices have the same label separated by dashed lines. Since each subgraph with label i also includes a root and leaves, it is a contribution to the correlation functions obtained for the case $N=1$, times h^i_k and vanishing jumps $B^{i,i}_{k,l}=0$. In the sum over graphs, one can thus replace every sum over such sub-graphs by vertices of corresponding genus weighted by the correlation function for $N=1$, which reads

$$\omega^{\vec{i}}_{g,n}(\vec{z}) = \sum_{\gamma\in\Gamma_{g,n}}\Omega(\gamma) \tag{5.79}$$

where

$$\Omega(\gamma) = \prod_{v\in V(\gamma)}\prod_{h\in H(v)}\operatorname*{res}_{Z_h\to 0}\omega^{\mathrm{KdV},i(v)}_{g(v),\mathrm{val}(v)}\left(\{Z_h\}_{h\in H(v)}\right)$$
$$\prod_{e\in E(\gamma)}\int^{Z_{h_1(e)}}\int^{Z_{h_2(e)}}B^{i(v_1(h)),i(v_2(h))}_{\mathrm{reg}}(Z_{h_1(e)},Z_{h_2(e)})\prod_{h\in L^*(\gamma)}\int^{Z_h}B^{i,j}(Z_h,z_h) \tag{5.80}$$

where $\omega_{g,n}^{\mathrm{KdV},i}(z_1,\dots,z_n)$ is the genus g, n-pointed correlation function obtained from the CEO-recursion in the case $N=1$ and the initial data:

$$\begin{cases} x(z) = z^2 + a_i \\ y(z) = \sum_{k=1}^{\infty} h_k^i z^k \\ B(z,z') = B_{\mathrm{KdV}}(z,z') = \frac{dz\otimes dz'}{(z-z')^2} \end{cases} . \tag{5.81}$$

As explained in the preceding section, it can be expressed in terms of intersection numbers:

$$\omega_{g,n}^{\mathrm{KdV},i}(z_1,\dots,z_n) = \left(-2h_1^i\right)^{2-2g-n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}$$
$$\sum_{\vec{\alpha}\in\mathbb{N}^{*m}} \prod_{k=1}^{m} (2\alpha_k+1)!! \frac{h_{2\alpha_k+1}^i}{h_1^i} \prod_{i=1}^{n} \frac{(2d_i+1)!!\,dz_i}{z_i^{2d_i+2}} \left\langle \prod_{j=1}^{n} \tau_{d_j} \prod_{k=1}^{m} \tau_{\alpha_k+1} \right\rangle_{g,n+m} \tag{5.82}$$

which can be made more symmetric under the exchange of the ordinary and dilation leaves by writing

$$\omega_{g,n}^{\mathrm{KdV},i}(z_1,\dots,z_n) = \sum_{m=0}^{\infty} (-2h_1)^{2-2g-n-m} \frac{1}{m!}$$
$$\sum_{\vec{\alpha}\in\mathbb{N}^{*m}} \prod_{k=1}^{m} (2\alpha_k-1)!! 2h_{2\alpha_k-1}^i \prod_{i=1}^{n} \frac{(2d_i+1)!!\,dz_i}{z_i^{2d_i+2}} \left\langle \prod_{j=1}^{n} \tau_{d_j} \prod_{k=1}^{m} \tau_{\alpha_k} \right\rangle_{g,n+m} . \tag{5.83}$$

Absorbing the factors of the form $\frac{(2d+1)!!dz}{z^{2d+2}}$ into the corresponding half-edge contribution, the weight of an inner edge becomes

$$\operatorname*{res}_{z_1\to 0} \operatorname*{res}_{z_2\to 0} \int^{z_1}\int^{z_2} B_{\mathrm{reg}}^{i,j}(z_1,z_2) \frac{(2d_1+1)!!dz_1}{z_1^{2d_1+2}} \frac{(2d_2+1)!!dz_1}{z_2^{2d_2+2}} \tag{5.84}$$

which is equal to

$$\check{B}_{d_1,d_2}^{i,j} := B_{2d_1,2d_2}^{i,j}\,(2d_1-1)!!\,(2d_2-1)!!\ , \tag{5.85}$$

while the weight of the ordinary leaves becomes

$$d\xi_d^i(z_\alpha,j) := \operatorname*{res}_{z\to 0} \frac{(2d+1)!!dz}{z^{2d+2}} \int^z B^{i,j}(z,z_\alpha), \tag{5.86}$$

where one has to consider both the singular and non-singular part of $B^{i,j}(z,z_\alpha)$. Collecting these contributions together proves the theorem. □

5.3.4 Change of scales

An important property of the correlation functions built in this way is their homogeneity property which reads

$$\forall\lambda\in\mathbb{C}\,,\ \omega_{g,n}(\vec{z}_N|x,\lambda y,B) = \lambda^{2-2g-n}\omega_{g,n}(\vec{z}_N|x,y,B) \tag{5.87}$$

One can thus get an additional factor λ^i by replacing $h_k^i \to \lambda^i h_k^i$ resulting in a rescaling of the weight of the vertices by $\left(\lambda^{\mathrm{i}(v)}\right)^{2-2g(v)-\mathrm{val}(v)}$.

5.3.5 Weights, Laplace transform and recursive definition

It is interesting to note that the weights of the edges are the coefficients of the Laplace transform of B;

$$\check{B}^{i,j}(u,v) := \sum_{(k,l)\in\mathbb{N}^2} \check{B}^{i,j}_{k,l} u^{-k} v^{-l} \tag{5.88}$$

is equal to

$$\check{B}^{i,j}(u,v) \;=\; \delta_{i,j}\frac{uv}{u+v} \;+\; \frac{\sqrt{uv}e^{ua_i+va_j}}{2\pi}\int_{x(z)-a_i\in\mathbb{R}^+}\int_{x(z')-a_j\in\mathbb{R}^+} B^{i,j}(z,z')e^{-ux(z)-vx(z')}. \tag{5.89}$$

In [34], it was proved that, if $\mathrm{d}x$ is a meromorphic form defined on a Riemann surface, $\check{B}^{i,j}(u,v)$ can be factorized and expressed in terms of some basic functions. Here, we will consider the converse and build $B^{i,j}_{k,l}$ by induction in such a way that there exist a set of functions $\{f_{i,j}(u)\}_{i,j=1}^N$ such that

$$\check{B}^{i,j}(u,v) = \frac{uv}{u+v}\left(\delta_{i,j} - \sum_{k=1}^N f_{i,k}(u)f_{k,j}(v)\right). \tag{5.90}$$

Let us define the coefficients $B^{i,j}_{k,l}$ recursively in terms of the initial data $B^{i,j}_{k,0}$ by imposing that

$$\xi^i_{d+1}(z,j) := -2\frac{\mathrm{d}\xi^i_d(z,j)}{\mathrm{d}x^{[j]}(z)} - \sum_{k=1}^{\mathfrak{b}} \check{B}^{i,k}_{d,0}\,\xi^k_0(z,j), \tag{5.91}$$

or, in terms of the Laplace transform

$$\begin{aligned} f^i_d(u,j) &:= \frac{\sqrt{u}}{2\sqrt{\pi}}\int_{x(z)-a_j\in\mathbb{R}^+} e^{-u(x(z)-a_j)}\mathrm{d}x^j(z)\,\xi^i_d(z) \\ &= \delta_{i,j}(-1)^d u^d - \sum_{d'} \check{B}^{i,j}_{d,d'}u^{-d'-1} \end{aligned} \tag{5.92}$$

this reads

$$f^i_{d+1}(u,j) := -2uf^i_d(u,j) - \sum_{k=1}^N \check{B}^{i,k}_{d,0}\,f^k_0(z,j). \tag{5.93}$$

With this definition, one has

$$\check{B}^{i,j}(u,v) = \frac{uv}{u+v}\left(\delta_{i,j} - \sum_{k=1}^{\mathfrak{b}} f^k_0(u,i)f^k_0(v,j)\right). \tag{5.94}$$

5.4 Identification of the two theories

In this section we show how to find a local spectral curve corresponding to any semi-simple conformal Frobenius manifold.

Suppose some local spectral curve is given. For any $i\in\{1,\ldots,N\}$ and $k\in\mathbb{Z}_{\geq 0}$ define

$$W^i_k(z) := \sum_{j=1}^N \mathrm{d}\left(\left(-\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z}\right)^k \xi^i_0(z,j)\right). \tag{5.95}$$

Theorem 5.16. *Let R be some series of operators on an N-dimensional vector space V as in Section 5.2. Let $Z = \hat{R}\hat{\Delta}\mathcal{T}$, where $\mathcal{T}$ is a product of N KdV τ-functions, be the partition function of the corresponding semi-simple cohomological field theory.*

Define a local spectral curve by the following data

$$\check{B}^{i,j}_{p,q} := [z^p w^q] \frac{\delta^{ij} - \sum_{s=1}^N R^i_s(-z)R(-w)^j_s}{z+w} \tag{5.96}$$

and

$$\check{h}^i_k := [z^{k-1}] \left(-R(-z)\right)^i_{\mathbf{1}} \tag{5.97}$$

$$h^i_1 := -\frac{1}{2\sqrt{\Delta^i}}. \tag{5.98}$$

Let $\omega_{g,n}$ be the genus g, n-pointed CEO-recursion invariant of this spectral curve and denote by

$$\Omega(\{v^{d,i}\}) = \left(\sum_{g,d} \omega_{g,d}(z_1, \dots, z_d)\Big|_{W^i_d(z_m)=v^{d,i}} \hbar^{g-1} \right) \tag{5.99}$$

their sum after a change of variables $W^i_k(z_m) \leftrightarrow v^{d,i}$ for all m. Then the partition function of the cohomological field theory and the CEO-recursion invariants agree in the following sense:

$$Z(\{v^{d,i}\}) = \exp\left(\Omega(\{v^{d,i}\})\right). \tag{5.100}$$

Proof. In Sections 5.2 and 5.3 we have given representations of Z and $\omega_{g,n}$ as sums over the set Γ (in fact, in the case of $\omega_{g,n}$ this set is $\check{\Gamma}$ rather than Γ, but after changing the variables $W^i_k(z_m) \leftrightarrow v^{d,i}$ we can take the sum over orderings and arrive at the sum over Γ acquiring an additional factor of $n!$, which cancels with the corresponding factor in (5.57)). We prove the theorem by showing that the contribution of each individual graph to Z is equal to the contribution to Ω.

Let $\gamma \in \Gamma$ be some graph. Note that on both sides we assign the same weight to the vertices of γ, namely to a vertex labelled (g, i) with n half-edges attached to it labelled $d_1, \dots, d_n$ we associate

$$(-2h^i_1)^{2-2g-n} \left\langle \tau_{d_1} \cdots \tau_{d_n} \right\rangle_{g,n}. \tag{5.101}$$

Furthermore, by equation (5.96), any edge in γ contributes the same to Z and Ω.

Let l be an ordinary leaf of γ labelled by k attached to a vertex labelled by (g, i). We use induction on k to show that the contribution to Z is the same as the contribution to Ω.

The contribution of l to Z is given by

$$\mathcal{L}^i_k(l) = [z^k] \left(\sum_d (R(-z))^i_j v^{d,j} z^d \right) = \sum_{d=0}^{k} (-1)^{k-d} (R_{k-d})^i_j v^{d,j}. \tag{5.102}$$

When $k = 0$, the contribution of l to Ω is given by

$$\sum_j \mathrm{d}\xi^i_0(z_j, j) = W^i_0. \tag{5.103}$$

Since $(R_0)^i_j = \delta^i_j$, the contributions to Z and Ω agree when $k = 0$.

Now suppose that they agree for some $k \in \mathbb{Z}_{\geq 0}$. That is, suppose that

$$\sum_j \mathrm{d}\xi^i_k(z^j, j) = \sum_{l=0}^{k} (-1)^{k-l} (R_{k-l})^i_s W^s_l. \tag{5.104}$$

Then, using Equation (5.91), the contribution of the leaf to Ω for the index $k+1$ is given by

$$\begin{aligned}
\sum_j \mathrm{d}\xi^i_{k+1}(z^j, j) &= \sum_j \mathrm{d}\left(-2\frac{\partial \xi^i_k(z^j, j)}{\partial x^j} - \sum_{t=1}^{N} \check{B}^{i,t}_{k,0} \xi^t_0(z^j, j) \right) \\
&= \sum_j \mathrm{d}\left(-\frac{1}{z^j}\frac{\partial}{\partial z^j} \xi^i_k(z^j, j) - \sum_{t=1}^{n} -(-1)^{k+1} (R_{k+1})^i_t \xi^t_0(z^j, j) \right) \\
&= \sum_{l=0}^{k} (-1)^l (R_l)^i_t W^t_{k+1-l} + (-1)^{k+1} (R_{k+1})^i_t W^t_0 = \sum_{l=0}^{k+1} (-1)^l (R_l)^i_t W^t_{k+1-l},
\end{aligned} \tag{5.105}$$

where we used equation (5.96) to write

$$\check{B}^{i,t}_{k,0} = -(-1)^{k+1} (R_{k+1})^i_t. \tag{5.106}$$

This completes the induction, and since it is clear that the dilaton leaves contribute the same in both cases, it also completes the proof of the theorem. □

Remark 5.17. The theorem above deals with the potential of a cohomological field theory written in terms of formal variables $v^{d,i}$ corresponding to normalized canonical basis. In order to pass to flat coordinates one can change the variables in the following way:

$$v^{d,i} = \Psi^i_\mu t^{d,\mu}. \tag{5.107}$$

On the spectral curve side it will correspond to changing the variables W^i_k in the following way:

$$W^i_k = \Psi^i_\mu V^\mu_k. \tag{5.108}$$

Thus, the theorem holds in the same form for the potential of cohomological field theory written in terms of formal variables $t^{d,\mu}$, only one should identify $t^{d,\mu}$ with V^μ_d.

Remark 5.18. Above we established the correspondence between cohomological field theories and symplectic invariants of spectral curves. However, as noted in Remark 5.5, in the case of Gromov-Witten theories we cannot disregard quadratic terms. So, in the formula for the total descendent potential an additional operator $\hat{S}$ appears. In some cases, again see Remark 5.5, it performs only a linear change of formal variables $t^{d,\mu}$ on which the potential depends. Thus, to establish the correspondence in this case, one has to change the variables W^i_k in precisely the same way, and then identify the resulting variables with $t^{d,\mu}$, similar to the case of previous remark. Occasionally, the changes of variables preformed by $\hat{\Psi}$ and $\hat{S}^{-1}$ can be a re-expansion of $\omega_{g,n}$ in a new coordinate on the spectral curve. In Chapter 6 we explain this procedure in detail for the case of $\mathbb{P}^1$.

Remark 5.19. The system of equations obtained via a Laplace transform from the equations of Givental for the R-matrix (that is, the so-called equations of deformed flat connection) is studied in detail in [28, Section 5]. This gives, in particular, a recipe to reconstruct the two-point function directly from the Frobenius structure bypassing the reconstruction of the R-matrix. This also explains why we call the critical values $a_1, \ldots, a_N$ of x the canonical coordinates.

Part III

Applications of Givental-CEO correspondence

–6– The Norbury-Scott conjecture

As a first application of Theorem 5.16, in this chapter we recall and prove the Norbury-Scott conjecture on the stationary sector of the Gromov-Witten theory of $\mathbb{P}^1$.

6.1 Gromov-Witten theory of $\mathbb{P}^1$

The Gromov-Witten theory of $\mathbb{P}^1$ is discussed from the geometric point of view in many sources, see e. g. [86]. Givental proved in [46] that his formula for the formal Gromov-Witten potential coincides with the geometric Gromov-Witten potential of $\mathbb{P}^1$, so we discuss it here only from the Givental point of view, ignoring the geometric background. One can find the same computations in [101, 102].

The underlying structure of Frobenius manifold is determined by the following solution of the WDVV equation

$$\frac{1}{2}(t^1)^2t^2 + e^{t^2}, \tag{6.1}$$

and the scalar product given by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{6.2}$$

All ingredients of the Givental formula depend on a particular choice of the point on the Frobenius manifold, and in this case we choose the point $(0,0)$ in the coordinates (t_1,t_2).

We perform a direct computation following the recipe of Givental in [45], see also Section 5.2.2. As a possible choice of the canonical coordinates, we use

$$u^1 = t^1 + 2\exp(t^2/2); \tag{6.3}$$

$$u^2 = t^1 - 2\exp(t^2/2). \tag{6.4}$$

In particular, for $t^1 = t^2 = 0$ we have $u^1 = -u^2 = 2$. Then,

$$\Delta_1^{-1} = \left\langle \frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^1} \right\rangle = \frac{\exp(-t^2/2)}{2}; \tag{6.5}$$

$$\Delta_2^{-1} = \left\langle \frac{\partial}{\partial u^2}, \frac{\partial}{\partial u^2} \right\rangle = \frac{-\exp(-t^2/2)}{2}, \tag{6.6}$$

so we can choose the square roots as

$$\Delta_1^{-1/2} = \frac{\exp(-t^2/4)}{\sqrt{2}}; \tag{6.7}$$

$$\Delta_2^{-1/2} = \frac{-i\exp(-t^2/4)}{\sqrt{2}}. \tag{6.8}$$

For this choice we have the following matrix of transition from the basis given by $(\partial/\partial t^1, \partial/\partial t^2)$ to the normalized canonical basis:

$$\Psi = \begin{pmatrix} \frac{\exp(-t^2/4)}{\sqrt{2}} & \frac{-i\exp(-t^2/4)}{\sqrt{2}} \\ \frac{\exp(t^2/4)}{\sqrt{2}} & \frac{i\exp(t^2/4)}{\sqrt{2}} \end{pmatrix}. \tag{6.9}$$

It is the matrix $\Psi = \Psi^i_\alpha$, where α labels the rows and corresponds to the flat basis, while i labels the columns and corresponds to the normalized canonical basis.

The recipe of reconstruction of the matrix R from [45] gives at the origin the matrix $R(\zeta) = \sum_{k=0}^\infty R_k\zeta^k$, where

$$R_k = \frac{(2k-1)!!(2k-3)!!}{2^{4k}k!} \cdot \begin{pmatrix} -1 & (-1)^{k+1}2ki \\ 2ki & (-1)^{k+1} \end{pmatrix} \tag{6.10}$$

The S matrix is given by the derivatives of the deformed flat coordinates, computed in [29, Example 3.7.9] At the origin we have:

$$\begin{aligned} S(\zeta^{-1}) &= \mathbf{I} + \zeta^{-1} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ &+ \sum_{k=1}^{\infty} \frac{\zeta^{-2k}}{(k!)^2} \begin{pmatrix} 1 - 2k\left(\frac{1}{1} + \cdots \frac{1}{k}\right) & 0 \\ 0 & 1 \end{pmatrix} \\ &+ \sum_{k=1}^{\infty} \frac{\zeta^{-2k-1}}{(k!)^2} \begin{pmatrix} 0 & -2\left(\frac{1}{1} + \cdots \frac{1}{k}\right) \\ \frac{1}{k+1} & 0 \end{pmatrix}. \end{aligned} \tag{6.11}$$

(Note once again that we are using the convention that the matrices are acting on vector rows, opposite to the standard one).

The unit vector at the origin in the normalized canonical basis is equal to

$$e = (1,0) \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} = \left(\frac{1}{\sqrt{2}}, \frac{-i}{\sqrt{2}}\right). \tag{6.12}$$

Therefore, the dilaton leaves (cf. Equation (5.26)) in the Givental formula for $\mathbb{P}^1$ at the origin are

$$(\mathcal{L}^\bullet)^1_{k+1} = \frac{1}{\sqrt{2}} \cdot \frac{(-1)^{k+1}\left((2k-1)!!\right)^2}{k!2^{4k}}; \tag{6.13}$$

$$(\mathcal{L}^\bullet)^2_{k+1} = \frac{i}{\sqrt{2}} \cdot \frac{\left((2k-1)!!\right)^2}{k!2^{4k}} \tag{6.14}$$

for $k \geq 0$.

Proposition 6.1. *The Gromov-Witten potential of* $\mathbb{P}^1$,

$$Z_{\mathbb{P}^1}(\hbar, \{t^{\ell,1}, t^{\ell,2}\}_{\ell=0}^\infty), \tag{6.15}$$

is obtained from $\hat{R}\hat{\Delta}Z_{\mathrm{KdV}}^{\otimes 2}$ *(understood as a sum over graphs in the sense of Section 5.2.3 and written down in the normalized canonical basis, that is, in the variables* $v^{d,i}$, $d \geq 0$, $i = 1,2$*) via a linear change of variables given by*

$$\sum_{m\geq k}^{\infty} \left(t^{m,1}, t^{m,2}\right) S_k\zeta^{m-k} = \sum_{\ell=0}^{\infty} \left(v^{\ell,1}, v^{\ell,2}\right)\zeta^\ell \cdot \Psi^{-1}, \tag{6.16}$$

and a correction of the unstable terms (that is, (g,n)*-correlators with* $2g-2+n \leq 0$*).*

Proof. In order to get the Gromov-Witten potential of $\mathbb{P}^1$ as given by the Givental formula, we have to apply the $\hat{\Psi}$- and $\hat{S}^{-1}$-action to the expression in terms of graphs discussed in Section 5.2.3 that corresponds to $\hat{R}\hat{\Delta}Z_{\mathrm{KdV}}^{\otimes 2}$. The $\hat{\Psi}$-action is just a linear change of variable by definition. The general S-action is discussed in [44, Section 4.2]. It is a combination of a shift of variables that vanishes in our case (indeed, $(1,0)S_1 = (0,0)$), the linear change of variables that we have in the statement of Proposition, and a correction of unstable terms that is not essential for us. □

6.2 The Norbury-Scott conjecture

Norbury and Scott [84] propose the following construction. They consider a spectral curve given by

$$\begin{cases} x = z + \frac{1}{z}; \\ y = \log z, \end{cases} \tag{6.17}$$

and the standard two-point function

$$B(z,z') = \frac{\mathrm{d}z \otimes \mathrm{d}z'}{(z-z')^2}. \tag{6.18}$$

Via CEO-recursion they obtain the n-forms $\omega_{g,n}$ that they consider in the global variable x, and they conjecture the following theorem (they prove it for $g = 0,1$):

Theorem 6.2. *For $2g-2+n>0$, we have:*

$$\prod_{j=1}^{n}\left(-\underset{x_j=\infty}{\mathrm{res}} \frac{1}{(a_j+1)!} x_1^{a_j+1}\right)\omega_{g,n}(x_1,\dots,x_n) = \langle \prod_{j=1}^{n} \tau_{2,a_j}\rangle_g, \tag{6.19}$$

where $\langle\prod_{j=1}^{n}\tau_{2,a_j}\rangle_g$ is the corresponding correlator in $Z_{\mathbb{P}^1}$, that is, the coefficient of

$$\hbar^{g-1}\prod_{j=1}^{n} t_{2,a_j}/|Aut(a_1,\dots,a_n)| \tag{6.20}$$

in $\log Z_{\mathbb{P}^1}$.

In the rest of this section we prove this theorem, identifying all ingredients of the CEO-recursion with the corresponding parts of the Givental formula.

