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The Mathematical Beauty of Symmetry Proceedings of the 2010 Zacatecas Workshop on Mathematical Physics II, México,

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Foreword

In his response to the question of the philosophy of physics, *Paul Dirac* posted a remarkable statement: "**Physical Laws Should have Mathematical Beauty**". The era of the Proceedings of the 2010 Zacatecas Workshop on Mathematical Physics II, México, showed the beauty of mathematical physics through the symmetry, supersymmetry, geometry, special and general relativity, Poincaré Gauge invariance, noncommutative cosmology, inflation, quantum cosmology, field theory and its symmetries, quantum Chromodynamics, Higgs Bosons, up to biophysics.

All the papers of this proceedings have gone through the standard peer review process, the hard work of editors and referees are extremely important to ensure the quality of this proceedings.

I am grateful to the people who supported this workshop, authors, speakers, referees, and the Editors for their reviewing and fruitful suggestions. In particular, Profs. *Alberto Molgado, Valeriy V. Dvoeglazov, Jose Luis Lópe*z *Bonilla*, and *Ignazio Licata.*

I hope that this proceedings will make more physicists and mathematicians aware of the current research of Mathematical and Theoretical Physics.

Ammar Sakaji, EJTP Co-Editor

Editorial Introduction

Valeriy V. Dvoeglazov
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Proceedings of the 2010 Zacatecas Workshop on Mathematical Physics II, México, December 2010

The second Workshop on Symmetries has been held in Zacatecas from the 9th to the 11th of December 2010. There were 54 participants with 26 invited talks. Most of presentations were in the form of 40 minutes lectures. The relaxed and friendly hospitality of Zacatecas, the elegant atmosphere given by the antiguous building of "La casa de la moneda", which now occupies the Zacatecas Institute for Culture, provided an exceptionally interactive Meeting for attenders. The aim of the Workshop was to showcase the advances of the mathematical physics group in the Unidad Academica de Fisica of the Universidad Autonoma de Zacatecas (UAF-UAZ), and to increase the academic interactions of our faculties with the rest of Mexico and the world.

On the inaugural ceremony presided Dr. José de Jesús Araiza Ibarra (the Director of the UAF-UAZ), Dr. Gema Mercado Sánchez (the General Director of the COZCyT), and I. Q. Armando Silva Cháirez (the General Secretary of the UAZ). The scientific programme has been opened by the exciting talk by Prof. A. Aranda from the Universidad de Colima. The first day has been finished by the "bar talk" of Prof. O. Obregón, who is one of the most renown researchers in the country, and many members of the gravitational scientific community of Mexico consider him as their Teacher. The Conference continued with the presentations of well-known physicists and mathematicians as Profs. M. Agüero, E. Ayón-Beato, A. Balankin, A. Herrera, M. Kirchbach, J. López-Domínguez, E. Mena, M. Montesinos, Z. Oziewicz, H. Quevedo, E. Rojas, J. Socorro, M. Socolovsky, L. Ureña, J. A. Vallejo, and J. D. Vergara, to mention some.

This is not the first event organized by the Zacatecas physicists. The International Workshop "Lorentz Group, CPT and Neutrinos" (1999), the Summer School on Theoretical Physics (2000), the 1st Mini-Workshop "Symmetries" (2005), the Mini-Colloquium "Año Internacional de Física" (2005), have been organized previously by the mathematical physics group of the Physics Faculty of the Zacatecas University. Among the principal themes of the conferencies were group theory, supersymmetry, gauge theories, origin of mass, spin and quantization, differential geometry, non-commutative geometry, non-associative algebras, etc.

The Proceedings of the first Workshop have been published in the Ukranian Journal "Electromagnetic Phenomena", http://www.emph.com.ua/17/.

Complete information about the 2nd Workshop can be found on the web page http://planck.reduaz.mx/˜congreso/sim10/bienvenidos.html.

In this Proceedings special issue we present the papers by; V. Dvoeglazov, H. Quevedo and M. N. Quevedo, N. Barbosa-Cendejas, A. Herrera-Aguilar, K. Kanakoglou and J. E. Paschalis, Z. Oziewicz and W. Page, M. Socolovsky *et al*, E. Mena and M. Cano, C. Escamilla-Rivera, O. Obregón and L. A. Ureña-López, L. A. Ureña-López and E. Torres-Lomas, A. Molgado, S. I. Kruglov, M. Kirchbach and C. B. Compean, P. Castañeda-Almanza and A. Gutiérrez-Rodríguez, and O. Osorio and M. A. Agüero. The topics of their presentations range from the supersymetric extension of the action of general relativity to the relations between non-linear physics and the DNA dynamics. The review by H. Quevedo and M. N. Quevedo results very interesting since it accounts for the mathematical methods necessary to obtain a geometric version of any physical theory, including thermodynamics. The electrovacuum theory in terms of complex potentials has been presented by A. Herrera *et al.* In this paper, they also analyzed some solutions which are somehow linked to gravitational physics. The most fundamental questions of the Relativity Theory have been considered in the paper by Z. Oziewicz and W. S. Page. We still consider that the discussion of relevant experiments would be highly desirable in these frameworks. Moreover, from our previous discussions in our events it seems to be clear that the question of if the acceleration is absolute or relative is still obscure. Socolovsky *et al.* present a brief review on the present status of the Einstein-Cartan torsion. The non-commutative cosmology on using the WKB approximation was presented by E. Mena and M. Cano. The paper by O. Obregón *et al* is significant due to its relation to the fundamental physics of scalar fields within the supersymmetric cosmology scenario. Also, a couple of scalar fields serves as a model to explain the preheating phenomena in the article by L. Ureña-López and E. Torres-Lomas. The paper by A. Molgado is related to group theoretical issues in the quantization of constrained models for cosmology. In addition, Kruglov works within QED formalism in the covariant gauge, concluding that, as a consequence of the fact that the photon has two degrees of freedom only, it is possible to introduce additional non-covariant "gauge" related to curvature. Is this an old or new idea? The manuscript by M. Kirchbach and C. Compean continues with the idea of missing barions and the spin-parity multiplets (now within the conformally compactified AdS_5). The paper of P. Castañeda-Almanza and A. Gutiérrez-Rodríguez is rather phenomenological, as it considers various scenarios for the decay $H \to Z\gamma$. Finally, the paper by O. Osorio and M. A. Agüero describes within the generalized coherent states approach the quasi-spin model of the DNA molecule. We have also included, as an Appendix, the abstracts of the talks whose participants, due to various reasons, did not submit a written version of his/her presentation.

We are planning to continue organizing Workshops on symmetries for the mathematical physics community in the future.

We are grateful to our sponsors for their support: Universidad Autónoma de Zacatecas, and its Unidad Académica de Física, Universidad Veracruzana (through the personal project of E. Rojas), Instituto Zacatecano de Cultura "Ramón Lopez Velarde", Consejo Zacatecano de Ciencia, Tecnología e Innovación and Programa de Mejoramiento del Profesorado under the projects UAZ-PTC-086 and UAZ-PTC-116.

Valeriy Dvoeglazov
Alberto Molgado
Carlos Ortiz
The Organizers and the Editors
Cuerpo Académico de Física-Matemática y Gravitación
Universidad Autónoma de Zacatecas

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Fundamentals of Geometrothermodynamics

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Abstract: We present the basic mathematical elements of geometrothermodynamics which is a formalism developed to describe in an invariant way the thermodynamic properties of a given thermodynamic system in terms of geometric structures. First, in order to represent the first law of thermodynamics and the general Legendre transformations in an invariant way, we define the phase manifold as a Legendre invariant Riemannian manifold with a contact structure. The equilibrium manifold is defined by using a harmonic map which includes the specification of the fundamental equation of the thermodynamic system. Quasi-static thermodynamic processes are shown to correspond to geodesics of the equilibrium manifold which preserve the laws of thermodynamics. We study in detail the equilibrium manifold of the ideal gas and the van der Waals gas as concrete examples of the application of geometrothermodynamics.

Keywords: Geometrothermodynamics, contact geometry, phase transitions
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1. Introduction

Differential geometry is a very important tool of mathematical physics with many applications in physics, chemistry and engineering. As an example, one can mention the case of the four known interactions of nature which can be described in terms of geometrical concepts. Indeed, Einstein proposed in 1915 the astonishing principle "field strength = curvature" to understand the physics of the gravitational field (see, for instance, Ref. [1]). In an attempt to associate a geometric structure to the electromagnetic field, Yang

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and Mills [2] used in 1953 the concept of a principal fiber bundle with the Minkowski spacetime as the base manifold and the symmetry group $U(1)$ as the standard fiber to demonstrate that the Faraday tensor can be interpreted as the curvature of this particular fiber bundle. Today, it is well known [1] that the weak interaction and the strong interaction can be represented as the curvature of a principal fiber bundle with a Minkowski base manifold and the standard fiber $SU(2)$ and $SU(3)$, respectively. In this work, we will show that it is possible to interpret the thermodynamic interaction as the curvature of a Legendre invariant Riemannian manifold. It should be mentioned that our interpretation of the thermodynamic interaction is based upon the standard statistical approach to thermodynamics in which all the properties of the system can be derived from the explicit form of the corresponding Hamiltonian and partition function [3], and in which the interaction between the particles of the system is described by the potential part of the Hamiltonian. Consequently, if the potential vanishes, we say that the system has a zero thermodynamic interaction.

In very broad terms, one can say that in a thermodynamic system, all the known forces act among the particles that constitute the system. Due to the large number of particles involved in the system, only a statistical approach is possible, from which average values for the physical quantities of interest are derived. Although the laws of thermodynamics are based entirely upon empirical results which are satisfied under certain conditions in almost any macroscopic system, the geometric approach to thermodynamics has proved to be very useful. One can say that the following three branches of geometry have found sound applications in equilibrium thermodynamics: analytic geometry, Riemannian geometry, and contact geometry.

Probably, one of the most important contributions of analytic geometry to the understanding of thermodynamics is the identification of points of phase transitions with extremal points of the surface determined by the corresponding state equation. For a more detailed description of these contributions see, for instance, [4, 5]. Riemannian geometry was first introduced in statistical physics and thermodynamics by Rao [6], in 1945, by means of a metric whose components in local coordinates coincide with Fisher's information matrix. Rao's original work has been followed up and extended by a number of authors (see, e.g., [7] for a review). On the other hand, Riemannian geometry in the space of equilibrium states was introduced by Weinhold [8] and Ruppeiner [9, 10], who defined metric structures as the Hessian of the internal energy and the entropy, respectively. Both metrics have been used intensively to study the geometry of the thermodynamics of ordinary systems and black holes; however, several inconsistencies and contradictions have been found [11-19]. It is now well established that these puzzling results are a consequence of the fact that Weinhold and Ruppeiner metrics are not invariant with respect to Legendre transformations [20]. Furthermore, contact geometry was introduced by Hermann [21] into the thermodynamic phase space in order to formulate in a consistent manner the geometric version of the laws of thermodynamics.

In order to incorporate Legendre invariance in Riemannian structures at the level of the phase space and the equilibrium space, the formalism of geometrothermodynamics

(GTD) was recently proposed by Quevedo [20]. The main motivation for introducing the formalism of GTD was to formulate a geometric approach which takes into account the fact that in ordinary thermodynamics the description of a system does not depend on the choice of the thermodynamic potential, i. e., it is invariant with respect to Legendre transformations. One of the main goals of GTD has been to interpret in an invariant manner the curvature of the equilibrium space as a manifestation of the thermodynamic interaction. This would imply that an ideal gas and its generalizations with no mechanic interaction correspond to a Riemannian manifold with vanishing curvature. Moreover, in the case of interacting systems with non-trivial structure of phase transitions, the curvature should be non-vanishing and reproduce the behavior near the points where phase transitions occur. These intuitive statements represent concrete mathematical conditions for the metric structures of the phase and equilibrium spaces. In the present work, we present geometric structures which satisfy these conditions for systems with no thermodynamic interaction as well as for systems characterized by interaction with first and second order phase transitions.

In this work, we present the formalism of GTD by using Riemannian contact geometry for the definition of the thermodynamical phase manifold and harmonic maps for the definition of the equilibrium manifold. We will see that this approach allows us to interpret any thermodynamic system as a hypersurface in the equilibrium space completely determined by the field theoretical approach of harmonic maps. This paper is organized as follows: In Section 2., we introduce the main concepts of Riemannian contact geometry that are necessary to define the phase manifold. Section 3. is dedicated to the description of the equilibrium manifold as resulting from a harmonic map in which the target space is the phase manifold. Section 4. contains a discussion of the quasi-static thermodynamic processes which are interpreted as geodesics preserving the laws of thermodynamics. In Section 5., we present the main geometric properties of the ideal and the van der Waals gas. Finally, Section 6. is devoted to discussions of our results and suggestions for further research. Throughout this paper, we use units in which $G = c = k_B = \hbar = 1$.

2. The thermodynamic phase manifold

Consider a $(2n+1)-$dimensional differential manifold $\mathcal{T}$ and its tangent manifold $T(\mathcal{T})$. Let $\mathcal{V} \subset T(\mathcal{T})$ be an arbitrary field of hyperplanes on $\mathcal{T}$. It can be shown that there exists a non-vanishing differential 1-form Θ on the cotangent manifold $T^*(\mathcal{T})$ such that the field $\mathcal{V}$ can be associated with the kernel of Θ, i. e., $\mathcal{V} = \ker \Theta$. If the Frobenius integrability condition $\Theta \wedge d\Theta = 0$ is satisfied, the hyperplane field $\mathcal{V}$ is completely integrable. On the contrary, if $\Theta \wedge d\Theta \neq 0$, then $\mathcal{V}$ is non-integrable. In the limiting case $\Theta \wedge (d\Theta)^n \neq 0$, the hyperplane field $\mathcal{V}$ becomes maximally non-integrable and is said to define a contact structure on $\mathcal{T}$. The pair $(\mathcal{T}, \mathcal{V})$ determines a contact manifold [22] and is sometimes denoted as $(\mathcal{T}, \Theta)$ to emphasize the role of the contact form Θ. Consider G as a non-degenerate metric on $\mathcal{T}$. The set $(\mathcal{T}, \Theta, G)$ defines a Riemannian contact manifold. It should be noted that the condition $\Theta \wedge (d\Theta)^n \neq 0$ is independent of Θ; in fact, it is a property of $\mathcal{V} = \ker\Theta$. If another 1-form Θ' generates the same $\mathcal{V}$, it must be of the form $\Theta' = f\Theta$, where $f : \mathcal{T} \to \mathbb{R}$ is a smooth non-vanishing function. This implies that the contact manifold $(\mathcal{T}, \Theta)$ is uniquely defined up to a smooth function $f : \mathcal{T} \to \mathbb{R}$.

Let us choose a particular set of coordinates of $\mathcal{T}$ as $Z^A = \{\Phi, E^a, I^a\}$ with $a = 1, ..., n$, and $A = 0, 1, ..., 2n$. Here, Φ represents the thermodynamic potential used to describe the system whereas the coordinates E^a correspond to the extensive variables and I^a to the intensive variables. Notice that since in the phase manifold $\mathcal{T}$ all the coordinates Φ, E^a and I^a must be completely independent, it is not possible to describe thermodynamic systems in $\mathcal{T}$ which are usually defined in terms of equations of state that relate different thermodynamic variables. An important ingredient of GTD is the concept of Legendre transformations that in general are defined as [23]

$$\{Z^A\} \longrightarrow \{\widetilde{Z}^A\} = \{\tilde{\Phi}, \tilde{E}^a, \tilde{I}^a\} \ , \tag{1}$$

$$\Phi = \tilde{\Phi} - \delta_{kl}\tilde{E}^k\tilde{I}^l \ , \quad E^i = -\tilde{I}^i, \quad E^j = \tilde{E}^j, \quad I^i = \tilde{E}^i, \quad I^j = \tilde{I}^j \ , \tag{2}$$

where $i \cup j$ is any disjoint decomposition of the set of indices $\{1, ..., n\}$, and $k, l = 1, ..., i$. In particular, for $i = \{1, ..., n\}$ and $i = \emptyset$, we obtain the total Legendre transformation and the identity, respectively.

In the particular coordinates $Z^A = \{\Phi, E^a, I^a\}$, the contact 1–form can be written as

$$\Theta = d\Phi - \delta_{ab}I^a dE^b \ , \quad \delta_{ab} = \mathrm{diag}(1, 1, ..., 1) \ , \tag{3}$$

where we assume the convention of summation over repeated indices. This expression for the 1-form Θ is manifestly invariant with respect to the Legendre transformations given in Eq.(2), i. e., under a Legendre transformation it transforms as $\Theta \to \tilde{\Theta} = d\tilde{\Phi} - \delta_{ab}\tilde{I}^a d\tilde{E}^b$. Consequently, the contact manifold $(\mathcal{T}, \Theta)$ is a Legendre invariant structure. Furthermore, if we demand the Legendre invariance of the metric G, the Riemannian contact manifold $(\mathcal{T}, \Theta, G)$ is Legendre invariant. Any Riemannian contact manifold $(\mathcal{T}, \Theta, G)$ whose components are Legendre invariant is called a thermodynamic *phase manifold* and constitutes the starting point for a description of thermodynamic systems in terms of geometric concepts. We would like to emphasize the fact that Legendre invariance is an important condition that guarantees that the description does not depend on the choice of the thermodynamic potential, a property that is essential in ordinary thermodynamics.

From the above description if follows that the only freedom in the construction of the phase manifold is in the choice of the metric G. Although Legendre invariance implies a series of algebraic conditions for the metric components G_{AB} [20], and it can be shown that these conditions are not trivially satisfied, the metric G cannot be fixed uniquely. It is important to mention that a straightforward computation shows that the flat metric $G = \delta_{AB}dZ^A dZ^B$ is not invariant with respect to the Legendre transformations given in Eq.(2). It then follows that the phase manifold is necessarily curved. We performed a detailed analysis of the Legendre invariance conditions and found as a solution the metric

$$G = (d\Phi - I_a dE^a)^2 + \Lambda\,(E_a I_a)^{2k+1} dE^a dI^a \ , \quad E_a = \delta_{ab}E^b \ , \quad I_a = \delta_{ab}I^b \ , \tag{4}$$

where Λ is an arbitrary Legendre invariant real function of E^a and I^a, and k is an integer. To our knowledge, this is the most general metric satisfying the conditions of Legendre invariance.

If we limit ourselves to the case of total Legendre transformations, we find that there exists a class of metrics,

$$G = (d\Phi - I_a dE^a)^2 + \Lambda\left(\xi_{ab}E^a I^b\right)\left(\chi_{cd}dE^c dI^d\right) \tag{5}$$

parametrized by the diagonal constant tensors ξ_{ab} and χ_{ab}, which is invariant for several choices of these free tensors. In fact, since ξ_{ab} and χ_{ab} must be constant and diagonal it seems reasonable to express them in terms of the usual Euclidean and pseudo-Euclidean metrics $\delta_{ab} = \text{diag}(1, ..., 1)$ and $\eta_{ab} = \text{diag}(-1, 1, ..., 1)$, respectively. Then, for instance, the choice

$$\xi_{ab} = \delta_{ab} \ , \quad \chi_{ab} = \delta_{ab} \tag{6}$$

corresponds to a Legendre invariant metric which has been used to describe the geometric properties of systems with first order phase transitions [20, 24]. Moreover, the choice

$$\xi_{ab} = \delta_{ab} \ , \quad \chi_{ab} = \eta_{ab} \tag{7}$$

turned out to describe correctly second order phase transitions especially in black hole thermodynamics [24-27]. The additional choice

$$\xi_{ab} = \frac{1}{2}\left(\delta_{ab} - \eta_{ab}\right) \ , \quad \chi_{ab} = \eta_{ab} \tag{8}$$

can be used to handle in a geometric manner second order phase transitions and also the thermodynamic limit $T \to 0$. Obviously, for a given thermodynamic system it is very important to choose the appropriate metric in order to describe correctly the thermodynamic properties in terms of the geometric properties in GTD.

3. The equilibrium manifold

Consider the (smooth) harmonic map $\varphi : \mathcal{E} \to \mathcal{T}$, where $\mathcal{E}$ is a subspace of the phase manifold $(\mathcal{T}, \Theta, G)$ and $\dim(\mathcal{E}) = n$, where n is the number of independent degrees of freedom of the thermodynamic system, i. e., the number of independent thermodynamic variables which are necessary to describe a thermodynamic system. Let us assume that the extensive variables $\{E^a\}$ can be used as the coordinates of the base space $\mathcal{E}$. Then, in terms of coordinates, the harmonic embedding map reads $\varphi : \{E^a\} \longmapsto \{Z^A(E^a)\} = \{\Phi(E^a), E^a, I^a(E^a)\}$. Since the phase manifold is endowed with a Legendre invariant nondegenerate metric G, the pullback φ^* of the harmonic map induces canonically a thermodynamic metric g on $\mathcal{E}$ by means of

$$g = \varphi^*(G) \ , \quad \text{i.e.} \quad g_{ab} = \frac{\partial Z^A}{\partial E^a}\frac{\partial Z^B}{\partial E^b} G_{AB} = Z^A_{,a} Z^B_{,b} G_{AB} \ . \tag{9}$$

If we assume that the metric of the base manifold coincides with the induced metric $g = \varphi^*(G)$, the action of the harmonic map [28] can be expressed as

$$S = \frac{1}{2}\int d^nE\sqrt{|\det(g)|}\; G_{AB}\frac{\partial Z^A}{\partial E^a}\frac{\partial Z^B}{\partial E^b}g^{ab} = \frac{n}{2}\int d^nE\sqrt{|\det(g)|}\ , \tag{10}$$

and turns out to correspond to the volume element of the submanifold $\mathcal{E} \subset \mathcal{T}$. Consequently, according to the definition of harmonic maps [28], the variation $\delta S = 0$, i. e., the field equations

$$\frac{1}{\sqrt{|\det(g)|}}\left(\sqrt{|\det(g)|}\; g^{ab}Z^A_{,a}\right)_{,b} + \Gamma^A_{\ BC}Z^B_{,b}Z^C_{,c}g^{bc} = 0\ , \tag{11}$$

represent the condition for $\mathcal{E}$ to be an extremal hypersurface in the phase manifold $\mathcal{T}$ [26]. Here, the symbols $\Gamma^A_{\ BC}$ represent the Christoffel symbols associated with the metric G_{AB} of the phase manifold, i. e.,

$$\Gamma^A_{\ BC} = \frac{1}{2}G^{AD}\left(\frac{\partial G_{DB}}{\partial Z^C} + \frac{\partial G_{CD}}{\partial Z^B} - \frac{\partial G_{BC}}{\partial Z^D}\right)\ . \tag{12}$$

The pair $(\mathcal{E}, g)$ is called equilibrium manifold if the harmonic map $\varphi : \mathcal{E} \to \mathcal{T}$ satisfies the condition

$$\varphi^*(\Theta) = \varphi^*(d\Phi - \delta_{ab}\, I^a\, dE^b) = 0\ . \tag{13}$$

The last condition implies that

$$d\Phi = I_a dE^a\ , \quad \frac{\partial \Phi}{\partial E^a} = I_a\ . \tag{14}$$

The first of these equations corresponds to the first law of thermodynamics whereas the second one is usually known as the condition for thermodynamic equilibrium [5].

We see that the harmonic map $\varphi : \mathcal{E} \to \mathcal{T}$ defines the equilibrium manifold $(\mathcal{E}, g)$ as an extremal submanifold of the phase manifold $(\mathcal{T}, \Theta, G)$ in which the first law of thermodynamics and the equilibrium conditions hold. This means that the thermodynamic systems are represented through the equilibrium manifold and that the phase manifold is an auxiliary geometric structure that allows us to handle correctly the Legendre transformations and to define the equilibrium manifold in an invariant manner. The harmonic map φ demands the existence of the function $\Phi = \Phi(E^a)$ that is known in ordinary thermodynamics as the fundamental equation from which all the equations of state can be obtained [5]. The second law of thermodynamics implies that the fundamental equation satisfies the condition

$$\pm\frac{\partial^2\Phi}{\partial E^a \partial E^b} \geq 0\ , \tag{15}$$

where the sign depends on the thermodynamic potential. For instance, if Φ is identified as the entropy, the sign must be positive whereas it is negative if Φ is the internal energy of the system [5].

The metric g of the equilibrium manifold is determined uniquely from the metric G by means of $g = \varphi^*(G)$. Therefore, the invariance of G under Legendre transformations implies the invariance of g. However, as mentioned above, Legendre transformations act only on the phase manifold and so to investigate the invariance of g it is necessary to apply Legendre transformations on the metric G in $\mathcal{T}$ that generates g. The pullback φ^* of the Legendre invariant metric (4) generates the following thermodynamic metric

$$g = \Lambda \left(E_a \Phi_a\right)^{2k+1} \delta^{ab} \Phi_{bc} dE^a dE^c \ , \tag{16}$$

where

$$\Phi_a = \frac{\partial \Phi}{\partial E^a} \ , \quad \Phi_{bc} = \frac{\partial^2 \Phi}{\partial E^b \partial E^c} \ , \tag{17}$$

which can be shown to be invariant with respect to arbitrary (partial and total) Legendre transformations. On the other hand, the metric (5) of the phase manifold generates the thermodynamic metric

$$g = \Lambda \left(\xi_a{}^b E^a \Phi_b\right) \left(\chi_a{}^b \Phi_{bc} dE^a dE^c\right) \ , \tag{18}$$

where

$$\xi_a{}^b = \xi_{ac} \delta^{bc} \ , \quad \chi_a{}^b = \chi_{ac} \delta^{bc} \ , \tag{19}$$

which is invariant with respect to total Legendre transformations. Notice that the explicit components of the thermodynamic metric g can be calculated in a straightforward manner once the fundamental equation $\Phi(E^a)$ is explicitly given.

4. Quasi-static thermodynamic processes

In ordinary thermodynamics, a quasi-static process is a thermodynamic process that happens infinitely slowly so that it can be ensured that the system will pass through a sequence of states that are infinitesimally close to equilibrium and, consequently, the system remains in quasi-static equilibrium. Since each point of the manifold $\mathcal{E}$ represents an equilibrium state, a quasi-static process can be interpreted as a sequence of points, i. e., as a curve in $\mathcal{E}$. In particular, the geodesic curves of $\mathcal{E}$ can represent quasi-static processes under certain conditions. A geodesic curve can be interpreted as a harmonic map from a 1-dimensional base space to the equilibrium manifold $(\mathcal{E}, g)$. The corresponding action represents a distance in $\mathcal{E}$ that we denote as the thermodynamic length $S = \int ds$ with $ds^2 = g_{ab} dE^a dE^b$. Then, the variation of the thermodynamic length leads to the geodesic equation

$$\frac{d^2 E^a}{d\tau^2} + \Gamma^a{}_{bc} \frac{dE^b}{d\tau} \frac{dE^c}{d\tau} = 0 \ , \tag{20}$$

where $\Gamma^a{}_{bc}$ are the Christoffel symbols of the thermodynamic metric g, and τ is an arbitrary affine parameter along the geodesic.

One can expect that not all the solutions of the geodesic equations must be physically realistic. Indeed, there could be geodesic curves connecting equilibrium states that are not compatible with the laws of thermodynamics. In particular, one would expect that the second law of thermodynamics imposes strong requirements on the solutions. In ordinary thermodynamics two equilibrium states are related to each other only if they can be connected by means of quasi-static process. Then, a geodesic that connects two physically meaningful states can be interpreted as representing a quasi-static process. Since a geodesic curve is a dense succession of points, we conclude that a quasi-static process can be seen as a dense succession of equilibrium states, a statement which coincides with the definition of quasi-static processes in equilibrium thermodynamics [5]. Furthermore, the affine parameter τ can be used to label all equilibrium states which belong to a geodesic. Since the affine parameter is defined up to a linear transformation, it should be possible to choose it in such a way that it increases as the entropy of a quasi-static process increases. This opens the possibility of interpreting the affine parameter as a "time" parameter with a specific direction which coincides with the direction of entropy increase.

5. Ordinary Thermodynamic systems

The mathematical tools presented in the last sections allow us to define geometric structures in an invariant way. In particular, the curvature of the thermodynamic metric g should represent the thermodynamic interaction independently of the thermodynamic potential. In fact, this is not a trivial condition from a geometric point of view. For instance, a geometric analysis of black hole thermodynamics by using metrics introduced *ad hoc* in the equilibrium manifold leads to contradictory results [11-19]. Using the induced thermodynamic metric g as defined in Section 3. for systems with second order phase transitions, the results are consistent and invariant. To illustrate the formalism of GTD we now investigate the geometric representation of some ordinary thermodynamic systems.

5.1 The ideal gas

As a concrete example of the application of GTD, we consider a mono-component ideal gas. This corresponds to the particular case $n = 2$ of the metrics given in the last section. The corresponding fundamental equation can be written as $U(S, V) = [\exp(S/\kappa)/V]^{2/3}$, where κ is a constant. In this particular case, it turns out that the entropy representation is more convenient for the investigation of the field equations. To transform the results of the previous sections into the entropy representation, we notice that in this case the first law of thermodynamics is written as $dS = (1/T)dU + (P/T)dV$ so that the fundamental equation must be given as $S = S(U, V)$, and the conditions of thermodynamic equilibrium are $1/T = \partial S/\partial U$ and $P/T = \partial S/\partial V$. Consequently, in the entropy representation, the 5-dimensional phase manifold can be described by means of the coordinates

$$Z^A = \left\{ S, U, V, \frac{1}{T}, \frac{P}{T} \right\} \tag{21}$$

and the Riemannian metric (4) takes the form

$$G = \left(dS - \frac{1}{T}dU - \frac{P}{T}dV \right)^2 + \Lambda \left[\left(\frac{U}{T} \right)^{2k+1} dU d\left(\frac{1}{T} \right) + \left(\frac{VP}{T} \right)^{2k+1} dV d\left(\frac{P}{T} \right) \right] . \tag{22}$$

Moreover, the explicit form of the Riemannian metric for the equilibrium manifold can be derived from Eq.(16). Then

$$g = \Lambda \left\{ \left(U\frac{\partial S}{\partial U}\right)^{2k+1} \frac{\partial^2 S}{\partial U^2} dU^2 + \left(V\frac{\partial S}{\partial V}\right)^{2k+1} \frac{\partial^2 S}{\partial V^2} dV^2 \right.$$
$$\left. + \left[\left(U\frac{\partial S}{\partial U}\right)^{2k+1} + \left(V\frac{\partial S}{\partial V}\right)^{2k+1} \right] \frac{\partial^2 S}{\partial U \partial V} dU dV \right\} . \tag{23}$$

It should be mentioned that this form of the thermodynamic metric is valid for any thermodynamic system with two degrees of freedom represented by the extensive variables U and V. It is only necessary to specify the fundamental equation $S = S(U, V)$ in order to completely determine the form of the metric. In the specific case of an ideal gas, the fundamental equation can be expressed as

$$S(U, V) = \frac{3\kappa}{2} \ln U + \kappa \ln V \ . \tag{24}$$

A straightforward computation leads to the metric

$$g = -\kappa^{2k+2} \Lambda \left[\left(\frac{3}{2} \right)^{2k+2} \frac{dU^2}{U^2} + \frac{dV^2}{V^2} \right] . \tag{25}$$

All the geometrothermodynamical information about the ideal gas must be contained in the metric (25). First, we must show that the subspace of equilibrium states $(\mathcal{E}, g)$ determines and extremal hypersurface in the phase manifold $(\mathcal{T}, G)$. The identification of the coordinates in $\mathcal{T}$ is as given in Eq.(21) so that the Christoffel symbols $\Gamma^A_{\ BC}$ for the metric components G_{AB} can be computed in a straightforward way. Then, the field equations can be reduced to

$$\frac{\partial \Lambda}{\partial U} + \frac{3\kappa}{2U^2} \frac{\partial \Lambda}{\partial Z^3} + 2(k+1) \frac{\Lambda}{U} = 0 \ , \tag{26}$$
$$\frac{\partial \Lambda}{\partial V} + \frac{\kappa}{V^2} \frac{\partial \Lambda}{\partial Z^4} + 2(k+1) \frac{\Lambda}{V} = 0 \ . \tag{27}$$

These are the conditions for the space of equilibrium states of the ideal gas to be an extremal hypersurface of the thermodynamic phase space. Clearly, the arbitrariness contained in the conformal factor Λ allows us to find many solutions to the above equation. For instance, if we choose $\Lambda =$ const. and $k = -1$, we obtain a particular solution which is probably the simplest one. This shows that the geometry of the ideal gas is a solution to the motion equations of GTD. This special choice leads to the metric

$$g = \frac{dU^2}{U^2} + \frac{dV^2}{V^2} \tag{28}$$

whose curvature scalar vanishes identically. This result agrees with our intuitive expectation that a thermodynamic metric with zero curvature should describe a system in which no thermodynamic interaction is present.

To continue the analysis of the geometry of the ideal gas we now investigate the geodesic equations. By means of the transformation $\xi = \ln U, \eta = \ln V$, the metric (28) takes the form

$$g = d\xi^2 + d\eta^2 \ , \tag{29}$$

where for simplicity we set the additive constants of integration such that $\xi, \eta \geq 0$. The solutions of the geodesic equations are then found as $\xi = \xi_1 \lambda + \xi_0$ and $\eta = \eta_1 \lambda + \eta_0$, where ξ_0, ξ_1, η_0 and η_1 are constants. This solution represents straight lines which on a $\xi\eta-$plane can be depicted by using the equation $\xi = c_1 \eta + c_0$, with constants c_0 and c_1. With our choice of integration constants, the only allowed range of values for ξ and η is within the quadrant determined by $\xi \geq 0$ and $\eta \geq 0$.

In this representation, the entropy becomes a simple linear function of the coordinates and can be expressed as $S = (3\kappa/2)\xi + \kappa\eta$. Since each point on the $\xi\eta-$plane can represent an equilibrium state, the geodesics should connect those states which are allowed by the laws of thermodynamics. For instance, consider all geodesics with initial state $\xi = 0$ and $\eta = 0$. Then, any straight line pointing outwards of the initial zero point and contained inside the allowed positive quadrant connect states with increasing entropy. This behavior is schematically depicted in Fig.1 where the arrows indicate the direction in which a quasi-static process can take place. A quasi-static process connecting states in the inverse direction is not allowed by the second law of thermodynamics. Consequently, the affine parameter τ along the geodesics can actually be interpreted as a time parameter and the direction of the geodesics indicates the "arrow of time". If the initial state is not at the origin of the $\xi\eta-$plane, the second law permits the existence of geodesics for which one of the coordinates, say η, can decrease as long as the other coordinate ξ increases in such a way that the entropy increases or remains constant. This is schematically depicted in Fig.1 which also contains the region that cannot be reached by geodesics.

5.2 The van der Waals gas

A more realistic model of a gas, which takes into account the size of the particles and a pairwise attractive force between the particles of the gas, is based upon the van der Waals fundamental equation

$$S = \frac{3\kappa}{2} \ln \left(U + \frac{a}{V} \right) + \kappa \ln(V - b) \ , \tag{30}$$

where a and b are constants. Usually, a is interpreted as being responsible for the thermodynamic interaction, whereas b plays a more qualitative role in the description of the interaction [5].

The Riemannian structure of the manifold $\mathcal{T}$ is as before determined by the metric (4). For the sake of simplicity, we limit ourselves to the case with $k = -1$. Then, introducing the fundamental equation (30) into the metric (16) with $k = -1$, the Riemannian

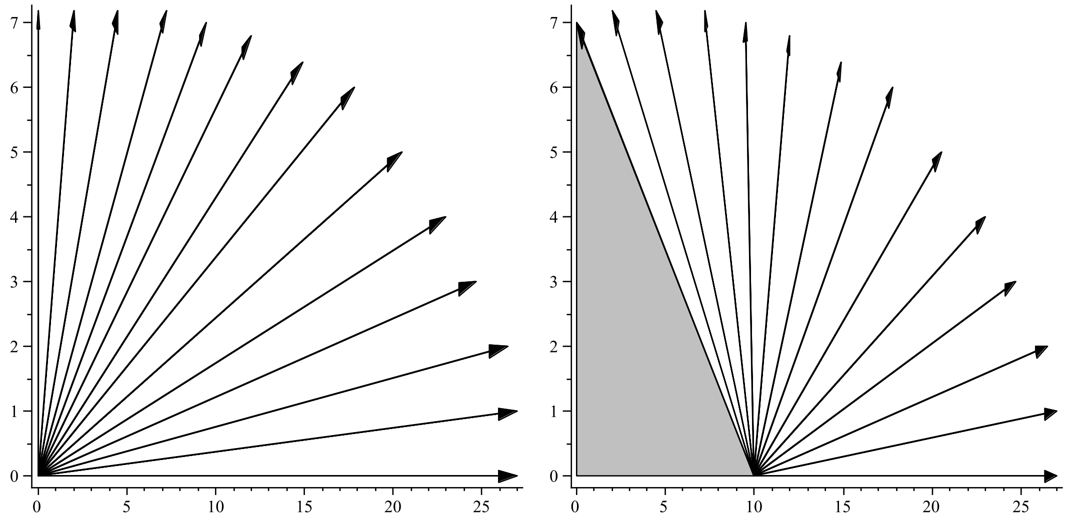

Fig. 1 Left figure: Geodesics in the space of equilibrium states of the ideal gas. All the states contained in the quadrant can be reached by only one geodesic which starts from the initial state that coincides with the origin of coordinates. The arrows show the direction in which entropy increases. Right figure: Geodesics with an initial state situated outside the origin of coordinates. The shadow region contains all the states that due to the second law cannot be reached by geodesics with the fixed initial state. In all the geodesics the "arrow of time" is a consequence of the second law.

structure of the manifold $\mathcal{E}$ is described by the metric

$$g = \frac{\Lambda}{U(U+a/V)}\left[-dU^2 + \frac{U}{V^3}\frac{a(a+2UV)(3b^2-6bV+V^2)-2U^2V^4}{(V-b)(3ab-aV+2UV^2)}dV^2 \right. \\ \left. +\frac{a}{V^2}\frac{3ab-aV-3bUV+5UV^2}{3ab-aV+2UV^2}dUdV \right] . \tag{31}$$

The curvature of this thermodynamic metric is in general non-zero, reflecting the fact that the thermodynamic interaction of the van der Waals gas is non-trivial. Furthermore, the scalar curvature of the above metric can be written in the form

$$R = \frac{a\mathcal{N}^{vdW}}{\left(PV^3 - aV + 2ab\right)^2} \tag{32}$$

where $\mathcal{N}^{vdW}$ is a function of U, and V that is well–behaved at the points where the denominator vanishes. We see that the scalar curvature diverges at the critical points determined by the algebraic equation $PV^3 - aV + 2ab = 0$. This is exactly the equation that determines the location of first order phase transitions of the van der Waals gas [5]. Consequently, a first order phase transition can be interpreted geometrically as a curvature singularity. This is in accordance with our intuitive interpretation of thermodynamic curvature.

The motion equations (11) can be derived explicitly for this case by using the phase manifold metric (4), with $k = -1$, and the metric (16) for the equilibrium manifold. It turns out that the motions equations reduce to only two first order partial differential

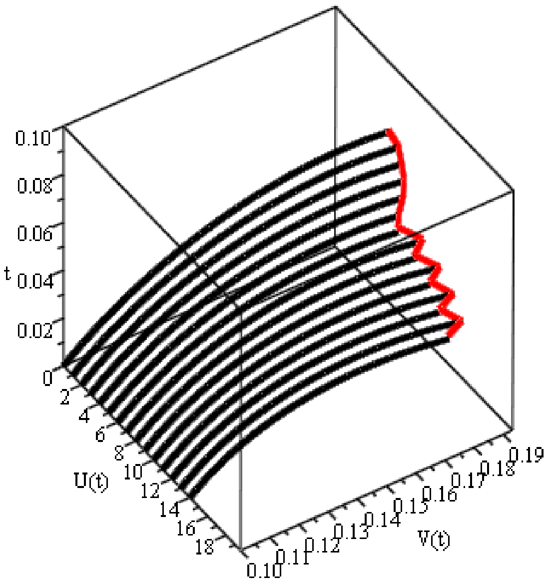

Fig. 2 Geodesics in the equilibrium manifold of the van der Waals gas for different initial values $U(\tau=0)$ and $V(\tau=0)=0.1$. The end points of the thermodynamic variables are associated with first order phase transitions.

equations that can be expressed as

$$\frac{\partial\Lambda}{\partial U} + F_3\frac{\partial\Lambda}{\partial Z^3} + F_4\frac{\partial\Lambda}{\partial Z^4} + F_0\Lambda = 0\ , \tag{33}$$

$$\frac{\partial\Lambda}{\partial V} + G_3\frac{\partial\Lambda}{\partial Z^3} + G_4\frac{\partial\Lambda}{\partial Z^4} + G_0\Lambda = 0\ , \tag{34}$$

where F_0, F_3, F_4, G_0, G_3, and G_4 are fixed rational functions of U and V. Because of the arbitrariness of the conformal factor Λ it is possible to find solutions to the above system of partial differential equations. We conclude that a family of non-flat thermodynamic metrics can be found that determines an extremal hypersurface in the phase space, and can be used to describe the geometry of the van der Waals gas.

The geodesic equations in the manifold described by the van der Waals metric (31) are highly non-trivial and require a numerical analysis [29]. The results are illustrated in Fig.2. The main observation is that the geodesics are incomplete, i.e, there exist a maximum value of the affine parameter τ_{max} for which the numerical integration delivers an end value of $U(\tau_{max})$ and $V(\tau_{max})$. We analyzed numerically the end points $U(\tau_{max})$ and $V(\tau_{max})$ and fount that at those points the relationship $PV^3 - aV + 2ab = 0$ is satisfied. We conclude that the geodesic incompleteness is due to the appearance of first order phase transitions. Since geodesic incompleteness is usually associated with the existence of curvature singularities (see, for instance, [30]) the above result result corroborates the fact that phase transitions correspond curvature singularities in the equilibrium space.

6. Conclusions

In this paper, we presented the most important mathematical elements of geometrothermodynamics (GTD), a formalism whose main goal is to describe in an invariant manner the properties of thermodynamic systems by using geometric concepts. We use the concepts of contact geometry to define the thermodynamic phase manifold and to handle correctly the first law of thermodynamics and the Legendre transformations. The phase manifold must be endowed with a Legendre invariant metric. We present the most general metric which is invariant with respect to partial and total Legendre transformations. If we limit ourselves to the case of total Legendre transformations there are several metrics that preserve this symmetry. It turns out that it is necessary to use different metrics to describe thermodynamic systems with either first order or second order phase transitions. We expect to explore in the near feature the cause of this difference.

The equilibrium manifold is defined by means of a harmonic map in which the target space is the phase manifold. In this context, the equilibrium manifold turns out to be an extreme submanifold of the phase manifold endowed with a Riemannian thermodynamic metric which is determined uniquely in terms of the Legendre invariant metric introduced *ad hoc* in the phase manifold. The construction is such that only the fundamental equation of the thermodynamic system is necessary in order to completely construct the geometry of the equilibrium manifold whose geometric properties are related to thermodynamic properties of the system. In particular, the thermodynamic interaction is described by means of the curvature, and phase transitions of the thermodynamic system correspond to true curvature singularities of the equilibrium manifold. In this work, it was shown explicitly that the curvature is a measure of the thermodynamic interaction in the case of the ideal gas and the van der Waals gas. This statement has been confirmed in all the cases in which GTD has been applied so far [25,31-36].

As concrete examples of the application of GTD, we present the thermodynamic metric of the ideal gas and the van der Waals gas. In the case of the ideal gas, the metric is flat as a result of the lack of thermodynamic interaction. In a particular coordinate system, the geodesics are represented as straight lines. Those geodesics which are in accordance with the laws of thermodynamics turn out to represent quasi–static processes. In the case of the van der Waals gas, the metric is curved, indicating the presence of mechanical thermodynamic interaction between the constituents of the gas. True curvature singularities are found at those points where the gas undergoes a first order phase transition. The geodesics of the equilibrium manifold of the van der Waals gas are shown to be incomplete at those points where phase transitions occur. This could be used as an alternative method to find critical points where phase transitions take place and curvature singularities exist.

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Nonlinear Hidden Symmetries in General Relativity and String Theory: a Matrix Generalization of the Ernst Potentials

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Abstract: In this paper we recall a simple formulation of the stationary electrovacuum theory in terms of the famous complex Ernst potentials, a pair of functions which allows one to generate new exact solutions from known ones by means of the so–called nonlinear hidden symmetries of Lie–Bäcklund type. This formalism turned out to be very useful to perform a complete classification of all 4D solutions which present two spacetime symmetries or possess two Killing vectors. Curiously enough, the Ernst formalism can be extended and applied to stationary General Relativity as well as the effective heterotic string theory reduced down to three spatial dimensions by means of a (real) matrix generalization of the Ernst potentials. Thus, in this theory one can also make use of nonlinear matrix hidden symmetries in order to generate new exact solutions from seed ones. Due to the explicit independence of the matrix Ernst potential formalism of the original theory (prior to dimensional reduction) on the dimension D, in the case when the theory initially has $D \geq 5$, one can generate new solutions like *charged* black holes, black rings and black Saturns, among others, starting from uncharged field configurations.

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1. Introduction

It is well known that both, the stationary action and the coupled field equations of the Einstein–Maxwell theory can be formulated in terms of a pair of very simple complex functions that were called *Ernst potentials* after their inventor [1, 2]. In the language of these potentials, the black holes of Schwarzschild and Kerr, Reissner–Nordström and Kerr–Newmann adopt a very simple form, as well as some cosmological models, among other exact solutions [1, 3]. Indeed, this formalism facilitates the general study of the symmetries of the theory and, hence, the construction of new exact solutions by means of very well-known solution–generating techniques (see, for instance, [4]).

It turns out that the Ernst formalism can be generalized to low–energy effective string theories and General Relativity with extra dimensions in terms of matrix potentials instead of complex functions (see [3,5-7], for instance). This matrix formalism also enables one to study the complete symmetry group of the underlying theory and to apply generalized solution–generating techniques with matrix charges involved [8-9]. In particular, this matrix formalism can be applied to the classification and construction of charged black holes, black rings and black Saturns in 5D and multiple black rings in $D \geq 6$ in the framework of such theories [10,11].

In this paper we first recall the derivation of the Ernst potentials for the stationary Einstein–Maxwell theory and write both field equations and the effective action in their language. We further refer to the stationary formulation of the low–energy heterotic string theory, and the corresponding field equations, in terms of a pair of matrix Ernst potentials that closely resembles the formulation of the stationary theory of electrovacuum in the language of the complex Ernst potentials. A fact that, in principle, allows one to generalize all the so far obtained results in the stationary Einstein–Maxwell theory to the realm of the stationary heterotic string theory.

As an extra bonus, within the framework of higher dimensional General Relativity and the low energy limit of heterotic string theory, the matrix Ernst potentials can be used to classify and construct exact solutions that corresponds to higher dimensional objects like black holes, black rings, black Saturns and multiple black rings. A sketch of how this program can be performed is given at the end of this paper.

2. Ernst potentials in the stationary Einstein–Maxwell theory

In this section we briefly review the derivation of the Ernst potentials within the framework of the stationary Einstein–Maxwell theory basically following the work given by [2].

Let us consider the 4D action of the electrovacuum theory

$$\mathcal{S}_{EM} = \int d^4x \mid G \mid^{\frac{1}{2}} \left({}^4R - \frac{1}{4} F_{mn}^2 \right), \tag{1}$$

where G is the determinant of the metric G_{mn}, $F_{mn} = \partial_m A_n - \partial_n A_m$, A_m is the gauge field, ${}^4R$ is the scalar curvature in 4D and $m, n, = 0, 1, 2, 3;\ \ \mu, \nu = 1, 2, 3.$

Consider now the stationary ansatz for the metric

$$ds^2 = G_{mn}dx^m dx^n = -f(dt + \omega_\mu dx^\mu)^2 + f^{-1}\gamma_{\mu\nu}dx^\mu dx^\nu, \tag{2}$$

where f, $\gamma_{\mu\nu}$ and ω_μ are quantities independent on t.

Indices of spatial coordinates are raised and lowered with the aid of the metric tensor $\gamma_{\mu\nu}$ and its inverse $\gamma^{\mu\nu}$, unless otherwise indicated through a left superindex $^{(0)}$.

Thus, if F_{mn} is a covariant tensor, then

$$F^{\alpha\beta} = \gamma^{\alpha\mu}\gamma^{\beta\nu}F_{\mu\nu} \quad \text{and} \quad {}^{(0)}F^{\alpha\beta} = g^{\alpha m}g^{\beta n}F_{mn}.$$

The three–dimensional vector ω_μ can always be dualized through an invariant torsion vector in the following form

$$f^{-2}\tau^\mu = -\gamma^{-1/2}\epsilon^{\mu\rho\sigma}\partial_\rho\omega_\sigma \tag{3}$$

or, equivalently,

$$f^{-2}\vec{\tau} = -\nabla\times\vec{\omega}, \tag{4}$$

by making use of the three–dimensional vectorial calculus which employs $\gamma_{\mu\nu}dx^\mu dx^\nu$ as background metric.

Let us now consider a stationary electromagnetic field $F_{mn} = \partial_m A_n - \partial_n A_m$ with the given metric.

The stationarity condition $\partial_0 A_m = 0$ for the electric field implies

$$F_{0\nu} = -\partial_\nu A_0, \tag{5}$$

while the sourceless Maxwell equations

$$\partial_\nu\left[(-g)^{1/2}\,{}^{(0)}F^{m\nu}\right] = 0 \tag{6}$$

in the case when $m = \mu$ provide us with the magnetic components

$${}^{(0)}F^{\mu\nu} = f\gamma^{-1/2}\epsilon^{\mu\nu\rho}\partial_\rho\psi, \tag{7}$$

in terms of the scalar magnetic potential ψ.

It turns out that all the remaining components can be expressed as functions of these six magnitudes; for instance,

$${}^{(0)}F^{0\nu} = \omega_\mu{}^{(0)}F^{\mu\nu} + \gamma^{\mu\nu}F_{0\mu}, \tag{8}$$

is an identity that is directly inferred from the stationary metric.

By substituting the relations (8), (7), (5) and (3) in the Maxwell equations (6) with $m = 0$ one gets

$$\nabla\left(f^{-1}\nabla A_0\right) = -f^{-2}\vec{\tau}\cdot\nabla\psi. \tag{9}$$

By rewriting $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ with the aid of the relations (5) and (7), and making use of the expression for the cyclic identity $\epsilon^{\mu\nu\rho}\partial_\rho F_{\mu\nu} = 0$, one obtains

$$\nabla\left(f^{-1}\nabla\psi\right) = f^{-2}\vec{\tau}\cdot\nabla A_0. \tag{10}$$

Now one is able to introduce the scalar complex potential

$$\Phi = A_0 + i\psi, \tag{11}$$

which is precisely the electromagnetic Ernst potential.

By combining (9) and (10) one obtains a single complex equation

$$\nabla\left(f^{-1}\nabla\Phi\right) = if^{-2}\vec{\tau}\cdot\nabla\Phi. \tag{12}$$

Thus, in this way we have reduced the stationary Maxwell equations to a single equation in terms of the complex electromagnetic Ernst potential.

On the other side, within the framework of the Einstein equations for the gravitational field, it turns out convenient to express the Ricci tensor

$$R_{mn} = \partial_m\Gamma^a_{\ na} - \partial_a\Gamma^a_{\ mn} + \Gamma^a_{\ bm}\Gamma^b_{\ an} - \Gamma^a_{\ ba}\Gamma^b_{\ mn} \tag{13}$$

in terms of a complex three–dimensional vector $\vec{G}$ defined by

$$2f\vec{G} = \nabla f + i\vec{\tau} \tag{14}$$

for the general case of the stationary metric.

In this way we can obtain the following relations

$$-f^{-2}R_{00} = \nabla\vec{G} + \left(\vec{G}^* - \vec{G}\right)\cdot\vec{G}, \tag{15}$$

$$-2if^{-2}\,{}^{(0)}R^\mu_0 = \gamma^{-1/2}\epsilon^{\mu\rho\sigma}\left(\partial_\sigma G_\rho + G_\rho G^*_\sigma\right), \tag{16}$$

$$f^{-2}\left(\gamma_{\rho\mu}\gamma_{\sigma\nu}\,{}^{(0)}R^{\mu\nu} - \gamma_{\rho\sigma}R_{00}\right) = R_{\rho\sigma}(\gamma) + G_\rho G^*_\sigma + G^*_\rho G_\sigma, \tag{17}$$

where $R_{\rho\sigma}(\gamma)$ stands for the Ricci tensor calculated through the three–dimensional metric $\gamma_{\mu\nu}dx^\mu dx^\nu$.

Thus, from the above obtained formulas, for the energy–momentum tensor of the electromagnetic field

$$-4\pi T_{mn} = g^{ab}F_{ma}F_{nb} - \frac{1}{4}g_{mn}F_{ab}F^{ab} \tag{18}$$

one gets the following relations

$$\frac{1}{2}F_{mn}F^{mn} = (\nabla\psi)^2 - (\nabla A_0)^2, \tag{19}$$

$$8\pi f^{-1}T_{00} = (\nabla\psi)^2 + (\nabla A_0)^2\,, \tag{20}$$

$$4\pi f^{-1}\,{}^{(0)}T^{\mu}_{0} = \gamma^{-1/2}\epsilon^{\nu\rho\sigma}\left(\partial_\rho\psi\right)\left(\partial_\sigma A_0\right), \tag{21}$$

$$-4\pi f^{-1}\,{}^{(0)}T^{\mu\nu} = \left(\partial^\mu\psi\right)\left(\partial^\nu\psi\right) + \left(\partial^\mu A_0\right)\left(\partial^\nu A_0\right) - \frac{1}{2}\,\gamma^{\mu\nu}\left[(\nabla\psi)^2 + (\nabla A_0)^2\right], \tag{22}$$

where $\partial^\mu = \gamma^{\mu\nu}\partial_\nu$.

By making use of the Einstein equations

$$R_{mn} = -8\pi T_{mn}, \tag{23}$$

from the relations (16) and (21) one obtains

$$\nabla\times\vec{\tau} = -4\nabla\psi\times\nabla A_0 = i\nabla\times\left(\Phi\nabla\Phi^* - \Phi^*\nabla\Phi\right). \tag{24}$$

In this way, the following equation

$$\vec{\tau} + i\left(\Phi^*\nabla\Phi - \Phi\nabla\Phi^*\right) = \nabla\chi \tag{25}$$

defines the scalar potential χ up to an additive constant.

Now let us define the complex scalar potential

$$E = f - \Phi\Phi^* + i\chi, \tag{26}$$

called gravitational Ernst potential.

