of the drop of the cross sections at high energies it would be extremely desirable to visualize (e.g. by diagrams) the mechanism responsible for this phenomenon, provided, of course, that it exists in reality.

In conclusion we wish to express our appreciation to V. N. Gribov, L. D. Landau, L. B. Okun and I. M. Shmushkevich for interesting discussions connected with this work.

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ON THE ASYMPTOTIC BEHAVIOR OF SCATTERING AMPLITUDES AT HIGH ENERGIES

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(presented by A. Rudik)

The asymptotic behavior of scattering amplitudes is investigated at high energies. It is shown that at high energies the usual diffraction pattern of scattering contradicts unitarity conditions and analytic properties of the scattering amplitude formulated with the help of Mandelstam's representations. The most natural behavior in terms of these conditions is a decrease of the cross section faster than $(\ln E)^{-1}$.

1. INTRODUCTION

Asymptotic behavior of scattering amplitudes in quantum field theory has been investigated in a number of works [1-3]. In all cases, however, only very weak restrictions on possible asymptotic behavior have been obtained. At present, mainly owing to Mandelstam's work [4], in studying asymptotic behavior it has become possible to make a more extensive use of the dispersion relations for the momentum transfer, and of unitarity conditions. Since we can, so far, operate only with two-body states under unitarity conditions, evidently we cannot expect a complete solution of the problem. Nevertheless, some limited information can be obtained as will be shown below.

The description of elastic scattering at high energies is based on the so-called diffraction picture. According to this picture, particles with an impact parameter ρ smaller than a certain R (of the order of $1/\mu$; μ is the meson mass) strongly interact with the scatterer and are emitted from the elastic channel, while particles with an essentially larger impact parameter would not be scattered. This results in a diffractional scattering which is characterized by two main features : the total cross-section (σ_T) and the differential cross section of the elastic scattering in a unit interval of the square of the momentum transfer $d\sigma/dt$ are energy-independent ($-t$ is the square of the momentum transfer).

The present work investigates the consistency of such a pattern with the requirements of unitarity and analyticity for scattering amplitudes. It is shown that energy independence of σ_T and $d\sigma/dt$ cannot be naturally fitted with the conditions of unitarity and analyticity. (In terms of these conditions, the most natural behavior is a faster than logarithmic decrease of σ_T and $d\sigma/dt$ at high energies).

V. Berestetsky and I. Pomeranchuk [5], investigating inelastic processes in the pole approximation, arrived at a similar inconsistency of diffractional representations with the general properties of amplitudes in quantum field theory. Assuming that the elastic scattering cross section remains constant, they found that the cross section of inelastic processes tends to infinity. An assumption about the decrease of the scattering cross section makes their results self-consistent.

2. GENERAL PROPERTIES OF SCATTERING AMPLITUDES

For simplicity we shall first consider scattering of two identical particles without spin (mesons) and assume that these particles are the lightest in our theory and are pseudo-scalar, to prevent virtual conversion of two particles into one.

The elastic scattering amplitude of these particles, A, will be treated as a function of the two invariant variables: the square of the center-of-mass energy, s, and the square of the momentum transfer, $-t$. $A(s, t)$ is normalized in such a way that

$$A_1(s,0) = \text{Im}\, A(s,0) = \frac{s}{16\pi}\sigma_T \tag{1}$$

We shall assume that $A(s, t)$ possesses the analytic properties formulated by Mandelstam [4] and we shall use the Mandelstam plane.

The functions $A(s, t)$ and $A_1(s, t)$ will interest us in the region I [$s>4\mu^2$] for $s\to\infty$, $-t\ll s$. In the $t<4\mu^2$ region, $A(s, t)$ and $A_1(s, t)$ can be represented by a Legendre polynomial series. In particular, $A_1(s, t)$ can be written down in the form

$$A_1(s,t) = \sqrt{\frac{s}{s-4\mu^2}}\sum_{l=0}^{\infty} a_l(s)(2l+1)P_l\left(1+\frac{t}{2k^2}\right) \tag{2}$$

where $0<a_l<2$; $k=\frac{1}{2}\sqrt{s-4\mu^2}$ is the particle momentum in the center-of-mass system.