6.3 Proof of the Norbury-Scott conjecture

6.3.1 Local coordinates near the branch points

We denote the local coordinates by $z_1 = \sqrt{x-2}$ and $z_2 = \sqrt{x+2}$. Then we have:

$$x = z_1^2 + 2 \text{ near } x = 2,\ z = 1,\ z_1 = 0; \tag{6.21}$$
$$x = z_2^2 - 2 \text{ near } x = -2,\ z = -1,\ z_2 = 0. \tag{6.22}$$

Therefore,

$$z = 1 + \frac{z_1^2}{2} \pm z_1\sqrt{1+\frac{z_1^2}{4}}; \tag{6.23}$$
$$z = -1 + \frac{z_2^2}{2} \pm iz_2\sqrt{1-\frac{z_2^2}{4}}. \tag{6.24}$$

In both cases we choose + for ±.

6.3.2 Expansion of y

Recall that $y = \log z$. A direct computation shows:

$$y = \int \frac{\mathrm{d}z_1}{\sqrt{1+\frac{z_1^2}{4}}}; \tag{6.25}$$

$$y = \int \frac{-i\,\mathrm{d}z_2}{\sqrt{1-\frac{z_2^2}{4}}}; \tag{6.26}$$

Note that

$$\frac{1}{\sqrt{1+\frac{z_1^2}{4}}} = 1 + \sum_{k=1}^{\infty} z_1^{2k} \cdot \frac{(-1)^k(2k-1)!!}{k!2^{3k}}; \tag{6.27}$$

$$\frac{-i}{\sqrt{1-\frac{z_2^2}{4}}} = -i + \sum_{k=1}^{\infty} z_2^{2k} \cdot \frac{(-i)\cdot(2k-1)!!}{k!2^{3k}}. \tag{6.28}$$

Therefore

$$y = z_1 + \sum_{k=1}^{\infty} z_1^{2k+1} \cdot \frac{(-1)^k(2k-1)!!}{k!2^{3k}(2k+1)}; \tag{6.29}$$

$$y = -iz_2 + \sum_{k=1}^{\infty} z_2^{2k+1} \cdot \frac{(-i)\cdot(2k-1)!!}{k!2^{3k}(2k+1)}. \tag{6.30}$$

Thus the coefficients $\check{h}^i_{k+1}$, $k \geq 0$, are given by the following formulas:

$$\check{h}^1_{k+1} = 2 \cdot \frac{(-1)^k\left((2k-1)!!\right)^2}{k!2^{3k}}; \tag{6.31}$$

$$\check{h}^2_{k+1} = 2 \cdot \frac{(-i)\cdot\left((2k-1)!!\right)^2}{k!2^{3k}}. \tag{6.32}$$

6.3.3 The matrix $f_{i,j}(w)$

We use the following definition of the matrix $f_{ij}(w)$ (cf. Equation (5.90)):

$$f_{ij}(w) = \delta_{ij} - w\check{B}^{[ij]}(0, w^{-1}), \tag{6.33}$$

where $w = v^{-1}$. We use $\tilde{B}^{ij}_{0,l} = (B^{ij}_{\mathrm{reg}})_{0,2l}(2l-1)!!$, and the following expressions:

$$B^{11}_{\mathrm{reg}}(0, z_1) = \left[\frac{\mathrm{d}z(z_1')\otimes\mathrm{d}z(z_1)}{(z(z_1')-z(z_1))^2} - \frac{\mathrm{d}z_1'\otimes\mathrm{d}z_1}{(z_1'-z_1)^2}\right]_{z_1'=0} \tag{6.34}$$

$$B^{12}_{\mathrm{reg}}(0, z_2) = \left[\frac{\mathrm{d}z(z_1')\otimes\mathrm{d}z(z_2)}{(z(z_1')-z(z_2))^2}\right]_{z_1'=0} \tag{6.35}$$

$$B^{21}_{\mathrm{reg}}(0, z_1) = \left[\frac{\mathrm{d}z(z_2')\otimes\mathrm{d}z(z_1)}{(z(z_2')-z(z_1))^2}\right]_{z_2'=0} \tag{6.36}$$

$$B^{22}_{\mathrm{reg}}(0, z_2) = \left[\frac{\mathrm{d}z(z_2')\otimes\mathrm{d}z(z_2)}{(z(z_2')-z(z_2))^2} - \frac{\mathrm{d}z_2'\otimes\mathrm{d}z_2}{(z_2'-z_2)^2}\right]_{z_2'=0} \tag{6.37}$$

Therefore,

$$B^{11}_{\mathrm{reg}}(0,z_1) = \frac{1}{z_1^2}\left(\frac{1}{\sqrt{1+\frac{z_1^2}{4}}} - 1\right) \tag{6.38}$$

$$B^{12}_{\mathrm{reg}}(0,z_2) = \frac{i}{4(1-\frac{z_2^2}{4})^{3/2}} \tag{6.39}$$

$$B^{21}_{\mathrm{reg}}(0,z_1) = \frac{i}{4(1+\frac{z_1^2}{4})^{3/2}} \tag{6.40}$$

$$B^{22}_{\mathrm{reg}}(0,z_2) = \frac{1}{z_2^2}\left(\frac{1}{\sqrt{1-\frac{z_2^2}{4}}} - 1\right) \tag{6.41}$$

So, we have the following expansions:

$$B^{11}_{\mathrm{reg}}(0,z_1) = \sum_{k=0}^{\infty} z_1^{2k} \cdot \frac{(-1)^{k+1}(2k+1)!!}{(k+1)!2^{3(k+1)}} \tag{6.42}$$

$$B^{12}_{\mathrm{reg}}(0,z_2) = \sum_{k=0}^{\infty} z_2^{2k} \cdot \frac{i(2k+1)!!}{(k)!2^{3k+2}} \tag{6.43}$$

$$B^{21}_{\mathrm{reg}}(0,z_1) = \sum_{k=0}^{\infty} z_2^{2k} \cdot \frac{i(-1)^k(2k+1)!!}{(k)!2^{3k+2}} \tag{6.44}$$

$$B^{22}_{\mathrm{reg}}(0,z_2) = \sum_{k=0}^{\infty} z_2^{2k} \cdot \frac{(2k+1)!!}{(k+1)!2^{3(k+1)}}. \tag{6.45}$$

The formulas for $f_{ij}(w)$ are then

$$f_{11}(w) = 1 + \sum_{k=1}^{\infty} w^k \cdot \frac{(-1)^{k+1}(2k-1)!!(2k-3)!!}{k!2^{3k}} \tag{6.46}$$

$$f_{12}(w) = \sum_{k=1}^{\infty} w^k \cdot \frac{-i(2k-1)!!(2k-3)!!}{(k-1)!2^{3k-1}} \tag{6.47}$$

$$f_{21}(w) = \sum_{k=1}^{\infty} w^k \cdot \frac{(-1)^k i(2k-1)!!(2k-3)!!}{(k-1)!2^{3k-1}} \tag{6.48}$$

$$f_{22}(w) = 1 + \sum_{k=1}^{\infty} w^k \cdot \frac{-(2k-1)!!(2k-3)!!}{k!2^{3k}} \tag{6.49}$$

This coincides with the formula for the $\sum_{k=0}^{\infty} R_k 2^k(-w)^k$ at the point $(0,0)$.

6.3.4 Comparison of the coefficient of (g,n,m)-vertex

In this section we consider a vertex of genus g with n attached half-edges or ordinary leaves, and m dilaton leaves, with an associated intersection number $\langle \prod_{i=1}^{n} \tau_{d_i} \prod_{i=1}^{m} \tau_{a_i+1} \rangle_{g,n+m}$. There are vertices of type 1 and type 2, depending on the canonical coordinate that we associate to the vertex. We compare the coefficients that we associate to these vertices in the Givental case, using the data from Section 6.1 in Formula (5.29), and in the case of local CEO-recursion, using the data from Sections 6.3.1-6.3.3 in Formula (5.57).

The coefficients that we have in Formula (5.57) (at the vertex of the type 1 and 2 resp.) are:

$$(-2)^{2-2g-n-m} \qquad \text{and} \qquad (2i)^{2-2g-n-m}. \tag{6.50}$$

Let us compute how these coefficients change if we take into account all the differences between R-matrix and the dilaton leaves. For convenience, from now on we rescale the differential forms on the leaves, $W^i_a \to 2^{-a}W^i_a$, $i = 1, 2$, $a = 0, 1, 2, \dots$. Observe that this rescaling, the extra factor of 2^k in R_k and, in addition, an extra factor of $\sqrt{2}$ that we have to put by hand on each ordinary leave give us together the extra factors of

$$2^{\sum_{i=1}^n d_i} 2^{n/2} \qquad \text{and} \qquad 2^{\sum_{i=1}^n d_i} 2^{n/2}. \tag{6.51}$$

Then the quotient of the contributions of the dilaton leaves gives us extra factors of

$$2^{\sum_{i=1}^m (a_i+1)} 2^{m/2} (-1)^m \qquad \text{and} \qquad 2^{\sum_{i=1}^m (a_i+1)} 2^{m/2} (-1)^m. \tag{6.52}$$

Let us assign by hand an extra factor of $(-1)^{2g-2+n}$ to each (g, n, m)-vertex. This way we get the following coefficients:

$$2^{g-1+n/2+m/2} \qquad \text{and} \qquad 2^{g-1+n/2+m/2} i^{2g-2+n+m}. \tag{6.53}$$

These coefficients are precisely

$$\left((\Delta_1)^{1/2}\right)^{2g-2+n+m} \qquad \text{and} \qquad \left((\Delta_2)^{1/2}\right)^{2g-2+n+m}. \tag{6.54}$$

Therefore, the coefficient of $\prod_{k=1}^{\hat{n}} 2^{-d_k} W^{i_k}_{d_k}$ in a graph of global genus $\hat{g}$ with $\hat{n}$ marked leaves in the Formula (5.57) for the set up of Norbury-Scott, multiplied by

$$2^{\hat{n}/2}(-1)^{2\hat{g}-2+\hat{n}} = (-\sqrt{2})^{\hat{n}}, \tag{6.55}$$

is equal to the coefficient of $\prod_{k=1}^{\hat{n}} t^{d_k, i_k}$ in the same graph in Formula (5.29). This extra factor will be taken into account via a rescaling of the variables by $-\sqrt{2}$.

6.3.5 The Ψ-action

Let us apply the Ψ-operator to the leaves. After comparing the R-action with the graph expansion given by formulas (5.29) and (5.57), and taking into account the extra factor of $-\sqrt{2}$, we have the following identification of the marking on the leaves:

$$\sum_{a-b=c} (t^{a,1}, t^{a,2}) S_b = \left(2^{-c}W^1_c, 2^{-c}W^2_c\right) \Psi^{-1}/(-\sqrt{2}). \tag{6.56}$$

Here

$$W^1_0 = \left.\frac{\mathrm{d}z}{(1-z)^2}\right|_{z=z(z_1)} + \left.\frac{\mathrm{d}z}{(1-z)^2}\right|_{z=z(z_2)} \tag{6.57}$$

$$W^2_0 = \left.\frac{i\,\mathrm{d}z}{(1+z)^2}\right|_{z=z(z_1)} + \left.\frac{i\,\mathrm{d}z}{(1+z)^2}\right|_{z=z(z_2)}, \tag{6.58}$$

and

$$2^{-c}W^i_c = \mathrm{d}\left(\left(-\frac{d}{dx}\right)^c \int W^i_0\right), \tag{6.59}$$

so we can work in the global coordinate z rather than in the local coordinates z_1, z_2.

Since

$$\Psi^{-1}/(-\sqrt{2}) = \begin{pmatrix} \frac{-1}{2} & \frac{-1}{2} \\ \frac{-i}{2} & \frac{i}{2} \end{pmatrix}, \tag{6.60}$$

we have:

$$\sum_{a-b=c} (t^{a,1}, t^{a,2}) S_b = \left(U^1_c, U^2_c\right), \tag{6.61}$$

where

$$U^1_0 = \frac{1}{2}\left(-\frac{dz}{(1-z)^2} + \frac{dz}{(1+z)^2}\right) \tag{6.62}$$

$$U^2_0 = \frac{-1}{2}\left(\frac{dz}{(1-z)^2} + \frac{dz}{(1+z)^2}\right) \tag{6.63}$$

and

$$U^i_c = \mathrm{d}\left(\left(-\frac{\partial}{\partial x}\right)^c \int U^i_0\right), \qquad i = 1, 2; c = 0, 1, 2, \ldots. \tag{6.64}$$

6.3.6 The S-action

The S-action is just a linear change of variables prescribed by Equation (6.61). This means that we replace each U^i_c with a linear combination of times $t^{a,j}$, $a \geq c$, where the coefficient of $t^{a,2}$ (this is the series of variables corresponding to the stationary sector) is equal to

$$\begin{cases} 0, & \text{if } a - c \text{ is even;} \\ \frac{1}{(k+1)\cdot(k!)^2}, & \text{if } a - c = 2k+1. \end{cases} \tag{6.65}$$

for $i = 1$, and

$$\begin{cases} \frac{1}{(k!)^2}, & \text{if } a - c = 2k; \\ 0, & \text{if } a - c \text{ is odd.} \end{cases} \tag{6.66}$$

for $i = 2$.

Norbury and Scott make the same kind of a linear change of variables, with the coefficient of $t^{a,2}$ in U^j_c, $j = 1, 2$, given by

$$-\operatorname*{res}_{x=\infty} \frac{1}{(a+1)!} x^{a+1} U^j_c = \frac{1}{(a+1)!} \operatorname*{res}_{z=0} \left(z + \frac{1}{z}\right)^{a+1} U^j_c. \tag{6.67}$$

In order to complete the proof of Theorem 6.2, we have to check two things: (1) that the Norbury-Scott formula for the contribution depends only on the difference $a - c$; (2) that for $c = 0$ Equation (6.67) gives exactly the same coefficients as we have in Equations (6.65) and (6.66).

The first thing follows directly from the formula. Indeed,

$$\begin{aligned} -\oint \frac{x^{a+1}}{(a+1)!} \mathrm{d}\left(\left(-\frac{\partial}{\partial x}\right)^c \int U^j_0\right) &= \oint \frac{x^a}{(a)!} \left(\left(-\frac{\partial}{\partial x}\right)^c \int U^j_0\right) \mathrm{d}x \\ &= \oint \frac{x^{a-c}}{(a-c)!} \left(\int U^j_0\right) \mathrm{d}x \\ &= -\oint \frac{x^{a+1-c}}{(a+1-c)!} \mathrm{d}\left(\int U^j_0\right). \end{aligned} \tag{6.68}$$

In particular, we see that the coefficient is equal to 0 if $a < c$.

Then, a direct computation shows that

$$\begin{aligned}
&\frac{1}{(a+1)!}\operatorname*{res}_{z=0}\left(z+\frac{1}{z}\right)^{a+1} U_0^1 \\
&= \frac{1}{(a+1)!}\operatorname*{res}_{z=0}\left(z+\frac{1}{z}\right)^{a+1} \frac{1}{2}\left(-\frac{\mathrm{d}z}{(1-z)^2}+\frac{\mathrm{d}z}{(1+z)^2}\right) \\
&= \frac{1}{(a+1)!}\operatorname*{res}_{z=0}\left(z+\frac{1}{z}\right)^{a+1} \frac{-2z\,\mathrm{d}z}{(1-z^2)^2} \\
&= \begin{cases} 0, & \text{if } a \text{ is even;} \\ \frac{-2}{(2k+2)!}\left(\binom{2k+2}{0}(k+1)+\binom{2k+2}{1}k+\cdots+\binom{2k+2}{k}1\right) & \text{if } a=2k+1. \end{cases} \\
&= \begin{cases} 0, & \text{if } a \text{ is even;} \\ \frac{-1}{(k+1)(k!)^2} & \text{if } a=2k+1. \end{cases}
\end{aligned} \tag{6.69}$$

and

$$\begin{aligned}
&\frac{1}{(a+1)!}\operatorname*{res}_{z=0}\left(z+\frac{1}{z}\right)^{a+1} U_0^2 \\
&= \frac{1}{(a+1)!}\operatorname*{res}_{z=0}\left(z+\frac{1}{z}\right)^{a+1} \frac{-1}{2}\left(\frac{\mathrm{d}z}{(1-z)^2}+\frac{\mathrm{d}z}{(1+z)^2}\right) \\
&= \frac{-1}{(a+1)!}\operatorname*{res}_{z=0}\left(z+\frac{1}{z}\right)^{a+2} \frac{z\,\mathrm{d}z}{(1-z^2)^2} \\
&= \begin{cases} \frac{-1}{(2k+1)!}\left(\binom{2k+2}{0}\cdot(k+1)+\binom{2k+2}{1}\cdot k+\cdots\binom{2k+2}{k}\cdot 1\right) & \text{if } a=2k; \\ 0, & \text{if } a \text{ is odd} \end{cases} \\
&= \begin{cases} \frac{-1}{(k!)^2} & \text{if } a=2k; \\ 0, & \text{if } a \text{ is odd.} \end{cases}
\end{aligned} \tag{6.70}$$

We see that there is an extra factor of (-1) in all coefficients. This means that the (g,n)-correlation functions of Norbury-Scott differ from the stationary Gromov-Witten invariants of $\mathbb{P}^1$ by a factor of $(-1)^n$. But this factor is exactly the difference we must have because Norbury and Scott are using a different convention of the sign in the CEO-recursion, cf. Remark 5.12. This completes the proof of Theorem 6.2.

–7– Polynomiality of Hurwitz numbers, Bouchard-Mariño conjecture, and a new proof of the ELSV formula

7.1 Introduction

Hurwitz numbers play an important role in the interaction of combinatorics, representation theory of symmetric groups, integrable systems, tropical geometry, matrix models, and intersection theory of the moduli spaces of curves. In this chapter we revisit two of the most remarkable properties of Hurwitz numbers.

The ELSV formula [31] gives an expression for connected Hurwitz numbers in terms of intersection numbers on the moduli space of curves:

$$h^{\circ}_{g,\mu} = m! \prod_{i=1}^{\ell(\mu)} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{\mathcal{M}}_{g,\ell(\mu)}} \frac{\Lambda_g^{\vee}(1)}{\prod_{i=1}^{\ell(\mu)}(1-\mu_i\psi_i)}. \tag{7.1}$$

The Bouchard-Mariño conjecture [10] (proved by now in several different papers) is also a relation of Hurwitz numbers to matrix models. Consider the spectral curve

$$x = ye^{-y} \tag{7.2}$$

equipped with the two-point function

$$\frac{\mathrm{d}y\mathrm{d}y'}{(y-y')^2}. \tag{7.3}$$

Then the n-point functions $\omega_{g,n}$ produced from this data via CEO-recursion [36] are equal to

$$\sum_{\mu_1,\dots,\mu_n} \frac{h^{\circ}_{g;\mu_1,\dots,\mu_n}}{m!}\, \mu_1\dots\mu_n\, x_1^{\mu_1-1}\dots x_n^{\mu_n-1}\mathrm{d}x_1\dots\mathrm{d}x_n, \tag{7.4}$$

These two statements are known to be equivalent [33]. In this chapter we revisit this equivalence and present this argument in a new way (see also Chapter 8 for a generalization).

Let us describe the existing proofs of both statements. All proofs of the ELSV formula [31, 52, 86, 74] are based, either directly or, as the original one, indirectly, on the computation of the Euler class of the fixed locus of the $\mathbb{C}^*$-action on the space of (relative stable) maps to $\mathbb{P}^1$. All mathematically rigorous proofs of the Bouchard-Mariño conjecture [35, 82] use the ELSV formula and the Laplace transform of the so-called cut-and-join equation for Hurwitz numbers, the basic equation that also allows to reconstruct them recursively. There is one more proof of the Bouchard-Mariño conjecture in [8] that goes through the construction of a matrix model for Hurwitz numbers and a direct derivation of the CEO-recursion, but it will require plenty of subtle analytic work to make it really mathematically rigorous. Of course, since the ELSV formula is proved independently, the fact [33, 116] that the two statements are equivalent implies the Bouchard-Mariño conjecture as well.

There is still a number of interesting questions on both statements. The first question is whether it is possible to prove the Bouchard-Mariño conjecture independently of the ELSV formula. The second question is whether there exists any way to derive the ELSV formula combinatorially, rather than via the computation of the Euler class mentioned above. For example, all Hurwitz numbers can be computed combinatorially, either using the character formula, or, equivalently, using the semi-infinite wedge formalism, or recursively via the cut-and-join equation. On the other hand, the intersection number in the ELSV formula can also be computed combinatorially. Indeed, we can use the Mumford formula [83] for the Chern characters of the Hodge bundle in order to reduce the intersection number in the ELSV formula to intersection numbers of ψ-classes, and any intersection number of ψ-classes can be computed using the Witten-Kontsevich theorem [107, 66]. The third question, posed e.g. in [105, 50], is the following. The structure of the ELSV formula implies some polynomiality property of Hurwitz numbers, that is

$$h^{\circ}_{g;\mu_1,\dots,\mu_n} = m! \left(\prod_{i=1}^{n} \frac{\mu_i^{\mu_i}}{\mu_i!} \right) P_{g,n}(\mu_1,\dots,\mu_n),$$

where $P_{g,n}(\mu_1,\dots,\mu_n)$ are some polynomials in $\mu_1,\dots,\mu_n$. Though this fact is completely combinatorial, the only way to prove it known up to now is to use the ELSV formula. So, the third question we consider here is whether it is possible to prove this polynomiality in some direct way, without any usage of the ELSV formula.

7.1.1 Organisation of the chapter

This chapter provides full answer to all three questions. It is organized in the following way. First, we prove in Section 7.2 the polynomiality of Hurwitz numbers directly from the definition in terms of the semi-infinite wedge formalism. Our argument is a refinement of an argument by Okounkov and Pandharipande in [87]. Then, using the polynomiality property of Hurwitz numbers we are able to derive in Section 7.3 the Bouchard-Mariño conjecture directly from the cut-and-join equation. Then, since we have an equivalence of the Bouchard-Mariño conjecture and the ELSV formula, we immediately derive the ELSV formula in a new way. In Section 7.4 we review the correspondence between the topological recursion and the Givental theory, with a special focus on the 1-dimensional case, and in Section 7.5 we provide a (slightly refined) proof of the equivalence of the ELSV formula and the Bouchard-Mariño conjecture.

7.2 Polynomiality of the Hurwitz numbers

In this section we prove the following theorem:

Theorem 7.1. *The Hurwitz numbers $h^{\circ}_{g;\mu_1,\dots,\mu_n}$ for $(g,n) \notin \{(0,1),(0,2)\}$ can be expressed as follows:*

$$h^{\circ}_{g;\mu_1,\dots,\mu_n} = (2g + |\mu| + n - 2)! \left(\prod_{i=1}^{n} \frac{\mu_i^{\mu_i}}{\mu_i!} \right) P_{g,n}(\mu_1,\dots,\mu_n), \tag{7.5}$$

where $P_{g,n}(\mu_1,\dots,\mu_n)$ is some polynomial in $\mu_1,\dots,\mu_n$.

Basically this theorem gives the form of the ELSV formula without specifying the precise formulas for the coefficients. This property (in a bit stronger form) was conjectured in [48] and then proved in [50], with the help of the ELSV formula. Still, the question whether this property can be derived without using the ELSV formula remained open [105]. This is precisely what we do here: we prove this statement without using the ELSV formula.