This potential allows one to obtain, from the relations (14) and (25), the following equality

$$f\vec{G} = \frac{1}{2}\nabla E + \Phi^*\nabla\Phi. \tag{27}$$

By substituting (27) in the gravitational field equations (15) and (20), and making use of the Maxwell equations (12), we obtain a single equation

$$f\nabla^2 E = \left(\nabla E + 2\Phi^*\nabla\Phi\right)\cdot\nabla E; \tag{28}$$

on the other hand, the relation (12) can be expressed in the following way:

$$f\nabla^2\Phi = \left(\nabla E + 2\Phi^*\nabla\Phi\right)\cdot\nabla\Phi. \tag{29}$$

It is evident that from the definition (26), one can obtain the following expression for the function f :

$$f = \frac{1}{2}\left(E + E^*\right) + \Phi\Phi^*. \tag{30}$$

Thus, relations (28) and (29) are the well–known differential Ernst equations for the stationary electrovacuum.

Finally, the gravitational field equations (17) and (22) reduce to the following expression

$$-f^2 R_{\mu\nu} = \frac{1}{2} E_{,(\mu} E^*_{,\nu)} + \Phi E_{,(\mu} \Phi^*_{,\nu)} + \Phi^* E^*_{,(\mu} \Phi_{,\nu)} - (E + E^*)\, \Phi_{,(\mu} \Phi^*_{,\nu)}, \tag{31}$$

where the symmetrization of indices are defined in the following form

$$2E_{,(\mu} E^*_{,\nu)} \equiv (\partial_\mu E)(\partial_\nu E^*) + (\partial_\nu E)(\partial_\nu E^*). \tag{32}$$

In this way, the field equations for the Ernst potentials (28) and (29), together with the Einstein equations (31), determine the dynamics of the field system of the stationary Einstein–Maxwell theory.

This system of self–consistent second order differential equations, despite their apparent simplicity, has no general solution at the moment. Only particular solutions are known in the literature and it is of great relevance to obtain new solutions possessing a coherent and consistent physical interpretation. It is worth noticing that precisely at this point is where the solution–generating techniques (which make use of nonlinear hidden symmetries to construct new solutions starting from seed ones) can be of great help towards this aim.

2.1 Effective action of the stationary EM theory and Ernst potentials

Now let us express the effective action of the stationary Einstein–Maxwell theory from which one can derive both the Einstein equations (31), and the Ernst equations (28) and (29) by the variational method.

By redefining the electromagnetic Ernst potential as follows

$$\Phi \equiv \frac{1}{\sqrt{2}} F, \tag{33}$$

the effective stationary action of the Einstein–Maxwell theory adopts the following form

$$^4\mathcal{S}_{EM} = \int d^3x \mid g \mid^{\frac{1}{2}} \left(-^3R + ^3\mathcal{L}_{EM} \right),$$

where the matter Lagrangian $^3\mathcal{L}_{EM}$ is given by

$$^3\mathcal{L}_{EM} = \frac{1}{2f^2} |\nabla E + F^* \nabla F|^2 - \frac{1}{f} |\nabla F|, \tag{34}$$

where now $f = \frac{1}{2}(E + E^* + FF^*)$. It is a straightforward exercise to vary this action and obtain the above quoted Einstein and Ernst equations.

3. Low energy effective action of heterotic string and matrix Ernst potentials

The effective action of the low–energy limit of the heterotic string at tree level takes into account just the massless modes of the theory and possesses the form [12, 13]

$$\mathcal{S}^{(D)} = \int d^{(D)}x \mid G^{(D)} \mid^{\frac{1}{2}} e^{-\phi^{(D)}} \left(R^{(D)} + \phi^{(D)}_{;M} \phi^{(D);M} - \right.$$

$$\frac{1}{12}H^{(D)}_{MNP}H^{(D)MNP}-\frac{1}{4}F^{(D)I}_{MN}F^{(D)IMN}\Bigg), \tag{35}$$

where

$$F^{(D)I}_{MN}=\partial_M A^{(D)I}_N-\partial_N A^{(D)I}_M, \qquad I=1,\,2,\,...,n;$$

$$H^{(D)}_{MNP}=\partial_M B^{(D)}_{NP}-\frac{1}{2}A^{(D)I}_M\,F^{(D)I}_{NP}+\text{cyclic perms. of M, N and P.}$$

Here $G^{(D)}_{MN}$ is the metric, $B^{(D)}_{MN}$ is the anti–symmetric Kalb–Ramond tensor field, $\phi^{(D)}$ is the dilaton and $A^{(D)I}_M$ is a set of $U(1)$ vector fields ($I=1,\,2,\,...,n$). D is the dimensionality of the spacetime and $M,N,P=1,2,3,...,10$. In the consistent critical case (where the quantum theory is free of anomalies) $D=10$ and $n=16$, but we shall leave these parameters arbitrary in our analysis for the sake of generality.

By following Maharana and Schwarz [12], and Sen [13], we further perform the dimensional reduction of this model on a $D-3=d$–torus. Thus, the resulting three–dimensional, stationary theory possesses the $SO(d+1,d+1+n)$ symmetry group and describes gravity in terms of the metric tensor

$$g_{\mu\nu}=e^{-2\phi}\left(G^{(D)}_{\mu\nu}-G^{(D)}_{p+3,\mu}G^{(D)}_{q+3,\nu}G^{pq}\right), \tag{36}$$

where the subscripts $p,q=1,2,...,d$; coupled to the following set of three–dimensional fields:

a) scalar fields

$$G=\left(G_{pq}=G^{(D)}_{p+3,q+3}\right), \qquad B=\left(B_{pq}=B^{(D)}_{p+3,q+3}\right),$$

$$A=\left(A^I_p=A^{(D)I}_{p+3}\right), \qquad \phi=\phi^{(D)}-\frac{1}{2}\ln|\det G|. \tag{37}$$

b) antisymmetric tensor field of second rank

$$B_{\mu\nu}=B^{(D)}_{\mu\nu}-4B_{pq}A^p_\mu A^q_\nu-2\left(A^p_\mu A^{p+d}_\nu-A^p_\nu A^{p+d}_\mu\right), \tag{38}$$

(hereafter we shall set $B_{\mu\nu}=0$ in order to remove the effective three–dimensional cosmological constant from our consideration).

c) vector fields $A^{(a)}_\mu=\left((A_1)^p_\mu,(A_2)^{p+d}_\mu,(A_3)^{2d+I}_\mu\right)$ $(a=1,...,2d+n)$

$$(A_1)^p_\mu=\frac{1}{2}G^{pq}G^{(D)}_{q+3,\mu}, \quad (A_3)^{I+2d}_\mu=-\frac{1}{2}A^{(D)I}_\mu+A^I_qA^q_\mu,$$

$$(A_2)^{p+d}_\mu=\frac{1}{2}B^{(D)}_{p+3,\mu}-B_{pq}A^q_\mu+\frac{1}{2}A^I_pA^{I+2d}_\mu. \tag{39}$$

In three dimensions all vector fields $A_\mu^{(a)}$, can be dualized on–shell with the aid of the pseudoscalar potentials u, v and s in the following form:

$$\begin{aligned}
\nabla \times \overrightarrow{A_1} &= \frac{1}{2}e^{2\phi}G^{-1}\left(\nabla u + (B + \frac{1}{2}AA^T)\nabla v + A\nabla s\right),\\
\nabla \times \overrightarrow{A_3} &= \frac{1}{2}e^{2\phi}(\nabla s + A^T\nabla v) + A^T\nabla \times \overrightarrow{A_1},\\
\nabla \times \overrightarrow{A_2} &= \frac{1}{2}e^{2\phi}G\nabla v - (B + \frac{1}{2}AA^T)\nabla \times \overrightarrow{A_1} + A\nabla \times \overrightarrow{A_3}.
\end{aligned} \tag{40}$$

Thus, the resulting effective three–dimensional theory describes the scalars G, B, A and ϕ and the pseudoscalars u, v and s coupled to the metric $g_{\mu\nu}$.

We further define the so–called *matrix* Ernst potentials (MEP) from all these scalar and pseudoscalar potentials in order to express the low–energy effective action of the heterotic string in a similar form to the formulation of the stationary Einstein–Maxwell theory in terms of the complex Ernst potentials [6]:

$$\mathcal{X} = \begin{pmatrix} -e^{-2\phi} + v^TXv + v^TAs + \frac{1}{2}s^Ts & v^TX - u^T \\ Xv + u + As & X \end{pmatrix} \quad \text{and} \quad \mathcal{A} = \begin{pmatrix} s^T + v^TA \\ A \end{pmatrix}, \tag{41}$$

where $X = G + B + \frac{1}{2}AA^T$. These potentials are of dimensions $(d+1) \times (d+1)$ and $(d+1) \times n$, respectively.

The physical meaning of their components are as follows: The relevant information about the gravitational field is encoded in the potential X, while its rotational nature is parameterized by the pseudoscalar u; ϕ is the dilatonic field; v is related to the multi–dimensional components of the antisymmetric tensor field of Kalb–Ramond. Finally, A and s represent electric and magnetic potentials.

3.1 Stationary effective action of heterotic string and field equations in the language of MEP

In terms of MEP the effective three–dimensional theory adopts the form [6]:

$$^3\mathcal{S} = \int d^3x \mid g \mid^{\frac{1}{2}} \{-^3R + ^3\mathcal{L}_{HS}\}, \tag{42}$$

where the matter Lagrangian is given by

$$\mathcal{L}_{HS} = \mathrm{Tr}\left[\frac{1}{4}\left(\nabla\mathcal{X} - \nabla\mathcal{A}\mathcal{A}^T\right)\mathcal{G}^{-1}\left(\nabla\mathcal{X}^T - \mathcal{A}\nabla\mathcal{A}^T\right)\mathcal{G}^{-1} + \frac{1}{2}\nabla\mathcal{A}^T\mathcal{G}^{-1}\nabla\mathcal{A}\right], \tag{43}$$

^3R is the three–dimensional curvature scalar and the matrix potential $\mathcal{X}$ is defined by $\mathcal{X} = \mathcal{G} + \mathcal{B} + \frac{1}{2}\mathcal{A}\mathcal{A}^T$.

The symmetric part of the potential is given by the matrix $\mathcal{G} = \frac{1}{2}\left(\mathcal{X} + \mathcal{X}^T - \mathcal{A}\mathcal{A}^T\right)$ and the antisymmetric one by $\mathcal{B} = \frac{1}{2}\left(\mathcal{X} - \mathcal{X}^T\right)$; these matrices are parameterized as follows:

$$\mathcal{G} = \begin{pmatrix} -e^{-2\phi} + v^T G v & v^T G \\ G v & G \end{pmatrix} \qquad \text{and} \qquad \mathcal{B} = \begin{pmatrix} 0 & v^T B - u^T \\ B v + u & B \end{pmatrix}. \tag{44}$$

By making use of the conventional method of variations, from the effective action (42) one obtains both the *Einstein equations*

$$^3R_{\mu\nu} = \mathrm{Tr}\left[\frac{1}{4}\left(\nabla_\mu \mathcal{X} - \nabla_\mu \mathcal{A}\mathcal{A}^T\right)\mathcal{G}^{-1}\left(\nabla_\nu \mathcal{X}^T - \mathcal{A}\nabla_\nu \mathcal{A}^T\right)\mathcal{G}^{-1} + \frac{1}{2}\nabla_\mu \mathcal{A}^T \mathcal{G}^{-1}\nabla_\nu \mathcal{A}\right], \tag{45}$$

as well as the *Ernst equations* for the potentials $\mathcal{X}$ and $\mathcal{A}$ which represent the matter sector of the theory:

$$\nabla^2 \mathcal{X} - 2\left(\nabla \mathcal{X} - \nabla \mathcal{A}\mathcal{A}^T\right)\left(\mathcal{X} + \mathcal{X}^T - \mathcal{A}\mathcal{A}^T\right)^{-1}\nabla \mathcal{X} = 0,$$

$$\nabla^2 \mathcal{A} - 2\left(\nabla \mathcal{X} - \nabla \mathcal{A}\mathcal{A}^T\right)\left(\mathcal{X} + \mathcal{X}^T - \mathcal{A}\mathcal{A}^T\right)^{-1}\nabla \mathcal{A} = 0,$$

as a matrix version of the equations of the stationary Einstein–Maxwell theory.

As we have pointed out above, these differential equations are not so simple to solve in a closed form. However, one can make use of the similarity which exists with respect to the equations of the stationary Einstein–Maxwell theory in order to guess and write down the solutions in a direct way or to perform nonlinear symmetries to generate new exact solutions from known ones (for some examples see [14]).

4. Heterotic string *vs.* Einstein–Maxwell

Thus, it has been shown that there exists a close relation between the stationary effective actions of the heterotic string and the Einstein–Maxwell theory:

$$\mathcal{X} \longleftrightarrow -E, \qquad \mathcal{A} \longleftrightarrow F, \tag{46}$$

$$\textit{matrix transposition} \quad \longleftrightarrow \quad \textit{complex conjugation}.$$

One can realize that the relation (46) allows us to generalize in a straightforward way the results obtained within the framework of the Einstein–Maxwell theory to the realm of the heterotic string (where a suitable physical interpretation will be needed since more fields are involved) by making use of the MEP formalism. Actually, the four–dimensional Einstein–Maxwell theory, being reduced to three dimensions, can be written as a special case of the MEP formalism with some peculiarities in terms of the complex Ernst potentials E and F [9].

Let us rewrite them in a less conventional form

$$-\mathcal{X}_{EM} = \mathrm{Re}E + \sigma_2\,\mathrm{Im}E, \quad \mathcal{A}_{EM} = \mathrm{Re}F + \sigma_2\,\mathrm{Im}F, \quad \text{where} \quad \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{47}$$

We can treat these matrices as the matrix Ernst potentials (41) of the $D = 4$ theory (35) with $\phi^{(4)} = B^{(4)}_{MN} = 0$. Then we conclude that we need two Abelian gauge fields $n = 2$ and that they should satisfy the following constraint

$$s^1 = A^2 = \mathrm{Re}F, \quad -s^2 = A^1 = \mathrm{Im}F. \tag{48}$$

Note, that s^I $(I = 1, 2)$ describe the magnetic potentials, whereas A^I are the electric ones. Thus, both Maxwell fields arising in the framework of the representation (41)–(43) and (47) turn out to be mutually conjugated (i.e. $F^{(4)2}_{MN} = \tilde{F}^{(4)1}_{MN}$ in four dimensions). Next, for the single extra metric component one has:

$$G = -\frac{1}{2}\left(E + E^* + FF^*\right) \equiv f, \qquad \text{and} \qquad u = \mathrm{Im}E. \tag{49}$$

By taking into account that $\mathcal{G} = G$, and by substituting equations (47) and (49) into the matter Lagrangian (43), we obtain

$$\mathcal{L}_{EM} = \frac{1}{2f^2}\left|\nabla E + F^*\nabla F\right|^2 - f^{-1}\left|\nabla F\right|^2. \tag{50}$$

As we already have seen, this is precisely the matter Lagrangian of the stationary Einstein–Maxwell theory. Thus, our MEP formulation of the heterotic string theory includes the Einstein–Maxwell theory as a special case.

It is worth noticing as well that the higher dimensional General Relativity theory can also be written in terms of a matrix Ernst potential when reduced to three dimensions. This fact corresponds to a special case in which the matter degrees of freedom of the low–energy heterotic string theory (35) vanish: the anti–symmetric Kalb–Ramond tensor field $B^{(D)}_{MN} = 0$, the dilaton $\phi^{(D)} = 0$ and the Abelian gauge fields $A^{(D)I}_{M} = 0$, so that the matrix Ernst potential is symmetric $\mathcal{X} = \mathcal{G}$ and $\mathcal{B} = \mathcal{A} = 0$. It should also be mentioned that the three-dimensional dilaton field must remain nontrivial since it is identified with the determinant of the extra dimensional metric according to the definitions (36) and (37).

Thus, this parametrization of the above mentioned higher dimensional theories in terms of the MEP can be very useful when performing a complete classification of the higher dimensional ($D \geq 5$) black objects (holes, rings, Saturns, etc.) obtained in the literature during last years (see [10] for a review).

5. Nonlinear hidden symmetries and their possible applications in $D \geq 5$

One of the advantages of the (matrix) Ernst potential formalism is that the study of symmetries (conservation laws) of the stationary effective action can be performed in a very straightforward way. It turns out that the complete symmetry group, apart from rescalings and shifts of the Ernst potentials, involves nonlinear symmetries that were initially called *hidden* in the framework of General Relativity; moreover, an infinite–dimensional double hidden symmetry structure was revealed for string effective actions

[15]. In particular, these symmetries act nontrivially in the charge space of a seed solution and can be used to generate new charged solutions from uncharged ones. There also other effects when applying this symmetries (see, for instance, [4, 5, 9, 16, 17].

Here we shall quote just the symmetries which preserve the asymptotic properties of the (matrix) Ernst potentials for physically meaningful field configurations of both the stationary Einstein–Maxwell and low–energy heterotic string theories. These symmetries possess the same form for both theories and allow one to generate similar solutions in both realms [9].

For the stationary Einstein–Maxwell theory we have:

$$E \to E, \qquad F \to e^{i\alpha} F; \qquad \text{(EMT)} \tag{51}$$

$$E \to \frac{E + i\epsilon}{1 + i\epsilon E}, \qquad F \to \frac{1 - i\epsilon}{1 + i\epsilon E} F; \qquad \text{(NET)} \tag{52}$$

$$E \to \frac{E + \frac{1}{2}\left|\lambda_{\mathcal{H}}\right|^2 - \bar{\lambda}_{\mathcal{H}} F}{1 - \bar{\lambda}_{\mathcal{H}} F + \frac{1}{2}\left|\lambda_{\mathcal{H}}\right|^2 E}, \qquad F \to \frac{\left(1 + \frac{1}{2}\left|\lambda_{\mathcal{H}}\right|^2\right) F - \lambda_{\mathcal{H}}\left(E + 1\right)}{1 - \bar{\lambda}_{\mathcal{H}} F + \frac{1}{2}\left|\lambda_{\mathcal{H}}\right|^2 E}, \qquad \text{(NHT)} \tag{53}$$

where EMT stands for Electric–Magnetic Transformation, NET for Normalized Ehlers Transformation and NHT for Normalized Harrison Transformation, the parameter $\lambda_{\mathcal{H}}$ is complex while the parameters α and ϵ are real. It is easy to check that when the parameters $\lambda_{\mathcal{H}}$, α and ϵ vanish, one recovers the original (seed) potentials.

On the other hand, for the stationary low–energy effective action of the heterotic string we have the following matrix symmetries:

$$\mathcal{X} \to \mathcal{X} + \lambda_{\mathcal{X}}, \qquad \mathcal{A} \to \mathcal{A} \qquad \text{with} \qquad \lambda_{\mathcal{X}}^T = -\lambda_{\mathcal{X}} \tag{54}$$

$$\mathcal{A} \to \mathcal{A} + \lambda_{\mathcal{A}}, \qquad \mathcal{X} \to \mathcal{X} + \mathcal{A}\lambda_{\mathcal{A}}^T + \frac{1}{2}\lambda_{\mathcal{A}}\lambda_{\mathcal{A}}^T \tag{55}$$

$$\mathcal{A} \to \mathcal{A}\mathcal{T}, \qquad \mathcal{X} \to \mathcal{X}, \qquad \text{where} \qquad \mathcal{T}\mathcal{T}^T = 1 \tag{56}$$

$$\mathcal{X} \to \mathcal{S}^T \mathcal{X} \mathcal{S}, \qquad \mathcal{A} \to \mathcal{S}^T \mathcal{A}, \qquad \text{with} \qquad \mathcal{S} \to (\mathcal{S}^T)^{-1}. \tag{57}$$

$$\begin{aligned} &\mathcal{A} \to \left(1 + \Sigma\lambda_{\mathcal{E}}\right)\left(1 + \mathcal{X}\lambda_{\mathcal{E}}\right)^{-1} \mathcal{A}, \qquad \text{(NET)} \\ &\mathcal{X} \to \left(1 + \Sigma\lambda_{\mathcal{E}}\right)\left(1 + \mathcal{X}\lambda_{\mathcal{E}}\right)^{-1} \mathcal{X}(1 - \lambda_{\mathcal{E}}\Sigma) + \Sigma\lambda_{\mathcal{E}}\Sigma. \end{aligned} \tag{58}$$

$$\mathcal{A} \to \left(1 + \frac{1}{2}\Sigma\lambda_{\mathcal{H}}\lambda_{\mathcal{H}}^T\right)\left(1 - \mathcal{A}\lambda_{\mathcal{H}}^T + \frac{1}{2}\mathcal{X}\lambda_{\mathcal{H}}\lambda_{\mathcal{H}}^T\right)^{-1} \times$$

$$(A - \mathcal{X}\lambda_{\mathcal{H}}) + \Sigma\lambda_{\mathcal{H}}, \qquad \text{(NHT)}$$

$$\mathcal{X} \to \left(1 + \frac{1}{2}\Sigma\lambda_{\mathcal{H}}\lambda_{\mathcal{H}}^T\right)\left(1 - \mathcal{A}\lambda_{\mathcal{H}}^T + \frac{1}{2}\mathcal{X}\lambda_{\mathcal{H}}\lambda_{\mathcal{H}}^T\right)^{-1} \times$$

$$\left[\mathcal{X} + \left(\mathcal{A} - \frac{1}{2}\mathcal{X}\lambda_{\mathcal{H}}\right)\lambda_{\mathcal{H}}^T\Sigma\right] + \frac{1}{2}\Sigma\lambda_{\mathcal{H}}\lambda_{\mathcal{H}}^T\Sigma. \tag{59}$$

where $\lambda_{\mathcal{E}}^T = -\lambda_{\mathcal{E}}$ and $\lambda_{\mathcal{H}}$ is a real rectangular matrix of dimension $(d+1) \times n$.

The last pair of nonlinear symmetries can be applied to construct new exact solutions starting from known (sometimes quite simple) field configurations in both theories. As an example one can cite the construction of the of the Reissner–Nordström solution starting from the Schwarzschild black hole one in the 4D Einstein–Maxwell theory.

We finally quote a procedure to construct new charged field configurations from known neutral solutions within the framework of theories like General Relativity and the effective low–energy action of the heterotic string with more than four dimensions (in the spirit of [16, 17]). Thus, this procedure can be applied to the construction of charged black holes, black rings and black Saturns if $D = 5$, and charged multiple black rings in $D = 6$:

(1) Write the exact solution of the uncharged field configuration (black ring or black Saturn, for instance) in the form of a generalized Weyl metric [18, 19] by making use of a suitable coordinate system.
(2) Identify the symmetric and antisymmetric parts of the matrix Ernst potential $\mathcal{X}$.
(3) Perform the nonlinear hidden symmetry NHT on the matrix Ernst potentials $\mathcal{X}$ and $\mathcal{A}$.
(4) Write the new higher–dimensional charged exact solution with the aid of $\mathcal{X}$ and $\mathcal{A}$.
(5) Physically interpret the new solution with the aid of the behaviour of the fields and their properties.

This procedure can be performed also in a wider class of higher–dimensional field configurations that have the form of a stationary axisymmetric seed solution (the so–called Weyl–Papapetrou class) [20] and it is interesting to see what kind of physical configurations arise after applying the MEP symmetry method.

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Concepts of Relative Velocity

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Abstract: The central concept of the theory of relativity is the **relativity** of velocity. The velocity of a material body is not an intrinsic property of the body; it depends on a free choice of reference system. Relative velocity is thus reference-dependent, it is not an absolute concept. We stress that even zero-velocity must be relative. Every reference system possesses its own zero-velocity relative only to that particular reference system. Does the theory of relativity formulated in terms of relative velocities, with many zero-velocities, imply the Lorentz isometry group? We discuss the many relative spaces of Galileo and Poincaré, as quotient spaces.

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1. Einstein's Relative Velocity is Ternary

In the present paper relativity means the historical term 'special relativity', where we drop 'special' because the theory of relativity, in our understanding, is *coordinate-free.*

The Lorentz isometry is frequently presented as the transformation of coordinates. However the concept of an isometry does not exist without coordinate-free metric tensor, *i.e.* without a scalar product.

Starting from the metric tensor Fock derived the following particular Lorentz-boost transformation [Fock 1955, 1959, 1961, 1964 §10 and §16; Jackson 1962, 1975 §11.3]. Our

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question is: Of what exactly is this a transformation?

$$\gamma \equiv \frac{1}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}, \qquad \mathbf{x}' = \mathbf{x} + \frac{\gamma^2}{\gamma+1}\frac{(\mathbf{v}\cdot\mathbf{x})}{c^2}\mathbf{v} - \gamma\mathbf{v}t, \tag{1}$$

$$t' = \gamma\left(t - \frac{\mathbf{v}\cdot\mathbf{x}}{c^2}\right) \qquad \Longleftrightarrow \qquad \frac{\mathbf{v}\cdot\mathbf{x}}{c^2} = t - \frac{t'}{\gamma} \tag{2}$$

Fock's expression needs the scalar product $\mathbf{v}\cdot\mathbf{x}$. One can ask where is this scalar product? In the spacetime? or in a three-dimensional space?

For two-dimensional spacetime (1)-(2) collapses to Einstein's expression below (3) [Einstein 1905], however we must stress that the naive generalization of Einstein's coordinate transformation to more dimensions is *not* an isometry,

$$\begin{array}{ccc} \text{isometry} & & \text{not isometry} \\ & \not\Rightarrow & \\ x' = \gamma\,(x - vt) & & \mathbf{x}' = \gamma\,(\mathbf{x} - \mathbf{v}t) \end{array} \tag{3}$$

We are interested in the precise definition and interpretation of all symbols in (1)-(2). What it is the meaning of the symbol $\mathbf{v}$, generating transformation of what? It is the relative velocity of what body relative to what reference? If $\mathbf{v}$ is a vector, on spacetime or on some space, then (1) implies that also $\mathbf{x}$ must be a vector, and not just a set of coordinates. If the symbol $\mathbf{x}$ denotes a vector, then (1) implies the vanishing of the Grassmann bivector

$$(1) \quad \Longrightarrow \quad (\mathbf{x}' - \mathbf{x}) \wedge \mathbf{v} = 0. \tag{4}$$

Where it is the above bivector (4), in four-dimensional spacetime? or in some three-dimensional space?

Inserting (2) into (1) allows us to express the velocity $\mathbf{v}$ in terms of a vector $\mathbf{x} - \mathbf{x}'$, this solves (4) explicitly. Still there is only an implicit $(\mathbf{x} - \mathbf{x}')$-dependence because of the Lorentz factor $\gamma = \gamma(\mathbf{v})$ is $\mathbf{v}$-dependent,

$$\mathbf{v} = \left(1 + \frac{1}{\gamma(\mathbf{v})}\right)\left(\frac{\mathbf{x} - \mathbf{x}'}{t + t'}\right). \tag{5}$$

In the Galilean limit, $c \longrightarrow \infty$, and $t' = t$, the above expression of relative velocity collapses to the widely accepted expression. Now we can insert (5) into the Lorentz factor γ (1), and this allows us to express γ in terms of the scalar product $(\mathbf{x} - \mathbf{x}')^2$,

$$\frac{\mathbf{v}^2}{c^2} = \left(1 + \frac{1}{\gamma}\right)\left(1 - \frac{1}{\gamma}\right) \qquad \Longrightarrow \qquad \gamma = \frac{1 + \frac{1}{c^2}\left(\frac{\mathbf{x}-\mathbf{x}'}{t+t'}\right)^2}{1 - \frac{1}{c^2}\left(\frac{\mathbf{x}-\mathbf{x}'}{t+t'}\right)^2} \tag{6}$$

Finally inserting $\gamma = \gamma((\mathbf{x} - \mathbf{x}')^2, \ldots)$ (6) into (5) gives the desired operational expression of the velocity in terms of the vector $\mathbf{x} - \mathbf{x}'$, *i.e.* the expression ready for experimental measurement of relative velocity within the Lorentz group-relativity,

$$\mathbf{v} = 2\left\{1 + \frac{1}{c^2}\left(\frac{\mathbf{x} - \mathbf{x}'}{t + t'}\right)^2\right\}^{-1}\left(\frac{\mathbf{x} - \mathbf{x}'}{t + t'}\right). \tag{7}$$

The above explicit expression (7) for the relative velocity parameterizing the Lorentz transformation (1)-(2), was derived by Urbantke in another way, using reflections, *i.e.* involutory isometries, [Urbantke 2003, p. 115, formula (7)]. Previously Ungar derived the same expression using gyration [Ungar 2001, p. 348, Theorem 11.16],

$$\mathbf{v} = \frac{\mathbf{x}'}{t'} \ominus \mathrm{gyr}\left[\frac{\mathbf{x}'}{t'}, \frac{\mathbf{x}}{t}\right] \frac{\mathbf{x}}{t}. \tag{8}$$

The above expression, Ungar (8), and Urbantke (7), can be adopted as the *definition* of relative velocity in Einstein's special relativity in terms of directly measured quantities. However, isometry implies a more general definition. There is also the question of who is actually measuring the relative velocity according to the above formulae (7)?

The expression (7) is an easy consequence of the Lorentz isometry transformation (1)-(2), and at least it should always be presented jointly with the Lorentz transformations. Just to verify, insertion of (7) into (1)-(2), gives an identity.

1.1 Exercise. Using symmetry (10) below, one can show that there are still other expressions for relative velocity that are equivalent to Ungar's and Urbantke's expression (7),

$$\mathbf{v} = \frac{t\,\mathbf{x} - t'\,\mathbf{x}'}{t^2 + \frac{\mathbf{x}'^2}{c^2}} \qquad \text{or} \qquad \mathbf{v} = \frac{t'\,\mathbf{x} - t\,\mathbf{x}'}{t'\,t + \frac{\mathbf{x}\cdot\mathbf{x}'}{c^2}} \tag{9}$$

Do we like this definition of relative velocity? The actual physical concept of relative velocity has not yet even been discussed.

The velocity of one material (or massive) body is always relative to another body, or, we could say that the velocity of a body is always relative to a free *choice* of reference system. In fact relativity theory is a theory of massive reference systems.

Most definitions of velocity, including (7)-(8)-(9), are **obscured** by imposing a coordinate system. These coordinate systems contain implicit, hidden or incomplete, obscure information about the material bodies involved.

If we do not define or make precise the meaning of material body, then the concept of velocity is meaningless. The exact concept of material body is crucial for understanding the concept of velocity. Georg Hegel (1770–1831) wrote: *no motion without matter*.

Light is massless and therefore can not be considered to define a reference system. The same applies to cosmic background radiation. **The velocity of light must *not* be considered a primary concept of the theory of relativity.** A similar opinion is shared by [Paiva and Ribeiro 2005] who claim that *special relativity does not depend on electromagnetism*[1].

[1] Relativity need not postulate the velocity of light. However it does seem to need an invertible metric tensor on spacetime and this metric tensor is involved in Maxwell electromagnetism. In particular the metric determines the constitutive properties of 'empty space', *i.e.* ε and μ which are related to the speed of light by Maxwell's theory.

2. Lorentz Transformation Without Metric? No.

The Lorentz group is a symmetry group of the metric tensor on spacetime (this tensor is sometimes strangely called the 'interval' in [Zakharov 2006]). What does this mean? We must stress

no metric $\implies$ no Lorentz group of isometries.

Our questions about the transformation of coordinates (1)-(2) are:

(1) Where is the metric tensor? Where is the scalar product on spacetime?
(2) What is the interpretation of each symbol in (1)?
(3) What is the physical meaning of the transformation vector $\mathbf{v}$ if we understand that isometries are generated by a bivector, not by a vector?

How could it be nowadays that so many textbooks[2] of 'special' relativity still present the Lorentz group in terms of coordinate transformations (1)-(2), without even mentioning the metric *tensor* on spacetime? We think that the omission of the metric tensor is a crime, and presenting this tensor in the diagonal form, something that it is possible only in particular basis, while tensors are basis-free, has brought even more misunderstanding.

The one textbook interpretation of (1)-(2) is that there are two reference systems and that both observe an event e. However time can not be stopped. An event as a point on spacetime manifold is not observable, it does not exist in nature. What can be observed is the world line of an event, *i.e.* 'event' must be a life history of a material particle, or better a time-like vector field E.

Textbooks interpret $\{\mathbf{x}, t\}$ and $\{\mathbf{x}', t'\}$, by two different observers in two different reference systems.

There are three actors: street s, bus b, and eagle e. What is the 'position' vector $\mathbf{x}$? Suppose that it is the 'position' vector of the eagle as seen from the street reference system, $\mathbf{x}(s, e)$. Analogously from the bus moving with respect to the street, $\mathbf{x}' = \mathbf{x}(b, e)$. The quantities in (1) depend on the motion of the eagle, and therefore the relative velocity of the bus relative to the street a priori should be a function of three variables, viz. $\mathbf{v}(s, b, e)$. It is not obvious how this 'relative' velocity can be independent of the eagle. In fact we should suppose that the eagle is a third reference system. Thus the relative velocity $\mathbf{v}$ in (7) is ternary, not binary.

The Lorentz-boost isometric transformation deduced by Fock, (1)-(2), presuppose the following symmetry of the scalar product,

$$-(ct)^2 + \mathbf{x}^2 = -(ct')^2 + \mathbf{x}'^2 = -1. \tag{10}$$

Fock started his deduction of (1) by first exhibiting the metric tensor g. This metric *tensor* is still implicit in the scalar form (10). We need to incorporate this metric tensor g explicitly. We will then re-derive (1)-(2) in a coordinate-free manner below.

A material reference system can be modeled in terms of a time-like vector field on space-time. This was proposed by Minkowski in 1908: a material reference system is a

[2] For example. In Wolfgang Rindler's *Essential Relativity*, Springer 1969, 1977, the Lorentz transformation is derived on pages 32-33, but the concept of the metric is introduced on page 62.

normalized time-like vector, a monad, and not some basis = tetrad. Within this philosophy the domain of the Lorentz isometry transformation **must** be the all vectors tangent to spacetime. Tensors and vectors (a vector is a tensor) are coordinate-free. A transformation of vectors induces the transformation of all tensors. Lorentz transformation operates on all vectors and tensors in a coordinate-free manner. All tensors are GL and Lorentz-covariant, but one tensor, the metric tensor will remain Lorentz-invariant. The Lorentz transformation is an isometry, and it is coordinate-free, when acting on vectors.

Let S be a time-like vector field, $S^2 \equiv g(S \otimes S) = -1$. The associated differential one-form '$-gS$' is said to be an S-proper-time form, and

$$(-gS)S = 1. \tag{11}$$

Therefore, $s \equiv S \otimes (-gS)$, is an idempotent, $s^2 = s$, $sS = S$, and $(\mathrm{id} - s)$ is also idempotent.

For any vector field E we have the following coordinate-free identity

$$E = sE + (\mathrm{id} - s)E. \tag{12}$$

Here, $sE = -(S \cdot E)S$, is time-like, and it is orthogonal to the space-like $(\mathrm{id} - s)E$,

$$S \cdot (\mathrm{id} - s)E = S \cdot \{E + (S \cdot E)S\} = 0, \quad E^2 = (sE)^2 + ((\mathrm{id} - s)E)^2. \tag{13}$$

Let a time-like vector field E represent the eagle, $E^2 = -1$, with an associated idempotent, $e \equiv E \otimes (-gE)$, $e^2 = e$.

In what follows time-like vector S represents the reference street, with an associated idempotent $s^2 = s$. Let moreover a time-like vector field B, $B^2 = -1$, represent the bus with associated idempotent $b^2 = b$. The eagle seen from the street and from the bus is as follows

$$E = sE + (\mathrm{id} - s)E = bE + (\mathrm{id} - b)E. \tag{14}$$

2.1 Notation[Eagle observed from street and from bus]. We introduce the following notation-conventions,

$$sE = ctS \quad i.e. \quad ct \equiv -S \cdot E \equiv -g(S \otimes E), \tag{15}$$

$$bE = ct'B \quad i.e. \quad ct' \equiv -B \cdot E \equiv -g(B \otimes E), \tag{16}$$

$$\mathbf{x} \equiv (\mathrm{id} - s)E, \qquad \mathbf{x}' \equiv (\mathrm{id} - b)E, \tag{17}$$

$$(\mathbf{x}' - \mathbf{x}) \wedge S \wedge B \equiv 0. \tag{18}$$

The difference of the position vectors, $(\mathbf{x}' - \mathbf{x})$ must be co-planar with a plane $S \wedge B$. With above notation, the metric symmetry (10) implies that the eagle must be represented as a time-like vector field $E^2 = -1$.

2.2 Notation[Eagle observing street and bus] Alternatively one can suppose that an eagle e is observing the motion of the bus relative to the street,

$$\begin{aligned} S &= eS + (\mathrm{id} - e)S = ct\,E\; + \mathbf{x}, \\ B &= eB + (\mathrm{id} - e)B = ct'E + \mathbf{x}'. \end{aligned} \tag{19}$$

$$(\mathbf{x}' - \mathbf{x}) \wedge E \wedge S \wedge B = 0. \tag{20}$$

Here we see that the difference of the position vectors, $(\mathbf{x}' - \mathbf{x})$ must be within a three-dimensional volume $E \wedge S \wedge B$.

3. Lorentz Transformation Without Bivectors? No.

We are interested in the concept of the velocity of the bus relative to the street, *i.e.* the velocity as measured by the eagle. Why eagle should be the involved in this concept?

We need to define the Lorentz-boost isometry-transformation of coordinate-free vectors.

The Lie algebra of the Lie group of isometries $\mathrm{Aut}(g) \simeq O(1,3)$, coincides with the vector space of Grassmann bi-vectors inside Clifford algebra. A Minkowski bivector $P \wedge Q$ generate an isometry

$$P \wedge Q \quad \hookrightarrow \quad L_{P\wedge Q} \in O(1,3), \tag{21}$$

$$\text{street} \quad \xrightarrow{\text{Lorentz-boost}} \quad \text{bus}, \tag{22}$$

$$S \quad \xrightarrow{L_{\text{bivector}}} \quad B = L_{\text{bivector}} S. \tag{23}$$

Here we arrive at what we consider the essence of special relativity theory, that isometries are generated by bivectors, not vectors.

3.1 Theorem [Isometry-link problem (Oziewicz 2007)] Given a massive three-body system in terms of three time-like normalized vectors $\{E, S, B\}$. Let a space-like Minkowski vector $\mathbf{w}$ be observed by E *i.e.* $E\cdot \mathbf{w} = 0$. Then the Lorentz-boost-link equation for the unknown $\mathbf{w}$,

$$L_{E\wedge\mathbf{w}}\, S = B \quad \text{with} \quad E\cdot \mathbf{w} = 0,$$

has a unique solution, $\gamma\frac{\mathbf{v}}{c} \equiv \mathbf{w} = \mathbf{w}(E, S, B)$.

We say that the unbounded velocity $\mathbf{w}$ of the bus B relative to the street S is observed by the eagle E. All isometric relative velocities are ternary [Oziewicz 2007, 2009; Celakoska 2008; Celakoska and Chakmakov 2010].

Elsewhere we derived the following general expression for the isometry generated by bivector $E \wedge \mathbf{w}$ with $E\cdot \mathbf{w} = 0$ [Oziewicz 2006-2009],

$$L_{E\wedge\mathbf{w}} S = S - \{(\gamma - 1)E\cdot S - \mathbf{w}\cdot S\}E - \left(E\cdot S - \frac{\mathbf{w}\cdot S}{\gamma + 1}\right)\mathbf{w} \tag{24}$$

Let $\mathbf{w}$ be unbound velocity of the bus B relative to street S as measured by eagle E, $B = L_{E\wedge\mathbf{w}} S$. Using Notation 2 we have

$$e L_{E\wedge\mathbf{w}} S = (\mathbf{w}\cdot S - \gamma E\cdot S)E \quad \Longleftrightarrow \quad t' = \gamma\left(t + \frac{\mathbf{v}\cdot\mathbf{x}}{c^2}\right). \tag{25}$$

This proves the Fock transformation (2), and clarifies that

- The scalar product $\mathbf{v}\cdot \mathbf{x}$ is a scalar product of vectors in spacetime - not space!
- The actual relative velocity $\mathbf{v}$ is eagle E-dependent [Oziewicz 2007].

proof. Now we will prove the Fock expression (2). Using Notation 2 we have

$$\mathbf{x}' \equiv (\mathrm{id} - e) L_{E \wedge \mathbf{w}} S = S + (E \cdot S) E - \left(E \cdot S - \frac{\mathbf{w} \cdot S}{\gamma + 1} \right) \mathbf{w}, \tag{26}$$

$$\mathbf{x} \equiv (\mathrm{id} - e) S = S + (E \cdot S) E, \tag{27}$$

$$\mathbf{x}' - \mathbf{x} = \frac{\gamma^2}{\gamma + 1} \frac{\mathbf{v} \cdot \mathbf{x}}{c^2} - \gamma \mathbf{v} t. \tag{28}$$

4. Space is not Physical Reality

All considerations above take place in spacetime. In spacetime there is a unique zero velocity. In order to introduce many zero relative velocities, each zero for each reference system, we must consider the mathematical conventions of Galileo and Poincaré concerning many relative spaces as quotient spaces.

Galileo stressed in 1632 that all velocities are relative and our everyday experience tells us the same thing. One can not discover one's own motion without looking outside for another reference system. The theory of relativity by Galileo 1632, and by Poincaré 1902, is all about the concept of relative velocity.

> ... treatises on mechanics do not clearly distinguish between what is experiment, what is mathematical reasoning, what is convention, and what is hypothesis.
>
> There is no absolute space, and we only conceive of relative motion; and yet in most cases mechanical facts are enunciated as if there is an absolute space to which they can be referred.
>
> Henri Poincaré (1854-1912), Science and Hypothesis
> Chapter 6: Classical Mechanics 1902

In Galilean and Poincaré relativity three-dimensional space does not exist as a physical reality, it is merely a mathematical *convention*.

> There is no entity 'physical space'; there is only the abstract space chosen by the physicist as a structure in which to plot phenomena; and some choices give simpler theorems than others (thus making the laws of nature look simpler).
>
> The essence of scientific freedom is the right to come to conclusions which differ from those of the majority.
>
> Edward Arthur Milne (1896-1950) [1951]

Neither Einstein 1905, nor Minkowski in 1908, made such explicit and clear statements about the relativity of space, about a choice of a rigid body.

> Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows and only a kind of union of two will preserve an independent reality.
>
> Hermann Minkowski 1908

The Minkowski 'union of two' could suggest incorrectly the uniqueness of space, and spacetime as a Cartesian product of space and time. The primary concern of the theory of relativity is the necessity of the relativity of *space*: that there are *many* spaces. This notion of the relativity of space is of metric-independent, and requires no concept of simultaneity. This was also observed by Ruggiero [2003], and by Arminjon and Reifler

[2010 §4 Discussion ii) on page 10].

The relativity of time is not a primary concern. We should not insist on the necessity of the relativity of time, because the only relative concept of time is the metric dependent proper time. Nature allows other metric-free conventions of simultaneity, such as radio-simultaneity, etc. For related explications we refer to [Poincaré 1902; Trautman 1970; Matolcsi 1994 Part I §3; Selleri 2010].

In spite of the above assertions of Galileo and Poincaré concerning the necessity of *many non-physical spaces*, it seems that the majority of the present-day scientific community believe in the existence of a unique three-dimensional physical space. In some publications the word 'space' is always used in the singular, understood as unique and therefore as a physical concept that one can experience in Nature. For example Jammer's monograph [1954] entitled 'Concepts of Space', avoids the plural 'spaces'. The unique space was exactly the point of view of Aristotel in ancient Greece, however the Galilean revolution of *many relative spaces,* each three-dimensional space as merely a mathematical convention, is still not widely accepted more than 400 years later!

	Space	Spacetime
Ancient Greeks and some present Scientific community	Space is unique and it has physical reality	Spacetime is non-physical mathematical abstraction. Not an essential part of Nature
Galileo 1632 Poincaré 1902	Space does not exist in Nature. There are many mathematical spaces as conventions	Four-dimensional spacetime is physical reality

5. Galilean Relativity of Spaces

In 1632 Galileo Galilei observed that to be in the same place is *relative*, *i.e.* a subjective concept, not objective. Place is observer-dependent. Galileo implicitly (conceptually) introduced a four-dimensional *physical* space-time of absolute events, after the experimentally confirmed observation that it is impossible to detect the motion of a boat without a choice of external reference system.

If the concept of place in a three-dimensional space needs an artificial choice of some physically irrelevant reference system, then three-dimensional space is an illusion. Different reference systems yield different three-dimensional spaces, and the only objective physical-arena is four-dimensional space-time (Galilean space-time or Minkowski space-time). Three-dimensional space is a *mathematical convention* that depends on a subjective choice of reference system. According to Galileo: there are as many three-dimensional spaces as there are reference systems, *i.e.* there does not exist any 'unique physical space'. Therefore space-time must not be seen as it was by the Aristotelan Greeks: an Earth-space moving in time. Aristotle has only one unique observer: the Earth. Galilean space-time allows infinite number of observers.

Galilean space-time is a fiber=simultaneity-bundle over one-dimensional time, without any preferred space [Trautman 1970]. Trautman claims that each fiber over a time-moment is 'isomorphic to Euclidean 3-space $\mathbb{R}^3$', that one can interpret (incorrectly) a fiber over time as (isomorphic to) a physical space of places. This is not the case! Each fiber is a set of simultaneous **events**, and not a set of places in a 'physical' space! There is no space concept within Galilean physical space-time, because the concept of the space needs an artificial *choice* of the reference system. Galilean space-time is *not* the cartesian product of time with some fixed space, because there does not exist a privileged space among the many spaces. There is not just a single space, there are infinite *many* spaces.

If some reference system is chosen, Earth or Sun?, then the corresponding space of *this* massive body is *not* a fiber in space-time, but it is rather a quotient-space = space-time/material-body,

$$\text{Space} \equiv \frac{\text{Space-time}}{\text{material body}} \qquad \text{Time} \equiv \frac{\text{Space-time}}{\text{Convention of simultaneity}}, \tag{29}$$

$$\text{Proper-Time} \equiv \frac{\text{Space-time}}{\text{Metric simultaneity of material body}}. \tag{30}$$

In our interpretation of Galileo Galilei: *physical* reality is a four-dimensional space-time of **events**. Time can never "stop", and the choice of three-dimensional space is no more than a *mathematical* convenience. The name *space-time*, introduced by Hermann Minkowski in 1908, is misleading, suggesting incorrectly that this concept is derived from two primitive concepts of 'space' and 'time'. It is just the opposite, the most primitive concept is the Galilean space-time of events, and space is a *derived* concept that needs an artificial *choice* of massive body, e.g. Earth or Sun, as a reference system (29). But any such choice is irrelevant for physical phenomena, it is no more then for example a convenience for a computer program.

The Galilean four-dimensional space-time does not possess an invertible metric tensor. The Minkowski version of Poincaré's and Einstein's special relativity added an invertible metric tensor, the Minkowski metric, to Galilean space-time.

Galilean relativity postulates an absolute *simultaneity relation*, denoted by τ on Figure

1. Composed with a clock-function it gives a coordinate of spacetime of events,

$$t = \text{clock} \circ \tau. \tag{31}$$

There is no need for another clock $t' = t$. Absolute simultaneity is compatible with Einstein and Minkowski special relativity where it can be identified as just one among many different conventions of synchronization, such as for example the radio-synchronization which gives simultaneity that is metric-free [Marinov 1975; de Abreu and Guerra 2005].
5.1 Definition [Place] Each reference system is completely defined in terms of an equivalence *relation* on events being in the *same* place.

Thus every observer-monad field, $V \in \text{der}\,\mathcal{F}$, gives rise to a surjective projection π_V from four-dimensional space-time of events, onto a three-dimensional relative quotient V-space of places. Two space-time events, e_1 and e_2, are in the same **place** for a π-observer if and only if, $\pi(e_1) = \pi(e_2)$.

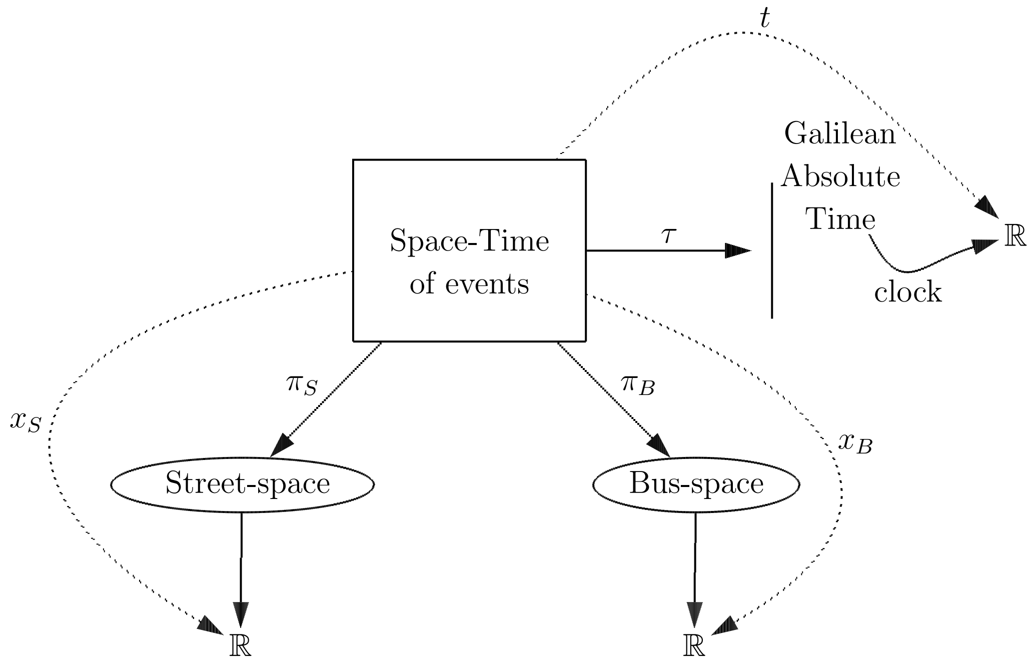

Fig. 1 Two-body system, {Street, Bus}. To be in the same place is relative. Quotient B-space is different from quotient S-space.

5.2 Example. We must see how two reference systems, say a bus B and a street S, in a mutual motion, are distinguished within the space-time of **events**. Let us denote a street by π_S-system, and a bus by π_B-system. Lets illustrate the relativity of space in terms of the following list of three events:

$e_1 =$ bus start from the bus stop 'Metro'

$e_2 =$ bus almost arrive to the next bus stop 'Center'

$e_3 =$ late passenger arrived to the bus stop 'Metro'

From the point of view of the bus driver, this is the π_B-system, the driver is in the same B-place inside of the bus, bus is at π_B-'rest':

$$\pi_B(e_1) = \pi_B(e_2), \quad \text{but} \quad \pi_B(e_3) \neq \pi_B(e_1). \tag{32}$$

From the point of view of the crowd standing on the street, the street is the π_S-system:

$$\pi_S(e_1) = \pi_S(e_3), \quad \text{but} \quad \pi_S(e_2) \neq \pi_S(e_1). \tag{33}$$

5.3 Example. Another example is a space of Sun and a space of Earth (Copernicus versus Ptolemy). The events are

$e_1 =$ Greg born (in Long Beach in July)

$e_2 =$ Bill born (in Long Beach in January)

$e_3 =$ Jamie born (in Washington in July)

Were any of them, Greg, Bill, Jamie, born 'in the same place'?

6. Conclusion: Galileo Galilei Still not Understood

Presented here opinion that relative motion is coordinate-free, but must be understood as relative motion of material bodies with respect to each other or with respect to material reference system, is often attributed to Gottfried Wilhelm Leibniz (1646-1716) or to Ernst Mach (1838-1916). However we are sure that must be in the first place attributed to Galileo Galilei (1564-1642).

In 1911 Langevin considered that acceleration must have an absolute meaning, independent of the reference system, independent of the choice of space. However if the concept of the relativity of velocity is not accepted a priori as an explicit function of the artificial material reference system, *i.e.* if the relative velocity is not accepted as a binary or ternary function of material reference systems, then we must not yet talk about acceleration.

If relative velocity is reference-system-dependent, then how can we be sure a priori that a change of velocity, the covariant derivative of velocity, that must involve the covariant derivative of any reference system, be reference-system-free, *i.e.* be absolute?

We conclude that the Galilean relativity of space of 1632 is not yet understood nor accepted by the scientific community in XXI century.

Science should be based on dissent. But as science becomes publicly funded, ideas become entrenched, and science becomes dogmatic. Textbooks extort only one unique absolute truth. Consensus, not dissent, is considered to be a good way to progress. Alternative ideas are derided, and not heard, frequently not accepted for publication by the referee system.

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Poincaré Gauge Invariance of General Relativity and Einstein-Cartan Theory

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Abstract: We present a simple proof of the Poincaré gauge invariance of general relativity and Einstein-Cartan theory, in the context of the corresponding bundle of affine frames.

Keywords: Poincaré gauge Invariance; General Relativity; Einstein-Cartan Theory; Mathematical Physics
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1. Introduction

It is well understood that general relativity (GR) and its extension with torsion, the Einstein-Cartan theory (E-C), are invariant under internal local Lorentz transformations ($\mathcal{L}_4$), the spin connection $\omega_{\mu ab}$ and the tetrads $e_\mu{}^a$ (coframes) (or rather the displaced fields $B_\mu{}^a = \delta_\mu{}^a - e_\mu{}^a$) being respectively the rotational and translational gravitational gauge potentials (Hehl, 1985; Hayashi, 1977). Then, the group of symmetry of both theories is the semidirect sum $\mathcal{L}_4 \odot \mathcal{D}$, with $\mathcal{D}$ the group of general coordinate transformations (O'Raifeartaigh, 1997).

However, the internal symmetry group is in fact larger, since translations $\mathcal{T}_4$ are *naturally* included, leading to $\mathcal{P}_4 = \mathcal{T}_4 \odot \mathcal{L}_4$, the Poincaré group. Then, the total symmetry of GR and E-C, as gauge theories, turns out to be $\mathcal{P}_4 \odot \mathcal{D}$ (Feynman, 1963; Hehl et al, 1976; Mc Innes, 1984; Hammond, 2002). The problem with the proof of this fact has been, historically, the apparent difficulty with the treatment of translations as part of the gauge group, that is, as *vertical* transformations of a bundle. If for a translation one writes $x^\mu \to x'^\mu = x^\mu + \xi(x)$, one is *not* considering it as a $\mathcal{P}_4$-gauge transformation, but

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instead as an element of $\mathcal{D}$. The appropiate treatment of gauge translations is in the framework of the *bundle of Poincaré frames* over space-time, $\mathcal{F}^P_{M^4}$: $\mathcal{P}_4 \to A^P M^4 \xrightarrow{\pi_P} M^4$.

