

Energy conditions in $f(R, G)$ gravity

K. Atazadeh · F. Darabi

Received: 26 August 2013 / Accepted: 5 January 2014 / Published online: 17 January 2014
© Springer Science+Business Media New York 2014

Abstract Modified gravity is one of the most favorable candidates for explaining the current accelerating expansion of the Universe. In this regard, we study the viability of an alternative gravitational theory, namely $f(R, G)$, by imposing energy conditions. We consider two forms of $f(R, G)$, commonly discussed in the literature, which account for the stability of cosmological solutions. We construct the inequalities obtained by energy conditions and specifically apply the weak energy condition using the recent estimated values of the Hubble, deceleration, jerk and snap parameters to probe the viability of the above-mentioned forms of $f(R, G)$.

Keywords Energy conditions · $f(R, G)$ modified gravity · Cosmological parameters

1 Introduction

As is well known in general relativity, the energy conditions are often required in the proofs of various important theorems about black holes, such as no hair theorem or laws of black hole thermodynamics [1]. The energy conditions are systematically obtained when one refers to the Raychaudhuri equation, in which the attractive character of gravity is reflected through the positivity condition $R_{\mu\nu}k^\mu k^\nu \geq 0$, where $R_{\mu\nu}$ is the Ricci tensor and k^μ is any null vector. In general relativity, through the Einstein field equation, this condition on geometry is translated to a corresponding condition on matter, namely null energy condition $T_{\mu\nu}k^\mu k^\nu \geq 0$, where $T_{\mu\nu}$ is the stress energy tensor. In particular, the weak energy condition (WEC) assumes that the local energy density is positive and states that $T_{\mu\nu}U^\mu U^\nu \geq 0$, for all timelike vectors U^μ . For instance, in the case of a perfect fluid

K. Atazadeh (✉) · F. Darabi
Department of Physics, Azarbaijan Shahid Madani University, 53714-161 Tabriz, Iran
e-mail: atazadeh@azaruniv.ac.ir

F. Darabi
e-mail: f.darabi@azaruniv.ac.ir

$$T_{\mu\nu} = (\rho + p)U^\mu U^\nu + pg_{\mu\nu}, \quad (1)$$

we have $\rho > 0$ and $\rho + p \geq 0$. By continuity, the WEC implies the null energy condition (NEC), $T_{\mu\nu}k^\mu k^\nu \geq 0$ [1].

The energy conditions have been widely studied in the context of modified gravity, such as $f(R)$, $f(G)$, $f(R, T)$ and $f(T)$ gravity [2–16]. The motivation for the study of modified theories of gravity is clear and strong. As we know, Einstein's theory of general relativity (GR) can not explain the late-time accelerated expansion of the Universe, unless an unknown dark energy element is introduced in the field equations [17–23]. In the context of GR, the attempts to explain the observed speed-up of the Universe, have introduced some modifications of Einstein–Hilbert action by adopting a general function of the scalar curvature in the gravitational Lagrangian density as $f(R)$ [24–39]. The motivation for this procedure consists of the analysis of strong gravitational fields near the curvature singularities and considering the consistent candidates of a fundamental theory of quantum gravity. Indeed, string/M-theory predicts that scalar field couplings with the Gauss–Bonnet invariant G are important in the appearance of non-singular early time cosmologies. These motivations may also be considered in the context of late-time acceleration of the Universe [40–43].

Recently, a new generalized modified Gauss–Bonnet gravity, whose action contains a general function of R and G as $f(R, G)$, has attracted considerable attention [44–60]. Besides its stability, this attention is due to its ability to describe the present acceleration of the universe as well as the phantom divide line crossing and transition from acceleration to deceleration phases. Two specific models of $f(R, G)$ gravity were constructed to account for the late-time cosmic acceleration [61], and the respective constraints of the parameters of the models were also analyzed in [61]. So, in order to proceed along with the interests of these models, in the present work we shall consider two forms of $f(R, G)$ introduced in [61]. In Ref. [61] the authors have studied the stability of de Sitter and power-law solutions in $f(R, G)$ gravity and have shown that gravitational action plays a very important role in the stability of the solutions both of which depending on the form of the $f(R, G)$ theory and the parameters of the model. In this context, we further consider the constraints imposed by the energy conditions and verify whether the parameter range of the proposed models considered in [61] are consistent with the energy conditions. More specifically, we define generalized energy conditions for $f(R, G)$ modified theories of gravity, and consider their realization for flat Friedmann cosmological models. In particular, we analyze whether the weak energy condition is satisfied by particular choices of $f(R, G)$ which were advocated in Ref. [61].

In Sect. 2, we introduce the gravitational field equations for the $f(R, G)$ gravity. In Sect. 3, we obtain the inequalities corresponding to the energy conditions. In Sect. 4, we consider two specific forms of $f(R, G)$, and analyze the constraints resulting from the energy conditions. The paper ends with a brief conclusions in Sect. 5. We use the units $c = G_N = 1$.

2 Field equations of $f(R, G)$ modified gravity

Let us start by writing the most general action for modified Gauss–Bonnet gravity

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(R, G) + S_M(g^{\mu\nu}, \psi), \tag{2}$$

where $S_M(g^{\mu\nu}, \psi)$ is the matter action, and $f(R, G)$ is a function of the Ricci scalar and Gauss–Bonnet invariant defined by

$$G \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}. \tag{3}$$

Variation of the action (2) with respect to the metric provides the following gravitational field equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu}^{(\text{mat})} + \Sigma_{\mu\nu}, \tag{4}$$

where $\Sigma_{\mu\nu}$ is defined by

$$\begin{aligned} \Sigma_{\mu\nu} = & \nabla_\mu \nabla_\nu f_R - g_{\mu\nu} \square f_R + 2R \nabla_\mu \nabla_\nu f_G - 2g_{\mu\nu} R \square f_G - 4R_\mu^\lambda \nabla_\lambda \nabla_\nu f_G \\ & - 4R_\nu^\lambda \nabla_\lambda \nabla_\mu f_G + 4R_{\mu\nu} \square f_G + 4g_{\mu\nu} R^{\alpha\beta} \nabla_\alpha \nabla_\beta f_G + 4R_{\mu\alpha\beta\nu} \nabla^\alpha \nabla^\beta f_G \\ & - \frac{1}{2} g_{\mu\nu} V + (1 - f_R) \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right). \end{aligned} \tag{5}$$

Note that

$$f_R \equiv \frac{\partial f(R, G)}{\partial R}, \quad f_G \equiv \frac{\partial f(R, G)}{\partial G}, \tag{6}$$

$V \equiv f_R R + f_G G - f(R, G)$, and $T_{\mu\nu}^{(\text{mat})}$ is the stress energy tensor describing the ordinary matter.

