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A DISPERSION THEORETIC STUDY OF PION FORM FACTOR*

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ABSTRACT

We show that the pion electromagnetic form factor satisfies a nonsubtracted dispersion relation, if one of the pions is taken to be massless. We point out that by quantizing the theory on the light-cone the mass extrapolation ambiguity due to one of the pions being massless can be overcome. We then use this result to establish an upper bound on the pion's charge radius.

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1. Introduction

The asymptotic dependence of the elastic form factors of hadrons plays a critical role in theories of large-angle scattering [1]. Physically the elastic form factor is the probability amplitude for a hadron to remain a single hadron after the transfer of momentum. Thus, in several models [2], the falloff of the exclusive scattering amplitude at large momentum transfer is controlled by the same physics that controls the falloff of the form factors.

The pion form factor, to which we will restrict ourselves in this work, is not that well known experimentally. But both timelike data (from the $e^+e^- \to \pi^+\pi^-$ experiment [3], Fig. 1) and spacelike data (from the $ep \to e\pi p$ experiment [4], Fig. 2) are consistent with a falloff, $F_\pi(t) \simeq t^{-1}$, or slightly faster. A previous important work [5], using a rigorous data analysis technique, shows that, if the asymptotic falloff is $F_\pi(t) \simeq t^{-n}$, on the average, for $|t| > 2\ \mathrm{GeV}^2$, then $n < 1.2 \pm 0.3$. Thus the pion form factor cannot fall asymptotically faster than $t^{-3/2}$.

There have been several attempts to understand the asymptotic behavior of the form factor [6, 7]. Of course to have an understanding of this an exact knowledge of the short distance structure of the hadrons is needed. Assuming the constituent form factor to be pointlike, Brodsky et al. [6] predict $F(t) \underset{t\to\infty}{\sim} t^{1-n}$ for the asymptotic dependence of the hadron form factor containing n elementary constituents. Physically this rule allows a factor t^{-1} for each additional quark line, which changes direction from along p to along p+q, where $t = q^2$. This model predicts $F_\pi(q^2) \sim q^{-2}$. Polyakov and Migdal [7] also get $F_\pi(q^2) \sim q^{-2}$ within a conformally invariant field theoretic scheme.

This work is yet another attempt to understand the asymptotic behavior of the pion form factor in a model-independent way. The price we pay is that our prediction is not as detailed as that of Ref. [6], for example, but certainly consistent.

2. Dispersion Relation for $F_\pi(t)$

The pion electromagnetic form factor nonrelativistically can be thought of as the Fourier transform of the radial charge distribution. Therefore the behavior of the form factor at $q^2 \to \infty$ corresponds to the charge structure around $\vec{x} = 0$ (deep inside). In the dispersion theoretic analysis it is the asymptotic behavior which determines the number of subtractions needed. By definition the electromagnetic form factor is proportional to the matrix element of the electromagnetic current between the vacuum and a $\pi^+\pi^-$ state, restricted by Lorentz invariance:

$$\begin{aligned} F_\mu(q^2, k_+^2, k_-^2) &\equiv \langle \pi^+(k^+), \pi^-(k^-) | J_\mu^{em}(0) | 0 \rangle \\ &= F_-(q^2, k_+^2, k_-^2) p_\mu + F_+(q^2, k_+^2, k_-^2) q_\mu \end{aligned} \quad (2.1)$$

where $q = k_+ + k_-$ and $p = k_+ - k_-$. Due to electric charge conservation, F_+ and F_- are not independent but related as

$$F_+ = -\left(\frac{q \cdot p}{q^2}\right) F_- .$$

Thus the pion electromagnetic form factor which satisfies Lorentz invariance and gauge invariance properties can be written in the general form

$$F_\mu(q^2, k_+^2, k_-^2) = \left(p_\mu - \frac{q \cdot p}{q^2} q_\mu\right) F(q^2, k_+^2, k_-^2) \quad (2.2)$$

If the form factor under consideration is physical, i.e., if both pions are on the mass-shell, then $q \cdot p = k_+^2 - k_-^2 = 0$, and we get back the familiar result, $F_\mu = p_\mu F$.