This has been discussed by several authors (Smrz, 1977; Gronwald, 1997, 1998), and it is the purpose of this note to present an even simpler proof of this fact. On the one hand, at the global level, we show, using general theorems of connections (Kobayashi-Nomizu, 1963), that there is a 1-1 correspondence between affine Poincaré connections ω_P in $\mathcal{F}^P_{M^4}$ and pairs (θ_L, ω_L), with θ_L the canonical form and ω_L a connection on the bundle of Lorentz frames $\mathcal{F}^L_{M^4}$: $\mathcal{L}_4 \to F^L M^4 \xrightarrow{\pi_L} M^4$. On the other hand, locally, we show the invariance under $\mathcal{P}_4$-gauge transformations of the Einstein-Hilbert action for pure gravity, and the Dirac-Einstein action for the coupling of gravity to the Dirac field.

In section 2, we describe basic features of a U_4 space-time. In section 3, in the language of tetrads and spin connection, we review the E-C equations for pure gravity and for gravity coupled to the Dirac field. Lorentz and Poincaré invariance are discussed and proved in sections 4 and 5, respectively. Finally, in section 6, we discuss the nature of a shifted tetrad field, and comment on the difficulty of interpreting the theory in terms of an interaction tetrads-spin connection.

2. The Space-time

We assume that space-time is a 4-dimensional Lorentzian manifold M^4 with a connection Γ compatible with the metric i.e. $D^\Gamma_\mu g_{\nu\rho} = 0$, but not necessarily symmetric: a U_4 space-time. Then, $\Gamma^\alpha_{\nu\mu} = (\Gamma_{LC})^\alpha_{\nu\mu} + K^\alpha_{\nu\mu}$, where Γ_{LC} is the Levi-Civita connection with coordinate components $(\Gamma_{LC})^\alpha_{\nu\mu} = \frac{1}{2} g^{\alpha\sigma}(\partial_\nu g_{\mu\sigma} + \partial_\mu g_{\nu\sigma} - \partial_\sigma g_{\nu\mu})$, and $K^\alpha_{\nu\mu} = (K_A)^\alpha_{\nu\mu} + (K_S)^\alpha_{\nu\mu}$ is the contortion tensor, where $(K_A)^\alpha_{\nu\mu} = T^\alpha_{\nu\mu} = -T^\alpha_{\mu\nu} = \frac{1}{2}(\Gamma^\alpha_{\nu\mu} - \Gamma^\alpha_{\mu\nu}) = \Gamma^\alpha_{[\mu,\nu]}$ is the *torsion* tensor, and K_S, its symmetric part, has components $(K_S)^\alpha_{\mu\nu} = g^{\alpha\rho}(T^\lambda_{\rho\mu} g_{\lambda\nu} + T^\lambda_{\rho\nu} g_{\lambda\mu})$.

In terms of the tetrads $e_a = e_a{}^\mu \partial_\mu$ and their dual coframes $e^a = e_\mu{}^a dx^\mu$, obeying $e_a{}^\mu e_\mu{}^b = \delta^b_a$ and $e_a{}^\mu e_\nu{}^a = \delta^\mu_\nu$, and the spin connection 1-form $\omega^a{}_b = \omega^a_{\mu b} dx^\mu$, Γ is given by $\Gamma^\sigma_{\mu\lambda} = e_a{}^\sigma \partial_\mu e_\lambda{}^a + e_c{}^\sigma e_\lambda{}^a \omega^c_{\mu a}$ with inverse $\omega^c_{\mu a} = e_\rho{}^c \partial_\mu e_a{}^\rho + e_\rho{}^c e_a{}^\nu \Gamma^\rho_{\mu\nu}$ (Carroll, 2004). For the metric, one has $g_{\mu\nu}(x) = \eta_{ab} e_\mu{}^a(x) e_\nu{}^b(x)$, where $x \in M^4$ and $\eta_{ab} = diag(1, -1, -1, -1)$ is the Lorentz metric. Each metric $g_{\mu\nu}$ is in 1-1 correspondence with an equivalence class of frames $[e_a{}^\mu]$: if $e_c{}'^\mu$ is in the class, then $e_a{}^\mu = h_a{}^c e_c{}'^\mu$ with $h_a{}^c$ in the Lorentz group $\mathcal{L}_4$; for the coframes $e_\mu{}^a = e_\mu{}'^c h^{-1a}_c$. Thus, the $e_a{}^\mu$'s and the $e_\mu{}^a$'s are both Lorentz vectors in the internal or gauge (latin) indices, and respectively vectors and 1-forms in the local coordinate (world) indices. The metric character of the connection implies $\omega_{ab} = -\omega_{ba}$ (for latin indices, $X_a = \eta_{ab} X^b$ and $X^b = \eta^{ba} X_a$). The torsion and the curvature of the connection are given by $T^a = de^a + \omega^a{}_b \wedge e^b = \frac{1}{2} T^a{}_{\mu\nu} dx^\mu \wedge dx^\nu$ with $T^a{}_{\mu\nu} = \partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a + \omega^a_{\mu b} e_\nu{}^b - \omega^a_{\nu b} e_\mu{}^b$, and $R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = \frac{1}{2} R^a{}_{b\mu\nu} dx^\mu \wedge dx^\nu$ with $R^a_{b\mu\nu} = \partial_\mu \omega^a_{\nu b} - \partial_\nu \omega^a_{\mu b} + \omega^a_{\mu c}\omega^c_{\nu b} - \omega^a_{\nu c}\omega^c_{\mu b}$.

3. Einstein-Cartan Equations

Consider first the case of *pure gravity* ("vacuum"). The Einstein-Hilbert action is

$$S_G = \int d^4x \; eR \tag{1}$$

where $e = \sqrt{-detg_{\mu\nu}} = det(e_\nu{}^a)$, and for the Ricci scalar one has

$$R = \eta^{bc} R^a{}_{b\mu\nu} e_a{}^\mu e_c{}^\nu. \tag{2}$$

Variation of S_G with respect to the spin connection $\omega^a_{\mu b}$ and the tetrads $e_a{}^\mu$ lead, respectively, to the Cartan equation for torsion and to the Einstein equation:

$\delta_\omega S_G = 0 \Longrightarrow$

$$T^\nu_{ac} + e_a{}^\nu T_c - e_c{}^\nu T_a = 0 \tag{3}$$

or

$$T^\nu_{\rho\sigma} + \delta^\nu_\rho T_\sigma - \delta^\nu_\sigma T_\rho = 0. \tag{3a}$$

$\delta_e S_G = 0 \Longrightarrow$

$$G^a{}_\mu = 0 \tag{4}$$

with

$$G^a{}_\mu = R^a{}_\mu - \frac{1}{2} R e_\mu{}^a, \tag{5}$$

where $R^a{}_\mu = \eta^{ab} R_{b\mu} = \eta^{ab} R^c{}_{b\nu\mu} e_c{}^\nu$. In vacuum $R = 0$, then

$$R^a{}_\mu = 0. \tag{5a}$$

In this case, *torsion vanishes*, since taking the $\nu - \sigma$ trace in (3a), for the torsion vector one obtains $T_\rho = T^\nu_{\rho\nu} = 0$ and therefore, by (3a) again,

$$T^\mu_{\nu\rho} = 0. \tag{6}$$

Thus, for the pure gravity case, E-C theory reduces to GR.

The coupling of gravity to Dirac fermions is described by the action

$$S_{D-E} = k \int d^4x \; eL_{D-E} = k \int d^4x \; e \; (\frac{i}{2}(\bar{\psi}\gamma^a(D_a\psi) - (\bar{D}_a\bar{\psi})\gamma^a\psi) - m\bar{\psi}\psi) \tag{7}$$

where

$$D_a\psi = (e_a - \frac{i}{4}\omega_{abc}\sigma^{bc})\psi = e_a{}^\mu(\partial_\mu - \frac{i}{4}\omega_{\mu bc}\sigma^{bc})\psi = e_a{}^\mu D_\mu\psi \tag{8}$$

and

$$\bar{D}_a\bar{\psi} = e_a\bar{\psi} + \frac{i}{4}\omega_{abc}\bar{\psi}\sigma^{bc} = e_a{}^\mu(\partial_\mu\bar{\psi} + \frac{i}{4}\omega_{\mu bc}\bar{\psi}\sigma^{bc}) = e_a{}^\mu \bar{D}_\mu\bar{\psi} \tag{9}$$

are the covariant derivatives of the Dirac field ψ and its conjugate $\bar{\psi} = \psi^\dagger\gamma_0$ with respect to the spin connection, which give the *minimal coupling* between fermions and gravity. $\sigma^{bc} = \frac{i}{2}[\gamma^b, \gamma^c]$, and the γ^a's are the usual numerical (constant) Dirac gamma matrices satisfying $\{\gamma^a, \gamma^b\} = 2\eta^{ab} I$, $\gamma^{0\dagger} = \gamma^0$ and $\gamma^{j\dagger} = -\gamma^j$. $k = -16\pi \frac{G}{c^4}$. Variation with respect to the spin connection,

$$\delta_\omega S_{D-E} = \frac{k}{8} \int d^4x \; e \; \bar{\psi} \{\gamma^\mu, \sigma^{bc}\} \psi \delta\omega_{\mu bc} = \frac{k}{2} \int d^4x \; e \; S^{\mu bc} \delta\omega_{\mu bc}$$

with $S^{\mu bc} = e_a{}^\mu S^{abc}$, where

$$S^{abc} = \frac{1}{4} \bar{\psi} \{\gamma^a, \sigma^{bc}\} \psi \tag{10}$$

is the *spin density tensor* of the Dirac field. S^{abc} is totally antisymmetric and therefore has 4 independent components: S^{012}, S^{123}, S^{230} and S^{301}.

Combining this result with the corresponding variation for the pure gravitational field, we obtain

$$0 = \delta_\omega (S_G + S_{D-E}) = \int d^4x \; e \; \delta\omega_\nu{}^{ac} (T^\nu_{ac} + e_a{}^\nu T_c - e_c{}^\nu T_a + \frac{k}{2} S^\nu_{ac}) \tag{11}$$

and therefore

$$T^\nu_{ac} + e_a{}^\nu T_c - e_c{}^\nu T_a = -\frac{k}{2} S^\nu_{ac},$$

the *Cartan equation*. Multiplying by $e_\rho{}^a e_\sigma{}^c$ one obtains

$$T^\nu_{\rho\sigma} + \delta^\nu_\rho T_\sigma - \delta^\nu_\sigma T_\rho = -\frac{k}{2} S^\nu_{\rho\sigma} \tag{12}$$

with

$$S^\nu_{\rho\sigma} = \frac{1}{4} \bar{\psi} \{\gamma^\mu, \sigma_{\rho\sigma}\} \psi.$$

The solution of (12) gives the *torsion in terms of the spin tensor*:

$$T^\nu_{\rho\sigma} = \frac{8\pi G}{c^4} (S^\nu_{\rho\sigma} + \frac{1}{2} (\delta^\nu_\rho S_\sigma - \delta^\nu_\sigma S_\rho)) \tag{13}$$

with $S_\rho = S^\nu_{\rho\nu}$. (In natural units, $G = c = \hbar = 1$ and so $T^\nu_{\rho\sigma} = 8\pi(S^\nu_{\rho\sigma} + \frac{1}{2}(\delta^\nu_\rho S_\sigma - \delta^\nu_\sigma S_\rho))$.)

Finally, variation with respect to the tetrads,

$$\delta_e S_{D-E} = k \int d^4x \; e \; (\frac{i}{2} (\bar{\psi} \gamma^a (D_\mu \psi) - (\bar{D}_\mu \bar{\psi}) \gamma^a \psi) - e_\mu{}^a L_{D-E}) \delta e_a{}^\mu.$$

For the Dirac fields which obey the equations of motion

$$\frac{\delta S_{D-E}}{\delta \bar{\psi}} = \frac{\delta S_{D-E}}{\delta \psi} = 0$$

i.e.

$$i\gamma^a (\bar{D}_a \bar{\psi}) + m\bar{\psi} = i\gamma^a D_a \psi - m\psi = 0$$

the Dirac-Einstein lagrangian vanishes i.e. $L_{D-E}|_{eq.\ mot.} = 0$. Then, combining this result with the corresponding variation for the pure gravitational field,

$$0 = \delta_e (S_G + S_{D-E}) = \int d^4x \; e \; (2R^a{}_\mu - R e_\mu{}^a + k\frac{i}{2} (\bar{\psi} \gamma^a (D_\mu \psi) - (\bar{D}_\mu \bar{\psi}) \gamma^a \psi)) \delta e_a{}^\mu, \tag{14}$$

and from the arbitrariness of $\delta e_a{}^\mu$,

$$R^a{}_\mu - \frac{1}{2} R e_\mu{}^a = -\frac{k}{2} T^a{}_\mu \tag{15}$$

with

$$T^a{}_\mu = \frac{i}{2}(\bar\psi \gamma^a (D_\mu \psi) - (\bar D_\mu \bar\psi)\gamma^a \psi) \tag{16}$$

the *energy-momentum tensor* of the Dirac field. Multiplying (15) by $e_a{}^\nu$ one obtains

$$R^\nu{}_\mu - \frac{1}{2} R \delta^\nu_\mu = -\frac{k}{2} T^\nu{}_\mu \quad or \quad R_{\lambda\mu} - \frac{1}{2} R g_{\lambda\mu} = -\frac{k}{2} T_{\lambda\mu}, \tag{15a}$$

the *Einstein equation* in local coordinates.

Note: For L_{D-E} one has

$$L_{D-E} = e_a{}^\mu T^a{}_\mu - m\bar\psi\psi$$

i.e. $T^a{}_\mu$ *couples to the tetrad*. On the other hand,

$$T^a{}_\mu = \theta^a{}_\mu + \omega_{\mu bc} S^{abc}$$

where

$$\theta^a{}_\mu = \frac{i}{2}(\bar\psi \gamma^a \partial_\mu \psi - (\partial_\mu \bar\psi)\gamma^a \psi)$$

is the *canonical energy-momentum tensor* of the Dirac field. Then,

$$L_{D-E} = e_a{}^\mu \theta^a{}_\mu + e_a{}^\mu \omega_{\mu bc} S^{abc} - m\bar\psi\psi = e_a{}^\mu \theta^a{}_\mu + \omega_{abc} S^{abc} - m\bar\psi\psi.$$

So, $\theta^a{}_\mu$ couples to the tetrad while spin couples to the spin connection; moreover, since S^{abc} is totally antisymmetric, *the Dirac field only interacts with the totally antisymmetric part of the connection.*

4. Lorentz Gauge Invariance

Under local Lorentz transformations $h_a{}^b(x)$, tetrads and coframes transform as indicated in section 2; as a consequence, the coordinate invariant volume element $d^4x\ e$ is also gauge invariant: in fact,

$$g_{\mu\nu}(x) = \eta_{ab} e_\mu{}^a(x) e_\nu{}^b(x) = \eta_{ab} e'_\mu{}^c h_c^{-1a} e'_\nu{}^d h_d^{-1b} = e'_\mu{}^c e'_\nu{}^d h_c^{-1a} \eta_{ab} h_d^{-1b} = e'_\mu{}^c e'_\nu{}^d \eta_{cd} = g'_{\mu\nu}(x)$$

implies $e'(x) = e(x)$, and since $x'^\mu = x^\mu$, then $d^4x\ e = d^4x'\ e'$.

On the other hand, the transformation of the spin connection is given by

$$\omega^c{}_a = h_c{}^d \omega'^r{}_d h_r^{-1c} + (dh_a{}^d) h_d^{-1c}, \tag{16}$$

which is not a Lorentz tensor. Its curvature, however, is a Lorentz tensor:

$$R^a{}_b = h_b{}^d h_c^{-1a} R'^c{}_d, \tag{17}$$

and therefore the Ricci scalar is also gauge invariant:

$$R = R^a{}_b e_a \eta^{bc} e_c = h_b{}^d R'^c{}_d h_c^{-1a} h_a{}^f e'_f \eta^{bg} h_g{}^l e'_l = R'^c{}_d \delta_c^f e'_f \eta^{dl} e'_l = R'^c{}_d e'_c \eta^{dl} e'_l = R'. \quad (18)$$

Then, S_G is Lorentz gauge invariant. (A direct and more explicit proof of the gauge invariance of R is given in Appendix 1.)

The part of the action corresponding to the coupling of gravity to the Dirac field, S_{D-E}, is automatically local Lorentz invariant, since it is written in terms of the covariant derivatives $D_a\psi$ and $\bar{D}_a\bar{\psi}$.

5. Poincaré Gauge Invariance

5.1 Global Analysis

The affine group $GA_4(\mathbb{R}) = \{\begin{pmatrix} g & \xi \\ 0 & 1 \end{pmatrix}, \ g \in GL_4(\mathbb{R}), \ \xi \in \mathbb{R}^4\}$ acts on the affine space $\mathbb{A}^4 = \{\begin{pmatrix} \lambda \\ 1 \end{pmatrix}, \ \lambda \in \mathbb{R}^4\}$ in the form

$$GA_4(\mathbb{R}) \times \mathbb{A}^4 \to \mathbb{A}^4, \ (\begin{pmatrix} g & \xi \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \lambda \\ 1 \end{pmatrix}) \mapsto \begin{pmatrix} g\lambda + \xi \\ 1 \end{pmatrix}. \quad (19)$$

Then, one has the following diagram of short exact sequences (s.e.s.'s) of groups and group homomorphisms:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{R}^4 & \xrightarrow{\mu} & GA_4(\mathbb{R}) & \underset{\overleftarrow{\ \rho\ }}{\xrightarrow{\nu}} & GL_4(\mathbb{R}) & \longrightarrow & 0 \\ & & Id\uparrow & & \uparrow\iota & & \uparrow\iota & & \\ 0 & \longrightarrow & \mathbb{R}^4 & \xrightarrow{\mu|} & \mathcal{P}_4 & \underset{\overleftarrow{\ \rho|\ }}{\xrightarrow{\nu|}} & \mathcal{L}_4 & \longrightarrow & 0 \end{array}$$

with $\mu(\xi) = \begin{pmatrix} I_4 & \xi \\ 0 & 1 \end{pmatrix}$ and $\nu(\begin{pmatrix} g & \lambda \\ 0 & 1 \end{pmatrix}) = g$. μ is 1-1, ν is onto, and $ker(\nu) = Im(\mu) = \{\begin{pmatrix} I_4 & \xi \\ 0 & 1 \end{pmatrix}, \ \xi \in \mathbb{R}^4\}$. We have also restricted μ and ν (respectively $\mu|$ and $\nu|$) to the connected components of the Poincaré ($\mathcal{P}_4$) and Lorentz ($\mathcal{L}_4$) groups. Both s.e.s.'s split, i.e. there exists the group homomorphism $\rho : GL_4(\mathbb{R}) \to GA_4(\mathbb{R})$, $g \mapsto \rho(g) = \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$ and its restriction $\rho|$ to $\mathcal{L}_4$, such that $\nu \circ \rho = Id_{GL_4(\mathbb{R})}$ and $\nu| \circ \rho| = Id_{\mathcal{L}_4}$. So

$$GA_4(\mathbb{R}) = \mathbb{R}^4 \odot GL_4(\mathbb{R}), \ \mathcal{P}_4 = \mathbb{R}^4 \odot \mathcal{L}_4 \quad (20)$$

with composition law

$$(\lambda', g')(\lambda, g) = (\lambda' + g'\lambda, g'g). \quad (20a)$$

The above s.e.s.'s pass to s.e.s.'s of the corresponding Lie algebras:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{R}^4 & \xrightarrow{\tilde{\mu}} & ga_4(\mathbb{R}) & \underset{\overleftarrow{\ \tilde{\rho}\ }}{\xrightarrow{\tilde{\nu}}} & gl_4(\mathbb{R}) & \longrightarrow & 0 \\ & & Id\uparrow & & \uparrow\iota & & \uparrow\iota & & \\ 0 & \longrightarrow & \mathbb{R}^4 & \xrightarrow{\tilde{\mu}|} & p_4 & \underset{\overleftarrow{\ \tilde{\rho}|\ }}{\xrightarrow{\tilde{\nu}|}} & l_4 & \longrightarrow & 0 \end{array}$$

with $gl_4(\mathbb{R}) = \mathbb{R}(4)$, $ga_4(\mathbb{R}) = \mathbb{R}^4 \odot gl_4(\mathbb{R})$ with Lie product

$$(\lambda', R')(\lambda, R) = (R'\lambda - R\lambda', [R', R]), \tag{21}$$

where $[R', R]$ is the Lie product in $gl_4(\mathbb{R})$ and $[\lambda', \lambda] = 0$ in $\mathbb{R}^4$, $\tilde{\mu}(\xi) = (\xi, 0)$, $\tilde{\nu}(\xi, R) = R$, and $\tilde{\rho}(R) = (0, R)$. $\tilde{\mu}$, $\tilde{\nu}$ and $\tilde{\rho}$ (and their corresponding restrictions $\tilde{\mu}|$, $\tilde{\nu}|$ and $\tilde{\rho}|$) are Lie algebra homomorphisms, with $\tilde{\nu} \circ \tilde{\rho} = Id_{gl_4(\mathbb{R})}$ and $\tilde{\nu}| \circ \tilde{\rho}| = Id_{l_4}$. The s.e.s.'s split only at the level of vector spaces i.e. if $(\lambda, R) \in ga_4(\mathbb{R})$, then $(\lambda, R) = \tilde{\mu}(\lambda) + \tilde{\rho}(R)$, but $(\lambda, R) \neq \tilde{\mu}(\lambda)\tilde{\rho}(R)$.

If $\mathcal{F}_{M^4} : GL_4 \to FM^4 \xrightarrow{\pi_F} M^4$ and $\mathcal{A}_{M^4} : GA_4 \to AM^4 \xrightarrow{\pi_A} M^4$ are respectively the bundles of linear and affine frames over M^4, where $FM^4 = \cup_{x \in M^4}(\{x\} \times (FM^4)_x)$ with $(FM^4)_x$ the set of ordered basis $r_x = (v_{1x}, \ldots, v_{4x})$ of T_xM^4, and $AM^4 = \cup_{x \in M^4}(\{x\} \times AM^4_x)$ with $AM^4_x = \{(v_x, r_x),\ v_x \in A_xM^4,\ r_x \in (FM^4)_x\}$, where A_xM^4 is the tangent space at x considered as an affine space (Appendix 3), then one has the following bundle homomorphism:

$$\begin{array}{ccc}
AM^4 \times GA_4 & \begin{array}{c}\xrightarrow{\beta \times \nu} \\ \xleftarrow{\gamma \times \rho}\end{array} & FM^4 \times GL_4 \\
\psi_A \downarrow & & \downarrow \psi_F \\
AM^4 & \begin{array}{c}\xrightarrow{\beta} \\ \xleftarrow{\gamma}\end{array} & FM^4 \\
\pi_A \downarrow & & \downarrow \pi_F \\
M^4 & \xrightarrow{Id} & M^4
\end{array}$$

where $\beta(x, (v_x, r_x)) = (x, r_x)$, $\gamma(x, r_x) = (x, (0_x, r_x))$, $0 \in T_xM^n$, $\psi_F((x, r_x), g) = (x, r_x g)$, and

$$\psi_A((x, (v_x, r_x)), (\xi, g)) = (x, (v_x + r_x\xi, r_x g)). \tag{22}$$

A *general affine connection* (g.a.c.) on M^4 is a connection in the bundle of affine frames $\mathcal{A}_{M^4}$; let ω_A be the 1-form of this connection, then $\omega_A \in \Gamma(T^*AM^4 \otimes ga_4)$. From the smoothness of γ, the pull-back $\gamma^*(\omega_A)$ is a ga_4-valued 1-form on FM^n:

$$\gamma^*(\omega_A) = \varphi \odot \omega_F, \tag{23}$$

where ω_F is a connection on FM^4, and φ is an $\mathbb{R}^4$-valued 1-form. There is a 1-1 correspondence between g.a.c.'s on AM^4 and pairs (ω_F, φ) on FM^4:

$$\{\omega_A\}_{g.a.c.} \longleftrightarrow \{(\omega_F, \varphi)\}. \tag{24}$$

ω_A is an *affine connection* (a.c.) on M^4 if φ is the soldering (canonical) form θ_{FM^4} (see Appendix 2) on FM^4. Then, if ω_A is an a.c. on AM^4,

$$\gamma^*(\omega_A) = \theta_{FM^4} \odot \omega_F. \tag{25}$$

There is then a 1-1 correspondence

$$\{\omega_A\}_{a.c.} \longleftrightarrow \{\omega_F\}, \tag{26}$$

since θ_{FM^4} is fixed. Also, if Ω_A is the curvature of ω_A, then

$$\gamma^*(\Omega_A) = D^{\omega_F}\theta \odot \Omega_F = T_F \odot \Omega_F \tag{27}$$

since $D^{\omega_F}\theta_{FM^4} = T_F$: the *torsion* of the connection ω_F on FM^4.

We now consider the following diagram of bundle homomorphisms:

$$\begin{tikzcd}
AM^4 \times GA_4 \arrow[d, "\psi_A"] & A^P M^4 \times \mathcal{P}_4 \arrow[l, "\iota\times\iota"'] \arrow[d, "\psi_A|"] \arrow[r, shift left, "\beta|\times\nu|"] & F^L M^4 \times \mathcal{L}_4 \arrow[l, shift left, "\gamma|\times\rho|"] \arrow[d, "\psi_F|"] \arrow[r, "\iota\times\iota"] & FM^4 \times GL_4 \arrow[d, "\psi_F"] \\
AM^4 \arrow[d, "\pi_A"] & A^P M^4 \arrow[l, "\iota"'] \arrow[d, "\pi_A|"] \arrow[r, shift left, "\beta|"] & F^L M^4 \arrow[l, shift left, "\gamma|"] \arrow[d, "\pi_F|"] \arrow[r, "\iota"] & FM^4 \arrow[d, "\pi_F"] \\
M^4 \arrow[r, "Id"] & M^4 \arrow[r, "Id"] & M^4 \arrow[r, "Id"] & M^4
\end{tikzcd}$$

where $\pi_A| = \pi_P$, $\pi_F| = \pi_L$, $\psi_A| = \psi_P$ and $\psi_F| = \psi_L$, where ψ_P and ψ_L are the group actions in the Poincaré and Lorentz bundles, respectively.

The facts that $A^P M^4$ is a subbundle of AM^4 and $F^L M^4$ is a subbundle of FM^4, with structure groups and Lie algebras the corresponding subgroups and sub-Lie algebras, and the existence of the restrictions $\beta| : A^P M^4 \to F^L M^4$ and $\gamma| : F^L M^4 \to A^P M^4$, allow us to obtain similar conclusions for the relations between affine connections on the Poincaré bundle and linear connections on the Lorentz bundle:

There is a 1-1 correspondence between affine Poincaré connections ω_P on $F^P M^4$ and Lorentz connections on $F^L M^4$:

$$\{\omega_P\} \longleftrightarrow \{\omega_L\} \tag{28}$$

with

$$\gamma|^*(\omega_P) = \theta_L \odot \omega_L \tag{29}$$

where $\theta_L = \theta_{FM^4}|_{F^L M^4}$ is the canonical form on $F^L M^4$. Also,

$$\gamma|^*(\Omega_P) = D^{\omega_L}\theta_L \odot \Omega_L = T_L \odot \Omega_L. \tag{30}$$

So, there is a 1-1 correspondence between curvatures of affine connections on $F^P M^4$ and torsion and curvature pairs on $F^L M^4$:

$$\{\Omega_P\} \longleftrightarrow \{(T_L, \Omega_L)\}. \tag{31}$$

For pure gravity governed by the Einstein-Hilbert action, $T_L = 0$.

5.2 Local Analysis: Invariance of the Actions S_G and S_{D-E}

To explicitly prove the Poincaré gauge invariance of GR and E-C theory, we have to consider as gauge transformations both the Lorentz part, already studied in the previous section, and the translational part. This last has to be done using the bundle of Poincaré frames $\mathcal{F}^P_{M^4}$.

A gauge transformation or vertical automorphism in an arbitrary principal G-bundle $\xi : G \to P \xrightarrow{\pi} B$, is a diffeomorphism $\alpha : P \to P$ such that i) $\alpha(pg) = \alpha(p)g$ and ii) $\pi(\alpha(g)) = \pi(p)$, for all $p \in P$ and $g \in G$. Therefore, from ii), $\alpha(p) = pk$ for some $k \in G$.

Then there is a bijection $Aut_{vert}(P) \xrightarrow{\Phi} \Gamma_{eq}(P,G)$, $\Phi(\alpha) = \gamma_\alpha$ with $\alpha(p) = p\gamma_\alpha(p)$ and $\gamma_\alpha(pg) = g^{-1}\gamma_\alpha(p)g$; for the inverse, $\gamma \mapsto \alpha_\gamma$ with $\alpha_\gamma(p) = p\gamma(p)$.

The action of $\mathcal{P}_4$ on $A^P M^4$ is given by

$$\psi_P : A^P M^4 \times \mathcal{P}_4 \to A^P M^4,\ \psi_P((x,(v_x,r_x)),(\xi,h)) \equiv (x,(v_x,r_x))(\xi,h) = (x,(v_x + r_x\xi, r_x h))$$

$$= (x,(v'_x, r'_x)), \tag{32}$$

where $r_x = (e_{ax})$, $a = 1,2,3,4$, is a Lorentz frame, $h \in \mathcal{L}_4$, and $\xi \in \mathbb{R}^4 \cong \mathbb{R}^{1,3}$ is a Poincaré gauge translation. For a pure translation, $h = I_L$ i.e. ${h_a}^b = {\delta_a}^b$ and therefore

$$(x,(v_x,r_x))(\xi,I_L) = (x,(v_x + r_x\xi, r_x I_L)) = (x,(v_x + r_x\xi, r_x))$$

i.e.

$$r'_x = r_x. \tag{33}$$

Therefore $e'_{ax} = e_{ax}$, $a = 1,2,3,4$, and then, from the definition of $\omega^a_{\mu b}$ in section 2,

$$\omega'^a_{\mu b} = \omega^a_{\mu b} \tag{34}$$

since $\Gamma^\mu_{\nu\rho} = (\Gamma_{LC})^\mu_{\nu\rho} + K^\mu_{\nu\rho}$ remains unchanged (in the case of pure gravity $K^\mu_{\nu\rho} = 0$). So the coordinate Ricci scalar R is also a gauge scalar, and therefore S_G is invariant.

By the same reason invoked in the case of Lorentz gauge invariance, S_{D-E} is also invariant under translations: in an arbitrary G-bundle P with connection ω, a section s of an associated bundle and its covariant derivative $D^\omega s$ transform in the same way.

The Poincaré bundle extends the symmetry group of GR and E-C theory to the semidirect sum

$$G_{GR/E-C} = \mathcal{P}_4 \odot \mathcal{D}, \tag{35}$$

with composition law

$$((\xi',h'),g')((\xi,h),g) = ((\xi',h')(g'(\xi,h)g'^{-1}), g'g). \tag{35a}$$

The left action of $\mathcal{D}$ on $\mathcal{P}_4$ is given by the commutative diagram

$$\begin{array}{ccc} A^P M^4 & \xrightarrow{(\xi,h)} & A^P M^4 \\ g\downarrow & & \downarrow g \\ A^P M^4 & \xrightarrow{(\xi',h')} & A^P M^4 \end{array}$$

with

$$g : A^P M^4 \to A^P M^4,\ (x,(v^\mu_x \frac{\partial}{\partial x^\mu}|_x, ({e_{ax}}^\nu \frac{\partial}{\partial x^\nu}|_x))) \mapsto (x,(v'^\mu_x \frac{\partial}{\partial x'^\mu}|_x, ({e_{ax}}'^\nu \frac{\partial}{\partial x'^\nu}))), \tag{36}$$

where $v'^\mu_x = \frac{\partial x'^\mu}{\partial x^\alpha}|_x v^\alpha_x$ and ${e_{ax}}'^\nu = \frac{\partial x'^\nu}{\partial x^\beta}|_x {e_{ax}}^\beta$.

6. Gravitational Potentials and Interactions

It is usually said that the coframes $e^a = e_\mu{}^a dx^\mu$ are the translational gravitational potentials (Hehl, 1985; Hehl et al, 1976; Hammond, 2002). This is *not* strictly true since these fields are not gauge potentials, but tensors, both in their Lorentz (a) and world (μ) indices: see section 2 and (Hayashi, 1977). The translational gauge potentials are the 1-form fields $B_\mu{}^a$ locally defined as follows (Hayashi and Nakano, 1967; Aldrovandi and Pereira, 2007):

$$B_\mu{}^a = e_\mu{}^a - \frac{\partial v_x^a}{\partial x^\mu} \quad or \quad B^a = e^a - dv_x^a, \tag{37}$$

where $v_x = \sum_{a=0}^3 v_x^a e_{ax} \in A_x M^4$ (section 5.1.); the v_x^a's are here considered the coordinates of the tangent space at x. A straightforward calculation leads to the following transformation properties:

Internal Lorentz:

$$B_\mu{}'^a = h_a{}^b B_\mu{}^b - \partial_\mu (h_b{}^a) v_x^b \quad or \quad B'^a = h_b{}^a B^b - (dh_b{}^a) v_x^b, \tag{38}$$

General coordinate transformations:

$$B_\mu{}'^a = \frac{\partial x^\nu}{\partial x'^\mu} B_\nu{}^a, \tag{39}$$

Internal translations:

$$B_\mu{}'^a = B_\mu{}^a - \partial_\mu \xi^a \quad or \quad B'^a = B^a - d\xi^a. \tag{40}$$

Then, $B = B_\mu dx^\mu = B_\mu{}^a dx^\mu b_a$, where b_a, $a = 0, 1, 2, 3$, is the canonical basis of $\mathbb{R}^4$, is the connection 1-form corresponding to the translations.

In terms of the $B_\mu{}^a$ fields and the spin connection, the Ricci scalar (2) is given by

$$R = (\frac{\partial v_x^a}{\partial x^\mu}\frac{\partial v_x^b}{\partial x^\nu} + \frac{\partial v_x^a}{\partial x^\mu} B_\nu{}^b + \frac{\partial v_x^b}{\partial x^\nu} B_\mu{}^a + B_\mu{}^a B_\nu{}^b)(\partial^\mu \omega_{ab}^\nu - \partial^\nu \omega_{ab}^\mu + \omega_{ac}^\mu \omega_b^{\nu c} - \omega_{ac}^\nu \omega_b^{\mu c}). \tag{41}$$

If one intends to use this Lagrangian density as describing a $(B_\mu{}^a, \omega_{bc}^\nu)$ (or $(e_\mu{}^a, \omega_{bc}^\nu)$) interaction (Randono, 2010), then immediately faces the problem that the $B_\mu{}^a$ (or $e_\mu{}^a$) does not have a free part (in particular a kinematical part), since all its powers are multiplied by ω's or $\partial\omega$'s. So an interpretation in terms of fields interaction seems difficult, and may be, impossible.

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Appendix 1

The Ricci scalar is given by

$$R = \eta^{bd} e_a{}^\mu e_d{}^\nu (\partial_\mu \omega_{\nu b}^a - \partial_\nu \omega_{\mu b}^a + \omega_{\mu c}^a \omega_{\nu b}^c - \omega_{\nu c}^a \omega_{\mu b}^c) \equiv \eta^{bd} e_a{}^\mu e_d{}^\nu ((\gamma) - (\delta) + (\alpha) - (\beta)),$$

with $(\gamma) = \partial_\mu \omega^a_{\nu b}$, $(\delta) = \partial_\nu \omega^a_{\mu b}$, $(\alpha) = \omega^a_{\mu c}\omega^c_{\nu b}$, and $(\beta) = \omega^a_{\nu c}\omega^c_{\mu b}$.

Under the transformation

$$\omega^a_{\mu c} = h_c{}^l \omega'^r_{\mu l} h_r^{-1a} + (\partial h_c{}^l) h_l^{-1a}$$

we have:

$(\alpha) = (a) + (b) + (c) + (d)$ with

$$(a) = h_c{}^l \omega'^r_{\mu l} h_r^{-1a} h_b{}^g \omega'^s_{\nu g} h_s^{-1c},\ (b) = h_c{}^l \omega'^r_{\mu l} h_r^{-1a} (\partial_\nu h_b{}^g) h_g^{-1c},$$

$$(c) = h_b{}^g \omega'^s_{\nu g} h_s^{-1c} (\partial_\mu h_c{}^l) h_l^{-1a},\ (d) = (\partial_\mu h_c{}^l) h_l^{-1a} (\partial_\nu h_b{}^g) h_g^{-1c};$$

$(\beta) = (e) + (f) + (g) + (h)$ with

$$(e) = h_c{}^g \omega'^s_{\nu g} h_s^{-1a} h_b{}^l \omega'^r_{\mu l} h_r^{-1c},\ (f) = h_c{}^g \omega'^s_{\nu g} h_s^{-1a} (\partial_\mu h_b{}^l) h_l^{-1c},$$

$$(g) = h_b{}^l \omega'^r_{\mu l} h_r^{-1c} (\partial_\nu h_c{}^g) h_g^{-1a},\ (h) = (\partial_\nu h_c{}^l) h_l^{-1a} (\partial_\mu h_b{}^g) h_g^{-1c};$$

$(\gamma) = [1] + [2] + [3] + [4]$ with

$$[1] = h_b{}^n h_t^{-1a} (\partial_\mu \omega'^t_{\nu n}),\ [2] = \omega'^t_{\nu n} \partial_\mu (h_b{}^n h_t^{-1a}),\ [3] = (\partial_\mu \partial_\nu h_b{}^n) h_n^{-1a},\ [4] = (\partial_\nu h_b{}^n)(\partial_\mu h_n^{-1a});$$

and $(\delta) = [5] + [6] + [7] + [8]$ with

$$[5] = h_b{}^l h_s^{-1a} (\partial_\nu \omega'^s_{\mu l}),\ [6] = \omega'^s_{\mu l} \partial_\nu (h_b{}^l h_s^{-1a}),\ [7] = (\partial_\nu \partial_\mu h_b{}^l) h_l^{-1a},\ [8] = (\partial_\mu h_b{}^l)(\partial_\nu h_l^{-1a}).$$

Now,

$$[3] - [7] = (\partial_\mu \partial_\nu h_b{}^n) h_n^{-1a} - (\partial_\nu \partial_\mu h_b{}^l) h_l^{-1a} = 0,$$

$$(b) + (c) = \omega'^r_{\mu l} h_r^{-1a} \partial_\nu h_b{}^l - \omega'^s_{\nu g} h_b{}^g \partial_\mu h_s^{-1a},$$

$$(f) + (g) = \omega'^s_{\nu g} h_s^{-1a} \partial_\mu h_b{}^g - \omega'^r_{\mu l} h_b{}^l \partial_\nu h_r^{-1a};$$

so

$$((b) + (c)) - ((f) + (g)) = \omega'^r_{\mu l} \partial_\nu (h_r^{-1a} h_b{}^l) - \omega'^s_{\nu g} \partial_\mu (h_s^{-1a} h_b{}^g);$$

also,

$$[2] - [6] = \omega'^s_{\nu g} \partial_\mu (h_b{}^g h_s^{-1a}) - \omega'^r_{\mu l} \partial_\nu (h_b{}^l h_r^{-1a});$$

then

$((b) + (c)) - ((f) + (g)) + ([2] - [6]) = 0.$

Also,

$[4] - [8] = (\partial_\nu h_b{}^l)(\partial_\mu h_l^{-1a}) - (\partial_\mu h_b{}^l)(\partial_\nu h_l^{-1a})$

and

$(d) - (h) = (\partial_\nu h_l^{-1a})(\partial_\mu h_b{}^l) - (\partial_\mu h_l^{-1a})(\partial_\nu h_b{}^l);$

so

$([4] - [8]) + ((d) - (h)) = 0.$

Finally,

$[1] - [5] + (a) - (e) = h_b{}^l h_s^{-1a}(\partial_\mu \omega'^s_{\nu l} - \partial_\nu \omega'^s_{\mu l} + \omega'^s_{\mu r}\omega'^r_{\nu l} - \omega'^s_{\nu r}\omega'^r_{\mu l}).$

Therefore,

$R = \eta^{bd} e_a{}^\mu e_d{}^\nu h_b{}^l h_s^{-1a}(\partial_\mu \omega'^s_{\nu l} - \partial_\nu \omega'^s_{\mu l} + \omega'^s_{\mu r}\omega'^r_{\nu l} - \omega'^s_{\nu r}\omega'^r_{\mu l}) = \eta^{lt} e_s{}'^\mu e_t{}'^\nu(\partial_\mu \omega'^s_{\nu l} - \partial_\nu \omega'^s_{\mu l} + \omega'^s_{\mu r}\omega'^r_{\nu l} - \omega'^s_{\nu r}\omega'^r_{\mu l})$

$= R'.$

Appendix 2

The *soldering* or *canonical* form on the frame bundle $\mathcal{F}_{M^n}$ of an n dimensional differentiable manifold, is the $\mathbb{R}^n$-valued differential 1-form on FM^n given by

$$\theta : FM^n \to T^*FM^n \otimes \mathbb{R}^n,\ (x, r_x) \mapsto \theta((x, r_x)) = ((x, r_x), \theta_{(x,r_x)}),$$

with

$$\theta_{(x,r_x)} : T_{(x,r_x)}FM^n \to \mathbb{R}^n,\ v_{(x,r_x)} \mapsto \theta_{(x,r_x)}(v_{(x,r_x)}) = \tilde{r}_x^{-1} \circ d\pi_F|_{(x,r_x)}(v_{(x,r_x)})$$

i.e.

$$\theta_{(x,r_x)} = \tilde{r}_x^{-1} \circ d\pi_F|_{(x,r_x)},$$

where π_F is the projection in the bundle $\mathcal{F}_{M^n} : GL_n(\mathbb{R}) \to FM^n \xrightarrow{\pi_F} M^n$ and $\tilde{r}_x$ is the vector space isomorphism

$$\tilde{r}_x : \mathbb{R}^n \to T_xM,\ (\lambda^1, \ldots, \lambda^n) \mapsto \tilde{r}_x(\lambda^1, \ldots, \lambda^n) = \sum_{i=1}^n \lambda^i v_{ix}$$

with inverse

$$\tilde{r}_x^{-1}(\sum_{i=1}^n \lambda^i v_{ix}) = (\lambda^1, \ldots, \lambda^n).$$

In local coordinates (x^ρ, X^μ_ν) on $\mathcal{F}_U$,

$$\theta^\mu = \sum_{\nu=1}^{n} (X^{-1})^\mu_\nu dx^\nu$$

with $(X^{-1})^\mu_\nu(x, r_x) = (X^\mu_\nu(x, r_x))^{-1} = (v^\mu_{\nu x})^{-1}$, where $r_x = (v_{1x}, \ldots, v_{nx})$ and $v_{\nu x} = \sum_{\mu=1}^n v^\mu_{\nu x} \frac{\partial}{\partial x^\mu}|_x$. Then $\theta^a = e_\mu{}^a \theta^\mu = e_\mu{}^a (X^{-1})^\mu{}_\nu dx^\nu = (X^{-1})^a{}_\nu dx^\nu = e_\nu{}^a dx^\nu = e^a$; so, if ω_F is a connection on $\mathcal{F}_{M^n}$, then $D^{\omega_F}\theta^a = d\theta^a + \omega^a_{F\,b} e^b = T^a_F$ is the torsion of ω_F.

Appendix 3

An *affine space* is a triple (V, φ, A) where V is a vector space, A is a set, and φ is a free and transitive left action of V as an additive group on A:

$$\varphi : V \times A \to A,\ (v, a) \mapsto v + a,$$

with

$$0 + a = a \ and\ (v_1 + v_2) + a = v_1 + (v_2 + a),\ for\ all\ a \in A\ and\ all\ v_1, v_2 \in V.$$

Then, given $a, a' \in A$, there exists a unique $v \in V$ such that $a' = v + a$. Also, if v_0 is fixed in V, $\varphi_{v_0} : A \to A$, $\varphi_{v_0}(a) = \varphi(v_0, a)$ is a bijection.

Example. $A = V$: The vector space itself is considered as the set on which V acts. In particular, when $V = T_xM^n$ and $A = T_xM^n$, the tangent space is called *affine tangent space* and denoted by A_xM^n. The points "a" of A_xM^n are the tangent vectors at x.

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Noncommutative Cosmological Solutions

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Abstract: In this work, we present several solutions to noncommutative cosmology using the WKB approximation in noncommutative quantum cosmology. We study the solutions in the context of the cosmological constant problem.

Keywords: Noncommutative Cosmology; WKB Approximation; Quantum Cosmology
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1. Introduction

There has been a lot of interest in the old idea of noncommutative space-time [1], and an immense amount of work has been done on the subject, this renewed interest is a consequence of the developments in M-Theory and String Theory [2, 3]. Along the lines of noncommutative gauge theory, noncommutative theories of gravity have been constructed [4], all versions of noncommutative gravity are highly nonlinear and calculations are incredibly difficult. One expects noncommutative effects to be present at the Planck scale but due to the UV/IR mixing [5, 6], the effects of noncommutative might be important at the cosmological scale.

In the last few years there have been several attempts to study the possible effects of noncommutativity in the classical cosmological scenario [7, 8]. In [9] the authors avoid the difficulties of analyzing noncommutative cosmological models, that arise when these are derived from the full noncommutative theory of gravity [4]. They introduces the effects of noncommutativity at the quantum level, namely quantum cosmology, by deforming the minisuperspace through a Moyal deformation of the Wheeler-DeWitt equation. It is then possible to proceed as in noncommutative quantum mechanics [10]. Following this

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idea several works has been done for different comological models [11].

In this work we apply the WKB type method to noncommutative quantum cosmology, and find the noncommutative classical solutions [12]. Finally the old cosmological constant is analyzed in the context of noncommutative cosmology [13]. This work is organized as follows. In section 2 we review several Quantum Cosmological models via the Wheeler-DeWitt equation (WDW) and find the corresponding wave function for most of these models, then we obtain the classical solutions using a WKB type approximation. In section 3 we repeat the same analysis using the noncommutative counterparts of the examples presented in section 2, this is achieved through a noncommutative deformation of the minisuperspace variables. In section 4 a toy model to study the old cosmological problem is presented. Finally, section 5 is devoted to discussion and outlook.

2. FRW cosmology with Scalar Field and Λ

Let us start with the homogeneous and isotropic universe, the so called Friedmann-Robertson-Walker (FRW) universe coupled to a scalar field and cosmological constant. The FRW metric is given by:

$$ds^2 = -N^2dt^2 + e^{2\alpha(t)}\left[\frac{dr^2}{1-\mathrm{k}r^2} + r^2(d\vartheta^2 + sin^2\vartheta d\varphi^2)\right], \tag{1}$$

where $a(t) = e^{\alpha(t)}$ is the scale factor, $N(t)$ is the lapse function, and k is the curvature constant that takes the values $0, +1, -1$, which correspond to a flat, closed and open universes, respectively. The Lagrangian is composed by the gravity sector and the matter sector, which for the FRW universe endowed with a scalar field and cosmological constant Λ is

$$\mathcal{L}_{tot} = \mathcal{L}_g + \mathcal{L}_\phi = e^{3\alpha}\left[6\frac{\dot{\alpha}^2}{N} - \frac{1}{2}\frac{\dot{\phi}^2}{N} - N\left(2\Lambda + 6ke^{-2\alpha}\right)\right], \tag{2}$$

the corresponding canonical momenta are

$$\Pi_\alpha = \frac{\partial\mathcal{L}}{\partial\dot{\alpha}} = 12\mathrm{e}^{3\alpha}\frac{\dot{\alpha}}{\mathrm{N}}, \qquad \Pi_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} - \mathrm{e}^{3\alpha}\frac{\dot{\phi}}{\mathrm{N}}, \tag{3}$$

we can get the WDW equation from the classical Hamiltonian. By the variation of (2) with respect to N, $\partial\mathcal{L}/\partial\mathrm{N} = 0$, implies the well-known result $\mathcal{H} = 0$.

$$e^{-3\alpha}N\left[-\frac{1}{24}\frac{\partial^2}{\partial\alpha^2} + \frac{1}{2}\frac{\partial^2}{\partial\phi^2} + e^{6\alpha}\left(2\Lambda + 6\mathrm{k}e^{-2\alpha}\right)\right]\Psi(\alpha,\phi) = 0. \tag{4}$$

Now that we have the complete framework and found the corresponding WDW equation, we can proceed to study different cases.

In table 1 we can see the different cases that we solved (the case $k \neq 0$, $\Lambda \neq 0$ does not have a closed analytical solution to the WDW equation), all of them are calculated by using the WKB type procedure, the classical solutions are the same we would get by solving Einstein's field equations. We can expect that this approximation includes all the gravitational degrees of freedom of the particular cosmological model under study. This almost trivial observation is central to the ideas we are presenting in the next section.

case	Quantum Solution	Classical Solution
k=0, $\Lambda \neq 0$	$\psi = e^{\pm i\nu \frac{\sqrt{3}}{2}\phi} K_{i\nu}\left(4\sqrt{\frac{\Lambda}{3}}e^{3\alpha}\right)$ and J_ν for $\Lambda < 0$	$\phi(t) = \phi_0 - P_{\phi_0} t$, $\alpha(t) = \frac{1}{6}\ln\left(\frac{P_{\phi_0}^2}{4\Lambda}\right) + \frac{1}{3}\ln\left(\mathrm{sech}\left[\frac{\sqrt{3}}{2}P_{\phi_0}(t-t_0)\right]\right)$.
k $\neq$0, $\Lambda = 0$	$\psi^{(1)} = e^{\pm i\frac{\nu}{\sqrt{3}}\phi} K_{i\nu}\left(6e^{2\alpha}\right)$, for k $= 1$, $\psi^{(2)} = e^{\pm i\frac{\nu}{\sqrt{3}}\phi} J_{\nu}\left(6e^{2\alpha}\right)$, for k $= -1$	$\phi(t) = \phi_0 - P_{\phi_0}(t-t_0)$, $\alpha(t) = \frac{1}{4}\ln\left[\frac{P_{\phi_0}^2}{12\mathrm{k}}\right] + \frac{1}{2}\ln\left(\mathrm{sech}\left[\frac{1}{\sqrt{3}}P_{\phi_0}(t-t_0)\right]\right)$,
k $\neq$0, $\Lambda \neq 0$	Unknown	$\phi(t) = \phi_0 - P_{\phi_0}(t-t_0)$, $\int \frac{d\alpha(t)}{\sqrt{P_{\phi_0} - 2e^{6\alpha}(2\Lambda + 6\mathrm{k}e^{-2\alpha})}} = \frac{1}{\sqrt{12}}(t-t_0)$.

Table 1: Classical and quantum solutions for the FRW universe coupled to a scalar field ϕ. For the case $\Lambda \neq 0$ k $\neq 0$, the classical solution for the scale factor is given in an implicit expression. We have fixed the lapse function to $N(t) = e^{3\alpha}$.

3. Noncommutative Quantum Cosmology and the WKB Type Approximation

In this section we construct noncommutative quantum cosmology for the examples presented in Table 1 and calculate the classical evolution via a WKB type approximation.

Finding the classical cosmological solutions for any cosmological model in noncommutative gravity [4] is a very difficult task, this is a consequence of the highly nonlinear character of the theory. To avoid these difficulties, we will follow the original proposals of noncommutative quantum cosmology that was developed in [9]. We start by presenting, in quite a general form the construction of noncommutative quantum cosmology and the WKB type method to calculate the classical evolution.

Let us start with a generic form for the commutative WDW equation, this is defined in the minisuperspace variables x, y. As mentioned in [9] a noncommutative deformation of the minisuperspace variables is assumed

$$[x, y] = i\theta, \tag{5}$$

this noncommutativity can be formulated in terms of noncommutative minisuperspace functions with the Moyal product of functions

$$f(x,y)\star g(x,y) = f(x,y)e^{i\frac{\theta}{2}\left(\overleftarrow{\partial_x}\overrightarrow{\partial_y}-\overleftarrow{\partial_y}\overrightarrow{\partial_x}\right)}g(x,y). \tag{6}$$

Then the noncommutative WDW equation can be written as

$$\left(-\Pi_{\rm x}^2+\Pi_{\rm y}^2-{\rm V(x,y)}\right)\star\Psi({\rm x,y})=0, \tag{7}$$

we know from noncommutative quantum mechanics [10], that the symplectic structure is modified changing the commutator algebra. It is possible to return to the original commutative variables and usual commutation relations if we introduce the following change of variables

$$x\to x+\frac{\theta}{2}\Pi_y \qquad \text{and} \qquad y\to y-\frac{\theta}{2}\Pi_x, \tag{8}$$

the efects of the Moyal star product are reflected in the WDW equation, only through the potential

$$V(x,y)\star\Psi(x,y)=V(x+\frac{\theta}{2}\Pi_y, y-\frac{\theta}{2}\Pi_x), \tag{9}$$

taking this into account and using the usual substitutions $\Pi_{{\rm q}^\mu}{=}{-}{\rm i}\partial_{{\rm q}^\mu}$ we arrive to

$$\left[\frac{\partial^2}{\partial{\rm x}^2}-\frac{\partial^2}{\partial{\rm y}^2}-{\rm V}\left({\rm x}-{\rm i}\frac{\theta}{2}\frac{\partial}{\partial{\rm y}},{\rm y}+{\rm i}\frac{\theta}{2}\frac{\partial}{\partial{\rm x}}\right)\right]\Psi({\rm x,y})=0, \tag{10}$$

this is the Noncommutative WDW equation (NCWDW) and its solutions give the quantum description of the noncommutative universe. We can use the NCWDW to find the temporal evolution of our noncommutative cosmology by a WKB type procedure. For this we propose that the noncommutative wave function has the form $\Psi_{NC}(\beta,\Omega)\approx e^{i(S_{NC1}(\beta)+S_{NC2}(\Omega))}$, which in the limit

$$\begin{aligned}\left|\frac{\partial^2 S_{NC1}(\beta)}{\partial\beta^2}\right| &<< \left(\frac{\partial S_{NC1}(\beta)}{\partial\beta}\right)^2,\\ \left|\frac{\partial^2 S_{NC2}(\Omega)}{\partial\Omega^2}\right| &<< \left(\frac{\partial S_{NC2}(\Omega)}{\partial\Omega}\right)^2,\end{aligned} \tag{11}$$

yielding the noncommutative Einstein-Hamilton-Jacobi equation (NCEHJ), that gives the solutions to S_{NC1} and S_{NC2}. After the identification $\Pi_{x_{NC}}=-\frac{\partial(S_{NC1})}{\partial x}$ and $\Pi_{y_{NC}}=-\frac{\partial(S_{NC2})}{\partial y}$ together with the definitions of the canonical momenta and equation (8) we can find the time dependent solutions for x and y.

