THE ALGEBRAIC METHOD IN REPRESENTATION THEORY

(ENVELOPING ALGEBRAS)

Anthony JOSEPH *)

Institut des Hautes Etudes Scientifiques
91440 - BURES-sur-YVETTE
(France)

*) Permanent address : Department of Physics and Astronomy, Tel-Aviv University, Ramit Avic, Israel

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1 Introduction

In Lie algebra theory a basic open problem is to classify all irreducible representations (up to equivalence). For the category of finite dimensional representations the answer is known and is classical. By contrast a full classification for infinite dimensional representations appears to be impossible. For example, this is evidenced by the work of Arnal and Pinczon [2] on $s\ell(2)$ and by the work of McConnell and Robson [34] on A_1 which can be used to show [5] that the Heisenberg Lie algebra [18], 4.6.1, admits infinitely many inequivalent irreducible representations all very different from the standard one.

One way out of this difficulty is to consider only representations which integrate to unitary (or just bounded) representations of the corresponding Lie group(s). This has physical justification through Wigner's theorem. We call it the analytic approach. In it the Lie algebra plays only a subservient role.

About ten years ago, Dixmier proposed a purely algebraic way out which has since then generated a new area of mathematics called enveloping algebras. The philosophy is to find a less refined classification than equivalence classes rather than to exclude representations. We call it the algebraic approach. Except for finite dimensional representations, or for nilpotent Lie algebras, the exact relationship between these two approaches is not yet known, though one can find many analogies. Consequently in a given application one must make a definite choice as to which to use.

It is my personnel conviction that the algebraic approach has a very real and important contribution to make in physics. First of all it gives an overall framework which I believe physicists have been unsucessfully groping at for some years. Secondly, new techniques have been developed which go far beyond the relatively naive (though often intricate) computations of physicists. Some of these were taken over from situations where the analytic approach could be reduced to purely Lie algebraic questions and in this sense notable early contributors to enveloping algebras are Gelfand, Kostant and Harish-Chandra (the acknowledgement here is Dixmier's [18], p. 5). Other techniques are inspired by commutative (and non-commutative) algebra and in particular localization plays a fundamental role.

I do not suggest that either approach is superior in its physical applicability; rather that a choice between the two should be dependent on the situation concerned. Apart from the question of taste (i.e. if one prefers algebra to analysis) two further principles should be kept in mind.

1) If the Lie algebra is given through the action of a Lie group on a manifold (such as the Poincaré group acting on space-time) then the analytic approach is favoured. Yet for internal symmetries (e.g. su(3) in particle physics) where the symmetry has no well-established geometric origin, then the algebraic approach is at least of equal importance. In fact the search for "realizations" of Lie algebras (especially in the context of spectrum generating algebras [36] and chiral Lagrangians [14]) constitutes what I believe to be the primordial component of enveloping algebras in physics.

2) Roughly speaking, the analytic approach is best adapted to fundamental questions in physics, whereas the algebraic approach applies best to a computational scheme.

In the next section we give a precise definition of the algebraic approach. For "representation space", it is convenient to use the more precise term - module [18] , 1.2.1. This in particular excludes the possibility that we are talking about representations in Hilbert space, though in this context we remark that the module should be regarded as a common dense domain for the generators of the Lie algebra (in the language of say field theorists). Moreover this notion can often be made precise and then has a highly non-trivial content [22,39].

Throughout we take the base field $\underline{k}$ to be commutative and of characteristic zero. For example, $\underline{k}$ can be the real field $\mathbb{R}$ or the complex field $\mathbb{C}$, the latter being algebraically closed. $\mathbb{N}$ denotes the natural numbers and $\mathbb{N}^+$ the positive integers.

