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Conformal Aspects of String Theory

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Abstract

This thesis deals with aspects of conformal field theory in string theory. It investigates two areas of string theory where two dimensional conformal field theory plays an important role: First, the dynamics of strings and branes are described by their worldsheet CFT and its boundary conditions. Second, the holographic dual of a string theory living on 2+1 dimensional anti-de Sitter space is also given by a two dimensional CFT.

The first part of this thesis investigates perturbations of the worldsheet CFT and the resulting renormalisation group flows: If a bulk theory is perturbed by inserting integrated marginal fields, a flow on the boundary may be induced which changes the brane configuration in such a way that the resulting theory is again conformal. Also, depending on the type of perturbation and the boundary condition, symmetries of the theory may or may not be broken. Conversely, the branes themselves may backreact on the bulk theory via contributions of higher order in the string coupling g_s. These effects are analysed and the corresponding RG equations are obtained.

In a second part, the role of dual CFTs in the framework of the AdS/CFT correspondence is investigated. First, a proposal is made for the dual CFT of a specific configuration of heterotic strings, NS5 branes and Kaluza-Klein monopoles. Second, the existence of extremal CFTs is investigated. Extremal theories are proposed as holographic duals to pure quantum gravity in 2+1 dimensions. Checking this proposal in the bosonic case, one can use the formalism of Zhu to derive differential equations to test the consistency of a given theory. In the $N = 2$ supersymmetric case, the mere existence of the elliptic genus is already a non-trivial consistency test. Such methods are used to make statements on the existence of extremal CFTs.

Zusammenfassung

Die vorliegende Dissertation behandelt Aspekte von konformen Feldtheorien in der Stringtheorie. Es werden zwei Bereiche der Stringtheorie untersucht, in denen zweidimensionale konforme Feldtheorien eine wichtige Rolle spielen: Einerseits wird die Dynamik von Strings und Branes durch ihre Weltflächen-CFT und die zugehörigen Randbedingungen beschrieben. Andererseits ist die duale Randtheorie zu Stringtheorien, die auf 2+1 dimensionalem anti-de Sitter-Räumen leben, ebenfalls wieder eine zweidimensionale CFT.

In einem ersten Teil werden Störungen der Weltflächen-CFT und die dabei entstehenden Renormierungsgruppen-Flüsse untersucht: Wenn die Theorie mittels eingesetzter, integrierter marginaler Felder gestört wird, kann ein Fluss auf dem Rand der Theorie auftreten, der die Konfiguration der Branes dahingehend verändert, dass die resultierende Theorie wieder konform ist. Zudem können abhängig von der Art der Störung und der Randbedingung Symmetrien der ursprünglichen Theorie gebrochen werden. Andererseits können die Branes auch eine Rückwirkung auf die Bulk-Theorie bewirken durch Beiträge von Diagrammen höherer Ordnung in der String-Kopplungskonstante g_s.

Im zweiten Teil wird die Rolle von dualen konformen Theorien im Rahmen der AdS/CFT-Korrespondenz untersucht. Es wird ein Vorschlag präsentiert für die duale CFT einer bestimmten Konfiguration von heterotischen Strings, NS5-Branen und Kaluza-Klein-Monopolen. Anschliessend wird die Existenz von extremalen CFTs untersucht. Solche Theorien werden aufgestellt als holographische Dualtheorien zu reinen Quantengravitationstheorien in 2+1 Dimensionen. Im bosonischen Fall kann man den Zhu'schen Formalismus benutzen, um Differentialgleichungen herzuleiten, die die Konsistenz einer gegebenen Theorie überprüfen. Wenn die Theorie hingegen $N = 2$-Supersymmetrie aufweist, ist schon die Existenz des elliptischen Genus ein nicht-trivialer Konsistenztest.

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Introduction

String theory

In nature, the interaction of all particles are governed by just four forces: gravity, electromagnetism, the weak nuclear force, and the strong nuclear force. The laws of gravity are given by Einstein's theory of general relativity, a classical field theory. The other three forces are described by quantum field theories (QFT). They are all explained by the *standard model* of high energy physics. The standard model is one of the most successful scientific theories of our time. It makes predictions on the interaction of particles which have been tested to an extremely high level of precision.