3.1 Noncommutative FRW Cosmology with Scalar Field and Λ

We can use the NCWKB type method to FRW universe coupled to a scalar field. Proceeding as before, the corresponding NCWDW equation is

$$\left[-\frac{1}{24}\frac{\partial^2}{\partial\alpha^2}+\frac{1}{2}\frac{\partial^2}{\partial\phi^2}+e^{6(\alpha-i\frac{\theta}{2}\frac{\partial}{\partial\phi})}\left(2\Lambda+6{\rm k}e^{-2(\alpha-{\rm i}\frac{\theta}{2}\frac{\partial}{\partial\phi})}\right)\right]\Psi=0. \tag{12}$$

From the NCWDW equation, we use the method developed in the previous section and calculate the classical evolution by applying the NCWKB type method. These results are presented in the next table

case	NC Quantum Solution	NC Classical Solution
k=0, $\Lambda \neq 0$	$\psi = e^{\pm i\nu \frac{\sqrt{3}}{2}\phi} K_{i\nu}\left[4\sqrt{\frac{\Lambda}{3}} e^{3\left(\alpha - \frac{3}{2}\nu\theta\right)}\right]$ and J_ν for $\Lambda < 0$	$\phi(t) = \phi_0 - P_{\phi_0} t$ $-\sqrt{3}\theta P_{\phi_0} \tanh\left(\frac{\sqrt{3}}{2} P_{\phi_0}(t - t_0)\right)$, $\alpha(t) = \frac{\theta}{2} P_{\phi_0} + \frac{1}{6}\ln\left(\frac{P_{\phi_0}^2}{4\Lambda}\right)$ $+\frac{1}{3}\ln\left(\mathrm{sech}\left[\frac{\sqrt{3}}{2} P_{\phi_0}(t - t_0)\right]\right)$.
k $\neq$0,	$\psi^{(1)} = e^{\pm i \frac{\nu}{\sqrt{3}}\phi} K_{i\nu}\left[6e^{2\left(\alpha - \frac{\theta}{2}\nu\right)}\right]$, for k = 1 $\psi^{(2)} = e^{\pm i\nu/\sqrt{3}\phi} J_\nu\left[6e^{2\left(\alpha - \frac{\theta}{2}\nu\right)}\right]$, for k = -1	$\phi(t) = \phi_0 - P_{\phi_0}(t - t_0)$ $-\sqrt{3}\theta P_{\phi_0} \tanh\left(\frac{P_{\phi_0}}{\sqrt{3}}(t - t_0)\right)$, $\alpha(t) = \frac{\theta}{2} P_{\phi_0} + \frac{1}{4}\ln\left[\frac{P_{\phi_0}^2}{12\mathrm{k}}\right]$ $+\frac{1}{2}\ln\left(\mathrm{sech}\left[\frac{1}{\sqrt{3}} P_{\phi_0}(t - t_0)\right]\right)$.
k $\neq$0, $\Lambda \neq 0$	Unknown	$\phi(t) = \phi_0 - P_{\phi_0}(t - t_0)$ $+6\theta \int e^{6\alpha}\left(\Lambda + 2e^{-2\alpha}\right) dt$, $\int \frac{d\alpha(t)}{\sqrt{P_{\phi_0} - 2e^{6\alpha + 3\theta P_{\phi_0}}\left(2\Lambda + 6\mathrm{k}e^{-2\alpha - \theta \mathrm{P}_{\phi_0}}\right)}}$ $= \frac{1}{\sqrt{12}}(t - t_0)$.

Table 2: Classical and quantum solutions for noncommutative FRW universe coupled to a scalar field. For these models noncommutativity is introduced in the gravitational and matter sectors. As in the commutative scenario, for $\Lambda \neq 0$ and k $\neq 0$ the noncommutative classical solution is given in an implicit form, and there is not a closed analytical quantum solutions. As in the commutative case we have fixed the value of the lapse function $N(t) = e^{3\alpha}$.

4. Noncommutativity and the Old Cosmological Constant Problem

Now let us examine in detail the case k = 0 $\Lambda \neq 0$ for the cosmological constant problem which has been addressed by means of different approaches for several years and still today remains as one of the central issues of not only modern day cosmology but also particle physics [14]. In a more precise manner, why is the effective cosmological constant Λ_{eff} so close to zero. The different contributions to the vacuum energy density, from ordinary particle physics should give a value for $\langle\rho\rangle$ of order M_p^4, which should be canceled by the bare value of Λ. This cancellation has to be better than 10^{-121} if we compare the zero-point energy of a scalar field, using the Planck scale as a cut-off, to the experimental value of $\langle\rho_{obs}\rangle \approx 10^{-47}(GeV)^4$. We will consider this case in our 4D space-time and noncommutativity in both the gravitational and matter sectors. We dont intent to explain the origin of the cosmological constant, however, we will show that by means of minisuperspace noncommutativity a small cosmological constant arise. Comparing the results of $\Lambda \neq 0$, k = 0 for $\alpha(t)$ for the commutative and noncommutative model we find that the classical evolution of this two universes are remarkably the same. From this we can be confident that the phenomenology described by the commutative model can also be explained by the noncommutative model, we can establish the relationship

$$\Lambda_{nc} = \Lambda e^{-3\theta P_{\phi_0}}, \tag{13}$$

so the expansion of the universe described by either the commutative or the noncommutative model is the same and the difference is the value of the cosmological constant. This is a very suggestive result, which implies that if we consider a noncommutative universe, the standard value of the cosmological constant is significantly reduced eliminating the necessity of the high degree of fine tuning.

The problem of the smallness of Λ actually means that the rate of $\langle\rho_{obs}\rangle$ to $\langle\rho_{vac}\rangle$ calculated from ordinary particle physics is of order 10^{-121}. With this in mind and given the behavior of Λ_{nc} we attempt to find the value of the vacuum energy density $\langle\rho_{vac}\rangle_{nc}$ in our noncommutative minisuperspace model. To calculate the vacuum energy one starts with the energy momentum relationship, write down the Fourier transform of the fields and integrate to a cut-off scale. Even though, as mentioned, we have not made use of a particular noncommutative theory of gravity [4], our procedure allow us from equations for $\alpha(t)$ and $\alpha_{nc}(t)$ and the definition of the scale factor to establish the relationship between the commutative and noncommutative scale factors $a_{nc}(t) = e^{P_{\phi_0}\theta/2}a(t)$. In this manner we can define a kind of noncommutative metric

$$g^{(nc)}_{\mu\nu} = diag(e^{3\theta P_{\phi_0}} g_{00}, e^{\theta P_{\phi_0}} g_{ij}). \tag{14}$$

Calculations should now be performed only with this metric, which in the linear limit gives the noncommutative equivalent to the Minkowski metric. In order to calculate the vacuum energy density, we must sum the zero point energies of quantum fields in our modified Minkowski metric. This is done as in the commutative case but yields a different coefficient which comes from the deformed metric

$$\langle\rho_{vac}\rangle_{nc} \approx e^{-\theta P_{\phi_0}} k^4_{max}, \tag{15}$$

where k_{max} is the fundamental cut-off scale. One may be tempted to use different cut-off energies, i.e. grand unification scale or the QCD scale. Because noncommutativity is assumed at the quantum regime of the universe it is expected to be present at Planck's length, then it makes sense to take $k_{max} \approx M_p$. As already stated, current observations put the energy density at a value $\langle\rho_{obs}\rangle \approx 10^{-47}$ and should be of the same order of magnitude as the vacuum energy density. If we consider that the universe is described by the noncommutative model, then we must analyze the ratio between the observed energy density and the vacuum energy density calculated in the noncommutative formalism, this gives the relationship

$$\frac{\langle \rho_{obs} \rangle}{\langle \rho_{vac} \rangle_{nc}} = e^{\theta P_{\phi_0}} \frac{\langle \rho_{obs} \rangle}{k^4_{max}}, \tag{16}$$

we note that the ratio of the observed energy density to the cut-off scale is regulated by the exponential $e^{\theta P_{\phi_0}}$. Considering the usual huge discrepancy of order 10^{-121} on the calculated and observed densities, a value of $\theta P_{\phi_0} \approx 240$ can easily suppress it. So, the usual quantum field theory calculation of the vacuum energy density is correct and gives the expected value in the noncommutative universe. Still, the fact that we need the appropriate initial conditions of the universe remains. Fortunately the effects of the minisuperspace noncommutativity are only reflected through a modified cosmological constant (13).

5. Conclusions and Outlook

In this work we have presented the NCWKB type method for noncommutative quantum cosmology and with this procedure, found the noncommutative classical solutions for several noncommutative quantum cosmological models.

Noncommutativity is a proposal that originally emerged at the quantum level, by this reason we incorporate noncommutativity in the minisuperspace variables in a similar manner as it is considered in standard quantum mechanics. By means of the WKB approximation on the corresponding NCWDW equation, one gets the noncommutative generalized Einstein-Hamilton-Jacobi equation (NCEHJ), from which the classical evolution of the noncommutative model is obtained. In the commutative scenario, that the classical solutions found from the WKB-type method are solutions to the corresponding Einsteins field equations. Due to the complexity of the noncommutative theories of gravity [4], classical solutions to the noncommutative field equations are almost impossible to find, but in the approach of noncommutative quantum cosmology and by means of the WKB-type procedure, they can be easily constructed. Also the quantum evolution of the system is not needed to find the classical behavior, from table 2 we can see that for the case $\Lambda \neq 0$ and k $\neq 0$ the wave function can not be analitacally calculated, but still the noncommutative effects can be incorporated and the classical evolution is found implicitly. This procedure gives a straightforward algorithm to incorporate noncommutative effects to cosmological models. In this approach the effects of noncommutativity are encoded in the potential through the Moyal product of functions equation (9). We only need the NCWDW equation and the WKB approximations, to get the NCEHJ and from it, the noncommutative classical behavior can easily be constructed.

Besides we show a toy model of noncommutative minisuperspace cosmology and the possible influence that noncommutativity could be in the evolution of our present universe is calculated. The old cosmological constant problem has been addressed, by showing that the value of Λ can drastically change to an appropriate small value. The fine tuning problem can be softened by considering now the ratio of the observed energy density to the calculated noncommutative vacuum energy equation (16). We also conjecture that the vacuum energy density drastically diminishes because of the "discreteness" of the noncommutative minisuperspace.

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Supersymmetric Classical Cosmology: A Quickly Review

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Abstract: In agreement with the most recent observations of the universe, there are two intrinsically different models of the early universe. In the first alternative, the universe was created at a big bang singularity, and after a period of exponential expansion, it was driven to a hot state which agrees perfectly with all cosmological observations presently at hand. In contrast, the second proposal argues, in the beginning, there was a transition from a contracting phase to an expanding one. Using an approach based on a supersymmetric quantum cosmology, we will describe a new formalism to explain the origin and evolution of the universe. For this, we study a supersymmetric and flat Friedmann-Robertson-Walker model in the superfield formulation. The WKB method is then applied, and we show that supersymmetry introduces extra terms in the Einstein-Klein-Gordon equations of motion. We prove that supersymmetry (or indirectly, the gravitinos) fixes a type of matter with a stiff equation of state at early stages in the evolution of the universe. Finally, we study the solutions of the equations of motion, their stability properties, and their cosmological consequences.

Keywords: Supersymmetric Classical Cosmology; Universe; Quantum Cosmology
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1. Introduction

How did the universe begin? Which is the universe made of many civilisations have asked and answered this question before, giving to different myths, religions or philosophical models. However, most civilizations seemed afraid of providing an empty answer to this

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question, giving little room to a different postulate like: was there really a creation? or is the universe eternal? In the case of physical cosmology, these questions represent a real challenge and, of course, a rational answer is expected. It is interesting to note that present technology has allowed the humankind to partially answer these questions with some important accuracy [1-3]. From the latest observations, we know that about 95% of matter in the Universe is of non baryonic nature, and the rest is made of radiation, baryons, neutrinos, etc., in agreement with Big Bang Nucleosynthesis (BBN) predictions. On the other hand, scalar fields are strong candidates to be the missing, non-baryonic, matter of the Universe [4-6]. The cosmological research of the last three decades has elevated the role of scalar fields in the description of various sides of nature. In Cosmology, we are already used to the presence of scalar fields: from the concept of quintessence to explain the dark energy, to models of inflation in the early universe [7-14].

On the other hand, the physics required to understand the early Universe is necessarily rooted in a theory of quantum gravity. Supersymmetric quantum cosmology has emerged as one of the most active areas of current research. In considering the quantum creation of the universe, we are of course dealing with the very earliest epochs of the Universe's existence, at which time it is believed that supersymmetry would have not been broken yet. The inclusion of supersymmetry could therefore be vital from the point of view physical consistency. The first model proposed in[15] was based on the fact that, shortly after the invention of supergravity[16], it was shown that this theory provides natural classical square root equations and their corresponding Hamiltonians. A second method was a superfield formulation, in which is possible to obtain the corresponding fermionic partners and also being able to incorporate matter in a simpler way [17-19]. The last method allows to define a *square root* of the potential, in the minisuperspace, of the cosmological model of interest, and consequently operators whose square results in the Hamiltonian[20, 21].

Then, in the same way as we seek a desirable scalar field potentials to explain the evolution (and early times) of the universe from a point of view of standard general relativity, we can reconcile these requirements along with the ideas of local supersymmetry using now *superpotentials*. For this purpose, we need a supersymmetric quantum cosmological model and find out what happens now with super-scalar fields. It is then important to see the influence of the *fermionic* variables in company of these super fields and how they might alter the usual cosmological landscapes.

In this work, a Hamiltonian for a homogeneous scalar supermultiplet (with four components and different signs) in supergravity $n = 2$, interacting with the super scalar factor (also a supermultiplet), is considered. We first promote this Hamiltonian to an operator, representing the Grassmann variables by matrices, then with help of the WKB process we find two classical evolution equations. The first one, associated with the scalar field, is obtained through Hamiltonian equations. This procedure gives to us a modified Einstein-Klein-Gordon (EKG) set of equations (that we call SUSY-EKG equations) due to indirect presence of *gravitinos*, and the fermionic variables corresponding to the scalar field which are inherently contained in each entry of the supermultiplets. From a phenomenological point of view, these new extra terms (which are proportional to the scale factor) in the model offer a different kind of components that behave, as we will show, as *dark stiff matter*.

On the theoretical side, if we extrapolate our SUSY-EKG equations back in time, we reach a point of infinite density and zero size, the big bang singularity. In one of our models, a negative cosmological constant appears, and the analytical solutions provide a scenario in which the universe collapses into a big crunch singularity, followed by another big bang, and so on[22]. The other landscape that we present here, and a beautiful example, is a cosmological model using an exponential superpotential. As we shall see, the analytic solution suggest a dark stiff matter period before the inflationary phase.

In addition, to fully understand the dynamics of these super cosmologies with super-scalar fields, we require numerical solutions. It is well know that information about the evolution of cosmological models can be retrieved using dynamical systems as a tool. Particularly, the asymptotic behavior of cosmological models are closely related to concepts like past and future attractors[14, 23].

This work considers that the Universe had an epoch dominated by supersymmetry at small radius and, accordingly, at large energy scales. The study focuses on two particular choices of models: with a constant and exponential superpotentials. In the first one, we found a cyclic model and the second one provide a pre-inflationary phase that we call dark stiff matter phase.

The manuscript has been organized in the following manner. In section II, we outline the basic mathematics that allows us to define the superfields associated with the expansion factor and the scalar field, and how we can generalize the usual FRW action with a scalar field that will be able to define supercharges. Next, in the subsections we pay special attention to interesting simple examples. The main section III is devoted to the analysis of an exponential superpotential, in this case we show that one can identify a dark stiff matter phase as a pre-inflationary epoch. The cosmological dynamical formulation is used to see whether there is any attractive mechanism in the cosmological solutions. Finally, section IV is dedicated to general comments.

2. The SUSY-EKG Equations for a FRW Universe

In this section we will describe some features of standard cosmology, starting, of course, with the so-called Cosmological Principle (CP), and then see what happens when SUSY is consider, giving us a new set of cosmological equations which we call SUSY-EKG equations. In order to generalize the WDW equation to its supersymmetric version, the superfield method outlined in the introduction will be used. For pedagogical reasons, we study two examples assuming a constant energy potential V_0.

2.1 The FRW Standard Mathematical Background

The remarkable thing is that the CP suffices to fix the metric of the spacetime a homogeneous and isotropic Universe must have. It is called the FRW metric, a metric with constant curvature, whose line element is usually written as (in units with $c = 1$)

$$ds^2 = -N^2(t)dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right], \tag{1}$$

where $a(t)$ is the (time-dependent) scale factor, $N(t)$ is the lapse function, and k is the curvature constant. Then, we write the total action that represents a (real) scalar field ϕ minimally coupled to gravity and endowed with a scalar field potential $V(\phi)$,

$$S = \int d^4x\sqrt{-g}\left[-\frac{R}{16\pi G} + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right]. \tag{2}$$

The equations of motion arising from action (2) are

$$\dot{H} = -\frac{\kappa^2}{6}\dot{\phi}^2, \tag{3a}$$

$$\ddot{\phi} = -3H\dot{\phi} - \frac{dV}{d\phi}, \tag{3b}$$

together with the (constraint) Friedmann equation

$$H^2 = \frac{\kappa^2}{3}\left(\dot{\phi}^2 + V(\phi)\right), \tag{4}$$

where $H = \dot{a(t)}/a(t)$ is the Hubble parameter, and $\kappa^2 = 8\pi G$. Eqs. (3a), (3b), and (4) are the representative equations of motion of a FRW Universe driven by a scalar field.

2.2 The SUSY FRW Mathematical Background

After the introduction of SUSY in the scalar field model presented above, the general form of the total action (2), corresponding to the FRW metric (1), takes the form

$$S = \frac{6}{8\pi G}\int\left(-\frac{a\dot{a}^2}{2N} + \frac{1}{2}kNa\right)dt + S_{mat}(\Phi), \tag{5}$$

where Φ is a matter field. The action (5) is invariant under the time reparametrization, $t' \to t + a(t)$, if the transformation of $a(t)$ and $N(t)$ are defined as

$$\delta a = a\dot{a}, \quad \delta N = \dot{(aN)}. \tag{6}$$

The variation with respect to $a(t)$ and $N(t)$ leads to the classical equations of motion for the scale factor $a(t)$, see Eqs. (3a) and (4), which generates the local reparametrization of $a(t)$ and $N(t)$. The constraint (4) leads to the standard WDW equation in quantum cosmology.

In order to obtain the superfield formulation of the action (5), we extend the transformations of time reparametrization to the $n = 2$ local SUSY of time $(t, \eta, \bar{\eta})$, where η and $\bar{\eta}$ represents the superpartners of time variable.

The Hamiltonian can then be calculated in the usual way. We have the classical canonical Hamiltonian

$$\mathcal{H}_{can} = NH + \frac{1}{2}\bar{\psi}S - \frac{1}{2}\psi\bar{S} + \frac{1}{2}\nu F, \tag{7}$$

where $\mathcal{H}$ is the Hamiltonian of the system, F is the $U(1)$ rotation generator, and S, and $\bar{S}$ are the supercharges with the following structure

$$S = A\lambda + B\lambda, \quad \bar{S} = A^{+}\bar{\lambda} + B^{+}\bar{\lambda}, \tag{8}$$

where

$$A = \frac{i\kappa a^{-\frac{1}{2}}\Pi_a}{3} - \frac{2\sqrt{k}a^{\frac{1}{2}}}{\kappa} + 2\kappa a^{\frac{3}{2}}g(\varphi) + \frac{\kappa a^{-\frac{3}{2}}\left[\bar{\chi}, \chi\right]}{4}\,, \tag{9a}$$

$$B = ia^{-\frac{3}{2}}\Pi_\varphi + 2a^{\frac{3}{2}}\frac{\partial g(\varphi)}{\partial\varphi}\,, \tag{9b}$$

and $g(\varphi)$ is the superpotential. The Grassmann variables λ, $\bar{\lambda}$ and χ, $\bar{\chi}$, satisfy the Clifford algebra

$$\{\lambda, \bar{\lambda}\} = -\frac{3}{2}, \quad \{\chi, \bar{\chi}\} = 1. \tag{10}$$

The momenta will be the usual differential operators $\Pi_a \to -i\frac{\partial}{\partial a}$ and $\Pi_\varphi \to -i\frac{\partial}{\partial\varphi}$.

To construct the quantum Hamiltonian $\mathcal{H}$, we must consider, at quantum level, the nature of the Grassmann variables; thus,

$$\begin{aligned}\mathcal{H} = & -\frac{\kappa^2}{12}a^{\frac{1}{2}}\Pi_a a^{\frac{1}{2}}\Pi_a - \frac{3ka}{\kappa^2} - \frac{1}{6}\frac{\sqrt{k}}{a}\left[\bar{\lambda}, \lambda\right] + \frac{\Pi_\varphi^2}{2a^3} - \frac{ik}{4a^3}\Pi_\varphi\left(\left[\bar{\lambda}, \chi\right] + \left[\lambda, \bar{\chi}\right]\right) - \frac{\kappa^2}{16a^3}\left[\bar{\lambda}, \lambda\right]\left[\bar{\chi}, \chi\right] \\ & + \frac{3\sqrt{k}}{4a}\left[\bar{\chi}, \chi\right] + \frac{\kappa^2}{2}g\left(\varphi\right)\left[\bar{\lambda}, \lambda\right] + 6\sqrt{k}g\left(\varphi\right)a^2 + a^3V\left(\varphi\right) + \frac{3}{4}\kappa^2 g\left(\varphi\right)\left[\bar{\chi}, \chi\right] + \frac{\partial^2 g\left(\varphi\right)}{\partial\varphi^2}\left[\bar{\chi}, \chi\right] \\ & + \frac{k}{2}\frac{\partial g\left(\varphi\right)}{\partial\varphi}\left(\left[\bar{\lambda}, \chi\right] - \left[\lambda, \bar{\chi}\right]\right)\,.\end{aligned} \tag{11}$$

On the other hand, the scalar field potential reads

$$V\left(\varphi\right) = 2\left(\frac{\partial g\left(\varphi\right)}{\partial\varphi}\right)^2 - 3\kappa^2 g^2\left(\varphi\right), \tag{12}$$

Notice that, in general, the scalar potential (12) is not positive semi-definite. Unlike the standard supersymmetric quantum mechanics, our model, describing the minisuperspace approach to supergravity coupled to matter, allows SUSY breaking when the vacuum energy is equal to zero $V(\varphi) = 0$. The relevant term in the Eq. (12) is $g(\varphi)$, which is related to the superpotential and whose form shall be chosen appropriately for the cosmological model under study.

To obtain the supersymmetric Hamiltonian operator it is necessary to find appropriate representations for the bosonic and fermionic variables. To realize the fermionic variable algebra (10) we will represent those variables as the tensorial product of 2×2 matrices,

$$\lambda = \sqrt{\frac{3}{2}}\sigma_- \otimes 1, \quad \lambda = \sqrt{\frac{3}{2}}\sigma_- \otimes 1\,, \tag{13a}$$

$$\chi = \sigma_3 \otimes \sigma_-, \quad \bar{\chi} = \sigma_3 \otimes \sigma_+\,, \tag{13b}$$

where $\sigma_\pm = \sigma_1 \pm i\sigma_2/2$, $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices. For (13b) we have the following matrices

$$\lambda = \frac{3}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \bar{\lambda} = -\frac{3}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{14a}$$

$$\chi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \bar{\chi} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{14b}$$

The eigenstates of Hamiltonian (11) have four components, so we can obtain a diagonal Hamiltonian operator $\hat{\mathcal{H}}$ using both (2.2) and the supercharges (8), (2.2). Nevertheless, the second and third Hamiltonian in the diagonal line are coupled; hence, we will only work with the first and fourth components of this Hamiltonian operator.

2.3 The classical Landscape

As we mentioned before, our objetive is to find the classical solutions of the (super)WDW equation, $\hat{\mathcal{H}}\Phi(a,\varphi) = 0$, and for that we will use the semiclassical limit or WKB method. This is achieved by taking

$$\Psi = e^{(S_a + S_\varphi)}, \tag{15}$$

and imposing the usual WKB conditions on $S_a + S_\varphi$. The variation of the Hamilton equations gives the SUSY-EKG classical equations,

$$\frac{\ddot{\varphi}}{N} = -3H\frac{\dot{\varphi}}{N} - N\frac{\partial V}{\partial \varphi} + \frac{\dot{N}}{N^2}\dot{\varphi} \pm \frac{3}{2}N\frac{\kappa^2}{a^3}\frac{\partial g(\varphi)}{\partial \varphi} - 6\sqrt{k}\frac{N}{a}\frac{\partial g(\varphi)}{\partial \varphi} \mp \frac{N}{a^3}\frac{\partial^3 g(\varphi)}{\partial \varphi^3}\,, \tag{16}$$

and

$$H^2 = \frac{\kappa^2\dot{\varphi}^2}{6} + \frac{\kappa^2}{3}V(\varphi) - \frac{k}{a^2} \pm \frac{\kappa^2\sqrt{k}}{3a^4} + \frac{\kappa^4}{32a^6} \mp \frac{\kappa^4}{2a^3}g(\varphi) + 2\frac{\kappa^2\sqrt{k}}{a}g(\varphi) \pm \frac{\kappa^2}{3a^3}\frac{\partial^2 g(\varphi)}{\partial \varphi^2}, \tag{17}$$

For simplicity, and without any loss of generality, we have henceforth set the lapse function to $N = 1$.

Note that both SUSY-EKG classical equations differ from the SGR equations (3a), (3b) and (4) only by the extra terms due to SUSY. The ($\pm$) signs in the equations above have the information of certain states related with our fermionic variable, the *gravitinos*. It is then reasonable to expect that this model is reduced to the SGR at high energies.

The extra terms in (16) and (17) are expected to be relevant only when the supersymmetric contribution is large. We can recognize in Eq.(17) extra contribution behaving like radiation (a^{-4}) and stiff matter (a^{-6}), which should be dominant at very early times, whereas other terms show a combination of scale factor powers mediated by the presence of the superpotential $g(\varphi)$ and its derivatives.

2.4 Simple Classical Examples

As first instance of the role played by SUSY terms in the classical equations of motion, we will consider the case of a free scalar field, $V(\varphi) = 0$, which also corresponds to a null superpotential, $g(\varphi) = 0$. According to this, the exact solutions of Eqs. (16) and (17) can be easily found,

$$\dot{\varphi}(t) = \dot{\varphi}_0 (a_0/a)^3 \,, \tag{18a}$$

$$a(t) = {a_0}^3 + 3\left(\frac{\kappa^2 \dot{\varphi}_0^2 {a_0}^6}{6} + \frac{\kappa^4}{32}\right)^{1/2} (t - t_0)\,, \tag{18b}$$

where t_0,$\dot{\varphi}_0$ and a_0 are integration constants. From this we conclude that the scale factor represents an indefinitely expanding model and which goes like t asymptotically and corresponds to stiff matter.

Another interesting case is that with a constant superpotential, $g(\varphi) = g_0$, that corresponds to a constant and negative definite scalar field potential, $V = -3\kappa^2 {g_0}^2$; as in the previous case of the free scalar field, the case is simplified because the derivatives of the superpotential disappear from the SUSY equations.

The cosmological solutions can be expressed as

$$\dot{\varphi}(t) = \dot{\varphi}_0 (a_0/a)^3 \,, \tag{19a}$$

$$a(t) = \frac{a_0}{4g_0}\left[\left(\sqrt{\frac{8{\dot{\varphi}_0}^2 {a_0}^{62}}{3\kappa^2} + \frac{9}{8}}\right) \sin\left(3\kappa^2 g_0 (t - t_0)\right) \mp 1\right]^{1/3} \,, \tag{19b}$$

Because the scale factor is a positive quantity, the only acceptable solution is when the factor of the sinus function is less or equal to one. It is then clear that the scale factor has a periodic solution in which a_0 is the amplitude at maximum expansion.

In other words, Eq. (19b) represents an oscillatory universe. This behaviour is also already present in the SGR, however, in this latter case the radius of the universe would would take negative values.

3. Dark Stiff Matter Arising from Exponential Superpotential

The possible cosmological roles of exponential potentials in scalar field models have been investigated thoroughtly in the specialize literature [6-8,14,23-25], almost always as a means of driving a period of cosmological inflation, but also as possible candidates for dark matter and dark energy.

Scalar fields cosmologies with an exponential potentials are, as compared to others, mathematically simple, and their solutions have many interesting features. For the purpose of this work, we only mention the possibility of having inflationary solutions and the appearance of the so-called scaling solutions.

The inflationary solution for exponential potential is the simple power law inflation which never ends and needs modifications to provide a graceful exit towards a Hot Big Bang model. On the other hand, the scaling solution arises whenever the scalar field is accompanied by another matter fields, so that both fields evolve with a fixed ratio of their energy densities.

In this section we explore in detail the type of solutions permitted by our (classical) SUSY cosmological model when the scalar field is endowed with an exponential potential. Our main interest will be to find inflationary and scaling solutions. Even though we are not considering extra matter fields apart from scalar field, the new terms in Eq. (11) will play the role of partner fields which should impose a non-trivial behavior upon the field φ.

Let us consider the following superpotential and potential, respectively,

$$g(\varphi) = g_0 e^{-\lambda\kappa\varphi/2} \,, \tag{20a}$$

$$V(\varphi) = V_0 e^{-\lambda\kappa\varphi} , \quad V_0 \equiv \frac{\kappa^2 {g_0}^2}{2} \left(\lambda^2 - 6\right) \,, \tag{20b}$$

where the potential parameters were chosen to ease their comparison with the standard case; notice that in order to avoid a negative definite potential we should impose the condition $\lambda > \sqrt{6}$. The equations of motion (16) and (17) with an exponential superpotential explicitly read

$$\ddot{\varphi} = -3\frac{\dot{a}}{a}\dot{\varphi} + \lambda\kappa V \pm \left(\lambda^2 - 6\right) \frac{\lambda\kappa^3 g}{8a^3} \,, \tag{21a}$$

$$H^2 = \frac{\kappa^2}{6}\dot{\varphi}^2 + \frac{\kappa^4}{3} V + \frac{\kappa^2}{32a^6} \pm \left(\lambda^2 - 6\right) \frac{\kappa^4 g}{12a^3} \,, \tag{21b}$$

where we have written the superpotential $g(\varphi)$ in all terms when needed.

3.1 Exact SUSY Scaling Solution

It can be noticed that there is a stiff matter term in Eq. (21b), and the superpotential g appears accompanied by factor a^{-3}. Thus, one can foresee that there must be a stiff matter solution of the equations of motion, so that $a \approx t^{1/3}$, as long as $g \approx a^{-3}$ and

$V \approx a^{-6}$. It can be shown, just by direct substitution in Eqs. (21), that the exact scaling solution is

$$a(t) = a_0(t/t_0)^{1/3}\,, \tag{22a}$$

$$\kappa\varphi(t) = \frac{2}{\lambda}\ln t/t_0\,, \tag{22b}$$

$$g(t) = g_0(t_0/t), \quad g_0 = \mp {a_0}^{-3} \tag{22c}$$

where a_0 is an appropriate integration constant.

The scaling solution corresponds to stiff fluid matter, as revealed by the power law behavior of the scale factor in Eq. (22a); this is probably not surprising, because we have already noticed the presence of a stiff-term in the SUSY Friedmann equation (17). This solution is not inflationary, but its existence indicates its possible importance in the early Universe.

3.2 An Initial Dynamical System

An interesting step the study of the evolution of four SUSY classical model in which the scalar field φ is endowed with an exponential potential. As in the standard case, it its possible perform a dynamical study of the cosmological model so that its physically relevant solutions are easily unveiled.

In order to construct a dynamical system for our cosmological model, we follow Ref.[14], see also [6, 24]. One first step is to introduce a set of conveniently chosen variables which allow rewriting the conservation equations and the evolution equation H as an autonomous phase system subject to a constraint arising from the Friedmann equation.

We choose the following variables,

$$x \equiv \frac{\kappa\dot{\varphi}}{\sqrt{6}H}, \quad y \equiv \frac{\kappa\sqrt{V}}{\sqrt{3}H} \quad z \equiv \frac{\kappa^2}{\sqrt{32}a^3 H}\,, \tag{23}$$

which render the Friedmann equation as

$$x^2 + y^2 + z^2 \pm 2\sqrt{(\lambda^2 - 6)/3}\,yz = 1. \tag{24}$$

The constraint equation (24) follows from Eq. (17), and we see that variable z plays the role of an extra fluid term which, contrary to the standard case, see Ref.[14], is not trivially coupled to the scalar field variables.

In what follows, we shall restric ourselves to the part of the phase space corresponding to $-\infty < y < \infty$, and $H \geq 0$, since our main concern are expanding universes. Combining expressions (16) and (23), the equations of motion read

$$x' = -3x - \frac{\dot{H}}{H^2}x - \sqrt{\frac{3}{2}}\lambda y^2 \pm \frac{\lambda\sqrt{\lambda^2-6}}{\sqrt{2}}yz\,, \tag{25a}$$

$$y' = \sqrt{\frac{3}{2}}\lambda xy - \frac{\dot{H}}{H^2}y\,, \tag{25b}$$

$$z' = -3z - \frac{\dot{H}}{H^2}z\,, \tag{25c}$$

where

$$\frac{\dot{H}}{H^2} = -3x^2 - 3z^2 \mp \sqrt{3\,(\lambda^2-6)}yz. \tag{26}$$

Here primes denote differentiation with respect to the logarithm of the scale factor, $N = \ln(a)$. The evolution of the phase space variables x, y, and z, takes place only on the constraint surface described by Eq. (24), which is an ellipsoid.

There are five critical points, in close similarity to the standard case, whose main features are described next.

- Stiff matter domination. The potential variable is null, $y = 0$, and then the dynamical system is equivalent to the standard case of stiff matter (a^{-6}) plus a free scalar field ($\dot{\phi} \approx a^{-3}$), so that $x^2 + z^2 = 1$.
- Scalar field domination. It is the coexistence of the (scalar) kinetic and potential energies, $x^2 + z^2 = 1$, and then the point is located in the unitary circumpherence on the plane $z = 0$. Notice, however, that the existence of this point requieres $\lambda^2 < 6$, which is in contradiction with our earlier assumption that $\lambda^2 > 6$.
- Scaling solution. This point corresponds to the scaling solution in section 3.1 and represents the coexistence of all energy terms in the equations of motion. It should be notice that, contrary to the present work, in the standard cosmological case the scaling solution in the presence of stiff fluid matter necessarily requires $y = 0$.

4. General Comments

In this work, we have briefly explained a new idea of the supersymmetric extension of the action of general relativity for a scalar field with the scale factor of the universe. For this purpose, we have introduced a superfield formulation in which fermionic degrees of freedom are associated to both the sacel factor and to the scalar field. By realizing the algebra of the fermionic variables and representing them as matrices, we get four equations for four components of the wave function. We focused our attention in two of them, and apply the WKB method in order to get two classical SUSY-cosmological equations. The associated equations of motion for the scalar field are obtained by means of Hamilton's equations.

In these supersymmetric Einstein-Klein-Gordon equations (SUSY-EKG), new contributions arise that behave like stiff matter, and some others in which the usual scalar field terms are modified by functions of the scale factor.

We perform an analysis of the dynamical system structure of the SUSY-EKG equations in order to find all relevant physical solutions, full details of which are provided in[26]. One of these solutions is only valid at early times, and it is for that reason that we need to understand the changes induced upon the dynamics of a scalar field endowed with an exponential potential. And these are possibilities that may modify again the phase structure of the solutions and the existence of inflationary solutions. This would require full solutions of the equations of motion beyond the dynamical system analysis, and this can be considered as a first step towards a more complete picture where the supersymmetry is a fundamental key of the early universe.

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Preheating in a Inflationary Chaotic Model

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Abstract: Inflationary cosmological models must include a process able to produce an evolution towards a phase of radiation domination so that the Friedman hot expanded state is retrieved satisfactory. At the end of inflation, the universe must enter to the stage known as (p)reheating during which cosmological reheating temperature should be increased in an explosive way. We study the process of preheating in a model that includes two scalar fields, one of which is the inflaton field, and four-leg interaction term in the Lagrangian. Analytically and numerically we analyze the production of particles by parametric excitations.

Keywords: Inflation; Cosmological Models; Chaotic Model; Preheating; Pre-inflationary Universe
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1. Introduction

During the period of inflation all energy is contained in a classical scalar field ϕ (inflaton) with negligible kinetic energy. Eventually, the inflaton field decays and transfers all its energy to relativistic particles causing that the universe evolves to a state of domination of radiation; this process should produce a hot Friedmann universe. This stage is known as **reheating** [2] and during its development (almost) all elementary particles were created that subsequently populated the universe. It is then important to analyze the processes that took place and the potential repercussions on the subsequent cosmological evolution.

The basic idea for cosmological reheating was proposed by A. D. Linde in [2], particle

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[2] The term *"reheating"* was derived from assuming that there was a pre-inflationary universe in a state with high temperature.

production occurs due to oscillations of the scalar field ϕ around the minimum of the self-interaction potential, immediately after the inflationary era ended. The particles interact with each other so that eventually reach an thermal equilibrium state with reheating temperature T_r. This process is considered complete once the inflaton has given all its energy to relativistic particles that have been created. This gradual reheating can be analyzed by perturbative methods and a eventual thermalization [3], but in many inflationary models the production of particles is carried out by non-perturbative processes in the so-called parametric excitation regime [4-7]. This process is known as **preheating** because these are carried out immediately after inflation has ended and before perturbative contributions are valid.

The preheating stage occurs in a very short time period in which relativistic particles are produced copiously, and is followed by turbulent interactions between different oscillation modes of the scalar fields. We present the analytical study for preheating in a model with two scalar fields, one of which is the inflation field with quadratic self-interaction potential, and the second scalar field is an "auxiliary" one with negligible bare mass. An interaction term of four-legs is included in the theory which induces resonant parametric solutions for the field equations.

Parametric production of scalar particles is analytically reviewed in this model both in the limit narrow parametric resonance and in the case of stochastic parametric resonance, without taking into account the rescattering of fields. Redispersion between the scalar fields becomes crucial for the dynamics of the universe after the energy density of the auxiliary scalar field begins to be comparable with the energy density of the inflaton field, so we present the results of numerical analysis for preheating in which the redispersion becomes clear, this prevents the inflaton scalar field from decay completely. We discuss the possibility for the system to achieve a possible thermal equilibrium. Finally we analyze the evolution of the equation of state to evaluate whether the universe evolves towards a state of radiation domination.

2. Graceful Exit from Inflation in the $V(\phi) = \frac{1}{2}m^2\phi^2$ Model

If the dynamics of the inflationary stage is determined by an inflaton scalar field ϕ, the relevant Lagrangian density must be written as

$$\mathcal{L} = -\frac{1}{16\pi G}R + \frac{1}{2}\phi^{,\mu}\phi_{,\mu} - V(\phi). \tag{1}$$

where $V(\phi)$ is the effective potential for inflaton field. The field equations obtained from (1) in a homogeneous and isotropic flat universe (FLRW), without a cosmological constant term, are

$$H^2 = \frac{8\pi G}{3}\rho, \tag{2}$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV(\phi)}{d\phi} = 0, \tag{3}$$

where $H \equiv \dot{a}/a$ is the Hubble parameter, the energy density and pressure are defined as

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi), \qquad p = \frac{1}{2}\dot{\phi} - V(\phi), \tag{4}$$

so that the energy-momentum tensor can be rewritten as a perfect fluid.

To analyze the behavior of the field ϕ during the graceful exit we will consider the change of variable

$$\dot{\phi} = \sqrt{\frac{3}{4\pi}}H\cos\theta, \quad m\phi = \sqrt{\frac{3}{4\pi}}H\sin\theta, \tag{5}$$

thus we have the following first order differential equations system for H y θ

$$\dot{H} = 3H^2\cos^2\theta, \tag{6}$$

$$\dot{\theta} = -m - \frac{3}{2}H\sin 2\theta. \tag{7}$$

In the limit $m \gg 1$, the solutions are

$$\phi \simeq \frac{\sin mt}{\sqrt{3\pi}mt}\left(1 - \frac{\sin 2mt}{2mt}\right) + O\left((mt)^{-3}\right), \tag{8}$$

$$a \propto t^{2/3}\left(1 + \frac{\cos(2mt)}{6m^2t^2} - \frac{1}{24m^2t^2}\cdots\right), \tag{9}$$

so the inflaton field asymptotically behaves as

$$\phi(t) \simeq \Phi(t)\sin(mt), \tag{10}$$

where $\Phi(t) \equiv 1/(\sqrt{3\pi}mt)$ is the oscillation amplitude.

The behavior $a \propto t^{2/3}$ is characteristic for a universe in "dust" domination, whose equation state is $p = 0$. This cold matter consists of a set of highly scalar field massive particles. For a complete evolution to a hot Friedmann universe this model must be able to produce a cosmological state whose temperature is sufficiently high [8-10].

3. Preheating

Consider the basic model of reheating with scalar field inflaton ϕ and auxiliary scalar field χ as follows,

$$\mathcal{L}_{mat} = \frac{1}{2}\phi_{,\mu}\phi^{,\mu} - V(\phi) + \frac{1}{2}\chi_{,\mu}\chi^{,\mu} - \frac{1}{2}m_\chi^2(0)\chi^2 - \frac{1}{2}g^2\phi^2\chi^2, \tag{11}$$

where g is the coupling constant between scalar fields, $m_\chi(0)$ is the bare mass of the field χ, and $V(\phi)$ is the potential effective potential of ϕ whose bare mass is m. For generality, we assume the vacuum expectation value of the inflaton field is σ, and that $V(\phi)$ near to the minimum is quadratic, $V(\phi) \sim \frac{1}{2}m^2(\phi - \sigma)^2$. If we rewrite at (11) around the minimum we have

$$\mathcal{L}_{mat} = \frac{1}{2}\phi_{,\mu}\phi^{,\mu} - \frac{1}{2}m^2\phi^2 + \frac{1}{2}\chi_{,\mu}\chi^{,\mu} - \frac{1}{2}\left(m_\chi^2(0) + g^2\sigma^2\right)\chi^2 - \frac{1}{2}g^2\phi^2\chi^2 - g^2\sigma\phi\chi^2 \tag{12}$$

where the effective mass for the χ-particles after the change $(\phi - \sigma) \to \phi$ is

$$m_\chi = \sqrt{m_\chi^2(0) + g^2\sigma}. \tag{13}$$

3.1 Preheating due Parametric Resonances

Field equations for *k-modes* of χ obtained from Eq. (12) are

$$\ddot{\chi}_k + 3H\dot{\chi}_k + \left(\frac{\mathbf{k}^2}{a^2} + m_\chi^2(0) + g^2\sigma^2 + g^2\phi^2 + 2g^2\sigma\phi\right)\chi_k = 0, \tag{14}$$

which is an equation for an oscillator with time-varying frequency. Consider that $m_\chi(0) = 0$, and that the amplitude of oscillations is such that $\Phi \ll \sigma$; also, as a first approximation, we ignore the expansion of the universe, i.e. $\dot{a} = 0$, and normalize with $a = 1$, so Eq. (14) can be written as

$$\ddot{\chi}_k + \left(k^2 + g^2\sigma^2 + 2g^2\sigma\Phi\sin(mt)\right)\chi_k = 0, \tag{15}$$

where $k = |\mathbf{k}|$. This equation describes a oscillator with frequency

$$\omega_k^2(t) = k^2 + g^2\sigma^2 + 2g^2\sigma\Phi\sin(mt). \tag{16}$$

If we define $mt = 2z - \pi/2$, $A_k \equiv 4\frac{k^2+g^2\sigma^2}{m^2}$ and $q \equiv \frac{4g^2\sigma\Phi}{m^2}$, Eq. (15) can be written as

$$\chi_k'' + (A_k - 2q\cos 2z)\,\chi_k = 0, \tag{17}$$

where prime denotes differentiation with respect to z. Eq. (17) is known as the **Mathieu equation**. In this part of the analysis, we are only interested for a qualitative study of behavior for the modes χ_k, and to determine if they lead to an important growth that may be associated with the production of particles of the field χ.

By Floquet theory, it is possible to know the stability of the solutions of the Mathieu equation. This can even be applied to a more general differential equation called the *Hill equation*, which has been studied for its applications in the study of classical nonlinear oscillators (see, for example, [11, 12]).

The behavior of the solutions of (17) is determined by the so-called *Floquet coefficient* $\mu_k = \mu_k(a_k, q)$, which is defined through

$$\chi_k \propto \exp(\mu_k z). \tag{18}$$

If $\mathrm{Re}(\mu_k) > 0$ the mode χ_k grows exponentially so it is called **parametric resonance**. It is useful to identify regions of stability-instability on the $A_k - q$ space, which are bounded by curves that can be calculated numerically [13, 14].[3]

Fig. 1 shows the structure of the first bands where parametric resonances occur(shaded tongues). One can see that each of the regions intersects the axis A on l^2, where l is an integer. This tongues are becoming wider in the direction in which q grows, while they become thinner as A increases. The first three polynomials that distinguish stable from

[3] The existence of bands of stability-instability is supported by a theorem whose proof can be found for example in *Magnus & Wrinkle, 1966*, which also determines the ordering of such bands.

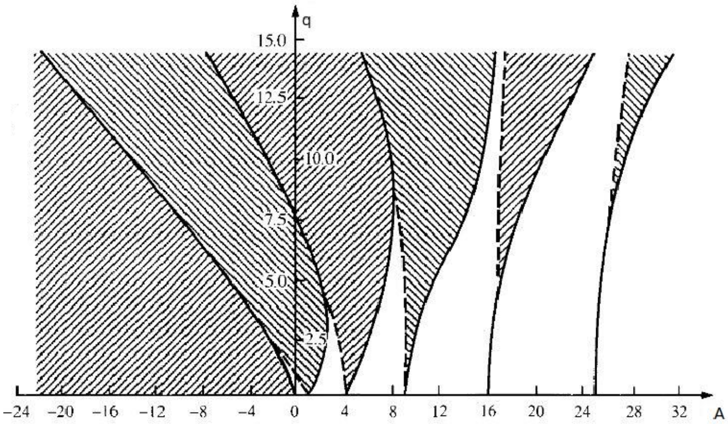

Fig. 1 The shaded areas represent the regions of instability of the solutions of the Mathieu equation Eq. (17). This parametric resonances are related with creation of particles in non-perturbative way. Figure taken from [11].

unstable regions are

$$A_0(q) = -\frac{q^2}{2} + \frac{7q^4}{128} - \frac{29q^6}{2304} + \frac{68687q^8}{18874368} + \cdots, \tag{19}$$

$$A_1(q) = 1 - q - \frac{q^2}{8} + \frac{q^3}{64} - \frac{q^4}{1536} - \frac{11q^5}{36864} + \frac{49q^6}{589824} - \frac{55q^7}{9437184} - \frac{83q^8}{35389440} \cdots \tag{20}$$

$$A_2(q) = 1 + q - \frac{q^2}{8} - \frac{q^3}{64} - \frac{q^4}{1536} + \frac{11q^5}{36864} + \frac{49q^6}{589824} + \frac{55q^7}{9437184} - \frac{83q^8}{35389440} \cdots \tag{21}$$

The existence of exponential instability $\chi \propto \exp(\lambda_k^{(l)} z)$ in a series resonance bands labeled by the integer index l induces exponential growth of occupation numbers n_k associated with each mode χ_k, this can be interpreted as particle production. This production for parametric resonance is essentially different from that proposed in which only production of particles in perturbative form were considered.

We consider the phenomenon of parametric resonance in the case where $q \ll 1$, called *narrow resonance*. Physically, this regime will occur when $g\Phi \ll g\sigma \ll m$. Resonance occurs in thin bands around $A_k \simeq l^2$, $l = 1, 2 \ldots$. Each band has a width of around $\Delta k \sim q^l$, so the first will be, in this case, the most important.

The first band is located roughly between the lines $A_k \sim 1 \pm q = 1 \pm 4g^2\sigma\Phi/m^2$. The factor that describes the growth in $\chi_k \propto \exp(\mu_k z)$ is [14]

$$\mu_k = \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{2k}{m} - 1\right)^2}. \tag{22}$$

Resonance occurs for $k = \frac{m}{2}(1 \pm q/2)$. The μ_k index vanishes at the edges of the resonance band, and takes its maximum values at $\mu_k = q/2 = 2g^2\sigma\Phi/m^2$ in $k = m/2$; the corresponding mode grows as

$$\chi_k \propto \exp(qz/2) = \exp\left(\frac{g^2\sigma\Phi t}{m}\right). \tag{23}$$

The growth of the modes χ_k produces an increase in the occupation number $n_k(t)$ defined as

$$n_k = \frac{\omega_k}{2}\left(\frac{|\dot{\chi}_k|^2}{\omega_k^2} + \omega_k^2|\chi_k|^2\right) - \frac{1}{2}, \tag{24}$$

then, for modes χ_k in parametric resonance the occupation number grows as

$$n_k \propto \exp\left(qz\right) \propto \exp\left(\frac{qmt}{2}\right) = \exp\left(\frac{2g^2\sigma\Phi t}{m}\right). \tag{25}$$

The fact that the resonance occurs near $k = m/2$ can be interpreted, in the limit $g\sigma \ll m$, as if the effective mass of χ-particles created is much smaller than m. This parametric production of particles with very large momentum can eventually contribute very significantly to increase the temperature of the universe.

Because chaotic inflation models do not impose any conditions on the initial values of the inflaton field, the amplitudes of the oscillations of the field ϕ can be large, even much larger than σ, in then the rest of the analysis consider the simplest theory of chaotic inflation without spontaneous symmetry breaking, with $V(\phi) = \frac{m^2}{2}\phi^2$. The interaction term is $\mathcal{L}_{int} = -\frac{1}{2}g^2\phi^2\chi^2$, so, if $H \ll 1$, the field equations are written as

$$\ddot{\chi}_k + \left(k^2 + g^2\Phi^2\sin^2 mt\right)\chi_k = 0, \tag{26}$$

which can be rewritten as a Mathieu equation defining $A_k = \frac{k^2}{m^2} + 2q$, $q = g^2\Phi^2/4m^2$, and $z = mt$.

For $g\Phi < m$, we have narrow parametric resonance as $q \ll 1$, so the analysis will be identical to the corresponding case in the previous subsection. In this regime resonance is more pronounced in the first band, for modes with $k^2 \sim m^2(1-2q \pm q)$. The modes χ_k with momentum corresponding to the center of the resonance band, $k \sim m$, grows as $e^{qz/2}$, and the occupation number for χ particles created grows as $e^{2\mu_k z} \sim \exp\left(g^2\Phi^2 t/4m\right)$. This process can be interpreted as the decay of 2 ϕ-particles with mass m into 2 χ-particles with momentum $k \sim m$.

On the other hand if the amplitude of oscillations Φ is large so that $q = g^2\Phi^2/4m^2 \gg 1$ we will be located in the so-called **broad resonance** regime. In this case, the production of particles becomes extremely efficient because the resonance occurs in a wide range of values of k.

Fig. 2(a) shows the growth of the occupation number in this regime obtained numerically, which is obviously different from the exponential behavior in the narrow resonance regime. The time dependence of the amplitude of oscillations of inflaton field produce each mode χ_k change of instability band so the analysis of broad parametric resonance in terms of calculating the Floquet coefficients are not useful in this case.

3.2 Stochastic Resonance

A this time we have analyzed the phenomenon of preheating in the limit that $H \equiv \dot{a}/a = 0$. When considering the expansion of the universe, the nature of preheating is much

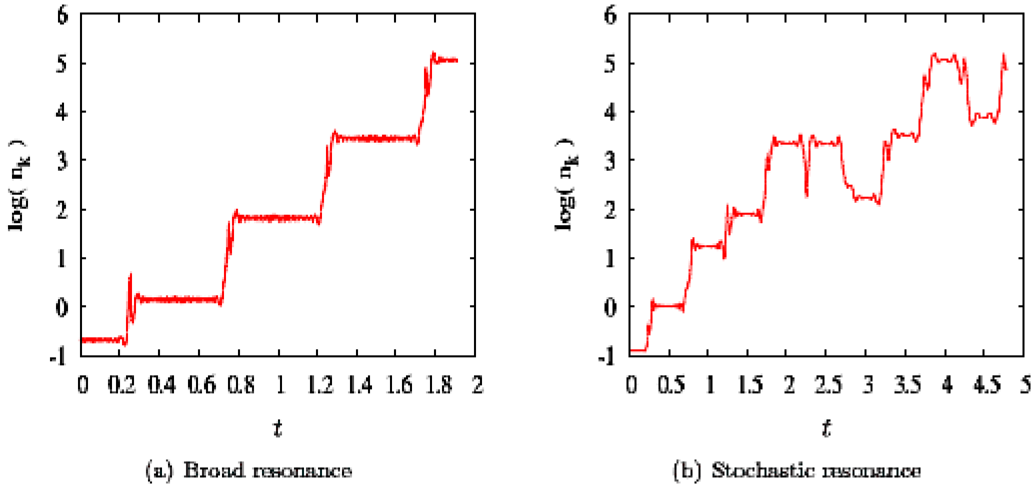

Fig. 2 Numeric solutions of Eq. (14) for the mode χ_k with k corresponding to the maximal speed of growth. Temporal scale is in $m/2\pi$ units and this is equal to the number of oscillations of the inflaton field ϕ. In broad parametric resonance in Minkowski space with $q \sim 10^2$, particle productions occurs only in small intervals when ϕ is small. In stochastic parametric resonance in an expanding univers with scale factor $a \propto t^{2/3}$ the occupation number typically increase but may decreases too in a random way.

more complicated [7]. Fig. 2(b) shows the growth of the occupation number in this regime, which is obviously different from the exponential behavior in the narrow and broad resonance regimes.

The next objective is to calculate the change in the number density for a single "jump" when $\phi(t)$ crosses a zero at some time t_j. We consider, in this case that ϕ^2 is small, so that the time dependence can be approximated by $(t-t_j)^2$. This process can be considered as the passage of a wave through parabolic time-dependence potential; for this we flow the method described in [7].