Now, we consider the flat FRW metric

$$ds^2 = -dt^2 + a^2(t) d\mathbf{x}^2, \tag{7}$$

where $a(t)$ is the scale factor. In the FRW background with a perfect fluid equation of state for ordinary matter, the field equations for $f(R, G)$ gravity are given by

$$\begin{aligned} f_R \dot{H} = & -\frac{\kappa}{2} (p^{(m)} + \rho^{(m)}) + \frac{1}{2} (H \dot{f}_R - \ddot{f}_R + 4H^3 \dot{f}_G - 8H \dot{H} \dot{f}_G - 4H^2 \ddot{f}_G), \\ f_R H^2 = & \frac{\kappa}{3} \rho^{(m)} + \frac{1}{6} (f_R R - f - 6H \dot{f}_R + G f_G - 24H^3 \dot{f}_G). \end{aligned} \tag{8}$$

where ρ and p are the energy density and pressure of ordinary matter, respectively, and the overdot denotes a derivative with respect to the time coordinate, t . Moreover, we have

$$R = 6 \left(2H^2 + \dot{H} \right), \tag{9}$$

$$G = 24H^2 \left(H^2 + \dot{H} \right), \tag{10}$$

and the gravitational field equations may be rewritten in the following form

$$\rho_{\text{eff}} = \frac{3}{\kappa} H^2, \quad p_{\text{eff}} = -\frac{1}{\kappa} (2\dot{H} + 3H^2), \tag{11}$$

where ρ_{eff} and p_{eff} are the effective energy density and pressure, respectively, defined by

$$\rho_{\text{eff}} = \frac{1}{f_R} \left[\rho^{(m)} + \frac{1}{2\kappa} (f_R R - f - 6H\dot{f}_R + Gf_G - 24H^3\dot{f}_G) \right], \tag{12}$$

$$p_{\text{eff}} = \frac{1}{f_R} \left[p^{(m)} + \frac{1}{\kappa} \left(2H\dot{f}_R + \ddot{f}_R + 8H^3\dot{f}_G + 8H\dot{H}\dot{f}_G + 4H^2\ddot{f}_G - \frac{1}{2}(Rf_R + Gf_G - f) \right) \right]. \tag{13}$$

Combining the above equations, we obtain the following useful relationship

$$\rho_{\text{eff}} + p_{\text{eff}} = \frac{1}{f_R} \left[\rho^{(m)} + p^{(m)} + \frac{1}{\kappa} \left(-H\dot{f}_R - 4H^3\dot{f}_G + \ddot{f}_R + 8H\dot{H}\dot{f}_G + 4H^2\ddot{f}_G \right) \right],$$

which will be used throughout the paper.

3 Energy conditions

In general, the energy conditions arise when one considers the Raychaudhuri equation given by

$$\frac{d\theta}{d\tau} = -\frac{1}{2}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}k^\mu k^\nu, \tag{14}$$

where θ , $\sigma^{\mu\nu}$ and $\omega_{\mu\nu}$ are the expansion, shear and rotation, respectively, associated with the congruences defined by the null vector field k^μ .

From Raychaudhuri equation it is seen that for spatial shear tensor with $\sigma^2 \equiv \sigma_{\mu\nu}\sigma^{\mu\nu} \geq 0$, and for any hypersurface orthogonal congruences, which imposes $\omega_{\mu\nu} \equiv 0$, the condition for attractive gravity, namely $d\theta/d\tau < 0$ reduces to $R_{\mu\nu}k^\mu k^\nu \geq 0$. In general relativity, using the Einstein field equations one can rewrite the above condition in terms of the stress-energy tensor given by $T_{\mu\nu}k^\mu k^\nu \geq 0$. However, in any other theory of gravity such as $f(R, G)$, one should know how to replace $R_{\mu\nu}$ in terms of $T_{\mu\nu}$, using the corresponding field equations.

Equation (4) may be written in the following effective form

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = T_{\mu\nu}^{\text{eff}}, \tag{15}$$

where the effective stress-energy tensor is given by

$$T_{\mu\nu}^{\text{eff}} = \kappa T_{\mu\nu}^{(\text{mat})} + \Sigma_{\mu\nu}. \tag{16}$$

The positivity condition $R_{\mu\nu}k^\mu k^\nu \geq 0$, through the modified gravitational field Eq. (15), provides the following form for the null energy condition

$$T_{\mu\nu}^{\text{eff}}k^\mu k^\nu \geq 0. \tag{17}$$

Moreover, it is plausible to impose the condition $T_{\mu\nu}^{(\text{mat})}k^\mu k^\nu \geq 0$ for ordinary matter, because it implies that the energy density of ordinary matter is positive in all local frames of references.

To deduce the energy conditions in the context of $f(R, G)$ modified gravity, we use the modified (effective) gravitational field equations and obtain the energy conditions as follows

$$\text{NEC} \iff \rho_{\text{eff}} + p_{\text{eff}} \geq 0, \tag{18}$$

$$\text{WEC} \iff \rho_{\text{eff}} \geq 0 \text{ and } \rho_{\text{eff}} + p_{\text{eff}} \geq 0, \tag{19}$$

$$\text{SEC} \iff \rho_{\text{eff}} + 3p_{\text{eff}} \geq 0 \text{ and } \rho_{\text{eff}} + p_{\text{eff}} \geq 0, \tag{20}$$

$$\text{DEC} \iff \rho_{\text{eff}} \geq 0 \text{ and } \rho_{\text{eff}} \pm p_{\text{eff}} \geq 0, \tag{21}$$

where the notation NEC, WEC, SEC and DEC stand for the null, weak, strong and dominant energy conditions, respectively.