7.2.1 Hurwitz numbers in the infinite wedge formalism

By $h_{g,\mu}$ we denote the Hurwitz numbers for possibly disconnected covering surfaces. From Proposition 2.23 we know the character formula for the disconnected Hurwitz numbers $h_{g,\mu}$ implies that (see also [87]):

$$h_{g,\mu} = \left\langle e^{\alpha_1} \mathcal{F}_2^m \prod_{i=1}^{\ell(\mu)} \frac{\alpha_{-\mu_i}}{\mu_i} \right\rangle. \tag{7.6}$$

Here $\ell(\mu)$ denotes the number of parts of μ, and

$$m := 2g + |\mu| + \ell(\mu) - 2. \tag{7.7}$$

We remind the reader of the difference of a factor $|\mathrm{Aut}(\mu)|$ between our disconnected Hurwitz numbers $h_{g,\mu}$ and the ones in [87] (which are denoted by $\mathsf{C}_g(\mu)$ there).

Definition 7.2. Define the genus-generating functions for the disconnected Hurwitz numbers and for the connected ones as well:

$$h_\mu(u) := \sum_{g=0}^{\infty} \frac{u^{2g-2}}{m!} h_{g,\mu} \tag{7.8}$$

$$h^\circ_\mu(u) := \sum_{g=0}^{\infty} \frac{u^{2g-2}}{m!} h^\circ_{g,\mu} \tag{7.9}$$

They are related to each other through the inclusion-exclusion formula.
We have

$$\begin{aligned} h_\mu(u) &= u^{-|\mu|-\ell(\mu)} \left\langle e^{\alpha_1} e^{u\mathcal{F}_2} \prod_{i=1}^{\ell(\mu)} \frac{\alpha_{-\mu_i}}{\mu_i} \right\rangle \\ &= u^{-|\mu|-\ell(\mu)} \left\langle e^{\alpha_1} e^{u\mathcal{F}_2} \left(\prod_{i=1}^{\ell(\mu)} \frac{\alpha_{-\mu_i}}{\mu_i} \right) e^{-u\mathcal{F}_2} e^{-\alpha_1} \right\rangle \\ &= u^{-|\mu|-\ell(\mu)} \left\langle \prod_{i=1}^{\ell(\mu)} \left(e^{\alpha_1} e^{u\mathcal{F}_2} \frac{\alpha_{-\mu_i}}{\mu_i} e^{-u\mathcal{F}_2} e^{-\alpha_1} \right) \right\rangle \end{aligned} \tag{7.10}$$

The second equality holds since $e^{-u\mathcal{F}_2}$ and $e^{-\alpha_1}$ fix the vacuum vector.

7.2.2 $\mathcal{A}$-operators

Now, following [87], we introduce certain operators that we use later on to rewrite the formula for Hurwitz numbers.

Definition 7.3. Define

$$\mathcal{A}(a,b) := \left(\frac{\zeta(b)}{b} \right)^a \sum_{k \in \mathbb{Z}} \frac{\zeta(b)^k}{(a+1)_k} \mathcal{E}_k(b), \tag{7.11}$$

where a and b are parameters and we use the standard notation:

$$(a+1)_k = \frac{(a+k)!}{a!} = \begin{cases} (a+1)(a+2)\cdots(a+k)\,, & k \geq 0\,, \\ (a(a-1)\cdots(a+k+1))^{-1}\,, & k \leq 0\,. \end{cases} \tag{7.12}$$

If $a \neq 0, 1, 2, \ldots$, the sum in (7.11) is infinite in both directions. If a is a nonnegative integer, the summands with $k \leq -a-1$ in (7.11) vanish.

Note that Proposition 3 of [87] implies that the correlator

$$\langle \mathcal{A}(z_1, uz_1) \ldots \mathcal{A}(z_n, uz_n) \rangle \tag{7.13}$$

is well-defined for all $(z_1, \ldots, z_n) \in \Omega \subset \mathbb{C}^n$ and sufficiently small u, where

$$\Omega = \left((z_1, \ldots, z_n) \,\middle|\, |z_k| > \sum_{i=1}^{k-1} |z_i|,\ k = 1, \ldots, n \right). \tag{7.14}$$

Definition 7.4. Define the *connected correlator* of $\mathcal{A}$-operators

$$\langle \mathcal{A}(z_1, uz_1) \ldots \mathcal{A}(z_n, uz_n) \rangle^\circ \tag{7.15}$$

through the disconnected ones via the inclusion-exclusion formula.

Proposition 7.5.

$$h^\circ_{g;\mu_1\ldots\mu_n} = m! \prod_{i=1}^{n} \left(\frac{\mu_i^{\mu_i}}{\mu_i!} \right) [u^{2g-2+n}] \frac{\langle \mathcal{A}(\mu_1, u\mu_1) \cdots \mathcal{A}(\mu_n, u\mu_n) \rangle^\circ}{\mu_1 \cdots \mu_n} \tag{7.16}$$

Proof. The main part of the proof follows [87]. Note that

$$e^{u\mathcal{F}_2}\, \alpha_{-m}\, e^{-u\mathcal{F}_2} = \mathcal{E}_{-m}(um) \tag{7.17}$$

which is easy to see since $\mathcal{F}_2$ acts diagonally. From the commutation relations for $\mathcal{E}_i$ we see that

$$e^{\alpha_1}\, \mathcal{E}_{-m}(s)\, e^{-\alpha_1} = \frac{\zeta(s)^m}{m!} \sum_{k \in \mathbb{Z}} \frac{\zeta(s)^k}{(m+1)_k}\, \mathcal{E}_k(s) \tag{7.18}$$

The previous two formulas imply the following, for $m \in \{1, 2, 3, \ldots\}$ (Lemma 2 of [87]):

$$e^{\alpha_1}\, e^{u\mathcal{F}_2}\, \alpha_{-m}\, e^{-u\mathcal{F}_2}\, e^{-\alpha_1} = \frac{u^m\, m^m}{m!}\, \mathcal{A}(m, um) \tag{7.19}$$

Now we can rewrite formula (7.10) as

$$h_{\mu_1\ldots\mu_n}(u) = u^{-n} \prod_{i=1}^{n} \left(\frac{\mu_i^{\mu_i}}{\mu_i!} \right) \frac{\langle \mathcal{A}(\mu_1, u\mu_1) \cdots \mathcal{A}(\mu_n, u\mu_n) \rangle}{\mu_1 \cdots \mu_n} \tag{7.20}$$

Recall that the connected Hurwitz numbers can be expressed though the disconnected ones with the help of the inclusion-exclusion formula. Since the relation between connected and disconnected Hurwitz numbers is the same as the one between connected and disconnected correlators, we have:

$$h^\circ_{\mu_1\ldots\mu_n}(u) = u^{-n} \prod_{i=1}^{n} \left(\frac{\mu_i^{\mu_i}}{\mu_i!} \right) \frac{\langle \mathcal{A}(\mu_1, u\mu_1) \cdots \mathcal{A}(\mu_n, u\mu_n) \rangle^\circ}{\mu_1 \cdots \mu_n} \tag{7.21}$$

Comparing the coefficients in front of the same powers of u on the right hand side and on the left hand side we directly obtain the statement of the proposition. ☐

Now we see that in order to prove Theorem 7.1 we only have to show that expressions

$$[u^{2g-2+n}]\frac{\langle \mathcal{A}(\mu_1, u\mu_1)\cdots\mathcal{A}(\mu_n, u\mu_n)\rangle^\circ}{\mu_1\cdots\mu_n} \tag{7.22}$$

are polynomial in $\mu_1,\dots,\mu_n$.

7.2.3 Further properties of $\mathcal{A}$-operators

In this subsection we modify an expression for the connected correlators of $\mathcal{A}$-operators in order to exclude possible so-called *unstable terms.*

Definition 7.6. Let $\mathcal{A}_k$ be the coefficients of the expansion of the operator $\mathcal{A}(z, uz)$ in powers of z:

$$\mathcal{A}(z, uz) = \sum_{k\in\mathbb{Z}} \mathcal{A}_k\, z^k\,. \tag{7.23}$$

We will use the following theorem, due to Okounkov and Pandharipande:

Theorem 7.7 (Okounkov-Pandharipande, [87])**.**

$$[\mathcal{A}_k, \mathcal{A}_l] = (-1)^l \delta_{k+l-1}\,. \tag{7.24}$$

Definition 7.8. Define

$$\mathcal{A}_+(z, uz) := \sum_{k=1}^{\infty} \mathcal{A}_k\, z^k\,. \tag{7.25}$$

Notation 7.9. For any operator $\mathcal{P}(u)$ define

$$\begin{aligned} \langle\mathcal{P}(u)\rangle_k &:= [u^k]\,\langle\mathcal{P}(u)\rangle && \text{(the coefficient of } u^k \text{ in } \langle\mathcal{P}(u)\rangle) \\ \langle\mathcal{P}(u)\rangle_k^\circ &:= [u^k]\,\langle\mathcal{P}(u)\rangle^\circ && \text{(the coefficient of } u^k \text{ in } \langle\mathcal{P}(u)\rangle^\circ) \end{aligned} \tag{7.26}$$

Definition 7.10. We denote by $\mathcal{Y}_{n,k}$ be the set of $\{1,\dots,n\}$ Young tableaux (i. e. Young diagrams of size n with each box labelled by a number from 1 to n such that no two boxes are labelled by the same number) with certain conditions and additional row labels.

Namely, let y be such a tableau. Let $c_{i,j}(y)$ be the number in the i-th row and j-th column. Let $h(y)$ be the number of rows, an let $l_i(y)$ be the length of the i-th row. Now we are ready to describe the conditions.

First, the numbers in the rows should be ascending, i. e. for any i and for any $j_1 < j_2$ we have $c_{i,j_1}(y) < c_{i,j_2}(y)$. Second, the numbers in the first column that correspond to rows of the same length should be ascending, i. e., if $l_{i_1}(y) = l_{i_2}(y)$ and $i_1 < i_2$, then $c_{i_1,1}(y) < c_{i_2,1}(y)$.

By $\lambda_i(y) \in \{-1, 0, 1, \dots\}$ we denote additional labels that are assigned to all rows, and we require that $\sum_{i=1}^{h(y)} \lambda_i(y) = k$.

Note that there is a one-to-one correspondence between the elements of $\mathcal{Y}_{n,k}$ and the terms in the expression for a disconnected correlator through the connected ones (the "inverse" inclusion-exclusion formula). Rows in y correspond to individual connected correlators in the product, while labels λ correspond to the Euler characteristics of these connected correlators. This can be expressed through the following formula:

$$\begin{aligned} &\langle\mathcal{A}(z_1, uz_1)\dots\mathcal{A}(z_n, uz_n)\rangle_k \\ &= \sum_{y\in\mathcal{Y}_{n,k}} \prod_{i=1}^{h(y)} \Big\langle \mathcal{A}(z_{c_{i,1}(y)}, uz_{c_{i,1}(y)})\dots\mathcal{A}(z_{c_{i,l_i(y)}(y)}, uz_{c_{i,l_i(y)}(y)})\Big\rangle^\circ_{\lambda_i(y)} \end{aligned} \tag{7.27}$$

The terms in this sum that contain either $\langle\mathcal{A}(z_i, uz_i)\rangle^\circ_{-1}$ or $\langle\mathcal{A}(z_i, uz_i)\mathcal{A}(z_j, uz_j)\rangle^\circ_0$ are called *unstable terms.* If we exclude all unstable terms, we obtain the following expression.

Proposition 7.11. *We have:*

$$
\begin{aligned}
&\langle \mathcal{A}_+(z_1, uz_1) \dots \mathcal{A}_+(z_n, uz_n)\rangle_k \\
&= \sum_{y \in \widetilde{\mathcal{Y}}_{n,k}} \prod_{i=1}^{h(y)} \left\langle \mathcal{A}(z_{c_{i,1}(y)}, uz_{c_{i,1}(y)}) \dots \mathcal{A}(z_{c_{i,l_i(y)}(y)}, uz_{c_{i,l_i(y)}(y)}) \right\rangle^\circ_{\lambda_i(y)}.
\end{aligned} \tag{7.28}
$$

Here

$$
\widetilde{\mathcal{Y}}_{n,k} = \left\{ y \in \mathcal{Y}_{n,k} \middle| \, l_i(y) = 1 \Rightarrow \lambda_i(y) \neq -1, \; l_i(y) = 2 \Rightarrow \lambda_i(y) \neq 0 \right\}. \tag{7.29}
$$

In other words, $\langle \mathcal{A}_+(z_1, uz_1) \dots \mathcal{A}_+(z_n, uz_n)\rangle_k$ *is equal to* $\langle \mathcal{A}(z_1, uz_1) \dots \mathcal{A}(z_n, uz_n)\rangle_k$ *with all the unstable terms dropped.*

Proof. Let us first compute the unstable factors, i. e. the genus-zero one- and two-point connected correlators.

Note that

$$
\langle \mathcal{A}(z, uz)\rangle = \frac{1}{uz} + \frac{z(z-1)}{24} u + O(u^2). \tag{7.30}
$$

This directly implies the following formula for the genus-zero one-point correlator:

$$
\langle \mathcal{A}(z, uz)\rangle^\circ_{-1} = \frac{1}{z}. \tag{7.31}
$$

The definition of the operator $\mathcal{A}$ implies that

$$
\langle 0 | \mathcal{A}(z, uz) = \frac{1}{uz} \langle 0 | + \langle 0 | \mathcal{A}_+(z, uz) \tag{7.32}
$$

The definition of the two-point connected correlators together with formulas (7.30), (7.32) and (7.24) implies the following formula for the genus-zero two-point connected correlator:

$$
\begin{aligned}
\langle \mathcal{A}(z_1, uz_1)\mathcal{A}(z_2, uz_2)\rangle^\circ_0 &= \langle \mathcal{A}(z_1, uz_1)\mathcal{A}(z_2, uz_2)\rangle_0 - \langle \mathcal{A}(z_1, uz_1)\rangle_{-1} \langle \mathcal{A}(z_2, uz_2)\rangle_1 \\
&\quad - \langle \mathcal{A}(z_1, uz_1)\rangle_1 \langle \mathcal{A}(z_2, uz_2)\rangle_{-1} \\
&= \langle \mathcal{A}_+(z_1, uz_1)\mathcal{A}(z_2, uz_2)\rangle_0 - \langle \mathcal{A}(z_1, uz_1)\rangle_1 \langle \mathcal{A}(z_2, uz_2)\rangle_{-1} \\
&= \langle \mathcal{A}(z_2, uz_2)\mathcal{A}_+(z_1, uz_1)\rangle_0 + z_1 \sum_{k=0}^{\infty} (-1)^k \left(\frac{z_1}{z_2}\right)^k \\
&\quad - \langle \mathcal{A}(z_2, uz_2)\rangle_{-1} \langle \mathcal{A}(z_1, uz_1)\rangle_1 \\
&= \langle \mathcal{A}_+(z_2, uz_2)\mathcal{A}_+(z_1, uz_1)\rangle_0 + z_1 \sum_{k=0}^{\infty} (-1)^k \left(\frac{z_1}{z_2}\right)^k \\
&= z_1 \sum_{k=0}^{\infty} (-1)^k \left(\frac{z_1}{z_2}\right)^k.
\end{aligned} \tag{7.33}
$$

Now we prove the statement of the proposition by induction over the number of operators n in the correlator on the left hand side. From the definition of the operator $\mathcal{A}$ it is easy to see that the statement holds for $n = 1$. Suppose that it holds for the correlator of any number of operators less than n. We will prove that it holds for n operators.

Taking into account (7.32), (7.24), (7.31) and (7.33) we see that

$$\begin{aligned}
&\langle \mathcal{A}(z_1,uz_1)\dots\mathcal{A}(z_n,uz_n)\rangle_k \qquad (7.34)\\
&= \frac{1}{z_1}\langle \mathcal{A}(z_2,uz_2)\dots\mathcal{A}(z_n,uz_n)\rangle_{k+1}\\
&\quad + \langle \mathcal{A}_+(z_1,uz_1)\mathcal{A}(z_2,uz_2)\dots\mathcal{A}(z_n,uz_n)\rangle_k\\
&= \langle \mathcal{A}(z_1,uz_1)\rangle^\circ_{-1}\langle \mathcal{A}(z_2,uz_2)\dots\mathcal{A}(z_n,uz_n)\rangle_{k+1}\\
&\quad + z_1\sum_{k=0}^{\infty}(-1)^k\left(\frac{z_1}{z_2}\right)^k\langle \mathcal{A}(z_3,uz_3)\dots\mathcal{A}(z_n,uz_n)\rangle_k\\
&\quad + \langle \mathcal{A}(z_2,uz_2)\mathcal{A}_+(z_1,uz_1)\mathcal{A}(z_3,uz_3)\dots\mathcal{A}(z_n,uz_n)\rangle_k\\
&= \langle \mathcal{A}(z_1,uz_1)\rangle^\circ_{-1}\langle \mathcal{A}(z_2,uz_2)\dots\mathcal{A}(z_n,uz_n)\rangle_{k+1}\\
&\quad + \langle \mathcal{A}(z_1,uz_1)\mathcal{A}(z_2,uz_2)\rangle^\circ_0\langle \mathcal{A}(z_3,uz_3)\dots\mathcal{A}(z_n,uz_n)\rangle_k\\
&\quad + \langle \mathcal{A}(z_2,uz_2)\rangle^\circ_{-1}\langle \mathcal{A}_+(z_1,uz_1)\mathcal{A}(z_3,uz_3)\dots\mathcal{A}(z_n,uz_n)\rangle_{k+1}\\
&\quad + \langle \mathcal{A}_+(z_1,uz_1)\mathcal{A}_+(z_2,uz_2)\mathcal{A}(z_3,uz_3)\dots\mathcal{A}(z_n,uz_n)\rangle_k
\end{aligned}$$

We continue with the same computation (replacing the leftmost operator $\mathcal{A}$ with $\mathcal{A}_+$ and commuting it to the right, collecting the emerging coefficients in the unstable correlators), finally arriving at the following expression.

$$\begin{aligned}
&\langle \mathcal{A}(z_1,uz_1)\dots\mathcal{A}(z_n,uz_n)\rangle_k = \langle \mathcal{A}_+(z_1,uz_1)\dots\mathcal{A}_+(z_n,uz_n)\rangle_k \qquad (7.35)\\
&+\sum_{p=3}^{n-1}\sum_{q=0}^{[\frac{n-p}{2}]}\sum_{y\in\widehat{\mathcal{Y}}^{p,q}_{n,k}}\big\langle \mathcal{A}_+(z_{c_{1,1}(y)},uz_{c_{1,1}(y)})\dots\mathcal{A}_+(z_{c_{1,p}(y)},uz_{c_{1,p}(y)})\big\rangle_{k+h(y)-q-1}\\
&\quad\times\prod_{i=2}^{q+1}\big\langle \mathcal{A}(z_{c_{i,1}(y)},uz_{c_{i,1}(y)})\mathcal{A}(z_{c_{i,2}(y)},uz_{c_{i,2}(y)})\big\rangle^\circ_0\prod_{i=q+2}^{h(y)}\big\langle \mathcal{A}(z_{c_{i,1}(y)},uz_{c_{i,1}(y)})\big\rangle^\circ_{-1}\\
&+\sum_{q=0}^{[\frac{n-2}{2}]}\sum_{y\in\widehat{\mathcal{Y}}^{2,q}_{n,k}}\Big\langle \mathcal{A}_+(z_{c_{s(y),1}(y)},uz_{c_{s(y),1}(y)})\mathcal{A}_+(z_{c_{s(y),2}(y)},uz_{c_{s(y),2}(y)})\Big\rangle_{k+h(y)-q-1}\\
&\quad\times\prod_{\substack{i=1\\ i\neq s(y)}}^{q+1}\big\langle \mathcal{A}(z_{c_{i,1}(y)},uz_{c_{i,1}(y)})\mathcal{A}(z_{c_{i,2}(y)},uz_{c_{i,2}(y)})\big\rangle^\circ_0\prod_{i=q+2}^{h(y)}\big\langle \mathcal{A}(z_{c_{i,1}(y)},uz_{c_{i,1}(y)})\big\rangle^\circ_{-1}\\
&+\sum_{q=0}^{[\frac{n-1}{2}]}\sum_{y\in\widehat{\mathcal{Y}}^{1,q}_{n,k}}\Big\langle \mathcal{A}_+(z_{c_{s(y),1}(y)},uz_{c(y)_{s(y),1}(y)})\Big\rangle_{k+h(y)-q-1}\\
&\quad\times\prod_{i=1}^{q}\big\langle \mathcal{A}(z_{c_{i,1}(y)},uz_{c_{i,1}(y)})\mathcal{A}(z_{c_{i,2}(y)},uz_{c_{i,2}(y)})\big\rangle^\circ_0\prod_{\substack{i=q+1\\ i\neq s(y)}}^{h(y)}\big\langle \mathcal{A}(z_{c_{i,1}(y)},uz_{c_{i,1}(y)})\big\rangle^\circ_{-1}
\end{aligned}$$

Here $\widehat{\mathcal{Y}}^{p,q}_{n,k}$ contains all elements y of $\mathcal{Y}_{n,k}$ such that there is precisely one row of length p labelled by $k+h(y)-q-1$, q rows of length 2 labelled by 0, and all other rows are of length 1 and labelled by -1. $s(y)$ stands for the position of the row with p elements labelled by $k+h(y)-q-1$. If $p=2$ and $k+h(y)-q-1=0$ or $p=1$ and $k+h(y)-q-1=-1$ one cannot determine $s(y)$ in this way, but this is not a problem since, due to the fact that $\langle \mathcal{A}_+(z,uz)\rangle_{-1}=0$ and

$\langle \mathcal{A}_+(z_1,uz_1)\mathcal{A}_+(z_2,uz_2)\rangle_0 = 0$, the corresponding term vanishes in any case. Also note that, obviously, for $p \geq 3$ we have $s(y) = 1$.

Note that the right hand side of formula (7.35) is equal to the correlator

$$\langle \mathcal{A}_+(z_1,uz_1)\dots\mathcal{A}_+(z_n,uz_n)\rangle_k \tag{7.36}$$

plus all possible unstable terms entering exactly once, since, by the induction hypothesis, the correlators of less than n operators $\mathcal{A}_+$ are equal to sums of all possible stable terms. This means that upon moving these terms to the left hand side and subtracting them from $\langle \mathcal{A}(z_1,uz_1)\dots\mathcal{A}(z_n,uz_n)\rangle_k$ we get precisely all possible stable terms. This proves the proposition. □

7.2.4 Polynomiality

In this subsection we establish polynomiality of some correlators, and this allows us to complete the proof of Theorem 7.1.

Proposition 7.12. *The series*

$$\frac{\langle \mathcal{A}_+(z_1,uz_1)\dots\mathcal{A}_+(z_n,uz_n)\rangle_k}{z_1\cdots z_n} \tag{7.37}$$

for $(n,k) \notin \{(1,-1),(2,0)\}$ *is a symmetric polynomial in* $z_1,\dots,z_n$.

Proof. From the definition of $\mathcal{A}_+$ it is easy to see that for every i the power of z_i in the series $\langle \mathcal{A}_+(z_1,uz_1)\dots\mathcal{A}_+(z_n,uz_n)\rangle_k$ is bounded from below by 1. From (7.24) it is clear that this series is symmetric in $z_1,\dots,z_n$. Let us prove that, for fixed k, the power of z_n in this series is bounded from above, following the proof of Proposition 9 of [87].

Note that

$$\langle \mathcal{E}_{k_1}(uz_1)\dots\mathcal{E}_{k_n}(uz_n)\rangle = \left\langle \frac{\mathcal{E}_{k_1}(uz_1)}{u^{k_1}}\dots\frac{\mathcal{E}_{k_n}(uz_n)}{u^{k_n}}\right\rangle, \tag{7.38}$$

holds since the correlator vanishes unless $\sum k_i = 0$.