If we use the dispersion relation for $A_1(s, t)$ as a function of t:

$$A_1(s,t) = \frac{1}{\pi}\int_{t_1(s)}^{\infty}\frac{\text{Im}\, A_1(s,t')}{t'-t}dt' - \frac{1}{\pi}\int_{-\infty}^{t_2(s)}\frac{\text{Im}\, A_1(s,t')}{t'-t}dt' \tag{3}$$

$$t_1(s)>4\mu^2, \quad t_2(s)<-s$$

then according to the work of Galanin *et al* [6],

$$a_l = \frac{4}{\pi\sqrt{s(s-4\mu^2)}}\int_{t_1(s)}^{\infty} Q_l\left(1+\frac{t'}{2k^2}\right)\text{Im}\, A_1(s,t')dt' \tag{4}$$

where $$Q_l(z) = \tfrac{1}{2}\int_{-1}^{+1}\frac{P_l(z')}{z'-z}dz'$$

From this formula, they easily derived the asymptotic behavior of $a_l(s)$ at $l\to\infty$, since at $l\gg 1$, $\frac{t'}{2k^2}\ll 1$,

$$Q_l\left(1+\frac{t'}{2k^2}\right)\approx K_0(\rho\sqrt{t'})$$

$\rho = l/k$ is the impact parameter, and $K_0(\chi)$ is McDonald's function; then

$$a_l(s) \equiv a(\rho, s) = \frac{4}{\pi}\int_{4\mu^2}^{\infty} K_0(\rho\sqrt{t'})\,\text{Im}\, A_1(s,t')\frac{dt'}{s} \tag{5}$$

$$s/(4\mu^2)\gg 1$$

decreases exponentially for $l\gg k/2\mu$ and any s, provided only that $\text{Im}\, A_1(s,t')$ does not change essentially when $t-4\mu^2\ll 4\mu^2$.

At this point the analytic properties of $A_1(s, t)$ are consistent with the diffraction picture. The exponential decrease of $a_l(s)$ for $l\gg k/2\mu$ means that the function $A_1(s, t)$ changes significantly only when t changes by a value greater or of the order of μ^2. This is a simple consequence of the uncertainty principle between l and the scattering angle θ.

As $s\to\infty$ there are three possibilities:

(i) $a(\rho, s)\to a(\rho)$; (6*a*)

(ii) $a(\rho, s)\to 0$; (6*b*)

(iii) $a(\rho, s)$ oscillates around the mean value of $a(\rho)$ (6c)

Passing from summation to integration in (2) and employing the relation

$$P_l\left(1+\frac{t}{2k^2}\right)=J_0\left(\frac{l}{k}\sqrt{t}\right)=J_0(\rho\sqrt{t}), \tag{7}$$

where $J_0(\chi)$ is the zeroth Bessel function, we shall obtain simple relations for $A_1(s, t)$ as $s\to\infty$ in each of these cases.

$$\begin{aligned} A_1(s,t) &= \frac{s}{4}\int a(\rho,s)2\rho d\rho J_0(\rho\sqrt{-t}) \qquad (t<0) \\ A_1(s,t) &= \frac{s}{4}\int a(\rho,s)2\rho d\rho I_0(\rho\sqrt{t}) \qquad (0<t<4\mu^2) \end{aligned} \tag{8}$$

[$I_0(\chi)$ is the Bessel function of imaginary argument.]

In the first case $A_1(s, t) = sf(t)$

$$\sigma_T=\frac{f(0)}{16\pi}=\text{constant}\ ; \tag{9}$$

in the second case

$$\frac{A_1(s,t)}{s}\to 0 \qquad \sigma_T\to 0\ ; \tag{10}$$

in the third case

$$A_1(s,t)=sf(s,t) \tag{11}$$

where, although $f(s, t)$ oscillates with s, as a function of t it changes essentially only when t changes by the order of μ^2; generally speaking $\sigma_T = f(s, 0)/16\pi$ also oscillates with energy. Only the first case is completely consistent with the usual diffraction picture.

Note that if, in this case, we assume (6a) to hold at ρ, then by virtue of (5), $\text{Im}\,A_1(s, t) = s\,\text{Im}\,f(t)$, at any rate at $t-4\mu^2\ll\mu^2$. (The author's attention was drawn to the fact that $A_1 = sf(t)$ corresponds to the diffraction picture and that this expression must agree with the dispersion relation on t by I. Pomeranchuk.)