In cosmological context, in addition to the Hubble parameter $H = \dot{a}/a$, it is appropriate to define the deceleration, jerk, and snap parameters as

$$q = -\frac{1}{H^2} \frac{\ddot{a}}{a}, \quad j = \frac{1}{H^3} \frac{\dddot{a}}{a}, \quad \text{and} \quad s = \frac{1}{H^4} \frac{\overset{\cdot\cdot\cdot}{a}}{a}, \tag{22}$$

respectively; in terms of which, we may consider the following definitions

$$\dot{H} = -H^2(1 + q), \tag{23}$$

$$\ddot{H} = H^3(j + 3q + 2), \tag{24}$$

$$\dddot{H} = H^4(s - 2j - 5q - 3), \tag{25}$$

respectively. By using (23)–(25) one may rewrite Eqs. (12) and (13) in the explicit forms, as follows

$$\begin{aligned} \rho_{\text{eff}} = \frac{1}{f_R} & \left[\rho^{(m)} + \frac{1}{2\kappa} \left(-24H^4 q f_G - 576H^8 f_{GG} (j + 2q^2 + 3q) \right. \right. \\ & - 1728H^6 f_{GR} (j + q^2 + q - 1) - 36H^4 f_{RR} (j - q - 2) \\ & \left. \left. + 6H^2(1 - q)f_R - f \right) \right], \tag{26} \end{aligned}$$

$$\begin{aligned} p_{\text{eff}} = \frac{1}{f_R} & \left[p^{(m)} + \frac{1}{\kappa} \left[576 (2q^2 + 3q + j)^2 (4f_{GGG}H^2 + f_{GGR}) H^{10} \right. \right. \\ & \left. \left. + 24(-2j(3q + 2) - q(2q(q + 6) + 5) + s + 3) (4f_{GG}H^2 + f_{GR}) H^6 \right] \right] \end{aligned}$$

$$\begin{aligned}
 &+ 144(j - q - 2) \left(2q^2 + 3q + j \right) \left(8f_{GGR}H^2 + 2f_{RRG} \right) H^6 \\
 &+ 36(j - q - 2)^2 \left(4f_{RRG}H^2 + f_{RRR} \right) H^6 \\
 &+ 24 \left(2q^2 + 3q + j \right) \left(8f_{GG}H^3 - 8(q + 1)f_{GR}H^3 + 2f_{GR}H \right) H^5 \\
 &+ 6 \left(4q^2 + 15q + 2j + s + 9 \right) \left(4f_{GG}H^2 + f_{RR} \right) H^4 \\
 &+ 6(j - q - 2) \left(-8(q + 1)f_{GR}H^3 + 8f_{GR}H^3 + 2f_{RR}H \right) H^3 \\
 &+ \frac{1}{2} \left(24qf_GH^4 - 6(1 - q)f_RH^2 + f \right) \Big] \Big], \tag{27}
 \end{aligned}$$

where $\rho^{(m)}$ and $p^{(m)}$ are the matter energy density and pressure, respectively. To derive the above equations we have replaced for \dot{f}_R , \dot{f}_G , \ddot{f}_G and \ddot{f}_R in Eqs. (12) and (13), for example as

$$\begin{aligned}
 \dot{f}_R(R, G) &= f_{RR}\dot{R} + f_{RG}\dot{G} = 6(4H\dot{H} + \ddot{H})f_{RR} \\
 &+ (96H^3\dot{H} + 48H\dot{H}^2 + 24H^2\ddot{H})f_{RG}. \tag{28}
 \end{aligned}$$

Using these definitions, the energy conditions (18)–(21) take on the following respective forms

NEC :

$$\begin{aligned}
 \rho_{\text{eff}} + p_{\text{eff}} &= \rho^{(m)} + p^{(m)} + \frac{1}{\kappa} \left[24H^6(-2j(3q + 2) - q(2q(q + 6) + 5) + s + 3) \right. \\
 &\times \left(4H^2f_{GG} + f_{GR} \right) + 24H^5(j + q(2q + 3)) \left(8H^3f_{GG} - 2Hf_{GR} \left(4H^2(q + 1) - 1 \right) \right) \\
 &- 288H^8f_{GG}(j + q(2q + 3)) + 6H^4 \left(2j + 4q^2 + 15q + s + 9 \right) \left(4H^2f_{GG} + f_{RR} \right) \\
 &+ 576H^{10}(j + q(2q + 3))^2 \left(4H^2f_{GGG} + f_{GGR} \right) \\
 &+ 288H^6(j - q - 2)(j + q(2q + 3)) \left(4H^2f_{GGR} + f_{RRG} \right) \\
 &- 864H^6f_{GR} \left(j + q^2 + q - 1 \right) - 12H^3(j - q - 2) \left(4H^3qf_{GR} - Hf_{RR} \right) \\
 &\left. - 18H^4f_{RR}(j - q - 2) + 36H^6(-j + q + 2)^2 \left(4H^2f_{RRG} + f_{RRR} \right) \right] \geq 0, \tag{29}
 \end{aligned}$$

WEC :

$$\begin{aligned}
 \rho_{\text{eff}} &= \rho^{(m)} + \frac{1}{2\kappa} \left[-f(R, G) - 24H^4qf_G - 576H^8f_{GG} \left(j + 2q^2 + 3q \right) \right. \\
 &\left. - 24^2H^6f_{GR} \left(j + q^2 + q - 1 \right) - 36H^4f_{RR}(j - q - 2) + 6H^2(1 - q)f_R \right] \geq 0, \\
 \rho_{\text{eff}} + p_{\text{eff}} &\geq 0, \tag{30}
 \end{aligned}$$

SEC :

$$\begin{aligned}
 \rho_{\text{eff}} + 3p_{\text{eff}} &= \rho^{(m)} + 3p^{(m)} + \frac{1}{\kappa} [3(0.5(24H^4qf_G + 6H^2(q - 1)f_R + f) \\
 &+ 24H^6(-2j(3q + 2) - q(2q(q + 6) + 5) + s + 3)(4H^2f_{GG} + f_{GR})
 \end{aligned}$$

$$\begin{aligned}
 &+ 24H^5(j + q(2q + 3))(8H^3 f_{GG} - 2Hf_{GR}(4H^2(q + 1) - 1)) \\
 &+ 6H^4(2j + q(4q + 15) + s + 9)(4H^2 f_{GG} + f_{RR}) \\
 &+ 576H^{10}(j + q(2q + 3))^2(4H^2 f_{GGG} + f_{GGR}) \\
 &+ 288H^6(j - q - 2)(j + q(2q + 3))(4H^2 f_{GGR} + f_{RRG}) \\
 &- 12H^4(j - q - 2)(4H^2 qf_{GR} - f_{RR}) + 36H^6(-j + q + 2)^2(4H^2 f_{RRG} + f_{RRR}) \\
 &+ \frac{1}{2}(6H^2(-4H^2 qf_G - 96H^6 f_{GG}(j + q(2q + 3)) - 288H^4 f_{GR}(j + q^2 + q - 1) \\
 &+ 6H^2 f_{RR}(-j + q + 2) - (q - 1)f_R - f)] \geq 0, \quad \rho_{\text{eff}} + p_{\text{eff}} \geq 0, \tag{31}
 \end{aligned}$$