The field equation in a universe with expansion can be written as

$$\ddot{X}_k + \omega_k^2 X_k = 0, \tag{27}$$

where $X_k(t) \equiv a^{3/2}(t)\chi_k(t)$, and

$$\omega_k^2 = \frac{k^2}{a(t)^2} + g^2\Phi^2 \sin^2 mt + \Delta, \qquad \Delta = m_\chi^2 - 3\frac{3}{4}\left(\frac{\dot{a}}{a}\right)^2 - \frac{3}{2}\frac{\ddot{a}}{a}, \tag{28}$$

the quantity Δ is usually negligible at the end of inflation.

In the semiclassical (adiabatic) representation the solutions of the Eq (27) can be written as

$$X_k(t) = \frac{\alpha_k(t)}{\sqrt{2\omega_k}} \exp\left(-i\int^t \omega_k dt\right) + \frac{\beta_k(t)}{\sqrt{2\omega_k}} \exp\left(+i\int^t \omega_k dt\right), \tag{29}$$

provided that

$$\dot{\alpha}_k = \frac{\dot{\omega}_k}{2\omega_k} \exp\left(+2i\int^t \omega_k dt\right)\beta_k, \qquad \dot{\beta}_k = \frac{\dot{\omega}_k}{2\omega_k} \exp\left(-2i\int^t \omega_k dt\right)\alpha_k, \tag{30}$$

and that the initial conditions at $t \to 0$ are $\alpha_k = 1$, $\beta_k = 0$, with normalization $|\alpha_k|^2 - |\beta_k|^2 = 1$. Replacing Eq. (29) in Eq. (24) we obtain

$$n_k = |\beta_k|^2 \,. \tag{31}$$

The vacuum expectation value for the for the density number of particles per comoving volume is then

$$\langle n_\chi \rangle = \frac{1}{2\pi^2 a^3} \int_0^\infty dk\, k^2 |\beta_k|^2. \tag{32}$$

If we consider the solutions of the equation Eq. (27) are adiabatic evolution from t_j between the instants $j = 1, 2, 3, \ldots$, where the inflaton field $\phi(t_j) = 0$ (twice in each period of oscillation). Non-adiabatic changes of $X_k(t)$ occur only in the neighborhood of t_j, therefore the adiabatic solution (29) is valid before dispersion occurs at the point t_j

$$X_k^j(t) = \frac{\alpha_k^j(t)}{\sqrt{2\omega_k}} e^{-i\int^t \omega_k dt} + \frac{\beta_k^j(t)}{\sqrt{2\omega_k}} e^{+i\int^t \omega_k dt}, \tag{33}$$

where the coefficients α_k^j, β_k^j are constants en the interval $t_{j-1} < t < t_j$. After dispersion $X_k(t)$ in the interval $t_j < t < t_{j+1}$ will be

$$X_k^{j+1}(t) = \frac{\alpha_k^{j+1}(t)}{\sqrt{2\omega_k}} e^{-i\int^t \omega_k dt} + \frac{\beta_k^{j+1}(t)}{\sqrt{2\omega_k}} e^{+i\int^t \omega_k dt}; \tag{34}$$

the coefficients α_k^{j+1}, β_k^{j+1} are constants in the interval $t_j < t < t_{j+1}$.

Coefficients α_k^{j+1} y β_k^{j+1} can be written in terms of α_k^j and β_k^jthrough the transmission D_k and reflection R_k amplitudes of dispersion that occurs in t_j:

$$\begin{pmatrix} \alpha_k^{j+1} e^{-i\theta_k^j} \\ \beta_k^{j+1} e^{+i\theta_k^j} \end{pmatrix} = \begin{pmatrix} \frac{1}{D_k} & \frac{R_k^*}{D_k^*} \\ \frac{R_k}{D_k} & \frac{1}{D_k^*} \end{pmatrix} \begin{pmatrix} \alpha_k^j e^{-i\theta_k^j} \\ \beta_k^j e^{+i\theta_k^j} \end{pmatrix}, \tag{35}$$

where $\theta_k^j = \int_0^{t_j} dt \omega(t)$ is the accumulated phase for the instant t_j.

To specify the dispersion, consider that the term $g^2\phi^2$ has parabolic behavior around each point t_j: $g^2\phi^2 \approx g^2\Phi^2 m^2 (t-t_j)^2 \equiv k_*^4 (t-t_j)^2$, so the field equation in each neighborhood t_j will be

$$\frac{d^2 X_k}{d\tau^2} + \left(\kappa^2 + \tau^2\right) X_k = 0, \tag{36}$$

where we are introduced the rescaled momentum $\kappa \equiv k/(ak_*)$ and a new time variable $\tau \equiv k_*(t - t_j)$.

An analitic general solution for (36) wil be a linear combination of parabolic cylindric funtions $W\left(-\kappa^2/2; \pm\sqrt{2}\tau\right)$ and the reflection R_k and transmission D_k coefficients for a parabolic dispersion can be written as [13]

$$R_k = -\frac{ie^{i\varphi_k}}{\sqrt{1 + e^{\pi\kappa^2}}}, \qquad D_k = \frac{e^{-i\varphi_k}}{\sqrt{1 + e^{\pi\kappa^2}}}, \tag{37}$$

with properties

$$R_k = -iD_k e^{-\frac{\pi}{2}\kappa^2}, \qquad |R_k|^2 + |D_k|^2 = 1. \tag{38}$$

The mapping of coefficients α_k^j y β_k^j into α_k^{j+1} y β_k^{j+1} will be

$$\begin{pmatrix} \alpha_k^{j+1} \\ \beta_k^{j+1} \end{pmatrix} = \begin{pmatrix} \sqrt{1+e^{\pi\kappa^2}}e^{i\varphi_k} & ie^{-\frac{\pi}{2}\kappa^2+2i\theta_k^j} \\ -ie^{-\frac{\pi}{2}\kappa^2-2i\theta_k^j} & \sqrt{1+e^{\pi\kappa^2}}e^{-i\varphi_k} \end{pmatrix} \begin{pmatrix} \alpha_k^j \\ \beta_k^j \end{pmatrix}. \tag{39}$$

The number of outgoing density particle after scattering at t_j can be calculated as $n_k^{j+1} = |\beta_k^{j+1}|^2$, so

$$n_k^{j+1} = e^{-\pi\kappa^2} + \left(1+2e^{-\pi\kappa^2}\right) n_k^j - 2e^{-\frac{\pi}{2}\kappa^2}\sqrt{1+e^{-\pi\kappa^2}}\sqrt{n_k^j(1+n_k^j)}\sin\theta_{tot}^j, \tag{40}$$

where $\theta_{tot}^j = 2\theta_k^j - \varphi_k + \arg\beta_k^j - \arg\alpha_k^j$. In the limit $n_k \gg 1$ we have

$$n_k^{j+1} \approx \left(1+2e^{-\pi\kappa^2} - 2\sin\theta_{tot}^j e^{-\frac{\pi}{2}\kappa^2}\sqrt{1+e^{-\pi\kappa^2}}\right) n_k^j. \tag{41}$$

If we define the growth index through the relation $n_k^{j+1} = n_k^j \exp(2\pi\mu_k^j)$, we obtain

$$\mu_k^j = \frac{1}{2\pi}\ln\left(1+2e^{-\pi\kappa^2} - 2\sin\theta_{tot}^j e^{-\frac{\pi}{2}\kappa^2}\sqrt{1+e^{-\pi\kappa^2}}\right). \tag{42}$$

which takes the maximum value when $k = 0$ and $\sin\theta_{tot}^j = -1$,

$$\mu^{max} = \frac{1}{\pi}\ln\left(1+\sqrt{2}\right) \approx 0.28, \tag{43}$$

and the typical value for $k = 0$ is

$$\bar{\mu} = \frac{1}{\pi}\ln\sqrt{3} \approx 0.175, \tag{44}$$

while for $\sin\theta_{tot}^j = 1$ the value of μ will be negative. Therefore the behavior of resonance is essentially due to value of θ_k^j as a function of k for differents t_j moments. In the case where $\Phi(t) = const$ and $a(t) = const$, the phase $\theta_k^j = \theta_k$ does not depend time, in this case we expect for the existence of k-bands of stability and instability. But as we consider the time dependence of θ_k analysis using k-bands will be obsolete.

To calculate the total number of particles created until the instant t_j, it is necessary to repeat the recurrence relation (40) j-times with initial conditions $\alpha_k^0 = 1$, $\beta_k^0 = 0$, $n_k^0 = 0$ and random initial phase θ_k^0 .

4. Simulation of Preheating

In this section we report parameters obtained through numerical simulation of preheating process which took place on a cubic lattice $256 \times 256 \times 256$ points. This simulation is performed through LATTICEEASY code [15]. The mass of the inflation field considered is $m = 10^{-6} M_{Pl}$ and the coupling constant is $g = 2.5 \times 10^{-7}$. The time scale is in units of m, so that a unit correspond to $10^{-37}s$.

Fig. 3(a) shows the graph of the temporal evolution of the density occupation number for different modes of vibration. It is noted that the modes with k/m lower, the infrared modes (IR), are excited more than ultraviolet modes (UV), modes with higher k/m. It is important to note that although that n_k^{χ} has some regions with exponential growing, from $t \approx 120/m$ onwards the number of occupation of each mode reaches an upper limit almost simultaneously, this moment marks the approximate end of the stage of mass production of χ-particles.

Fig. 3(b) displays the evolution of n_k^{ϕ} for 5 different values of k. The fact that ϕ-modes are excited abundantly at about $t = 50/m$ tells us that the field ϕ acts as a background field to earlier times when the redispersion is almost zero, and therefore the analytical developments above are valid in the range $0 \leq t \leq 50/m$.

Fig. 4(a) and Fig. 4(b) show the full spectrum that allows the simulation of the occupation numbers n_k^{χ} and n_k^{ϕ} respectively. It showns the spectrum ranges from early times (red graphs) to later times (blue graph) with a time interval $\delta t = 10/m$.

It is possible to distinguish in Fig. 4(a) as in Fig. 4(b) three important stages. The first is a stage in which the IR modes are quickly populate, in which n_k^{χ} and n_k^{ϕ} are very large numbers. In the second stage, redispersion produce spectral regions in which the occupation numbers even decrease. The latter is a stage in which the system are satured (blue graph).

During the second and third phases it can be seen a "migration" in the ocupation number from IR to UV regions, this pruduces increasingly smooth curves, what makes us assume that the system is evolving into a kinetic stable state.

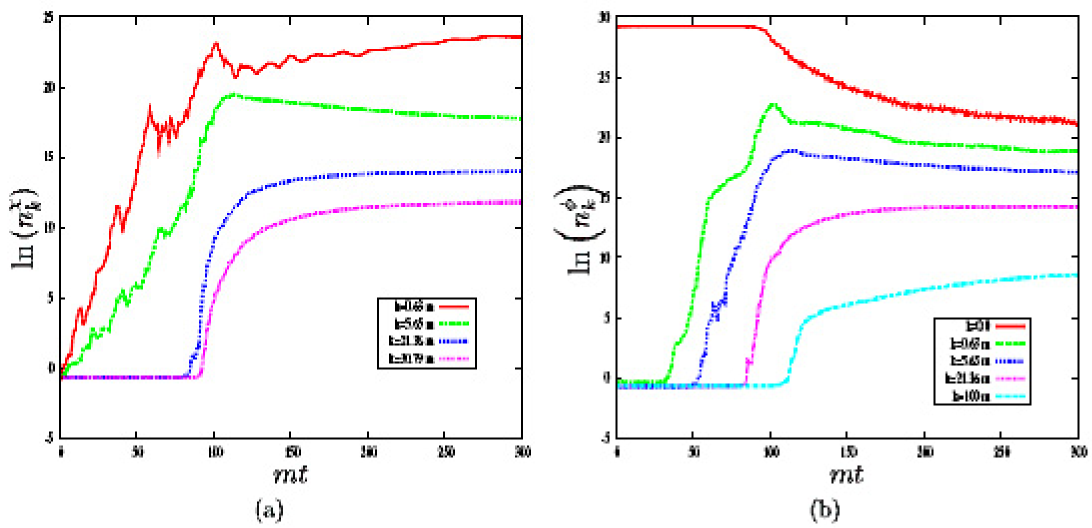

Fig. 3 Time evolution of occupation numbers of modes χ_k and ϕ_k. The χ_k modes are immediately excited after the simulation begins. The ϕ_k modes are exited at $t \approx 50/m$, then redispersion is important after this time.

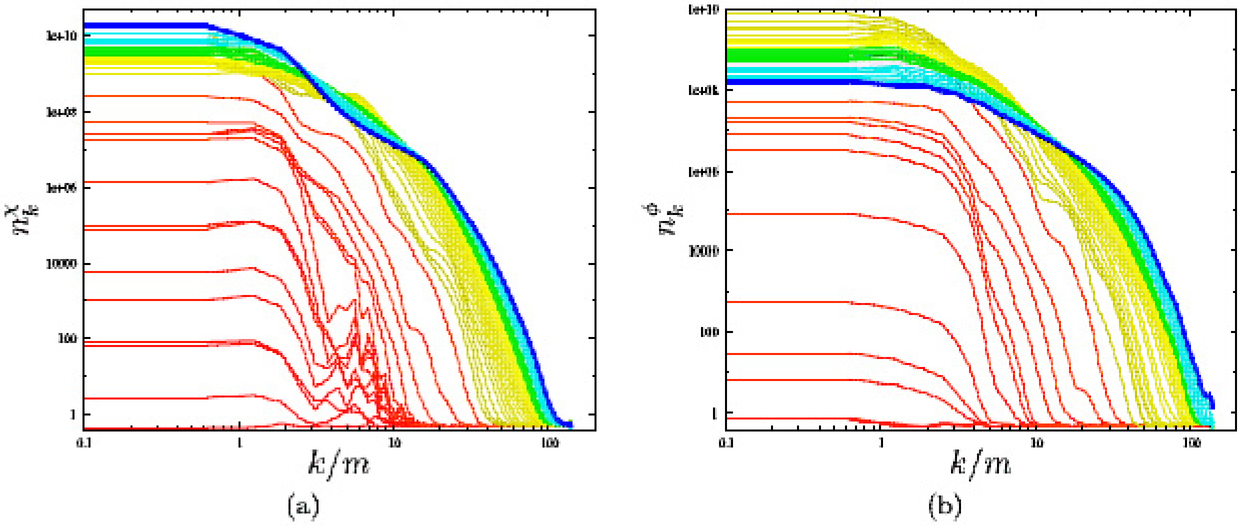

Fig. 4 Occupation number spectra for n_k^χ and n_k^ϕ . The spectrum evolves from small time (red graphs) at large times (blue graphs) with a spacing of $\Delta t = 10/m$. The IR modes are quickly populated at beginning of simulation and a "migration" towards UV modes are shown at later times.

An amount that can be more instructive than n_k is the product $n_k \omega_k$ where ω_k, is the energy per mode, since this product relates to the spectrum of Rayleigh-Jeans, namely

$$n_k \approx \frac{T_{eff}}{\omega_k}. \tag{45}$$

This correspond to the equipartition spectrum of clasiccal waves.

The combination $n_k \omega_k$ for the two fields χ and ϕ is shown in Fig. 5(a) and Fig. 5(b), respectively, each of which shows the graphs for six characteristic times. It shows a trend towards a state with an uniform temperature $T_{eff} \approx n_k \omega_k$, indicating that the model is viable even though the simulation did not show a total thermal equilibrium at the end of the run.

4.1 Equation of State

When we select the initial conditions of the simulation we assumed that at the end of inflation the universe is dominated by matter, and therefore the equation of state at that time must be $w(t_0) = p(t_0)/\rho(t_0) = 0$.

If this preheating model has aspirations to be a viable model it must be capable of shifting from a state of domination of matter into a state or radiation domination with equation of state $\omega(t_r) = 1/3$, where t_r is the time when preheating ends and a hot Friedmann desceleration universe is recovered. Fig. 6 displays the evolution of the equation of state $w(t) = p/\rho$ where p and ρ have been calculated according to the

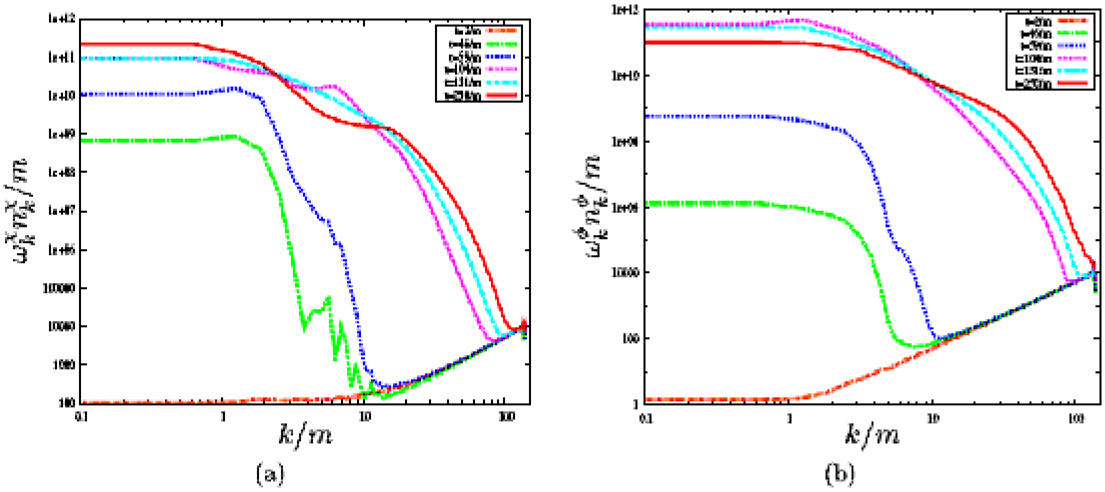

Fig. 5 Evolution of $\omega_k^\chi n_k^\chi$ y $\omega_k^\phi n_k^\phi$ this can be compared with the Rayleigh-Jeans spectrum $n_k \approx T_{eff}/\omega_k$. The effective temperature grows in explosive way only for infrared modes. The system shows a tendency towards thermalization only for IR modes.

expressions:

$$\rho = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\dot{\chi}^2 + \frac{1}{2a^2}|\nabla\phi|^2 + \frac{1}{2a^2}|\nabla\chi|^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{2}g^2\phi^2\chi^2, \tag{46}$$

$$p = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\dot{\chi}^2 - \frac{1}{6a^2}|\nabla\phi|^2 - \frac{1}{6a^2}|\nabla\chi|^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{2}g^2\phi^2\chi^2, \tag{47}$$

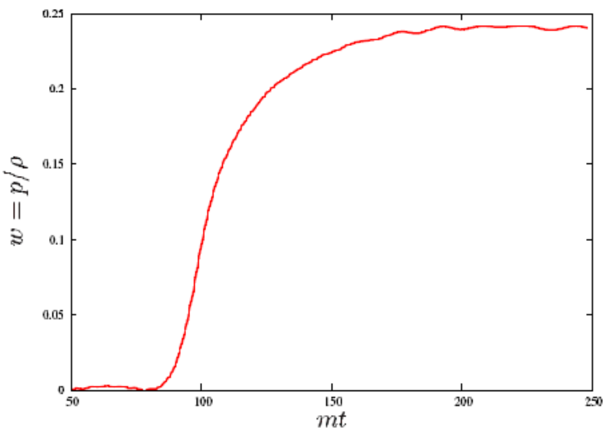

Fig. 6 Equation of state $\omega = p/\rho$ evolution. At beginning the system are in matter domination era characterized by $w = 0$. Simulation shows that this model produces a evolution towards a near radiation domination era.

The curve shows that the equation of state in the simulation grows to a maximum value $w_{max} = 0.25$, indicating that the model is able to evolve the universe from state of "dust" and bring it to a state close to radiation domination.

5. Conclusions

In this work we have analyzed the cosmological preheating process. This occurs through stochastic parametric resonances. The analytical study was carried out without considering the redispersion between scalar fields ϕ and χ. This assumption becomes invalid soon because of occupation number n_k^χ quickly causes redispersion.

The numerical simulation shows how modes of the inflaton field and of the field χ are excited. The results indicates, that due to the redispersion, the behavior of n_k^ϕ and n_k^χ is no always growing, but there are regions in both spectra in which after reaching a maximum there may even be a decline. The curves of spectra of the occupation numbers tend to be soft towards the final stages of the simulation. This results agree with others reported in [16].

Although the model does not show a trend towards an eventual complete thermalization it has been observed, through the evolution of the equation of state $w = p/\rho$, the system evolves showing an asymptotic tendency towards a state characterized by $w \approx 0.25$. This shows that the this preheating model, which consider that the cosmological dynamics relies only on two scalar fields, one of which is the inflaton field and whose interaction term is $\frac{1}{2}g^2\phi^2\chi^2$, is a viable model despite the equation of state does not reach the value associated with an era of radiation domination.

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Boundary Terms in Cosmological Models and their Quantization

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Abstract: We analyze the boundary terms emerging from a class of cosmological models which are linear either in the extrinsic or the scalar curvature, and involve up to second derivative terms in the Lagrangians describing them. In the classical theory we explicitly identify the constraints. The algebra of first-class constraints results isomorphic to the triangular lower algebra of $\mathrm{SL}(2,\mathbb{R})$ whose associated Lie group is non-unimodular. We then pursue to complete the quantization for our systems by considering recent proposals to canonically quantize non-unimodular groups by enlarging the group structure, which in turn, brings modifications to the quantum potentials appearing in the Wheeler–DeWitt equations. As stated, the modifications on these potentials are related to the boundary terms appearing in our approach. A comparison with naïve Dirac quantization is also analyzed.

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1. Introduction

With the advent of brane world universes, cosmology in the presence of extra dimensions has been the subject of intense research. In this article we analyze a couple of models related to brane cosmology.

Our first model is related to the original proposal by Dirac [1] to model the electron as a charged membrane. This model started an exhaustive study of relevant physical systems strongly tied to geometrical theories of surfaces moving in a spacetime. The spinless Dirac geometrical theory describes a dynamic membrane in the presence of an external fixed electromagnetic field where the non-electromagnetic forces are described

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by a constant surface tension. Over the years the model has been improved by taking into account the inclusion of second-order correction terms built from the extrinsic curvature of the worldvolume swept out by the membrane [2, 3]. In fact, such extrinsic curvature terms appear in several effective actions aimed to describe surfaces in diverse contexts which accommodate relativistic extended objects as notable realizations of interesting physical systems [4]. This model introduces interesting resemblances to cosmological brane models which deserves a careful analysis [5, 6].

Our second model is related to the idea that our universe could be a surface embedded in a higher dimensional spacetime, as was set up by Regge and Teitelboim (RT) [7]. The scope of such model is that gravitation can be described in a string-like fashion, as the worldvolume swept out by the motion of a three-dimensional spacelike brane evolving in a higher dimensional bulk spacetime. Recently, the RT brane model has been considered as one of the two main pillars of a unified brane-like theory [8], together with the Randall–Sundrum theory [9].

The cosmology that arise from these models is interesting since it provides an alternative route to understand better classical cosmology in extra dimensions, and also by supplying a compelling model to apply the canonical quantization methods. Both models are genuine second order derivative model in the field variables, which are, the embedding functions rather than the induced metric.

In the present paper we consider an alternative formulation for the cosmological models introduced which is strongly based in the Ostrogradski programme for higher-order derivative theories. We pay close attention to a Hamiltonian approach for our models, which in turn leads to the correct dynamics. In particular, it is of a great interest to use the full model straightforwardly for obtaining the quantum approach for brane cosmology. To illustrate our development we specialize our considerations to a minisuperspace model where it is evident the inherent gauge invariance under the reparametrization of time. We show that the canonical constraint quantization of this model casts into a satisfactory Wheeler–DeWitt (WDW) equation on the wave function for a brane-like universe. Our quantum treatment hence leads to potentials with the expected behaviour when naïve Dirac quantization is applied. However, in quantisation of constrained systems an interesting situation arises from the rather different senses in which diverse quantum schemes satisfies the Dirac constraints for unimodular and non-unimodular groups [10]. To guarantee that the would-be inner product provided is real, the physical states must be invariant under the group inverse. For a non-unimodular group, the left and right invariant Haar measures do not coincide, and neither is invariant under the group inverse, but their geometric average is. For systems amenable to algebraic or geometric quantization in both reduced and unreduced phase space, the way in which we define physical states is in fact the form of Dirac constraints equivalent to reduced phase space quantisation [10, 11]. Our non-unimodular gauge group G is the connected component of the lower triangular subgroup of $\mathrm{SL}(2,\mathbb{R})$. G is two-dimensional and non-Abelian, and hence isomorphic to every two-dimensional connected non-Abelian group.

This article is closely related to previous work detailed in [12-14].

2. Classical Cosmological Models

For our first model, we consider a 2-dimensional surface, Σ, evolving in a Minkowski 4-dimensional background spacetime with metric $\eta_{\mu\nu}$, described by the embedding $x^\mu = X^\mu(\xi^a)$ where x^μ are local coordinates for the background spacetime $(\mu,\nu = 0,1,2,3)$, ξ^a are local coordinates for the worldvolume, m, swept out by the surface $(a,b = 0,1,2)$ and X^μ are the embedding functions for m. We consider the following effective action underlying the dynamics of the surface Σ

$$S_{\mathrm{Dirac}}[X^\mu] = \int_m d^3\xi \,\left(-\mu\sqrt{-g} - \alpha\sqrt{-g}\,K + \beta\, j^a\, e^\mu{}_a\, A_\mu\right)\,, \tag{1}$$

where K is the mean extrinsic curvature of the worldvolume constructed with the extrinsic curvature tensor $K_{ab} = -\eta_{\mu\nu} n^\mu D_a e^\nu{}_b$ and g denotes the determinant of the induced worldvolume metric $g_{ab} = \eta_{\mu\nu} e^\mu{}_a e^\nu{}_b$, where $e^\mu{}_a = X^\mu{}_{,a}$ are the tangent vectors to the worldvolume; n^μ is the spacelike unit normal vector to the worldvolume. Further, $D_a = e^\mu{}_a D_\mu$, where D_μ is the background covariant derivative. The factors μ and α are constants related to the surface tension and the rigidity parameter of the surface Σ, respectively, and β is the form factor of the model. Furthermore, $A_\mu(x)$ is the gauge field living in the ambient spacetime, and j^a is a fixed electric charge current density continuously distributed over the worldvolume, responsible for the minimal coupling between the charged surface and the electromagnetic field A_μ. The action functional (1) is invariant under reparametrizations of the worldvolume m. This model was extensively investigated in [2, 3, 12, 13].

Analogously, for our second model, we consider a brane Σ of dimension d, evolving in a fixed Minkowski N dimensional background spacetime with metric $\eta_{\mu\nu}$. Its trajectory, or worldvolume m of dimension $d+1$, is described by the embedding $x^\mu = X^\mu(\xi^a)$, where x^μ are local coordinates for the background spacetime, ξ^a local coordinates for m, and X^μ the embedding functions $(\mu,\nu = 0,1,\ldots,N-1;\ a,b = 0,1,\ldots,d)$. We denote by $e^\mu{}_a = \partial_a X^\mu$ the tangent vectors to m. In this framework we introduce $N-d-1$ unit normal vectors to the worldvolume, denoted by $n^\mu{}_i$ $(i = 1,2,\ldots,N-d-1)$. These are defined implicitly by $n^i \cdot e_a = 0$ and we choose to normalize them as $n_i \cdot n_j = \delta_{ij}$. The RT model for a d-dimensional brane Σ is defined by the action functional

$$S_{\mathrm{RT}}[X] = \frac{\alpha}{2}\int_m d^{d+1}\xi\,\sqrt{-g}\,\mathcal{R} - \int_m d^{d+1}\xi\,\sqrt{-g}\,\Lambda\,, \tag{2}$$

where the constant α has dimensions $[L]^{(1-d)}$, g denotes the determinant of the induced metric $g_{ab} = \eta_{\mu\nu}\, e^\mu{}_a e^\nu{}_b = e_a \cdot e_b$. We have also included in this action a cosmological constant term, Λ. The extrinsic curvature of m is $K_{ab}{}^i = -n^i \cdot D_a e_b$, where $D_a = e^\mu{}_a D_\mu$ and D_μ is the covariant derivative in the bulk spacetime. The mean extrinsic curvature is given by the trace $K^i = g^{ab} K_{ab}{}^i$ where g^{ab} denotes the inverse of g_{ab}. The scalar curvature $\mathcal{R}$ of m can be obtained either directly from the induced metric g_{ab}, or, in terms of the extrinsic curvature, via the contracted Gauss–Codazzi equation, $\mathcal{R} = K^i K_i - K^i_{ab} K^{ab}_i$. Further details can be found in [7, 14].

We turn now to specialize the previous definitions to the description of spherical membranes Σ. From now on, we consider a background Minkowski spacetime described by $ds^2 = -dt^2 + da^2 + a^2 d\theta^2 + a^2 \sin^2\theta d\phi^2$ for our first model, while we consider a 3-brane Σ, evolving in a 5-dimensional Minkowski spacetime, $ds^2 = -dt^2 + da^2 + a^2\, d\Omega_3^2$, where $d\Omega_3^2$ stands for the metric of a unit 3-sphere, for our second model. For simplicity, we will consider a closed universe. Thus our membranes are described by the following parametric representations of the trajectory of Σ for each model: $x^\mu = X^\mu(\tau,\theta,\phi) = (t(\tau), a(\tau), \theta, \phi)$ and $x^\mu = X^\mu(\xi^a) = (t(\tau), a(\tau), \chi, \theta, \phi)$ respectively, so that the induced metric on the worldvolume is explicitly given by $ds^2 = g_{ab} d\xi^a d\xi^b = -N^2 d\tau^2 + a^2 d\theta^2 + a^2 \sin^2\theta d\phi^2$ where $N = \sqrt{\dot{t}^2 - \dot{a}^2}$ (analogously for the second case). Here the dot stands for derivative with respect to the parameter τ, and $a(\tau)$ is the scale factor. It is worth mentioning that N corresponds to the lapse function in the ADM Hamiltonian approach for branes [15]. The normal vector to the worldvolume is implicitly defined by $g_{\mu\nu} n^\mu e^\nu{}_a = 0$, and $g_{\mu\nu} n^\mu n^\nu = 1$.

From the actions (1) and (2) we see that the effective Lagrangians densities specialized to the associated membranes read[2]

$$L_{\mathrm{D}irac} = -\mu N a^2 - \alpha \frac{a^2}{N^2}\left(\ddot{a}\dot{t} - \dot{a}\ddot{t}\right) - 2\alpha a \dot{t} - \beta \frac{q^2 \dot{t}}{a}\,, \tag{3}$$

$$L_{\mathrm{RT}} = \frac{a\,\dot{t}}{N^3}\left(a\ddot{a}\dot{t} - a\dot{a}\ddot{t} + \dot{t}^3 - \dot{a}^2\dot{t}\right) - N a^3 H^2\,. \tag{4}$$

In addition to the velocities $\dot{t}$ and $\dot{a}$, these Lagrangians depend also on their corresponding accelerations $\ddot{t}$ and $\ddot{a}$, therefore we are dealing with genuine second order derivative theories. Note that both Lagrangians can be rewritten as $L = L_b + L_d$, where L_b is a boundary term and L_d gives a true dynamic term. As customary, the boundary term can be neglected without affecting the membrane evolution in time for the classical theory. However, our treatment will rely on considering explicitly both terms, the boundary and the dynamic, confronting us with a couple of Lagrangians depending up to the accelerations, hence evoking an Ostrogradski-Hamiltonian formalism.

From now on, we will complete the classical treatment for the first of our models as done in [12, 13], and we will refer to [14] for the analogous treatment of the second model, for which we only cite the main results when appropriate.

Following [15], the highest momentum spacetime vector can be rewritten as $P_\mu = -(\alpha a^2\, n_\mu)/N$. Note that the momentum P_μ is directed normal to the worldvolume. This is a general issue for this type of brane models as discussed in [15]. In addition, the conjugate momenta to the position variables, p_μ, are conveniently written in terms of the kinetic momentum, $\pi_\mu = p_\mu - \beta\, q\, A_\mu$, as follows: $\pi_\mu = (2\alpha a[NaH + \dot{t}]\,\dot{X}_\mu)/N^2$. Important to note is the fact that both momenta, p_t and p_a, are from a totally different nature. Indeed, while the momentum p_t is not influenced at all by the surface terms, the momentum p_a is obtained by two contributions: $\mathbf{p}_a$ coming from the ordinary dynamical theory (L_d) and $\mathfrak{p}_a$ coming from the boundary term (L_b). Also, the momentum p_t results

[2] Here we consider the electromagnetic potential on the spherical shell to take the specific form $A_\mu = \left(-\frac{q}{a}, 0, 0, 0\right)$ where q is the total electric charge on the shell, and we also fix the electric current as $j^\tau = q\sin\theta$

to be a conserved quantity for both systems, which we will denote by $-\Omega$, for convenience. Hitherto, the appropriate phase space of the system, $\Gamma = \{t, a, \dot{t}, \dot{a}; p_t, p_a, P_t, P_a\}$, has been explicitly identified. In order to complete the Ostrogradski-Hamiltonian programme in phase space Γ, we will consider the canonical Hamiltonian

$$
\begin{aligned}
H_0 &= p_a\,\dot{a} + p_t\,\dot{t} + P_a\,\ddot{a} + P_t\,\ddot{t} - L \\
&= \pi_a\,\dot{a} + \pi_t\,\dot{t} + \frac{N^2}{\dot{t}}\left(\Omega - \beta\frac{q^2}{a}\right) .
\end{aligned} \tag{5}
$$

As expected, the canonical Hamiltonian results a function only of the physical momentum π_μ. Here $\Omega := -p_t$ results the conserved internal energy.

Next, in order to get control over the model, we implement the following canonical transformation to a new set of phase space variables defined by $N := \sqrt{\dot{t}^2 - \dot{a}^2}$, $\Pi_N := (P\cdot\dot{X})/N$, $\nu := -(N(P\cdot n) + \alpha\, a^2)$, $\Pi_\nu := \mathrm{arctanh}(\dot{a}/\dot{t})$, together with the transformation $X^\mu := X^\mu$ and $\mathbf{p}_\mu := p_\mu + \{p_\mu, \nu\}\,\Pi_\nu$. Of relevance is the fact that under this canonical transformation the coordinates X^μ remain unaltered, while the dynamical momentum $\mathbf{p}_a$ is distinguished as the relevant momentum of the model. Such transformation can be physically interpreted as a Lorentz rotation in phase space.

For our model in question we have a set of four constraints which should be separated into subsets of first- and second-class constraints [16, 17]. For our system we have two first-class constraints and two second-class constraints. We judiciously choose them as

$$
\mathcal{F}_1 = N\Pi_N \approx 0\,, \tag{6}
$$

$$
\begin{aligned}
\mathcal{F}_2 = N\Big[&\left(p_t + \beta\frac{q^2}{a}\right)\cosh\Pi_\nu + (\mathbf{p}_a - 2\alpha a\Pi_\nu)\sinh\Pi_\nu \\
&+\mu a^2 + 2\alpha a\cosh\Pi_\nu\Big] \approx 0\,,
\end{aligned} \tag{7}
$$

$$
\mathcal{S}_1 = \nu \approx 0\,, \tag{8}
$$

$$
\mathcal{S}_2 = N\left[\left(p_t + \beta\frac{q^2}{a}\right)\sinh\Pi_\nu + (\mathbf{p}_a - 2\alpha a\Pi_\nu)\cosh\Pi_\nu\right] \approx 0\,, \tag{9}
$$

where the $\mathcal{F}$'s and the $\mathcal{S}$'s stand for the first- and the second-class constraints, respectively. As customary, the second-class constraints (8) and (9) may be taken as algebraic identities after implementing the Dirac bracket [16, 17]. Furthermore, these second-class identities will become auspicious at the quantum level since they enclose important operator identities. The model thus has $(8-2\times 2-2)/2 = 1$ physical degree of freedom. Note that as we have two linear independent first-class constraints, we will have the presence of two gauge transformations for this brane model.[3]

[3] We will consider throughout the following gauge conditions $\varphi_1 = N - 1 = \sqrt{\dot{t}^2 - \dot{a}^2} - 1 \approx 0$ and $\varphi_2 = N^2 + \dot{a}^2 - \gamma^2\, N^2 H^2 a^2 \approx 0$. From the geometric point of view, this set of gauge conditions is good enough since the matrix $(\{\mathcal{F}, \varphi_{1,2}\})$ is non-degenerate in the constraint surface. Here we note that for our first model the function $\gamma(a)$ is explicitly given as the solution to $\gamma(\gamma+1) = \frac{1}{2\alpha H^2 r^3}\left(\Omega - \frac{\beta q^2}{r}\right)$, while for the second model $\gamma(a)$ is related to the equation $\gamma(\gamma-1)^2 = \Omega^2/a^8 H^6$, difference which will be relevant for the behaviour of the models under consideration.

The constraint $\mathcal{F}_1$ is simply associated to the gauge transformations $N\partial_N - \Pi_N\partial_{\Pi_N}$ which only acts on the $N\Pi_N$-plane. As for the constraint $\mathcal{F}_2$ in equation (7), we can further transform it by expressing the hyperbolic functions in terms of the phase space variables as $\cosh\Pi_\nu = -(p_t+\beta q^2/a)/[2\alpha a^2 H(1+\gamma)]$ and $\sinh\Pi_\nu = (\mathbf{p}_a - 2\alpha a\Pi_\nu)/[2\alpha a^2 H(1+\gamma)]$, where γ is related to the evolution equation enclosed by the gauge condition φ_2. Thus, $\mathcal{F}_2$ is transformed into

$$\mathcal{F}_2 = \frac{N}{\mu a^2(1+\gamma)}\left[(\mathbf{p}_a - 2\alpha a\Pi_\nu)^2 - \left(p_t + \frac{\beta q^2}{a}\right)^2 + \mu^2 a^4(1+\gamma) \right. \\ \left. -2\alpha a\left(p_t + \frac{\beta q^2}{a}\right)\right] \approx 0\,, \tag{10}$$

and we have arrived to an expression quadratic in the physical momenta for the canonical Hamiltonian H_0, which is identified with the constraint $\mathcal{F}_2$ when the second-class constraint $\mathcal{S}_1$ is considered.

At this point we will note that, for our second model, we analogously will have the first class constraints

$$\mathcal{F}_1 = N\Pi_N \approx 0\,, \tag{11}$$

$$\mathcal{F}_2 = N\left\{\mathbf{p}_a^2 - a\left[-\left(\frac{\Omega}{(\gamma-1)a^3H^2}\right)p_t + \frac{\mathfrak{p}_a\dot{a}}{N} + a^3H^2 + \frac{1}{a^3}N^2\Pi_N^2 \right.\right. \\ \left.\left. -\frac{1}{a^3}\Pi_v^2\right] \times \left[\,(\gamma-1)a^2H^2 + 2\right]\right\} \approx 0\,. \tag{12}$$

These first-class constraint parallel the constraints (6) and (10) for the first system. We thus note that the first-class constraint $\mathcal{F}_2$ for the two models is of a totally different nature. Indeed, for our first model the first-class constraint is a quadratic function of the momenta p_a, p_t and Π_ν, but the momentum Π_N is absent, while for the second model, the first-class constraint is quadratic in all the momenta. This will be of great relevance while analyzing the quantum potentials involved in the corresponding Wheeler–DeWitt equations.

It is also important to mention that for both models constraints $\mathcal{F}_1$ and $\mathcal{F}_2$ form an algebra, namely, $\{\mathcal{F}_1, \mathcal{F}_2\} = -\mathcal{F}_2$ which reflects the invariance under reparametrizations of the models as a fundamental gauge symmetry. Indeed, this algebra results an isomorphism of the Lie algebra $\mathfrak{g}$ associated to the lower triangular subgroup of $SL(2,\mathbb{R})$ with positive diagonal elements, G. Such isomorphism is realized through the identification $\mathcal{F}_1 \mapsto h/2$ and $\mathcal{F}_2 \mapsto e^-$, which renders the algebra $\mathfrak{g}$ [18]. Among the relevant properties of the subgroup G we refer that G is two-dimensional, non-Abelian, connected, and non-unimodular. This last property will play an important role in our quantum theory, as developed below.

3. Quantization

In this section we study the quantum potentials emerging in the canonical quantization for our systems. Once again, we consider to some detail our first model, and only refer to the second model when appropriate. Also, we emphasize the totally dissimilar nature which first- and second-class constraints play in the quantum theory, and also, we explore the different senses in which the physical states for our models can be defined.

We start in the conventional way by promoting the classical constraints into operators, densely defined on a common domain in a proper Hilbert space. As it is well known, we can only achieve a consistent classical theory by implementation of the Dirac bracket. Once this is done, the second-class constraints are eliminated off the theory by converting them into strong identities. At the quantum level this is mirrored by defining the quantum commutator of two quantum operators as $[\hat{A},\hat{B}] := i\widehat{\{A,B\}^*}$, where the Dirac bracket $\{\cdot,\cdot\}^*$ is defined as usual [16, 17]. Thus the operators corresponding to second-class constraints are also enforced as operator identities. For our system, this yields the quantum operator expressions $\hat{\mathcal{S}}_1 := \hat{\nu} = 0$, and $\hat{\mathcal{S}}_2 := \hat{N}[(\widehat{p_t + \beta q^2/a})\widehat{\sinh\Pi_\nu} + (\widehat{\mathbf{p}_a - 2\alpha a \Pi_\nu})\widehat{\cosh\Pi_\nu}] = 0$, which, in particular, tell us the character of the quantum operators $\hat{\nu}$, $\widehat{\cosh\Pi_\nu}$ and $\widehat{\sinh\Pi_\nu}$. Also, we will represent the radial operator as $\hat{\mathbf{p}}_a := -i(\partial/\partial a)a$ since then the operator $\hat{\mathbf{p}}_a^2 = -(\partial^2/\partial a^2 + (2/a)\partial/\partial a)$ will be Hermitian in the inner product of states in a conventional Hilbert space, namely an L^2-space. For the rest of the variables, we choose to work on the "position" representation, where we consider the position operators by multiplication and their associated momenta operators by $-i$ times the corresponding derivative operator.

Next, we define the quantum first-class constraints as

$$\hat{\mathcal{F}}_1 := -iN\frac{\partial}{\partial N}\,, \tag{13}$$

$$\hat{\mathcal{F}}_2 := N\left[\left(\hat{\mathbf{p}}_a - 2\alpha a\hat{\Pi}_\nu\right)^2 - \left(\hat{p}_t + \frac{\beta q^2}{a}\right)^2 + \mu^2 a^4(1+\gamma) - 2\alpha a\left(\hat{p}_t + \frac{\beta q^2}{a}\right)\right]. \tag{14}$$

Note that the N factor in $\hat{\mathcal{F}}_2$ is necessary in order to maintain at the quantum level the classical algebraic structure between the two first-class constraints. We will work on the assumption that the commutators of these quantum constraints form a closed Lie algebra which will be also isomorphic to the algebra $\mathfrak{g}$. Quantization of the lower triangular subgroup of $SL(2,\mathbb{R})$ by algebraic methods was extensively studied in [19]. Below, we will explore the rather different senses in which the quantum constraints can be used to define physical states.

At this point, we will note that the first-class constraints for our second model are analogously given by

$$\hat{\mathcal{F}}_1 := -iN\frac{\partial}{\partial N}\,, \tag{15}$$

$$\hat{\mathcal{F}}_2 := N\left\{-\frac{\partial^2}{\partial a^2} - \left[\frac{i\Omega}{(\gamma-1)a^3H^2}\frac{\partial}{\partial t} + 2a(\gamma H^2a^2 - 1) + a(1-\gamma)H^2a^2 - \frac{1}{a^3}\left(N\frac{\partial}{\partial N}\right)^2\right] \times a\left[(\gamma-1)H^2a^2 + 2\right]\right\}, \tag{16}$$

which parallel the quantum constraints (15) and (16). However, recall that the functions $\gamma(a)$ are of a different nature depending on the model, as referred in the footnote on page 7.

3.1 Naïve Dirac constraints

First, we explore the Wheeler–DeWitt equation emerging by considering the physical states Ψ of the theory as those defined by naïve Dirac conditions

$$\hat{\mathcal{F}}_1\Psi = 0\,, \tag{17}$$
$$\hat{\mathcal{F}}_2\Psi = 0\,. \tag{18}$$

Equation (17) simply tells us that our physical states Ψ are not explicitly depending on the phase space variable N.[4] As classically, equation (18) is the most interesting for us, since it is related to the Hamiltonian operator $\hat{H}_0$, hence resulting in a Wheeler–DeWitt equation. Next, we will identify the quantum potentials associated to this equation. In order to do that, we first notice that the second-class constraint $\hat{\mathcal{S}}_1$ tell us that the variable ν is fixed to zero, thus getting rid of any possible dependence on this variable in Ψ, which in turn leads to the conclusion that the action of the operator $\hat{\Pi}_\nu$ on the Ψ states vanishes automatically. Further, we see that the t-dependence can be solved by assuming $\Psi(a,t) := e^{-i\Omega t}\psi(a)$, in agreement with the classical definition for Ω. Finally, we notice that $\psi(a)$ must satisfy the differential equation

$$\left[-\left(\frac{d^2}{da^2}+\frac{2}{a}\frac{d}{da}\right)+V_D(a)\right]\psi(a)=0\,, \tag{19}$$

where the quantum potential $V_D(a)$ is given by

$$V_D(a)=\mu^2a^4(1+\gamma)+\left(\Omega-\beta\frac{q^2}{a}\right)\left(2\alpha a-\Omega+\beta\frac{q^2}{a}\right)\,, \tag{20}$$

and $\gamma(a)$ was introduced in the evolution equation introduced by the second gauge (see footnote on page 7). Even though the potential is complicated by the inclusion of the γ term, it results analytic for generic points. The behavior of this potential is drawn in Figure 1. The real zeroes of $V_D(a)$ are located at $a = a_1 := \beta q^2/\Omega$ and approximately at $a = a_2 := 2\alpha/\mu$ while the global maximum tends to infinity as a tends to zero, the local minimum (maximum) is located at $a = a_1$ and the local maximum (minimum) is situated close to $a = a_3 := (-a_2^2/2a_1)(1-\sqrt{1+8a_1^2/b_2^2})$ subject to the condition $a_1 < a_2$ ($a_2 < a_1$). For the case $a_1 = a_2$ the point a_3 also equals a_1 and thus we do not have the presence of a local extrema on the potential $V_D(a)$. In any case, the potential is bounded from below at the constant value $-\Omega^2$.

For our second model, the quantum potential $V_{RT}(a)$ is given by the relation

$$V_{RT}(a)=a^2\left[(\gamma-1)H^2a^2+2\right]^2\left(1-\gamma H^2a^2\right)\,, \tag{21}$$

[4] A different factor ordering in $\hat{\mathcal{F}}_1$ could indeed bring an N-dependence on the states Ψ. However, this dependence can be eliminated by a proper gauge-fixing procedure.

Fig. 1 Quantum potential for the Dirac model. The potentials for the first system are equal under both quantizations.

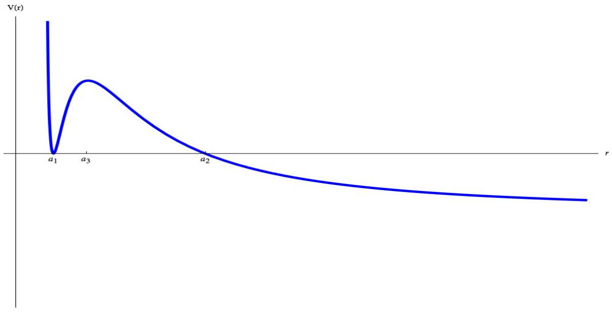
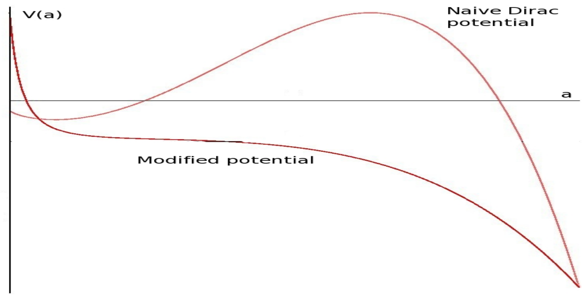


Fig. 2 Quantum potential for the RT model. A different potential has to be consider depending on the quantization procedure.



where the function $\gamma(a)$ is also described in the footnote on page 7, and is of a totally different character as compared to the $\gamma(a)$ for the first model. Once again, the potential $V_{RT}(a)$ results analytic for generic points. The potential is shown in Figure 2.

3.2 Modified Dirac Constraints

As discussed in [10, 11], there exists a procedure which allows to reduce non-unimodular groups to unimodular ones which in turn introduces a modification for the Dirac conditions on physical states. Let $\{\hat{C}_a\}$ be a set of quantum constraint operators that generate a non-unimodular gauge group with the commutators $[\hat{C}_a, \hat{C}_b] = if^c_{ab}\hat{C}_c$, where f^a_{bc} are the structure constants of the corresponding Lie algebra. Thus, the "unimodularization" procedure for non-unimodular groups dictates to consider the physical states $|\Psi\rangle$ as those satisfying $\hat{C}_a|\Psi\rangle = -(i/2)f^b_{ab}|\Psi\rangle$. Such modified Dirac conditions agree with the naïve Dirac constraints if, and only if, the group is unimodular for which $f^b_{ab} = 0$. For our systems, G is the lower triangular subgroup of $\mathrm{SL}(2,\mathbb{R})$ with positive diagonal elements. G is two-dimensional, non-Abelian and connected, properties which characterize G uniquely up to isomorphisms. As described in [19, 20], the left and right invariant Haar measures for G are, respectively, $d_Lg = e^{2\lambda}d\lambda d\mu$ and $d_Rg = d\lambda d\mu$. The adjoint action of G on $\mathfrak{g}$ reads $\mathrm{Ad}_g(h) = ghg^{-1} = h + 2\mu e^-$, $\mathrm{Ad}_g(e^-) = ge^-g^{-1} = e^{-2\lambda}e^-$. Hence the modular function is $\Delta(g) := \det(\mathrm{Ad}_g) = e^{-2\lambda}$. The symmetric measure, invariant under $g \mapsto g^{-1}$, is $d_0g = [\Delta(g)]^{1/2}d_Lg = [\Delta(g)]^{-1/2}d_Rg = e^{\lambda}d\lambda d\mu$.

In this way, for our models, the modified Dirac conditions for the gauge group invariant quantization of the system can be shown to be equivalent to

$$\left[\hat{\mathcal{F}}_1 - \frac{i}{2}\right]|\Psi\rangle = 0\,, \tag{22}$$

$$\hat{\mathcal{F}}_2|\Psi\rangle = 0\,, \tag{23}$$

which consequently define physical states $|\Psi\rangle$. Equation (22) is equivalent to the homogeneity condition $|\Psi(rN)\rangle = r^{-1/2}|\Psi(N)\rangle$ for $r > 0$ [19]. Further, (22) can be explicity solved by taking $|\Psi\rangle = \frac{A}{N^{1/2}}|\psi\rangle$, where A is a constant, and $|\psi\rangle$ is a function of the variables a and t. The t-dependence can be avoided by assuming $|\psi(a,t)\rangle := e^{-i\Omega t}|\varphi\rangle$, where $|\varphi\rangle$ is thought as a function of the scale factor a.

The main point to discuss then is on the consequences of the different nature of the first-class quantum operators $\hat{\mathcal{F}}_2$ for the two systems we are studying. As mentioned before, for our first model this constraint does not contain a quadratic term on the momenta $\hat{\Pi}_N$, and thus the quantum potential will be unaffected by the algebraic structure of the first-class constraints, which in turn implies that equation (23) and the corresponding quantum potential (20) will remain unaltered (see Figure 1). For the second model, however, the situation will be completely different as the momentum $\hat{\Pi}_N$ explicitly appears in both first-class constraints. In this way, for our second model, equation (22) modifies the Wheeler–DeWitt equation by switching the effective potential to $V_{\rm RT}^{\rm eff}(a)$ given by

$$V_{\rm RT}^{\rm eff}(a) = V_{RT}(a) + \frac{(\gamma-1)a^2H^2+2}{4a^2}\,, \tag{24}$$

where the potential $V_{RT}(a)$ was described in equation (21). Hence, we see that our modified quantum theory brings out an extra potential term into our Wheeler–DeWitt equation which succinctly differs from the one found with the naïve Dirac procedure.

We note that the extra term is purely emerging from the modified quantum Dirac equations (22) and (23), and it is completely absent while considering the naïve Dirac procedure. This term will be nonvanishing even in the Einstein limit ($\gamma \to 1$), where it goes as a^{-2}. Further studies about the possible physical implications of this term could be carried out. The behaviour of the modified potential is also drawn in Figure 2, where we can notably see that the central barrier potential present for the potential $V_{\rm RT}(a)$ is transformed into an infinite barrier at the origin. Although the first-class constraint symmetries suggest this modified prescription, the resulting unbounded potential is not realistic, until our knowledge.[5] Nevertheless, one can not resist the speculation of such possible quantum behaviour. Thus, rather than a nice potential, this time it is a more complicated function with distinct features notwithstanding the internal constraint symmetries that is demanded by the unimodularization procedure.

[5] Also note that we did not complete any quantization programme, as our primordial intention was only to analyze the quantum potentials.

4. Concluding Remarks

By making use of the Ostrogradski formalism we have developed an alternative Hamiltonian description of the cosmological models introduced. Our analysis keeps the original variables without the necessity of introducing non-dynamical variables. An important point to mention is that the formalism is rich enough to demonstrate the real role of the extra terms coming from the surface: the phase space constraints of the system impose identities for these quantities which are valid at both classical and quantum levels, hence eliminating the unphysical degrees of freedom.

Although our Wheeler–DeWitt equation for the scale factor is not analytically manageable, it is good enough to substract from it some interesting features. In particular, the quantum potential we found is exactly the same as those discussed on the literature. Also important are the different ways in which the physical states in the quantum theory are defined. In this respect, we have shown that even though for both models the first-class constraints form an algebra isomorphic to a non-unimodular algebra, the inner nature of the models allow one to incorporate a modification in the quantum potentials studied.

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A Note on Generalized Electrodynamics

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Abstract: The generalized Maxwell equations with arbitrary gauge parameter are considered in the 11×11-matrix form. The gauge invariance of such a model is broken due to the presence of a scalar field. The canonical and symmetrical Belinfante energy-momentum tensors are found. The dilatation current is obtained and we demonstrate that the theory possesses the dilatation symmetry. The matrix Schrödinger form of equations is derived. The non-minimal interaction in curved space-time is introduced and equations are considered in Friedmann−Robertson−Walker background. We obtain some solutions of equations for the vector field.