2 The Primitive Spectrum

Let $\underline{g}$ be a finite dimensional Lie algebra with enveloping algebra $U(\underline{g})$. Recall that if $\{X_i\}_{i=1}^n$ is a basis for $\underline{g}$, then $X_1^{k_1} X_2^{k_2} \ldots\ldots X_n^{k_n} : k_i \in \mathbb{N}$ is a basis for $U(\underline{g})$ and that for all $X,Y \in \underline{g}$, we have $[X,Y] = XY - YX$ in $U(\underline{g})$, where $[\ ,\ ]$ denotes the Lie bracket in $\underline{g}$. The importance of $U(\underline{g})$ derives from the fact that every $\underline{g}$ module extends (in the obvious fashion) to a $U(\underline{g})$ module. Given M a $U(\underline{g})$ module, we set $\mathrm{Ann}\ (M) = \{a \in U(\underline{g}) : aM = 0\}$. Ann(M) is a two-sided ideal in $U(\underline{g})$ and if ρ denotes the representation of $U(\underline{g})$ associated with M, then $\ker \rho = \mathrm{Ann}(M)$. If M_1, M_2 are isomorphic (i.e. equivalence of representations) as $U(\underline{g})$ modules, then $\mathrm{Ann}(M_1) = \mathrm{Ann}(M_2)$. By Wedderburn's theorem [23], Chap. 2, the converse holds in the category of finite dimensional simple $U(\underline{g})$ modules.

Yet the converse fails for infinite dimensional modules and this observation is the starting point of the algebraic approach. In more detail, set Prim $U(\underline{g}) = \{ \text{Ann}(M) : M \text{ a simple } U(\underline{g}) \text{ module}\}$. Then the fundamental goal of the algebraic approach is to classify Prim $U(\underline{g})$ (the primitive spectrum) for each $\underline{g}$. This is less refined than classification by equivalence classes and unlike the latter is proving quite feasible.

One should admit from the start that Prim $U(\underline{g})$ is not sufficiently refined for all physical purposes. Thus we shall see that the description of a primitive ideal $I \in$ Prim $U(\underline{g})$ entails the use of differential operators. Then a canonical transformation like $x \to x$, $d/dx \to d/dx + f(x)$, does not alter I. On the other hand (if these operators are applied to a fixed space of functions) this transformation does in general alter the equivalence class of the associated representation [19], Sect. 3. Moreover this transformation has been used for example by Zak to contrast the behaviour of a free electron with one in a magnetic field [7,21] . In a simply connected region one may solve the equation $g'(x) = f(x)$ and eliminate $f(x)$ through a gauge transformation. However this is not possible in general and different $f(x)$ can correspond to different physical situations as exemplified by the Bohm-Aharonov effect [1]. Again one might recall the work of Miller [35] who shows that a quite large variety of different physical situations can be encompassed within various realizations of one or two Lie algebras and in fact within a quite small class of (primitive) ideals. Yet again the realization of the symplectic algebra sp(2n) through symmetrized quadratic polynomials in x_i, $\partial/\partial x_i$: $i = 1,2,..n$, leads by its action on the polynomial ring $\underline{k}[x_1,x_2,\dots x_n]$ to two inequivalent irreducible representations (the Weil representations) spanned by polynomials of odd and even degree. These have the same annihilator in $U(\underline{g})$ and both have enjoyed enormous popularity in physical models [32] , p. 69.

Nonetheless it should be emphasized that the description of a primitive ideal through differential operators allows one to recover practically all the representations of Lie algebras discussed in physics. In fact I know of no examples for which this is not so.

3 Goldie's Theorem and the Gelfand-Kirillov Conjecture.

There are two main problems in the study of Prim $U(\underline{g})$. One, its description as a set and secondly the description of a given primitive ideal. First we give two basic properties of the latter which highly motivates its physical interest.

Given $I \in$ Prim $U(\underline{g})$, we set $U = U(\underline{g})/I$. The structure of the factor algebra U should be considered as the generic property of the set of simple $U(\underline{g})$ modules with annihilator I and is hence of fundamental importance. In general U is not integral and so we set $S = \{ a \in U : ab = 0,\ b \in U \text{ implies } b = 0\}$.

Example 1 . Let I be the annihilator of a simple finite dimensional $U(\underline{g})$ module. Then $\dim U < \infty$ and for some $m \in \mathbb{N}^+$, it is isomorphic to the full matrix algebra M_m over $\underline{k}$. Then S is the set of all invertible matrices in M_m.

Application of Goldie's theorem [18], 3.6.2, shows that $\text{Fract } U = \{ a^{-1}b : a \in S,\ b \in U\}$ is an algebra under the obvious identifications and furthermore that Fract U is isomorphic for some $m \in \mathbb{N}^+$ to M_m over some non-commutative field K (which is an extension of the base field $\underline{k}$). More precisely we have

THEOREM -

(i). S <u>is an Ore set for</u> U.