Now let us assume that $F(t, k_+^2, k_-^2)$, thought of as a function of $q^2 = t$, satisfies a once-subtracted dispersion relation. Choosing the subtraction point at $t = 0$, and using the charge renormalization relation, $F(0) = 1$, we can write

(2.3) $$F(t,k_+^2,k_-^2) = 1 + \frac{t}{\pi} \int_{4\mu^2}^{\infty} dt' \frac{\operatorname{Im} F(t',k_+^2,k_-^2)}{t'(t' - t + i\epsilon)}$$

In writing (2.3) we used the well-known [8] fact that

(a) $F_\pi(t)$ is an analytic function of t, in the complex t-plane, with a cut on the positive real axis from $4\mu^2$ to ∞ (μ is the pion mass), and

(b) $F_\pi(t)$ is real on the negative real axis, due to the hermiticity of the electromagnetic current.

Next we shall show that, by assuming one of the pions, say π^-, massless,

$$1 = \frac{1}{\pi} \int_{4\mu^2}^{\infty} dt' \frac{\operatorname{Im} F(t',k_+^2,k_-^2)}{t'} \quad ;$$

thus $F(t,\mu^2,0)$ satisfies a nonsubtracted dispersion relation. Later we shall give arguments in support of the fact that extrapolation from the unphysical point $k_-^2 = 0$ to physical point $k_-^2 = \mu^2 = 0.02\ \text{GeV}^2$ does not change the behavior of the form factor significantly.

Since we have Lorentz invariance, we can pick any frame which is most convenient for computational purposes. We shall choose the frame in which π^+ is at rest: $k_+ = (\mu,0,0,0)$. Since π^- is massless, then $k_- = (E,0,0,-E)$. The relevant physical quantities are

(2.4) $$q^2 = (k_+ + k_-)^2 = \mu^2 + 2\mu E$$
$$q \cdot p = k_+^2 - k_-^2 = \mu^2$$

Multiplying both sides of Eq. (2.1) by k_-, we get

(2.5) $$EF_{0+3} = \mu E\left(\frac{q^2 - \mu^2}{q^2}\right) F(t,\mu^2,0)$$

or

(2.6) $$F(t,\mu^2,0) \underset{t>>\mu^2}{\simeq} \frac{1}{\mu} F_{0+3}(t,\mu^2,0) \;.$$

Since F(t) has an imaginary part only in the timelike region $t > 4\mu^2$, the condition $t>>\mu^2$ is satisfied, to a very good approximation, along the physical cut, and we can write

(2.7) $$F(t,\mu^2,0) = \frac{1}{\mu} <\pi^+(k^+),\ \pi^-(k^-)|J^{em}_{0+3}|0> \;.$$

Now let us reduce massless π^- in (2.1), by using the standard LSZ-reduction technique.

(2.8) $$F_\mu = i\int d^4x\, e^{ik_-\cdot x} (\Box+\mu^2) <\pi^+(k^+)|T(\phi^-(x)J^{em}_\mu(0))|0> \;.$$

Using PCAC-theorem, we can relate the pion interpolating field to the divergence of an axial vector current:

(2.9) $$F_\mu = i\kappa\mu^2\int d^4x\, e^{iE(x_0+x_3)} <\pi^+|T(\partial_\nu A^\nu(x)J^{em}_\mu(0))|0> \,.$$

The imaginary part of F_μ effectively comes from the $\theta(x_0)$ term:

(2.10) $$\text{Im}\, F_\mu(t,\mu^2,0) = \frac{1}{2} i\kappa\mu^2\int d^4x\, e^{iE(x_0+x_3)} <\pi^+|[\partial_\nu A^\nu(x)\; J^{em}_\mu(0)]|0> \,.$$

From (2.6) and (2.10) it is clear that light-cone (LC) coordinates [9] are most suitable for carrying out this discussion further:

(2.11) $$A^\pm = \frac{1}{\sqrt{2}}(A^0 \pm A^3)$$
$$A^i = (A^1, A^2) \equiv \vec{A}_\perp$$
$$A^2 \equiv 2A^+A^- - \vec{A}_\perp^{\,2} \,.$$

It is not only computational advantage that suggests the use of LC coordinates. It is clear from the beautiful analysis of Ref. 10 that LC quantization is especially useful for the soft-pion problems, which involve mass extrapolations. Their most important result relevant to our analysis is that "When quantized on light-like hyperplanes, fields with different masses become unitarily equivalent (whereas they are inequivalent on space-like planes)". This statement means that our results would not depend on π^- being massless. So there is not any extrapolation ambiguity involved in our results. In LC coordinates, (2.10) becomes

$$(2.12) \quad \mathrm{Im} F_\mu(t,\mu^2,0) = i\kappa\mu^2 \int d^2x_\perp \, d\tau \, dz \, e^{i\sqrt{2}E\tau} \langle \pi^+ | [\partial_\nu A^\nu(x), J_\mu^{em}(0)] | 0 \rangle$$