DEC :

$$\begin{aligned}
 \rho_{\text{eff}} - p_{\text{eff}} = &\rho^{(m)} - p^{(m)} - \frac{1}{\kappa} \left[24H^4 qf_G - 24H^6(-2j(3q + 2) \right. \\
 &- q(2q(q + 6) + 5) + s + 3) \left(4H^2 f_{GG} + f_{GR} \right) \\
 &- 24H^5(j + q(2q + 3)) \left(8H^3 f_{GG} - 2Hf_{GR} \left(4H^2(q + 1) - 1 \right) \right) \\
 &- 288H^8 f_{GG}(j + q(2q + 3)) - 6H^4(2j + q(4q + 15) + s + 9) \left(4H^2 f_{GG} + f_{RR} \right) \\
 &- 576H^{10}(j + q(2q + 3))^2 \left(4H^2 f_{GGG} + f_{GGR} \right) \\
 &- 288H^6(j - q - 2)(j + q(2q + 3)) \left(4H^2 f_{GGR} + f_{RRG} \right) \\
 &- 864H^6 f_{GR} \left(j + q^2 + q - 1 \right) + 12H^4(j - q - 2) \left(4H^2 qf_{GR} - f_{RR} \right) \\
 &+ 18H^4 f_{RR}(-j + q + 2) - 6H^2(q - 1)f_R \\
 &\left. - 36H^6(-j + q + 2)^2 \left(4H^2 f_{RRG} + f_{RRR} \right) - f \right] \geq 0, \quad \rho_{\text{eff}} + p_{\text{eff}} \geq 0, \quad \rho_{\text{eff}} \geq 0. \tag{32}
 \end{aligned}$$

4 Constraints on $f(R, G)$ gravity

In this section, we consider the viable $f(R, G)$ modified theories of gravity which were used in [61] to study the stability of cosmological solutions. Stability of power-law solutions for $f(R, G)$ models were given by expressions (33) and (34), bellow. In the cases where no fluids are considered (vacuum) it is seen that stability of the cosmological solutions can be achieved by appropriate choices of the parameters space [61]. However, in this section we consider the constraints imposed by the energy conditions and verify the consistency between the parameters ranges of the models considered in [61] and the energy conditions in $f(R, G)$ flat Friedman cosmological models obtained here as (29)–(32).

In this regard, we consider two classes of viable $f(R, G)$ given by [61]

$$f_1(R, G) = \mu R^\beta G^\gamma, \tag{33}$$

$$f_2(R, G) = k_1 R + k_2 R^n G^m, \tag{34}$$

where $k_1, k_2, \mu, n, m, \beta,$ and γ are constants and n, β are assumed to be positive. The Ricci scalar and Gauss–Bonnet invariant, defined in Eqs. (9) and (10) in terms of the Hubble and deceleration parameters, can also be expressed as

$$R = 6H^2(1 - q), \tag{35}$$

$$G = -24H^4q, \tag{36}$$

respectively.

Since the inequalities (29)–(32) imposed by the energy conditions in $f(R, G)$ gravity are so lengthy, for simplicity we only consider the WEC in the following analysis. Moreover, we take the following observed values for the deceleration, jerk and snap parameters [62, 63]: $q_0 = -0.81 \pm 0.14, j_0 = 2.16^{+0.81}_{-0.75},$ and $s_0 = -0.22^{+0.21}_{-0.19}.$

4.1 $f_1(R, G) = \mu R^\beta G^\gamma$

To impose energy conditions on $f_1(R, G)$ gravity, for simplicity we consider the vacuum, i.e., $\rho^{(m)} = p^{(m)} = 0.$ Hence, the WEC constraints for the vacuum, $\rho_{\text{eff}}^{(v)} \geq 0$ and $\rho_{\text{eff}}^{(v)} + p_{\text{eff}}^{(v)} \geq 0,$ are given respectively by

$$2^{\beta+3\gamma-1} 3^{\beta+\gamma} (-H^2(q-1))^\beta (-H^4q)^\gamma \left[\frac{(-j+q+2)(\beta-1)\beta}{(q-1)^2} - \frac{12(q^2+q+j-1)\gamma\beta}{(q-1)q} + \beta - \frac{(j+q(2q+3))(\gamma-1)\gamma}{q^2} + \gamma - 1 \right] \mu \geq 0, \tag{37}$$

$$2^{\beta+3\gamma-3} 3^{\beta+\gamma-1} (-H^2(q-1))^\beta (-H^4q)^\gamma \times \left[-\frac{4(\beta-1)\beta(q(\beta+\gamma-2)-\gamma)(-j+q+2)^2}{(q-1)^3q} + \frac{12(\beta-1)\beta(-j+q+2)}{(q-1)^2} - \frac{8\beta(\beta-q\gamma+\gamma-1)(-j+q+2)}{(q-1)^2} + \frac{8(j+q(2q+3))\beta\gamma(-\gamma+q(\beta+\gamma-2)+1)(-j+q+2)}{H^2(q-1)^2q^2} - \frac{144(q^2+q+j-1)\beta\gamma}{(q-1)q} - \frac{12(j+q(2q+3))(\gamma-1)\gamma}{q^2} - \frac{8(j+q(2q+3))\gamma(q((4H^2(q+1)-1)\beta-\gamma+1)+\gamma-1)}{(q-1)q^2} + \frac{(2j+q(4q+15)+s+9)((\gamma-1)\gamma(q-1)^2+4H^2q^2(\beta-1)\beta)}{H^2(q-1)^2q^2} - \frac{4(j+q(2q+3))^2(\gamma-1)\gamma(-\gamma+q(\beta+\gamma-2)+2)}{(q-1)q^3} - \frac{4(j(6q+4)+q(2q(q+6)+5)-s-3)\gamma(-\gamma+q(\beta+\gamma-1)+1)}{(q-1)q^2} \right] \mu \geq 0. \tag{38}$$