(ii). Fract U = $M_m(K)$, <u>up to isomorphism</u>.

Observe that this assigns a number m to I known as its Goldie rank. M_m describes in the generic sense, the finite dimensional piece of the representation associated with I. If $\underline{g}$ is solvable, then m is always one [18], 3.7.2, and this generalizes Lie's theorem to the infinite dimensional case ! Now K can be thought of as carrying the infinite dimensional part of the representation and it is a remarkable fact that in the enveloping algebra context K is apparently always a Weyl field. In more detail, recall that the Weyl algebra A_n of index n (over $\underline{k}$) is isomorphic to the associative algebra generated by x_i , $\partial/\partial x_i$: $i = 1,2,\ldots.\ n$, which physicists should consider as creation and annihilation operators (and correspondingly $\underline{k}[x_1,x_2,\ldots..\ x_n]$ as Fock space). Set $K_n = \text{Fract } A_n = \{ a^{-1}b: a \neq 0, b \in A_n\}$. Let K be defined by (ii) above.

CONJECTURE - <u>Suppose that</u> $\underline{g}$ <u>is a split algebraic Lie algebra</u>. <u>Then</u>

(i). <u>For some</u> $n \in \mathbb{N}$, $K = K_n$, <u>up to isomorphism</u>.

(ii). $n = 1/2 \ \text{Dim } U$.

The above conjecture, in a slightly different form, was proposed by Gelfand and Kirillov [20]. It has been established for $\underline{g}$ solvable [9, 33] and for the minimal primitive ideal (see Sect. 6) of a semisimple Lie algebra [16], Cor. 10.5. In [28] I have shown that the origin of the Weyl field derives from nilpotent action. In (ii), Dim denotes Gelfand-Kirillov dimension. This is roughly the same as transcendence degree; but we refer the reader to [11] for its exact definition and principle properties. We remark that $\underline{g}$ is always split if $\underline{k}$ is algebraically closed. The requirement that $\underline{g}$ be algebraic [8] is of a technical nature and we could equally well have extended slightly the class of possible candidates for K (c.f. [33]). Most Lie algebras in physics are algebraic.

We should like to emphasize three points.

1) A further integer, namely Dim U is associated with I. It is called the Gelfand-Kirillov dimension of I. Whilst it is an integer, it is not obviously even. However, this has been shown for $\underline{g}$ solvable (and algebraic) and for $\underline{g}$ semi-simple [11], 7.1.

2) The identity $K = K_n$ has the physical interpretation that the infinite part of the representation is carried by Fock or a Fock-like space (depending on the amount of localization necessary in the embedding of U in K_n). The conjecture and the theorem can be regarded as a precise way of saying that the matrix generalization of quantum realizations provides the general setting for Lie algebra representations. The representation space (in the generic sense) is spanned by finite component wave functions such as those encountered in the Dirac or Majorana equations.

3) It should be emphasized that the algebraic approach is not just a transition from the study of representations to the study of realizations. This would have little more than heuristic value since the latter problem is equally intractable (though some progress was made with the realization of $s\ell(2)$ in A_1 [26]). Rather one classifies all the possible relations between U relative to M and for this the description of U in terms of differential operators is particularly useful.

Example 2 Let $\underline{b}$ be a subalgebra of $\underline{g}$ and σ a finite dimensional irreducible representation of $\underline{b}$ of dimension m. Let ind (σ , $\underline{b} \uparrow \underline{g}$) [18], Chap. 5, denote the representation induced to $\underline{g}$ and set I = ker ind (σ, $\underline{b} \uparrow \underline{g}$). Under favourable circumstances (see Sects. 4-6) , I is a primitive ideal of Goldie rank m, and Gelfand-Kirillov dimension n, where n = dim $\underline{g}$ - dim $\underline{b}$. It is said to be an induced ideal.