Keywords: Electrodynamics; Maxwell Equations; Generalized Maxwell Equations; Classical Field Theory
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1. Introduction

We have investigated the first-order formulation of the generalized Maxwell equations which describe massless vector fields with an additional scalar field in [1], [2] (see also [3]). Such a model is not a gauge-invariant and can be treated as a Maxwell theory in the definite gauge. Gradient terms were introduced in Maxwell equations by many authors (see references in [3] and [4]). Here, we take into account a gauge parameter which allows us to consider different gauges. It should be mentioned that gauge parameter is physical value in our scheme, contrarily to classical electrodynamics, which can contribute to gravity interaction. Therefore, schemes with difference gauges are not equivalent each other. As was mentioned in [2] the reason for leaving a scalar field in the spectrum is the application of such a non-gauge-invariant model in astrophysics. We have stressed [2] that the additional degree, a scalar field, can play an important role in the inflation

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theory of universe. Later, authors of the work [6] considered a scalar field of generalized electrodynamics as a source of dark energy. Dark energy is introduced to explain the acceleration of expanded universe at the present time and results in the negative pressure. Dark energy interacts only gravitationally representing weakly coupled substance. Scenario of inflationary universe [5] allows us to understand observable data: our universe is homogeneous and isotropic for scales > 100 Mpc (1 pc = 3.26 light years) and expands in accordance with the Hubble law. In the chaotic inflation model a massive scalar field (quintessence) minimally coupled to gravity is responsible for slow-roll inflation and plays the role of dark energy. But, in this model, the potential terms should be fine tuned to have the acceleration of universe at the definite time. Another phenomenological way to describe dark energy is to introduce a cosmological constant Λ into the Einstein equation (the term $(-\Lambda g_{\mu\nu})$ in the left side of the Einstein equation). One implies the existence of vacuum energy by introducing the cosmological constant. If $\Lambda > 0$ the additional cosmological term leads to anti-gravity. But in this case the difficulty arises: a vacuum solution is not a Minkowski space-time. In addition, there is no physical explanation of a coincidence problem: matter and dark energy densities possess the same orders of values at the present time and had big difference in magnitudes in previous eras. Today, approximately 70% of the energy density of the universe is in the form of dark energy, and the rest 30% is in the form of non-relativistic matter. Cosmological constant, which gives the energy density, remains constant during the expansion of the universe, but the energy density of matter and radiation decreases in time. Thus, the nature of dark energy is one of the most important problems in astrophysics.

In the scenario suggested in [6], the time component of a field in generalized electrodynamics grows in time and becomes dominant explaining the acceleration of universe. Therefore, it is of great interest further investigation of the generalized electrodynamics where the addition degree, the scalar state of the field, can play the role of dark energy.

The paper is organized as follows. In Sec.2, the generalized Maxwell equations with arbitrary gauge parameter are formulated in the matrix form. Matrices of the relativistic wave equation (RWE) obey the generalized Duffin −Kemmer−Petiau (DKP) algebra. We obtain, in Sec.3, canonical and the Belinfante dilatation currents which are not conserved. The conserved modified dilatation current is also found demonstrating the scale invariance of the theory of massless fields. We obtain the Schrödinger form of equations and the quantum-mechanical Hamiltonian in Sec.4. The minimal equation for the matrix Hamiltonian is found. In Sec.5, a novel non-minimal interaction in curved space-time is introduced and equations are considered in Friedmann−Robertson−Walker (FRW) background. The solution of equations for the time component of the four-potential is found which grows in time. We discuss the results obtained in Sec.6. The quantum-mechanical Hamiltonian is found from relativistic wave equation in Appendix A. Starting with the second-order formulation of the theory, we obtain canonical and symmetrical energy-momentum tensors and dilatation currents in Appendix B. In Appendix C the quantization of fields is performed in the second-order formalism.

The Euclidean metric is used in Sec.1-4 and Appendixes A, B and C, and four-vectors are $x_\mu = (x_m, x_4) = (x_m, ix_0)$, and x_0 is a time; Greek letters run $1, 2, 3, 4$ and Latin letters run $1, 2, 3$. We use natural units $\hbar = c = 1$.

2. First-order form of equations

In [7], we considered the general Lagrangian form of massive vector fields. For the case of neutral massless vector fields it reduces to

$$\mathcal{L} = \delta_{\mu\nu,\sigma\rho} \left(\partial_\mu A_\sigma\right) \left(\partial_\nu A_\rho\right), \tag{1}$$

where $\delta_{\mu\nu,\sigma\rho} = a\delta_{\mu\nu}\delta_{\sigma\rho} + b\delta_{\mu\sigma}\delta_{\nu\rho} + c\delta_{\mu\rho}\delta_{\sigma\nu}$. To have the standard kinetic term, we put $a = -1/2$. The Euler-Lagrange equations follow from (1):

$$\partial_\mu^2 A_\nu - 2\left(b + c\right) \partial_\nu \partial_\mu A_\mu = 0. \tag{2}$$

One can see from Eq.(2) that only one parameter $(b + c)$ remains in the equations of motion. It is convenient to choose: $c = 1/2$, $2b = -\xi$, where ξ defines the gauge. Then Eq.(1) and (2) become

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{2}\xi \left(\partial_\nu A_\nu\right)^2, \tag{3}$$

$$\partial_\nu F_{\mu\nu} - \xi \partial_\mu \partial_\nu A_\nu = 0, \tag{4}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength. Eq.(4) can be treated as the Maxwell equations with the additional gauge parameter. In QED the physical values do not depend on the gauge [8], but in our scheme, we expect the dependence on ξ because the scalar state presents in the spectrum. In [1], [2], we have chosen the gauge $\xi = 1$. At $\xi = 0$, one arrives at standard Maxwell equations. Here we imply that $\xi \neq 0$. Introducing notations $\psi_{[\mu\nu]} = (1/\kappa)F_{\mu\nu}$, $\psi_\mu = A_\mu$, $\psi_0 = -(\xi/\kappa)\partial_\nu A_\nu$, where κ is the mass parameter, second order equation (4) can be represented as a system of the first-order equations:

$$\begin{gathered}
\partial_\nu \psi_{[\mu\nu]} + \partial_\mu \psi_0 = 0, \\
\partial_\nu \psi_\mu - \partial_\mu \psi_\nu + \kappa \psi_{[\mu\nu]} = 0, \\
\partial_\mu \psi_\mu + \frac{\kappa}{\xi} \psi_0 = 0.
\end{gathered} \tag{5}$$

We note that fields ψ_A ($A = 0$, μ, $[\mu\nu]$) have the same dimension. Introducing wave function $\Psi(x) = \{\psi_A(x)\}$, and using elements of the entire matrix algebra $\varepsilon^{A,B}$ obeying equations: $\left(\varepsilon^{A,B}\right)_{CD} = \delta_{AC}\delta_{BD}$, $\varepsilon^{A,B}\varepsilon^{C,D} = \delta_{BC}\varepsilon^{A,D}$, Eq.(5) can be written in the first-order matrix form

$$\left[\alpha_\nu \partial_\nu + \kappa \left(\frac{1}{\xi} P_s + P_t\right)\right] \Psi(x) = 0, \tag{6}$$

where

$$\begin{gathered}
\alpha_\mu = \beta_\mu^{(1)} + \beta_\mu^{(0)}, \quad \beta_\mu^{(1)} = \varepsilon^{\nu,[\nu\mu]} + \varepsilon^{[\nu\mu],\nu}, \quad \beta_\mu^{(0)} = \varepsilon^{\mu,0} + \varepsilon^{0,\mu}, \\
P_s = \varepsilon^{0,0}, \qquad P_t = \frac{1}{2}\varepsilon^{[\mu\nu],[\mu\nu]}.
\end{gathered} \tag{7}$$

At the Feynman gauge $\xi = 1$, Eq.(6) is simplified because $P_s + P_t = \varepsilon^{0,0} + (1/2)\varepsilon^{[\mu\nu],[\mu\nu]}$ is the projection operator but $(1/\xi)P_s + P_t$ is not. The 11×11 Hermitian matrices α_μ obey the generalized Duffin−Kemmer−Petiau algebra [1]:

$$\alpha_\mu\alpha_\nu\alpha_\alpha + \alpha_\alpha\alpha_\nu\alpha_\mu + \alpha_\mu\alpha_\alpha\alpha_\nu + \alpha_\nu\alpha_\alpha\alpha_\mu + \alpha_\nu\alpha_\mu\alpha_\alpha + \alpha_\alpha\alpha_\mu\alpha_\nu = \\ = 2\left(\delta_{\mu\nu}\alpha_\alpha + \delta_{\alpha\nu}\alpha_\mu + \delta_{\mu\alpha}\alpha_\nu\right), \tag{8}$$

and P_s, P_t are the projection matrices, $P_s^2 = P_s$, $P_t^2 = P_t$, and extract the scalar ant tensor parts of the wave function, respectively. One can verify the relations:

$$P_s\beta_\mu^{(0)} + \beta_\mu^{(0)}P_s = \beta_\mu^{(0)}, \qquad P_t\beta_\mu^{(1)} + \beta_\mu^{(1)}P_t = \beta_\mu^{(1)},$$

$$P_s\beta_\mu^{(1)} = \beta_\mu^{(1)}P_s = 0, \qquad P_t\beta_\mu^{(0)} = \beta_\mu^{(0)}P_t = 0.$$

Introducing the Hermitianizing matrix η [1]:

$$\eta = -\varepsilon^{0,0} + \varepsilon^{m,m} - \varepsilon^{4,4} + \varepsilon^{[m4],[m4]} - \frac{1}{2}\varepsilon^{[mn],[mn]}, \tag{9}$$

the "conjugated" equation reads

$$\overline{\Psi}(x)\left[\alpha_\nu\overleftarrow{\partial}_\nu - \kappa\left(\frac{1}{\xi}P_s + P_t\right)\right] = 0, \tag{10}$$

where $\overline{\Psi} = \Psi^+\eta = \left(-\psi_0, \psi_\mu, -\psi_{[\mu\nu]}\right)$. Matrices α_μ and η satisfy equations: $\eta\alpha_m = -\alpha_m^+\eta^+$, $\eta\alpha_4 = \alpha_4^+\eta^+$. In the first-order formalism the Lagrangian can be written as follows:

$$\mathcal{L} = -\overline{\Psi}(x)\left[\alpha_\nu\partial_\nu + \kappa\left(\frac{1}{\xi}P_s + P_t\right)\right]\Psi(x). \tag{11}$$

Eq.(6) follows from Lagrangian (11) by varying the corresponding action. Similar to the Dirac theory, Lagrangian (11) vanishes for fields Ψ, obeying RWE (6). It should be noted that in the second-order formalism, based on the equations (4), Lagrangian (3) vanishes only within four-divergence. Lagrangian (11), with the help of Eq.(7) becomes

$$\mathcal{L} = \psi_0\partial_\mu\psi_\mu - \psi_\mu\partial_\mu\psi_0 - \psi_\rho\partial_\mu\psi_{[\rho\mu]} + \psi_{[\rho\mu]}\partial_\mu\psi_\rho + \kappa\left(\frac{1}{\xi}\psi_0^2 + \frac{1}{2}\psi_{[\rho\mu]}^2\right). \tag{12}$$

One may verify that Lagrangian (12) vanishes for fields obeying equations of motion (5) and within four-divergence, which does not influence on the equations of motion, can be represented as

$$\mathcal{L} = -\frac{1}{2\kappa}\left(\partial_\mu\psi_\nu - \partial_\nu\psi_\mu\right)^2 - \frac{\xi}{\kappa}\left(\partial_\mu\psi_\mu\right)^2. \tag{13}$$

As κ has the dimension of the mass, we need to renormalize the fields, and under the replacement $\psi_\mu \to (\sqrt{\kappa}/\sqrt{2})A_\mu$, Lagrangian (13) coincides with (3). Of course, one could define fields ψ_A according to this replacement from the very beginning. It was pointed on the importance of normalization in [9].

3. Energy-momentum tensors and dilatation currents

Now, we investigate the scale invariance in the model with arbitrary gauge parameter ξ. For this purpose, one needs to obtain energy-momentum tensors and dilatation currents. The conserved canonical energy-momentum tensor (see [2]) is

$$\begin{aligned} T^c_{\mu\nu} &= \left(\partial_\nu \overline{\Psi}(x)\right) \alpha_\mu \Psi(x) \\ &= \psi_0 \partial_\nu \psi_\mu - \psi_\mu \partial_\nu \psi_0 - \psi_\rho \partial_\nu \psi_{[\rho\mu]} + \psi_{[\rho\mu]} \partial_\nu \psi_\rho, \end{aligned} \tag{14}$$

so that $\partial_\mu T^c_{\mu\nu} = 0$. We took here into account that Lagrangian (11) vanishes on the solutions of equations of motion. The canonical dilatation current [12] becomes

$$D^c_\mu = x_\alpha T^c_{\mu\alpha}, \tag{15}$$

with its non-zero four-divergence

$$\partial_\mu D^c_\mu = T^c_{\mu\mu} = \kappa \overline{\Psi} \left(\frac{1}{\xi} P_s + P_t\right) \Psi = -\kappa \left(\frac{1}{\xi} \psi_0^2 + \frac{1}{2} \psi^2_{[\mu\nu]}\right). \tag{16}$$

The appearance of the gauge parameter ξ here is due to using equations of motion (5). The conserved symmetric Belinfante energy-momentum tensor is given by [2]

$$\begin{aligned} T^B_{\mu\alpha} &= 2\kappa \psi_{[\lambda\mu]} \psi_{[\alpha\lambda]} - 2\psi_\mu \partial_\alpha \psi_0 - 2\psi_\alpha \partial_\mu \psi_0 \\ &+ \delta_{\alpha\mu} \partial_\beta \left(\psi_0 \psi_\beta\right) - \delta_{\alpha\mu} \partial_\beta \left(\psi_\lambda \psi_{[\lambda\beta]}\right). \end{aligned} \tag{17}$$

We find non-zero trace of the symmetric Belinfante energy-momentum tensor (17):

$$T^B_{\mu\mu} = 4\partial_\mu \left(\psi_0 \psi_\mu\right). \tag{18}$$

A modified dilatation current [2] is given as follows:

$$D^B_\mu = x_\alpha T^B_{\mu\alpha} + \psi_\lambda \psi_{[\lambda\mu]} - 3\psi_0 \psi_\mu. \tag{19}$$

One may verify that the divergence of the Belinfante dilatation current (19) coincides with the divergence of the canonical dilatation current (16), $\partial_\mu D^B_\mu = \partial_\mu D^c_\mu \neq 0$. As the trace of the Belinfante dilatation current (18) is a four-divergence, we may define new conserved dilatation current

$$D_\mu = x_\alpha T^B_{\mu\alpha} - 4\psi_0 \psi_\mu, \tag{20}$$

and $\partial_\mu D_\mu = 0$. Thus, new dilatation current (20) is conserved, and strictly speaking, the dilatation symmetry is not broken and the model possesses the scale invariance. We have change the conclusion made in [2] about the scale invariance because of obtaining new conserved dilatation current (20). Relations found in this section are the generalization of formulas obtained in [2] on the case of arbitrary gauge ξ.

4. Schrödinger form of equations

It should be noted that in some cases the Schrödinger equation has advantages for the investigation of interacting field problems. To obtain the Schrödinger form of equations and quantum mechanical Hamiltonian, we have to exclude the non-dynamical components from Eq.(5). For this purpose, Eq.(5) can be represented as follows:

$$\begin{gathered}
\kappa\psi_{[4m]} = \partial_4\psi_m - \partial_m\psi_4, \qquad \partial_4\psi_{[m4]} + \partial_n\psi_{[mn]} + \partial_m\psi_0 = 0, \\
\psi_{[mn]} = \frac{1}{\kappa}\left(\partial_m\psi_n - \partial_n\psi_m\right), \\
\partial_n\psi_{[4n]} + \partial_4\psi_0 = 0, \qquad \partial_4\psi_4 + \partial_m\psi_m + \frac{\kappa}{\xi}\psi_0 = 0.
\end{gathered} \tag{21}$$

Third equation in (21) possesses only spatial derivatives and, therefore, $\psi_{[mn]}$ are non-dynamical (auxiliary) components. Excluding $\psi_{[mn]}$ from Eq.(21), we arrive at the system of equations containing only dynamical components

$$\begin{gathered}
i\partial_t\psi_0 = \partial_n\psi_{[4n]}, \qquad i\partial_t\psi_m = -\kappa\psi_{[4m]} - \partial_n\psi_4, \\
i\partial_t\psi_4 = \partial_m\psi_m + \frac{\kappa}{\xi}\psi_0, \qquad i\partial_t\psi_{[m4]} = \frac{1}{\kappa}\left(\partial_m\partial_n\psi_n - \partial_n^2\psi_m\right) + \partial_m\psi_0.
\end{gathered} \tag{22}$$

Let us introduce the 8-component wave function

$$\Phi(x) = \begin{pmatrix} \psi_0(x) \\ \psi_\mu(x) \\ \psi_{[m4]}(x) \end{pmatrix}. \tag{23}$$

Exploring the elements of the matrix algebra, Eq.(22) can be represented as follows:

$$\begin{gathered}
i\partial_t\Phi(x) = \Bigg[\kappa\left(\frac{1}{\xi}\varepsilon^{4,0} + \varepsilon^{m,[m4]}\right) + \left(\varepsilon^{0,[4m]} - \varepsilon^{[4m],0} + \varepsilon^{4,m} - \varepsilon^{m,4}\right)\partial_m \\
+\frac{1}{\kappa}\left(\varepsilon^{[m4],n}\partial_n\partial_m - \varepsilon^{[m4],m}\partial_n^2\right)\Bigg]\Phi(x).
\end{gathered} \tag{24}$$

Then Eq.(24) takes the Schrödinger form:

$$i\partial_t\Phi(x) = \mathcal{H}\Phi(x), \tag{25}$$

where the Hamiltonian is given by

$$\begin{gathered}
\mathcal{H} = \kappa\left(\frac{1}{\xi}\varepsilon^{4,0} + \varepsilon^{m,[m4]}\right) + \left(\varepsilon^{0,[4m]} - \varepsilon^{[4m],0} + \varepsilon^{4,m} - \varepsilon^{m,4}\right)\partial_m \\
+\frac{1}{\kappa}\left(\varepsilon^{[m4],n}\partial_n\partial_m - \varepsilon^{[m4],m}\partial_n^2\right).
\end{gathered} \tag{26}$$

In the momentum space the Hamiltonian becomes:

$$\mathcal{H} = \kappa\left(\frac{1}{\xi}\varepsilon^{4,0} + \varepsilon^{m,[m4]}\right) + ik_m\left(\varepsilon^{0,[4m]} - \varepsilon^{[4m],0} + \varepsilon^{4,m} - \varepsilon^{m,4}\right)$$
$$+\frac{1}{\kappa}k_m k_n\left(\varepsilon^{[k4],k}\delta_{mn} - \varepsilon^{[m4],n}\right). \tag{27}$$

The 8-component wave function (23) describes fields with four spin states with positive and negative energies, and there are only dynamical components in the wave function. The matrix Hamiltonian (27) obeys the minimal equation as follows:

$$\mathcal{H}^2\left(\mathcal{H}^2 - \mathbf{k}^2\right)^2\left(\mathcal{H}^2 - 2\mathbf{k}^2\right) = 0. \tag{28}$$

There are three eigenvalues of the Hamiltonian: 0, $\mathbf{k}^2$, $2\mathbf{k}^2$. The physical eigenvalue is $\mathbf{k}^2$ so that $k_0^2 = \mathbf{k}^2$. In Appendix A, one can find the Hamiltonian (26) expressed through the matrices (7) of equation (6).

5. Interaction with gravitation

5.1 Equations

Now, we consider the non-minimal interaction of the massless vector field with gravity. Some aspects of interactions of vector fields with gravity where investigated in [13-21]. Let us consider the novel action

$$S = \int d^4x\sqrt{-g}\left[-\frac{R}{16\pi G} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\xi\left(\nabla_\nu A^\nu\right)^2 + \lambda R\nabla_\nu A^\nu\right], \tag{29}$$

where G is the gravitational (Newton) constant, λ is a coupling constant, and ∇_μ are covariant derivatives. In Eq.(29), we have introduced the coupling of a scalar curvature with the vector field A_μ. Within the four-divergence, which does not change equations of motion, the non-minimal interaction term in action (29) also can be represented as

$$S_{int} = -\lambda\int d^4x\sqrt{-g}A^\nu\frac{\partial R}{\partial x^\nu}.$$

Such term can be added for any vector-tensor theory of gravity but it will lead to the higher derivative model. In [22] other couplings to gravity were investigated. Let us consider FRW space-time with the flat spatial part with the metric

$$g_{00} = g^{00} = 1, \quad g_{11} = g_{22} = g_{33} = -a(t)^2, \quad g^{11} = g^{22} = g^{33} = -\frac{1}{a(t)^2}, \tag{30}$$

where $a(t)$ is a scale factor. Nonzero components of the Christoffel symbols and curvatures are given by

$$\Gamma^1_{10} = \Gamma^1_{20} = \Gamma^3_{30} = \frac{\dot{a}}{a} = H, \qquad \Gamma^0_{11} = \Gamma^0_{22} = \Gamma^0_{33} = \dot{a}a = a^2H,$$

$$R_{00} = -3\frac{\ddot{a}}{a}, \quad R_{ik} = \delta_{ik}\left(a\ddot{a} + 2\dot{a}^2\right), \quad R = -6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right], \tag{31}$$

where $H = \dot{a}(t)/a(t)$ is the Hubble parameter; $R_{\mu\nu}$, R are the Ricci and scalar curvatures, respectively. Varying action (29) with respect to vector field A_μ yields the equations of motion as follows:

$$\nabla_\nu F^{\mu\nu} - \xi\nabla^\mu\nabla_\nu A^\nu + \lambda\nabla^\mu R = 0. \tag{32}$$

For homogeneous electromagnetic fields ($\partial_i A_\mu = 0$) the $\mu = 0$ component of Eq.(32) in FRW background gives

$$\nabla_\nu F^{0\nu} - \xi\nabla^0\nabla_\nu A^\nu + \lambda\nabla^0 R = -\xi\left(\ddot{A}^0 + 3\dot{H}A^0 + 3H\dot{A}^0\right) + \lambda\dot{R} = 0. \tag{33}$$

Taking into account Eq.(31), one obtains from Eq.(33) the equation for A_0:

$$\left[\ddot{A}_0 + \partial_t\left(3HA_0 - \frac{\lambda}{\xi}R\right)\right] = 0. \tag{34}$$

The equation for spatial components are given by

$$\ddot{A}_m + H\dot{A}_m = 0. \tag{35}$$

Variation of action (29) with respect of the metric leads to generalized Einstein's equation.

5.2 Solutions

Let us obtain solutions to novel equation (34). Integrating Eq.(34), one finds

$$\dot{A}_0 + 3HA_0 - \frac{\lambda}{\xi}R = C_1, \tag{36}$$

where C_1 is the integration constant. Eq.(36) with the help of Eq.(31) becomes the first-order non-homogeneous differential equation:

$$\dot{A}_0 + 3\frac{\dot{a}}{a}A_0 = C_1 - \frac{6\lambda}{\xi}\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right]. \tag{37}$$

We can take $H = p/t$, with $p = 1/2$ for radiation and $p = 2/3$ for matter eras respectively. In this case, we obtain the solution to Eq.(37):

$$A_0 = C_1 t + C_2 t^{-3p} - \frac{6\lambda p(2p-1)}{\xi(3p-1)}t^{-1}. \tag{38}$$

The last term in Eq.(38) is due to non-minimal interaction introduced in Eq.(29). The first term of A_0-component grows with the cosmic time for any p [6], and the last term decays. The A_i-component grows but the temporal component (A_0) dominates [6]. As was noted in [6], the term $\nabla_\mu A^\mu$ plays the role of a cosmological constant during the evolution of the universe.

6. Conclusion

The generalized Maxwell equations with arbitrary gauge parameter are formulated in the first-order formalism. The gauge U(1)-symmetry of a model is broken. As a result, the scalar state of the field presents in the spectrum of the theory. If one introduces the four-current J_μ in the right side of Eq.(4), then due to the conservation of the current, $\partial_\mu J_\mu = 0$, the equation $\partial_\mu^2 \partial_\nu A_\nu = 0$ holds. It means that the scalar state of the field A_μ does not interact with charges and currents. But this scalar state can interact with gravity via the coupling (29), we have introduced.

As the matrices of the RWE obey the generalized DKP algebra, one can apply covariant methods for finding solutions for definite spin (one and zero), spin projections and energy-momentum [1], [2]. Although the canonical and Belinfante dilatation currents, found within the first-order formalism, are not conserved, we have obtained the conserved modified dilatation current. This demonstrates the scale invariance of the massless fields theory. The Schrödinger form of equations obtained possesses some advantages because it contains only dynamical components of fields. The found quantum-mechanical Hamiltonian can be used for investigation of problems with interacting fields. RWE, as well as Hamiltonian, are simplified for the choice $\xi = 1$ which was used in [1], [2]. In addition, the commutation relations (C5) take the canonical normalized form at $\xi = 1$. In our opinion the value of the parameter $\xi = 1$ is natural. The consistency of the model will be studied in subsequent papers.

We suggest a novel non-minimal interaction of fields in the FRW background. The solution of equations for the time component of the four-potential grows in time in the same manner as in [6]. Therefore, the model considered has the similar behavior. Although there are difficulties with the unbounded Hamiltonian and indefinite metrics (see Appendix C), the model has attractive features. We leave the detailed analysis of the model based on the action (29) for further investigations.

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Appendix A

Let us obtain the Schrödinger equation and quantum-mechanical Hamiltonian from relativistic wave equation (6). One can find from (6) the equation as follows:

$$i\alpha_4 \partial_t \Psi(x) = \left[\alpha_a \partial_a + \kappa\left(\frac{1}{\xi} P_s + P_t\right)\right]\Psi(x). \qquad (A1)$$

Taking into account the relation $\alpha_4\left(\alpha_4^2 - 1\right) = 0$, which follows from algebra (8), one can introduce the projection operators:

$$\Lambda \equiv \alpha_4^2 = \varepsilon^{0,0} + \varepsilon^{\mu,\mu} + \varepsilon^{[m4],[m4]}, \qquad \Pi \equiv 1 - \alpha_4^2 = \frac{1}{2}\varepsilon^{[mn],[mn]}, \qquad (A2)$$

with the properties $\Lambda = \Lambda^2$, $\Pi^2 = \Pi$, $\Lambda\Pi = \Pi\Lambda = 0$, $\Lambda + \Pi = I_{11}$ (I_{11} is unit 11×11-matrix). Operator Λ extracts 8-dimensional subspace of dynamical components and operator Π acts in 3-dimensional subspace of non-dynamical components of the wave function Ψ. Thus, we introduce dynamical, $\phi(x)$, and non-dynamical, $\chi(x)$, functions:

$$\phi(x) = \Lambda\Psi(x), \qquad \chi(x) = \Pi\Psi(x). \tag{A3}$$

After multiplying Eq.(A1) by the matrices α_4 and Π, one finds equations

$$i\partial_t\phi(x) = \alpha_4 \left[\alpha_a\partial_a + \kappa\left(\frac{1}{\xi}P_s + P_t\right)\right](\phi(x) + \chi(x)), \tag{A4}$$

$$0 = \left(\alpha_4^2 - 1\right)\left[\alpha_a\partial_a + \kappa\left(\frac{1}{\xi}P_s + P_t\right)\right](\phi(x) + \chi(x)). \tag{A5}$$

In these equations we imply that the direct sum of functions $\phi(x)$ and $\chi(x)$ is Ψ ($\phi(x) + \chi(x) = \Psi$). One can verify equations

$$\left(\frac{1}{\xi}P_s + P_t\right)\Pi = \Pi\left(\frac{1}{\xi}P_s + P_t\right) = \Pi, \qquad \Pi\alpha_a\Pi = 0,$$

and obtain from Eq.(A5) the function $\chi(x)$:

$$\chi(x) = -\frac{1}{\kappa}\Pi\alpha_a\partial_a\phi(x). \tag{A6}$$

Excluding the $\chi(x)$ from Eq.(A4) with the help of (A6) and using the relation $\alpha_4\Pi = 0$, we find the Schrödinger equation $i\partial_t\phi(x) = \mathcal{H}\phi(x)$, with the Hamiltonian

$$\mathcal{H} = \alpha_4\left[\alpha_a\partial_a + \kappa\left(\frac{1}{\xi}P_s + P_t\right)\right] - \frac{1}{\kappa}\alpha_4\alpha_a\Pi\alpha_b\partial_a\partial_b. \tag{A7}$$

Although the matrices α_μ are 11×11-matrices, the Hamiltonian (A7) acts in 8-dimension subspace. One can check that Hamiltonian (A7) coincides with Hamiltonian (26). Thus, we have obtained here the Hamiltonian in terms of matrices of relativistic wave equation (7). It should be noted that Eq.(A6) is equivalent to Eq.(21) for non-dynamical components, ψ_{mn}. The Schrödinger equation with Hamiltonian (A7) does not contain non-dynamical components and can be used for solving some problems of interacting fields.

Appendix B

Now, we obtain canonical and symmetrical energy-momentum tensors and dilatation currents starting with the second order formulation based on the Lagrangian (3). The canonical energy-momentum tensor found from equation

$$\Theta^c_{\mu\alpha} = \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu A_\beta\right)}\partial_\alpha A_\beta - \delta_{\mu\alpha}\mathcal{L}$$

is given by

$$\Theta^c_{\mu\nu} = -F_{\mu\beta}\partial_\nu A_\beta - \xi\left(\partial_\nu A_\mu\right)\left(\partial_\alpha A_\alpha\right) + \delta_{\mu\nu}\left[\frac{1}{4}F^2_{\rho\sigma} + \frac{\xi}{2}\left(\partial_\alpha A_\alpha\right)^2\right], \qquad (B1)$$

and is conserved: $\partial_\mu \Theta^c_{\mu\nu} = 0$. The trace of the energy-momentum tensor is non-zero and reads

$$\Theta^c_{\mu\mu} = \frac{1}{2}F^2_{\rho\sigma} + \xi\left(\partial_\alpha A_\alpha\right)^2. \qquad (B2)$$

The dilatation current is given as follows [12]:

$$\begin{gathered} D^c_\mu = x_\alpha \Theta^c_{\mu\alpha} + \Pi_{\mu\alpha}A_\alpha, \qquad \Pi_{\mu\alpha} = \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu A_\alpha\right)} = -F_{\mu\alpha} - \xi\delta_{\mu\alpha}\left(\partial_\nu A_\nu\right), \\ D^c_\mu = x_\alpha\Theta^c_{\mu\alpha} - F_{\mu\alpha}A_\alpha - \xi A_\mu\left(\partial_\alpha A_\alpha\right). \end{gathered} \qquad (B3)$$

One can check with the help of equations of motion (4) that the dilatation current (B3) is conserved, $\partial_\mu D^c_\mu = 0$. Thus, the the scale invariance is valid.

The canonical energy-momentum tensor (B1) is not symmetrical. To obtain the symmetrical Belinfante tensor, we use the formulas [12]:

$$\begin{gathered} \Theta^B_{\mu\alpha} = \Theta^c_{\mu\alpha} + \partial_\beta X_{\beta\mu\alpha}, \\ X_{\beta\mu\nu} = \frac{1}{2}\left[\Pi_{\beta\alpha}\left(\Sigma_{\mu\nu}\right)_{\alpha\sigma}A_\sigma - \Pi_{\mu\alpha}\left(\Sigma_{\beta\nu}\right)_{\alpha\sigma}A_\sigma - \Pi_{\nu\alpha}\left(\Sigma_{\beta\mu}\right)_{\alpha\sigma}A_\sigma\right], \end{gathered} \qquad (B4)$$

where the matrix elements of the generators of the Lorentz group $\Sigma_{\mu\nu}$, in Euclidian space-time, are given by

$$\left(\Sigma_{\mu\nu}\right)_{\alpha\sigma} = \delta_{\mu\alpha}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\alpha}. \qquad (B5)$$

From Eq.(B4), with the help of Eq.(B5), we obtain

$$X_{\beta\mu\nu} = F_{\mu\beta}A_\nu + \xi\left(\partial_\alpha A_\alpha\right)\left(\delta_{\beta\nu}A_\mu - \delta_{\mu\nu}A_\beta\right), \qquad (B6)$$

so that $X_{\beta\mu\nu}$ is antisymmetric in indexes β,μ, and $\partial_\beta\partial_\mu X_{\beta\mu\nu} = 0$. The symmetric and conserved Belinfante energy-momentum tensor, using Eq.(B4),(B6), becomes

$$\begin{gathered} \Theta^B_{\mu\nu} = -F_{\mu\beta}F_{\nu\beta} + \xi\left(A_\nu\partial_\mu + A_\mu\partial_\nu\right)\left(\partial_\alpha A_\alpha\right) \\ +\delta_{\mu\nu}\left[\frac{1}{4}F^2_{\rho\sigma} - \frac{\xi}{2}\left(\partial_\alpha A_\alpha\right)^2 - \xi A_\beta\partial_\beta\left(\partial_\alpha A_\alpha\right)\right]. \end{gathered} \qquad (B7)$$

At $\xi = 0$ Eq.(B7) converts into energy-momentum tensor of electrodynamics. Energy-momentum tensor similar to (B7) was found in [23] by varying action on the metric tensor. A modified dilatation current is given by [12]

$$D^B_\mu = x_\alpha\Theta^B_{\mu\alpha} + V_\mu, \qquad (B8)$$

where the field-virial V_μ, in our case, becomes

$$V_\mu = \Pi_{\mu\alpha} A_\alpha - \Pi_{\alpha\beta} \left(\Sigma_{\alpha\mu}\right)_{\beta\sigma} A_\sigma = 2\xi A_\mu \left(\partial_\alpha A_\alpha\right). \qquad (B9)$$

One can obtain the trace of the Belinfante tensor (B7):

$$\Theta^B_{\mu\mu} = -2\xi \partial_\mu \left[A_\mu \left(\partial_\alpha A_\alpha\right)\right]. \qquad (B10)$$

From Eq.(B8),(B10), we obtain

$$\partial_\mu D^B_\mu = \Theta^B_{\mu\mu} + \partial_\mu V_\mu = 0,$$

and the modified dilatation current is conserved. Thus, the scale invariance takes place with the conserved currents (B3) and (B8).

Appendix C

Let us consider the quantization of fields for given Lagrangian (3) (see also [24]). Conjugated momenta for generalized "coordinates" $A_\mu(x)$ are given by

$$\pi_0(x) = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = -\xi \partial_\mu A_\mu, \qquad \pi_m(x) = \frac{\partial \mathcal{L}}{\partial \dot{A}_m} = \dot{A}_m + \partial_m A_0. \qquad (C1)$$

Then the density of the Hamiltonian becomes

$$\begin{aligned} H = \pi_m \dot{A}_m - \pi_0 \dot{A}_0 - \mathcal{L} = \dot{A}_m \left(\dot{A}_m + \partial_m A_0\right) \\ + \frac{1}{4} F^2_{\mu\nu} - \xi \dot{A}_0 \partial_\mu A_\mu + \frac{1}{2} \xi \left(\partial_\nu A_\nu\right)^2. \end{aligned} \qquad (C2)$$

One can verify that the equality $H = \Theta^c_{44}$ holds where the canonical energy-momentum tensor $\Theta^c_{\mu\nu}$ is given by (B1). It should be noted that the classical energy $\mathcal{E} = \int H d^3x$ is not bounded from below and the system is unstable. Therefore, we need to introduce indefinite metrics for quantization. With the help of standard commutation relations for canonical variables $[A_\mu(\mathbf{x}, t), \pi_\nu(\mathbf{y}, t)] = i\delta_{\mu\nu}\delta\left(\mathbf{x} - \mathbf{y}\right)$, one obtains

$$\begin{aligned} \left[A_n(\mathbf{x}, t), \dot{A}_m(\mathbf{y}, t) + \partial_m A_0(\mathbf{y}, t)\right] = i\delta_{mn}\delta\left(\mathbf{x} - \mathbf{y}\right), \\ \left[A_0(\mathbf{x}, t), \partial_\mu A_\mu(\mathbf{y}, t)\right] = -\frac{i}{\xi}\delta\left(\mathbf{x} - \mathbf{y}\right). \end{aligned} \qquad (C3)$$

In the momentum space the real fields A_μ read

$$A_\mu(x) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2Vk_0}} \left[a_\mu(\mathbf{k}) e^{ik_\mu x_\mu} + a^+_\mu(\mathbf{k}) e^{-ik_\mu x_\mu}\right], \qquad (C4)$$

where $k^2_\mu = \mathbf{k}^2 - k^2_0 = 0$, V is the normalization volume. It should be noted that the field A_μ possesses four independent components: two components are transverse, one component is longitudinal, and one component corresponds to the scalar polarization.

In [1], [2], we found four independent solutions to equations of motion in the form of matrix-dyads for the gauge $\xi = 1$. The fields (C4) satisfy commutators (C3) if creation and annihilation operators obey the commutation relations as follows:

$$\left[a_m(\mathbf{k}), a_n^+(\bar{\mathbf{k}})\right] = \delta_{mn}\delta\left(\mathbf{k} - \bar{\mathbf{k}}\right), \qquad \left[a_0(\mathbf{k}), a_0^+(\bar{\mathbf{k}})\right] = -\frac{1}{\xi}\delta\left(\mathbf{k} - \bar{\mathbf{k}}\right). \qquad (C5)$$

For the Feynman gauge $\xi = 1$ (which was used in [2]) the RWE (6), (10) (and (A7)) are simplified. In this case the "wrong" sign $(-)$ in the commutator for $a_0(\mathbf{k})$, $a_0^+(\bar{\mathbf{k}})$ in (C5) indicates on the necessity of introducing the indefinite metrics. For any gauge ξ there are difficulties with the presence of the ghosts if one considers four polarizations of the field A_μ to be physical. In QED the photon fields possess only two polarizations and physical values do not depend on the gauge ξ due to the restriction on the physical Hilbert space [24].

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Degeneracy Patterns in Baryon Spectra from Space-Time Symmetries

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Abstract: We review recent work on the degeneracy phenomenon in the light flavors hadron sector of QCD. Specifically, attention is drawn to the fact that the nucleon and the Δ spectra carry each quantum numbers characteristic for an unitary representation of the conformal group. We show that the above phenomenon is well explained for baryons whose internal structure is dominated by a quark-diquark configuration that resides in a conformally compactified Minkowski space time, $\mathbf{R}^1 \otimes S^3$, and is described by means of the conformal scale equation there. The $\mathbf{R}^1 \otimes S^3$ space-time represents the boundary of the conformally compactified AdS_5, on which one expects to encounter a conformal theory in accord with the gauge-gravity duality. Within this context, our model is congruent with $\mathrm{AdS}_5/\mathrm{CFT}_4$. It has furthermore the advantage to allow for a Fourier transformation to momentum space.

Keywords: degeneracy phenomenon; Baryon Spectra; Space-Time Symmetries ; Quantum Chromodynamics
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1. The Riddle of the Baryon Spectra

Understanding the baryon spectra is still a challenge despite the long history of the related studies [1], [2]. The great riddle concerns the number of resonances with masses below ∼2500 MeV. The traditional quark model based upon the full Hilbert space of three quark degrees of freedom and the non-relativistic $\mathrm{SU}(6)_{SF} \otimes O(3)_L$ classification scheme, predicts far more states than have been experimentally confirmed so far [3]. The respec-

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tive resonance deficit, termed to as "missing" states, is still awaiting for explanation. Quark-diquark (q–(qq)) models [4] based on a diquark with limited angular momentum values, carry reduced spin-flavor degrees of freedom, and are obvious candidates for providing a lesser number of "missing" states, an option taken into consideration by several authors. Additional restrictions on the quantum numbers of the q-(qq) excitations can come from imposing on the spatial wave-functions a symmetry higher than $\mathrm{O}(3)_L$. Natural candidates are the exact Lorentz and the approximate conformal symmetries, the latter having relevance for the lightest flavors. It is the goal of the present work to examine consequences of these symmetries for the systematic of the N and Δ spectra and construct corresponding radial wave-functions of a q–(qq) system. We implement conformal symmetry into a quark Hamiltonian describing quarks in a position space of a finite volume [5]. This is achieved in placing the q-(qq) system directly on the AdS_5 boundary, which is the AdS_5 cone, considered as conformally compactified to $S^1 \otimes S^3$ [6], or, $\mathbf{R}^1 \otimes S^3$, at a microscopic scale [7]. According to [7], correlation functions of CFT_4 on regular Minkowski spacetime, $\mathcal{M}$=$\mathbf{R}^{1+3}$, can be analytically continued to the full Einstein universe because $\mathbf{R}^{1+3}$ can be conformally mapped on $\mathbf{R}^1 \otimes S^3$. The implication of this important observation is that each CFT_4 state on $\mathbf{R}^1 \otimes S^3$ can be brought into unique correspondence with a state of the brane theory on $AdS_5 \otimes S^5$. Consequences on thermal states have been worked out in ref. [8]. We here examine consequences for the systematics of the N and Δ spectra.
The paper is organized as follows. In the next section we briefly review the space time symmetries of QCD and relate them to the observed degeneracy phenomenon in the N and Δ. In section 3 we present the conformal equation of ref. [9] and apply it to a quark-diquark (q-qq) model of light baryons. There, we also review calculations of the N and Δ spectra and the proton electric charge form-factor. Section 4 contains the momentum space formulation of the model. The paper closes with concise conclusions.

2. The Degeneracy Phenomenon in N and Δ Spectra

The exact fundamental internal and external (space-time) symmetries of the QCD Lagrangian are the color-gauge, and the global relativistic Lorentz symmetries, respectively. They are mathematically expressed in terms of the invariance of the Lagrangian under local $SU(3)_c$, and global $SO(1,3)$ group transformations. In addition, in the light flavors sector, one considers some approximate internal and external symmetries, the most important being the three flavor $SU(3)_F$, and the conformal $SO(2,4)$ symmetries. Relativity requires fermionic high-spin states to emerge as part of Lorentz group representations, not necessarily single spin valued. The most important representations are described by means of totally symmetric Lorentz tensors of rank K with Dirac spinor components and are generated in colliding bosonic projectiles with fermionic targets. Consider for example the excitation of a proton target, a Dirac spinor Ψ, by a photon beam, in the four-vector A_μ, equivalently, $(1/2, 1/2)$ representation. In effect of this (γ, p) collision, the two incoming states described by the above two Lorentz representations merge to their direct product, $A_\mu \otimes \Psi$, which gives rise to a four-vector spinor,

$$A_\mu \otimes \Psi = \Psi_\mu . \tag{1}$$

The spin- and parity content of Ψ_μ is well known and given by,

$$\frac{1}{2}^+ \oplus \left(\frac{1}{2}^-, \frac{3}{2}^-\right) \in \Psi_\mu . \tag{2}$$

The Ψ_μ representation is reducible and its irreducible parts in standard notations [10] read,

$$\left(\frac{1}{2}^-, \frac{3}{2}^-\right) \in \{1, \frac{1}{2}\} \oplus \{\frac{1}{2}, 1\}, \quad \frac{1}{2}^+ \in \{\frac{1}{2}, 0\} \oplus \{0, \frac{1}{2}\}. \tag{3}$$

Were the beam to be in the single-spin valued $(1,0) \oplus (0,1)$ representation, as is the electromagnetic field strength tensor, $F_{\mu\nu}$, the final state would be the totally anti-symmetric Lorentz tensor of rank 2 with Dirac spinor components, $\Psi_{[\mu\nu]}$, [11] which is reducible too,

$$F_{\mu\nu} \otimes \Psi = \Psi_{[\mu\nu]} = \Psi_\eta \oplus \{\frac{3}{2}, 0\} \oplus \{0, \frac{3}{2}\}. \tag{4}$$

The spin-parity content of $\Psi_{[\mu\nu]}$ is,

$$\frac{1}{2}^+ \oplus \left(\frac{1}{2}^-, \frac{3}{2}^-\right) \oplus \frac{3}{2}^+ \in \Psi_{[\mu\nu]} . \tag{5}$$

The example shows that $\Psi_{[\mu\nu]}$ can serve as a vehicle for the single-spin valued $\{\frac{3}{2}, 0\} \oplus \{0, \frac{3}{2}\}$, provided, one would be able to project out Ψ_η covariantly. Following this logic, it is not difficult to show that a spin-$\left(K + \frac{1}{2}\right)^P$ field (with K=0,1,2,3,...) of parity $P = \pm$, naturally emerges as the maximal spin in $\Psi_{\mu_1 \ldots \mu_K}$,

$$J^P_{\mathrm{max}} = \left[K + \frac{1}{2}\right]^P \in \Psi_{\mu_1 \ldots \mu_K} . \tag{6}$$

The spin and parity content of $\Psi_{\mu_1 \ldots \mu_K}$ is,

$$\left(\frac{1}{2}^\pm, \frac{3}{2}^\pm, \left[K - \frac{1}{2}\right]^\pm, \left[K + \frac{1}{2}\right]^P\right) \in \Psi_{\mu_1 \ldots \mu_K}, \tag{7}$$

with the sign of $P = (-1)^K$, or $P = (-1)^{K+1}$, for $\Psi_{\mu_1 \ldots \mu_K}$ being a tensor, or pseudo-tensor, respectively. Take as an example $K = 3$. The pseudo-tensor of rank-3 is reducible as,

$$\begin{aligned} \Psi_{\mu_1\mu_2\mu_3} &= \left[\{1, \frac{3}{2}\} \oplus \{\frac{3}{2}, 1\}\right] \oplus \left[\{\frac{3}{2}, 2\} \oplus \{2, \frac{3}{2}\}\right] \\ &\longrightarrow \left(\frac{1}{2}^+, \frac{3}{2}^-, \frac{5}{2}^+\right) \oplus \left(\frac{1}{2}^-, \frac{3}{2}^+, \frac{5}{2}^-, \frac{7}{2}^+\right). \end{aligned} \tag{8}$$

The latter equation implies that the states belonging to the natural parity spin cascade, $\left(\frac{1}{2}^+, \frac{3}{2}^-, \frac{5}{2}^+\right)$, being part of the same irreducible Lorentz representation, will have same mass, say, M_1. Same is valid for the states of the accompanying unnatural parity cascade, $\left(\frac{1}{2}^-, \frac{3}{2}^+, \frac{5}{2}^-, \frac{7}{2}^+\right)$, which will have in general an other mass, say, M_2. In this way, $\Psi_{\mu_1\mu_2\mu_3}$ is a two mass state,

$$M_1: \quad \left(\frac{1}{2}^+, \frac{3}{2}^-, \frac{5}{2}^+\right), \qquad M_2: \quad \left(\frac{1}{2}^-, \frac{3}{2}^+, \frac{5}{2}^-, \frac{7}{2}^+\right). \tag{9}$$

In consequence, relativity of boson-fermion collisions favors higher spin nucleon excitations to emerge in groups of $(2K+1)$, and as part of $\Psi_{\mu_1...\mu_K}$, rather than as single spins. Were higher spins to be fundamental, Lorentz symmetry would require exact degeneracy among the members of anyone of the irreducible pieces of $\Psi_{\mu_1...\mu_K}$. For example, the negative parity states in eq. (2) had to have equal masses, while the positive parity state would be allowed to have a different mass. The $\Psi_{\mu_1...\mu_K}$ tensors are referred to as Rarita-Schwinger fields. As shown above, high spins always appear accompanied by lower-spin states. For example, the lowest fractional spin $\frac{3}{2}^-$ appears accompanied by just one spin $\frac{1}{2}^-$ state, as illustrated by eqs. (2)-(3). It would be natural to describe this group of states as a whole, a path taken by ref. [12]. Instead, the canonical opinion is that one has to find a method to restrict $\{1, \frac{1}{2}\} \oplus \{\frac{1}{2}, 1\}$ to a single spin-$\frac{3}{2}^-$. As long as the representation under consideration is irreducible, finding such methods is by no means an easy task and if not done with care, employing irreducible multi-spin parity representations of the Lorentz group in the description of single spins may result inconsistent. Precisely this happens when the restriction of $\Psi_{\mu_1...\mu_K}$ to a single spin-$\left[K + \frac{1}{2}\right]^P$ is executed along the line of the Rarita-Schwinger prescription [13]. According to the latter, it is conjectured that the highest spin is described by means of a Dirac equation for any Lorentz index, μ_i, supplemented by auxiliary conditions,

$$(\not{p} - m)\Psi_{\mu_1..\mu_i..\mu_K} = 0, \;\; p^{\mu_i}\Psi_{\mu_1..\mu_i..\mu_K} = 0, \quad \gamma^{\mu_i}\Psi_{\mu_1..\mu_i..\mu_K} = 0. \tag{10}$$

It has been known for a long time that this prescription is plagued by several inconsistencies. Specifically, upon electromagnetic gauging, a superluminal propagation can occur (so called Velo-Zwanziger problem). Because of this and other inconsistencies, the requirement of the relativity that higher spins should emerge as $\Psi_{\mu_1...\mu_K}$, i.e. as groups of K pairs of states of opposite parities and one unpaired state of maximal spin (c.f. eq. (7)), has rarely been taken into consideration. In so doing, the important point has been missed that Lorentz symmetry prescribes the type of degeneracy between resonances with rising spins and alternating parities illustrated by eq. (9).

However, in the recent work [14] significant progress in the consistent description of spin-$\frac{3}{2}$ in terms of the four-vector spinor has been achieved, and the venue hit toward a liberation of $\Psi_{\mu_1...\mu_K}$ from the stigma of being plagued by pathologies. The study in ref. [14] executes the idea of ref. [15] that the highest spin in $\Psi_{\mu_1...\mu_K}$ has to be pinned down by a projector consistent with the Poincaré symmetry (i.e. the Lorentz symmetry extended by translations). Such covariant projectors, $\mathcal{P}^{(MJ)}_{\mu\nu}$, are built up from the two

Casimir operators of the Poincaré group, the squared operators of the linear momentum, P^2, and the Pauli-Lubanski vector, $\mathcal{W}^2$, whose eigenvalue upon a state, $|MJ\rangle$ of given mass M and spin J equals, $\mathcal{W}^2|MJ\rangle = -M^2J(J+1)|MJ\rangle$. Specifically for a spin-$\frac{3}{2}$ residing in Ψ_μ such projectors have been designed in [15],

$$\begin{aligned} \mathcal{P}^{(M,3/2)}_{\mu\nu}\psi^{(M,3/2)\,\nu} &= \psi^{(M,3/2)}_\mu \,, \\ \mathcal{P}^{(M,3/2)}_{\mu\nu} &= -\frac{P^2}{M^2}\frac{1}{3}\left[\frac{1}{P^2}\mathcal{W}^2 + \frac{3}{4}1_4\otimes 1_4\right]_{\mu\nu} \,, \end{aligned} \tag{11}$$

and shown to lead to fermion wave equations quadratic in the momenta. The spin-$\frac{3}{2}$ wave equation worked then out in [14] along the line of [15] reads,

$$\begin{aligned} \Big(\left(\pi^2 - m^2\right) g_{\alpha\beta} - i g_{\frac{3}{2}} \left(\frac{\sigma_{\mu\nu}\pi^\mu\pi^\nu}{2} g_{\alpha\beta} - e\, F_{\alpha\beta}\right) + \frac{1}{3}\left(\gamma_\alpha \not\pi - 4\pi_\alpha\right)\pi_\beta \\ + \frac{1}{3}(\pi_\alpha \not\pi - \gamma_\alpha\pi^2)\gamma_\beta\Big)\psi^\beta = 0 \,, \quad g_{\frac{3}{2}} = 2. \end{aligned} \tag{12}$$

It has been proved to be free from the Velo-Zwanziger pathology provided the gyromagnetic factor takes the universal value of $g = 2$ and not as in the Rarita-Schwinger framework, the inverse spin value, $g = 1/J$. In this manner the consistent description of spin-$\frac{3}{2}^-$ in terms of Ψ_μ has been taken under control at least at the classical level. Later on, the method of the covariant Poincaré projectors has been successfully applied also to the spin-1 case in [16]. With that the venue toward a consistent description of high spins has been hit and the confidence in the possible observability of mass degenerate states belonging to a $\Psi_{\mu_1..\mu_K}$ in accord with relativity, regained.

Now it is quite instructive to check whether data on the N and Δ spectra are compatible with the degeneracies predicted in eq. (9). We here focus for the sake of concreteness on the Δ spectrum. We observe that the resonances with masses around 1900 MeV fit perfectly well in the $\Psi_{\mu_1\mu_2\mu_3}$ field from eq. (9). Specifically, the three states from $\{1, \frac{3}{2}\} \oplus \{\frac{3}{2}, 1\}$ are identified with the $\Delta(1910)$, $\Delta(1940)$, and $\Delta(1905)$ resonances of an averaged mass $M_1 = 1918$ MeV. The remaining states belong to $\{\frac{3}{2}, 2\} \oplus \{2, \frac{3}{2}\}$, and they can be identified with the $\Delta(1900)$, $\Delta(1920)$, $\Delta(1930)$, and $\Delta(1950)$ states of an averaged mass $M_2 = 1925$ MeV. Though the degeneracy phenomenon by itself is well pronounced, the mass separation between the two irreducible representations constituting $\Psi_{\mu_1\mu_2\mu_3}$ appears insignificant. Rather, it seems that $\Psi_{\mu_1\mu_2\mu_3}$ as a whole can be characterized by the mass of the underlying $\left(\frac{K}{2}, \frac{K}{2}\right)$ representation around which the physical masses of the states then spread due to relativistic fine-splitting effects. This type of degeneracy can also be interpreted as due to a symmetry group larger than $SO(1,3)$ such as the conformal group, $SO(2,4)$. The latter has the Lorentz group as a subgroup, and an unitary representation which is an infinite tower of levels which carry precisely all the quantum numbers of $\Psi_{\mu_1...\mu_K}$ in their rest frames. Within this context, all the states belonging to same $\Psi_{\mu_1...\mu_K}$ are to leading order mass degenerate modulo the relativistic effect of the fine splitting. Stated differently, according to eq. (7), and modulo the fine-splitting, one expects to observe degeneracy among parity pairs with spins rising from $\frac{1}{2}^\pm$ to $\left[K - \frac{1}{2}\right]^\pm$, and the unpaired state of maximal spin $\left[K+\frac{1}{2}\right]^P$. The values of K have to be determined by comparison to data.

The nucleon and the Δ spectra have been analyzed in terms of $\Psi_{\mu_1..\mu_K}$ already in ref. [17]. There, it has been found that the reported N and Δ spectra can be organized each in three $\Psi_{\mu_1...\mu_K}$ states with K=1,3, and 5, from which a total of solely five resonances is "missing".

