Geometric Quantization of Riemann Ellipsoids

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INTRODUCTION

A Riemann ellipsoid is a fluid with an ellipsoidal boundary whose velocity field is a linear function of position. The theory of Riemann ellipsoids forms a highly satisfying and coherent structure which unifies and draws upon ideas across traditional disciplinary lines:

1. Riemann ellipsoids are applied to diverse areas of physics from rotating stars to superdeformed nuclei.
2. The theory uses ideas from classical fluid mechinics, Lie groups, geometric quantization, Hamiltonian dynamics, bifurcation theory, and quantum dynamical symmetry.
3. Riemann ellipsoids may be realized as classical or quantum models of rotating systems from a single unified viewpoint.

During the 1960s, Chandrasekhar and Lebovitz systematically investigated the equilibrium and stability of rotating stars by modeling them as Riemann ellipsoids bound by their gravitational self-energy [1]. Bohr and Wheeler in their seminal 1939 paper on nuclear fission proposed viewing rapidly rotating nuclei as irrotational fluid droplets, whose stability is determined by a competition between an attractive surface tension and the Coulomb repulsion between the protons [2]. Rigid body rotation was studied for nuclei by Rosenkilde [3] and Cohen, Plasil and Swiatecki [4], who both emphasized a unified approach with astrophysics.

In 1968, Cusson initiated an independent line of research by proposing the application of classical linear velocity fields to low-energy rotational bands in nuclei [5]. However, in this domain, an explanation of rotational nuclear states must be fully quantum mechanical. By constructing the irreducible unitary representations of Lie algebras of purely collective observables, quantum models of collective motion have been derived. For rigid rotation, the rotational algebra rot(3) is relevant [6-7]. This Lie algebra is a semidirect sum of so(3) with an abelian five dimensional ideal spanned by the traceless inertia tensor, $\mathrm{rot}(3)=[\mathbb{R}^5]\mathrm{so}(3)$. For incompressible linear flow in the low energy domain, Weaver and Biedenharn introduced the irreducible unitary representations of $\mathrm{sl}(3,\mathbb{R})$, whose noncompact generator is the time rate of change of the traceless inertia tensor [8]. But, in order to allow for a potential energy which is a function of the size and deformation of the ellipsoid, $\mathrm{sl}(3,\mathbb{R})$ must be augmented with the inertia tensor itself, thereby forming the special collective motion algebra $\mathrm{scm}(3)=[\mathbb{R}^6]\mathrm{sl}(3,\mathrm{R})$ or, for compressible flow, the general collective motion algebra $\mathrm{gcm}(3)=[\mathbb{R}^6]\mathrm{gl}(3,\mathbb{R})$ [9-10]. In order to achieve compatibility with the harmonic oscillator shell model, the collective motion algebras must be extended further to the symplectic Lie algebra $\mathrm{sp}(3,\mathbb{R})$, which is the subject of M. Moshinsky's paper.

In this paper, the quantum theory of collective motion is unified with the classical theory of Riemann ellipsoids, via the group GCM(3). The classical Riemann ellipsoids live on the co-adjoint orbits of GCM(3), while the corresponding quantum models are produced by its irreducible unitary representations. The connection between the classical and quantum models is achieved by geometric quantization. This elegant unified picture is aesthically pleasing. It is similar to the case of the relativistic free particle presented by Souriau [11].

There are two important by-products of this analysis. The first is the observation that the GCM(3) Casimir invariant $\mathcal{L}^2$ is just the square of the Kelvin circulation vector, which is expected generally to be conserved. Hence, the constancy of $\mathcal{L}^2$ on the co-adjoint orbits and irreducible representations of GCM(3) is an advantageous feature of the algebraic approach with a firm basis in fluid mechanics. The second is the demonstration, apparently for the first time, that the classical

Riemann ellipsoids form a Hamiltonian dynamical system on the co-adjoint GCM(3) orbits. This is the reduced phase space of the system for which conservation of circulation has been exploited fully.

The plan of this article is to first consider linear velocity fields, establish some notation, and then show that this kinematics is characterized uniquely by the gcm(3) observables. Next, the co-adjoint orbits of GCM(3) are enumerated, the Poisson bracket defined, and the Hamiltonian dynamics of Riemann ellipsoids is presented. Finally, the geometric quantization of the co-adjoint orbits is reviewed.