So far, we have concerned ourselves only with analytic properties of $A_1(s, t)$ as a function of t and employed the unitarity condition in the physical region on a limited scale (the $0<a_l<2$ condition). However, as Mandelstam first showed, unitarity conditions in other physical regions also impose essential restrictions on amplitudes by virtue of the analytic property of $A(s, t)$ as a function of s.

Let us consider the unitarity condition in region III [$t>4\mu^2$]. When $4\mu^2<t<16\mu^2$, inelastic processes in channel t being forbidden, the unitarity condition has a simple form:

$$A_3(s,t)=\frac{1}{4\pi}\sqrt{\frac{t-4\mu^2}{t}}\int\frac{dz_1dz_2}{\sqrt{1+2zz_1z_2-z^2-z_1^2-z_2^2}}\times \\ \times A(s_1,t)A^*(s_2,t) \tag{12}$$

where $A_3(s, t) = \text{Im}A(s, t)$ in region III, and $z_i=1+\dfrac{2s_i}{t-4\mu^2}$ is the cosine of the scattering angle in the center-of-mass system of channel III.

This form of unitarity condition differs from the usual one only by a replacement of the variable ϕ by z_2. Integration is performed over the area in which the square root is real. It was shown by Mandelstam that this relation can be continued from region III ($|z|<1$) to region (I, III) ($|z|>1$). In continuing (12) sufficiently far to region (I, III) expressions to the right and to the left become complex. Calculating their imaginary part and taking into account that $\text{Im}A_1(s, t) = \text{Im}A_3(s, t) = \rho(s, t)$ we shall find, according to Mandelstam, that

$$\text{Im}\,A_1(s,t)=\frac{1}{\pi}\sqrt{\frac{t-4\mu^2}{t}}\int\frac{dz_1dz_2}{\sqrt{z^2-2zz_1z_2+z_1^2+z_2^2-1}}\times \\ \times A_1(s_1,t)A_1^*(s_2,t) \tag{13}$$

when $4\mu^2<t<16\mu^2$. Integration is performed over the domain

$$z_1z_2+\sqrt{(z_1^2-1)(z_2^2-1)}\leqslant z \tag{14}$$

Note that this equation is valid irrespective of the asymptotic behavior of $A_1(s, t)$ as $s\to\infty$.

3. INVESTIGATION OF THE BEHAVIOR OF $A_1(s, t)$ AT LARGE s WITH THE HELP OF THE UNITARITY CONDITION

Since $A_1 = sf(t)$ is the most natural expression for $A_1(s, t)$ obtained on the basis of the diffraction picture, it is necessary, first of all, to substitute it into the unitary condition and to check whether it can be fulfilled. It can be done simply since, with such an asymptotic behavior of $A_1(s, t)$ and at $z \gg 1$, the essential contribution to the integrand is made by $z_1 \gg 1, z_2 \gg 1$. Therefore, it is possible to substitute $A_1(s_1, t) = s_1 f(t)$ and $A_2^*(s_2, t) = s_2 f^*(t)$ into the right-hand side of Eq. (13). Then, integrating, we obtain

$$\text{Im } A_1(s, t) = \frac{1}{6\pi}\sqrt{\frac{t-4\mu^2}{t}}(t-4\mu^2)s \ln s|f(t)|^2 + sO(1) \tag{15}$$

Since the left-hand side must equal $s \text{ Im} f(t)$ by assumption, then in view of the presence of $\ln s$ in the right-hand side we come to the conclusion that $A_1(s, t) \neq sf(t)$ at arbitrary large s. This means that the diffraction picture described in the introduction contradicts the unitarity condition.