We remark that the inducing construction in the algebraic context [16], Sect. 5, or [27], Sect. 3, exactly coincides with what physicists called non-linear or induced realizations of Lie algebras [14, 24] . Moreover the above decomposition of the representation into finite and infinite dimensional pieces was given [14] the physical interpretation of being the decomposition of the particle spectrum into baryonic (finite) and mesonic (infinite) parts. (This distinction arose because baryon number is conserved and for fixed baryon number, the baryon spectrum is finite. Also it is easy to give the finite part the correct statistics through the use of operators satisfying the canonical anticommutation relations). An important fact is that not all primitive ideals are obtained by induction from a finite (or even infinite) dimensional representation of a proper subalgebra (c.f. Sects. 5,6). In particular this is true of the annihilator of the Weil representations. I have succeeded in giving some new constructions for non-induced primitive ideals and I anticipate that these may have some useful physical applications.

4 Geometric Quantization and the Solvable Case

To describe Prim $U(\underline{g})$ as a set, we should like to relate it to some simple geometric object. Suppose (for technical simplicity) that $\underline{g}$ is an algebraic Lie algebra [8]. Recall that the adjoint group G acts by transposition in the dual space $\underline{g}^*$ of $\underline{g}$ and let $\underline{g}^*/G$ denote the corresponding orbit space. After Kirillov-Kostant each G orbit in $\underline{g}^*$ is a symplectic manifold [6], Chap. II, and hence in physicists' language provides a classical realization of $\underline{g}$. Through the conjecture of Sect. 3, primitive ideals are associated with quantum realizations. This suggests that one can construct a bijection between $\underline{g}^*/G$ and Prim $U(\underline{g})$ which we should naturally call (geometric) quantization. Actually this term is usually reserved for the rather specific procedure initiated by Kirillov [31] for the nilpotent case and generalized by Auslander and Kostant [3,4] to the solvable case. This was put in the enveloping algebra framework by Dixmier for the nilpotent case and by Borho, Gabriel and Rentschler [9] , [18], Chap. 6, in the solvable case. We sketch the method below. For its wider context we refer the reader to Simms' article in this series.

Given $f \in \underline{g}^*$, let Gf denote the G orbit containing f. A polarization $\underline{b}$ for f is a subalgebra of $\underline{g}$ satisfying

(i). $\dim \underline{g} - \dim \underline{b} = \frac{1}{2} \dim Gf$ (recall Ex. 2).

(ii). $(f, [\underline{b},\underline{b}]) = 0.$

The set of all polarizations for f is denoted by P(f).

By (ii) the map $f : \underline{b} \to (\underline{b}, f)$ is a one-dimensional representation of $\underline{b}$ which we simply denote by f. Set $I(f,\underline{b}) = \ker \operatorname{ind} (f, \underline{b} \uparrow \underline{g})$. Ideally this is the required element of Prim $U(\underline{g})$. For $\underline{g}$ split solvable, $I(f,\underline{b})$ is in fact primitive [18], 6.1.1 ; but may also depend on the polarization $\underline{b}$ chosen. In the above case this difficulty can be removed by replacing ind by $\operatorname{ind}^{\sim}$ - namely, the twisted induced representation [18], 5.2. Indeed [18], 6.1.4, THEOREM 1 - ($\underline{g}$ split solvable) . Let $f \in \underline{g}^*$, $\underline{b}_1$, $\underline{b}_2$ $\in P(f)$. Then $\ker \operatorname{ind}^{\sim}(f, \underline{b}_1 \uparrow \underline{g}) = \ker \operatorname{ind}^{\sim}(f, \underline{b}_2 \uparrow \underline{g})$.

From now on we take $\underline{g}$ solvable and $\underline{k}$ algebraically closed. Following Theorem 1, we set $I(f) = \ker \operatorname{ind}^{\sim}(f, \underline{b} \uparrow \underline{g}) : \underline{b} \in P(f)$. Then Theorem 2, [18], 6.1.7, asserts that all primitive ideals are obtained in this way.

THEOREM 2 - The map $f \to I(f)$ of $\underline{g}^*$ into Prim $U(\underline{g})$ is surjective.

Let $\mathfrak{G}$ denote the algebraic adjoint group of $\underline{g}$ [18], 6.1.5. (One has $\mathfrak{G} = G$ if $\underline{g}$ is algebraic). Then $I(f_1) = I(f_2)$ if $f_2 \in \mathfrak{G} f_1$. Extend $f \to I(f)$ to a map $\overline{I}$ of $\underline{g}^*/\mathfrak{G}$ into Prim $U(\underline{g})$ by passage to the quotient (i.e. in the obvious manner). Theorem 3, [18], 6.5.12, shows that different $\mathfrak{G}$ orbits give different ideals.