Let us apply this transformation to the correlator of $\mathcal{A}$ operators:

$$\langle \mathcal{A}(z_1,uz_1)\dots\mathcal{A}(z_n,uz_n)\rangle = \left\langle \widetilde{\mathcal{A}}(z_1,uz_1)\dots\widetilde{\mathcal{A}}(z_n,uz_n)\right\rangle \tag{7.39}$$

Here $\widetilde{\mathcal{A}}$ stands for the operator $\mathcal{A}$ where the substitution $\mathcal{E}_k \mapsto u^{-k}\mathcal{E}_k$ was made. Note that each term in each $\widetilde{\mathcal{A}}$ is then regular and nonvanishing at $u = 0$, except for the term $\dfrac{1}{\zeta(uz)}$ coming from $\mathcal{E}_0$, which has a simple pole. Let us write the following:

$$\begin{aligned} &\widetilde{\mathcal{A}}(z_n,uz_n)\left|0\right\rangle \\ &= \left(\frac{\zeta(uz_n)}{uz_n}\right)^{z_n} \sum_{k\in\mathbb{Z}} \frac{\zeta(uz_n)^k}{(z_n+1)_k}\,\frac{\mathcal{E}_k(uz_n)}{u^k}\left|0\right\rangle \\ &= \left(\frac{\zeta(uz_n)}{uz_n}\right)^{z_n} \sum_{k=0}^{\infty} \frac{u^k}{\zeta(uz_n)^k} z_n\dots(z_n-k+1)\,\mathcal{E}_{-k}(uz_n)\left|0\right\rangle \\ &= \left(\frac{\zeta(uz_n)}{uz_n}\right)^{z_n} \sum_{k=0}^{\infty} \left(\frac{uz_n}{\zeta(uz_n)}\right)^k \left(1-\frac{1}{z_n}\right)\dots\left(1-\frac{k-1}{z_n}\right)\mathcal{E}_{-k}(uz_n)\left|0\right\rangle \end{aligned} \tag{7.40}$$

It is easy to see that z_n and u enter this expression in such a way that for all terms with a fixed power of u the power of z_n is bounded from above. Since in (7.39) this expression is multiplied by operators $\widetilde{\mathcal{A}}(z_i, uz_i)$, $i \in \{1, \dots, n-1\}$, which have at most simple poles in u, the whole correlator (7.39) is bounded from above in powers of z_n, for a fixed power of u.

From the definition of $\mathcal{A}_+$ it is clear that the fact that the power of z_n in

$$\langle \mathcal{A}(z_1, uz_1) \dots \mathcal{A}(z_n, uz_n) \rangle_k$$

is bounded from above for a fixed k immediately implies that the power of z_n in

$$\langle \mathcal{A}_+(z_1, uz_1) \dots \mathcal{A}_+(z_n, uz_n) \rangle_k$$

is bounded from above for a fixed k as well.

The symmetricity of $\langle \mathcal{A}_+(z_1, uz_1) \dots \mathcal{A}_+(z_n, uz_n) \rangle_k$ then implies that for fixed k the power of z_i in this expression is bounded from above for any i, which implies that

$$\frac{\langle \mathcal{A}_+(z_1, uz_1) \cdots \mathcal{A}_+(z_n, uz_n) \rangle_k}{z_1 \cdots z_n} \tag{7.41}$$

is polynomial in $z_1, \dots, z_n$. □

Proposition 7.13. *For* $(n,k) \notin \{(1,-1),(2,0)\}$ *the series*

$$\frac{\langle \mathcal{A}(z_1, uz_1) \cdots \mathcal{A}(z_n, uz_n) \rangle_k^\circ}{\mu_1 \cdots \mu_n} \tag{7.42}$$

is a symmetric polynomial in $z_1, \dots, z_n$.

Proof. Let us prove the statement of this proposition by induction in n, the number of operators in the correlator. It is clear that for $n = 1$ the statement holds. Suppose that it holds for any number of operators less than n. We will prove that it then holds for n operators as well.

Formula (7.28) can be rewritten as

$$\begin{aligned} \frac{\langle \mathcal{A}(z_1, uz_1) \cdots \mathcal{A}(z_n, uz_n) \rangle_k^\circ}{z_1 \cdots z_n} &= \frac{\langle \mathcal{A}_+(z_1, uz_1) \cdots \mathcal{A}_+(z_n, uz_n) \rangle_k}{z_1 \cdots z_n} \\ - \sum_{y \in \widetilde{\mathcal{Y}}'_{n,k}} \prod_{i=1}^{h(y)} & \frac{\left\langle \mathcal{A}(z_{c_{i,1}(y)}, uz_{c_{i,1}(y)}) \cdots \mathcal{A}(z_{c_{i,l_i(y)}(y)}, uz_{c_{i,l_i(y)}(y)}) \right\rangle^\circ_{\lambda_i(y)}}{z_{c_{i,1}(y)} \cdots z_{c_{i,l_i(y)}(y)}} \end{aligned} \tag{7.43}$$

Here, naturally, $\widetilde{\mathcal{Y}}'_{n,k}$ is equal to $\widetilde{\mathcal{Y}}_{n,k}$ with the single-row Young tableau thrown away.

By Proposition 7.12, the first term on the right hand side of (7.43) is polynomial in $z_1, \dots, z_n$. By induction hypothesis, all the terms in the sum on the right hand side of (7.43) are polynomial as well, since they are finite products of connected correlators of the lower number of operators (and, by definition of $\widetilde{\mathcal{Y}}'_{n,k}$, correlators with $(n_i, k_i) \in \{(1,-1),(2,0)\}$ never appear). This implies the statement of the proposition. □

Taking into account formula (7.16), we see that Proposition 7.13 directly implies the statement of Theorem 7.1.

7.3 Proof of the Bouchard-Mariño conjecture

In the present section we give a new proof of the Bouchard-Mariño conjecture using the polynomiality result from the previous section and not using the ELSV formula.

This conjecture was already proved in [8] and [35]. The first of these papers provides a "physical" proof through the study of the corresponding matrix model. Unfortunately, we were not able to attribute precise mathematical meaning to all of the statements of that paper (see Sections 3.3 and 3.4 in Chapter 3 for a discussion). In the second paper the Bouchard-Mariño formula is derived directly from the known cut-and-join recursion relation for Hurwitz numbers, with the help of the ELSV formula.

Here we follow the ideas of the proof of [35] presenting them in a simplified way, with one essential modification: we do not use the ELSV formula in this proof, using instead just the polynomiality property.

7.3.1 Generating function for Hurwitz numbers

Let us introduce the generating function for the connected Hurwitz numbers $h^{\circ}_{g;\mu}$ in the following way:

$$H^{\circ}_{g,n} := \sum_{\mu_1,\dots,\mu_n \in \{1,2,\dots\}} \frac{h^{\circ}_{g;\mu_1,\dots,\mu_n}}{m!} x_1^{\mu_1} \dots {}_n^{\mu_n} \tag{7.44}$$

Theorem 7.1 implies that, for $(g,n) \notin \{(0,1),(0,2)\}$,

$$H^{\circ}_{g,n} = \sum_{\substack{k_1,\dots,k_n \in \\ \{0,1,\dots,K_{g,n}\}}} c_{k_1\dots k_n} \prod_{i=1}^{n} \sum_{\mu_i=1}^{\infty} \frac{\mu_i^{\mu_i+k_i}}{\mu_i!} x_i^{\mu_i}, \tag{7.45}$$

where $c_{k_1\dots k_n}$ are the coefficients of the polynomials $P_{g,n}$ from Theorem 7.1, and $K_{g,n}$ is the highest power appearing in $P_{g,n}$.

Define

$$\rho_k(x) := \sum_{m=1}^{\infty} \frac{m^{m+k}}{m!} x^m \tag{7.46}$$

Now we can rewrite (7.45) as

$$H^{\circ}_{g,n} = \sum_{\substack{k_1,\dots,k_n \in \\ \{0,1,\dots,K_{g,n}\}}} c_{k_1\dots k_n} \prod_{i=1}^{n} \rho_{k_i}(x_i) \tag{7.47}$$

Consider the following change of variables:

$$x_i = \left(1 + \frac{1}{t_i}\right) e^{-1-\frac{1}{t_i}} \tag{7.48}$$

We see that the generating function $H_{g,n}$ is a polynomial in variables t_i (in all but two 'unstable' cases when $g = 0$ and $n \leq 2$) after the above substitution (we treat this substitution as a power series expansion at the point $t_i = -1$). For the unstable cases we have

$$\begin{aligned} H^{\circ}_{0,1} &= \sum_{a=1}^{\infty} \frac{a^{a-2}}{a!} x_1^a = \rho_{-2}(x_1) = \frac{1}{2} - \frac{1}{2t_1^2}, \\ H^{\circ}_{0,2} &= \sum_{a,b} \frac{a^a}{a!} \frac{b^b}{b!} \frac{x_1^a x_2^b}{a+b} = \log\left(\frac{\frac{1}{t_2+1} - \frac{1}{t_1+1}}{\frac{1}{x_1} - \frac{1}{x_2}}\right) \end{aligned} \tag{7.49}$$

The formula of Bouchard and Mariño is a recursion relation for these polynomials. In order to present it in a more closed form it is convenient to introduce another family of polynomials $W_{g,n}(t_1,\dots,t_n)$ obtained by the above substitution from the series

$$\Big(\prod x_k\partial_{x_k}\Big)H^\circ_{g,n}=\sum_{\mu_1,\dots,\mu_n}\frac{h^\circ_{g;\mu_1,\dots,\mu_n}}{m!}\,\mu_1\dots\mu_n\,x_1^{\mu_1}\dots x_n^{\mu_n},\tag{7.50}$$

i. e., for $(g,n)\notin\{(0,1),(0,2)\}$,

$$W_{g,n}(t_1,\dots,t_n)=\sum_{\substack{k_1,\dots,k_n\in\\ \{0,1,\dots,K_{g,n}\}}}c_{k_1\dots k_n}\prod_{i=1}^n\rho_{k_i+1}(t_i).\tag{7.51}$$

In the unstable cases we define the functions $W_{g,n}$ by setting explicitly

$$W_{0,1}(t_1)=0,\tag{7.52}$$

$$W_{0,2}(t_1,t_2)=\frac{t_1^2(t_1+1)t_2^2(t_2+1)}{(t_2-t_1)^2}\tag{7.53}$$

Define also auxiliary functions $\widetilde{W}_{g,n}(u,v;t_2,\dots,t_n)$ by

$$\begin{aligned}\widetilde{W}_{g,n}(u,v;t_{L'}):=&\,W_{g-1,n+1}(u,v,t_{L'})\\ &+\sum_{g_1+g_2=g}\ \sum_{A\sqcup B=L'}W_{g_1,|A|+1}(u,t_A)W_{g_2,|B|+1}(v,t_B)\end{aligned}\tag{7.54}$$

We denote here by $L'=\{2,\dots,n\}$ the index set, $t_{L'}=(t_2,\dots,t_n)$; the summation is taken over the set of all possible partitions of the index set into a disjoint union of two subsets, A and B.

Theorem 7.14 (Bouchard-Mariño conjecture)**.** *The polynomias $W_{g,n}$ can be determined by the either of the following recursive formulas*

$$\begin{aligned}W_{g,n}(t_1,t_{L'})&=-\operatorname*{res}_{z=0}\Big(K(z,t_1)\,\widetilde{W}_{g,n}\big(\tfrac{1}{z},\tfrac{1}{z};t_{L'}\big)\Big)\\ &=\operatorname*{res}_{z=0}\Big(K(z,t_1)\,\widetilde{W}_{g,n}\big(\tfrac{1}{z},\tfrac{1}{\sigma(z)};t_{L'}\big)\Big)\\ &=-\operatorname*{res}_{z=0}\Big(K(z,t_1)\,\widetilde{W}_{g,n}\big(\tfrac{1}{\sigma(z)},\tfrac{1}{\sigma(z)};t_{L'}\big)\Big)\end{aligned}\tag{7.55}$$

where

$$K(z,t_1)=\frac{t_1^2(1+t_1)}{2(1-z\,t_1)(1-s(z)\,t_1)}\,\frac{z\,\mathrm{d}z}{z+1}\tag{7.56}$$

and the series $\sigma(z)=-z+\frac{2}{3}z^2-\frac{4}{9}z^3+\dots$ is defined in the next subsection.

The second equality is a reformulation of the Bouchard-Mariño conjecture. Experiments show, however, that the first formula is more efficient for practical computations.

Analytically, the meaning of this theorem is as follows. The function $H_{g,n}$ is defined originally as a formal power series expansion at $x_i=0$. It turns out, however, that this series has a finite radius of convergence with respect to each variable x_i (to be precise, the radius of convergence is e^{-1}). An attempt to prolongate it beyond the radius of convergence meets difficulties: the function becomes multivalued with ramification at $x_i=e^{-1}$. Therefore, it is more natural to consider $H_{g,n}$ as a function on the product $\mathcal{S}\times\dots\times\mathcal{S}$ where $\mathcal{S}$ is the curve given by the equation $x=\left(1+\frac{1}{t}\right)e^{-1-\frac{1}{t}}$. When treated in this way, it becomes single-valued and even rational. The recursive relation of the theorem is formulated in terms of the analysis of the behavior of the function $H_{g,n}$ (and closely related to it function $W_{g,n}$) in a neighborhood of the ramification point $x_1=e^{-1}$ which is different from the origin.

7.3.2 The Lambert curve

The *Lambert curve* is a curve in $\mathbb{C}^2$ defined by the equation

$$x = y\, e^{-y}. \tag{7.57}$$

We consider this affine curve as an open part of its compactification $\mathcal{S} = \mathbb{P}^1$. We regard y as a rational coordinate on $\mathcal{S}$ and the projection to the x-line as a holomorphic function with an essential singularity at the point $y = \infty$. In addition to y we use other convenient rational coordinates on $\mathcal{S}$. In particular, we keep the notations z and t for the rational coordinates related to y by

$$y = 1 + z = 1 + \frac{1}{t}, \quad t = \frac{1}{z}. \tag{7.58}$$

There are two points on $\mathcal{S}$ of special interest for us: the *origin* O corresponding to the coordinates $y = 0$, $z = -1$, $t = -1$, and the *branching point* P with the coordinates $y = 1$, $z = 0$, $t = \infty$. The point P is a Morse critical point for the function x. It means that the projection to the x-line considered as a branched cover has ramification of order two at P.

Consider also the function $w = \log x$. It is multivalued, however, its differential is a well-defined meromorphic differential on C,

$$\mathrm{d}w = \frac{\mathrm{d}x}{x} = \frac{1-y}{y}\mathrm{d}y = -\frac{z}{z+1}\mathrm{d}z = \frac{\mathrm{d}t}{t^2(t+1)}. \tag{7.59}$$

Denote also by D the vector field dual to this 1-form,

$$D = x\partial_x = \frac{y}{1-y}\partial_y = -\frac{z+1}{z}\partial_z = t^2(t+1)\partial_t. \tag{7.60}$$

We regard (7.59) and (7.60) as a single meromorphic form and a single vector field on $\mathcal{S}$ respectively, whose coordinate presentation depends on the chosen local coordinate. Remark that the form $\mathrm{d}w$ vanishes at P, while the field D has a simple pole at this point.

The inversion of (7.57) near the origin is given [21, 30] by the expansion

$$y = \sum_{\mu=1}^{\infty} \frac{\mu^{\mu-1}}{\mu!} x^{\mu}. \tag{7.61}$$

It follows from (7.60) that for any integer k the series

$$\rho_k = \sum_{\mu=1}^{\infty} \frac{\mu^{\mu+k}}{\mu!} x^{\mu} = D^{k+1} y \tag{7.62}$$

is a rational function on C. More explicitly, in the t-coordinate it is given for $k \geq 0$ by the recursion

$$\rho_0(t) = -1 - t, \qquad \rho_{k+1}(t) = t^2(t+1)\frac{\partial}{\partial t}(\rho_k(t)). \tag{7.63}$$

It is a polynomial in t:

$$\rho_k(t) = -k!\, t^{k+1} - \cdots - (2k-1)!!\, t^{2k+1}. \tag{7.64}$$

The degree of this polynomial is $2k+1$. Equivalently, one can say that ρ_k considered as a meromorphic function on $\mathcal{S}$ has pole of order $2k+1$ at P. It follows that the linear span of the polynomials ρ_k form a subspace of 'approximately half' dimension in the space of all polynomials in t. This subspace has a nice characterization that we describe now.

Denote by σ the involution interchanging the sheets of the ramification defined by the function x near the point P. The function σ is holomorphic in a neighborhood of P and its Taylor expansion can be computed from the equation

$$(1+z)\,e^{-z} = (1+\sigma(z))\,e^{-\sigma(z)}. \tag{7.65}$$

Here are the first few terms of this expansion written in the coordinates z and t, respectively:

$$\sigma(z) = -z + \frac{2}{3}z^2 - \frac{4}{9}z^3 + \frac{44}{135}z^4 - \frac{104}{405}x^5 + \frac{40}{189}z^6 + \dots \tag{7.66}$$

$$\tilde{\sigma}(t) = \frac{1}{\sigma(1/t)} = -t - \frac{2}{3} - \frac{4}{135t^2} + \frac{8}{405t^3} - \frac{8}{567t^4} + \dots \tag{7.67}$$

Lemma 7.15. *For any $k \geq 0$ the principal part of the pole of $\rho_k(t)$ at the point P is odd with respect to the involution σ. In other words, the function $\rho_k(t) + \rho_k(\tilde{\sigma}(t))$ is holomorphic at P.*

Proof. For $k = 0$ the assertion is obvious since the principal part of any simple pole is odd. Now, arguing by induction, we assume that ρ_k is represented in the form

$$\rho_k(t) = \eta_k(t) + F_k(t) \tag{7.68}$$

where $\eta_k(t) = \frac{1}{2}\big(\rho_k(t) - \rho_k(\tilde{\sigma}(t))\big)$ is odd and $F_k(t) = \frac{1}{2}\big(\rho_k(t) + \rho_k(\tilde{\sigma}(t))\big)$ is even and holomorphic at P. Then, by definition,

$$\rho_{k+1}(t) = D(\rho_k(t)) = D(\eta_k(t)) + D(F_k(t)). \tag{7.69}$$

The field D is invariant with respect to the involution, therefore, it preserves the parity. It follows that $D(\eta_k(t))$ is odd, and $D(F_k(t))$ is even and the order of its pole at P is at most 1. It follows that $D(F_k(t))$ is, in fact, holomorphic at P, which proves the lemma. □

7.3.3 The cut-and-join equation

Yet another way to collect Hurwitz numbers into a generating series is given by the expansion

$$G_{g,n}(p_1, p_2, \dots) = \frac{1}{n!} \sum_{\mu_1,\dots,\mu_n} \frac{h^\circ_{g;\mu_1,\dots,\mu_n}}{m!}\, p_{\mu_1} \dots p_{\mu_n}. \tag{7.70}$$

The series $G_{g,n}$ involves an infinite collection of variables $p_1, p_2, \dots$ and all its term are homogeneous of degree n. The relation between the two series $G_{g,n}$ and $H^\circ_{g,n}$ is obvious. In particular, $G_{g,n}$ can be obtained from $\frac{1}{n!}H^\circ_{g,n}$ by replacing every monomial $x_1^{\mu_1} \dots x_n^{\mu_n}$ by the corresponding monomial $p_{\mu_1} \dots p_{\mu_n}$.

The cut-and-join equation is a recursion on Hurwitz numbers obtained through the analysis of the cyclic type of the result of multiplication of a given permutation by a single transposition. In its original form [49], it is written as

$$2u\frac{\partial e^G}{\partial u} + \sum_{i=1}^{\infty}(i+1)\,p_i\frac{\partial e^G}{\partial p_i} = \frac{1}{2}\sum_{a,b}\Big((a+b)p_a p_b \frac{\partial e^G}{\partial p_{a+b}} + u\,ab\,p_{a+b}\frac{\partial^2 e^G}{\partial p_a \partial p_b}\Big) \tag{7.71}$$

where

$$G = \sum_{g,n} u^{g-1} G_{g,n}. \tag{7.72}$$

The same equation written in terms of the individual components $G_{g,n}$ is

$$(2g-2+n)G_{g,n} + \sum_{i=1}^{\infty} i\, p_i \frac{\partial G_{g,n}}{\partial p_i} \tag{7.73}$$
$$= \frac{1}{2}\sum_{a,b}\Big((a+b)p_a p_b \frac{\partial G_{g,n-1}}{\partial p_{a+b}} + ab\, p_{a+b}\Big(\frac{\partial^2 G_{g-1,n+1}}{\partial p_a \partial p_b} + \sum_{\substack{g_1+g_2=g\\ n_1+n_2=n+1}} \frac{\partial G_{g_1,n_1}}{\partial p_a}\frac{\partial G_{g_2,n_2}}{\partial p_b}\Big)\Big).$$

Let us rewrite this equation in terms of the functions $H^\circ_{g,n}$. The operator $\sum i\, p_i \partial_{p_i}$ from the left hand side of the equation corresponds to the operator $\sum_{i=1}^n D_i$ acting on $H^\circ_{g,n}$ where

$$D_i = x_i \partial_{x_i} = t_i^2(t_i+1)\partial_{t_i}. \tag{7.74}$$

The action of the 'cut' operator $\sum (a+b)p_a p_b \partial_{p_{a+b}}$ in terms of the series $H^\circ_{g,n}$ results in the replacement of any monomial x_m^ℓ by the sum

$$\sum_{a+b=\ell}(a+b)x_j^a x_k^b = \ell \frac{x_k x_j (x_k^{\ell-1} - x_j^{\ell-1})}{x_k - x_j}$$
$$= \frac{x_j}{x_k - x_j} x_k \frac{\partial (x_k^\ell)}{\partial x_k} + \frac{x_k}{x_j - x_k} x_j \frac{\partial (x_j^\ell)}{\partial x_j} = \frac{x_j}{x_k - x_j} D_k(x_k^\ell) + \frac{x_k}{x_j - x_k} D_j(x_j^\ell). \tag{7.75}$$

In a similar way, the action of the 'join' operator $ab\, p_{a+b}\frac{\partial^2}{\partial p_a \partial p_b}$ results in the replacement of any monomial $x_j^a x_k^b$ by the monomial

$$ab\, x_m^{a+b} = \Big(x_m \frac{\partial (x_m^a)}{\partial x_m}\Big)\Big(x_m \frac{\partial (x_m^b)}{\partial x_m}\Big) = D_m(x_m^a) D_m(x_m^b). \tag{7.76}$$

The relation between the indices k, j, and m in the above considerations is not essential. One should only take care that the result is symmetric with respect to the permutations of the variables $x_1, \dots, x_n$.

The relation obtained from (7.73) in this way is presented below. In this relation L denotes the collection of indices $L = \{1, 2, \dots, n\}$, and $t_L = (t_1, \dots, t_n)$.

$$(2g-2+n)H^\circ_{g,n}(t_L) + \sum_{k=1}^n D_k H^\circ_{g,n}(t_L) \tag{7.77}$$
$$= \frac{1}{2}\sum_{k\neq j} 2\frac{x_j}{x_k - x_j} D_k H^\circ_{g,n-1}(t_{L\setminus\{j\}})$$
$$+ \frac{1}{2}\sum_{k=1}^n \Big(D_k D_{n+1} H^\circ_{g-1,n+1}(t_L, t_{n+1})\ \big|_{t_{n+1}=t_k}$$
$$+ \sum_{g_1+g_2=g}\ \sum_{A\sqcup B = L\setminus\{k\}} D_k H^\circ_{g_1,|A|+1}(t_k, t_A) D_k H^\circ_{g_2,|B|+1}(t_k, t_B)\Big),$$

where the last summation is taken over the set of all possible partitions of the index set $L\setminus\{k\} = \{1, \dots, k-1, k+1, \dots, n\}$ into a disjoint union of two subsets, A and B.