In view of the fact that $A_1(s, t) = sf(t)$ is impossible, it is necessary to clarify of what character a function satisfying Eq. (13) can generally be. For this purpose, we shall apply the Mellin transformation to Eq. (13), i.e., multiply the right- and the left-hand side by $z^{-(p+1)}$ and integrate over z. $A(s, t)$ does not have an essential singularity at $s \to \infty$, therefore $A_1(s, t)$ can be written in the form

$$A_1(s, t) = s^{q(t)} B_t(s) \tag{16}$$

where $B_t(s)$ neither increases nor decreases power-like as $s \to \infty$. Because of this, the Mellin transformation makes sense for any $p > q$. At $z = 0$ there is no divergence, since $A_1(s, t) = 0$ for $z > z_0$, $z_0 > 1$. Integrating over z we obtain

$$\int \text{Im } A_1(s, t) z^{-(p+1)} dz = \frac{1}{\pi}\sqrt{\frac{t-4\mu^2}{t}} \int dz_1 (z_1^2 - 1)^{-\frac{p+1}{2}} \times$$

$$\times A_1(s_1, t) \int dz_2 (z_2^2 - 1)^{-\frac{p+1}{2}} A_2(s_2, t) C(\alpha) \tag{17}$$

$$\alpha = \frac{z_1 z_2}{\sqrt{(z_1^2-1)(z_2^2-1)}} \qquad C(\alpha) = \int_1^\infty dx \frac{(x+\alpha)^{-(p+1)}}{\sqrt{x^2-1}}$$

Let us now consider this equation for p in the neighborhood of q, $p = q + \delta$. In this case, if $B_t(s)$ does not decrease as $s \to \infty$, then on the right-hand side of Eq. (17), the regions $z_1 \gg 1$ and $z_2 \gg 1$ become most important if $\delta \ll 1$. Under these conditions $\alpha \to 1$. Denoting

$$\psi(p) = \int z^{-(p+1)} A_1(s, t) dz \tag{18}$$

we obtain then

$$\text{Im } \psi(p) = \frac{1}{\pi}\sqrt{\frac{t-4\mu^2}{t}} C(1) |\psi(p)|^2 \tag{19}$$

But since $|\text{Im } \psi(p)| < |\psi(p)|$ it follows from Eq. (19) that

$$|\psi(p)| < \frac{\pi}{C(1)}\sqrt{\frac{t}{t-4\mu^2}} \tag{20}$$

for $\delta \ll 1$. This means that

$$\int z^{-\delta} z^{-q} A_1(s, t) \frac{dz}{z} < \infty$$

for any $\delta \ll 1$, i.e.,

$$\int^\infty B_t(s) \frac{ds}{s} < \infty \tag{21}$$

Integral (21) can converge either due to the decrease of $B_t(s)$ or due to oscillations. It can be proved easily by direct substitutions that an oscillatory solution does not satisfy Eq. (13). Therefore, we shall interpret condition (21) as a condition that $B_t(s)$ decreases faster than $1/\ln s$. Since apart from this $B_t(s)$ by definition does not contain a power dependence at $s \to \infty$, it is more convenient to regard B_t as a function of $\xi = \ln s$ and instead of (16), write down:

$$A_1(s, t) = s^q B_t(\xi) \; ; \quad \int^\infty B_t(\xi) d\xi < \infty \tag{16a}$$

Thus, we find that in order to make Eq. (13) valid it is necessary for $A_1(s, t)$ to have the form of Eq. (16*a*).

In this section we have so far been concerned with the behavior of $A_1(s, t)$ as a function of s. Now we shall find out what can be said on the basis of Eq. (16a) about the behavior of $A_1(s, t)$ as a function of t. If the power in Eq. (16*a*) is a function of t then $A_1(s, t)$ will essentially change with a change of t not by a value of the order μ^2, but by a value of the order μ^2/ξ, i.e., at sufficiently great s, arbitrarily rapidly. Such behavior of $A_1(s, t)$ should hardly be considered to be possible, because firstly, for such behavior of the amplitudes, $a_l(s)$ calculated by means of Eq. (5) would not, generally speaking, decrease exponentially when $l \gg k/2\mu$; and secondly, it seems extremely artificial that a function which in the region $t<4\mu^2$ changes essentially only with a change of t by a value of the order μ^2, begins to change arbitrarily fast when analytically continued beyond the branch point into the region $t>4\mu^2$. In any case, it seems reasonable to postpone the treatment of such rapidly changing functions until a more detailed investigation, and to assume that q is independent of t and is equal to unity. Assuming that $B_t(\xi)$ also changes essentially only with a change of t of the order μ^2, we shall come to the conclusion that the third case (6c), is also impossible, and so we conclude

$$\frac{A_1(s,t)}{s}\to 0 \quad \text{as} \quad s\to 0 \tag{22}$$

So far, we have shown only that the decrease of $B_t(\xi)$ is a necessary condition for $A_1(s, t)$ to satisfy Eq. (13). It is easy to show that a solution of such a structure exists. We shall assume that $B_t(\xi)$ is a power type function of ξ, i.e., it possesses the property that $B_t(\xi_1+\xi_2)\approx B_t(\xi_1)$ if $\xi_1 \gg \xi_2$; and we shall take for simplicity $q=1$. (The same result is easily arrived for arbitrary q).