THEOREM 3 - $\overline{I}$ is a bijection.

$\overline{I}$ is continuous in a natural manner. Indeed take the Jacobson topology [18], 3.2, in Prim $U(\underline{g})$ and the Zariski topology in $\underline{g}^*$ (and the corresponding quotient topology in $\underline{g}^*/\mathfrak{G}$). We remark that the Jacobson topology is the non-commutative analogue of the Zariski topology and that the latter is generally used in algebraic geometry to discuss algebraic curves. It is coarser than the usual metric topology. For $\underline{g}$ solvable, $\overline{I}$ is continuous [18], 6.4.4 and [9], Sect. 6, and for $\underline{g}$ nilpotent, it is known to be a homeomorphism [15].

To establish the connection with the analytic approach, let $\mathcal{G}$ be the connected, simply connected real Lie group with Lie algebra $\underline{g}$ over $\mathbb{R}$ and $\hat{\mathcal{G}}$ the set of classes of irreducible unitary representations of $\mathcal{G}$. When $\underline{g}$ is nilpotent, $\underline{g}^*/\operatorname{ad}(\mathcal{G}) \to \hat{\mathcal{G}}$ is a bijection [31]and [38], Thm. 2.10. Now in this case $\underline{g}$ is algebraic, so $\operatorname{ad}(\mathcal{G}) = \mathfrak{G} = G$. Furthermore the appropriate generalization of theorem 3 when $\underline{k}$ is not algebraically closed (and $\underline{g}$ is nilpotent) obtains by replacing Prim $U(\underline{g})$ by Rat $U(\underline{g})$, the set of rational ideals [18], 4.5.8. Then by [18], 6.2.4 and the above, Rat $U(\underline{g}) \to \hat{\mathcal{G}}$ is a bijection. Thus the algebraic and analytic approaches can be considered equivalent in the nilpotent case. For general solvable Lie groups the unitary representation theory has not yet been fully worked out. However for type 1 solvable Lie groups (which in particular includes the algebraic case) one has [3], Thm. 2, [4], Thm. V. 3.3,

$$\hat{\mathcal{G}} = \bigcup_{\mathfrak{S} \in \underline{g}^*/\operatorname{ad}(\mathcal{G})} \mathcal{H}_{\mathfrak{S}},$$

where $\mathcal{H}_{\mathfrak{S}}$ is the character group of the fundamental group of $\mathfrak{S}$ and has the structure of a torus of suitable dimension. Thus $\underline{g}^*/\operatorname{ad}(\mathcal{G}) \to \hat{\mathcal{G}}$ is a bijection only if all orbits are simply connected and this may fail even if $\underline{g}$ is algebraic, as exemplified by the oscillator group [41], Sect. 6. It remains to determine whether or not the algebraic approach exhibits a similar phenomenon (probably not).

5 Difficulties with Quantization in the Semisimple Case.

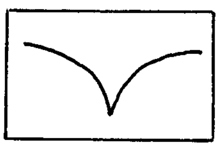
By contrast with the solvable case, quantization encounters many difficulties in the semisimple case. These we summarize below. For simplicity we take $\underline{k} = \mathbb{C}$.

1) Every simple Lie algebra, excepting $s\ell(n+1) : n \in \mathbb{N}^+$ has at least one orbit which does not admit polarization.

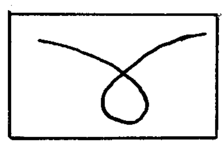
2) Theorem 1 fails. That is different polarizations of a given $f \in \underline{g}^*$ can give rise to different primitive ideals. This was remarked [10] by Borho and Rentschler for sp(4).

3) Theorem 2 fails. That is not all primitive ideals are induced. The first counterexample was given by Conze and Dixmier [17], Ex. 3. It turned out that this was exactly the annihilator of the Weil representations for sp(4). I have since then shown [25,29] that every simple Lie algebra, excluding $s\ell(n)$, admits at least one primitive ideal which is not induced. For sp(2n) my construction gives the annihilator of the Weil representations, whereas for the other cases it is entirely new both to mathematicians and to physicists. I believe it may have application to spectrum generating algebras and to the classification of nuclear states.