Here

$$(2.11') \qquad \tau = \frac{1}{\sqrt{2}}(x_0 + x_3), \text{ and } z = \frac{1}{\sqrt{2}}(x_0 - x_3)$$

and

$$(2.12') \qquad \partial_\nu A^\nu = \frac{\partial A^+}{\partial \tau} + \frac{\partial A^-}{\partial z} - \vec{\nabla}_\perp \cdot \vec{A}_\perp$$

It is only the first term in (2.12') which contributes after partial integration; the second term vanishes after partial integration, and the third term vanishes as a result of the two-dimensional Gauss theorem.

After these partial integrations we get

$$(2.13) \quad \mathrm{Im}\, F(t,\mu^2,0) = \frac{1}{\mu} \mathrm{Im}\, F_{0+3}(t,\mu^2,0)$$
$$= \frac{\mu E}{\sqrt{2}} \int d^2x_\perp \, d\tau \, dz \, e^{i\sqrt{2}E\tau} \langle \pi^+ | [A^+(\tau, z, x_\perp), J^+(0)] | 0 \rangle ,$$

where we have dropped the internal indices, which are (1-i2) on A, and (em) on J. Multiplying both sides of (2.13) by $\frac{dt}{t} = \frac{dE}{E}$ and integrating (and recalling that J^{em} is purely vector), we get

(2.14) $\int \frac{dt}{t} \operatorname{Im} F(t,\mu^2,0) = \kappa\mu\pi \int d^2x_\perp \, d\tau \, dz \, <\pi^+ | [A^+(0,z,x_\perp), V^+(0)] | 0>$.

The commutator in the right-hand side is computed in several models [11]. It is the simplest commutator which is independent of the details of the models, and, up to Schwinger terms (putting in the right internal indices), is given by

(2.15) $[A^+_{(1-i2)}(\tau,z,x_\perp), V^+_{(0+3)}]\delta(\tau) = \delta(\tau)\delta(z)\delta^2(x_\perp)A^+_{(1-i2)}(\tau,z,x_\perp)$.

Substituting (2.15) in (2.14), we get

(2.16) $\int \frac{dt}{t} \operatorname{Im} F(t,\mu^2,0) = \kappa\mu\pi <\pi^+ | A^+_{(1-i2)}(0) | 0> = \pi\kappa f_\pi \mu^2$.

Putting in $\kappa = 1/f_\pi \mu^2$, we finally obtain the desired result:

(2.17) $$\int \frac{dt}{t} \operatorname{Im} F(t,\mu^2,\mu^2) = \pi .$$

In the last step we made the extrapolation back to $k_-^2 = \mu^2$ without further ado, in the light of the arguments given above [10].

So we have proven that the pion form factor does not need any subtraction. The most immediate mathematical implication of this result is

(2.18) $$|\operatorname{Im} F(t)| \xrightarrow[|t| \to \infty]{} 0 .$$

That we do not get a more detailed prediction than this is understandable in view of the fact that no explicit model is used in getting our main result (2.17), other than some very general field theoretic theorems. Of course our result (2.17) or (2.18) is consistent with that of Brodsky et al. [6], which predicts $F_\pi(t) \sim 1/t$. Especially their very transparent physical arguments make it clear that only integer powers in $F(t) \sim t^{-n}$ make sense. Therefore we can claim without further reservation that our result (2.17) means $F_\pi(t) \sim t^{-1}$, which is supported beautifully by the data. (See Figs. 1 and 2.)

3. An Upper Bound for Pion's Charge Radius

The absolute bounds on strong interaction amplitudes are usually derived from the nonlinear character of the unitarity relations [12]. Weak and electromagnetic amplitudes are also subject to unitarity constraints. To first order in perturbation theory, the unitarity relation is linear, and hence provides no absolute bounds on the amplitudes. Because unitarity provides no absolute bounds to the first order weak or electromagnetic processes, one can find relative bounds in terms of other processes to first order in coupling [13]. Obviously this method does not have comparable rigor to the absolute bounds already existing in the literature [14]. It may only be thought of as an amusing elementary application of the results obtained in the previous section to get a rough order of magnitude estimate.