Contrary to Eq. (15), in the case of asymptotic behavior Eq. (16*a*) the regions $z_1 \gg 1, z_2 \sim 1$; $z_1 \sim 1$, $z_2 \gg 1$ prove to be most important for the integrand in the right-hand side of Eq. (13) at $z \gg 1$. Therefore, an asymptotic expression for $A_1(s, t)$ cannot be directly substituted into it.

We shall write down Eq. (13) in the following way:

$$\operatorname{Im} A_1(s,t)=\frac{1}{\pi}\sqrt{\frac{t-4\mu^2}{t}}\int\limits_{z_0^2}^{u_m(z)} du \int\limits_{z_0}^{u/z_0}\frac{dz_1}{z_1}\times$$

$$\times\frac{A_1(z_1)A_1{}^*(u/z_1)}{\sqrt{z^2-2zu+u^2/z_1^2+z_1^2-1}}, \tag{23}$$

$$u=z_1z_2\ ;\quad A_1(z)\equiv A_1(s,t)\ ;\quad A_1(z)=0 \ \text{ for } \ z<z_0 .$$

The main contribution to the integral occurs when $u \sim z_0$. Therefore, the integral over z_1 can be broken into two parts: from z_0 to λ and from λ to u/z_0; and λ can be chosen in such a way that

$$z_0 \ll \lambda \ll u/z_0 .$$

In the integral from z_0 to λ, $u/z_0 \gg 1$ and therefore we may substitute for $A^*{}_1(u/z_1)$ the asymptotic expression:

$$\frac{u}{z_1}\cdot\frac{1}{2}(t-4\mu^2)B_t{}^*\left(\ln\frac{u}{z_1}\right)=\frac{u}{z_1}\cdot\frac{1}{2}(t-4\mu^2)B_t{}^*(\ln u) .$$

Then we obtain

$$\tfrac{1}{2}(t-4\mu^2)uB_t{}^*(\ln u)\int\limits_{z_0}^{\lambda}\frac{dz_1}{z_1}\,\frac{A_1(z_1)}{z_1\sqrt{z^2-2zu+u^2/z_1^2+z_1^2-1}}$$

Since $\int \frac{dz_1}{z_1^2}A(z_1)$ converges according to the assumption, the main contribution to the integral from z_0 to λ is made by $z_1 \sim 1$, and therefore one can neglect z_1^2-1 compared with the remaining terms, and integrate not up to λ but up to the zero of the argument of the square roots.

In the same way we may treat the integral from λ to u/z_0. It will yield a complex conjugate expression. Transforming to the original variables, Eq. (22) can be written in the form

$$\operatorname{Im} A_1(s,t)=\frac{1}{\pi}\sqrt{\frac{t-4\mu^2}{t}}\int\frac{dz_1dz_2}{\sqrt{z^2-2zz_1z_2+z_2^2}}A_1(s_1,t)\times$$

$$\times s_2B_t^*(\ln s_2)+\frac{1}{\pi}\sqrt{\frac{t-4\mu^2}{t}}\int\frac{dz_1dz_2}{\sqrt{z^2-2z_1z_2+z_1^2}}\times$$

$$\times s_1B_t(\ln s_1)A^*{}_1(s_2,t) . \tag{24}$$

Integration is respectively performed over the regions:

$$z_2(z_1+\sqrt{z_1^2-1})\leqslant z;\ z_1(z_2+\sqrt{z_2^2-1})\leqslant z .$$

After this transformation, the first integral is easily integrated over z_2 and the second over z_1 (certainly asymptotically at $s\to\infty$).