4) Unlike the solvable case the map $\underline{g}^*/G \to \text{Prim } U(\underline{g})$ is not always continuous [40]. In particular for $s\ell(3)$, the space of subregular orbits (those of dimension 4) form a cusp, whereas the corresponding family of primitive ideals (namely those of Gelfand-Kirillov dimension 4 and Goldie rank 1) form a less singular loop. This is illustrated below.



space of subregular orbits orbits.



space of corresponding primitive ideals.

It is very striking that the passage from orbits to ideals exactly corresponds in this case to what algebraic geometers call a resolution of a singularity. Actually the mechanism is quite simple and derives from the $(\rho,\rho) : \rho = \frac{1}{2}$(sum of positive roots), term present in the spectrum of the Casimir invariants. This term is analogous to zero point energy and it would be very nice to find a resolution of singularity in the passage from classical to quantum mechanics. For $s\ell(3)$, Borho and I have shown [12] that the primitive ideals of higher Goldie rank form similar loops except that these have finitely many missing points where they intersect with loops of lower Goldie rank.

5) Theorem 3 fails. That is different orbits can give rise to the same ideal [10].

6) Goldie rank can be greater than one. Hence it is not sufficient to induce from a one-dimensional representation as outlined in Sect. 4. Of course for each $f \in \underline{g}^*$, one can simply induce from a finite dimensional irreducible representation σ of $\underline{b} \in P(f)$. However it can happen [12] that over a certain (finite ?) number of orbits, the resulting ideal is not primitive. Again it is not known if the Goldie rank of ker ind $(\sigma, \underline{b} \uparrow \underline{g})$ always coincides with dim σ, [18], Prob. 12.

We remark that these difficulties are not restricted to the algebraic approach and in fact become even more serious in the analytic approach.

A natural problem posed by 3) is to give a criterion for when an ideal is induced. I propose the following. First recall [27], Sect. 3, that through induction each $X \in \underline{g}$ is realized as a first-order differential operator with coefficients in $S(\underline{m})^\wedge$, where $\underline{m} = \underline{g} \ominus \underline{b}$ and $S(\underline{m})^\wedge$ denotes the formal power series completion of the symmetric algebra. (Of course this is also the essential fact behind polarization). Since each X is first order, $\underline{g}$ leaves $S(\underline{m})^\wedge$ stable. Under favourable circumstances [27], Prop. 3.5, [16], Cor. 10.5, we can replace $S(\underline{m})^\wedge$ by $R(\underline{m})$ = Fract $S(\underline{m})$ and furthermore identify $R(\underline{m})$ as a subfield of Fract $U(\underline{g})/I$. Obviously $R(\underline{m})$ is commutative. Set $L = R(\underline{m})$. Given $I \in$ Prim $U(\underline{g})$, set $U = U(\underline{g})/I$. Suppose that $L \subset U$ and let L' denote its commutant in U. Through the discussion of Sect. 3, one expects to have $L' = M_m(L)$, where m is the Goldie rank of I. In particular Dim L = Dim L' and $L = L'$ if $m = 1$. This motivates the following definition. Call a subfield L of U induced if (i) L is commutative, (ii) $[\underline{g},L] \subset L$, (iii) Dim L = Dim L'.

CONJECTURE - A primitive ideal I is induced if and only if U admits an induced subfield.

Assume that $\underline{g}$ is split solvable. Extending the results of [37] and in fact with some simplifications, I have shown that for all $I \in$ Prim $U(\underline{g})$, (i) $U(\underline{g})/I$ admits an induced field, (ii) $L = L'$ (recall that the Goldie rank equals one in this case). Combined with Theorem 2, this proves the conjecture for $\underline{g}$ solvable and $\underline{k}$ algebraically closed. An easy application of [16], Cor. 10.5, shows that the conjecture also holds for the minimal primitive ideal of a split semisimple Lie algebra (see Sect. 6). A possible generalization of quantization and its role in the Gelfand-Kirillov conjecture is discussed in [30]. (I should add that a certain primitive ideal is incorrectly identified in [30], pp. 232-233, and these pages should be ignored. For sp(4,$\mathbb{C}$) a correct description is given in [12]).

6 The Semisimple Case

In view of Sect. 5, new techniques must be devised to handle the semisimple case. Here we have space only to summarize some of the main results.