Nevertheless, we know that the standard model does not describe nature completely. In particular, it does not incorporate gravity. There have been many attempts to find a theory which includes both the standard model and gravity, and so unifies all four forces. Most candidate theories that have been found suffer from so-called *non-renormalisability*: when calculating physical quantities, they give infinite expressions. For non-renormalisable theories, these infinities cannot be removed by the usual prescriptions. It thus remains an open problem to find a unified theory.

The most promising candidate for such a fully unified theory is *string theory* — see *e.g.* [114, 166] for an introduction and overview of the subject. String theory contains gravity in a very natural way. It also circumvents the problem of non-renormalisability by abandoning the idea of pointlike particles. Instead, one-dimensional objects are introduced: *strings*. There are two kinds of strings, *closed strings* and *open strings*. The endpoints of open strings can be fixed to a geometric surface, a *D-brane*. If one follows a string through time, it carves out a two-dimensional area of spacetime, the *worldsheet*. The dynamics of a string is described by the two dimensional field theory that lives on its worldsheet. This worldsheet theory will be the main focus of the first part of this thesis.

Conformal field theory

The field theory on the worldsheet not only has the usual Poincaré symmetries, but in addition also possesses *conformal symmetry*. Such theories are called conformal field theories (CFTs). They are invariant under rescaling, or, more precisely, under any map that preserves angles. This means that such theories do not contain any intrinsic length scale and no massive particles.

The size of the symmetry group depends greatly on the number of dimensions. For $d > 2$ the number of symmetry generators is $\frac{1}{2}(d+1)(d+2)$. Apart from the usual rotations

R and translations **T**, they also contain dilatations **D** and so-called special conformal transformations **S**. It is for $d = 2$ that the full power of the conformal symmetry group appears. Writing the worldsheet as the complex plane $\mathbb{C}$ with coordinate z, all analytic maps $z \mapsto f(z)$ are conformal. If we write down the Laurent series of $f(z)$ we see that that the symmetry group has an infinite number of generators. This gives an infinite number of constraints on the correlation functions. Some of these theories are thus exactly solvable by symmetry considerations alone. For an introduction to two dimensional CFTs, see *e.g.* [105, 48].

The second advantage of the two dimensional case is that it is possible to divide the theory into two sectors which are essentially independent of each other: the right-moving sector and left-moving sector, or alternatively, holomorphic and anti-holomorphic sector. Invariance under dilatation requires that the energy-momentum tensor T be traceless. Moreover, we can separate it into its holomorphic part $T(z)$ and its anti-holomorphic part $\bar{T}(\bar{z})$. We can then expand $T(z)$ in terms of its modes,

$$T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-h} \ .$$

The Virasoro generators L_n satisfy the Virasoro algebra,

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n} \ .$$

These commutation relations encode the conformal symmetry of the theory. In particular, all states decompose into representations of the Virasoro algebra.

The second term on the right-hand side of the commutation relation, the *central term*, can be viewed as a quantum effect. The central charge c is a very important quantity when one tries to construct the worldsheet CFT of a string theory. It can be shown [114] that bosonic string theory requires $c = 26$, whereas the superstring requires $c = 15$. In a free theory, each boson gives a contribution of $c = 1$, whereas a fermion contributes $c = \frac{1}{2}$. In the bosonic theory, we thus need 26 bosons, whereas in the supersymmetric theory we need 10 bosons and 10 fermions. This is thus another way of phrasing the famous result that string theory is consistent only in 26 or in 10 dimensions.

The CFTs discussed so far serve as worldsheet theories of closed strings. If we want to describe worldsheet theories of open strings, we need to introduce boundaries into the CFT. The boundary condition then relates the left-moving fields to the right-moving fields, and the left-moving and right-moving Virasoro algebras are glued together so as to produce a single Virasoro algebra. If the theory is to remain conformal, the boundary condition must satisfy certain constraints. Nevertheless, there are in general many allowed conformal boundary conditions. In open string theory, these families of boundary conditions describe different configurations of D-branes. They form a moduli space, parametrised by open string moduli.

We will also consider theories which have more symmetries than conformal symmetry. Such additional symmetries are described by the chiral algebra, *i.e.* by fields which only have a left- or right-moving component. Again, if a symmetry is to be preserved in a theory with boundary, then the boundary condition must satisfy certain properties. Many of these issues will be discussed in more detail in chapter 2.