As a result we obtain

$$\operatorname{Im} A_1(s,t)=sB_t(\xi)\phi^*(t)+sB_t{}^*(\xi)\phi(t) \tag{25}$$

where

$$\phi(t)=\frac{1}{2\pi}\sqrt{\frac{t-4\mu^2}{t}}\int\limits_{z_0}^{\infty}dz'A_1(s',t)\left[z'\ln\left(\frac{z'+1}{z'-1}\right)-2\right]=$$

$$=\frac{1}{2\pi}\sqrt{\frac{t-4\mu^2}{t}}\int\limits_{-1}^{+1}zdz\int\limits_{z_0}^{\infty}\frac{A_1(s',t)}{z'-z}dz' . \tag{26}$$

Integral (26) converges provided condition (16) is fulfilled. Eq. (25) shows that (13) is really satisfied if $A_1(s, t)$ has the form of Eq. (16a). Simultaneously, an equation for $B_t(s)$ as a function of t emerges:

$$\text{Im } B_t(\xi) = B_t(\xi)\phi^*(t) + B^*_t(\xi)\phi(t) . \qquad (27)$$

This equation holds only at $4\mu^2 < t < 16\mu^2$ but if we assume it to be valid at any t we immediately obtain:

$$B_t(\xi) = B(\xi) \exp\left[(t-4\mu^2)\int_{4\mu^2}^{\infty}\frac{\delta(t')}{t'-t}\,\frac{dt'}{t'-4\mu^2}\right] \qquad (28)$$

where $\exp[2i\delta(t)] = (1-2i\phi(t))(1+2i\phi^*(t))^{-1}$; it is assumed that $\delta(t)$ does not decrease at $t\to\infty$. Thus, in particular, we have in the physical region:

$$A_1(s, t) = sB(\xi)f(t) ;$$

$$\sigma_n = \frac{B(\xi)f(0)}{16\pi} < \frac{C}{\xi} \text{ as } s\to\infty ; \qquad (29)$$

$\frac{1}{\sigma_n}\frac{d\sigma}{dt}$ is independent of s.

If, from the outset, we had proceeded from the assumption that $A_1(s, t) = \phi(s)f(t)$, then without making any further assumptions we would have arrived at the fact that $\phi(s) = sB(\xi)$ and that $f(t)$ satisfies the equation

$$\text{Im } f(t) = f(t)\phi^*(t) + f^*(t)\phi(t) . \qquad (29a)$$

It is interesting to note that the value $\phi(t)$ given by Eq. (26) is somehow related to the scattering phase shifts at the center-of-mass system energy. Indeed, the scattering amplitude in region (III) can be written as

$$A(z_1 t) = \frac{1}{\pi}\int_{z_0}^{\infty}\frac{A_1(z', t)}{z'-z}dz' - \frac{1}{\pi}\int_{-\infty}^{-z_0}\frac{\text{Im } A_1(z', t)}{z'-z}dz' \qquad (30)$$

where z is the cosine of the scattering angle in the center-of-mass system. If we wanted to evaluate the p-wave amplitude, multiplying $A_1(z, t)$ by z and integrating over z we would arrive at a zero by virtue of the symmetry of $A(z, t)$. However, if we take only the first term in Eq. (30) and multiply it by z, carrying out the integration we obtain $\phi(t)$ to within a factor.

4. ON ENERGY VALUES AT WHICH THE CROSS SECTION DECREASE MAY PROVE TO BE ESSENTIAL

It is very difficult to make a definite estimate of energies at which the cross section decrease may prove to be significant since we can use the unitary condition only in the $4\mu^2 < t < 16\mu^2$ interval which does not contain contribution from inelastic processes in the third channel.

However, one may attempt to make a rough estimate of the critical energy. It is rather interesting that such an estimate involves a numerical parameter which renders this energy very great.