Take $\underline{g}$ simple and $\underline{k} = \mathbb{C}$. Let $\underline{h}$ be a fixed Cartan subalgebra and Δ (resp. Δ+) a choice of non-zero (resp. positive) roots. Set $\mathfrak{D} = \{ \lambda \in \underline{h}^*: (\lambda,\alpha) > 0$, for all $\alpha \in \Delta^+ \}$. $\mathfrak{D}$ is called the Weyl chamber and plays a fundamental role in the representation theory of $\underline{g}$. Let $\overline{\mathfrak{D}}$ denote the closure of $\mathfrak{D}$ (in the metric topology). Then the walls $\overline{\mathfrak{D}} \setminus \mathfrak{D}$ of the Weyl chamber form a set of reflection planes which generates the Weyl group W of $\underline{g}$. Let $Z(\underline{g})$ denote the centre of $U(\underline{g})$. Given $I \in \text{Prim } U(\underline{g})$, then $I \cap Z(\underline{g}) = Z_\lambda$ is a maximal ideal in $Z(\underline{g})$ and after Harish-Chandra [18], 7.4.3, we can regard λ as an element of $\overline{\mathfrak{D}}$. That is each $\lambda \in \overline{\mathfrak{D}}$ specifies the eigenvalues of the Casimir invariants of $\underline{g}$ in some well-defined fashion, and is called the central character of I. Set $\underline{X}(\lambda) = \{ I \in \text{Prim } U(\underline{g}) : I \cap Z(\underline{g}) = Z_\lambda \}$. Then

1) $1 \le \text{card } \underline{X}(\lambda) < \infty$, for all $\lambda \in \overline{\mathfrak{D}}$, [18], 8.4.4, 8.5.7 (b).

2) Order $\underline{X}(\lambda)$ by inclusion. This is not in general a total ordering; but $\underline{X}(\lambda)$ does have a unique minimal element $I_{min}(\lambda)$ and a unique maximal element $I_{max}(\lambda)$, [18], 8.4.4, 8.5.8 (a).

3) $I_{min}(\lambda) = U(\underline{g})\, Z_\lambda$, [18], 8.4.3.

4) Card $\underline{X}(\lambda) = 1$, if and only if $2\,(\lambda,\alpha)/(\alpha,\alpha) \notin \mathbb{N}^+$, for each $\alpha \in \Delta^+$, [18], 8.5.8 (a).

The sets $\{\lambda \in \mathfrak{D} : 2\,(\lambda,\alpha)/(\alpha,\alpha) \in \mathbb{N}^+$, for some $\alpha \in \Delta^+\}$ are called the exceptional hyperplanes. By 3) and 4), the study of Prim $U(\underline{g})$ reduces to what happens on these planes. A similar phenomenon occurs in the analytic approach. We remark that a bounded representation gives rise to a Harish-Chandra module M [18], p. 277, for $U(\underline{g})$. Now whereas $I_{min}(\lambda)$ is the annihilator of a Verma module $M(\lambda)$, [18], Chap. 7, one cannot have $M = M(\lambda)$, except on the hyperplanes defined by the compact roots. This excludes the obvious interrelation. Again for a finite dimensional representation one must have $2\,(\lambda,\alpha)/(\alpha,\alpha) \in \mathbb{N}$, for all $\alpha \in \Delta^+$.

5) Given $I \in \text{Prim } U(\underline{g})$, then Dim $U(\underline{g})/I$ equals the dimensional of some nilpotent orbit [11], 7.1, and is hence even (c;f; Sect. 3).

6) Let $I_1, I_2 \in \text{Prim } U(\underline{g})$, then $I_1 \supset I_2$ implies $\text{Dim } U(\underline{g})/I_1 \le \text{Dim } U(\underline{g})/I_2$, with equality if and only if $I_1 = I_2$, [11], 3.6.

7) By 2), 5), 6) and [16], Cor. 10.5, $\text{Dim } U(\underline{g})/I = \frac{1}{2}(\dim \underline{g} - \text{rank } \underline{g})$, if

and only if $I = I_{min}(\lambda)$, for some $\lambda \in \bar{\mathfrak{D}}$. Again $\mathrm{Dim}\, U(\underline{g})/I = 0$, if and only if $\dim U(\underline{g})/I < \infty$, that is if the associated representation is finite dimensional.