KINEMATICS

A linear velocity field is defined if each infinitesimal element of the fluid centered at time t at the position vector $\underline{X}(t)$ in some inertial frame is constrained to an orbit of $Gl(3,\mathbb{R})$,

$$\underline{X}(t) = g(t)\, y_0 , \qquad g(t) \in Gl(3,\mathbb{R}) \tag{1}$$

where y_0 specifies the fluid's position at t=0. Since at time t the location of each fluid element is defined by the same matrix g(t), the fluid elements move in tandem, or collectively. The implications of the $Gl(3,\mathbb{R})$ constraint are explicated for ellipsoids by writing g as the product of two rotations, R and S, sandwiching a diagonal matrix A,

$$g = {}^{t}R\cdot A\cdot S . \tag{2}$$

Choose $R \in SO(3)$ to align the principal axes with the Cartesian axes, let $A = \mathrm{diag}(a_1,a_2,a_3)$ be composed of the lengths of the principal semi-axes, and leave $S \in SO(3)$ to determine the internal rotational motion, or vorticity. Then, the angular velocity $\underline{\omega}$ and the vortex velocity $\underline{\lambda}$ are defined by the antisymmetric matrices

$$\begin{aligned} \Omega &= \dot{R}\cdot {}^{t}R & \Lambda &= \dot{S}\cdot {}^{t}S \\ \omega_i &= 1/2\ \epsilon_{ijk}\Omega_{jk} & \lambda_i &= 1/2\ \epsilon_{ijk}\Lambda_{jk} . \end{aligned} \tag{3}$$

The components of the angular momentum $\underline{L}$ and the Kelvin circulation $\underline{\mathcal{L}}$ projected onto the axes of the body-fixed frame are given by

$$L_k=(M/5)\left[(a_i^2 + a_j^2)\omega_k - 2a_i a_j\lambda_k\right] \qquad \mathcal{L}_k=(M/5)\left[2a_i a_j\omega_k - (a_i^2 + a_j^2)\lambda_k\right] \tag{4}$$

for i,j,k cyclic.

Observables

Our goal is to show that the kinematics of Riemann ellipsoids is characterized uniquely by the gcm(3) = $[\mathbb{R}^6]gl(3,\mathbb{R})$ observables. The ideal $\mathbb{R}^6$ is spanned by the inertia tensor in the laboratory frame

$$\Omega^L_{ij} = \int \rho\, X_i\, X_j\, d^3X \ . \tag{5}$$

With respect to the body-fixed frame $\underset{\sim}{x}=R\underset{\sim}{X}$, the inertia tensor takes the form

$$\Omega_{ij} = \int \rho\, x_i\, x_j\, d^3x \ . \tag{6}$$

If the fluid has constant density ρ and total mass M,

$$\Omega^L = {}^tR\cdot\Omega\cdot R \qquad\qquad \Omega_{ij} = (M/5)\ a_i^2\ \delta_{ij} \ . \tag{7}$$

Thus, the eigenvectors of Ω^L define the instantaneous directions of the principal axes and its eigenvalues define their lengths. We conclude that the $\mathbb{R}^6$ observables give complete information about the spatial configuration of the ellipsoid.

As one would expect, the $gl(3,\mathbb{R})$ observables uniquely specify the motion of a linear velocity field. With respect to the body-fixed frame coordinate $\underset{\sim}{x} = R\underset{\sim}{X}$ and velocity $\underset{\sim}{U} = R\dot{\underset{\sim}{X}}$, the generators of $gl(3,\mathbb{R})$, known as the shear tensor, take the form

$$\mathfrak{N}_{ij} = \int \rho\, x_i\, U_j\, d^3x \ , \tag{8}$$

which is evaluated to be

$$\mathfrak{N} = (M/5)\ [A\cdot\dot{A} + A^2\cdot\Omega - A\cdot\Lambda\cdot A] \ . \tag{9}$$

The diagonal components of the shear tensor define the vibrations of the axes lengths, $\mathfrak{N}_{ii} = (M/5)a_i\dot{a}_i$, and the off-diagonal entries specify the angular momentum and circulation

$$L_k = \epsilon_{kij}\ \mathfrak{N}_{ij} \qquad\qquad \mathcal{L}_k = \epsilon_{kij}\ (\Omega^{-1/2}\cdot\mathfrak{N}\cdot\Omega^{-1/2})_{ij}, \tag{10}$$

which, in turn, define the angular and vortex velocities.