We shall find an upper bound for pion charge radius in terms of the e^+e^- annihilation total cross section, a phenomenon which became the focus of high energy physics recently, after the very exciting discoveries of the ψ family and charmed mesons [15].

Let us recall that the photon spectral function is defined as [16]

$$(3.1)\quad J_{\mu\nu} = \sum_n (2\pi)^3\delta^4(p_n - q) \langle 0|J_\mu^{em}(0)|n\rangle\langle n|J_\nu^{em}(0)|0\rangle \equiv \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right)J(q^2).$$

$J(q^2)$ is by definition a positive definite function, and therefore $J(q^2) > J_{(2\pi)}(q^2)$ for example. Calculating the contribution of the 2π-state, we get

$$(3.2)\qquad J_{(2\pi)}(q^2) = \frac{1}{6\pi^2} q^2 |F_\pi(q^2)|^2 \left[\frac{q^2 - 4\mu^2}{4q^2}\right]^{3/2} \theta(q^2 - 4\mu^2)$$

$J(q^2)$ is always greater than this; therefore

$$(3.3)\qquad |F_\pi(t)|^2 \leq 6\pi^2\left(\frac{4t}{t-4\mu^2}\right)^{3/2} \frac{J(t)}{t} .$$

Now let us employ the result of the previous section and write a dispersion relation for $F_\pi(t)$ without any subtraction. From (3.3) it is clear that

$$(3.4) \qquad |\mathrm{Im}\ F(t)| \leq |F(t)| \leq \sqrt{6\pi}\left(\frac{4t}{t-4\mu^2}\right)^{3/4}\left(\frac{J(t)}{t}\right)^{1/2} .$$

We wish to compute our bounds in the spacelike region; therefore we shall understand that $t < 0$ in the following. From (2.17) and (3.4)

$$(3.5) \quad |F(t)| \leq \frac{1}{\pi}\int_{4\mu^2}^{\infty} \frac{|\mathrm{Im}\ F(t')|}{t'+|t|}\, dt' \leq \sqrt{\frac{6}{\pi}}\int_{4\mu^2}^{\infty} \frac{dt'}{t'+|t|}\left(\frac{4t'}{t'-4\mu^2}\right)^{3/4}\left(\frac{J(t')}{t'}\right)^{1/2} .$$

There exists a famous relation [15] between the total e^+e^- annihilation cross section (in the one photon exchange approximation) and the spectral function:

$$(3.6) \qquad \sigma_T(t) = 2\pi e^2 \frac{J(t)}{t^2} .$$

Substituting this in the right-hand side of (3.5), we get

$$(3.7) \qquad |F(t)|_{t<0} \leq \frac{\sqrt{3}}{\pi e}\int_{4\mu^2}^{\infty} \frac{dt'}{t'+|t|}\left(\frac{4t'}{t'-4\mu^2}\right)^{3/4}\sqrt{t'\sigma_T(t')} .$$

Before proceeding further, let us make a mathematical consistency check of this inequality. That $F_\pi(t)$ satisfies a nonsubtracted dispersion relation means $F_\pi(t) \underset{|t|\to\infty}{\longrightarrow} 0$. Also, since the potentially divergent region of the integration is $t' \simeq 4\mu^2$, we can take the limit $|t| \to \infty$ under the integral sign and get

$$(3.8) \quad \frac{1}{|t|}\int_{4\mu^2}^{\infty} \frac{dt'}{t'}\left(\frac{4t'}{t'-4\mu^2}\right)^{3/4}\sqrt{t'\sigma_T(t')} \underset{|t|\to\infty}{\geq} 0 ,$$

or, since the coefficient of the $\frac{1}{|t|}$ term should be finite, we should have

$$(3.9) \qquad \int_{4\mu^2}^{\infty} \frac{dt'}{t'}\left(\frac{4t'}{t'-4\mu^2}\right)^{3/4}\sqrt{t'\sigma_T(t')} \geq 1 .$$

For scale invariant theories [17],

$$(3.10) \qquad t\sigma_T(t) \underset{t\to\infty}{\longrightarrow} \text{const.}$$

Thus the integral in (3.9) diverges logarithmically, and this inequality is trivially satisfied. A slight decrease like $t\sigma_T(t) \underset{|t|\to\infty}{\simeq} t^{-\epsilon}$ (where $\epsilon > 0$ is an arbitrarily small number), which cannot be ruled out by the present data [15], would render this integral finite, and, to satisfy this inequality, a careful adjustment of the parameters is needed.

