

Integrable Cosmological Models in Diverse Dimensions

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Abstract. Recent results on integrable cosmological models with different matter sources in DD and 4D are presented and their main properties related to basic problems of modern cosmology are described, with billiards, S-branes, with arbitrary potentials and possible variations of effective G in particular.

The aim of multidimensional models study now is the solving of basic problems of cosmology and black hole physics and elaboration of possible observable effects of these theories due to extra dimensions, scalar fields and p-branes (acceleration and coincidence problems, possible variation of constants, non-singular models, etc.)

At present the main motivation for studying multidimensional models of gravitation and cosmology [2, 3, 4] is the unification of all known fundamental physical interactions: electromagnetic, weak, strong and gravitational ones and a new cosmological challenge of the universe acceleration, dark matter and dark energy. During the recent decades there has been significant progress in unifying weak and electromagnetic interactions, and some more modest achievements in GUT, supersymmetric, string and superstring theories.

Now, theories of membranes, p -branes and M-theory are studied intensively. At the moment there are no a self-consistent successful theory of unification, hence, it is desirable to study the common features of these theories and their applications.

Multidimensional gravitational models, as well as scalar-tensor theories of gravity, are theoretical frameworks for describing possible temporal and range variations of fundamental physical constants [1, 5, 6, 7]. These ideas originated from the earlier papers of Milne (1935) and Dirac (1937) on relations between the phenomena of micro- and macro-worlds, and up until now they have been thoroughly studied both theoretically and experimentally.

Moreover, there appeared some data on possible variations of the fine structure constant.

In our approach we adopt a general class of models based on multidimensional Einstein equations with or without sources of a different nature: cosmological constant, perfect and viscous fluids, scalar and electromagnetic fields, fields of antisymmetric forms (related to p -branes), etc. Our main objective is to obtain exact self-consistent solutions for these models and then to analyze them in cosmological and spherically symmetric cases. In our view this is natural and most reliable way to study highly nonlinear systems. Here we review only our exact solutions with scalar fields and antisymmetric forms.

The history of the multidimensional approach begins with the well known papers of Kaluza and Klein on five-dimensional theories. These ideas were continued by Jordan who suggested

considering the more general case $g_{55} \neq \text{const}$, leading to a theory with an additional scalar field. They inspired a well-known paper of Brans and Dicke on a scalar-tensor gravitational theory. After their work many investigations were performed models with material or fundamental scalar fields, both conformal and non-conformal (see, for example, [1]).

A revival of the ideas of many dimensions started in the 1970s and continues now, mainly due to the development of unified theories. In the 1970s interest in multidimensional gravitational models was stimulated mainly by (i) the ideas of gauge theories leading to a non-Abelian generalization of the Kaluza-Klein approach and (ii) by supergravitational theories. In the 1980s the supergravitational theories were extended by superstring models. After 1995 some expectations are connected with the hypothetical M-theory. In all these theories, four-dimensional gravitational models with extra fields were obtained from some multidimensional model by a dimensional reduction based on the decomposition of the manifold

$$M = M^4 \times M_{\text{int}},$$

where M^4 is a four-dimensional manifold and M_{int} is some internal manifold or orbifold.

The large variety of papers on multidimensional gravity and cosmology dealt with multidimensional Einstein equations in vacuum or with different matter sources with a block-diagonal cosmological or spherically symmetric metric, defined on the manifold $M = \mathbf{R} \times M_0 \times \dots \times M_n$ of the form

$$g = -dt \otimes dt + \sum_{r=0}^n a_r^2(t) g^r$$

where (M_r, g^r) are Einstein spaces, $r = 0, \dots, n$. In some of them a cosmological constant and simple scalar fields were also used.

Such models are usually reduced to pseudo-Euclidean Toda-like systems with the Lagrangian

$$L = \frac{1}{2} G_{ij} \dot{x}^i \dot{x}^j - \sum_{k=1}^m A_k e^{u_k^i x^i}$$

and the zero-energy constraint $E = 0$. These systems are not well studied yet. Nevertheless there exists a special class of equations of state that gives rise to Euclidean Toda models [?].