This relation can be regarded as a relation on the functions in either x or t-variables, where x_i and t_i are related by (7.48). We consider this relation as the 'preliminary form' of the required cut-and-join equation. The final form is obtained by extracting unstable terms from the last summation corresponding to the functions $H^\circ_{0,1}$ and $H^\circ_{0,2}$ and combining these terms with the corresponding terms of the previous sums. Using (7.49), we find the coefficients of the recombined terms

$$1 - D_1 H^\circ_{0,1}(t_1) = -\frac{1}{t_1}, \tag{7.78}$$

$$\frac{x_2}{x_1 - x_2} + D_1 H^\circ_{0,2}(t_1, t_2) = \frac{t_1^2(1+t_2)}{t_1 - t_2} \tag{7.79}$$

We obtain thus the final form of the cut-and-join equation in the t-coordinates, see more details in [82]:

$$\begin{aligned}
&(2g-2+n)H^\circ_{g,n}(t_L) + \sum_{k=1}^{n}\Big(-\frac{1}{t_k}\Big) D_k H^\circ_{g,n}(t_L) \qquad (7.80)\\
&= \sum_{k\neq j} \frac{t_k^2(1+t_j)}{t_k - t_j} D_k H^\circ_{g,n-1}(t_{L\setminus\{j\}})\\
&\quad + \frac{1}{2}\sum_{k=1}^{n}\Big(D_k D_{n+1} H^\circ_{g-1,n+1}(t_L, t_{n+1})\,\big|_{t_{n+1}=t_k}\\
&\quad + \sum_{g_1+g_2=g}\ \sum_{A\sqcup B = L\setminus\{k\}}^{\text{stable}} D_k H^\circ_{g_1,|A|+1}(t_k, t_A) D_k H^\circ_{g_2,|B|+1}(t_k, t_B)\Big)
\end{aligned}$$

It is remarkable that the 'non-polynomial' summands are cancelled out, and both sides of the relation proved to be polynomial in t-variables. As it is pointed out in [82], selecting the highest and the lowest degree terms of this formula one gets immediately the Virasoro constrains for the intersection numbers of ψ-classes on the moduli spaces of curves [22, 66, 107] and the relation of the λ_g-formula [40, 42], respectively.

7.3.4 Reduction by symmetrization

The cut-and-join equation (7.80) can be used to determine $H^\circ_{g,n}$ inductively. However, in the presented form it is not very convenient since it is not clear how to invert the operator on the left hand side of the equation. It is not even obvious that the function $H^\circ_{g,n}$ obtained by this recursion is polynomial in t-variables. The following two key observations of [35] lead to a considerable simplification of (7.80):

1. The function $H^\circ_{g,n}$ is polynomial in each variable t_i, therefore, the whole information about this function is contained in the principal part of its pole at the point P with respect to t_i.

2. The principal part of the pole of $H_{g,n}$ is odd with respect to the involution σ on each t_i-line (as it follows from Lemma 7.15).

Consider the *even* summand of the principal part of the pole at P of each term in (7.80) *with respect to the first variable* t_1. It follows that most of the terms will give trivial contribution to the result so that the whole equation will be considerably simplified.

It is more convenient for us to use a slight modification of this idea. Namely, set

$$\eta(t_1) = \sigma\Big(\frac{1}{t_1}\Big) - \frac{1}{t_1}. \tag{7.81}$$

This function is holomorphic at P and odd with respect to the involution. Now, for any meromorphic function $f(t_1)$ we denote by

$$\left\lfloor \frac{f(t_1)}{\eta(t_1)} \right\rfloor_1^- \tag{7.82}$$

the *odd residueless principal part* of the pole of the quotient f/η at the point P. More explicitly, if we write the Laurent expansion

$$\frac{f(t_1) + f(\tilde\sigma(t_1))}{2\,\eta(t_1)} = \sum_{-\infty < i \le N} a_i\, t_1^i \tag{7.83}$$

at P, then we set, by definition,

$$\lfloor f/\eta \rfloor_1^- = \sum_{i=2}^{N} a_i\, t_1^i. \tag{7.84}$$

From this definition we see that $\lfloor f/\eta \rfloor_1^-$ is a polynomial in t_1 divisible by t_1^2.

We apply the transformation $f(t_1) \mapsto \lfloor 2f/\eta \rfloor_1^-$ to both sides of (7.80). This transformation annihilates any function in t_1 whose pole at P has odd principal part. In particular, it annihilates $H^\circ_{g,n}(t_L)$ on the left hand side as well as all terms on both sides of the equality corresponding to the summation index k different from 1.

Let us compute the action of this transformation on the therm $-\frac{1}{t_1} D_1 H^\circ_{g,n}$ on the left hand side. For any meromorphic function $f(t_1)$ which is odd with respect to the involution we have

$$\frac{\frac{-f(t_1)}{t_1} - \frac{f(\tilde\sigma(t_1))}{\tilde\sigma(t_1)}}{\eta(t_1)} = f(t_1)\,\frac{-\frac{1}{t_1} + \frac{1}{\tilde\sigma(t_1)}}{\eta(t_1)} = f(t_1). \tag{7.85}$$

Therefore, $\left\lfloor -\frac{2f(t_1)}{\eta(t_1)\,t_1} \right\rfloor_1^- = \lfloor f(t_1) \rfloor_1^-$. The function $D_1 H^\circ_{g,n}$ differs from such a function by a holomorphic summand that gives trivial contribution to the transformation. This implies

$$\left\lfloor -\frac{2}{\eta(t_1)\,t_1} D_1 H^\circ_{g,n} \right\rfloor_1^- = \lfloor D_1 H^\circ_{g,n} \rfloor_1^- = D_1 H^\circ_{g,n}. \tag{7.86}$$

We obtain finally the equation

$$D_1 H^\circ_{g,n} = \left\lfloor \frac{1}{\eta} \left(\begin{array}{l} \displaystyle\sum_{j=2}^{n} \frac{t_1^2(1+t_j)}{t_1 - t_j} D_1 H^\circ_{g,n-1}(t_1, t_{L'\setminus\{j\}}) \\ + D_1 D_{n+1} H^\circ_{g-1,n+1}(t_1, t_{L'}, t_{n+1}) \,\big|_{t_{n+1}=t_1} \\ + \displaystyle\sum_{g_1+g_2=g} \sum_{A\sqcup B = L'}^{\text{stable}} D_1 H^\circ_{g_1,|A|+1}(t_1, t_A) D_1 H^\circ_{g_2,|B|+1}(t_1, t_B) \end{array} \right) \right\rfloor_1^- \tag{7.87}$$

where $L' = L \setminus \{1\} = \{2, \dots, n\}$.

In order to represent this equation in a more readable form, let us apply $\prod_{k=2}^n D_k$ to both its sides and observe that the expression inside the square brackets becomes algebraic with respect to the functions $W_{g,k}$ defined by (7.51). Moreover, the first term on the right hand side can be formally included into the last since we defined the contribution of the unstable terms as in Equations (7.52) and (7.53). With this notation, the result of the application of $\prod_{k=2}^n D_k$ to both sides of Equation (7.87) takes the form of the following recursive relation on $W_{g,n}$.

Proposition 7.16. *The function $W_{g,n}$ defined by* (7.51)–(7.53) *satisfies the recursive equation*

$$W_{g,n}(t_1, t_{L'}) = \left\lfloor \frac{1}{\eta(t_1)} \widetilde{W}(t_1, t_1; t_{L'}) \right\rfloor_1^-, \tag{7.88}$$

where $L' = \{2, \dots, n\}$, $t_{L'} = (t_2, \dots, t_n)$, *and*

$$\begin{aligned} \widetilde{W}_{g,n}(u, v; t_{L'}) &= W_{g-1,n+1}(u, v, t_{L'}) \\ &+ \sum_{g_1+g_2=g} \sum_{A \sqcup B = L'} W_{g_1,|A|+1}(u, t_A) W_{g_2,|B|+1}(v, t_B). \end{aligned} \tag{7.89}$$

Remark 7.17. If $f(t_1)$ is a meromorphic function whose pole at P has odd principal part then for any other function g we have

$$\left\lfloor \frac{f(t_1)g(t_1)}{\eta(t_1)} \right\rfloor_1^- + \left\lfloor \frac{f(t_1)g(\tilde\sigma(t_1))}{\eta(t_1)} \right\rfloor_1^- = \left\lfloor f(t_1) \frac{g(t_1) + g(\tilde\sigma(t_1))}{\eta(t_1)} \right\rfloor_1^- = 0 \tag{7.90}$$

since $(g(t_1) + g(\tilde\sigma(t_1)))/\eta(t_1)$ is odd. Therefore, $W_{g,n}$ can equivalently be obtained by the either of the following relations

$$\begin{aligned} W_{g,n}(t_1, t_{L'}) &= - \left\lfloor \frac{1}{\eta(t_1)} \widetilde{W}_{g,n}(t_1, \tilde\sigma(t_1); t_{L'}) \right\rfloor_1^- \\ &= \left\lfloor \frac{1}{\eta(t_1)} \widetilde{W}_{g,n}(\tilde\sigma(t_1), \tilde\sigma(t_1); t_{L'}) \right\rfloor_1^- . \end{aligned} \tag{7.91}$$

7.3.5 Residual formalism

The coefficient f_k of the meromorphic function $f(t_1) = \sum_{-\infty < i \le N} f_i t_1^i$ can be extracted by taking the residue

$$f_k = \operatorname*{res}_{z=0} \Big(f\big(\tfrac{1}{z}\big) z^{k-1} \, \mathrm{d}z \Big). \tag{7.92}$$

It follows that the whole residuless principal part of the pole of f is given by

$$\sum_{k=2}^{N} f_k t_1^k = \operatorname*{res}_{z=0} \Big(f\big(\tfrac{1}{z}\big) \sum_{k=2}^{\infty} t_1^k z^{k-1} \, \mathrm{d}z \Big) = \operatorname*{res}_{z=0} \Big(f\big(\tfrac{1}{z}\big) \frac{t_1^2 z}{1 - t_1 z} \, \mathrm{d}z \Big). \tag{7.93}$$

Similarly, for the function $\bar f(t_1) = f(\tilde\sigma(t_1)) = \sum_{-\infty < i \le N} \bar f_i t_1^i$ we get

$$\begin{aligned} \sum_{k=2}^{N} \bar f_k t_1^k &= \operatorname*{res}_{z=0} \Big(f\big(\tfrac{1}{s(z)}\big) \frac{t_1^2 z}{1 - t_1 z} \, \mathrm{d}z \Big) \\ &= \operatorname*{res}_{z=0} \Big(f\big(\tfrac{1}{z}\big) \frac{t_1^2 \sigma(z)}{1 - t_1 \sigma(z)} \frac{z}{1+z} \frac{1 + \sigma(z)}{\sigma(z)} \, \mathrm{d}z \Big). \end{aligned} \tag{7.94}$$

We used here the equality

$$\frac{z \, \mathrm{d}z}{1+z} = \frac{\sigma(z) \, \mathrm{d}\sigma(z)}{1 + \sigma(z)} \tag{7.95}$$

that follows from Equation (7.65).

Combining (7.93) and (7.94) we obtain a residual formula for the odd residueless principal part of the pole of a function:

$$\lfloor f(t_1)/\eta(t_1) \rfloor_1^- = - \operatorname*{res}_{z=0} \big(K(z, t_1) \, f(1/z) \big) \tag{7.96}$$

where

$$K(z,t_1) = \frac{1}{2\eta(1/z)}\Big(\frac{t_1^2 z}{1-t_1 z} - \frac{t_1^2\sigma(z)}{1-t_1\sigma(z)}\frac{z}{1+z}\frac{1+\sigma(z)}{\sigma(z)}\Big)\,\mathrm{d}z \qquad (7.97)$$
$$= \frac{t_1^2(1+t_1)}{2(1-z\,t_1)(1-\sigma(z)\,t_1)}\,\frac{z\,\mathrm{d}z}{z+1}.$$

This, substituted into the recursive formulas of Proposition 7.16 and Remark 7.17, directly gives Theorem 7.14.

7.4 Topological Recursion/Givental correspondence revisited

In this section we review the correspondence between topological recursion and Givental theory established in [113]. We use it in the next section to prove the equivalence between the Bouchard-Mariño conjecture and the ELSV formula. This way we obtain a new proof of the ELSV formula, using the new independent proof of the Bouchard-Mariño conjecture from the previous section.

7.4.1 Givental formula

Let H be a Frobenius algebra, that is, a finite-dimensional commutative associative algebra over $\mathbb{C}$ with a unit denoted by $\mathbb{1} \in H$, equipped with a linear function $\ell : H \to \mathbb{C}$ such that the symmetric bilinear form given by $\langle a, b\rangle = \ell(a\,b)$ is nondegenerate. A typical example is the (even part of the) cohomology ring of a complex compact manifold. Its dimension will be denoted by $N = \dim H$. Fix a basis $e_1, \ldots, e_N$ in H.

Consider also an element of the *Givental upper triangular twisted loop group*, that is, a formal series of the form

$$R(z) = 1 + \sum_{k=1}^{\infty} R_k z^k, \quad R_k \in \mathrm{End}(H), \qquad (7.98)$$

satisfying

$$R(z)\,R^*(-z) = 1. \qquad (7.99)$$

In terms of the Lie algebra element $r(z) = \log(R(z))$, $R(z) = \exp r(z)$, the last relation can be equivalently rewritten as $r(z) + r^*(-z) = 0$.

To this data (a Frobenius algebra and an element R of the upper triangular group) Givental associates a *formal Gromov-Witten potential* F, a formal series in an infinite number of variables $t_{k\nu}$, $k = 0, 1, 2, \ldots$, $\nu = 1, 2, \ldots, N$, and one extra variable $\hbar$, defined by the formula

$$e^{\frac{1}{\hbar}F} = \widehat{R}\,e^{\frac{1}{\hbar}F^{\mathrm{top}}}, \quad \widehat{R} = e^{\widehat{r}}, \qquad (7.100)$$

where F^{top} is the potential of the topological field theory associated with the Frobenius algebra H, and $\widehat{r}$ is a second-order differential operator obtained from $r(z)$ by a procedure of 'quantization of quadratic Hamiltonians', see details in [46].

A choice of basis in H is not essential. A change of the basis leads to a linear change of variables in the potential of the form $t_{k\nu} \longrightarrow \sum_{\mu=1}^{N} \Psi^{\mu}_{\nu} t_{k\mu}$ where Ψ is the matrix of the change of basis. In other words, we can treat F as a formal function on $H \otimes H \otimes \ldots$.

It was observed in [44, 62] that the potential F constructed this way is, in fact, a descendant potential of a certain cohomological field theory. Moreover, it is proved in [103] that the descendant potential of any semi-simple cohomological field theory can be represented in such form.

7.4.2 Topological recursion

Topological recursion is a formal procedure leading to a family of certain differentials $\omega_{g,n}$ associated with a plane complex curve. They were introduced originally for particular curves in relation to matrix models in mathematical physics, then the procedure was formalized for arbitrary abstract curves.

Let $C \subset \mathbb{C}^2$ be a smooth complex curve on the plane with coordinates x, y. Let $a_1, \dots, a_N \in C$ be the critical points of the coordinate function x. The construction of the differentials $\omega_{g,n}$ requires the study of the curve in a neighbourhood of these points, therefore, it is sufficient to assume that instead of C we have a union of N small discs centered at the points a_i, $i = 1, \dots, N$, or even the union of formal neighbourhoods of these points. Respectively, by a function or differential form (holomorphic or meromorphic) on C we mean a collection of germs of functions or differential forms at the points a_i or even a collection of formal Laurent series at these points.

Assume that each point a_i is a Morse critical point of the function x, that is, x is a ramified covering with a ramification of order 2 at a_i. Let σ be the holomorphic involution on C interchanging the branches of the function x near a_i. In order to simplify notations, for any function or differential form α we denote $\overline{\alpha} = \sigma^*\alpha$. With this notation the involution is given by $\sigma\colon (x, y) \mapsto (x, \overline{y})$. Remark that this bar sign has nothing to do with the complex conjugation in the present context. Remark also that the form $\overline{\alpha}$ is defined in a neighbourhood of the point a_i only, even if the form α is globally defined.

On top of that, assume that we are given a 2-*point differential* $B(z_1, z_2)$ (called in some papers by *Bergman kernel*), that is, a meromorphic symmetric 2-differential on $C \times C$ representable near $a_i \times a_j \in C \times C$ in the form

$$B(z_1, z_2) = \delta_{i,j} \frac{dz_1^{(i)} dz_2^{(j)}}{(z_1^{(i)} - z_2^{(j)})^2} + B_{\mathrm{reg}}^{(ij)}(z_1^{(i)}, z_2^{(j)}) \tag{7.101}$$

where $z^{(i)}$ is a local coordinate on C near a_i and where $B_{\mathrm{reg}}^{(ij)}(z_1^{(i)}, z_2^{(j)})$ is holomorphic at $a_i \times a_j$.

The Eynard-Orantin invariants $\omega_{g,n}$, $g \geq 0$, $n \geq 1$, are meromorphic n-differentials on $C^{\times n}$ defined inductively by the following formulas:

$$\omega_{0,1}(z) = 0, \qquad \omega_{0,2}(z_1, z_2) = B(z_1, z_2), \tag{7.102}$$

and for $2g - 2 + n > 0$,

$$\omega_{g,n}(z, z_2, \dots, z_n) = -\sum_{i=1}^{N} \operatorname*{res}_{z'=a_i} \left(\frac{\widetilde{\omega}_{g,n}(z', \bar{z}', z_2, \dots, z_n)}{2\,\mu(z')} \int_{z'}^{\bar{z}'} B(z, \cdot) \right), \tag{7.103}$$

where μ is the 1-form $\mu := y\,dx - \overline{y}\,dx$ defined in a neighborhood of the union of points a_i, and where

$$\widetilde{\omega}_{g,n}(z', z'', z_K) = \omega_{g-1,n+1}(z', z'', z_K) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = K}} \omega_{g_1,|I|+1}(z', z_I)\, \omega_{g_2,|J|+1}(z'', z_J). \tag{7.104}$$

We used here notation $K = \{2, \dots, n\}$, and $u_I = (u_{i_1}, \dots, u_{i_{|I|}})$ for any subset

$$I = \{i_1, \dots, i_{|I|}\} \subset K. \tag{7.105}$$

Remark 7.18. We collect here several important remarks clarifying the meaning of all these formulas.

1. Consider the following operator $\alpha \mapsto P\alpha$ acting on the space of meromorphic 1-forms,

$$(P\alpha)(z) = \sum_{i=1}^{N} \operatorname*{res}_{z'=a_i} \left(\frac{\alpha(z')}{2} \int_{z'}^{\bar{z}'} B(z,\cdot) \right). \tag{7.106}$$

Denote by L the image of this operator. Then *the operator P is the projection to the subspace L*, that is, it is identical on L. The kernel of P is generated by holomorphic and by even (in the sense of local automorphism σ) meromorphic 1-forms.

2. It follows that the 1-form in z on the right hand side of (7.103) belongs to L. In other words, the invariants $\omega_{g,n}$ can be regarded as tensors $\omega_{g,n} \in L^{\otimes n}$ (for $(g,n) \neq (0,2)$). These tensors are *symmetric* and *polynomial*. The last property means that $\omega_{g,n}$ belongs to the corresponding tensor product space itself, not to just its completion.

3. The data contained in the collection of invariants $\omega_{g,n}$ can be collected in a single *potential* $F = \sum \hbar^g F_g$ such that the symmetric tensor $\omega_{g,n}$ is identified with the nth homogeneous term of the Taylor expansion of F_g,

$$\omega_{g,n} = \sum_{\alpha_1,\dots,\alpha_n} \frac{\partial^n F_g}{\partial t_{\alpha_1} \dots \mathrm{d}t_{\alpha_n}} \Big|_{t=0} \mathrm{d}\xi_{\alpha_1} \otimes \dots \otimes \mathrm{d}\xi_{\alpha_n}. \tag{7.107}$$

Here $\{\mathrm{d}\xi_a\}_{\alpha\in\mathcal{A}}$ is some chosen basis in L, and $t = \{t_\alpha\}_{\alpha\in\mathcal{A}}$ is the set of formal variables labeled by the same set of indices. The coordinate expression of the potential F depends on a choice of the basis in L. A different choice of the basis leads to the corresponding linear change of coordinates in F. Otherwise, F can be regarded as a formal function on the infinite dimensional space L^*; with this treatment the potential is invariantly defined and independent of any basis.

4. The dual space $V = L^*$ can be identified with the space of *odd holomorphic* 1-forms. The pairing is given by

$$(\alpha,\beta) = \sum_{\nu=1}^{N} \operatorname*{res}_{z=a_\nu} (\alpha \int \beta), \quad \alpha \in L. \quad \beta \in V. \tag{7.108}$$

If $\{\mathrm{d}\xi_\alpha\}_{\alpha\in\mathcal{A}}$ is any basis in L and $\{\mathrm{d}\xi^\alpha\}_{\alpha\in\mathcal{A}}$ is the dual basis in $V = L^*$, then there is an asymptotic expansion

$$\frac{1}{2}(B(z,w) - B(z,\overline{w})) = \sum_{\alpha\in\mathcal{A}} \mathrm{d}\xi_\alpha(z)\mathrm{d}\xi^\alpha(w). \tag{7.109}$$

This expansion takes place as $w \to a_i$, $|w - w(a_i)| \ll |z - z(a_i)|$.

5. It follows, in particular, that the subspace L is spanned by the coefficients of the Taylor expansion of the antisymmetrized Bergman kernel $\frac{1}{2}(B(z,w) - B(z,\overline{w}))$ with respect to the second argument w at the points a_i.

7.4.3 Givental action as topological recursion

Here we formulate in a refined way the result of [113] in the case $N = 1$.

Let C be a curve on the (x, y)-plane as above. Consider the following operator acting in the space of meromorphic 1-forms,

$$\mathcal{D} : \alpha \mapsto \mathrm{d}\Big(\frac{\alpha}{\mathrm{d}x}\Big). \tag{7.110}$$

This operator commutes with the action of the involution σ, $\mathcal{D}\overline{\alpha} = \overline{\mathcal{D}\alpha}$. Set

$$\mathrm{d}\xi^k := \mathcal{D}^{-k}\mathrm{d}y, \quad k = 0, 1, 2, \ldots. \tag{7.111}$$

The forms $\mathrm{d}\xi^k$ are holomorphic in a neighborhood of the point a_1. There is an ambiguity in the choice of integration constants appearing in the inversion of D. Different choices of these constants lead to forms that differ by a holomorphic and *even* (with respect to the involution σ) summand. It follows that the odd parts of these forms

$$\frac{1}{2}\big(\mathrm{d}\xi^k - d\overline{\xi}^k\big), \quad k = 0, 1, 2, \ldots \tag{7.112}$$

are independent of any choice. Moreover, these odd forms form a basis in the space of odd holomorphic forms. Let us take the antisymmetrized Bergman kernel $\frac{1}{2}\big(B(z, w) - B(z, \overline{w})\big)$, develop it over the obtained basis, and denote by $\mathrm{d}\xi_k$ the coefficients of this expansion:

$$\frac{1}{2}\big(B(z, w) - B(z, \overline{w})\big) = \sum_{k=0}^{\infty} \mathrm{d}\xi_k(z) \frac{\mathrm{d}\xi^k(w) - \mathrm{d}\overline{\xi}^k(w)}{2}. \tag{7.113}$$

This asymptotic expansion takes place as $w \to 0$, $|w| \ll |z|$ where z is a local holomorphic coordinate on C near the point a_1. The form $\mathrm{d}\xi_k$ defined by this expansion is meromorphic with a pole of order $2k + 1$ at $z = 0$.