In the nonrelativistic interpretation, the pion's average charge radius is given by $\langle r_\pi^2 \rangle = -6F'_\pi(t)\big|_{t=0}$, in the normalization $F_\pi(0) = 1$. Taking the derivative of (3.7), we get

$$(3.11) \qquad \langle r_\pi^2 \rangle \leq \left(\frac{6\sqrt{3}}{\pi e}\right) \int_{4\mu^2}^{\infty} \frac{dt'}{t'^2} \left(\frac{4t'}{t'-4\mu^2}\right)^{3/4} \sqrt{t'\sigma_T(t')}$$

which can be fitted by the present e^+e^- data to get a rough estimate for $\langle r_\pi^2 \rangle$. We are not going to follow this route, since there are already very rigorous absolute bounds on $\langle r_\pi \rangle$ [12]. Or we can look at (3.11) as a sum rule for $t\sigma_T(t)$ by putting the best experimental value for $\langle r_\pi^2 \rangle$ in the left-hand side.

4. Discussion

The weakest point in the above very general field theoretic arguments probably was taking one of the pions massless. Though this made it possible to use PCAC theorem without hesitation, it also brought up the usual headache of mass extrapolation from $k_-^2 = 0$ to $k_-^2 = 0.02\ \text{GeV}^2$. Fortunately, due to a very important theorem that "different mass theories are unitarily equivalent, when quantized on light-like slabs," this is not a problem at all, and our result that the dispersion relation for pion form factor does not need a subtraction is free from extrapolation ambiguities.

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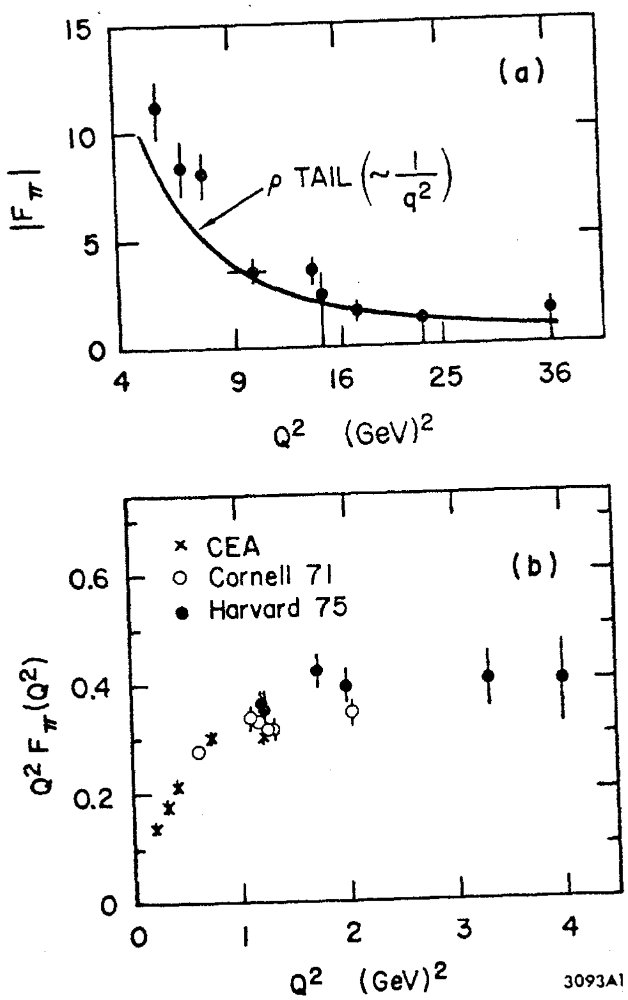
Figure Captions

1. The pion form factor in the timelike region. (From Ref. 3)
2. The pion form factor multiplied by t in the spacelike region. (From Ref. 4)



15
10
5
0
(a)
|F$_\pi$|
ρ TAIL (~ 1/q^2)
4
9
16
25
36
Q^2 (GeV)^2
× CEA
○ Cornell 71
● Harvard 75
(b)
0.6
0.4
0.2
0
Q^2 F$_\pi$(Q^2)
0
1
2
3
4
Q^2 (GeV)^2

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