It is well known that cosmological solutions are closely related to the solutions exhibiting spherical symmetry, and relevant schemes to obtain these solutions are quite similar to those applied in the cosmological approach [2]. The first multidimensional generalization of such a type was considered by Kramer. In our papers (see [2, 3]) first the Schwarzschild solution was generalized to the case of n internal Ricci-flat spaces, showing that a black hole configuration takes place when the scale factors of internal spaces are constants. Another important feature was that a minimally coupled scalar field is incompatible with the existence of black holes. Additionally, an analogous generalization of the Tangherlini solution was obtained, and an investigation of singularities was performed. These solutions were also generalized by us to the electrovacuum case with and without a scalar field, where it was proved that BHs exist only when a scalar field is switched off. Deviations from the Newton and Coulomb laws were obtained depending on mass, charge and number of dimensions.

A theorem was proved in that “cuts” all non-black-hole configurations as being unstable under even monopole perturbations. The extremely charged dilatonic black hole solution was generalized to a multicenter (Majumdar-Papapetrou) case, when the cosmological constant is non-zero.

We note that for $D = 4$ the pioneering Majumdar-Papapetrou solutions with a conformal scalar field and an electromagnetic field were considered in our papers in 1970 (see [1]).

1. Exact solutions with "branes"

Later many classes of the exact solutions for the multidimensional gravitational model governed by the Lagrangian

$$\mathcal{L} = R[g] - 2\Lambda - h_{\alpha\beta}g^{MN}\partial_M\varphi^\alpha\partial_N\varphi^\beta - \sum_a \frac{1}{n_a!} \exp(2\lambda_{a\alpha}\varphi^\alpha)(F^a)^2, \quad (1)$$

were considered. Here g is metric, $F^a = dA^a$ are forms of ranks n_a and φ^α are scalar fields and Λ is a cosmological constant (the matrix $h_{\alpha\beta}$ is invertible).

Supergravities. For certain field contents with distinguished values of total dimension D , ranks n_a , dilatonic couplings λ_a and $\Lambda = 0$ such Lagrangians appear as "truncated" bosonic sectors (i.e. without Chern-Simons terms) of certain supergravitational theories or low-energy limit of superstring models. For $D = 11$ supergravity (that is considered now as a low-energy limit of a conjectured M -theory we have a metric and 4-form in the bosonic sector. For $D = 10$ one may consider type I supergravity with metric, scalar field and 3-form, type IIA supergravity with bosonic fields of type I supergravity called as Neveu-Schwarz-Neveu-Schwarz (NS-NS) sector and additionally 2-form and 4-form Ramond-Ramond (R-R) sector, type IIB supergravity with bosonic fields of type I supergravity (NS-NS sector) and additionally 1-form, 3-form and (self-dual) 5-form (R-R sector). It is now believed that all five string theories (I, IIA, IIB and two heterotic ones with gauge groups $G = E_8 \times E_8$ and $\text{Spin}(32)/Z_2$) as well as 11-dimensional supergravity are limiting case of M -theory. All these theories are conjectured to be related by a set of duality transformations: $S-$, $T-$ (and more general $U-$) dualities.

It was proposed earlier that IIB string may have its origin in a 12-dimensional theory, known as F -theory, for which a low energy effective (bosonic) Lagrangian for was also suggested. The field content of this 12-dimensional field model is the following one: metric, one scalar field (with negative kinetic term), 4-form and 5-form. In our work [8] a chain of so-called B_D -models in dimensions $D = 11, 12, \dots$ was suggested. B_D -model contains $l = D - 11$ scalar fields with negative kinetic terms (i.e. scalar fields are so-called "phantom" fields) coupled to $(l+1)$ different forms of ranks $4, \dots, 4+l$. These models were constructed using p -brane intersection rules that will be discussed below. For $D = 11$ ($l = 0$) B_D -model coincides with the truncated bosonic sector of $D = 11$ supergravity. For $D = 12$ ($l = 1$) it coincides with truncated $D = 12$ model of F -theory. It was conjectured by us in [8] that these B_D -models for $D > 12$ may correspond to low energy limits of some unknown F_D -theories (analogues of $M-$ and F -theories).