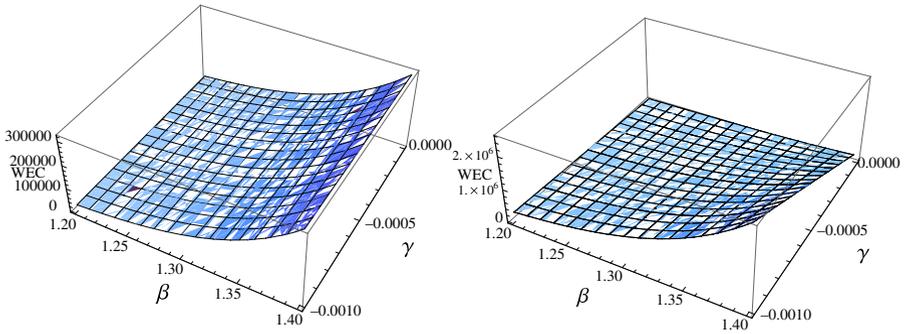


Fig. 1 Plots of the weak energy condition for the specific form of $f_1(R, G) = \mu R^\beta G^\gamma$. The left and right plots correspond respectively to $\rho_{\text{eff}}^{(v)} \geq 0$ and $\rho_{\text{eff}}^{(v)} + p_{\text{eff}}^{(v)} \geq 0$. The positivity requirement of the weak energy condition is satisfied in the plots for the parameter range considered

The exact analytical expressions for the parameter ranges of the constants β and γ can not be explicitly obtained because the constraints provided by the inequalities (37)–(38) are so complex, so we consider specific values for some of the parameters. In [61] the authors have shown that the vacuum solutions are stable for $\beta > 1$ and $\gamma < \frac{1}{8}(5 - 4\beta)$. To verify this result from the energy condition’s point of view, we plot the WEC as a function of β and γ , as is depicted in Fig. 1. It is seen that in the left (right) figure, the weak energy condition $\rho_{\text{eff}}^{(v)} \geq 0$ ($\rho_{\text{eff}}^{(v)} + p_{\text{eff}}^{(v)} \geq 0$) is satisfied for the parameter ranges $\beta \geq 1$ and $\gamma < 0$, in the specific form of $f_1(R, G)$ gravity by Eq. (33).

In order to compare with the particular case $f(R)$, we put $\gamma = 0$ in Eq. (33). We find that Eq. (33) can be reduced to $f_1(R) = \mu R^\beta$, which is a class of $f(R)$ -gravity. The energy conditions of such theories have been studied in [13–16]. Therefore, considering $\gamma = 0$ in the current model the WEC constraints, i.e., $\rho_{\text{eff}}^{(v)} \geq 0$ and $\rho_{\text{eff}}^{(v)} + p_{\text{eff}}^{(v)} \geq 0$, are given respectively by

$$\frac{1}{(q - 1)^2} \left[2^{\beta-1} 3^\beta \left(-H^2(q - 1) \right)^\beta (\beta - 1) \left((q - 1)^2 + (-j + q + 2)\beta \right) \right] \geq 0, \tag{39}$$

$$-\frac{1}{(q - 1)^3} \left[6^{\beta-1} \left(-H^2(q - 1) \right)^\beta (\beta - 1)\beta \left((\beta - 2)j^2 + 3(q + 3)j - 2(q + 2)\beta j + s - q(2q(2q + 7) + s + 3) + q(q + 4)\beta + 4\beta + 3 \right) \right] \geq 0. \tag{40}$$

For more understanding the above relations, we have plotted them in terms of β . Obviously Fig. 2 shows that the WEC is valuable for $\beta \geq 1$, and this is in agreement with result obtained in [13–16].

4.2 $f_2(R, G) = k_1 R + k_2 R^n G^m$

In this case, as in the previous one, we consider the vacuum, i.e., $\rho^{(m)} = p^{(m)} = 0$. This modified gravity model accounts for the stability of the cosmological solutions

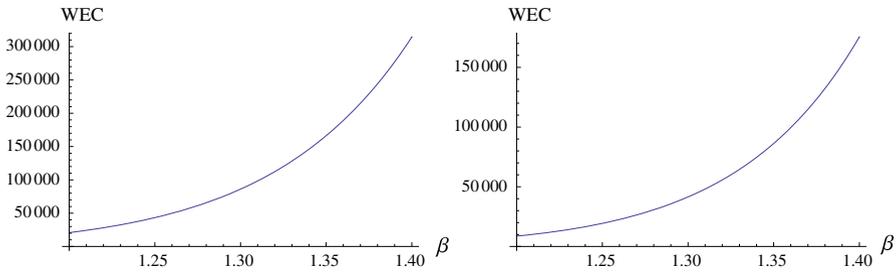


Fig. 2 The plots depict the weak energy condition for the specific form of $f_1(R) = \mu R^\beta$. The left plot corresponds to $\rho_{\text{eff}}^{(v)} \geq 0$; the right plot corresponds to $\rho_{\text{eff}}^{(v)} + p_{\text{eff}}^{(v)} \geq 0$

for late-time cosmic acceleration, and the stability of solutions depends on the values of the coupling constants k_1 and k_2 . In fact, for $k_1 > 2 \left(\frac{1152}{25}\right)^{3/2} k_2$ the perturbations grow exponentially, and the de Sitter solution becomes unstable. For other values of k_1 and k_2 than mentioned above, the perturbations behave as damped oscillations decaying to zero, hence the solution becomes stable. To verify whether the parameter range of the stability of solutions are consistent with the energy conditions, we again consider the weak energy condition.