In order to make the mentioned estimate we shall assume that starting with $s = \Lambda$ the cross section becomes constant, and $A_1 = sf(t)$. Substituting into the unitarity condition Eq. (13) we shall arrive at Eq. (15). To estimate up to what energy $A_1 \approx sf(t)$ it is necessary to determine up to what energy the first term in Eq. (15) containing $\ln s$ is small as compared with the entire $\text{Im } A_1$. We shall obtain then that the cross section may remain constant up to energies at which

$$\frac{1}{6\pi}\sqrt{\frac{t-4\mu^2}{t}}(t-4\mu^2)|f|^2 \ln\left(\frac{s_0}{\Lambda}\right) \ll \text{Im} f(t) . \qquad (31)$$

If we now put $\text{Im} f \sim f$, neglect the fact that $f(t)$ increases with t in the interval from 0 to $4\mu^2$, and substitute $\sigma_n/16\pi$ for $f(t)$, then instead of Eq. (31) we obtain:

$$\frac{1}{96}\sqrt{\frac{t-4\mu^2}{t}}\,\frac{(t-4\mu^2)\sigma_n}{\pi^2}\ln\left(\frac{s_0}{\Lambda}\right) \ll 1 . \qquad (32)$$

Eq. (32) makes sense up to $t = 16\mu^2$; therefore if $\sigma_n \sim \frac{1}{\mu^2}$, Eq. (32) roughly means that

$$\ln\left(\frac{s_0}{\Lambda}\right) \ll 100 . \qquad (33)$$

In spite of the fact that the value $W_0 = \sqrt{s_0}$ proves to be very great and that the cross section possibly vanishes only at unattainably high energies, already with the change of energy from 10^9 eV to 10^{13} eV the first term in Eq. (15) may result in a 10% change of the cross section.

5. PION-PION SCATTERING AT HIGH ENERGIES

So far, we have considered scattering of neutral particles without spins. In this section we shall show that the same reasoning can be applied to the case of pion-pion scattering.

In a further paper it will be shown that the same holds for pion-nucleon and nucleon-nucleon scattering. The scattering amplitude describing scattering of π-mesons with momentum p_1 in isotopic state α by a π-meson with momentum p_2 and isotopic state β into π-mesons with momenta $-p_3$, $-p_4$ and isotopic states γ, δ can be written in the form

$$T_{\alpha\beta,\,\gamma\delta} = \delta_{\alpha\beta}\delta_{\gamma\delta}A(s_{12}, s_{13}, s_{23}) + \\ + \delta_{\alpha\gamma}\delta_{\beta\delta}A(s_{13}, s_{12}, s_{23}) + \delta_{\alpha\delta}\delta_{\beta\gamma}A(s_{23}, s_{12}, s_{13}). \quad (34)$$

By virtue of crossing symmetry,

$$A(x; y, z) = A(x; z, y). \quad (35)$$

We shall consider the behavior of $T_{\alpha\beta,\,\gamma\delta}$ as $s_{12}\to\infty$ and when $s_{13}\sim -\mu^2$, and shall proceed at first from the assumption that

$$\begin{aligned} &A(s_{12}; s_{13}, s_{23})\to s_{12}F_3(s_{13}) \\ &A(s_{13}; s_{23}, s_{12})\to s_{12}F_2(s_{13}) \quad (36) \\ &A(s_{23}; s_{12}, s_{13})\to s_{12}F_1(s_{13}). \end{aligned}$$

As is well known the Feynman amplitude $A(s_{12}; s_{13}, s_{23})$ is not an analytic function of s_{12} in the upper half plane. It is the function corresponding to the retarded commutators that are analytic in the upper half plane. Therefore if we pass through the upper half plane from s_{12} to $-s_{12}$ we shall obtain not

$$A(-s_{12}; s_{13}, 4\mu^2+s_{12}-s_{13})$$

but

$$A^*(-s_{12}; s_{13}, 4\mu^2+s_{12}-s_{13}).$$

However

$$A^*(-s_{12}, s_{13}, 4\mu^2+s_{12}-s_{13})\to s_{12}F_1^*(s_{13}) \text{ as } s_{12}\to\infty.$$

Hence, in view of the fact that $s_{12}F_3(s_{13})$ transforms into $-s_{12}F_3(s_{13})$ as $s_{12}\to -s_{12}$ we obtain that

$$F_1^*(s_{13}) = -F_3(s_{13}) \quad (37)$$

and in a similar way we shall obtain for $A(s_{13}, s_{12}, s_{23})$

$$F_2(s_{13}) = -F_2^*(s_{13}). \quad (38)$$

If both functions $F_1(s_{13})$ and $F_2(s_{13})$ were non-zero it would follow from Eq. (34) firstly that backward scattering is of the same order as forward scattering for any α, β; and secondly that the cross section of the forward charge exchange is of the same order as elastic scattering. The second is evident; to make the first evident it is sufficient to consider $T_{\alpha\beta,\,\gamma\delta}$ at $s_{12}\to\infty$ and $s_{23}\sim -\mu^2$ and to put $\alpha=\gamma$, $\beta=\delta$. At $s_{12}\to\infty$ and $s_{23}\sim -\mu^2$ we shall have from Eq. (34):