2. Description of the models

In our reviews [37, 4, 45] certain classes of p -brane solutions to field equations corresponding to the Lagrangian (1) were presented.

These solutions have a block-diagonal metrics defined on D -dimensional product manifold, i.e.

$$g = e^{2\gamma}g^0 + \sum_{i=1}^n e^{2\phi^i}g^i, \quad M_0 \times M_1 \times \dots \times M_n, \quad (2)$$

where g^0 is a metric on M_0 and g^i are fixed Ricci-flat (or Einstein) metrics on M_i ($i > 0$). The moduli γ, ϕ^i and scalar fields φ^α are functions on M_0 and fields of forms are also governed by several scalar functions on M_0 . Any F^a is supposed to be a sum of monoms, corresponding to electric or magnetic p -branes (p -dimensional analogues of membranes), i.e. the so-called composite p -brane ansatz was considered (in non-composite case we have no more than one monom for each F^a); $p = 0$ corresponds to a particle, $p = 1$ to a string, $p = 2$ to a membrane etc. The p -brane worldvolume (worldline for $p = 0$, worldsurface for $p = 1$ etc) is isomorphic to some product of internal manifolds: $M_I = M_{i_1} \times \dots \times M_{i_k}$ where $1 \leq i_1 < \dots < i_k \leq n$ and has dimension $p + 1 = d_{i_1} + \dots + d_{i_k} = d(I)$, where $I = \{i_1, \dots, i_k\}$ is a multiindex describing the

location of p -brane and $d_i = \dim M_i$. Any p -brane is described by the triplet (p -brane index) $s = (a, v, I)$, where a is the color index labeling the form F^a , $v = e$ (*lectric*), m (*agnetic*) and I is the multiindex defined above. For the electric and magnetic branes corresponding to form F^a the worldvolume dimensions are $d(I) = n_a - 1$ and $d(I) = D - n_a - 1$, respectively. The sum of this dimensions is $D - 2$. For $D = 11$ supergravity we get $d(I) = 3$ and $d(I) = 6$, corresponding to electric $M2$ -brane and magnetic $M5$ -brane .

3. Sigma model representation

In our paper [11] the model under consideration was reduced to gravitating self-interacting sigma-model with certain constraints imposed. The sigma-model representation for non-composite electric case was obtained earlier in [9, 10], for electric composite case see also [12]). Recently, sigma model representation for non-block-diagonal metrics and two (intersecting) branes was obtained in [20].

The σ -model Lagrangian has the form [11]

$$\mathcal{L}_\sigma = R[g^0] - \hat{G}_{AB} g^{0\mu\nu} \partial_\mu \sigma^A \partial_\nu \sigma^B - \sum_s \varepsilon_s \exp(-2U^s) g^{0\mu\nu} \partial_\mu \Phi^s \partial_\nu \Phi^s - 2V, \quad (3)$$

where $(\sigma^A) = (\phi^i, \varphi^\alpha)$, V is a potential, (\hat{G}_{AB}) are components of (truncated) target space metric, $\varepsilon_s = \pm 1$,

$$U^s = U_A^s \sigma^A = \sum_{i \in I_s} d_i \phi^i - \chi_s \lambda_{a_s \alpha} \varphi^\alpha$$

are linear functions, Φ^s are scalar functions on M_0 (corresponding to forms), and $s = (a_s, v_s, I_s)$. Here parameter $\chi_s = +1$ for the electric brane ($v_s = e$) and $\chi_s = -1$ for the magnetic one ($v_s = m$).

A pure gravitational sector of the sigma-model was considered earlier in [34] and in our paper [35]. For p -brane applications g^0 is Euclidean, (\hat{G}_{AB}) is positive definite (for $d_0 > 2$) and $\varepsilon_s = -1$, if pseudo-Euclidean (electric and magnetic) p -branes in a pseudo-Euclidean space-time are considered. The sigma-model (3) may be also considered for the pseudo-Euclidean metric g^0 of signature $(-, +, \dots, +)$ (e.g. in investigations of gravitational waves). In this case for a positive definite matrix (\hat{G}_{AB}) and $\varepsilon_s = 1$ we get a non-negative kinetic energy terms.