For the form of $f_2(R, G)$ considered by Eq. (34), the WEC constraints, i.e., $\rho_{\text{eff}}^{(v)} \geq 0$ and $\rho_{\text{eff}}^{(v)} + p_{\text{eff}}^{(v)} \geq 0$, are given by

$$\begin{aligned}
 & -\frac{2^{3m+n-1}3^{m+n}k_2}{(q-1)^2q^2} \left[\left(-H^2(q-1)\right)^n \left(-H^4q\right)^m \left(j\left(m^2(q-1)^2\right.\right. \right. \\
 & \quad \left. \left. + m((12n-1)q+1)(q-1) + (n-1)nq^2\right) + q\left(m^2(2q+3)(q-1)^2\right.\right. \\
 & \quad \left. \left. + 3m\left(-q^2+4n\left(q^2+q-1\right)+1\right)(q-1) - (n-1)q\left((q-1)^2+n(q+2)\right)\right) \right] \geq 0, \tag{41}
 \end{aligned}$$

$$\begin{aligned}
 & 2^{3m+n-3}3^{m+n-1}(-H^2(q-1))^n(-H^4q)^mk_2 \\
 & \times \left[-\frac{4(n-1)n((m+n-2)q-m)(-j+q+2)^2}{(q-1)^3q} \right. \\
 & \quad \left. + \frac{8mn(-m+(m+n-2)q+1)(j+q(2q+3))(-j+q+2)}{H^2(q-1)^2q^2} \right. \\
 & \quad \left. + \frac{12(n-1)n(-j+q+2)}{(q-1)^2} - \frac{4(m-1)m(-m+(m+n-2)q+2)(j+q(2q+3))^2}{(q-1)q^3} \right. \\
 & \quad \left. + \frac{8n(j-q-2)(-qm+m+n-1)}{(q-1)^2} - \frac{144mn(q^2+q+j-1)}{(q-1)q} \right. \\
 & \quad \left. - \frac{12(m-1)m(j+q(2q+3))}{q^2} \right. \\
 & \quad \left. - \frac{8m(j+q(2q+3))(-qm+m+q+nq(4H^2(q+1)-1)-1)}{(q-1)q^2} \right]
 \end{aligned}$$

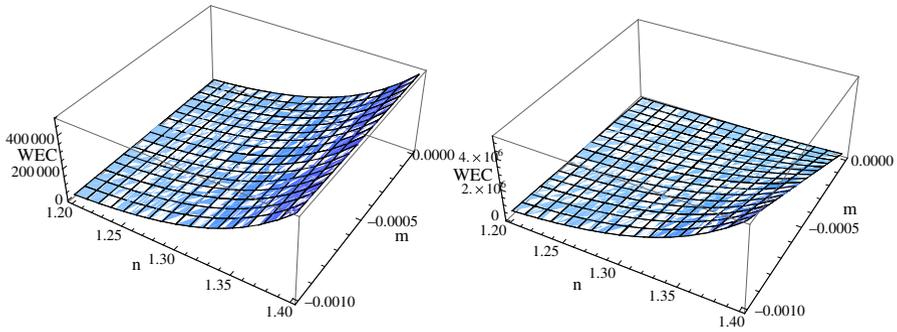


Fig. 3 The plots depict the weak energy condition for the specific form of $f_2(R, G) = k_1 R + k_2 R^n G^m$. The left plot corresponds to $\rho_{\text{eff}}^{(v)} \geq 0$; the right plot corresponds to $\rho_{\text{eff}}^{(v)} + p_{\text{eff}}^{(v)} \geq 0$

$$\begin{aligned}
 & \frac{4m(-m + (m + n - 1)q + 1)(j(6q + 4) + q(2q(q + 6) + 5) - s - 3)}{(q - 1)q^2} \\
 & + \left. \frac{(m^2(q - 1)^2 - m(q - 1)^2 + 4H^2(n - 1)nq^2)(2j + q(4q + 15) + s + 9)}{H^2(q - 1)^2q^2} \right] \geq 0, \tag{42}
 \end{aligned}$$

respectively.

It is seen that the weak energy condition dose not depend on k_1 . Therefore, it turns out that WEC do not interfere with the stability conditions. As in the previous example, considering the complex constraints provided by the inequalities (41)–(42), finding exact analytical expressions for the parameter ranges of the constants m and n is not an easy task. Hence, we consider specific values for the parameters to find WEC condition as a function of the parameters m and n . It is seen in the left (right) Fig. 3 that the weak energy condition $\rho_{\text{eff}}^{(v)} \geq 0$ ($\rho_{\text{eff}}^{(v)} + p_{\text{eff}}^{(v)} \geq 0$) is satisfied for the parameter ranges $n \geq 1$ and $m < 0$, in the specific form of $f_2(R, G)$ gravity by Eq. (34).

Again, in order to compare with the particular case $f(R)$ we put $m = 0$ in Eq. (34) to obtain $f_2(R) = k_1 R + k_2 R^n$. The energy conditions of such theories have been studied in [13–16]. Thus, considering $m = 0$ in our model, the WEC constraints, i.e., $\rho_{\text{eff}}^{(v)} \geq 0$ and $\rho_{\text{eff}}^{(v)} + p_{\text{eff}}^{(v)} \geq 0$, are given respectively by

$$\frac{1}{(q - 1)^2} \left[2^{n-1} 3^n (n - 1) \left(-H^2(q - 1) \right)^n \left((q - 1)^2 + n(-j + q + 2) \right) \right] \geq 0, \tag{43}$$

$$\begin{aligned}
 & - \frac{1}{(q - 1)^3} \left[6^{n-1} (n^2 - n) \left(-H^2(q - 1) \right)^n \left((n - 2)j^2 + (3(q + 3) - 2n(q + 2))j \right. \right. \\
 & \left. \left. + n(q + 2)^2 + s - q(2q(2q + 7) + s + 3) + 3 \right) \right] \geq 0. \tag{44}
 \end{aligned}$$

We have plotted Eqs. (43) and (44) in terms of β . Figure 4 shows that the WEC is valuable for $n \geq 1$, and this is in agreement with the results obtained in [13–16].

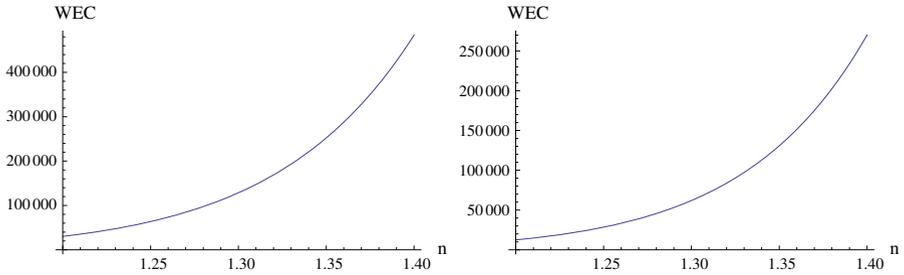


Fig. 4 The plots depict the weak energy condition for the specific form of $f_2(R) = k_1 R + k_2 R^n$. The left plot corresponds to $\rho_{\text{eff}}^{(v)} \geq 0$; the right plot corresponds to $\rho_{\text{eff}}^{(v)} + p_{\text{eff}}^{(v)} \geq 0$