In Fig. 1 the Δ levels with $K = 3$ and 5 are depicted for illustrative purposes. On the (J, M) plane, with the states of positive and negative parity being placed below and above the dashed line indicating the averaged mass, the $\Psi_{\mu_1...\mu_K}$'s appear shaped after fern-grass, a reason for which we here coin the notion of a "F E R N E O N" for them. As long as the $\Psi_{\mu_1..\mu_k}$ representations are uniquely identified by their maximal spins, $J_{\rm max}$, defined in eq. (6), the specification "spin-$J_{\rm max}$ ferneons" will be used occasionally. The figure shows that all the Δ resonances reported so far in that region by the Particle Data Group [3] fit pretty well into spin-7/2 and spin-11/2 ferneon levels of a conformal representation. We here challenge the above degeneracy phenomenon and search for reasons for its occurrence. In the next section we shall motivate relevance of conformal symmetry for the spectra of the lightest flavor baryons and develop a quantitative model for its description.

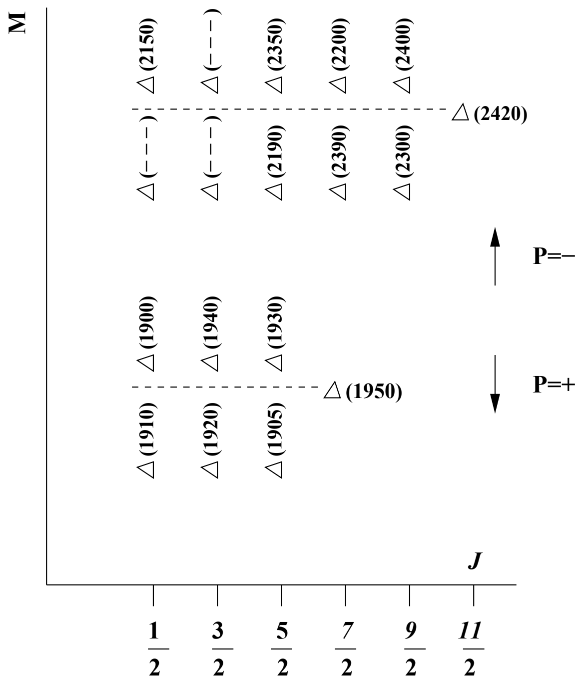

Fig. 1 Schematic presentation of the spin-parity degeneracy phenomenon in the Δ spectrum on the spin (J)-mass (M) plane. The positive parity states have been placed below, the negative parity– above the dashed line marking the averaged mass. The graphical shape of the spectrum designed in this fashion resembles of fern, a reason for which we hereafter shall referrer to $\Psi_{\mu_1..\mu_K}$ as a "*spin-*$\left(K+\frac{1}{2}\right)$ *ferneon*". Resonances marked by $\Delta(---)$ are "missing" from the spin-11/2 ferneon. The spin-7/2 ferneon is complete. The ferneons carry same quantum numbers as the levels of the $SO(2,4)$ unitary infinite tower known from the H atom. In this way one identifies conformal patterns in the Δ spectrum. Same applies to the N spectrum (shown elsewhere [5]).

The conformal symmetry is to a good approximation a symmetry of QCD in the lightest flavor, u and d, sector and in the ultraviolet, i.e. in the regime of the asymptotic freedom where the the u and d parton masses are negligible and the strong coupling constant starts approaching zero, $\alpha_s(q^2) \stackrel{q^2\to\infty}{\longrightarrow} 0$. It is well known that several sum rules for "hard" processes can be derived from conformal symmetry arguments. In the infrared, the $SO(2,4)$ invariance of the QCD Lagrangian would require $\alpha_s(q^2)$ to approach a fixed point, so called "conformal window", $\alpha_s(q^2) \stackrel{q^2\to 0}{\longrightarrow} \alpha_s^*$=const. Such a behavior has been predicted for example in theories with many flavors [18]. However, ref. [19] has also admitted for the possibility of having a conformal window for a number of flavors between 2 and 3, just as is the case in QCD. Recent measurements of the (e,p) scattering performed at the Jefferson Laboratory are strongly indicative of the opening of a conformal window in the infrared [20], and it is justified to expect from the conformal symmetry to leave a footprint in the spectra of the lightest flavor baryons, the nucleon and the Δ, and in support of the ferneon patterns in Fig. 1 advocated here. Moreover, according to the gauge-gravity duality [21], QCD is expected to share several features in common with a conformal string theory residing at the AdS_5 boundary. In the next section we shall present a quark model approach to QCD that captures the essential aspects of a conformal theory on the AdS_5 boundary and explains the degeneracy phenomenon in the light flavor baryon spectra.

3. Finite Volume Conformal Quark Model(FV-CQM)

The goal is to develop a potential quark model that describes the conformal patterns in the nucleon and Δ spectra and at same time captures correctly the aspects of the quark-gluon dynamic in all three regimes. As long as in the perturbative regime the $q-q$ interaction is well described by an inverse distance potential, and the non-perturbative flux-tube regime requires a linear potential, while the regime of the asymptotic freedom is adequately described by means of an infinite well, the potential of interest has to interpolate between a Coulombic potential and the infinite well while passing through a region of a linear growth. The cotangent potential, $(-\cot r)$, is an interaction of the required properties. Indeed, the first terms in its Taylor decomposition, $-\cot r = -1/r + r/3 + ...$ coincide in form with the Cornell potential predicted by Lattice QCD [22]. The higher terms can be viewed as a phenomenological parametrization of non-perturbative corrections [23]. The cotangent potential is better known under the name of the "curved" Coulomb potential on the closed Einsteinian Universe, $\mathbf{R}^1 \otimes S^3$, a notion indicating that the cotangent function of the second polar angle, χ, parameterizing the three dimensional surface of a constant positive curvature, the hypersphere S^3, is a harmonic function there. According to potential theory it can serve as an interaction on S^3, in a way similar as $1/r$ serves as a potential in flat three space, $\mathbf{R}^3$. The above considerations indicate that the most adequate geometry for employing the cotangent interaction is the closed Einstein Universe. The latter emerges naturally within the context of the conformally compactified AdS_5 boundary. Indeed, the AdS_5 boundary, defined by the light cone in

2+4 dimensions,

$$u^2 + v^2 - x^2 - y^2 - z^2 - x_4^2 = 0, \tag{13}$$

can be compactified to $S^1 \otimes S^3$ as,

$$S^1 \otimes S^3: \quad u^2 + v^2 = R^2 = x^2 + y^2 + z^2 + x_4^2, \tag{14}$$

locally equivalent to $\mathbf{R}^1 \otimes S^3$ at a microscopic scale. The latter manifold is a conformally compactified Minkowski space time. According to [7], correlation functions of CFT$_4$ on regular Minkowski spacetime, $\mathcal{M}$=$\mathbf{R}^{1+3}$, can be analytically continued to the full Einstein universe, this because $\mathbf{R}^{1+3}$ can be conformally mapped on $\mathbf{R}^1 \otimes S^3$. The implication of this important observation is that each CFT$_4$ state on $\mathbf{R}^1 \otimes S^3$ can be brought into unique correspondence with a state of the brane theory on $AdS_5 \otimes S^5$. We here wish to explore consequences of this for the spectra of the lightest flavor baryons, the nucleon and the Δ, assumed as two-body quark-diquark systems for the reasons explained in the introduction. We place this system on a compactified Minkowski space time, a manifold of a finite volume, and therefore well suited for confinement description [24]. Finally, we use the scale conformal equation in $\mathbf{R}^1 \otimes S^3$ gauged by the cotangent interaction. The scale conformal equation can be found in [9], [25] and reads:

$$\begin{aligned}
&\frac{\hbar^2}{R^3}\left(\left(\frac{d}{d\tau}\right)^2 - (\mathcal{K}^2+1)\right)R\Phi\left(\frac{x}{R}\right) = 0,\\
&\tau = \ln R, \quad \mathcal{K}^2 = -\frac{1}{\sin^2\chi}\frac{\partial}{\partial\chi}\sin^2\chi\frac{\partial}{\partial\chi} - \frac{L^2}{\sin^2\chi}.
\end{aligned} \tag{15}$$

Here, $\ln R$ behaves as time coordinate, and the scale equation describes free geodesic motion on S^3. Replacing $d/dlnR$ by E, transforms the wave equation for $\Phi(x)$ into the eigenvalue problem of the squared 4D angular momentum, $\mathcal{K}^2$,

$$\hbar^2\mathcal{K}^2\psi_{Klm} = E^2\psi_{Klm}, \;\; E^2 = \hbar^2[K(K+2)+1]. \tag{16}$$

Noticing that $K(K+2) = K(K+D-2)|_{D=4}$, and $1{=}\left(\frac{D-2}{2}\right)^2|_{D=4}$, one realizes that the scale dimension of the field, (D-2)/2, has become "mass", while the " anomalous dimensions", K(K+D-2), have become $\mathcal{K}^2$ eigenvalues. The $SO(2,4)$ generators on $\mathbf{R}^1 \otimes S^3$ have been constructed by [26]. The interaction on S^3 is then concluded from the Green functions obtained in [27] as,

$$\begin{aligned}
\mathcal{K}^2 G(\chi,\chi') &= \delta(\chi-\chi') - \frac{1}{2\pi^2},\\
G(0,\chi) \equiv G_0 &= \frac{1}{4\pi^2}(\pi-\chi)\cot\chi + c',\\
G(\pi,\chi) \equiv G_\pi &= -\frac{1}{4\pi^2}\chi\cot\chi + c''.
\end{aligned} \tag{17}$$

Here, G_π and G_0 refer to sources placed on the respective North, and South poles. They can be equivalently obtained from one-cusp Wilson loops giving rise to cusp anomalous dimensions [28]. As a potential in the quantum-mechanical eigenvalue problem on S^3 one employs the "electric" one-form,

$$V(\chi) = -4\pi A(G_\pi - G_0) = -A\cot\chi, \tag{18}$$

with $\mathcal{K}^2 \cot\chi = 0$, because of its property to make the Dirichlet energy functional on S^3 stationary. Its derivative, $\mathcal{D} = \frac{d}{d\chi}V(\chi) = A\frac{1}{\sin^2\chi}$, gives $A\csc^2\chi$ the meaning of a dipole potential on S^3, while $\cot\chi$ acts as the "curved" Coulomb potential there. We consider a q-(qq) configuration on S^3 bound within a cotangent confinement potential introduced as a gauge interaction, $i\hbar\partial_\tau \longrightarrow i\hbar\partial_\tau - [(-2G\sqrt{\kappa})\cot\chi + \bar{a}]$. The resulting exactly solvable equation is:

$$[-\kappa\hbar^2\frac{\mathrm{d}^2}{\mathrm{d}\chi^2} + \mathcal{V}(\chi)]\Psi(\chi) = ((E-\bar{a})^2 - c_0)\Psi(\chi), \tag{19}$$

$$\mathcal{V}(\chi) = -2G\sqrt{\kappa}(E-\bar{a})\cot\chi + \hbar^2\kappa\alpha(l)(\alpha(l)\ +\ 1)\csc^2\chi, \tag{20}$$

$$c_0 = \mu^2 - \hbar^2\kappa + \bar{a}^2 + (2G\sqrt{\kappa})^2, \quad \mu^2 = \frac{1}{6R^2}, \tag{21}$$

$$\alpha(l) = -\frac{1}{2} + \sqrt{\left(l+\frac{1}{2}\right)^2 - \frac{(2G)^2}{\hbar^2}} = l + \Delta\, l,\ \ \Delta\, l \approx \frac{1}{2}\frac{(2G)^2}{\hbar^2}\frac{(-1)}{l+\frac{1}{2}}, \tag{22}$$

with Δl standing for the kinematic $(P_{2I,1} - S_{2I,1})$ fine splitting. The wave functions are:

$$\Psi_{Kl}(\chi) = N_{Kl}e^{-a\chi}(\sin\chi)^{K+1+\Delta l}R_n^{(a,b)}(\cot\chi),$$

$$a = \frac{2G(E-\bar{a})}{\sqrt{\kappa}\hbar^2(K+1+\Delta l)}, \quad b = -(K+1+\Delta l), \quad K = n+l, \tag{23}$$

Here, $R_n^{(a,b)}(\cot\chi)$ are the non-classical orthogonal polynomials of Routh-Romanovski, rediscovered in [29], and reviewed in [30]. The mass formula is obtained as,

$$(E-\bar{a})^2 = \frac{c_0 + \hbar^2\kappa(K+1)^2}{1+\frac{4G^2}{\hbar^2(K+1)^2}} + 2\Delta l\Bigg(\frac{\hbar^2\kappa(K+1)}{1+\frac{4G^2}{\hbar^2(K+1)^2}} - \frac{4G^2\left(\frac{c_0}{(K+1)^3} + \frac{\hbar^2\kappa}{K+1}\right)}{\hbar^2\left(1+\frac{4G^2}{\hbar^2(K+1)^2}\right)^2}\Bigg). \tag{24}$$

The Δ spectrum of conformal patterns has been generated by a best fit to eq. (24) and presented in Fig. 2 next to the calculation of the proton electric charge form-factor as an illustration of the quality of the wave functions. The partial $F_{35}(1905) - F_{37}(1950) - P_{33}(1920) - P_{31}(1910)$ degeneracy of the positive parity states from the spin-7/2 ferneon has been noticed independently in LF-QCD [31], [32].

4. Formulating FV-CQM in Momentum Space

A further advantage of the finite volume conformal quark model (FV-CQM) is that it allows for a formulation in a momentum space. Upon expressing the second polar angle,

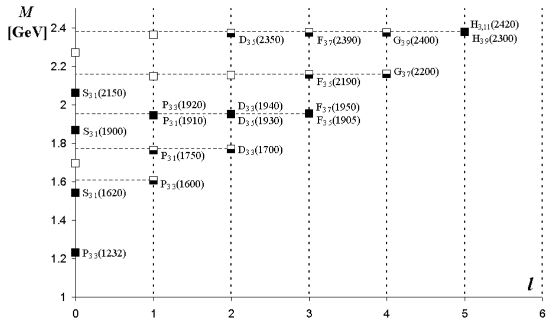

Fig. 2 The Δ spectrum from the Finite Volume Conformal Quark Model (FV-CQM). For the N spectrum and more details see [5].

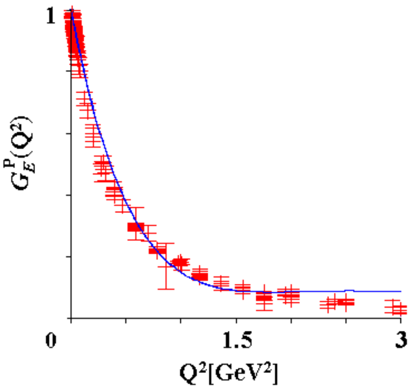

Fig. 3 The proton electric charge form factor from the FV-CQM in comparison to data (for details, see [5]).

χ, parameterizing S^3 in terms of the constant radius, R, of the sphere, and the relative distance, r, from origin of a point from the equatorial disk, $\mathcal{D}_3$, i.e. upon substitution $\chi = \sin^{-1} r/R$, the equations (19)–(22) equivalently rewrite to

$$\Big[-\frac{\hbar^2}{R^2}\frac{\sqrt{1-\frac{r^2}{R^2}}}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\sqrt{1-\frac{r^2}{R^2}}r^2\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\hbar^2}{R^2}\frac{\alpha(l)(\alpha(l)+1)}{r^2} - 2G(E-\bar{a})\frac{\sqrt{1-\frac{r^2}{R^2}}}{r}\Big]\psi\left(\frac{r}{R}\right)$$
$$= \left((E-c_0)^2 - c_0\right)\psi\left(\frac{r}{R}\right), \qquad \psi\left(\frac{r}{R}\right) \stackrel{\mathrm{df}}{:=} \Psi\left(\sin^{-1}\left(\frac{r}{R}\right)\right). \tag{25}$$

Following the prescription of [33], equation (25) is Fourier transformed to

$$\left(\mathbf{k}^2 + \mathbf{k}\cdot\nabla_{\mathbf{k}}\mathbf{k}\cdot\nabla_{\mathbf{k}} + 4\,\mathbf{k}\cdot\nabla_{\mathbf{k}} + 3\right)\phi(\mathbf{k}) - \frac{16\pi}{(2\pi)^{\frac{3}{2}}}b^2\frac{\mathrm{Si}|\mathbf{k}|}{|\mathbf{k}|}$$
$$+\frac{b}{2(2\pi)^{\frac{3}{2}}}\int \mathrm{d}^3\mathbf{k}'\phi(\mathbf{k}')\int \mathrm{d}^3\mathbf{k}''\mathrm{Sinc}^2\left(\frac{|\mathbf{q}|}{2}\right)\frac{J_0(|\mathbf{k}''|) - 2J_1(|\mathbf{k}''|)|)}{(\mathbf{k}'')^2} = \epsilon^2\phi(\mathbf{k}), \tag{26}$$

where we have switched to the dimensionless constants, ϵ, and b obtained in their turn upon dividing $(E-\bar{a})^2 - c_0$, and $G\sqrt{\kappa}(E-\bar{a})$, by $\hbar^2\kappa$. Furthermore, the notation, $\mathbf{q} = \mathbf{k} - \mathbf{k}' - \mathbf{k}''$, has been used.

5. Conclusions

We developed a model describing the conformal symmetry patterns in the N and Δ spectra. The model favors a strong coupling constant running to a fixed point rather than becoming singular in the infrared, and in accord with the observational data of ref. [20]. The Finite Volume Conformal Quark Model (FV-CQM) advocated here, provides an exactly solvable quantum mechanical description of QCD physics in the infrared, similarly to the Light Front QCD [31], [32], which provides an exactly solvable first approximation

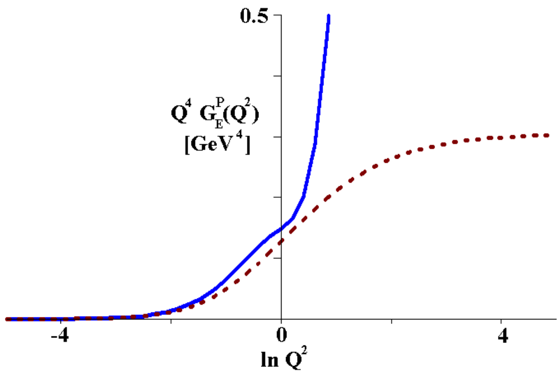

Fig. 4 The large Q^2 behavior of G_E^p calculated with the Schrödinger wave functions from [34] (dashed line), and the Klein-Gordon wave function in eq. (19) (solid line). The Schrödinger formalism turns to provide an adequate description of the large Q^2 scaling of G_p^E, presumably because it averages out well the relativistic effects. In contrast, it seems that the wave functions in eq. (19) do not capture realistically the relativistic effects. We expect improvement in that regard from future employment of the Dirac formalism on $R^1 \otimes S^3$.

to QCD in the ultraviolet, also in terms of a Schrödinger eigenvalue problem, though in reference to an infinite volume confinement.

The degeneracy phenomenon is innate to both the FV-CQM and LF-QCD whose common denominator is the conformal symmetry of the respective exactly solvable wave equations. These two approaches can be viewed as a new generation of AdS/CFT (first principles) motivated conformal $SO(2,4)$ quark models of a small number of parameters, versus conventional $SU(6)_{SF} \otimes O(3)_L$ quark model machineries with dozens of free parameters and a plethora of guessed interactions. Besides the spectra, also form-factors are adequately described (Fig. 3). We furthermore tested the UV behavior of our model in calculating the proton electric charge form factor for high momentum transfer first using the Schrödinger wave functions from [34], and then those in eq. (23). We found that the Schrödinger wave functions describe quite well the scaling behavior of the from-factor under investigation. The result is presented in Fig. 4.

In effect, we find conformal symmetry relevant for the spectroscopy of the baryons of the lightest flavors, the nucleon and the Δ.

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Higgs Boson Decay $H \to Z\gamma$ in the Context of Effective Lagrangian

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Abstract: We have performed a tree-level calculation of the decay width as well as of the Branching ratio for the $H \to Z\gamma$ reaction in the context of effective lagrangian for Higgs boson masses $100 \leq M_H \leq 200\ GeV$. We find that the decay width and the Branching ratio increase due to the anomalous couplings $h_1^{Z\gamma}$ and $h_2^{Z\gamma}$.

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1. Introduction

In the Standard Model (SM) of electroweak interactions there are no couplings at the tree level among three neutral bosons such as $HZ\gamma$. These couplings only appear at the one-loop level through fermion and charged vector bosons [1-3]. In the SM it is dominated by W gauge boson and top quark loops and the branching ratio for the decay mode $H \to Z\gamma$ reaches its maximum value of order 10^{-3} for an intermediate-mass Higgs boson ($115 < M_H < 140\ GeV$) [3]. The study of this vertex has attracted much attention because its strength can be sensitive to scales beyond the SM. The interest in this type of couplings thus lies in the additional contributions that may appear in extensions of the SM. For example, new charged scalar and vector bosons in Left-Right (L-R) symmetric gauge models [4], or Two Higgs Doublet Models (THDM) [5, 6], as well as charginos and neutralinos in the Minimal Supersymmetric Standard Model (MSSM) [5, 6]. The SM and LR symmetric models predict an anomalous $HZ\gamma$ vertex of order 10^{-4} [1, 2, 3], the MSSM may induce a suppression effect [5, 6] but an effective Lagrangian approach leaves room for an enhancement effect [5, 7]. It has been found also that the QCD corrections

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in the SM are well under control [8]. A measurement of this vertex thus may be used to distinguish among theories beyond the SM.

The sensitivity to the $HZ\gamma$ vertex has been studied in processes like $e^-\gamma \to e^-H$ and $e^+e^- \to H\gamma$ [11-13], rare Z and H decays [11, 12, 13], pp collisions via the basic interaction $qq \to qqH$ [13] and the annihilation process $e^+e^- \to HZ$ [10, 14, 15]. It has been found that the latter reaction with polarized beams may lead to the best sensitivity to the $HZ\gamma$ vertex [14] while an anomalous $HZ\gamma$ coupling may enhance partial Higgs decays widths by several orders of magnitude that would lead to measurable effects in Higgs signals at the LHC [13].

The general aim of the present paper is to analyzed the Higgs boson decay mode $H \to Z\gamma$ in the context of effective lagrangian. In Fig. 1, of Ref. [16] is shows the Feynman diagram for the $H \to Z\gamma$ reaction.

The paper is organized as follows. In Section II we present the HVV vertex. In Section III we presented the calculation of the respective width decay and in Section IV we presented our results and conclusions.

2. Generalized HVV Vertex with Dimension Six Operators

In our study, we adopted the effective lagrangian of the Higgs boson and the gauge bosons with operators up to mass dimension six:

$$L_{eff} = L_{SM} + \sum_i \frac{f_i^{(6)}}{\Lambda^2} O_i^{(6)}, \tag{1}$$

where L_{SM} denotes the renormalizable SM lagrangian and $O_i^{(6)}$ are the gauge invariant operators of mass dimension 6.The index i runs over all operators of the given mass dimension. The mass scale is set by Λ, and the coefficients $f_i^{(6)}$ are dimensionless parameters, which are determined once the full theory is known. Excluding the dimension 5 operators for the neutrino majorana masses, and the dimension 6 operators with quarks and lepton fields, we are left with the following eight CP even operators that affect the HVV coupling. Notation of the operators are taken from the reference [16]:

$$\begin{aligned}
O_{WW} &= \Phi^\dagger \overline{W}^{\mu\nu} \overline{W}_{\mu\nu} \Phi, && (2)\\
O_{BB} &= \Phi^\dagger \overline{B}^{\mu\nu} \overline{B}_{\mu\nu} \Phi, && (3)\\
O_{BW} &= \Phi^\dagger \overline{B}^{\mu\nu} \overline{W}_{\mu\nu} \Phi, && (4)\\
O_W &= (D^\mu\Phi)^\dagger \overline{W}_{\mu\nu} (D^\nu\Phi), && (5)\\
O_B &= (D^\mu\Phi)^\dagger \overline{B}_{\mu\nu} (D^\nu\Phi), && (6)\\
O_{\phi 1} &= \left[(D_\mu\Phi)^\dagger \Phi\right] \left[\Phi^\dagger (D^\mu\Phi)\right], && (7)\\
O_{\phi 4} &= (\Phi^\dagger\Phi)(D_\mu\Phi)^\dagger (D^\mu\Phi), && (8)\\
O_{\phi 2} &= \frac{1}{2}\partial_\mu(\Phi^\dagger\Phi)\partial^\mu(\Phi^\dagger\Phi). && (9)
\end{aligned}$$

Here Φ denotes the Higgs doublet field with the hypercharge $Y = \frac{1}{2}$, and the covariant derivate is $D_\mu = \partial_\mu + i\hat{g}_W T^a \hat{W}^a_\mu + i\hat{g}_Y Y \hat{B}_\mu$, where the gauge couplings and the gauge fields with a caret represent those of the SM, in the absence of higher dimensional operators. The gauge covariant and invariant tensor $\overline{W}_{\mu\nu}$ and $\overline{B}_{\mu\nu}$, respectively, are $\overline{W}_{\mu\nu} = i\hat{g}_W T^a \hat{W}^a_\mu$ and $\overline{B}_{\mu\nu} = i\hat{g}_Y Y \hat{B}_\mu$. The coefficients of the (1)-(8), which are denoted as $f_i^{(6)}/\Lambda^2$ in the effective lagrangian of Eq. (1), should give us information about physics beyond the SM.

The bilinear part of the effective lagrangian Eq. (1) is expressed as:

$$\begin{aligned} L_{eff} = & -\frac{1}{2} W^+_{\mu\nu} W^{-\mu\nu} - \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} - \frac{1}{4} A_{\mu\nu} A^{\mu\nu} + \frac{g_W g_Y v^2}{8\Lambda^2} f_{BM} B_{\mu\nu} W^{3\mu\nu} \\ & + m_W^2 W^+_\mu W^{-\mu} + \frac{m_Z^2}{2} Z_\mu Z^\mu - \frac{1}{2}(\partial_\mu H)(\partial^\mu H) - \frac{1}{2} m_H^2 H^2 + \cdots . \end{aligned} \tag{10}$$

The terms describing the HVV couplings in the effective lagrangian are now expressed as

$$\begin{aligned} L_{eff}^{HVV} = & (1 + c_{1WW}) g m_W H W^+_\mu W^{-\mu} + (1 + c_{1ZZ}) \frac{g_Z m_Z}{2} H Z_\mu Z^\mu \\ & + \frac{g_Z}{m_Z} \left[c_{2WW} H W^+_{\mu\nu} W^{-\mu\nu} + \frac{c_{3WW}}{2} \left\{ \Big((\partial_\mu H) W^-_\nu - (\partial_\nu H) W^-_\mu \Big) W^{+\mu\nu} + h.c \right\} \right] \\ & + \frac{g_Z}{m_Z} \left[\frac{c_{2ZZ}}{2} H Z_{\mu\nu} Z^{\mu\nu} + \frac{c_{3ZZ}}{2} \Big((\partial_\mu H) Z_\nu - (\partial_\nu H) Z_\mu \Big) Z^{\mu\nu} \right] + \frac{g_Z}{m_Z} \left[\frac{c_{2\gamma\gamma}}{2} H A_{\mu\nu} A^{\mu\nu} \right] \\ & + \frac{g_Z}{m_Z} \left[c_{2Z\gamma} H Z_{\mu\nu} A^{\mu\nu} + c_{3Z\gamma} \Big((\partial_\mu H) Z_\nu - (\partial_\nu H) Z_\mu \Big) A^{\mu\nu} \right], \end{aligned} \tag{11}$$

where the 9 dimensionless couplings, c_i, parameterize all the non-standard HVV interactions:

From the effective lagrangian of Eq. (1), we obtain the Feynman rule for $V_1^\mu(p_1) - V_2^\nu(p_2) - H(p_H)$ vertex as [16]

$$\Gamma^{HV_1V_2}_{\mu\nu}(p_H, p_1, p_2) = g_Z m_Z \left[h_1^{V_1V_2} g_{\mu\nu} + \frac{h_2^{V_1V_2}}{m_Z^2} p_{2\mu} p_{1\nu} \right], \tag{12}$$

where $g_Z = e/\cos\theta_W \sin\theta_W$ and all three momenta are incoming, $p_1 + p_2 + p_H = 0$, as shown in the Fig. 1 of Ref. [16]. V_1 and V_2 can be $(V_1V_2) = (ZZ), (Z\gamma), (\gamma Z), (W^+W^-)$ o (W^-W^+). The coefficients $h_i^{V_1V_2}(p_1, p_2)$ [16] are:

$$h_1^{Z\gamma}(p_1, p_2) = \frac{p_1^2 + p_2^2 - m_H^2}{m_Z^2} c_{2Z\gamma} - \frac{p_1^2 - p_2^2 - m_H^2}{m_Z^2} c_{3Z\gamma}, \tag{13}$$

$$h_2^{Z\gamma}(p_1, p_2) = 2(c_{2Z\gamma} - c_{3Z\gamma}), \tag{14}$$

for the $HZ\gamma$ coupling.

3. Width Decay of $H \to Z\gamma$

In this section we presented the width decay of the reaction $H \to Z\gamma$ in the context of effective lagrangian.

3.1 Transition Amplitude

The expression for the transition amplitude with $HZ\gamma$ anomalous couplings (Eq. (12)) is given by

$$M(H \to Z\gamma) = M_{\mu\nu}\varepsilon^{\mu}(p_1, \lambda_1)\varepsilon^{\nu}(p_2, \lambda_2), \tag{15}$$

where

$$M_{\mu\nu} = \Gamma^{HZ\gamma}_{\mu\nu}(p_H, p_1, p_2). \tag{16}$$

The transition amplitude squared is

$$\begin{aligned}\sum_{\lambda} |M|^2 &= \sum_{\lambda} MM^* \\ &= M_{\mu\nu}M^*_{\alpha\beta}\sum_{\lambda}\varepsilon^{\mu}(p_1, \lambda_1)\varepsilon^{*\alpha}(p_1, \lambda_1)\varepsilon^{\nu}(p_2, \lambda_2)\varepsilon^{*\beta}(p_2, \lambda_2),\end{aligned} \tag{17}$$

and of the follows properties for the polarization vectors

$$\sum_{\lambda_1}\varepsilon^{\mu}(p_1, \lambda_1)\varepsilon^{*\alpha}(p_1, \lambda_1) = -g^{\mu\alpha} + \frac{p_1^{\mu}p_1^{\alpha}}{m_Z^2}, \tag{18}$$

$$\sum_{\lambda_2}\varepsilon^{\nu}(p_2, \lambda_2)\varepsilon^{*\beta}(p_2, \lambda_2) = -g^{\nu\beta}, \tag{19}$$

we obtain

$$\begin{aligned}\sum_{\lambda} |M|^2 &= M_{\mu\nu}M^*_{\alpha\beta}\left(- g^{\mu\alpha} + \frac{p_1^{\mu}p_1^{\alpha}}{m_Z^2}\right)(-g^{\nu\beta}) \\ &= M_{\mu\nu}M^*_{\alpha\beta}\left[g^{\mu\alpha}g^{\nu\beta} - \frac{p_1^{\mu}p_1^{\alpha}}{m_Z^2}g^{\nu\beta}\right].\end{aligned} \tag{20}$$

On the other hand,

$$
\begin{aligned}
M_{\mu\nu}M^*_{\alpha\beta} &= \Gamma^{HZ\gamma}_{\mu\nu}\Gamma^{HZ\gamma^*}_{\alpha\beta} \\
&= g_Z m_Z\left[h_1^{Z\gamma}g_{\mu\nu} + \frac{h_2^{Z\gamma}}{m_Z^2}p_{2\mu}p_{1\nu}\right]g_Z m_Z\left[h_1^{Z\gamma}g_{\alpha\beta} + \frac{h_2^{Z\gamma}}{m_Z^2}p_{2\alpha}p_{1\beta}\right],
\end{aligned} \tag{21}
$$

substituting Eq. (21) in (20) we obtain

$$
\begin{aligned}
\sum_\lambda |M|^2 = g_Z^2 m_Z^2\Bigg[&(h_1^{Z\gamma})^2 g_{\mu\nu}g_{\alpha\beta} + \frac{h_1^{Z\gamma}h_2^{Z\gamma}}{m_Z^2}g_{\mu\nu}p_{2\alpha}p_{1\beta} + \frac{h_1^{Z\gamma}h_2^{Z\gamma}}{m_Z^2}g_{\alpha\beta}p_{2\mu}p_{1\nu} + \frac{(h_2^{Z\gamma})^2}{m_Z^4}p_{2\mu}p_{1\nu}p_{2\alpha}p_{1\beta}\Bigg] \\
&\times\left[g^{\mu\alpha}g^{\nu\beta} - \frac{p_1^\mu p_1^\alpha}{m_Z^2}g^{\nu\beta}\right],
\end{aligned} \tag{22}
$$

and performing the appropriate operations

$$
\begin{aligned}
\sum_\lambda |\mathcal{M}|^2 = g_Z^2 m_Z^2\Bigg[&(h_1^{Z\gamma})^2 g_{\mu\nu}g_{\alpha\beta}g^{\mu\alpha}g^{\nu\beta} - (h_1^{Z\gamma})^2\frac{p_1^\mu p_1^\alpha}{m_Z^2}g_{\mu\nu}g_{\alpha\beta}g^{\nu\beta} \\
&+ \frac{h_1^{Z\gamma}h_2^{Z\gamma}}{m_Z^2}g_{\mu\nu}p_{2\alpha}p_{1\beta}g^{\mu\alpha}g^{\nu\beta} - \frac{h_1^{Z\gamma}h_2^{Z\gamma}}{m_Z^4}g_{\mu\nu}p_{2\alpha}p_{1\beta}p_1^\mu p_1^\alpha g^{\nu\beta} \\
&+ \frac{h_1^{Z\gamma}h_2^{Z\gamma}}{m_Z^2}g_{\alpha\beta}p_{2\mu}p_{1\nu}g^{\mu\alpha}g^{\nu\beta} - \frac{h_1^{Z\gamma}h_2^{Z\gamma}}{m_Z^4}g_{\alpha\beta}p_{2\mu}p_{1\nu}p_1^\mu p_1^\alpha g^{\nu\beta} \\
&+ \frac{(h_2^{Z\gamma})^2}{m_Z^4}p_{2\mu}p_{1\nu}p_{2\alpha}p_{1\beta}g^{\mu\alpha}g^{\nu\beta} - \frac{(h_2^{Z\gamma})^2}{m_Z^6}p_{2\mu}p_{1\nu}p_{2\alpha}p_{1\beta}p_1^\mu p_1^\alpha g^{\nu\beta}\Bigg].
\end{aligned} \tag{23}
$$

From the relativistic energy-momenta relation

$$p_1^2 = m_Z^2, \tag{24}$$

$$p_2^2 = m_\gamma^2 = 0, \tag{25}$$

$$p_H^2 = m_H^2, \tag{26}$$

we obtain

$$
\begin{aligned}
p_H &= -(p_1 + p_2), \\
p_H^2 &= (p_1 + p_2)^2 = p_1^2 + 2(p_1\cdot p_2) + p_2^2,
\end{aligned}
$$

$$
\begin{aligned}
(p_1\cdot p_2) &= \frac{m_H^2 - m_Z^2}{2}, \\
(p_1\cdot p_2)^2 &= \frac{(m_H^2 - m_Z^2)^2}{4}.
\end{aligned} \tag{27}
$$

Finally, the squared of the amplitude of transition for the $H \to Z\gamma$ reaction is given by

$$\sum_{\lambda} |M|^2 = g_Z^2 m_Z^2 \left[3(h_1^{Z\gamma})^2 - \frac{(h_2^{Z\gamma})^2}{4m_Z^4}(m_H^2 - m_Z^2)^2 \right]. \tag{28}$$

3.2 The Width Decay of the Reaction $H \to Z\gamma$

The decay rate for a two-body process is given by

$$d\Gamma(H \to Z\gamma) = \frac{1}{32\pi^2}|M|^2 \frac{|\mathbf{p}|}{m_H^2} d\Omega, \tag{29}$$

where

$$|\mathbf{p}| = \frac{m_H^2 - m_Z^2}{2m_H}. \tag{30}$$

Explicitly, performing the integrals the decay rate is

$$\int d\Gamma(H \to Z\gamma) = \int \frac{1}{32\pi^2}|M|^2 \frac{|\mathbf{p}|}{m_H^2} d\Omega,$$

$$\Gamma(H \to Z\gamma) = \frac{g_Z^2 m_Z^2}{64\pi m_H^3}(m_H^2 - m_Z^2) \left[12(h_1^{Z\gamma})^2 - \left(\frac{m_H^2 - m_Z^2}{m_Z^2} \right)^2 (h_2^{Z\gamma})^2 \right]. \tag{31}$$

The anomalous couplings $h_1^{Z\gamma}$ and $h_2^{Z\gamma}$ are relations for Eqs. (13) and (14), then Eq. (31) can be written in terms of $h_1^{Z\gamma}$ only as in Eq. (2) of Ref. [17]. In our case we want to see the dependence of $\Gamma(H \to Z\gamma)$ with $h_1^{Z\gamma}$ and $h_2^{Z\gamma}$, respectively.

3.3 Width Decay of $H \to f\bar{f}$

In this section we presented the decay width of the reaction $H \to f\bar{f}$.

The expression for the amplitude of transition is given by

$$M = \bar{u}(p_1) \left[\frac{-iem_f}{2\sin\theta_W m_W} \right] v(p_2), \tag{32}$$

and the complex conjugate of Eq. (32) is

$$M^\dagger = \frac{iem_f}{2\sin\theta_W m_W} \bar{v}(p_2) u(p_1). \tag{33}$$

From Eqs. (32) and (33) the square of the amplitude is obtained by summing and averaging over the spins of the final state of the fermions, then

$$\sum_{\lambda} |M|^2 = \sum_{\lambda} M M^\dagger = \frac{e^2 m_f^2}{4\sin^2\theta_W m_W^2} \left[\bar{u}(p_1) v(p_2) \right] \left[\bar{v}(p_2) u(p_1) \right] \tag{34}$$

where we use the properties

$$\sum_s u(p_1)\bar{u}(p_1) = (\not{p}_1 + m_1),\ \text{for particles} \tag{35}$$

$$\sum_s v(p_2)\bar{v}(p_2) = (\not{p}_2 - m_2),\ \text{for anti-particles} \tag{36}$$

replacing we obtain

$$\sum_s |M|^2 = \frac{e^2 m_f^2}{4\sin^2\theta_W m_W^2}(\not{p}_1 + m_1)(\not{p}_2 - m_2),$$

Explicitly, we obtain

$$\sum_s |M|^2 = \frac{e^2 m_f^2}{4\sin^2\theta_W m_W^2}\left[P_{1\mu}P_{2\nu}Tr(\gamma^\mu\gamma^\nu) - m_2 P_{1\mu}Tr(\gamma^\mu) + m_1 P_{2\nu}Tr(\gamma^\nu) - m_1 m_2 Tr(I),\right.$$

and applying the properties of the theorems of traces of Dirac gamma matrices

$$\sum_s |M|^2 = \frac{e^2 m_f^2}{\sin^2\theta_W m_W^2}\left[\left(p_1 \cdot p_2\right) - m_1 m_2\right],$$

where $m_1 = m_2 = m_f$, then

$$\sum_s |M|^2 = \frac{e^2 m_f^2}{\sin^2\theta_W m_W^2}\left[\left(p_1 \cdot p_2\right) - m_f^2\right]. \tag{37}$$

From momentum conservation we obtain

$$p_H = p_1 + p_2,$$

then

$$\left(p_1 \cdot p_2\right) = \frac{m_H^2 - 2m_f^2}{2}, \tag{38}$$

and finally the square of the amplitude of transition is

$$\sum_s |M|^2 = \frac{e^2 m_f^2}{\sin^2\theta_W m_W^2}\left(\frac{m_H^2 - 4m_f^2}{2}\right). \tag{39}$$

The total decay width of the reaction $H \to f\bar{f}$ is given by

$$\Gamma(H \to f\bar{f}) = \frac{N_C g^2 m_f^2 m_H}{32\pi m_W^2}\left(1 - \frac{4m_f^2}{m_H^2}\right)^{\frac{3}{2}}, \tag{40}$$

where the color factor $N_C = 1$ for leptons and 3 for quarks.

3.4 Branching Ratio of the Process $H \to Z\gamma$

In this section we present the branching ratio of reaction $H \to Z\gamma$.

For Higgs boson mass in the range of 100-140 GeV the dominant mode is $H \to b\bar{b}$, in this case the branching ratio is given by

$$Br(H \to Z\gamma) = \frac{\Gamma(H \to Z\gamma)}{\Gamma(H \to b\bar{b})}, \tag{41}$$

where $\Gamma(H \to Z\gamma)$ and $\Gamma(H \to b\bar{b})$ are given by Eqs. (31) and (40) respectively.

4. Results and Conclusions

In this section we present our results and conclusions to the $H \to Z\gamma$ reaction in the context of effective lagrangian.

For the numerical computation, we have adopted the following parameters: the angle of Weinber $\sin^2\theta_W = 0.232$, the mass ($m_b = 4.5\ GeV$) of the bottom quark, the mass ($m_Z = 91.2\ GeV$) of the Z boson and the mass ($100 \leq M_H \leq 200\ GeV$) of the Higgs boson.

To illustrate our results we show the partial decay width in Fig. 1 as a function of the Higgs boson mass m_H for the values of $h_1^{Z\gamma} = 0.047$ and $h_2^{Z\gamma} = 0.081$ given in Ref. [17]. We observe from this figure that the partial decay width of the $H \to Z\gamma$ reaction decreases with increasing Higgs boson mass M_H, and crease to $h_1^{Z\gamma}$ and $h_2^{Z\gamma}$ given.

Figure 2 shows the branching ratio for the partial decay width $\Gamma(H \to Z\gamma)$. In this case, the branching ratio crease due to the anomalous couplings $h_1^{Z\gamma}$ and $h_2^{Z\gamma}$. We obtain an improvement of about an order of magnitude compared to the result reported in the literature [18] for the case of the standard model at one loop level.

In conclusion, we have analyzed the partial decay width and the branching ratio of the $H \to Z\gamma$ reaction with anomalous couplings $h_1^{Z\gamma}$ and $h_2^{Z\gamma}$. Our results are consistent with those reported in the literature and improved by an order of magnitude over the limits obtained in the context of the standard model [18].

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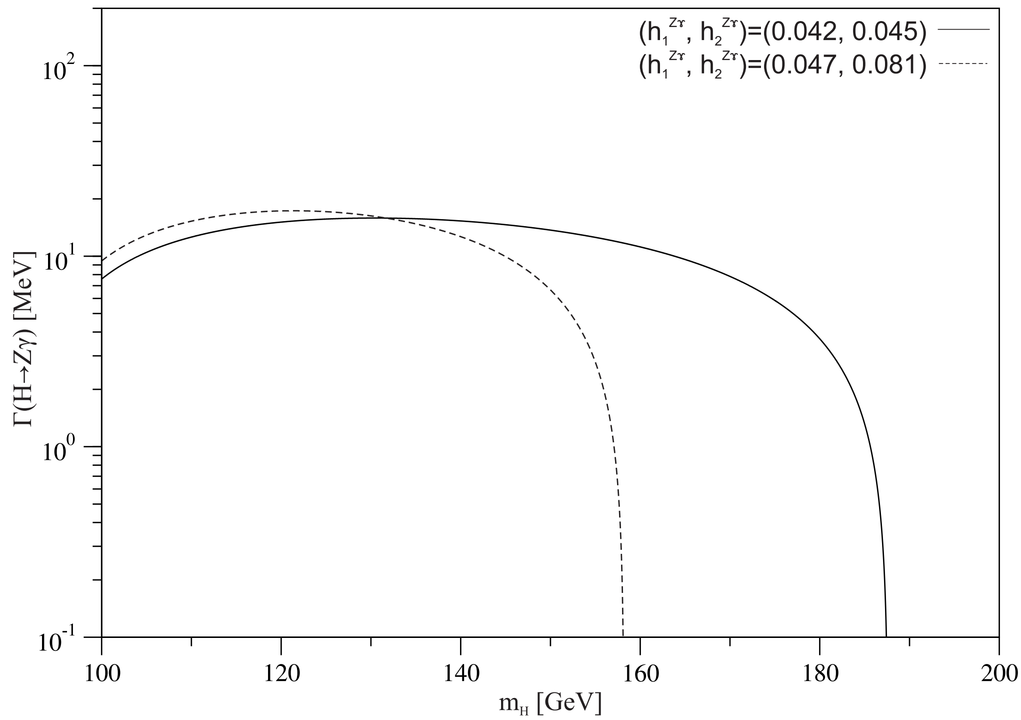

Fig. 1 The decay rate for the $H \to Z\gamma$ reaction as a function of the Higgs boson mass m_H.

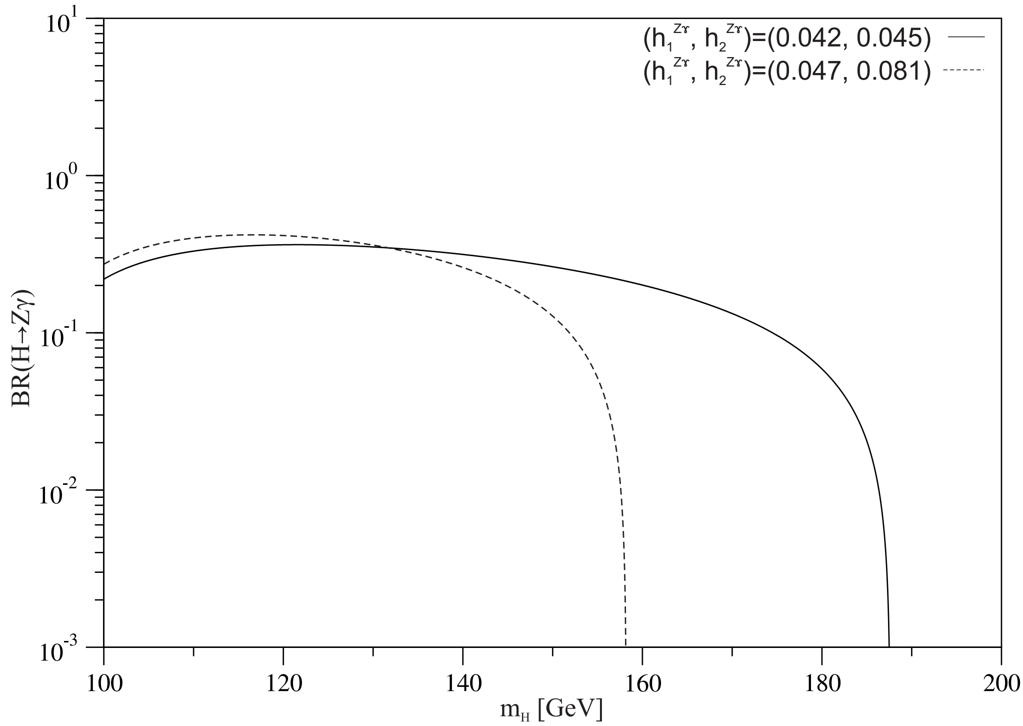

Fig. 2 The branching ratio for the $H \to Z\gamma$ reaction as a function of the Higgs boson mass m_H.

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Generalized Coherent States and Compactons

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Abstract: A generalized nonlinear Schrödinger equation was derived for the quasi-spin model of DNA macromolecule proposed by Takeno and Homma. This model considered the action of external agents such as a protein and also a bath of phonons. The analysis was done by using the Generalized Coherent States approach (GCS). We restricted our study to the case of weakly saturating approximation and found several important collectively formed stretching and unfolding traveling structures have been obtained for angle deviations. Their corresponding hydrogen bond displacements show the unexpected behavior of well defined compacton anti-compacton pairs.

Keywords: BioPhysics; DNA; Generalized Coherent States
PACS (2010): 91.62.Mn; 87.18.Vf; 87.18.-h;87.15.-v; 87.14.gk; 05.45.-a; 03.65.-w

1. Introduction

As is very well known the DNA macromolecule dynamics with soliton like structures traveling along the chain has been attracting the attention of many nonlinear researchers lately. It is assumed that open states of the DNA are the principal requirements for arising other important features of this molecule, specifically for transcription and replication processes. Therefore, it is obvious that not only the open states are important for the cell machinery to function, but also the inverse process "dual" to the former one that could repair the appearance of these open states, has the reason to exist. Consequently, it is important to understand how nonlinear waves influence the interactions of the units that conform the two strands of the DNA chain. Different approaches for variety of models of the DNA molecule have been applied. After the first proposal made by Englander

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[1], mainly two versions of modeling the DNA macromolecule have appeared among others. The first one is being deal with rotational degrees of freedom, and the first one was proposed by Yomosa [2]. The second type model is related with the dynamics of the hydrogen bonds along the chain, and the principal model in this direction is due to the Peyrard - Bishop model [3]. These two version of modeling DNA received a great improvements by many important contributions that one can find for example in the papers [4-7] and citation therein. In his pioneering work Yomosa [2] presented an important DNA model which includes the rotation of pair bases along the spiral model proposed by Watson and Crick. This model takes into consideration a dynamic of a plane base rotation perpendicular to the helical axis z around the backbone structure. Further improvements were done in the papers [7,8].

On the other order of things, the study of many body problems whose Hamiltonian has been explicitly written in terms of spin or quasi spin operators, can be reduced by appropriated approximations to a quasi-classical treatment. For doing this, certain reduction procedure from the quasi - spin operator description to a classical one is needed. This procedure consists in choosing trial functions which can be used for averaging the Hamiltonian. Then it is natural to choose for this aim, coherent states since these states are the most classical and minimize important uncertainty relations [9-11].

In this contribution we study a dynamical model related to the appearing of traveling waves in the DNA molecule by using the quasi-classical approximation due to the generalized coherent states approach. This treatment was possible to implement because of the quasi spin character of the Hamiltonian of the DNA that is written in terms of the Spin operators **S** proposed by Takeno-Homma (TH) in [7]. The approach to derive analytical solutions was implemented explicitly for the case of traveling waves. In the next section we briefly expose the main features of the Takeno-Homma model. The second section is devoted to the generalized coherent state approach applied to the lattice Hamiltonian of DNA discrete quasi-spin model. The nonlinear lattice equation and is reported in the third section. The compacton anti-compacton pair is analyzed in the fourth section. Comments and conclusion are done in the last section.

2. The Takeno - Homma Model

Let us review here the main aspects of the Takeno-Homma Model presented in the paper [7]. Here we will avoid details, for complete version of this model one can consult the papers cited above and the followings [12,13]. Thus, it is considered for studying the B form of the DNA double helical chain. Suppose the coordinates of P_n and $P_n^{'}$ be the points where the bases in the nth base pair are attached to the strands **A** and **B** respectively. By (θ_n, ϕ_n) and $(\theta_n^{'}, \phi_n^{'})$ we denote the angles of rotation of the bases in the nth base pair around the points P_n and $P_n^{'}$ in the XZ and XY planes respectively. From the heuristic argument it is assumed that the main contribution of the inter strand base-base interaction i.e. the hydrogen bonding energy of the DNA is closely related to the distance between the bases. Thus, taking in mind this argument the distance between the edges of the arrows $(Q_nQ_n^{'})^2$ is written as:

$$(Q_nQ_n^{'})^2 = 2 + 4r^2 + 2[S_n^x S_n^{'x} + S_n^y S_n^{'y} - S_n^z S_n^{'z}] - 4r[S_n^x + S_n^{'x}] \tag{1}$$

Where the well known expressions for the quasi-spin operators have been used

$$\mathbf{S_n} = (S_x, S_y, S_z) = (\sin\theta_n \cos\phi_n, \sin\theta_n \sin\phi_n, \cos\theta_n)$$

$$\mathbf{S_n^{'}} = (S_x^{'}, S_y^{'}, S_z^{'}) = (\sin\theta_n^{'} \cos\phi_n^{'}, \sin\theta_n^{'} \sin\phi_n^{'}, \cos\theta_n^{'})$$

The equation for $QQ^{'}$ coincides with the Hamiltonian for a generalized Heisenberg spin model. For the case of isotropically homogeneous coupled quasi-spin chain model, in the nearest neighboring interaction the Hamiltonian can be written as

$$H_1 = -\sum_{\Sigma} n + \mathbf{S_n^{'}} \cdot \mathbf{S_n^{'}}) + \mu(\mathbf{S_n} \cdot \mathbf{S_n^{'}}) \tag{2}$$

The first term of the Hamiltonian (4) corresponds to the stacking nearest neighbor interactions along the chain for each strand, the second term represents the inter strand interaction at the $n-$th site. Let us denote by X_n as the displacement of the bases along the hydrogen bond at the $n-$th site and by$p_n = m_1 X_n$ the corresponding momentum of the displacement X_n. Since the functions of DNA are switched on under the biological temperature, then it is necessary to include thermal surrounding phonon contribution and the coupling between the oscillation of the hydrogen atom due to thermal fluctuation and the rotation of bases. Therefore this contribution takes the form

$$H_2 = \sum_{\Sigma} n \tag{3}$$

Similarly, when the DNA macromolecule is interacting with other molecule like a protein for instance, at the first approximation this interaction could be modeled by adding a new contribution to the Hamiltonian in the following manner

$$H_3 = \sum_{\Sigma} n \tag{4}$$

Here y_n denotes the displacement of the $n-$th peptide group in the protein molecule. The first term of the potential part represents the oscillation of proteins sites and the second term represents the interaction between the protein and the DNA sites. Consequently, the total Hamiltonian that will be analyzed is written as follows:

$$H = \sum_n [-J(\mathbf{S_n} \cdot \mathbf{S_{n+1}} + \mathbf{S_n} \cdot \mathbf{S_n^{'}}) - (\mu - \alpha_1(X_{n+1} - X_{n+1}))(\mathbf{S_n} \cdot \mathbf{S_n^{'}}) +$$

$$+\alpha_2(y_{n+1} - y_{n-1}) S_n^z S_n^{z'} + \frac{p_n^2}{2m_1} + k_1(X_n - X_{n+1})^2 + \frac{q_n^2}{2m_1} + k_2(y_{n+1} - y_{n+1})^2] \tag{5}$$

3. Generalized Coherent State Approach

The fundamental task for finding the relation between the collective nonlinear effects in classical and quasi-quantum models appears. This correspondence has been discussed by many authors, for example see [14]. On the other hand the famous Heisenberg model which is the basis for the theoretical study of a various class of studies as in ferromagnetism, exciton models in crystals, etc; was studied by means of various reduction procedures [15]. Generally speaking, this reduction procedure could be done in case when the Hamiltonian of a certain many body problem is written in terms of a spin or quasi-spin operators and contains physical parameters like exchange integrals, constant of anisotropy, atom spin values etc. The reduction or transition procedures from quantum or quasi-quantum description to its classical or quasi-classical level has been realized by different important ways. The first one is the formal replacement of the spin o quasi spin operators with classical vectors. The next method uses the Holstein - Primakov transformation for bozonization the quasi spin model. This method allows to rewrite the initial quasi-spin Hamiltonian in terms of Bose operators and subsequently transforms the Hamiltonian to a classical one. The third version consists on averaging the quantum or quasi-quantum Hamiltonian over some states that minimizes certain uncertainties.