Definition 7.19. The Bergman kernel is said to be *compatible* with the operator $\mathcal{D}$ if the introduced meromorphic forms $\mathrm{d}\xi_k$ are given explicitly by $\mathrm{d}\xi_k = (-1)^{k+1}\mathcal{D}^{k+1}\mathrm{d}\xi^0$.

The following criterium simplifies the verification of the compatibility condition.

Lemma 7.20. *Assume that the Bergman kernel satisfies the identity*

$$(\mathcal{D}_z + \mathcal{D}_w)B(z, w) = -\mathcal{D}_z\mathrm{d}\xi^0(z)\, \mathcal{D}_w\mathrm{d}\xi^0(w). \tag{7.114}$$

Then it is compatible with $\mathcal{D}$.

Proof. Applying the expansion (7.113) we get

$$\begin{aligned} 0 &= (\mathcal{D}_z + \mathcal{D}_w)\frac{B(z, w) - B(z, \overline{w})}{2} + \mathcal{D}_z\mathrm{d}\xi^0(z)\, \mathcal{D}_w\frac{\mathrm{d}\xi^0(w) - \mathrm{d}\overline{\xi}^0(w)}{2} \\ &= \sum_{k=0}^{\infty}(\mathcal{D}_z\mathrm{d}\xi_k(z) + \mathrm{d}\xi_{k+1}(z))\frac{\mathrm{d}\xi^k(w) - \mathrm{d}\overline{\xi}^k(w)}{2} \\ &\quad + (\mathcal{D}_z\mathrm{d}\xi^0(z) + \mathrm{d}\xi_0(z))\, \mathcal{D}_w\frac{\mathrm{d}\xi^0(w) - \mathrm{d}\overline{\xi}^0(w)}{2}. \end{aligned} \tag{7.115}$$

This equality is equivalent to the system of equations $\mathrm{d}\xi_0 = -\mathcal{D}\mathrm{d}\xi^0$, $\mathrm{d}\xi_{k+1} = -\mathcal{D}\mathrm{d}\xi_k$, that is, $\mathrm{d}\xi_k = (-1)^{k+1}\mathcal{D}^{k+1}\mathrm{d}\xi^0$, as required. □

Now, assume that the Bergman kernel is compatible with $\mathcal{D}$. Introduce the local coordinate s on the curve near the point a_1 from the relation $\mathrm{d}x = s\,\mathrm{d}s$, that is,

$$s = \sqrt{2(x - x(a_1))}. \tag{7.116}$$

This coordinate is defined up to a sign, and the involution in this coordinate is given simply by $\overline{s} = -s$. Consider the expansion of the odd part of the form $\mathrm{d}y$ in this coordinate,

$$\frac{1}{2}(\mathrm{d}y - \mathrm{d}\overline{y}) = \mathrm{d}s + \sum_{k=1}^{\infty} R_k \frac{s^{2k}\,\mathrm{d}s}{(2k-1)!!}. \tag{7.117}$$

We can now formulate the main result of [113] for the case of $N = 1$.

Theorem 7.21. *If the Bergman kernel is compatible with the operator $\mathcal{D}$, then the spectral curve n-point functions are the n-point correlator functions of a certain formal GW potential $F(t_0, t_1, \dots) = \sum \hbar^g F_g$,*

$$\omega_{g,n} = \sum_{k_1,\dots,k_n} \frac{\partial^n F_g}{\partial t_{k_1} \dots dt_{k_n}}\Big|_{t=0} \mathrm{d}\xi_{k_1} \otimes \dots \otimes \mathrm{d}\xi_{k_n}. \tag{7.118}$$

Moreover, this GW potential is given by the Givental formula (7.100) *with the Witten-Kontsevich potential for F^{top} and with the element $R(z) = 1 + R_1 z + R_2 z^2 + \dots$ of the upper triangular group whose components R_k are determined by the expansion* (7.117).

7.5 New proof of the ELSV formula

In the present section we prove the equivalence of the Bouchard-Mariño formula and the ELSV formula with the help of the Givental-topological recursion correspondence reviewed in the previous section. Note that this equivalence was already proved by Eynard in [33], and a generalization is described in Chapter 8.

From this equivalence, using our new proof of the Bouchard-Mariño conjecture (Theorem 7.14), we obtain a new proof of the ELSV formula.

7.5.1 Hodge class

The total Hodge class $\Lambda_g = 1 - \lambda_1 + \dots + (-1)^g \lambda_g \in H^*(\mathcal{M}_{g,n})$ provides the simplest non-trivial example of a cohomological field theory (of dimension $N = 1$). It follows that its potential, the generating function for Hodge integrals,

$$F(\hbar, t_0, t_1, \dots) = \sum_{g,n} \frac{\hbar^g}{n!} \sum_{k_1,\dots,k_n} \int_{\mathcal{M}_{g,n}} \Lambda_g\, \psi_1^{k_1} \dots \psi_n^{k_n}\, t_{k_1} \dots t_{k_n} \tag{7.119}$$

is a formal GW potential. Indeed, Mumford's formula [83] for the Chern characters of the Hodge bundle rewritten in terms of intersection numbers has exactly the form (7.100) with the Witten-Kontsevich potential for the series F^{top} and the following element of the upper triangular group

$$R(z) = \exp\left(\sum_{n=1}^{\infty} \frac{B_{2n}}{2n\,(2n-1)} z^{2n-1}\right) = 1 + \frac{1}{12}z + \frac{1}{288}z^2 - \frac{139}{51840}z^3 + \dots, \tag{7.120}$$

where B_n is the n^{th} Bernoulli number. The operator $\widehat{R} = \exp\left(\sum_{n=1}^{\infty} \frac{B_{2k}}{2n\,(2n-1)} \widehat{z^{2n-1}}\right)$ corresponding to this element acts by

$$\widehat{z^{2n-1}} = -\frac{\partial}{\partial t_{2n}} + \sum_{i=0}^{\infty} t_i \frac{\partial}{\partial t_{i+2n-1}} - \frac{1}{2} \sum_{i+j=2n-2} (-1)^i \frac{\partial^2}{\partial t_i \partial t_j}. \tag{7.121}$$

In the definition of this operator, we use convention which differs by the sign from that of the original paper [46].

7.5.2 BM-ELSV equivalence

Consider the Lambert curve (7.57)

$$\tilde{x} = \tilde{y} - \log(1+\tilde{y}), \qquad \mathrm{d}\tilde{x} = \frac{\tilde{y}\,\mathrm{d}\tilde{y}}{1+\tilde{y}}, \tag{7.122}$$

which is given here in logarithmic coordinates

$$\begin{aligned}\tilde{x} &= -1 - \log x \\ \tilde{y} &= -1 + y\end{aligned}$$

For this curve, the standard Bergman kernel $B(\tilde{y}_1, \tilde{y}_2) = \frac{\mathrm{d}\tilde{y}_1 \mathrm{d}\tilde{y}_2}{(\tilde{y}_1 - \tilde{y}_2)^2}$ is compatible with the operator $\mathcal{D}$. Indeed, we have

$$\begin{aligned}(\mathcal{D}_{\tilde{y}_1} + \mathcal{D}_{\tilde{y}_2}) \frac{\mathrm{d}\tilde{y}_1 \mathrm{d}\tilde{y}_2}{(\tilde{y}_1 - \tilde{y}_2)^2} &= \mathrm{d}_{\tilde{y}_1} \frac{(1+\tilde{y}_1)\,\mathrm{d}\tilde{y}_2}{\tilde{y}_1(\tilde{y}_1 - \tilde{y}_2)^2} + \mathrm{d}_{\tilde{y}_2} \frac{(1+\tilde{y}_2)\,\mathrm{d}\tilde{y}_1}{\tilde{y}_2(\tilde{y}_1 - \tilde{y}_2)^2} \\ &= -\frac{\mathrm{d}\tilde{y}_1 \mathrm{d}\tilde{y}_2}{\tilde{y}_1^2 \tilde{y}_2^2} \\ &= -\mathcal{D}_{\tilde{y}_1} \mathrm{d}\tilde{y}_1\, \mathcal{D}_{\tilde{y}_2} \mathrm{d}\tilde{y}_2.\end{aligned} \tag{7.123}$$

Therefore, by Lemma 7.20 and Theorem 7.21, the spectral curve n-point functions in this case are the correlation functions of a certain formal GW potential. Moreover, this potential is obtained from the Kontsevich-Witten potential by the action of the element $R(z) = 1 + \sum R_k z^k$ of the Givental group whose coefficients are determined by the expansion

$$\frac{\partial}{\partial s} \frac{\tilde{y}(s) - \tilde{y}(-s)}{2} = 1 + \sum_{k=1}^{\infty} R_k \frac{s^{2k}}{(2k-1)!!}, \tag{7.124}$$

where the function $\tilde{y}(s)$ is given by the implicit equation

$$s = \sqrt{2\,(\tilde{y} - \log(1+\tilde{y}))}. \tag{7.125}$$

It is proved in [11] that these coefficients are the same as those given by the expansion (7.120).

This means that for our spectral curve we have

$$\begin{aligned}\omega_{g,n} &= \sum_{k_1,\dots,k_n} \frac{\partial^n F_g}{\partial t_{k_1} \dots \mathrm{d}t_{k_n}} \Big|_{t=0} (\mathrm{d}\xi_{k_1})_1 \dots (\mathrm{d}\xi_{k_n})_n \\ &= \sum_{k_1,\dots,k_n} \left(\int_{\overline{\mathcal{M}}_{g,n}} \Lambda_g \psi_1^{k_1} \dots \psi_n^{k_n} \right) \prod_{i=1}^{n} \sum_{\mu_i=1}^{\infty} \frac{\mu_i^{\mu_i+k_i+1}}{\mu_i!} x_i^{\mu_i-1} \mathrm{d}x_i \\ &= \sum_{\mu_1,\dots,\mu_n} \left(\int_{\overline{\mathcal{M}}_{g,n}} \frac{\Lambda_g}{\prod_{i=1}^{n}(1-\mu_i\psi_i)} \right) \prod_{i=1}^{n} \frac{\mu_i^{\mu_i+1}}{\mu_i!} x_i^{\mu_i-1} \mathrm{d}x_i\end{aligned} \tag{7.126}$$

Here we used the fact that in our case

$$\begin{aligned} \mathrm{d}\xi_k = (-1)^{k+1}\mathcal{D}^{k+1}\mathrm{d}\tilde{y} &= \mathrm{d}\left(\left(x\frac{\partial}{\partial x}\right)^{k+1} y\right) \\ &= \mathrm{d}\left(\left(x\frac{\partial}{\partial x}\right)^{k+1}\sum_{\mu=1}^{\infty}\frac{\mu^{\mu-1}}{\mu!}x^{\mu}\right) = \sum_{\mu=1}^{\infty}\frac{\mu^{\mu+k+1}}{\mu!}x^{\mu-1}\mathrm{d}x \end{aligned} \tag{7.127}$$

Note that the Bouchard-Mariño conjecture may be written as

$$\omega_{g,n} = \sum_{\mu_1,\ldots,\mu_n}\frac{h^{\circ}_{g;\mu_1,\ldots,\mu_n}}{m!}\,\mu_1\ldots\mu_n\,x_1^{\mu_1-1}\ldots x_n^{\mu_n-1}\mathrm{d}x_1\ldots\mathrm{d}x_n, \tag{7.128}$$

while the ELSV formula states that

$$h^{\circ}_{g;\mu_1,\ldots,\mu_n} = m!\left(\int_{\overline{\mathcal{M}}_{g,n}}\frac{\Lambda_g}{\prod_{i=1}^{n}(1-\mu_i\psi_i)}\right)\prod_{i=1}^{n}\frac{\mu_i^{\mu_i}}{\mu_i!} \tag{7.129}$$

We immediately see that formula (7.126) directly implies the following

Theorem 7.22. *The Bouchard-Mariño conjecture and the ELSV formula are equivalent.*

This means that we have a new proof of the ELSV formula, since we proved the Bouchard-Mariño conjecture independently in Section 7.3. Note that the Bouchard-Mariño conjecture as given in Theorem 7.14 is equivalent to formula (7.128), if one takes into account the topological recursion formula for $\omega_{g,n}$, given by Equations (7.103) and (7.104).

–8– The r-ELSV formula and the r-BM conjecture

8.1 Introduction

In this chapter we discuss two conjectures related to Hurwitz numbers with completed cycles that generalize two important theorems in the theory of ordinary Hurwitz numbers. The first is a generalization of the celebrated ELSV formula relating Hurwitz numbers and Hodge integrals [31] that we call the r-ELSV conjecture. The second is a generalization of the (now proved) Bouchard-Mariño conjecture [10, 35, 8] relating Hurwitz numbers to the CEO-recusion procedure of Chekhov, Eynard, and Orantin [15, 36], that we call the r-BM conjecture. We prove that these two conjectures are equivalent. This completes our evidence for the r-BM conjecture started in Chapter 3, while simultaneously providing evidence for the r-ELSV conjecture.

8.1.1 The space of r-spin structures

The r-spin structures on smooth curves

Fix an integer $r \geq 1$. Let C be a (smooth compact connected) complex curve of genus g with n marked (pairwise distinct numbered) points $x_1, \ldots, x_n$. Denote by K the canonical (cotangent) line bundle over C. Choose n integers $a_1, \ldots, a_n \in \{0, \ldots, r-1\}$ such that $2g - 2 - \sum a_i$ is divisible by r.

Definition 8.1. An *r-spin structure* on C is a line bundle $\mathcal{L}$ together with an identification

$$\mathcal{L}^{\otimes r} \xrightarrow{\sim} K\left(-\sum a_i x_i\right). \tag{8.1}$$

The *moduli space* $\mathcal{M}^{1/r}_{g,n;a_1,\ldots,a_n}$ is the space of all r-spin structures on all smooth curves.

We will denote this space by $\mathcal{M}^{1/r}$ omitting the other indices if this does not lead to ambiguity.

The element $K(-\sum a_i x_i) \in \mathrm{Pic}(C)$ can be divided by r in r^{2g} different ways. Hence there are exactly r^{2g} different r-spin structures on every smooth curve C. Thus the natural morphism $\pi : \mathcal{M}^{1/r} \to \mathcal{M}_{g,n}$ to the moduli space of smooth curves is actually a nonramified r^{2g}-sheeted covering.

The space $\mathcal{M}^{1/r}$ has a structure of orbifold or smooth Deligne-Mumford stack. The stabilizer of a generic r-spin structure is $\mathbb{Z}/r\mathbb{Z}$ (except for several cases with small g and n where it can be bigger), because every line bundle $\mathcal{L}$ has r automorphisms given by the multiplication by rth roots of unity. In particular, the degree of π is not r^{2g} but r^{2g-1}.

The compactification

A natural compactification $\overline{\mathcal{M}}^{1/r}_{g,n;a_1,\ldots,a_n}$ of $\mathcal{M}^{1/r}_{g,n;a_1,\ldots,a_n}$ was constructed in [58], [2], [12], [16]. It has the structure of an orbifold or a smooth Deligne-Mumford stack. There are three different constructions that involve different versions of the universal curve $\overline{C}^{1/r}_{g,n;a_1,\dots,a_n}$, but the moduli space $\overline{\mathcal{M}}^{1/r}_{g,n;a_1,\dots,a_n}$ is the same.

In the geometrically most natural construction the universal curve over the compactification has orbifold fibers and $\mathcal{L}$ is extended into a line bundle in the orbifold sense. The second construction is a fiberwise coarsification of the first one. After the coarsification the universal curve becomes a singular orbifold with A_{r-1} singularities and $\mathcal{L}$ becomes a rank 1 torsion-free sheaf rather than a line bundle. Finally, the third construction is obtained from the second one after resolving its singularities by a sequence of blow-ups. In the third construction the universal curve is smooth and $\mathcal{L}$ is again a line bundle, but the morphism $\mathcal{L}^{\otimes r} \to K(-\sum a_i x_i)$ has zeros on the exceptional divisors. The third construction is most convenient for intersection theoretic computations.

On a singular stable curve with an r-spin structure, every branch of every node carries an index $a \in \{0, \dots, r-1\}$ such that the sum of indices at each node equals -2 modulo r. The divisibility condition $2g-2-\sum a_i = 0 \mod r$ is satisfied on each irreducible component of the curve. Thus for a separating node the indices of the branches are determined uniquely, while for a nonseparating node there are r choices.

From now on we write simply $\overline{\mathcal{M}}^{1/r}$, and $\overline{C}^{1/r}$ for the compactified space of r-spin structures and its universal curve, if this does not lead to ambiguity.

Remark 8.2. The interest in $\overline{\mathcal{M}}^{1/r}$ was initially motivated by Witten's conjecture asserting that some natural intersection numbers on this space can be arranged into a generating series that satisfies the r-KdV (or rth higher Gelfand-Dikii) hierarchy of partial differential equations [108]. The conjecture is now proved [44] and other beautiful results on the intersection theory of $\overline{\mathcal{M}}^{1/r}$ have been obtained [17, 100].

One of the main ingredients in Witten's conjecture is the so-called "Witten top Chern class" whose definition uses the sheaves $R^0p_*\mathcal{L}$ and $R^1p_*\mathcal{L}$ on $\overline{\mathcal{M}}^{1/r}$ in a rather involved way (see [93] or [18]). The r-ELSV conjecture, on the other hand, uses the more straight-forward total Chern class $c(R^1p_*\mathcal{L})/c(R^0p_*\mathcal{L})$. Therefore it is not at all clear if the r-ELSV formula can be related to Witten's conjecture. However, it would provide a link between the intersection theory on $\overline{\mathcal{M}}^{1/r}$ and integrable hierarchies via Hurwitz numbers, see Theorem 8.12.

The map $\pi\colon \mathcal{M}^{1/r} \to \mathcal{M}_{g,n}$ extends to the compactifications; $\pi\colon \overline{\mathcal{M}}^{1/r} \to \overline{\mathcal{M}}_{g,n}$. Thus the classes $\psi_1, \dots, \psi_n \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ that one usually defines on $\overline{\mathcal{M}}_{g,n}$ (see, for instance, [104], Section 3.5) can be lifted to $\overline{\mathcal{M}}^{1/r}$. By abuse of notation we will denote the lifted classes by ψ_i instead of $\pi^*(\psi_i)$.

The projection $p : \overline{C}^{1/r} \to \overline{\mathcal{M}}^{1/r}$ induces a push-forward of $\mathcal{L}$ in the sense of derived functors:

$$R^\bullet p_*\mathcal{L} = R^0p_*\mathcal{L} - R^1p_*\mathcal{L}. \tag{8.2}$$

Notation 8.3. We denote by $\mathcal{S}$ the total Chern class

$$\mathcal{S} = c(-R^\bullet p_*\mathcal{L}) = c(R^1p_*\mathcal{L})/c(R^0p_*\mathcal{L}). \tag{8.3}$$

8.1.2 The r-ELSV conjecture

Let $\overline{\mathcal{M}}^{\mathrm{rel}} = \overline{\mathcal{M}}_{g,m;k_1,\dots,k_n}(\mathbb{P}^1)$ be the space of stable genus g maps to $\mathbb{P}^1$ relative to $\infty \in \mathbb{P}^1$ with profile $(k_1, \dots, k_n)$ and with m marked points in the source curve. It is a compactification of the space of meromorphic functions on genus g curves with n numbered poles of orders $k_1, \dots, k_n$ and m more marked points. See [104], Section 5 for a precise definition and main properties. Let $\omega \in H^2(\mathbb{P}^1)$ be the Poincaré dual class of a point.

We write every k_i in the form $k_i = rp_i + (r-1-a_i)$, where p_i is the quotient and $r-1-a_i$ the remainder of the division of k_i by r. Let $m = (\sum k_i + n + 2g - 2)/r$.

Denote by $h_{g,r;k_1,\dots,k_n}$ the following Gromov-Witten invariants

$$h_{g,r;k_1,\dots,k_n} := (r!)^m \int_{\overline{\mathcal{M}}^{\mathrm{rel}}} \mathrm{ev}_1^*(\omega)\psi_1^r \cdots \mathrm{ev}_n^*(\omega)\psi_m^r. \tag{8.4}$$

Okounkov and Pandharipande [88] studied Gromov-Witten invariants as above and proved that they are equal to the Hurwitz numbers with completed cycles $h^{(r)}_{g,k_1,\dots,k_n}$ that where introduced in Chapter 2.

Introduce the following integrals over the space of r-spin structures:

$$f_{g,r;k_1,\dots,k_n} = m!\, r^{m+n+2g-2} \prod_{i=1}^{n} \frac{\left(\frac{k_i}{r}\right)^{p_i}}{p_i!} \times \int_{\overline{\mathcal{M}}^{1/r}_{g,n;a_1,\dots,a_n}} \frac{\mathcal{S}}{\left(1-\frac{k_1}{r}\psi_1\right)\dots\left(1-\frac{k_n}{r}\psi_n\right)}. \tag{8.5}$$

Chiodo [17] studied the class $\mathcal{S}$ expressing it in terms of standard cohomology classes on the moduli space, the coefficients being equal to values of Bernoulli polynomials at rational points with denominator r. His work allows one to compute the numbers $f_{g,r;k_1,\dots,k_n}$.

Conjecture 8.4 (r-ELSV)**.** *We have*

$$h_{g,r;k_1,\dots,k_n} = f_{g,r;k_1,\dots,k_n}. \tag{8.6}$$

This conjecture suggests a hidden equality between virtual fundamental classes. More precisely, if we consider the space of relative stable maps f such that $\mathrm{d}f$ has an r-th root, the conjecture seems to imply that its virtual fundamental class is obtained from the virtual fundamental class of the space of all stable maps after multiplication by rth powers of ψ-classes. However this is not elucidated so far and our evidence for this conjecture is not geometric.

The r-ELSV formula was first conjectured in [111]. Since both sides are computable, the conjecture can be tested on a computer, and there is virtually no doubt that it is correct. The conjecture is proved in genus 0, and also for $(g,n) = (1,1)$ and arbitrary r (unpublished).

8.1.3 The r-Bouchard-Mariño conjecture

In [15, 36] Chekhov, Eynard, and Orantin assigned to every plane analytic complex curve $\mathcal{S}$ a series of invariants $\omega_{g,n}(\mathcal{S})$ called *correlation n-forms.* Each $\omega_{g,n}$ is a meromorphic n-form on $\mathcal{S}^n$. The construction is described in detail in Chapter 1.