Brane U -vectors. The co-vectors U^s play a key role in studying the integrability of the field equations [11, 16, 30] and possible existence of stochastic behaviour near the singularity, see our paper [36]. An important mathematical characteristic here is the matrix of scalar products $(U^s, U^{s'}) = \hat{G}^{AB} U_A^s U_B^{s'}$, where $(\hat{G}^{AB}) = (\hat{G}_{AB})^{-1}$. The scalar products for co-vectors U^s were calculated in [11] (for electric case see [9, 10, 12])

$$(U^s, U^{s'}) = d(I_s \cap I_{s'}) + \frac{d(I_s)d(I_{s'})}{2-D} + \chi_s \chi_{s'} \lambda_{a_s \alpha} \lambda_{a_{s'} \beta} h^{\alpha\beta},$$

where $(h^{\alpha\beta}) = (h_{\alpha\beta})^{-1}$; $s = (a_s, v_s, I_s)$, $s' = (a_{s'}, v_{s'}, I_{s'})$. They depend upon brane intersections (first term), dimensions of brane worldvolumes and total dimension D (second term), scalar products of dilatonic coupling vectors and electro-magnetic types of branes (third term). As will be shown below the so-called ‘‘intersections rules’’ (i.e. relations for $d(I_s \cap I_{s'})$) are defined by scalar products of U^s -vectors.

4. Cosmological and spherically symmetric solutions

A family of general cosmological type p -brane solutions with n Ricci-flat internal spaces was considered in our paper [30], where also a generalization to the case of $(n - 1)$ Ricci-flat spaces and one Einstein space of non-zero curvature (say M_1) was obtained. These solutions are defined up to solutions to Toda-type equations and may be obtained using the Lagrange dynamics

following from our sigma-model approach [8]. The solutions from [30] contain a subclass of spherically symmetric solutions (for $M_1 = S^{d_1}$). Special solutions with orthogonal and block-orthogonal sets of U -vectors were considered earlier in our works [8] and [25, 26] respectively (for non-composite case, see [14, 15]) and references therein).

5. Toda solutions

In [8] the reduction of p -brane cosmological type solutions to Toda-like systems was first performed (see also [30]). General classes of p -brane solutions (cosmological and spherically symmetric ones) related to Euclidean Toda lattices associated with Lie algebras (mainly \mathbf{A}_m , \mathbf{C}_m ones) were obtained by us [24, 28, 29, 30, 32, 33]. Special p -brane configurations were considered earlier by some other authors.

In [41] a class of space-like brane (S -brane) solutions (related to Toda-type systems) with product of Ricci-flat internal spaces was considered. S -brane solutions with special orthogonal intersection rules were considered also in [45, 47, 48] and solutions with accelerated expansion (e.g. with power-law and exponential behaviour of scale factors) were singled out.

6. Black brane solutions

In our papers [32, 33] a family of spherically-symmetric solutions from [30] was investigated and a subclass of black-hole configurations related to Toda-type equations, with certain asymptotical conditions imposed, was singled out. These black hole solutions are governed by functions $H_s(z) > 0$, defined on the interval $(0, (2\mu)^{-1})$, where $\mu > 0$ is the extremality parameter, and obey a set of differential equations (equivalent to Toda-type ones)

$$\frac{d}{dz} \left(\frac{(1 - 2\mu z)}{H_s} \frac{d}{dz} H_s \right) = \bar{B}_s \prod_{s'} H_{s'}^{-A_{ss'}},$$

with the following boundary conditions imposed: (i) $H_s((2\mu)^{-1} - 0) = H_{s0} \in (0, +\infty)$; (ii) $H_s(+0) = 1$, $s \in S$. Here $\bar{B}_s \neq 0$ and $(A_{ss'})$ is a quasi-Cartan matrix.