DYNAMICS

The dynamical description of Riemann ellipsoids depends upon the Lie structure of gcm(3):

$$\left\{\Omega^L_{ij},\Omega^L_{kl}\right\}=0,\quad \left\{\mathfrak{N}^L_{ij},\mathfrak{N}^L_{kl}\right\}=\delta_{il}\mathfrak{N}^L_{kj}-\delta_{jk}\mathfrak{N}^L_{il},\quad \left\{\Omega^L_{ij},\mathfrak{N}^L_{kl}\right\}=\delta_{il}\Omega^L_{jk}+\delta_{jl}\Omega^L_{ik} \tag{11}$$

This structure is utilized in two essential ways:

1. The reduced phase space of a Riemann ellipsoid is a co-adjoint GCM(3) orbit, $\mathcal{O}_{\mathcal{L}}$. The orbits are indexed by the total circulation $\mathcal{L}$.
2. Hamiltonian dynamics, as defined by the natural Poisson bracket on $\mathcal{O}_{\mathcal{L}}$, is identical to the Euler fluid equations of motion as given by Chandrasekar [1].

In order to compute the co-adjoint orbits explicitly, consider a faithful matrix representation of the Lie algebra gcm(3) by 6×6 real matrices,

$$\mathrm{gcm}(3) \cong \left\{(\Xi,X) = \begin{bmatrix} X & 0 \\ \Xi & -{}^tX \end{bmatrix} \;,\; {}^t\Xi = \Xi \right\} \tag{12}$$

where $\Omega^L_{ij} \to e_{i+3,j} + e_{j+3,i}$, $\mathfrak{N}^L_{ji} \to e_{ij} - e_{3+j,3+i}$, and e_{km} denotes the 6×6 matrix with 1 at the intersection of row k with column m and zero elsewhere. The group GCM(3) is represented by exponentiation of the gcm(3) matrices,

$$\mathrm{GCM}(3) \cong \left\{(\Delta,g) = \begin{bmatrix} g & 0 \\ \Delta\cdot g & {}^tg^{-1} \end{bmatrix} \;,\; {}^t\Delta = \Delta \right\} \tag{13}$$

with the semidirect product multiplication law

$$(\Delta_1,g_1)\cdot(\Delta_2,g_2) = (\Delta_1 + {}^tg_1^{-1}\cdot\Delta_2\cdot g_1^{-1} \;,\; g_1\cdot g_2) \;. \tag{14}$$

The dual space to the Lie algebra gcm(3) is given by the space of pairs of 3×3 real matrices

$$\mathrm{gcm}(3)^* \cong \left\{(\phi,\eta) \mid {}^t\phi = \phi \right\} , \tag{15}$$

where the linear function (ϕ,η) is defined by the pairing

$$<(\phi,\eta)\,|\,(\Xi,X)> = \frac{1}{2}\,\mathrm{tr}\,(\phi\cdot\Xi) + \mathrm{tr}(\eta\cdot X) \;. \tag{16}$$

The co-adjoint action is computed to be

$$\mathrm{Ad}^*_{(\Delta,g)}(\phi,\eta) = (g\cdot\phi\cdot{}^t g\ ,\ g\cdot\eta\cdot g^{-1}-(g\cdot\phi\cdot{}^t g)\cdot\Delta)\ . \tag{17}$$

The physical interpretation of the points of the abstract dual space is provided by the classical observables, i.e. smooth functions on the co-adjoint orbits in the dual space. In general, for a Lie algebra $\mathfrak{g}$, each element $Z \in \mathfrak{g}$ is mapped canonically into a classical observable $\Theta(Z) \in C^\infty(\mathfrak{g}^*)$,

$$\Theta(Z)(\eta) = \langle\eta|Z\rangle \tag{18}$$

for $\eta \in \mathfrak{g}^*$. Applying this general prescription to $\mathfrak{g} = gcm(3)$ gives

$$\begin{aligned}\Theta(\Omega^L_{ij})\ (\phi,\eta) &= \phi_{ij}\\ \Theta(\mathfrak{N}^L_{ij})\ (\phi,\eta) &= \eta_{ij}.\end{aligned} \tag{19}$$

Therefore, the points of the dual space are interpreted as $(\Omega^L,\mathfrak{N}^L) \in gcm(3)^*$. Unless required for clarity, distinctions between the coordinate function $\Theta(Z)$, the coordinate Z, and the Lie algebra element Z are not made explicit.