In the next section, we will investigate the particle-like localized nonlinear excitation in the quasi-spin Hamiltonian for DNA model obtained by Takeno-Homma [7] by making use of the third variant of reduction procedure, based on the generalized coherent states (GCS) on the group (SU(2)/U(1)).

Let us briefly describe the main aspects of the generalized coherent states (GCS) on the group (SU(2)/U(1)). Let G be an arbitrary Lie Group and $\hat{T}$ be its irreducible unitary representation acting in the Hilbert space H. Any vector of this space is denoted by the symbol $|\psi>$, the scalar product of the vectors $|\psi>$, and $|\phi>$, linear on $|\psi>$, and antilinear on $|\phi>$ by the symbol $<\phi|\psi>$, and the projection operator on the vector $|\psi>$, by $|\psi><\phi|$. Let $|\psi_0>$, be some fixed vector in the space H. Consider the set of vectors $\{|\psi_g>\}$, for that $|\psi_g>=T(g)|\psi_o>$ where g goes over all group G. It is easy to see that two vectors $|\psi_{g_1}>$ and $|\psi_{g_2}>$ will differ from one another only by a phase factor ($|\psi_{g_1}>=e^{i\alpha}|\psi_{g_2}>, |e^{i\alpha}|=1$), or in other words will determine the same state only if $T(g_2^{-1}.g_1)|\psi_0>=e^{i\alpha}|\psi_0>$.

Let $P=\{p\}$ be the set of elements of the group G such that $T(p)|\psi_0>=e^{i\alpha(p)}|\psi_0>$. It is evident that P is a subgroup of the group G and we denote it as the stationary group of the state $|\psi_0>$.This set is a stationary subgroup of the vector $|\psi_0>$.

From the above assumptions it is easily seeing that vectors $|\psi(g)>$ being embedded in the left adjoin class $g_{1p}\in g_1P$ will differ each other only in the phase. It means that they define the same state. Then one concludes that different vectors (states) will correspond to elements g_m that belong to the factor space $M=G/P$. This way in order to describe the set of different states it is enough to take one element of each class. From the geometrical point of view, the group G is treated as fiber-bundle space with a base $M=G/P$ and layer P. Then the choosing of g_m corresponds to some section of this fiber

-bundle space. The set of vectors $|\psi_m >= T(g_m)|\psi_0>$ with $g_m \in G/P$ we call a system of Generalized Coherent States (GCS) on the group G with a referent vector $|\psi_0>$. Usually, the choose of referent vector $|\psi_0>$ is determined by thinking on simplicity and with the states being nearly classical.

For our aims the group $SU(2)$ plays a crucial role. This group will be the group G of the above described scheme. It is known that the system of spin coherent states (GCS constructed on the SU(2)/U(1) space) may be written as

$$|\psi >= T(g)|\psi_0 >= e^{\alpha S^+ - \hat{\alpha} S^-}|0 >= \left(1+|\psi|^2\right)^{-j} e^{\psi S^+}|0>; \tag{6}$$

with $\hat{S}^{\pm} = \hat{S}^{x} + i\,\hat{S}^{y}$, $\psi = \frac{\alpha}{|\alpha|}Tan|\alpha|$, α, ψ are complex numbers, $|0>=|j,-j>$ and j defines the unitary representation of the group $SU(2)$. The set of trial functions (1) is seen to have the symmetry of sphere.

For $j=1$ the corresponding coherent states read

$$|\psi >= \frac{1}{1+|\psi|^2}\{|0>+\sqrt{2}\psi|1>+\psi^2|2>\} \tag{7}$$

with $(|i>, i=0,1,2)$ being the pure spin states (down, middle and up states as usual). The components of the classical spin vector, $\vec{S}=(S^x,S^y,S^z)=<\psi|\hat{S}|\psi>$ and of the quadrupole moment Q^{ij} for the GCS in the coset space $SU(2)/U(1)$ for any value of j are

$$S^+ = \bar{S}^- = 2j\frac{\bar{\psi}}{1+|\psi|^2}, \quad S^z = -j\frac{1-|\psi|^2}{1+|\psi|^2}, \quad Q^{zz} = \frac{j^2\left(1-|\psi|^2\right)+2j|\psi|^2}{\left(1+|\psi|^2\right)^2} \tag{8}$$

and the averaged Casimir operator is

$$<\hat{C}^2>=\frac{1}{2}[<\hat{S}^+\hat{S}^- > + <\hat{S}^-\hat{S}^+>] + <(\hat{S}^z)^2>=m=j(j+1)$$

4. Classical Lattice Nonlinear Equation

We will consider that all exchange integrals in the Hamiltonian (7) are constant and do no differ from one point to another, i.e. we have a homogeneous system. As is know, the choice of GCS method is dictated by the Hamiltonian symmetry [10]. The conditions for using this reduction procedure are fulfilled. First of all we have a quasi-spin Hamiltonian (2.) with zero anisotropy, because the exchange integrals in the x, y, z direction are the same. In this case the model as can be easily seen has the SU(2) symmetry. The demonstration of this fact was done by Makhankov in [16]. Second, the easy axis of the model is considered the axis z, that is the direction of the helical axis as an axis of "magnetization" in the spin chain anisotropy. Additionally the quasi-spin part of the Hamiltonian (2.) is formally written in terms of spin operators with spin value $s=1/2$. Therefore, in this case, it is not necessary to carry out the bozonization procedure of the Hamiltonian, since both the spin Hamiltonian and the generalized coherent states are constructed on the operators of the same group SU(2)/U(1).

Let us pass from quasi-quantum system to classical one. Before doing this, let us remind the reader that generalized coherent states on the SU(2) group in complex parametrization has the form

$$|\psi_j >= (1+|\psi_j|^2)^{-S} e^{\psi_j \widehat{S_j^+}} |S, -S>_j$$

Next, we consider the fact that since spin operators $\widehat{S_j^+}$ commute in neighboring sites of one DNA strands, the generalized coherent state for all the lattice is the direct product of GCS taken at separate sites.

$$|\psi >= \prod_{\Pi} j; j = 1, 2, 3, ...N$$

Thus we have for the spin averaging

$$< \psi|\widehat{S_j^+}\widehat{S_{j+1}^+}|\psi >=< \psi_j|\widehat{S_j^+}|\psi_j >< \psi_{j+1}|\widehat{S_{j+1}^+}|\psi_{j+1} >$$

The averaged values of quasi-spin operators $\mathbf{S} = (S^x, S^y, S^z)$ by using the SU(2) GCS can be written in the following stereographic projection forms, that subsequently will be used for averaging the lattice Hamiltonian (2.)

$$S^+ =< \psi|\widehat{S^+}|\psi >= \frac{\overline{\psi}}{1+|\psi|^2}; S^- =< \psi|\widehat{S^-}|\psi >= \frac{\psi}{1+|\psi|^2}; S^z =< \psi|\widehat{S^z}|\psi >= -\frac{1-|\psi|^2}{2(1+|\psi|^2)} \tag{9}$$

with $S^+ = S^x + iS^y$ and $S^- = S^x - iS^y$. The quantities $\psi_n, \theta_n and \phi_n$ from the Eq. (2) are interrelated by the formula

$$\psi_n = tan\,(\theta_n/2)\, e^{i\phi_n} \tag{10}$$

As one can easily verify, similar expressions can be constructed for the second strand of the DNA molecule. In this case GCS will be parameterized by the field ξ with its corresponding angles $\theta^{'}$ and $\phi^{'}$. After averaging the quasi-spin Hamiltonian (2.) with the GCS (9) we obtain the classical discrete Hamiltonian in the following form:

$$\begin{aligned} H \; &= \sum_n -\frac{J}{2} \frac{(\overline{\psi_n}\psi_{n+1} + \overline{\psi_n}\psi_{n+1}) + (1-|\psi_n|^2)(1-|\psi_{n+1}|^2)}{(1+|\psi_n|^2)(1+|\psi_{n+1}|^2)} + \\ &\frac{(\overline{\xi_n}\xi_{n+1} + \overline{\xi_n}\xi_{n+1}) + (1-|\xi_n|^2)(1-|\xi_{n+1}|^2)}{(1+|\xi_n|^2)(1+|\xi_{n+1}|^2)} \\ &-\frac{1}{4}(\mu - \alpha_1(X_{n+1} - X_{n-1}))\frac{2\overline{\psi_n}\xi_n + 2\overline{\xi_n}\psi_n + (1-|\xi_n|^2)(1-|\psi_n|^2)}{(1+|\xi_n|^2)(1+|\psi_n|^2)} \\ &+\frac{\alpha_2}{4}(y_{n+1} - y_{n-1})\frac{(1-|\xi_n|^2)(1-|\psi_n|^2)}{(1+|\xi_n|^2)(1+|\psi_n|^2)} + \frac{p_n^2}{2m_1} + \frac{q_n^2}{2m_2} + k_1(X_n - X_{n+1})^2 + k_2(y_n - y_{n+1})^2 \end{aligned} \tag{11}$$

Because of the appreciable length of excitations in DNA is much greater than the inter-site distance a between neighboring nucleotides, we can pass for making an approximation of continuous limit by standard procedures. For this we introduce as usual the fields $X_n \to X(z,t), y_n \to y(z,t)$ with $z = na$ and make the standard expansions

$$\psi_{n\pm1} = \psi(z,t) \pm a\psi_z + \frac{a^2}{2!}\psi_{zz} + ..., |\psi_{n\pm1}|^2 = |\psi|^2 \pm a(|\psi|)_z + \frac{a^2}{2!}(|\psi|)_{zz} + ..., \sum_n \longrightarrow \int \frac{dz}{a}$$

The similar expansions can be done directly for the variables $X_{n\pm1}$, $y_{n\pm1}$, $\xi_{\pm1}$ and $|\xi_{n\pm1}|$. After some algebra we obtain the new classical Hamiltonian

$$H = \int \left\{ \frac{aJ}{2}\left(\frac{|\psi_z|^2}{1+|\psi|^2} + \frac{|\xi_z|^2}{1+|\xi|^2}\right) - \frac{1}{4a}(\mu - 2\alpha_1 a X_z)\left(\frac{2(\psi\overline{\xi} + \overline{\psi}\xi) + (1-|\xi|^2)(1-|\psi|^2)}{(1+|\xi|^2)(1+|\psi|^2)}\right)\right.$$

$$\left. + \frac{\alpha_2}{2}\frac{(1-|\xi|^2)(1-|\psi|^2)}{(1+|\xi|^2)(1+|\psi|^2)}y_z + \frac{p^2}{2am_1} + \frac{q^2}{2am_2} + k_1 a(X_z)^2 + k_2 a(y_z)^2 \right\} dz + const. \quad (12)$$

The equation of motion for $X(z,t)$ and $y(z,t)$ can be obtained from the Eq.(12) by the standard Hamiltonian equations of motion $\dot{X} = \frac{\partial H}{\partial p}$ and $\dot{y} = \frac{\partial H}{\partial q}$ and their canonical conjugate counterparts. The equation of motion for ψ and ξ variables is obtained by using the variational derivative [17], in the following form

$$i\xi_t = -(1+|\xi|^2)^2\frac{\delta H}{\delta\overline{\xi}}$$

The same sort of equations are directly built also for the second field variable ψ. After several algebraic calculations, finally, we obtain the following system of nonlinear differential equations:

$$i\psi_t = -\frac{Ja}{2}\psi_{zz} + aJ\frac{2\psi_z^2\overline{\psi}}{1+|\psi|^2} - \left(\frac{\mu - 2a\alpha_1 X_z}{2a}\right)\left(\frac{\xi - \psi + \psi|\xi|^2 - \overline{\xi}\psi^2}{1+|\xi|^2}\right)$$
$$-\alpha_2 y_z\left(\frac{1-|\xi|^2}{1+|\xi|^2}\right)\psi, \quad (13)$$

$$i\xi_t = -\frac{Ja}{2}\xi_{zz} + aJ\frac{2\xi_z^2\overline{\xi}}{1+|\xi|^2} - \left(\frac{\mu - 2a\alpha_1 X_z}{2a}\right)\left(\frac{\psi - \xi + \xi|\psi|^2 - \overline{\psi}\xi^2}{1+|\psi|^2}\right)$$
$$-\alpha_2 y_z\left(\frac{1-|\psi|^2}{1+|\psi|^2}\right)\xi, \quad (14)$$

$$m_1 X_{tt} = 2ak_1 X_{zz} + \frac{\alpha_1}{2}\left(\frac{2(\psi\overline{\xi} + \overline{\psi}\xi) + (1-|\xi|^2)(1-|\psi|^2)}{(1+|\xi|^2)(1+|\psi|^2)}\right)_z \quad (15)$$

$$m_2 y_{tt} = 2ak_2 y_{zz} + \frac{\alpha_2}{2}\left(\frac{(1-|\xi|^2)(1-|\psi|^2)}{(1+|\xi|^2)(1+|\psi|^2)}\right)_z \quad (16)$$

By adding and subtracting equations (19) and (20) making $\psi = -\xi$, these two equations transform to

$$i\psi = -\frac{Ja}{2}\psi_{zz} + aJ\frac{\psi_z^2\overline{\psi}}{1+|\psi|^2} + \left(\frac{2\mu}{a} - (4\alpha_1 X_z + \alpha_2 y_z)\right)\frac{1-|\psi|^2}{1+|\psi|^2}\psi \quad (17)$$

Next, for the other unknown variables $X(z,t)$ and $y(z,t)$ one can find the new system of equations

$$m_1 X_{tt} = 2ak_1 X_{zz} + \frac{\alpha_1}{2}\left\{\frac{1+|\psi|^4-6|\psi|^2}{(1+|\psi|^2)^2}\right\}_z \tag{18}$$

$$m_2 y_{tt} = 2ak_2 y_{zz} + \frac{\alpha_2}{2}\left\{\left(\frac{1-|\psi|^2}{1+|\psi|^2}\right)^2\right\}_z \tag{19}$$

Let us analyze these two last equations. By applying the same procedure due to Vasumathi and Daniel in their paper [8] we denote $X_z = W(z,t)$ and $y_z = Q(z,t)$ and further we will look forward for traveling wave solutions by changing the variables$\sigma = z - vt$. After some algebra and integrating twice the new obtained equation for the variable σ and making the both constant of integration equal to zero we have obtained the next system

$$W = \frac{\alpha_1}{2(m_1 v^2 - 2k_1)}\frac{1+|\psi|^4-6|\psi|^2}{(1+|\psi|^2)^2} \tag{20}$$

$$Q = \frac{\alpha_2}{2(m_2 v^2 - 2k_2)}\left(\frac{1-|\psi|^2}{1+|\psi|^2}\right)^2 \tag{21}$$

Replacing the equations (20) and (21) in to the generalized nonlinear Schrödinger equation Eq. (17) one obtains

$$i\psi_t = -\frac{Ja}{2}\psi_{zz} + aJ\frac{\psi_z^2\overline{\psi}}{1+|\psi|^2} + \tag{22}$$
$$+\left(\frac{2\mu}{a} - \beta\frac{1+|\psi|^4-6|\psi|^2}{(1+|\psi|^2)^2} - \gamma\left(\frac{1-|\psi|^2}{1+|\psi|^2}\right)^2\right)\frac{1-|\psi|^2}{1+|\psi|^2}\psi$$

with the following parameter values

$$\beta = \frac{2\alpha_1^2}{m_1 v^2 - 2ak_1}, \gamma = \frac{\alpha_2^2}{2(m_2 v^2 - 2ak_2)} \tag{23}$$

Further, we will make several simplifications in order to find analytical solutions. First of all we will make a parametric change $z = s\sqrt{\frac{Ja}{2}}$. Second, we will assume that in the first approximation the parameter μ that is the exchange integral between strands and the separation of nucleotides satisfies the strong inequality μ/a<<1. Thus, after these restrictions the nonlinear generalized Schrödinger equation is transformed to the following saturable one:

$$i\psi_t = -\psi_{ss} + \frac{2\mu}{a}\left(\frac{1-|\psi|^2}{1+|\psi|^2}\right)\psi - \left(\beta\frac{1+|\psi|^4-6|\psi|^2}{(1+|\psi|^2)^2} + \gamma\left(\frac{1-|\psi|^2}{1+|\psi|^2}\right)^2\right)\frac{1-|\psi|^2}{1+|\psi|^2}\psi \tag{24}$$

5. Compactons for Hydrogen Bond Displacements

For the case when the external interaction disappears when $\alpha_1 = 0$ and $\alpha_2 = 0$, applying the relations (23) to the equation (24) gives us a new reduced nonlinear Schrodinger type equation that can be solved numerically for instance. The corresponding solutions for displacements of hydrogen bonds and for displacements inside the protein interacting molecule, can be found and they are no more than plain waves that was reported in the work [8]. Similar nonlinear equation appears also in the Heisenberg ferromagnetic theory [9].

Let us now analyze the equation (24) for the interesting case of weakly saturating approximation. We proceed here in the same way as in nonlinear optics and as in Bose-Einstein condensation theory [9, 18]. We have in the equation (24) terms proportional to $G = F(I)(1+I)^{-1}$ with $I = |\psi|^2$. Therefore, for the resulting equation not lose saturation properties, we will use the expansion $G = F(I)(1-I)$. Next, we make the following redefinition $\frac{\mu}{a} \to \mu$. Making such approximation after some algebra we obtain the Cubic-Quintic Nonlinear Schrödinger Equation (CQNSE):

$$i\psi_t + \psi_{ss} + \kappa_1\psi + \kappa_3|\psi|^2\psi - \kappa_5|\psi|^4\psi = 0 \tag{25}$$

with the parameter relations

$$\kappa_1 = -2\mu + \beta + \gamma, \kappa_3 = 4\mu - 10\beta - 6\gamma, \kappa_5 = 4\mu - 34\beta - 18\gamma \tag{26}$$

with $\kappa_3\kappa_5 < 0$, for ensuring saturation of nonlinearity. The values of β and γ are defined through the equations (23). As is well known, the CQNES does not belong to the class of integrable nonlinear equation in the sense of Lax pairs treatment or Hirota methods for finding its complete set of solutions. Instead, there are important particular solutions that can be used for analyzing the nonlinear excitations along the DNA macromolecule. For doing this, let us make a transformation for a better treatment of the equation (32) in such a way that we can distinguish the vacuum states of the CQNSE. It can be easily confirmed by some calculation, that the model's "potential" has several vacuum states. The term "vacuum" does not mean a state in the Hilbert space, but rather it is a classical configuration with minimal energy. Thus, if the field configuration has finite energy, the solutions of the equation of motion that lead to asymptotic values of $\psi(x)$ have to coincide with minimum potentials. Making the following change of variables and parameter transformations

$$x = \eta s, \tau = \varsigma t, \psi = \varrho\Phi(x,\tau)$$

being η, ς, ϱ constants. After some calculation we are able to find a new version of the CQNSE with explicit vacuum states inside the equation,

$$i\Phi_\tau + \Phi_{xx} - (3|\Phi|^2 - (2A + \rho_0))(|\Phi|^2 - \rho_0)\Phi = 0 \tag{27}$$

However, the new parameters $A, \rho_0, \kappa_1, \kappa_3, \kappa_5$ have to satisfy the relations

$$\varsigma = \eta^2 = \frac{1}{3}\kappa_5\varrho^4, \varrho^2 = \frac{3}{2(A+2\rho_0)}\frac{\kappa_3}{\kappa_5}, \frac{A}{\rho_0} = -2 - \frac{3}{4}\frac{\kappa_3^2}{\kappa_1\kappa_5}\left(1 \pm \sqrt{1 + 4\frac{\kappa_1\kappa_5}{\kappa_3^2}}\right) \tag{28}$$

Without lost of generality we can fix the value of $\rho_0 = 1$, this because, the solution properties depend explicitly only on the combination on A/ρ_0. The parameter A can acquire positive or negative values depending on the signs of parameter κ_i, with $i = 1, 5$ defined by the relations Eq. (26).

For obtaining some information on the DNA properties let us take the regular solutions obtained for the case when $A > 1$. Thus, for the case of trivial or drop boundary condition when $\psi \to 0$ whether $x \to \pm\infty$ the corresponding solution of the CQNSE is a non topological traveling drop soliton similar to that obtained in the book [9].

$$\Phi_d = e^{i(\frac{v}{2}x - \frac{v^2}{4}\tau + \theta_0)}(-4B)^{\frac{1}{2}} \left[1 + \sqrt{1 + \frac{16}{3}B} \cosh\left(\sqrt{-B}\,(x - v\tau - x_0)\right)\right]^{-\frac{1}{2}} \tag{29}$$

with

$$B = -\frac{3}{4}\frac{1+2A}{(A+2)^2} = \frac{\kappa_1\kappa_5}{\kappa_3^2}$$

Next, it is easy to check that when $B \to -\frac{3}{16}$ the integral of motion $N = \int dx\, |\psi|^2$, increases indefinitely and the soliton amplitude will turn to become a constant $\Phi \to \frac{\sqrt{3}}{2}$. This fact applied to the DNA molecular dynamics should imply that a collective wave of bubble type forming a certain open state for the angle deviations ϕ and ϕ' defined by $|\psi|^2 = (tan(\theta/2)^2$ should propagate along the chain due to the interaction of the DNA molecule with thermal phonons and protein surrounding molecule. This because the collective wave is constructed by the two types of drop solutions for ψ and for ξ field variables correspondingly. For the existence of this type of solitary wave solutions the parameter B have to satisfy the restriction $-3/16 \leq B \leq 0$, that consequently leads to the parameter restriction $A > 4$. An interesting traveling shape could be created when $B \to -\frac{3}{16}$. This structure will tend to maintain a certain constant amplitude along a appreciable segment of DNA in such a manner that along the DNA chain should be created a cigarette like bubble, for the angle θ and ϕ' deviations.

For calculating the displacements X along the hydrogen bonds we can use the analytical solution (28) and replace it into the equation (20) and integrate once with respect to the variable $\zeta = x - v\tau$. After integration we obtain a traveling solution for the displacements $X(\zeta)$ with an arbitrary constant of integration X_0. This constant of integration can be obtained by some initial condition imposed to the nonlinear equation (26). Further, we will also apply the same boundary conditions that has been applied for the drop soliton solutions founded above. This means that far from the zone of excitation $z \to \pm\infty$ the displacements tend to be zero. It has been performed the integration and we were able to found a compacton like structure see the Fig. 1 for displacements along the hydrogen bonds in the following form

$$X(\zeta) = X(x - v\tau), \text{for } -\zeta_0 \leq \zeta \leq \zeta_0 \tag{30}$$

$$X(\zeta) = 0, \text{elsewhere}$$

This solution represents a pair of traveling compacton and anti-compacton structures fused together. Let us remainder that the positive values of hydrogen displacements represent the separation of the two strands of DNA molecule, while the negative ones, represent the compression, the stretching of these bonds along the DNA chain. Specifically, in the sector of negative values of $\zeta \leq 0$ we have a compact solution, and in the region $\zeta \geq 0$ the anti-compacton lives, see Fig. 1. We can observe an interesting behavior of this solution, that we resume as follows. When this pair solution travels along the chain, its compacton part forms open states in opposite to the anti-compacton part that kills this deformation. This anti-compacton behaves like a solution that cure the open states; therefore, it could be named as "'healon". These fused antagonistic waves that following each other along the chain, resemble in certain manner the Cooper pairs profusely studied in condensed matter and field theories.

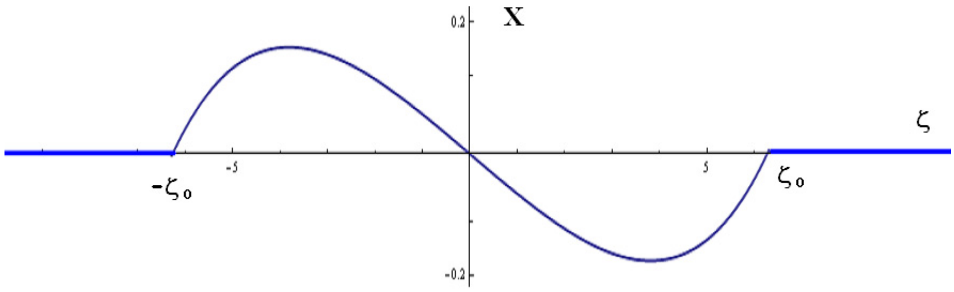

Fig. 1 Numerically integrated solution for the Eq. (26) that takes non-vanishing values X on the segment $[-\zeta_0, \zeta_0]$, beyond of this segment the displacement values vanish. The positive values of X represent the elongation of displacements along the hydrogen bonds, while the negative ones, represent the shrinking of that bonds.

Regarding the solution of Eq. (21) we can say a few words. We were unable to find physical interesting traveling solutions for this equation. Thus, in the protein that is interacting with DNA , could not be formed traveling waves along its structure, for this specific model, perhaps there could be another type of solutions not necessarily the traveling ones.

6. Conclusions

As is well known the relation of the quasi-classical field variable of the GCS approach ψ and the angle θ being the deviation of the angle of the classical spin $\mathbf{S}$ see Eq. (2) from the OZ axis is the following $|\psi|^2 = tan^2(\theta/2)$ [9]. Thus the using of the generalized coherent state approach, provides us directly with the possibility to calculate the angle deviation of the "classical" spin with respect to the chosen direction z.

Generally speaking, from the results obtained in this work we can infer the following. When the parameter "temperature" A is restricted to the segment $1 < A < 4$, the solutions with non vanishing boundary conditions, the rotational angle θ and θ' deviation bubbles, that collectively will build the stretching wave, appears along the DNA chain. Regarding the hydrogen bond displacements, we have been obtained crucial compacton anti-compacton pair solutions for both cases when the boundary conditions of two type were applied to the angle deviation θ and θ'. Depending on which collectively formed traveling solutions for angle deviation will be formed in certain segment of DNA, automatically the traveling compacton pair of positive and negative hydrogen bonds will be switched on. As soon as the nonlinear excitations in whichever segment of the DNA molecule is being activated by the presence of a protein for example, nonlinear waves of two types automatically could appear as a dual mechanism for maintaining the initial structure of the chain, indeed the nonlinear waves that tends to separate the both strands and their counterparts that tend to maintain them together. The last one could be interpreted as healing waves that cure the open states of the DNA segments. Similar pair of entities are profusely investigated in Physics.

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EJTP - **Zacatecas Proc. II** (2011) 157–162 *Electronic Journal of Theoretical Physics*

Discrete Symmetries for Truly Neutral Particles

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Abstract: We present a realization of a quantum field theory, envisaged many years ago by Gelfand, Tsetlin, Sokolik and Bilenky. Considering the special case of the $(1/2, 0) \oplus (0, 1/2)$ field and developing the Majorana construct for neutrino we show that a fermion and its antifermion can have the same properties with respect to the intrinsic parity (P) operation. The transformation laws for C and T operations have also been given. The construct can be applied to explanation of the present situation in neutrino physics. The case of the $(1, 0) \oplus (0, 1)$ field is also considered.

Keywords: Discrete Symmetries; Self/anti-self Charge Conjugate States; Relativistic Equation
PACS (2010): 03.65.Pm; 11.15.-q; 11.30.Cp; 11.30.Er

During the 20th century various authors introduced *self/anti-self* charge-conjugate 4-spinors (including in the momentum representation), see [1-4]. Later, Lounesto *et al*, Dvoeglazov, Kirchbach *etc* studied these spinors [5-8], they found dynamical equations, gauge transformations and other specific features of them. Recently, in [8] it was claimed that "for imaginary C parities, the neutrino mass can drop out from the single β decay trace and reappear in $0\nu\beta\beta$,... in principle experimentally testable signature for a non-trivial impact of Majorana framework in experiments with polarized sources" (see also Summary of the cited paper). Thus, phase factors can have physical significance in quantum mechanics. So, the aim of my talk is to remind what several researchers presented in the 90s concerning with the neutrino description.

We define the *self/anti-self* charge-conjugate 4-spinors in the momentum space[1]

$$C\lambda^{S,A}(p^\mu) = \pm\lambda^{S,A}(p^\mu)\,, C\rho^{S,A}(p^\mu) = \pm\rho^{S,A}(p^\mu)\,, \tag{1}$$

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[1] In [8] a bit different notation was used referring to [2].

where

$$\lambda^{S,A}(p^\mu) = \begin{pmatrix} \pm i\Theta\phi_L^*(p^\mu) \\ \phi_L(p^\mu) \end{pmatrix}, \rho^{S,A}(p^\mu) = \begin{pmatrix} \phi_R(p^\mu) \\ \mp i\Theta\phi_R^*(p^\mu) \end{pmatrix}. \tag{2}$$

The Wigner matrix is $\Theta_{[1/2]} = -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and ϕ_L, ϕ_R are the Ryder (Weyl) left- and right-handed 2-spinors

$$\phi_R(p^\mu) = \Lambda_R(\mathbf{p} \leftarrow \mathbf{0})\phi_R(\mathbf{0}) = \exp(+\sigma\cdot\varphi/2)\phi_R(\mathbf{0}), \tag{3}$$
$$\phi_L(p^\mu) = \Lambda_L\mathbf{p} \leftarrow \mathbf{0})\phi_L(\mathbf{0}) = \exp(-\sigma\cdot\varphi/2)\phi_L(\mathbf{0}), \tag{4}$$

with $\varphi = \mathbf{n}\varphi$ being the boost parameters:

$$cosh\varphi = \gamma = \frac{1}{\sqrt{1-v^2/c^2}}, \; sinh\varphi = \beta\gamma = \frac{v/c}{\sqrt{1-v^2/c^2}}, \; tanh\varphi = v/c. \tag{5}$$

As we have shown the 4-spinors λ and ρ are NOT the eigenspinors of helicity. Moreover, λ and ρ are NOT the eigenspinors of the parity $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} R$, as opposed to the Dirac case. Such definitions of 4-spinors differ, of course, from the original Majorana definition in x-representation:

$$\nu(x) = \frac{1}{\sqrt{2}}(\Psi_D(x) + \Psi_D^c(x)), \quad a_\sigma(\mathbf{p}) = \frac{1}{\sqrt{2}}(b_\sigma(\mathbf{p}) + d_\sigma^\dagger(\mathbf{p})), \tag{6}$$

$$\nu(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_p} \sum_\sigma [u_\sigma(\mathbf{p})a_\sigma(\mathbf{p})e^{-ip\cdot x} + v_\sigma(\mathbf{p})[\lambda a_\sigma^\dagger(\mathbf{p})]e^{+ip\cdot x}], \tag{7}$$

$C\nu(x) = \nu(x)$, that represents the positive real $C-$ parity field operator. However, the momentum-space Majorana-like spinors open various possibilities for description of neutral particles (with experimental consequences, see [8]).

The 4-spinors λ and ρ are NOT the eigenspinors of helicity. Moreover, λ and ρ are NOT the eigenspinors of the parity $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} R$, as opposed to the Dirac case. Such definitions of 4-spinors differ, of course, from the original Majorana definition in x-representation. They are eigenstates of the chiral helicity quantum number introduced in the 60s, $\eta = -\gamma^5 h$. While

$$Pu_\sigma(\mathbf{p}) = +u_\sigma(\mathbf{p}), Pv_\sigma(\mathbf{p}) = -v_\sigma(\mathbf{p}), \tag{8}$$

we have

$$P\lambda^{S,A}(\mathbf{p}) = \rho^{A,S}(\mathbf{p}), P\rho^{S,A}(\mathbf{p}) = \lambda^{A,S}(\mathbf{p}), \tag{9}$$

for the Majorana-like momentum-space 4-spinors on the first quantization level.

One can use the generalized form of the Ryder relation for zero-momentum spinors:

$$\left[\phi_L^h(\mathbf{0})\right]^* = (-1)^{1/2-h}\, e^{-i(\vartheta_1^L+\vartheta_2^L)}\, \Theta_{[1/2]}\, \phi_L^{-h}(\mathbf{0}), \tag{10}$$

in order to derive the dynamical equations [6]:

$$i\gamma^\mu\partial_\mu\lambda^S(x) - m\rho^A(x) = 0\,, \tag{11}$$
$$i\gamma^\mu\partial_\mu\rho^A(x) - m\lambda^S(x) = 0\,, \tag{12}$$
$$i\gamma^\mu\partial_\mu\lambda^A(x) + m\rho^S(x) = 0\,, \tag{13}$$
$$i\gamma^\mu\partial_\mu\rho^S(x) + m\lambda^A(x) = 0\,. \tag{14}$$

These are NOT the Dirac equations (cf. [9]). Similar formulation has been presented by A. Barut and G. Ziino [3]. The group-theoretical basis for such doubling has been first given in the papers by Gelfand, Tsetlin and Sokolik [10] and other authors. Hence, the Lagrangian is

$$\begin{aligned}\mathcal{L} = \frac{i}{2}\Big[&\bar\lambda^S\gamma^\mu\partial_\mu\lambda^S - (\partial_\mu\bar\lambda^S)\gamma^\mu\lambda^S + \bar\rho^A\gamma^\mu\partial_\mu\rho^A - (\partial_\mu\bar\rho^A)\gamma^\mu\rho^A + \\ &\bar\lambda^A\gamma^\mu\partial_\mu\lambda^A - (\partial_\mu\bar\lambda^A)\gamma^\mu\lambda^A + \bar\rho^S\gamma^\mu\partial_\mu\rho^S - (\partial_\mu\bar\rho^S)\gamma^\mu\rho^S\Big] - \\ &-m(\bar\lambda^S\rho^A + \bar\rho^A\lambda^S - \bar\lambda^A\rho^S - \bar\rho^S\lambda^A)\,.\end{aligned} \tag{15}$$

The connection with the Dirac spinors has been found. For instance [4, 6],

$$\begin{pmatrix}\lambda^S_\uparrow(p^\mu)\\ \lambda^S_\downarrow(p^\mu)\\ \lambda^A_\uparrow(p^\mu)\\ \lambda^A_\downarrow(p^\mu)\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & i & -1 & i\\ -i & 1 & -i & -1\\ 1 & -i & -1 & -i\\ i & 1 & i & -1\end{pmatrix}\begin{pmatrix}u_{+1/2}(p^\mu)\\ u_{-1/2}(p^\mu)\\ v_{+1/2}(p^\mu)\\ v_{-1/2}(p^\mu)\end{pmatrix}\,. \tag{16}$$

See also ref. [10, 3].

It was shown [6] that the covariant derivative (and, hence, the interaction) can be introduced in this construct in the following way:

$$\partial_\mu \to \nabla_\mu = \partial_\mu - ig\text{Ł}^5 B_\mu\,, \tag{17}$$

where $\text{Ł}^5 = \mathrm{diag}(\gamma^5 \quad -\gamma^5)$, the 8×8 matrix. With respect to the chiral phase transformations the spinors retain their properties to be self/anti-self charge conjugate spinors and the proposed Lagrangian [6, p.1472] remains to be invariant. This tells us that while self/anti-self charge conjugate states has zero eigenvalues of the ordinary (scalar) charge operator but they can possess the axial charge (cf. with the discussion of [3] and the old idea of R. E. Marshak and others).[2] Next, because the transformations

$$\lambda'_S(p^\mu) = \begin{pmatrix}\Xi & 0\\ 0 & \Xi\end{pmatrix}\lambda_S(p^\mu) \equiv \lambda^*_A(p^\mu) \quad , \tag{18}$$

$$\lambda''_S(p^\mu) = \begin{pmatrix}i\Xi & 0\\ 0 & -i\Xi\end{pmatrix}\lambda_S(p^\mu) \equiv -i\lambda^*_S(p^\mu) \quad , \tag{19}$$

$$\lambda'''_S(p^\mu) = \begin{pmatrix}0 & i\Xi\\ i\Xi & 0\end{pmatrix}\lambda_S(p^\mu) \equiv i\gamma^0\lambda^*_A(p^\mu) \quad , \tag{20}$$

$$\lambda^{IV}_S(p^\mu) = \begin{pmatrix}0 & \Xi\\ -\Xi & 0\end{pmatrix}\lambda_S(p^\mu) \equiv \gamma^0\lambda^*_S(p^\mu) \tag{21}$$

[2] In fact, from this consideration one can recover the Feynman-Gell-Mann equation (and its charge-conjugate equation).

with the 2×2 matrix Ξ defined as (ϕ is the azimuthal angle related to $\mathbf{p}\to\mathbf{0}$)

$$\Xi = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \quad , \quad \Xi \Lambda_{R,L}(0 \leftarrow p^\mu) \Xi^{-1} = \Lambda^*_{R,L}(0 \leftarrow p^\mu) \ , \tag{22}$$

and corresponding transformations for λ^A do *not* change the properties of bispinors to be in the self/anti-self charge conjugate spaces, the Majorana-like field operator ($b^\dagger \equiv a^\dagger$) admits additional phase (and, in general, normalization) $SU(2)$ transformations:

$$\nu^{ML\,\prime}(x^\mu) = [c_0 + i(\tau\cdot\mathbf{c})]\, \nu^{ML\,\dagger}(x^\mu) \quad , \tag{23}$$

where c_α are arbitrary parameters. The conclusion is: a non-Abelian construct is permitted, which is based on the spinors of the Lorentz group only (cf. with the old ideas of T. W. Kibble and R. Utiyama) . This is not surprising because both $SU(2)$ group and $U(1)$ group are the sub-groups of the extended Poincaré group (cf. [12]).

The Dirac-like and Majorana-like field operators can be built from both $\lambda^{S,A}(p^\mu)$ and $\rho^{S,A}(p^\mu)$, or their combinations. The anticommutation relations are the following ones (due to the *bi-orthonormality*):

$$[a_{\eta'}(p'^\mu), a^\dagger_\eta(p^\mu)]_\pm = (2\pi)^3 2E_p \delta(\mathbf{p}-\mathbf{p}')\delta_{\eta,-\eta'} \tag{24}$$

and

$$[b_{\eta'}(p'^\mu), b^\dagger_\eta(p^\mu)]_\pm = (2\pi)^3 2E_p \delta(\mathbf{p}-\mathbf{p}')\delta_{\eta,-\eta'} \tag{25}$$

Other (anti)commutators are equal to zero: ($[a_{\eta'}(p'^\mu), b^\dagger_\eta(p^\mu)] = 0$).

In the Fock space the operations of the charge conjugation and space inversions can be defined through unitary operators. The time reversal operation should be defined through *an antiunitary* operator. We further assume the vacuum state to be assigned the even P- and C-eigenvalue and, then, proceed as in ref. [13]. As a result we have very different properties with respect to the space inversion operation, comparing with the Dirac states (the case was also regarded in [3]):

$$U^s_{[1/2]}|\mathbf{p},\uparrow>^+ = +i|-\mathbf{p},\downarrow>^+, U^s_{[1/2]}|\mathbf{p},\uparrow>^- = +i|-\mathbf{p},\downarrow>^- \tag{26}$$

$$U^s_{[1/2]}|\mathbf{p},\downarrow>^+ = -i|-\mathbf{p},\uparrow>^+, U^s_{[1/2]}|\mathbf{p},\downarrow>^- = -i|-\mathbf{p},\uparrow>^- \tag{27}$$

For the charge conjugation operation in the Fock space we have two physically different possibilities. The first one, in fact, has some similarities with the Dirac construct. The action of this operator on the physical states are

$$U^c_{[1/2]}|\mathbf{p},\uparrow>^+ = +\,|\mathbf{p},\uparrow>^- \,,\, U^c_{[1/2]}|\mathbf{p},\downarrow>^+ = +\,|\mathbf{p},\downarrow>^- \,, \tag{28}$$

$$U^c_{[1/2]}|\mathbf{p},\uparrow>^- = -\,|\mathbf{p},\uparrow>^+ \,,\, U^c_{[1/2]}|\mathbf{p},\downarrow>^- = -\,|\mathbf{p},\downarrow>^+ \ . \tag{29}$$

But, one can also construct the charge conjugation operator in the Fock space which acts, *e.g.*, in the following manner:

$$\widetilde{U}^c_{[1/2]}|\mathbf{p},\uparrow>^+ = -\,|\mathbf{p},\downarrow>^- \,,\, \widetilde{U}^c_{[1/2]}|\mathbf{p},\downarrow>^+ = -\,|\mathbf{p},\uparrow>^- \,, \tag{30}$$

$$\widetilde{U}^c_{[1/2]}|\mathbf{p},\uparrow>^- = +\,|\mathbf{p},\downarrow>^+ \,,\, \widetilde{U}^c_{[1/2]}|\mathbf{p},\downarrow>^- = +\,|\mathbf{p},\uparrow>^+ \ . \tag{31}$$

This is due to corresponding algebraic structures of self/anti-self charge-conjugate spinors. Finally, the time reversal *anti-unitary* operator in the Fock space should be defined in such a way that the formalism to be compatible with the CPT theorem. We obtain for the $\Psi(x^\mu)$:

$$V^T a_\uparrow^\dagger(\mathbf{p})(V^T)^{-1} = a_\downarrow^\dagger(-\mathbf{p})\,,\; V^T a_\downarrow^\dagger(\mathbf{p})(V^T)^{-1} = -a_\uparrow^\dagger(-\mathbf{p})\,, \tag{32}$$
$$V^T b_\uparrow(\mathbf{p})(V^T)^{-1} = b_\downarrow(-\mathbf{p})\,,\; V^T b_\downarrow(\mathbf{p})(V^T)^{-1} = -b_\uparrow(-\mathbf{p})\,. \tag{33}$$

In the $(1,0)\oplus(0,1)$ representation space one can define the $\Gamma^5 C$ self/anti-self charge conjugate 6-component objects.

$$\Gamma^5 C_{[1]}\lambda(p^\mu) = \pm\lambda(p^\mu)\,, \tag{34}$$
$$\Gamma^5 C_{[1]}\rho(p^\mu) = \pm\rho(p^\mu)\,. \tag{35}$$

The $C_{[1]}$ matrix is constructed from dynamical equations for charged spin-1 particles. No self/anti-self charge-conjugate states are possible. They are also NOT the eigenstates of the parity operator (except for $\lambda_\rightarrow$):

$$P\lambda_\uparrow^S = +\lambda_\downarrow^S\,, P\lambda_\rightarrow^S = -\lambda_\rightarrow^S\,, P\lambda_\downarrow^S = +\lambda_\uparrow^S\,, \tag{36}$$
$$P\lambda_\uparrow^A = -\lambda_\downarrow^A\,, P\lambda_\rightarrow^A = +\lambda_\rightarrow^A\,, P\lambda_\downarrow^A = +\lambda_\uparrow^A\,. \tag{37}$$

The dynamical equations are

$$\gamma_{\mu\nu}p^\mu p^\nu \lambda_{\uparrow\downarrow}^S - m^2\lambda_{\downarrow\uparrow}^S = 0, \quad \gamma_{\mu\nu}p^\mu p^\nu \lambda_{\uparrow\downarrow}^A + m^2\lambda_{\downarrow\uparrow}^A = 0\,, \tag{38}$$
$$\gamma_{\mu\nu}p^\mu p^\nu \lambda_{\rightarrow}^S + m^2\lambda_{\rightarrow}^S = 0, \quad \gamma_{\mu\nu}p^\mu p^\nu \lambda_{\rightarrow}^A - m^2\lambda_{\rightarrow}^A = 0\,. \tag{39}$$

On the secondary quantization level we obtained similar results as in the spin-1/2 case.

The conclusions are: The momentum-space Majorana -like spinors are considered in the $(j,0)\oplus(0,j)$ representation space. They have different properties from the Dirac spinors even on the classical level. It is convenient to work in the 8-dimensional space. Then, we can impose the Gelfand-Tsetlin-Sokolik (Bargmann-Wightman-Wigner) prescription of 2-dimensional representation of the inversion group. Gauge transformations are different. The axial charge is possible. Experimental differencies have been recently discussed (the possibility of observation of the phase factor/eigenvalue of the C-parity), see [8]. (Anti)commutation relations are assumed to be different from the Dirac case (and the $2(2j+1)$ case) due to the bi-orthonormality of the states (the spinors are self-orthogonal). The $(1,0)\oplus(0,1)$ case has also been considered. The $\Gamma^5 C$-self/anti-self conjugate objects have been introduced. The results are similar to the $(1/2,0)\oplus(0,1/2)$ representation. The 12-dimensional formalism was introduced. The field operator can describe both charged and neutral states.

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EJTP - **Zacatecas Proc. II** (2011) 163–170 *Electronic Journal of Theoretical Physics*

Abstracts of the Talks not Included in the Proceedings

Proceedings of the 2010 Zacatecas Workshop on Mathematical Physics II, México, December 2010

Speakers: Alfredo Aranda; Eloy Ayón-Beato; Alexander Balankin; Aldrin Cervantes; Miguel Ángel Cruz; Valeriy Dvoeglazov; Jaime Keller; Julio López-Domínguez; Merced Montesinos; Carlos Ignacio Pérez; Efraín Rojas; José Socorro; Luis Alberto Torres; José Antonio Vallejo; José David Vergara; and Victor Villanueva.

Keywords: Mathematical Physics; Symmetry

Simetrías y masa
Alfredo Aranda

Comentaré sobre el uso de simetrías para tratar de entender los patrones observados en las masas y ángulos de mezcla de los fermiones del Modelo Estándar. Presentaré brevemente las generalidades de las principales propuestas, así como algunas ideas recientes que estamos tratando de explorar.

Analytic Lifshitz black holes in higher dimensions
Eloy Ayón-Beato

We generalize the four dimensional $\mathbb{R}^2$-corrected $z = 3/2$ Lifshitz black hole to a two-parameter family of black hole solutions with asymptotic Lifshitz symmetry for any dynamical exponent z and for any dimension D. For a particular relation between the parameters, we find the first example of an extremal Lifshitz black hole. An asymptotically Lifshitz black hole with a logarithmic decay is also exhibited for a specific critical exponent depending on the dimension. Additionally, we show how to generalize the two-parameter family of black holes to include other horizon topologies. We extend this analysis to the more general quadratic curvature corrections for which we present three new families of higher-dimensional $D \geq 5$ analytic Lifshitz black holes for generic z. One of these higher-dimensional families contains as critical limits the $z = 3$ three-dimensional Lifshitz black hole and a new $z = 6$ four-dimensional black hole. The variety of analytic solutions presented here encourages to explore these gravity models within the context of non-relativistic holographic correspondence.

Geometry and Physics of random folding
Alexander Balankin
Folded configurations of thin matter are very common in nature, ranging form the microscopic level-folded proteins and nanoparticle membranes to the macroscopic level folded paper and fault-related geological formations. In mathematics, Riemann has used a crumpled ball of paper with bookworms to explain the hidden dimensions in non-Euclidean geometry. In this presentation we discuss the topological and geometrical properties of folded configurations and the physics of folding. Special attention is paid to the fractal geometry and the thermodynamics of randomly folded thin sheets.

Superficies relativistas con cota en la aceleración
Aldrin Cervantes
Proponemos un modelo efectivo, geométrico, para describir la dinámica de objetos extendidos con una cota en la aceleración, los cuales evolucionan en un espacio-tiempo tipo Minkowski. El modelo efectivo involucra a la curvatura extrínseca de la trayectoria generada por el objeto durante su evolución. El Lagrangiano que describe ésta teoría es de segundo orden en las derivadas y en consecuencia las ecuaciones de movimiento son de cuarto orden en las coordenadas. Mostramos que en el caso de codimensión uno, las ecuaciones de movimiento semeja una ecuación tipo Klein-Gordon. Por ilustración, se estudia la dinámica de una $(3+1)$-superficie esférica, observando su expansión acelerada donde se nota una cota en dicha aceleración. Describiremos su posible aplicación al contexto de la cosmología en dimensiones extra como una alternativa dinámica para explicar la expansión acelerada del Universo.

Ecuación de Friedmann para un modelo cosmológico rígido
Miguel Ángel Cruz
Se propone un modelo cosmológico en el contexto de dimensiones extras con una corrección lineal en la traza de la curvatura. Se obtiene la correspondiente ecuación de Friedmann y se estudia de manera general el potencial clásico asociado a éste modelo.

The Bargmann-Wigner formalism for higher spins (up to 2)
Valeriy Dvoeglazov
On the basis of our recent modifications of the Dirac formalism we generalize the Bargmann-Wigner formalism for higher spins to be compatible with other formalisms for bosons. Relations with dual electrodynamics, with the Ogievetskii-Polubarinov notoph and the Weinberg $2(2s+1)$ theory are found. Next, we proceed to derive the equations for the symmetric tensor of the second rank on the basis of the Bargmann-Wigner formalism in a strightforward way. The symmetric multispinor of the fourth rank is used. It is constructed out of the Dirac 4-spinors. Due to serious problems with the interpretation of the results obtained on using the standard procedure we generalize it and obtain the

spin-2 relativistic equations, which are consistent with the previous one. We introduce the dual analogues of the Riemann tensor and derive corresponding dynamical equations in the Minkowski space. Relations with the Marques-spehler chiral gravity theory are discussed. The importance of the 4-vector field (and its gauge part) is pointed out.

Radiation and matter-like properties of 4-dimensional elasticity
Jaime Keller

We construct a mathematical theory of matter in space-time. The basic feature is to consider a special 4-dimensional elastic medium where a screw dislocation in the additional (fifth) coordinate causes a field of action over space-time. The properties of this dislocations are the mathematical model for elementary particles. The generated fields are linear in the sources and their squares are proportional to the energy stored in those fields. The waves in 3-D space, in the model, are the propagation of the properties of this field of action and carry energy, momentum and energy-momentum besides the information on the type of matter generating dislocations. If two screw dislocations approach each other their respective fields a and b add up and an interaction energy results from the additional terms in the square of the sum: the energy related to the interaction is the difference between $(a+b)^2$ and a^2+b^2, difference that can be negative for attractive interactions, or positive for repulsive interaction.

Generalización supersimétrica del agujero negro de Schwarzschild
Julio López-Domínguez

La ecuacin de Wheeler-DeWitt para el modelo cosmolgico de Kantowski-Sachs puede ser entendida como una ecuacin cuantica para el agujero negro de Schwarschild, debido al difehomorfismo que existe entre las dos soluciones a las ecuaciones de Relatividad General. En este trabajo se supersimetriza la ecuacin de Wheeler-DeWitt aplicando el metodo de Graham y se realiza una aproximacion semiclasica tipo WKB y asi obtener una solucin para el lmite clasico supersimetrico. Se analizan algunas propiedades de esta solucin clasica supersimetrica.

Boundary terms and Dirac constraints
Merced Montesinos

Time boundary terms usually added to action principles are systematically handled in the framework of Diracs canonical analysis. The procedure begins with the introduction of the boundary term into the integral action and then the resulting action is interpreted as a Lagrangian one to which Diracs method is applied. Once the general theory is developed, the current procedure is implemented and illustrated in various examples which are originally endowed with different types of constraints.

$SU(2)$ **Monopoles and braids**
Carlos Ignacio Pérez

In the realm of classical gauge theories, we review some topological (mainly homotopy theoretical) invariants of the critical space of Yang-Mills functional with group $SU(2)$. The space of time invariant antiselfdual and selfdual connections (monopoles), i.e., the critical space of Yang-Mills-Higgs functional, is also considered and a relationship with Artins braid groups is finally described.

Ostrogradski Hamiltonian approach for geodetic brane gravity
Efraín Rojas
We present an alternative Hamiltonian description of a branelike universe immersed in a flat background spacetime. This model is named geodetic brane gravity. We set up the Regge- Teitelboim model to describe our Universe where such field theory is originally thought as a second order derivative theory. We refer to an Ostrogradski Hamiltonian formalism to prepare the system to its quantization. This approach comprize the manage of both first- and second-class constraints and the counting of degrees of freedom follows accordingly.

Campos escalares en la evolución del Universo: teorías
José Socorro
Se presenta la cuestión de porque hay que incluir campos escalares para estudiar la evolución del Universo, de acuerdo a los datos observacionales. Se mencionan algunas teorías autoconsistentes que contienen de manera natural estos campos, los cuales son incluídos para resolver problemas particulares de la cosmología moderna.

Modelo cosmológico y las modificaciones a la gravitación
Luis Alberto Torres
Se considera que el modelo cosmológico actual tiene buenos cimientos en el panorama de la relatividad general, en el camino se han ido incorporando ingredientes extra, por ejemplo la presencia de la constante cosmológica o la inflación como un modelo de condiciones iniciales en lo que a perturbaciónes cosmológicas se refiere. Se dará un bosquejo de la tendencia a poder extender la teoría de gravitación, desde recetas con perspectiva fenomenológica (MOND), hasta la extensión del sector de curvatura (gravedades $f(R)$) pasando por invitaciones de campos escalares en el camino. Intentaré hacer énfasis en la posibilidad de vincular modificaciones a la gravitación con observaciones astrofísicas vinculadas preferentemente a problemas como formación de estructura y energía oscura.

Cuantización: de Dirac a Fedosov
José Antonio Vallejo
El problema de la cuantización (esto es, describir el paso de un sistema dinámico clásico a su equivalente cuántico mediante un conjunto de reglas universales) es de fácil enunciado, difícil solución y además, en la práctica no tiene ninguna utilidad. Estas características lo hacen irresistible. En la plática haremos un repaso de los métodos de cuantización que se han propuesto desde el nave de Dirac al más sofisticado de Fedosov-Kontsevich, examinando sus aciertos, sus fallos y, sobre todo, los interesantes conceptos a los que han

dado lugar. Sólo en la última parte (los últimos 10 minutos) veremos algo más técnico: la aplicación de las ideas de Fedosov al caso de las supervariedades y la conexión que este problema tiene con otros que han aparecido recientemente en Física.

Teorías de orden superior y no conmutatividad
José David Vergara
Se muestra que partiendo de una teoría con derivadas temporales de orden superior es posible obtener una teoría no conmutativa. Para establecer claramente esta relación se utiliza un modelo mecánico del tipo Chern-Simmons el cual presenta inicialmente no conmutatividad en las velocidades pero no en las coordenadas. Sin embargo, al cuantizar el modelo presenta estados de norma negativa, para evitar el problema se utiliza un método perturbativo el cual elimina los estados de norma negativa y a su vez crea la no conmutatividad en las coordenadas.

Cuantización de Espín 3/2: Más allá de Rarita-Schwinger con mecanismo de Higgs
Victor Villanueva

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