In refs. [31, 32, 33] the following hypothesis was suggested: the functions H_s are polynomials when intersection rules

$$d(I_s \cap I_{s'}) = \frac{d(I_s)d(I_{s'})}{D - 2} - \chi_s \chi_{s'} \lambda_{a_s \alpha} \lambda_{a_{s'} \beta} h^{\alpha\beta} + \frac{1}{2} (U^{s'}, U^s) A_{ss'}, \quad s \neq s'. \quad (4)$$

correspond to semisimple Lie algebras, i.e. when $(A_{ss'})$ is a Cartan matrix.

Here

$$(A_{ss'}) \equiv \begin{pmatrix} 2(U^s, U^s) \\ (U^{s'}, U^s) \end{pmatrix},$$

$s, s' \in S$, is a quasi-Cartan matrix.

This hypothesis was verified for Lie algebras: \mathbf{A}_m , \mathbf{C}_{m+1} , $m = 1, 2, \dots$, in [32, 33]. It was also confirmed by special black-hole "block orthogonal" solutions considered earlier in [19, 25, 27]. An analogue of this conjecture for extremal black holes was considered by us earlier. In our papers [31, 32, 33] explicit formulas for the solution corresponding to the algebra \mathbf{A}_2 are presented. These formulas are illustrated by two examples of \mathbf{A}_2 -dyon solutions: a dyon in $D = 11$ supergravity (with $M2$ and $M5$ branes intersecting at a point) and Kaluza-Klein dyon. Extremal configurations (e.g. with multi-black-hole extension) were also obtained.

We note, that special black hole solutions with orthogonal U -vectors were considered in [13] (for non-composite case) and [8]. These solutions have analogues in models with multicomponent perfect fluid [38, 43, 44].

The black brane solution, corresponding to Lie algebras \mathbf{C}_2 and \mathbf{A}_3 were obtained by us in [39] and [40], respectively.

In [13, 19] some propositions related to i) interconnection between the Hawking temperature and the singularity behaviour, and ii) multitemporal configurations were proved.

7. Cosmological models

Scalar fields play an essential role in modern cosmology. They are attributed to inflationary models of the early universe and models describing the present stage of the accelerated expansion as well. There is no unique candidate for the potential of the minimally coupled scalar field. Typically a potential is a sum of exponents. Such potentials appear quite generically in a large class of theories: multidimensional, Kaluza-Klein models, supergravity and string/M - theories.

Single exponential potential was extensively studied within Friedmann-Robertson-Walker (FRW) model containing both a minimally coupled scalar field and a perfect fluid with the linear barotropic equation of state. The attention was mainly focussed on the qualitative behaviour of solutions, stability of the exceptional solutions to curvature and shear perturbations and their possible applications within the known cosmological scenario such as inflation and scaling ("tracking"). In particular, it was found by a phase plane analysis that for "flat" positive potentials there exists a unique late-time attractor in the form of the scalar dominated solution. It is stable within homogeneous and isotropic models with non-zero spatial curvature with respect to spatial curvature perturbations and provides the power-law inflation. For "intermediate" positive potentials a unique late-time attractor is the scaling solution, where the scalar field "mimics" the perfect fluid, adopting its equation of state. The energy-density of the scalar field scales with that of the perfect fluid. Problem of integrability by quadratures of the model was studied in our paper [55]. Four classes of general solutions, when the parameter characterizing the steepness of the potential and the barotropic parameter obey some relations, were found by applying our DD approach to 4D. Inflationary solutions and tracking ones were singled out. Solutions for the model with a scalar field and multiple potential and dust was given in [56, 58, 59], where acceleration, recollapse and tracking was obtained.

As to scalar fields with multiple exponential potential in DD, a wide class of exact solutions was obtained in our paper [42], where solutions with acceleration were singled out. In our recent work [46] a behavior near the singularity of multi-scalar model with exponential potentials was studied using a billiard approach suggested earlier in our paper of 1994. S-brane solutions with maximal number of branes and several scalar fields were studied in [61, 62]. Solutions with variable equations of state in 4D and DD are in [57]. Nonsingular (bouncing) solutions may be found in [60].

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