8) Suppose $I = \ker \mathrm{ind}\,(\sigma, \underline{p} \uparrow \underline{g})$, where σ is a finite dimensional representation of a parabolic subalgebra $\underline{p}$ of $\underline{g}$. Then [13], Sect. 2, $\mathrm{Dim}\, U(\underline{g})/I = 2(\dim \underline{g} - \dim \underline{p})$.

9) By 2), 8) and [17], Prop. 2 , we have card $\underline{X}(\lambda) = 2$, if λ lies on just one exceptional hyperplane [13].

Given $\lambda \in \underline{h}^*$, let $M(\lambda)$ be the Verma module with highest weight $\lambda - \rho$: $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$, [18], Chap. 7. (Note that the $M(\omega\lambda)$: $\omega \in W$, are <u>not</u> isomorphic). Now $M(\lambda)$ admits a largest non-trivial submodule L , [18], 7.1.11 (ii) and the quotient module $M(\lambda)/L$ is simple. Hence $I(\lambda) = \ker M(\lambda)/L \in \mathrm{Prim}\, U(\underline{g})$ and if we choose $\omega \in W$ such that $\omega\lambda \in \bar{\mathfrak{D}}$, then $I(\lambda) \in \underline{X}(\omega\lambda)$. For each $\lambda \in \bar{\mathfrak{D}}$, let $\underline{X}^o(\lambda)$ denote the subset of all $I \in \underline{X}(\lambda)$ having the above form. Observe that card $\underline{X}^o(\lambda)$ is bounded above by the order of the Weyl group.

BASIC OPEN PROBLEM - <u>Is</u> $\underline{X}(\lambda) = \underline{X}^o(\lambda)$, <u>for all</u> $\lambda \in \bar{\mathfrak{D}}$?

10) $I_{min}(\lambda)$, $I_{max}(\lambda) \in \underline{X}^o(\lambda)$, [18], 7.6.24 and 8.4.4, and 8.5.8 (b).

Finally Borho and I have given [12] a complete description of $\mathrm{Prim}\, U(\underline{g})$ for $\underline{g} = s\ell(3)$ and $sp(4)$. Space does not permit a description of the results or the methods; but I should add that many aspects of the analysis admit a straightforward generalization to arbitrary simple $\underline{g}$. Also the annihilator of the Weil representations (which has Goldie rank one) generalizes giving ideals of Goldie rank $\frac{1}{2}(\ell + 1)(2k + \ell + 2)$: $k, \ell \in \mathbb{N}$. This is a new class of primitive ideals none of which are induced. It must surely be true that such a natural generalization of the Weil representations will play an equally important role in physical models. In any case our construction is very explicit.

7 <u>Decomposition Theory</u>

Let M be an arbitrary $U(\underline{g})$ module and set $I = \mathrm{Ann}(M)$. Physicists often require a decomposition theory for M especially in the discussion of coupling coefficients. Taking the algebraic approach to its logical conclusion we should attempt to do this in terms of I alone. Here there are some initial difficulties. First I can be primitive even if M is not simple. For example, the annihilator of the Verma module is always primitive [18], 8.4.4, even though the module itself is not always simple [18], Chap. 7. Also the direct sum of the Weil representations has a primitive annihilator. Again the algebraic approach does not distinguish between decomposability and reducibility. For example, $I = I_1 \cap I_2$ can mean either that $M = N_1 \oplus N_2$, where $I_j = \mathrm{Ann}(N_j)$: $j = 1,2.$, or that M is indecomposable and admits the submodule N with $I_1 = \mathrm{Ann}(N)$, $I_2 = \mathrm{Ann}(M/N)$. Nevertheless one does have a decomposition theory for I which generalizes a situation from algebraic geometry. Thus I can be written as a finite product of not necessarily distinct prime ideals [18], 3.1, which in the commutative case corresponds to the decomposition of the zero set of I into irreducible algebraic varieties one for each distinct prime ideal. Then, at least for $\underline{g}$ solvable [9], 10.8 and 13.4, each prime ideal can be written as (in general infinite) intersection of primitive ideals having different central character. It then remains (for physicists !) to develop a version of the Wigner-Eckart theorem.

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