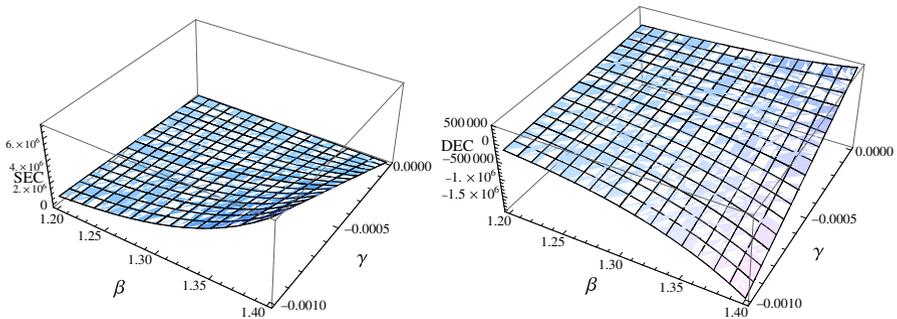


Fig. 5 The plots depict the strong and dominant energy conditions for the specific form of $f_1(R, G) = \mu R^\beta G^\gamma$. The left plot corresponds to SEC; the right plot corresponds to DEC

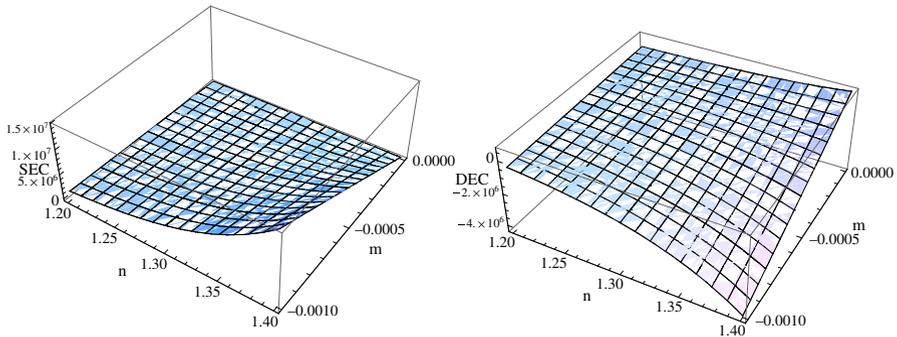


Fig. 6 The plots depict the strong and dominant energy conditions for the specific form of $f_2(R, G) = k_1 R + k_2 R^n G^m$. The left plot corresponds to SEC; the right plot corresponds to DEC

To complete our discussions, we have plotted SEC and DEC in the Figs. 5 and 6 for two suggested $f(R, G)$ models (33) and (34). From these figures it can be seen that in both models SEC holds but DEC does not. This is in agreement with the results obtained previously in the study of energy conditions in $f(R)$ gravity [13–16].

Note that for simplicity, we have examined the vacuum case $p^{(m)} = \rho^{(m)} = 0$. If we add regular matter to our models, the general results of the paper will not change,

because with no loss of generality we may always add a positive energy density or pressure of matter satisfying the WEC to the vacuum case such that $\rho_{\text{eff}} \geq 0$, $\rho_{\text{eff}} + p_{\text{eff}} \geq 0$.

5 Conclusions

In this paper, we have studied the viability of an interesting alternative gravitational theory, namely, $f(R, G)$ gravity. We have considered two realistic forms of $f(R, G)$, analyzed in the literature, accounting for the late-time cosmic acceleration and the stability of the cosmological solutions [61]. The general inequalities imposed by the energy conditions are obtained. To be specific and for simplicity, we have focused on the weak energy condition and used the recent estimated values of the Hubble, deceleration, jerk and snap parameters. We have shown the consistency of the above-mentioned forms of $f(R, G)$ with the weak energy condition.

We have just examined the vacuum case for which $p^{(m)} = \rho^{(m)} = 0$. Actually this is not a physically relevant case bearing in mind that the universe contains matter. This simplification, however, does not change the general results of the paper, if we add matter to our models. This is because we can add a positive energy density or pressure of regular matter satisfying the WEC ($\rho^{(m)} \geq 0$ and $\rho^{(m)} + p^{(m)} \geq 0$) to the vacuum case ($\rho^{(v)} \geq 0$ and $\rho^{(v)} + p^{(v)} \geq 0$) such that $\rho_{\text{eff}} \geq 0$, $\rho_{\text{eff}} + p_{\text{eff}} \geq 0$, where $\rho_{\text{eff}} = \rho^{(m)} + \rho^{(v)}$ and $p_{\text{eff}} = p^{(m)} + p^{(v)}$.

Acknowledgments This work has been supported financially by Research Institute for Astronomy and Astrophysics of Maragha (RIAAM) under research project No. 1/2782-58.

References

1. Hawking, S.W., Ellis, G.F.R.: The Large Scale Structure of Spacetime. Cambridge University Press, Cambridge, MA (1973)
2. Kung, J.H.: Phys. Rev. D **52**, 6922 (1995)
3. Kung, J.H.: Phys. Rev. D **53**, 3017 (1996)
4. Perez, S.E.: Bergliaffa. Phys. Lett. B **642**, 311 (2006)
5. Santos, J., Alcaniz, J.S.: Phys. Lett. B **619**, 11 (2005)
6. Visser, M.: Science **276**, 88 (1997)
7. Visser, M.: Phys. Rev. D **56**, 7578 (1997)
8. Santos, J., Alcaniz, J.S., Rebouças, M.J.: Phys. Rev. D **74**, 067301 (2006)
9. Santos, J., Alcaniz, J.S., Pires, N., Rebouças, M.J.: Phys. Rev. D **75**, 083523 (2007)
10. Santos, J., Alcaniz, J.S., Rebouças, M.J., Pires, N.: Phys. Rev. D **76**, 043519 (2007)
11. Atazadeh, K., Khaleghi, A., Sepangi, H.R., Tavakoli, Y.: Int. J. Mod. Phys. D **18**, 1101 (2009)
12. Sharif, M., Zubair, M.: J. Phys. Soc. Jpn. **82**, 014002 (2013)
13. Santos, J., Alcaniz, J.S., Rebouças, M.J., Carvalho, F.C.: Phys. Rev. D **76**, 083513 (2007)
14. García, N.M., Harko, T., Lobo, F.S.N., Mimoso, J.P.: Phys. Rev. D **83**, 104032 (2011)
15. Liu, D., Rebouças, M.J.: Phys. Rev. D **86**, 083515 (2012)
16. Alvarenga, F.G., Houndjo, M.J.S., Monwanou, A.V., Chabi Orou, J.B.: J. Mod. Phys. **4**, 130 (2013)
17. Perlmutter, S., et al.: Astrophys. J. **517**, 565 (1999)
18. Riess, A.G., et al.: Astron. J. **116**, 1009 (1998)
19. Riess, A.G., et al.: Astrophys. J. **607**, 665 (2004)
20. Grant, A., et al.: Astrophys. J. **560**, 49–71 (2001)
21. Perlmutter, S., Turner, M.S., White, M.: Phys. Rev. Lett. **83**, 670–673 (1999)
22. Bennett, C.L., et al.: Astrophys. J. Suppl. **148**, 1 (2003)
23. Copeland, E.J., Sami, M., Tsujikawa, S.: Int. J. Mod. Phys. D **15**, 1753 (2006)
24. De Felice, A., Tsujikawa, S.: Living Rev. Relativ. **13**, 3 (2010)