Definition 8.5. The *n-point functions of genus g* for r-Hurwitz numbers and for r-spin integrals are defined as

$$H^{(r)}_{g,n}(x_1,\dots,x_n) = \sum_{k_1,\dots,k_n} \frac{h^{(r)}_{g;k_1,\dots,k_n}}{m!} \exp(k_1x_1 + \dots + k_nx_n). \tag{8.7}$$

$$F_{g,r}(x_1,\dots,x_n) = \sum_{k_1,\dots,k_n} \frac{f_{g,r;k_1,\dots,k_n}}{m!} \exp(k_1x_1 + \dots + k_nx_n). \tag{8.8}$$

We also let, for any function $f(x_1,\dots,x_n)$,

$$Df = \frac{\partial^n f}{\partial x_1 \cdots \partial x_n} \mathrm{d}x_1 \cdots \mathrm{d}x_n. \tag{8.9}$$

Conjecture 8.6 (r-BM)**.** *Let $\omega_{g,n}$ be the n-point correlation forms of the plane curve $x = -y^r + \log y$. Then we have*

$$DH^{(r)}_{g,n}(x_1,\dots,x_n) = \omega_{g,n}. \tag{8.10}$$

8.1.4 Equivalence of the two conjectures

The goal of this chapter is to prove the following theorem.

Theorem 8.7. *Let $\omega_{g,n}$ be the n-point correlation forms of the plane curve $x = -y^r + \log y$. Then we have*

$$DF_{g,r}(x_1, \dots, x_n) = \omega_{g,n}. \tag{8.11}$$

It shows that the r-ELSV conjecture is equivalent to r-BM conjecture. Since there is independent evidence for both conjectures, this theorem transfers that evidence between them. It is also interesting in it's own right, providing a relation between to a priori very different aspects of the theory of Hurwitz numbers.

8.1.5 Plan of the chapter

In Section 8.2 we describe once again the completed cycles and completed Hurwitz numbers involved in the r-ELSV conjecture.

In Section 8.3 we review the work of Chiodo showing that the cohomology classes $\mathcal{S}$ form a cohomological field theory. We determine that cohomological field theory and the Givental R-matrix assigned to $\mathcal{S}$.

Section 8.4 contains the proof of the main theorem. It is based on the identification [113] of the Givental-Teleman theory of semi-simple cohomological field theories with the Chekhov-Eynard-Orantin topological recursion theory. We check that the R-matrix for the class $\mathcal{S}$ coincides with the R-matrix assigned to the curve $x = -y^r + \log y$.

8.1.6 Notation

The following notation is used consistently throughout this chapter.

- $r \geq 1$ is an integer; we work with the space of r-spin structures.
- I is the imaginary unit, $\mathrm{I}^2 = -1$.
- J is the primitive rth root of unity, $\mathrm{J}^r = 1$.
- $g \geq 0$ is the (arithmetic) genus of the curves C under consideration.
- $n \geq 1$ is the number of marked points on C; the points themselves are denoted by $x_1, \dots, x_n$.
- $k_1, \dots, k_n$ is an n-tuple of positive integers such that $m = (\sum k_i + n + 2g - 2)/r$ is also an integer. We consider meromorphic functions on C with n poles of orders $k_1, \dots, k_n$ at the marked points $x_1, \dots, x_n$. Such a function represents a ramified covering of $\mathbb{P}^1$ of degree $\sum k_i$ with a ramification point of type $(k_1, \dots, k_n)$ at ∞. We also require the covering to have exactly m additional ramification points of multiplicity r.
- Each k_i determines a unique couple of integers p_i, a_i such that $k_i = rp_i + (r - 1 - a_i)$ and $a_i \in \{0, \dots, r-1\}$. Thus p_i is the quotient and a_i the "reversed remainder" of the division of k_i by r. The r-spin structures we consider will be rth roots of the line bundle $K(-\sum a_i x_i)$.

The integers g, n, r, $k_1, \dots, k_n$ determine the integers m, p_i, and a_i uniquely.

8.2 Completed cycles and completed Hurwitz numbers

Okounkov and Panharipande proved in [88] that the left-hand side of Conjecture 8.4 is equal to Hurwitz numbers with completed cycles. To make this chapter more independent from the rest of the text, we introduce them once more in a way that is slightly different from what we did in Chapter 2. Note that in this chapter, as well as in Chapter 3, we use a different normalization from what we did in Chapter 2 (see Remark 2.5).

8.2.1 Completed cycles

A *partition* λ of an integer q is a non-increasing finite sequence $\lambda_1 \geq \cdots \geq \lambda_l$ such that $\sum \lambda_i = q$.

It is known that the irreducible representations ρ_λ of the symmetric group S_q are in a natural one-to-one correspondence with the partitions λ of q. On the other hand, to a partition λ of q one can assign a central element $C_{p,\lambda}$ of the group algebra $\mathbb{C}S_p$ for any positive integer p. The coefficient of a given permutation $\sigma \in S_p$ in $C_{p,\lambda}$ is defined as the number of ways to choose and number l cycles of σ so that their lengths are $\lambda_1, \dots, \lambda_l$, and the remaining $p-q$ elements are fixed points of σ. Thus the coefficient of σ vanishes unless its cycle lengths are $\lambda_1, \dots, \lambda_l, 1, \dots, 1$. In particular, $C_{p,\lambda} = 0$ if $p < q$. Thus $C_{p,\lambda}$ is *the sum of permutations with l numbered cycles of lengths $\lambda_1, \dots, \lambda_l$ and any number of non-numbered fixed points.*

The collection of elements $C_{p,\lambda}$ for $p = 1, 2, \dots$ is called a *stable center element*[1] C_λ. For example, the stable element C_2 is the sum of all transpositions in $\mathbb{C}S_p$, which is well-defined for each p, (in particular, equal to zero for $p = 1$).

Let λ be a partition of q and μ a partition of p. Since $C_{p,\lambda}$ lies in the center of $\mathbb{C}S_p$, it is represented by a scalar (multiplication by a constant) in the representation ρ_μ of S_p. Denote this scalar by $f_\lambda(\mu)$. Thus to a stable center element C_λ we have assigned a function f_λ defined on the set of all partitions. We are interested in the vector space spanned by the functions f_λ.

To study this space, one defines some new functions on the set of partitions as follows[2]:

$$\mathbf{p}_{r+1}(\mu) = \frac{1}{r+1} \sum_{i \geq 1} \left[(\mu_i - i + \frac{1}{2})^{r+1} - (-i + \frac{1}{2})^{r+1} \right] \qquad (r \geq 0). \tag{8.12}$$

Theorem 8.8 (Kerov, Olshansky [65]). *The vector space spanned by the functions f_λ coincides with the algebra generated by the functions $\mathbf{p}_1, \mathbf{p}_2, \dots$.*

As a corollary, to each stable center element C_λ we can assign a polynomial in $\mathbf{p}_1, \mathbf{p}_2, \dots$ and, conversely, each $\mathbf{p}_{r+1}$ corresponds to a linear combination of stable center elements C_λ.

Definition 8.9. The linear combination of stable center elements corresponding to $\mathbf{p}_{r+1}$ is called the *completed $(r+1)$-cycle* and denoted by $\overline{C}_{r+1}$.

The first completed cycles are:

$$\begin{aligned} \overline{C}_1 &= C_1, \\ \overline{C}_2 &= C_2, \\ \overline{C}_3 &= C_3 + C_{1,1} + \tfrac{1}{12} C_1, \\ \overline{C}_4 &= C_4 + 2C_{2,1} + \tfrac{5}{4} C_2, \\ \overline{C}_5 &= C_5 + 3C_{3,1} + 4C_{2,2} + \tfrac{11}{3} C_3 + 4C_{1,1,1} + \tfrac{3}{2} C_{1,1} + \tfrac{1}{80} C_1. \end{aligned} \tag{8.13}$$

For reasons that will become clear later, we say that a stable center element C_λ involved in the completed cycle $\overline{C}_{r+1}$ has *genus defect* $[r + 2 - \sum(\lambda_i + 1)]/2$.

[1] This has nothing to do with stable curves.

[2] The standard definition involves certain additive constants that we have dropped to simplify the expression, since these constants play no role here.

8.2.2 Completed Hurwitz numbers

Let $K = \sum k_i$. Recall that the completed $(r+1)$-cycle can be considered as a central element of the group algebra $\mathbb{C}S_K$. An *r-factorization of type* $(k_1, \ldots, k_n)$ in the symmetric group S_K is a factorization

$$\sigma_1 \ldots \sigma_m = \sigma \tag{8.14}$$

such that (i) the cycle lengths of σ equal $k_1, \ldots, k_n$ and (ii) each permutation σ_i enters the completed $(r+1)$-cycle with a nonzero coefficient. The product of these coefficients for i going from 1 to m is called the *weight* of the r-factorization.

Choose m points $y_1, \ldots, y_m \in \mathbb{C}$ and a system of m loops $s_i \in \pi_1(\mathbb{C} \setminus \{y_1, \ldots, y_m\})$, s_i going around y_i. Then to an r-factorization one can assign a family of stable maps from nodal curves to $\mathbb{P}^1$. This is done in the following way. (i) Consider the covering of $\mathbb{P}^1$ ramified over $y_1, \ldots, y_m$, and ∞ with monodromies given by $\sigma_1, \ldots, \sigma_m$ and σ^{-1} (relative to the chosen loops). (ii) If σ_i has l_i distinguished cycles and genus defect g_i, glue a curve of genus g_i with l_i marked points to the l_i preimages of the ith ramification point that correspond to the distinguished cycles. The covering mapping is extended on this new component by saying that it is entirely projected to the ith ramification point. (iii) Among the newly added components, contract those that are unstable.

One can easily check that the arithmetic genus of the curve C constructed in this way is equal to g. The complex structure on the newly added components of C can be chosen arbitrarily, which implies that in general we obtain not a unique stable map, but a family of stable maps.

An r-factorization is called *transitive* if the curve C assigned to the factorization is connected, in other words if one can go from every element of $\{1, \ldots, K\}$ to any other by applying the permutations σ_i and jumping from one distinguished cycle of σ_i to another one.

Definition 8.10. The *completed Hurwitz number* $h_{g,r;k_1,\ldots,k_n}$ is the sum of weights of the transitive r-factorizations of type $(k_1, \ldots, k_n)$.

Theorem 8.11 (Okounkov, Pandharipande [88])**.** *The relative Gromov-Witten invariant*

$$(r!)^m \int_{\overline{\mathcal{M}}^{\mathrm{rel}}} \mathrm{ev}_1^*(\omega)\psi_1^r \cdots \mathrm{ev}_n^*(\omega)\psi_m^r \tag{8.15}$$

is equal to the corresponding completed Hurwitz number.

Thus there is no clash of notation if we denote both by $h_{g,r;k_1,\ldots,k_n}$.

8.2.3 A digression on Kadomtsev-Petviashvili

The Hurwitz numbers can be arranged into a generating series

$$G_r(\beta; p_1, p_2, \ldots) = \sum_{g,n} \frac{1}{n!} \sum_{k_1,\ldots,k_n} h_{g,r;k_1,\ldots,k_n} \frac{\beta^m}{m!} p_{k_1} \ldots p_{k_n}. \tag{8.16}$$

(One can prove that $h_{g,r;k_1,\ldots,k_n} = 0$ whenever $m = (\sum k_i + n + 2g - 2)/r$ is not an integer.)

Theorem 8.12. *The series G_r is a solution of the Kadomtsev-Petviashvili (KP) hierarchy in variables p_i, β being a parameter.*

The proof of this theorem is a straightforward generalization of [63] and [85].

8.3 The class $\mathcal{S}$

8.3.1 Chiodo's formula

Chiodo computed the Chern characters

$$h_k^\circ(R^\bullet p_*\mathcal{L}) = h_k^\circ(R^0 p_*\mathcal{L}) - h_k^\circ(R^1 p_*\mathcal{L}) \tag{8.17}$$

and obtained the following expression.

Theorem 8.13 (Chiodo [17], Theorem 1.1.1)**.** *We have*

$$\begin{aligned} h_k^\circ(R^\bullet p_*\mathcal{L}) =& \frac{B_{k+1}(\frac{1}{r})}{(k+1)!}\kappa_k - \sum_{i=1}^n \frac{B_{k+1}(\frac{a_i+1}{r})}{(k+1)!}\psi_i^k \\ &+ \frac{r}{2}\sum_{a=0}^{r-1} \frac{B_{k+1}(\frac{a_i+1}{r})}{(k+1)!}(j_a)_*\frac{(\psi')^k + (\psi'')^k}{\psi' + \psi''}. \end{aligned} \tag{8.18}$$

Here j_a is the push-forward from the boundary components with a chosen node and a chosen branch of index a.

Chiodo's formula makes it possible to compute any class of the form

$$\exp\left(\sum_{k\geq 1} s_k h_k^\circ(R^\bullet p_*\mathcal{L})\right). \tag{8.19}$$

Specifically, we will need

$$c(-R^\bullet p_*\mathcal{L}) = \exp\left(-\sum_{k\geq 1}(k-1)! h_k^\circ(R^\bullet p_*\mathcal{L})\right), \tag{8.20}$$

with $s_k = -(k-1)!$.

8.3.2 Topological field theory and R-matrix

In this section we use the notion of a *cohomological field theory*, *topological field theory* and R-matrix action. See [92] or [97] for an introduction.

Let $\Omega_{g,n}(a_1,\dots,a_n) = \pi_*(\mathcal{S})$ be the push-forward of the class $\mathcal{S} = c(R^1p_*\mathcal{L})/c(R^0p_*\mathcal{L})$ from the space of r-spin structures to $\overline{\mathcal{M}}_{g,n}$. Further, denote by $\omega_{g,n}$ the degree 0 part of $\Omega_{g,n}$.

The topological field theory

The degree 0 class $\omega_{g,n}$ is equal to the push-forward of the cohomology class 1 from the space of r-spin structures to the space of stable curves. Thus it is east to compute:

$$\omega_{g,n}(a_1,\dots,a_n) = r^{2g-1}\delta_{2g-2-\sum a_i \bmod r}, \tag{8.21}$$

because the degree of $\pi : \overline{\mathcal{M}}^{1/r} \to \overline{\mathcal{M}}_{g,n}$ equals r^{2g-1}. The classes $\omega_{g,n}$ form a topological field theory with unit on the vector space $\langle e_0,\dots,e_{r-1}\rangle$ with quadratic form

$$\eta_{ab} = \frac{1}{r}\delta_{a+b+2 \bmod r} \tag{8.22}$$

and unit e_0.

The 3-point correlators in genus 0 are equal to

$$\langle e_a e_b e_c \rangle = \frac{1}{r}\delta_{a+b+c+2 \mod r}. \tag{8.23}$$

Therefore the quantum product is given by

$$e_a \bullet e_b = e_{a+b \bmod r}. \tag{8.24}$$

The idempotents of the quantum product are

$$\frac{1}{r}\sum_{a=0}^{r-1} \mathrm{J}^{ai} e_a, \tag{8.25}$$

where J is the primitive rth root of unity and $0 \le i \le r-1$.

The R-matrix and the correlators

It follows from Chiodo's formula that the family of cohomology classes $\Omega_{g,n}$ can be obtained from $\omega_{g,n}$ by Givental's R-matrix action, where

$$R(z) = \exp\left[-\sum_{k\ge 1} \frac{\mathrm{diag}_{a=0}^{r-1} B_{k+1}\left(\frac{a+1}{r}\right)}{k(k+1)} z^k \right] \tag{8.26}$$

in basis (e_a).

This fact has been observed in [20] in the presence of Witten's class, then in [19] in the setting we use here. It follows that the classes $\Omega_{g,n}$ form a semi-simple cohomological field theory.

We define the *correlators* $\left\langle \tau_{d_1}^{a_1} \cdots \tau_{d_n}^{a_n} \right\rangle_g^{\widetilde{\text{coh}}}$ of this cohomological field theory:

$$\left\langle \tau_{d_1}^{a_1} \cdots \tau_{d_n}^{a_n} \right\rangle_g^{\widetilde{\text{coh}}} := \int_{\overline{\mathcal{M}}_{g,n}} \Omega_{g,n}(a_1, \dots, a_n)\, \psi_1^{d_1} \cdots \psi_n^{d_n} = \int_{\overline{\mathcal{M}}^{1/r}_{g,n;a_1,\dots,a_n}} \mathcal{S}\, \psi_1^{d_1} \cdots \psi_n^{d_n}, \tag{8.27}$$

which are just the coefficients of the n-point function

$$F_{g,r}(x_1, \dots, x_n) = \sum_{\substack{d_1,\dots,d_n \ge 0 \\ 0 \le a_1,\dots,a_n \le r-1}} \left\langle \tau_{d_1}^{a_1} \cdots \tau_{d_n}^{a_n} \right\rangle_g^{\widetilde{\text{coh}}} r^{2g+2n-2+\frac{2g-2-\sum_{i=1}^n a_i}{r} - \sum_{i=1}^n d_i}$$
$$\prod_{i=1}^n \sum_{p_i=0}^{\infty} \frac{(rp_i + r - a_i - 1)^{p_i+d_i}}{p_i!} e^{(rp_i+r-a_i-1)x_i}. \tag{8.28}$$

8.4 Equivalence of the r-ELSV and r-BM conjectures

In this section we derive a formula for the n-point differentials $\omega_{g,n}$ of the curve $\mathcal{S}^{(r)}\colon x = -y^r + \log y$ and prove Theorem 8.7. In particular, this implies that the r-ELSV conjecture and the r-BM conjecture are equivalent.

To do that, we perform a number of local computations with the spectral curve, following the recipe of [113] in order to present the cohomological field theory corresponding to the spectral curve as a particular R-matrix action in the sense of Givental, applied to a topological field theory equal to a direct sum of r one-dimentional topological field theories, properly rescaled. We prove that the cohomological field theory that we obtain this way is a multiple of $\{\Omega_{g,n}\}$ defined in Section 8.3.2 rewritten in the basis of normalized idempotents.

Proof of Theorem 8.7. Our goal is to compare the coefficients of the n-differentials $\omega_{g,n}$ with the correlators $\widetilde{\left\langle\tau_{d_1}^{a_1}\cdots\tau_{d_n}^{a_n}\right\rangle_g^{\mathrm{coh}}}$ of the cohomological field theory described in Section 8.3.2 in terms of the Givental group action. In [113] the coefficients of the differentials $\omega_{g,n}$ are, under some conditions and modulo some extra factors, expressed in terms of correlators of a cohomological field theory. We use the result of [113] and compare the cohomological field theories and the extra factors that appear in the statement of the theorem and in the identification formula in [113].

Let us outline this comparison. We have the following formula for $\omega_{g,n}$ given in [113]:

$$\omega_{g,n} = \sum_{\substack{i_1,\dots,i_n \\ d_1,\dots,d_n}} \left\langle\tau_{d_1}^{i_1}\cdots\tau_{d_n}^{i_n}\right\rangle_g^{\text{t.r.}} D\left(\prod_{j=1}^{n}\left(-2\frac{\partial}{\partial x_j}\right)^{d_i}\xi_{i_j}(x_j)\right), \tag{8.29}$$

where ξ_i are some auxilliary functions of the coordinate x on the curve, we recall them below. The correlators $\left\langle\tau_{d_1}^{i_1}\cdots\tau_{d_n}^{i_n}\right\rangle_g^{\text{t.r.}}$ are represented as a sum over Givental graphs, whose structure is described in [118, 113].

Remark 8.14. The local computations that we do in this section in order to specify all ingredients of Equation (8.29) are a direct generalization of the computations of Eynard in [34, Section 8.2] for the curve $x = -y + \log y$ that shows that the usual ELSV formula is equivalent to the Bouchard-Mariño conjecture for the usual Hurwitz numbers.

Note that the indices $i_1,\dots,i_n$ that we have in Equation (8.29) refer to a summation in the basis of the normalized idempotents (cf. Section 8.3.2), while the indices $a_1,\dots,a_n$ in Theorem 8.7 refer to a summation in the natural flat basis of the cohomological field theory.

Notation 8.15. Throughout this text, when we write an object with a tilde we mean refer to this object expressed in the natural flat basis, while when we write the same object without a tilde it should be interpreted as expressed in the basis of canonical idempotents. Furthermore, $\langle\cdot\rangle^{\text{t.r.}}$ refers to the correlators of the cohomological field theory associated to the the spectral curve $\mathcal{S}^{(r)}$ by the methods of [113], whereas $\langle\cdot\rangle^{\text{coh}}$ refers to the correlators of the cohomological field theory $\Omega_{g,n}$.

Changing the coordinates, we can rewrite Equation (8.29) in the following form:

$$\omega_{g,n} = \sum_{\substack{a_1,\dots,a_n \\ d_1,\dots,d_n}} \widetilde{\left\langle\tau_{d_1}^{a_1}\cdots\tau_{d_n}^{a_n}\right\rangle_g^{\text{t.r.}}} D\left(\prod_{j=1}^{n}\left(-2\frac{\partial}{\partial x_j}\right)^{d_i}\tilde{\xi}_{a_j}(x_j)\right), \tag{8.30}$$

where

$$\widetilde{\left\langle\tau_{d_1}^{a_1}\cdots\tau_{d_n}^{a_n}\right\rangle_g^{\text{t.r.}}} := \sum_{i_1,\dots,i_n} \left\langle\tau_{d_1}^{i_1}\cdots\tau_{d_n}^{i_n}\right\rangle_g^{\text{t.r.}} \prod_{j=1}^{n} \mathrm{J}^{(a_j+1)i_j}, \tag{8.31}$$

and

$$\tilde{\xi}_a := \sum_{i=0}^{r-1} r^{-1}\mathrm{J}^{-(a+1)i}\xi_i . \tag{8.32}$$

Below, in Lemma 8.18, we prove that

$$\tilde{\xi}_a = \mathrm{I}\sqrt{2}r^{\frac{1}{2}-\frac{a+1}{r}}\sum_{n=0}^{\infty}\frac{(rn+r-a-1)^n}{n!}e^{(rn+r-a-1)x}. \tag{8.33}$$

Further, in Lemma 8.19 we prove that

$$\widetilde{\left\langle \tau_{d_1}^{a_1} \cdots \tau_{d_n}^{a_n} \right\rangle_g^{\text{t.r.}}} \prod_{j=1}^{n} \mathrm{I}\sqrt{2r} \cdot r^{\frac{-(a_j+1)}{r}} (-2)^{d_n} \tag{8.34}$$
$$= \widetilde{\left\langle \tau_{d_1}^{a_1} \cdots \tau_{d_\ell}^{a_\ell} \right\rangle_g^{\text{coh}}} r^{2g+2\ell-2+\frac{2g-2-\sum_{i=1}^{\ell} a_i}{r} - \sum_{i=1}^{\ell} d_i}$$

Substituting these two expression in Equation (8.30), using equation (8.28), we obtain the equality of Theorem 8.7, which proves the theorem. □

The rest of the section consists of a step-by-step computation of the expansions of various local objects on the curve $x = -y^r + \log y$ that are needed to formulate and prove Lemmas 8.18 and 8.19. We follow the scheme of computations proposed in [113] and use the same notation as there.

$$y_i := r^{-1/r} \mathrm{J}^i, \qquad i = 0, \dots, r-1. \tag{8.35}$$

The critical values of the function x at these points are

$$x_i := -\frac{1}{r} + \frac{\mathrm{I}2\pi i}{r} - \frac{\log r}{r}, \qquad i = 0, \dots, r-1. \tag{8.36}$$

We denote by z_i the local coordinates near the critical points, that is, $z_i^2 + x_i = x$, $i = 0, 1, \dots, r-1$. Let us make a choice of the expansion of function y in z_i. One of the possible choices, that fixes $y = y(z_i)$ unambiguously is

$$y(z_i) = y_i + y_{i1} z_i + O(z_i^3), \tag{8.37}$$

where

$$y_{i1} := \mathrm{I}\sqrt{2} r^{-\frac{1}{2}-\frac{1}{r}} \mathrm{J}^i, \qquad i = 0, \dots, r-1. \tag{8.38}$$

8.4.1 Reciprocal Gamma function

We are using the following expansion of the reciprocal Gamma function:

$$\frac{1}{\Gamma(w+v)} = \frac{\mathrm{I}}{2\pi} \int_{C_+} (-t)^{-w-v} e^{-t} \mathrm{d}t \tag{8.39}$$
$$\sim \frac{w^{-w-v+\frac{1}{2}} e^w}{\sqrt{2\pi}} \exp\left(\sum_{j=1}^{\infty} \frac{B_{j+1}(v)}{j(j+1)} (-w)^j \right),$$

where C_+ is the Hankel contour [76] that goes around the positive real numbers, and $B_j(v)$ are the Bernoulli polynomials defined by

$$\frac{w e^{wv}}{e^w - 1} = \sum_{j=0}^{\infty} B_j(v) \frac{w^j}{j!}. \tag{8.40}$$

Note that $B_j(1-v) = (-1)^j B_j(v)$, $j = 0, 1, \dots$.