$$T_{\alpha\beta,\,\alpha\beta} = s_{12}[F_1(s_{23}) + \delta_{\alpha\beta}F_3(s_{23}) + \delta_{\alpha\beta}F_2(s_{23})] \quad (39)$$

It follows from Eq. (30) that in order to prevent particles of different charges from backward scattering with the same amplitude as for forward scattering it is necessary that $F(s_{23}) = 0$. Then it will follow from Eq. (37) that $F_3(s_{13}) = 0$. In this case, we obtain

$$T_{\alpha\beta,\,\gamma\delta} = \delta_{\alpha\gamma}\delta_{\beta\delta}s_{12}F_2(s_{13}) \text{ as } s_{12}\to\infty, s_{13}\sim -\mu^2 \quad (40)$$

i.e., only scattering without charge-exchange occurs in the forward direction. Thus if we assume that backward scattering of particles with different charges is comparatively small we come to the conclusion that pion-pion scattering is characterized at high energies by a single scalar function $A(s_{13}, s_{23}, s_{12})$ symmetrical with respect to s_{12} and s_{23} in exactly the same way as for neutral particles without spins.

The unitarity condition for the channel in which s_{13} is the energy is written in a manner analogous to Eq. (17) for each of the three amplitudes corresponding to definite isotopic spins 0,1 and 2.

$$\begin{aligned} T_0 &= 3A(s_{13}, s_{23}, s_{12}) + A(s_{12}, s_{13}, s_{23}) + A(s_{23}, s_{12}, s_{13}) \\ T_1 &= A(s_{12}, s_{13}, s_{23}) - A(s_{23}, s_{12}, s_{13}) \\ T_2 &= A(s_{12}, s_{13}, s_{23}) + A(s_{23}, s_{12}, s_{13}). \quad (41) \end{aligned}$$

Each of these relations can be continued into region $|z|>1$. However for $z\gg 1$, since $A(s_{12}, s_{23}, s_{13})$ and $A(s_{23}, s_{12}, s_{13})$ make a small contribution to scattering at high energy, it is sufficient to consider only the relation for T_0 which is identical with Eq. (13) and will yield the same results as for neutral particles.

One more interesting consequence of these results is to be observed. Under the same assumptions as in section 3 we find that

$$A_1(s_{13}, s_{12}, s_{23}) = sB(\xi)f(t); \quad s = s_{12}, t = s_{13} \quad (42)$$

If $B(\xi)$ were absent it would follow from Eq. (38) that

$$A = isf(t) \tag{43}$$

i.e., the scattering amplitude is purely imaginary (the real part could be of the order $1/s$ as compared with the imaginary part).

In the presence of $B(\xi)$ the situation changes drastically. With the substitution of s for $-s$, $B(\xi)$ will be replaced by $B(\xi+i\pi)$. It is easy to show that in this case the correct expression for the whole amplitude, instead of Eq. (43), will have the form

$$A(s,t) = \text{is}\,B[\ln(-is)]f(t) \tag{44}$$

The function $A(s, t)$ in this case cannot be purely imaginary. If, as $s \to \infty$, $B(\xi) \sim \xi^{-q}$, then

$$B[\ln(-is)] \sim \xi^{-q}\left[1 + \frac{i\pi q}{2\xi}\right]. \tag{45}$$

Thus, the real part of $A(s, t)$ is only logarithmically small as compared with the imaginary part.

The main results obtained in this work can be summarized as follows :

(1) a representation of the scattering amplitude at high energies in the form $sf(t)$ which holds if the scattering is of diffractional character, contradicts unitarity condition;

(2) it follows from the conditions of unitary and analyticity under natural assumptions that the total cross section for the scattering decreases at high energies.

Our treatment does not claim to be mathematically rigorous, but in our opinion it is convincing enough from the physical standpoint to encourage further investigation.

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