25. Buchdahl, H.A.: *Mon. Not. Roy. Astron. Soc.* **150**, 1 (1970)
26. Kerner, R.: *Gen. Relativ. Gravit.* **14**, 453 (1982)
27. Duruisseau, J.P., Kerner, R., Eysseric, P.: *Gen. Relativ. Gravit.* **15**, 797 (1983)
28. Barrow, J.D., Ottewill, A.C.: *J. Phys. A: Math. Gen.* **16**, 2757 (1983)
29. Starobinsky, A.A.: *Phys. Lett. B* **91**, 99 (1980)
30. Sokolowski, L.M.: *Class. Quant. Gravit.* **24**, 3391 (2007)
31. Cognola, G., Gastaldi, M., Zerbini, S.: *Int. J. Theor. Phys.* **47**, 898 (2008)
32. Böhmer, C.G., Hollenstein, L., Lobo, F.S.N.: *Phys. Rev. D* **76**, 084005 (2007)
33. Carloni, S., Dunsby, P.K.S., Troisi, A.: *Phys. Rev. D* **77**, 024024 (2008)
34. Ananda, K.N., Carloni, S., Dunsby, P.K.S.: *Phys. Rev. D* **77**, 024033 (2008)
35. Olmo, G.J.: *Phys. Rev. D* **75**, 023511 (2007)
36. Capozziello, S., Cardone, V.F., Troisi, A.: *JCAP* **0608**, 001 (2006)
37. Capozziello, S., Cardone, V.F., Troisi, A.: *Mon. Not. R. Astron. Soc.* **375**, 1423 (2007)
38. Borowiec, A., Godlowski, W., Szydłowski, M.: *Int. J. Geom. Meth. Mod. Phys.* **4**, 183 (2007)
39. Martins, C.F., Salucci, P.: *Mon. Not. Roy. Astron. Soc.* **381**, 1103 (2007)
40. Nojiri, S., Odintsov, S.D., Sasaki, M.: *Phys. Rev. D* **71**, 123509 (2005)
41. Nojiri, S., Odintsov, S.D.: *Int. J. Geom. Meth. Mod. Phys.* **4**, 115 (2007)
42. Cognola, G., Elizalde, E., Nojiri, S., Odintsov, S.D., Zerbini, S.: *Phys. Rev. D* **73**, 084007 (2006)
43. Nojiri, S., Odintsov, S.D., Gorbunova, O.G.: *J. Phys. A: Math. Gen.* **39**, 6627 (2006)
44. Nojiri, S., Odintsov, S.D., Sami, M.: *Phys. Rev. D* **74**, 046004 (2006)
45. Nojiri, S., Odintsov, S.D.: *J. Phys. Conf. Ser.* **66**, 012005 (2007)
46. Cognola, G., Elizalde, E., Nojiri, S., Odintsov, S., Zerbini, S.: *Phys. Rev. D* **75**, 086002 (2007)
47. Li, B., Barrow, J.D., Mota, D.F.: *Phys. Rev. D* **76**, 044027 (2007)
48. De Felice, A., Tsujikawa, S.: *Phys. Lett. B* **675**, 1 (2009)
49. Alimohammadi, M., Ghalee, A.: *Phys. Rev. D* **79**, 063006 (2009)
50. Boehmer, C.G., Lobo, F.S.N.: *Phys. Rev. D* **79**, 067504 (2009)
51. Uddin, K., Lidsey, J.E., Tavakol, R.: *Gen. Relativ. Gravit.* **41**, 2725 (2009)
52. Zhou, S.Y., Copeland, E.J., Saffin, P.M.: *JCAP* **0907**, 009 (2009)
53. Goheer, N., Goswami, R., Dunsby, P.K.S., Ananda, K.: *Phys. Rev. D* **79**, 121301 (2009)
54. Goheer, N., Goswami, R., Dunsby, P.K.S., Ananda, K.: *Phys. Lett. B* **679**, 302 (2009)
55. De Felice, A., Tsujikawa, S.: *Phys. Rev. D* **80**, 063516 (2009)
56. Mohseni, M.: *Phys. Lett. B* **682**, 89 (2009)
57. Sadjadi, H.M.: *Europhys. Lett.* **92**, 50014 (2010)
58. Bazeia, D., Carneiro da Cunha, B., Menezes, R., Petrov, A.Y.: *Phys. Lett. B* **649**, 445 (2007)
59. Bazeia, D., Menezes, R., Petrov, A.Y.: *Eur. Phys. J. C* **58**, 171 (2008)
60. Elizalde, E., Myrzakulov, R., Obukhov, V.V., Sáez-Gómez, D.: *Class. Quant. Gravit.* **27**, 095007 (2010)
61. de la Cruz-Dombriz, Á., Sáez-Gómez, D.: *Class. Quant. Gravit.* **29**, 245014 (2012)
62. Rapetti, D., Allen, S.W., Amin, M.A., Blandford, R.D.: *Mont. Not. R. Soc.* **375**, 1510 (2007)
63. Poplawski, N.J.: *Class. Quant. Gravit.* **24**, 3013 (2007)