8.4.2 Local expansions of y

Let us fix $0 \le i \le r-1$. We are interested in the expansion of the odd part of $y = y(z_i)$ in z_i, that is, we consider $(y(z_i) - y(-z_i))/2$. We consider also the coordinate $t := ry^r$. It is easy to see that

$$y = t^{\frac{1}{r}} r^{-\frac{1}{r}} \mathrm{J}^i; \tag{8.41}$$

$$-x_i - y^r + \log y = \frac{1}{r} - \frac{t}{r} + \frac{\log t}{r}; \tag{8.42}$$

$$\mathrm{d}y = t^{\frac{1-r}{r}} r^{-1-\frac{1}{r}} \mathrm{J}^i \mathrm{d}t. \tag{8.43}$$

Note also that t runs along the negative Hankel contour when z_i runs from $-\infty$ to $+\infty$.

Lemma 8.16. *Denote the coefficients of the odd part of $y(z_i)$ by:*

$$\frac{y(z_i) - y(-z_i)}{2} = \mathrm{I}\sqrt{2} r^{-\frac{1}{2}-\frac{1}{r}} \mathrm{J}^i z_i \sum_{j=0}^{\infty} \frac{V_j (2r)^j z_i^{2j}}{(2j+1)!!} \ . \tag{8.44}$$

Then we have

$$\sum_{j=0}^{\infty} V_j \zeta^j = \exp\left(- \sum_{i=1}^{\infty} \frac{B_{j+1}(\frac{1}{r})}{j(j+1)} \zeta^j \right) \ . \tag{8.45}$$

Proof. The Laplace method gives the following asymptotic expansion:

$$\int_{-\infty}^{\infty} y'(z_i) \exp(-rwz_i^2) \mathrm{d}z_i = \mathrm{I}\sqrt{2\pi} r^{-1-\frac{1}{r}} \mathrm{J}^i w^{-\frac{1}{2}} \sum_{j=0}^{\infty} V_j w^{-j} \ . \tag{8.46}$$

On the other hand,

$$\begin{aligned} \int_{-\infty}^{\infty} y'(z_i) e^{-rwz_i^2} \mathrm{d}z_i &= \int_{\tilde{C}_-} e^{-rw(-x_i - y^r + \log(y))} \mathrm{d}y \\ &= r^{-\frac{1+r}{r}} \mathrm{J}^i \int_{C_-} e^{-w(1-t+\log t)} t^{\frac{1-r}{r}} \mathrm{d}t \\ &= r^{-\frac{1+r}{r}} \mathrm{J}^i e^{-w} \int_{C_-} e^{wt} t^{-w+\frac{1-r}{r}} \mathrm{d}t \\ &= -r^{-\frac{1+r}{r}} \mathrm{J}^i e^{-w} w^{w-\frac{1}{r}} \int_{C_+} e^{-p} (-p)^{-w+\frac{1-r}{r}} \mathrm{d}p \end{aligned} \tag{8.47}$$

(we used the substitution $p = -wt$ that transforms the negative Hankel contour C_- to the Hankel contour C_+). Thus we see that

$$\begin{aligned} \int_{-\infty}^{\infty} y'(z_i) e^{-rwz_i^2} \mathrm{d}z_i &= \frac{2\pi \mathrm{I} r^{-\frac{1+r}{r}} \mathrm{J}^i e^{-w} w^{w-\frac{1}{r}}}{\Gamma(w+1-\frac{1}{r})} \\ \sim \frac{2\pi \mathrm{I} r^{-\frac{1+r}{r}} \mathrm{J}^i e^{-w} w^{w-\frac{1}{r}} w^{-w+\frac{1}{r}-\frac{1}{2}} e^{w}}{\sqrt{2\pi}} &\exp\left(\sum_{j=1}^{\infty} \frac{B_{j+1}(1-\frac{1}{r})}{j(j+1)} (-w)^{-j} \right) \\ = \sqrt{2\pi} \mathrm{I} r^{-\frac{1+r}{r}} \mathrm{J}^i w^{-\frac{1}{2}} &\exp\left(- \sum_{j=1}^{\infty} \frac{B_{j+1}(\frac{1}{r})}{j(j+1)} w^{-j} \right) \ , \end{aligned} \tag{8.48}$$

which proves

$$\sum_{j=0}^{\infty} V_j w^{-j} = \exp\left(- \sum_{j=1}^{\infty} \frac{B_{j+1}(\frac{1}{r})}{j(j+1)} w^{-j} \right) \ . \tag{8.49}$$

$\square$

8.4.3 Two-point function

Now we consider the two-point function. According to [34], since $\mathrm{d}x$ is a meromorphic 1-form in y, we know that the Laplace transform of the two-point function is represented as a Givental-type edge contribution. So, we have to compute only the even coefficients of the local expansion of half of the two-point function in order to specify the Givental operator imposed by the topological recursion, namely, we are interested in the coefficients of the function

$$\mathrm{Y}_{i_1 i_2} := \frac{y_{i_1 1} y'(z_{i_b})}{(y_{i_1} - y(z_{i_2}))^2} = \frac{\delta_{i_1 i_2}}{z^2} + O(1). \tag{8.50}$$

Lemma 8.17. *We have:*

$$\frac{\mathrm{Y}_{i_1 i_2}(z_{i_2}) + \mathrm{Y}_{i_1 i_2}(-z_{i_2})}{2} = -\sum_{k=0}^{\infty} \frac{(U_k)_{i_1 i_2} (2r)^i z_{i_2}^{2k-2}}{(2k-3)!!} \ , \tag{8.51}$$

where $(U_k)_{i_1 i_2}$ is given by

$$\sum_{k=0}^{\infty} (U_k)_{i_1 i_2} z^k = \frac{1}{r} \sum_{c=0}^{r-1} \exp\left(-\sum_{k=1}^{\infty} \frac{B_{k+1}(\frac{c}{r}) z^k}{k(k+1)} \right) \mathrm{J}^{c i_2 - c i_1} \ . \tag{8.52}$$

Proof. Observe that

$$\int_{-\infty}^{\infty} \left(\frac{y_{i_1 1} y'(z_{i_2})}{(y_{i_1} - y(z_{i_2}))^2} \right) e^{-rwz_{i_2}^2} \mathrm{d}z_{i_2} \sim -2\sqrt{\pi} r^{\frac{1}{2}} w^{\frac{1}{2}} \sum_{k=0}^{\infty} (U_k)_{i_1 i_2} w^{-k}. \tag{8.53}$$

On the other hand,

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{y_{i_1 1} y'(z_{i_2})}{(y_{i_1} - y(z_{i_2}))^2} e^{-rwz_{i_2}^2} \mathrm{d}z_{i_2} \\ &= rwy_{i_1 1} \int_{-\infty}^{\infty} \frac{1}{y_{i_1} - y(z_{i_2})} e^{-rwz_{i_2}^2} 2z_{i_2} \mathrm{d}z_{i_2} \\ &= rwy_{i_1 1} \int_{C_-} \frac{y(z_{i_2})^r - \frac{1}{r}}{y(z_{i_2}) - y_{i_1}} e^{-w(1-t+\log t)} \frac{\mathrm{d}t}{t} \ . \end{aligned} \tag{8.54}$$

Here we can use that

$$\begin{aligned} ry_{i_1 1} \frac{y(z_{i_2})^r - \frac{1}{r}}{y(z_{i_2}) - y_{i_1}} &= r\mathrm{I}\sqrt{2} r^{-\frac{1}{2} - \frac{1}{r}} \mathrm{J}^{i_1} \frac{(t^{\frac{1}{r}} r^{-\frac{1}{r}} \mathrm{J}^{i_2})^r - r^{-1}}{t^{\frac{1}{r}} r^{-\frac{1}{r}} \mathrm{J}^{i_2} - r^{-\frac{1}{r}} \mathrm{J}^{i_1}} \\ &= \mathrm{I}\sqrt{2} r^{-\frac{1}{2}} \sum_{c=0}^{r-1} \mathrm{J}^{c i_2 - c i_1} t^{\frac{c}{r}} \ . \end{aligned} \tag{8.55}$$

Therefore,

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{y_{i_1 1} y'(z_{i_2})}{(y_{i_1} - y(z_{i_2}))^2} e^{-rwz_{i_2}^2} \mathrm{d}z_{i_2} \\ &= \mathrm{I}\sqrt{2} r^{-\frac{1}{2}} w e^{-w} \sum_{c=0}^{r-1} \mathrm{J}^{c i_2 - c i_1} \int_{C_-} e^{wt} t^{-w-1+\frac{c}{r}} \mathrm{d}t \\ &= -\mathrm{I}\sqrt{2} r^{-\frac{1}{2}} w^{w+1} e^{-w} \sum_{c=0}^{r-1} \mathrm{J}^{c i_2 - c i_1} w^{-\frac{c}{r}} \int_{C_+} e^{-p} (-p)^{-w-1+\frac{c}{r}} \mathrm{d}p \\ &\sim -2\sqrt{\pi} r^{-\frac{1}{2}} w^{\frac{1}{2}} \sum_{c=0}^{r-1} \mathrm{J}^{c i_2 - c i_1} \exp\left(-\sum_{n=1}^{\infty} \frac{B_{n+1}(\frac{c}{r})}{n(n+1)} w^{-c} \right) \ . \end{aligned} \tag{8.56}$$

Thus we see that

$$\sum_{k=0}^{\infty}(U_k)_{i_1 i_2} z^k = \frac{1}{r}\sum_{c=0}^{r-1}\exp\left(-\sum_{k=1}^{\infty}\frac{B_{k+1}(\frac{c}{r})z^k}{k(k+1)}\right)\mathrm{J}^{ci_2-ci_1} . \tag{8.57}$$

□

8.4.4 Functions on the leaves

According to [113], the auxilliary function $\xi_i(x)$ that we put on the leaves in the graph expression for the correlation forms of the spectral curve are given by the following formula:

$$\xi_i := \frac{\mathrm{I}\sqrt{2}r^{-\frac{1}{2}-\frac{1}{r}}\mathrm{J}^i}{r^{-\frac{1}{r}}\mathrm{J}^i - y}. \tag{8.58}$$

Here the index i corresponds to the basis of normalized idempotents, so in the standard flat basis we have to consider the functions $\tilde{\xi}_a$ given by Equation (8.32):

$$\tilde{\xi}_a := \sum_{i=0}^{r-1} r^{-1}\mathrm{J}^{-(a+1)i}\xi_i . \tag{8.59}$$

Lemma 8.18. *We have:*

$$\tilde{\xi}_a = \mathrm{I}\sqrt{2}r^{\frac{1}{2}-\frac{a+1}{r}}\sum_{n=0}^{\infty}\frac{(rn+r-a-1)^n}{n!}e^{(rn+r-a-1)x} . \tag{8.60}$$

Proof. First, observe that

$$\begin{aligned}\tilde{\xi}_a &= \sum_{i=0}^{r-1} r^{-1}\mathrm{J}^{-(a+1)i}\xi_i = \mathrm{I}\sqrt{2}r^{-1-\frac{1}{2}}\sum_{i=0}^{r-1}\frac{\mathrm{J}^{-(a+1)i}}{1-r^{\frac{1}{r}}\mathrm{J}^{-i}y} \\ &= \mathrm{I}\sqrt{2}r^{-\frac{1}{2}}\left(\frac{r^{\frac{r-a-1}{r}}y^{r-a-1}}{1-ry^r}\right).\end{aligned} \tag{8.61}$$

Following [21], we define the Lambert function

$$W(z) := -\sum_{n=1}^{\infty}\frac{n^{n-1}}{n!}(-z)^n , \tag{8.62}$$

and use its property

$$\left(\frac{W(z)}{z}\right)^{\alpha} = \sum_{n=0}^{\infty}\frac{\alpha(n+\alpha)^{n-1}}{n!}(-z)^n . \tag{8.63}$$

We have the following equation: $e^x = ye^{-y^r}$. This equation implies (cf. [114])

$$y = \left(\frac{W(-re^{rx})}{-r}\right)^{\frac{1}{r}} \tag{8.64}$$

and

$$\frac{\mathrm{d}y^{r-a-1}}{\mathrm{d}x} = \frac{(r-a-1)y^{r-a-1}}{1-ry^r} . \tag{8.65}$$

Therefore,

$$\begin{aligned}\frac{(r-a-1)y^{r-a-1}}{1-ry^r} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{W(-re^{rx})}{-r}\right)^{\frac{r-a-1}{r}} \\ &= (-r)^{-\frac{r-a-1}{r}}\frac{\mathrm{d}}{\mathrm{d}x}(-re^{rx})^{\frac{r-a-1}{r}}\sum_{n=0}^{\infty}\frac{\frac{r-a-1}{r}(n+\frac{r-a-1}{r})^{n-1}}{n!}(re^{rx})^n \\ &= (r-a-1)\frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=0}^{\infty}\frac{(rn+r-a-1)^{n-1}}{n!}e^{(rn+r-a-1)x} \\ &= (r-a-1)\sum_{n=0}^{\infty}\frac{(rn+r-a-1)^{n}}{n!}e^{(rn+r-a-1)x}\end{aligned} \tag{8.66}$$

So, we see that

$$\tilde{\xi}_a = \mathrm{I}\sqrt{2}r^{\frac{1}{2}-\frac{a+1}{r}}\sum_{n=0}^{\infty}\frac{(rn+r-a-1)^{n}}{n!}e^{(rn+r-a-1)x} \tag{8.67}$$

□

8.4.5 Comparison of the correlators

Consider the n-point correlators $\widetilde{\left\langle\tau_{d_1}^{a_1}\cdots\tau_{d_n}^{a_n}\right\rangle_g^{\text{t.r.}}}$ introduced in Equation (8.30). They are obtained via a linear change of the indices from the correlators $\left\langle\tau_{d_1}^{i_1}\cdots\tau_{d_n}^{i_n}\right\rangle_g^{\text{t.r.}}$, where the latter ones are defined in [113] as the sum over Givental graphs. The structure constants of these graphs, that is, the parameters that we put on vertices, edges, leaves, and dilaton leaves, are defined in terms of local data of the curve $x=-y^r+\log y$ at the ramification points. More precisely, they are defined via the coefficients of the expansions of the function $y(z_i)$ in the local coordinate z_i and the components of the Bergman kernel $Y_{i_1i_2}$ in the local coordinate z_{i_2}.

In this Section we prove that the correlators $\widetilde{\left\langle\tau_{d_1}^{a_1}\cdots\tilde{\tau}_{d_n}^{a_n}\right\rangle_g^{\text{t.r.}}}$ are equal, up to some multiplicative factors, to the correlators $\widetilde{\left\langle\tau_{d_1}^{a_1}\cdots\tau_{d_n}^{a_n}\right\rangle_g^{\text{coh}}}$ of the cohomological field theory described in Section (8.3.2). Namely, we prove the following lemma.

Lemma 8.19.

$$\begin{aligned}&\sum_{i_1,\dots,i_\ell}\left\langle\tau_{d_1}^{i_1}\cdots\tau_{d_n}^{i_n}\right\rangle_g^{\text{t.r.}}\prod_{k=1}^{n}r^{\frac{1}{2}}\mathrm{J}^{(a_k+1)i_k}\prod_{k=1}^{n}\mathrm{I}\sqrt{2}r^{\frac{-(a_k+1)}{r}}(-2)^{d_k} \\ &= \widetilde{\left\langle\tau_{d_1}^{a_1}\cdots\tau_{d_n}^{a_n}\right\rangle_g^{\text{coh}}}r^{2g+2n-2+\frac{2g-2-\sum_{k=1}^n a_k}{r}-\sum_{k=1}^n d_i}\end{aligned} \tag{8.68}$$

Proof. The proof follows from the comparison of the ingredients of the Givental graph expressions on both sides, following the identification theorem in [113]. Let us rewrite both sides of the equality in the basis of normalized idempotents:

$$\left\langle\tau_{d_1}^{i_1}\cdots\tau_{d_n}^{i_n}\right\rangle_g^{\text{t.r.}}\prod_{k=1}^{n}(-2r)^{d_k+\frac{1}{2}} = \left\langle\tau_{d_1}^{i_1}\cdots\tau_{d_n}^{i_n}\right\rangle_g^{\text{coh}}r^{2g+n-2+\frac{2g+n-2}{r}}, \tag{8.69}$$

where

$$\left\langle\tau_{d_1}^{i_1}\cdots\tau_{d_n}^{i_n}\right\rangle_g^{\text{coh}} := \sum_{a_1,\dots,a_n}\widetilde{\left\langle\tau_{d_1}^{a_1}\cdots\tau_{d_n}^{a_n}\right\rangle_g^{\text{coh}}}\prod_{j=1}^{n}\mathrm{J}^{-(a_j+1)i_j} \tag{8.70}$$

The result of Chiodo (see Theorem 8.13) implies that the generating function of the correlators $\left\langle \tau_{d_1}^{i_1}\cdots\tau_{d_n}^{i_n}\right\rangle_g^{\mathrm{coh}}$ is obtained from the r properly normalized copies of the Kontsevich-Witten tau function by application of the quantization of the operator

$$R_i^j(\zeta) := \exp\left(-\sum_{k=1}^{\infty}\frac{\zeta^k}{r}\sum_{a=0}^{r-1}\mathrm{J}^{ai-aj}\frac{B_{k+1}\left(\frac{a}{r}\right)}{k(k+1)}\right) . \tag{8.71}$$

In particular (we use it below), we have:

$$\begin{aligned} R_i^{\mathbf{1}}(\zeta) &:= \sum_{j=0}^{r-1}\sum_{a=0}^{r-1}\frac{\mathrm{J}^j}{r}\frac{\mathrm{J}^{ai-aj}}{r}\exp\left(-\sum_{k=1}^{\infty}\zeta^k\frac{B_{k+1}\left(\frac{a}{r}\right)}{k(k+1)}\right) \\ &= \frac{\mathrm{J}^i}{r}\exp\left(-\sum_{k=1}^{\infty}\zeta^k\frac{B_{k+1}\left(\frac{1}{r}\right)}{k(k+1)}\right) . \end{aligned} \tag{8.72}$$

The weight of the correlators $\langle\prod_{i=1}^p\tau_{a_i}\rangle_q$ of the i-th copy of the Kontsevich-Witten tau function (or, in other words, the weight of the vertex labelled by i in the graphical representation of the Givental formula, as in [113]) is equal to

$$r^{2q-1}\sum_{\substack{a_1,\dots,a_p:\\ r|2g-2-a_1+\cdots+a_p}}\prod_{j=1}^{p}\mathrm{J}^{-(a_j+1)i} = r^{2q+p-2}\mathrm{J}^{-(2q+p-2)i} . \tag{8.73}$$

Let us compare that with the formula we get from the topological recursion, following the lines of [113]. We compare the coefficients of the expansion of y and the two-point function in the coordinate z_i, $i=0,\dots,r-1$ (that determine the ingredients of the graphs in the formula for $\left\langle \tau_{d_1}^{i_1}\cdots\tau_{d_n}^{i_n}\right\rangle_g^{\mathrm{t.r.}}$) with the corresponding formulas in terms of the operator $R_i^j(\zeta)$ that are used in the Givental graphical formula for $\left\langle \tau_{d_1}^{i_1}\cdots\tau_{d_n}^{i_n}\right\rangle_g^{\mathrm{coh}}$, in the same way as it is done in [113, Theorem 4.1].

Lemma 8.16 implies that, in the notation of [113],

$$\begin{aligned} \check{h}_{k+1}^i &= \mathrm{I}2\sqrt{2}r^{-\frac{1}{2}-\frac{1}{r}}\mathrm{J}^i(2r)^k[\zeta^k]\exp\left(-\sum_{i=1}^{\infty}\frac{B_{i+1}(\frac{1}{r})}{i(i+1)}\zeta^i\right) \\ &= \mathrm{I}\sqrt{2r}r^{-1-\frac{1}{r}}(-2r)^{k+1}[\zeta^k]\left(-R_i^{\mathbf{1}}(-\zeta)\right) . \end{aligned} \tag{8.74}$$

Lemma 8.17 implies that, also in notation of [113],

$$\check{B}_{0,k}^{ji} = -(2r)^{k+1}[\zeta^{k+1}]R_i^j(\zeta) = (-2r)^{k+1}[\zeta^k]\left(\frac{1-R(-\zeta)}{\zeta}\right). \tag{8.75}$$

The vertex labelled by $\langle\prod_{i=1}^p\tau_{a_i}\rangle_q$ and an extra index i is also multiplied by (again in the notation of [113])

$$(-2h_1^i)^{2-2q-p} = (-\mathrm{I}2\sqrt{2}r^{-\frac{1}{2}-\frac{1}{r}}\mathrm{J}^i)^{2-2q-p}. \tag{8.76}$$

This all together (including the factors $(-2r)^{d_n+\frac{1}{2}}$ that we have on the global leaves in Equation (8.69)) gives the following extra factor for the vertex labelled by $\langle\prod_{i=1}^p\tau_{a_i}\rangle_q$, with p_d attached dilaton leaves and p_o ordinary leaves and/or half-edges ($p=p_d+p_o$), and an extra index i:

$$\frac{(-2r)^{a_1+\cdots+a_p}(\mathrm{I}\sqrt{2r})^p r^{-p_d-\frac{p_d}{r}}}{(-\mathrm{I}2\sqrt{2}r^{-\frac{1}{2}-\frac{1}{r}}\mathrm{J}^i)^{2q+p-2}} = r^{(1+\frac{1}{r})(2q+p_o-2)}r^{(2q+p-2)}\mathrm{J}^{(2q+p-2)i} \tag{8.77}$$

(we used that $a_1+\cdots+a_p = 3q-3+p$). Note that the factor $r^{(2q+p-2)}\mathrm{J}^{(2q+p-2)i}$ coinsides with the weight that we have in the Givental's presentation of Chiodo's formula, cf. Equation (8.73). Meanwhile, the sum of the exponents $2q+p_o-2$ over all vertices in a graph of genus g with n global leaves is equal to $2g+n-2$. Therefore, the product of the factors $r^{(1+\frac{1}{r})(2q+p_o-2)}$ over all vertices of a graph is exactly equal to the extra factor $r^{2g+n-2+\frac{2g+n-2}{r}}$ that we have on the right-hand side of Equation (8.69). $\square$

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