Co-adjoint Orbits

Each co-adjoint orbit of GCM(3) intersects exactly one point from the transversal

$$T = \left\{(I,\mathbb{L}) \in gcm(3)^*\ \middle|\ \mathbb{L}_{ij} = 1/2\ \mathcal{L}\ \epsilon_{ij3},\ \mathcal{L} \ge 0\right\}\ . \tag{20}$$

At each transveral point, $\Omega^L=I$, $\mathfrak{N}^L=\mathbb{L}$ and the circulation $\mathcal{L}_k=\mathcal{L}\delta_{k3}$, in dimensionless units. Thus, the net circulation uniquely distinguishes the co-adjoint orbits and provides their physical interpretation.

It has been shown by Gulshani and Rowe [12] that the Casimir invariant of gcm(3) is

$$\mathcal{L}^2 = \mathrm{tr}(\Omega^{-1}\cdot\mathfrak{N}\cdot\Omega\cdot{}^t\mathfrak{N}) - \mathrm{tr}(\mathfrak{N}^2)\ , \tag{21}$$

which is just the square of the circulation vector. Hence, the circulation must be constant on each orbit $\mathcal{O}_{\mathcal{L}}$ whose points are given by

$$\mathrm{Ad}^*_{(\Delta,g)}(I,\mathbb{L}) = (g\cdot{}^t g,\ g\cdot\mathbb{L}\cdot g^{-1}-g\cdot{}^t g\cdot\Delta)\ . \tag{22}$$

After evaluating the isotropy subgroups at the transversal points, one obtains

$$\mathcal{O}_{\mathcal{L}} = \begin{cases} GCM(3)/SO(2), \ \dim = 14, \ \mathcal{L} \neq 0 \\ GCM(3)/SO(3), \ \dim = 12, \ \mathcal{L} = 0. \end{cases} \tag{23}$$

Poisson Bracket

For any Lie algebra $\mathfrak{g}$, there is a natural Poisson bracket structure on each co-adjoint orbit

$$\left\{\theta(Y), \ \theta(Z)\right\} = \theta\left[[Y,Z]\right] \tag{24}$$

for $Y,Z \in \mathfrak{g}$.

Hamiltonian Dynamics

Using the Poisson bracket, Hamiltonian dynamics may be considered. Let the Hamiltonian H be the sum of the kinetic energy T and potential energy V. But, the kinetic energy is given by

$$T = \frac{1}{2} \int \rho \ \underset{\sim}{U} \cdot \underset{\sim}{U} \ d^3x = \frac{1}{2} \mathrm{tr}({}^t\mathfrak{N} \cdot \Omega^{-1} \cdot \mathfrak{N}), \tag{25}$$

which shows that T is a function of the gcm(3) observables [5]. Suppose that V is a scalar function of the inertia tensor, i.e. V is a function of the semi-axes lengths a_i of the ellipsoid. In this case, H is also a function of the gcm(3) generators and Hamiltonian dynamics is well-defined by

$$\dot{\Omega}^L_{ij} = \left\{\Omega^L_{ij}, \ H\right\} \qquad\qquad \dot{\mathfrak{N}}^L_{ij} = \left\{\mathfrak{N}^L_{ij}, \ H\right\} . \tag{26}$$

It has been proven by the author that this Hamiltonian system on $\mathcal{O}_{\mathcal{L}}$ is identical to the Euler equation of motion studied by Chandrasekhar [13].

GEOMETRIC QUANTIZATION

Since the geometric quantization of the co-adjoint orbits of SCM(3) has been determined by Ihrig and the author [14] and Guillemin and Sternberg [15] and the GCM(3) case is similar, only the three key results will be reviewed here. Firstly, the Bohr-Sommerfeld quantization condition demands that $\mathcal{L}$ be a nonnegative integer. Thus, the circulation is quantized. Secondly, a complex polarization is required if the circulation is nonvanishing, viz. the subalgebra spanned by $\mathbb{R}^6$ with the complex Borel subalgebra of $so(3)^{\mathbb{C}}$ generated by so(2) and the raising operator $L_+ = L_x + iL_y$. If the circulation vanishes, then the polarization is the real subalgebra $[\mathbb{R}^6]so(3)$. Thirdly, quantization yields all the irreducible unitary representations of GCM(3) as determined independently by Mackey inducing [9].

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