One symmetry which plays an important role in string theory is *supersymmetry*. Unlike ordinary symmetries, it relates fermionic degrees of freedom to bosonic ones, and vice versa. In ordinary quantum field theory, supersymmetry is very useful because it leads to cancellations between fermionic and bosonic Feynman graphs and so eliminates some of the divergences of the theory. In two dimensional CFT, $N = (2, 2)$ supersymmetry plays a special role; $N = (2, 2)$ here signifies that there are two left- and two right-moving supersymmetry charges. Combining the supersymmetry generators with the Virasoro generators, one obtains the superconformal algebra. One important feature of this algebra is the appearance of a $U(1)$ current J, which corresponds to the R-charge of the supersymmetry. Each state then not only has a definite conformal weight h, but also a charge Q.

From the string point of view, $N = (2, 2)$ supersymmetry on the worldsheet is intimately linked to supersymmetry in spacetime [87, 14], which is phenomenologically very important. On the other hand, from a purely CFT point of view supersymmetry places strong constraints on the theory. Chapter 6 will make use of this extensively.

Perturbed CFT and moduli in string theory

The first part of this thesis deals with perturbed conformal field theory and its application to string theory. Let us first motivate the interest in such questions from the point of view of moduli and moduli stabilisation in string theory.

The original motivation for string theory is to construct models which reproduce the standard model at low energies. To achieve this, one has to reconcile the fact that in the real world there are only 4 dimensions, whereas (super-) string theory needs 10 dimensions. The most common solution to this apparent paradox is to assume that 6 dimensions live on very small compact manifolds. There are many families of possible manifolds and backgrounds. They are parametrised by moduli, so that to each value of the moduli corresponds a viable string model. Unfortunately, these moduli become massless scalars in the low-energy theory. As we observe no such particles in nature, to obtain a realistic string model we thus need to stabilise all moduli. For reviews on this question see [60, 19].

Most backgrounds of interest also involve D-branes, so that there are two kinds of moduli to consider: the D-brane moduli that describe the different D-brane configurations in a given closed string background, and the closed string moduli that characterise the closed string background. These two moduli spaces are not independent of one another: the moduli space of D-branes depends on the closed string background, and thus on the closed string moduli. On the other hand, the D-branes 'back-react' on the background, and thereby modify the original closed string background in which they were placed. In order to make progress with stabilising all moduli in string theory, it is therefore important to understand the interplay between these two moduli spaces better.

From the perspective of the worldsheet CFT, the moduli space is given by the space of all conformal theories. The moduli are given by those deformations of the theory and its boundary condition which preserve its conformal symmetry. Since the theory is deformed by inserting perturbing operators, it is necessary to employ methods of perturbation theory in CFT. A CFT can be subjected to a perturbation by inserting an operator in the bulk of the theory and integrating over the entire plane. Similar to the usual perturbation theory used in QFT, this integral diverges. To render it finite, one has to introduce a regulator. This regulator breaks scale invariance, which may mean that the perturbed theory is no longer conformal.

More concretely, we add to the original action a term of the form

$$\Delta S = \lambda \int \phi(z, \bar{z})\, d^2 z \ ,$$

where λ is the coupling constant of the perturbation. If ϕ has conformal weight $(h, \bar{h}) = (1, 1)$, then ΔS is classically invariant under conformal transformations. Such operators are called *marginal*. As mentioned above, the integrals that appear in the resulting expressions are divergent if two fields come close to each other and must therefore be regularised. The simplest way to do this is to cut out small circles of radius ℓ around each field. Note that by that we have introduced a length scale; we are thus no longer guaranteed that the resulting theory is conformal. If we change ℓ, we can compensate the resulting change in the physical observables by changing the value of the coupling constant λ. The *renormalisation group equation*

$$\dot{\lambda} = \beta(\lambda)$$

is a differential equation that describes the dependence of λ on ℓ. If $\beta(\lambda)$ does not vanish, then the theory flows; in particular, it is no longer conformal, since it now depends on a length scale. If we let ℓ go to infinity, *i.e.* if we flow to the infrared, then we expect to encounter a fixed point where the theory is again conformal.

On the other hand, if the β-function vanishes, then the perturbed theory is again conformal. Operators ϕ for which this is the case are called *exactly marginal*. In the terminology of string theory, such an operator is a closed string modulus, that is, a modulus of the bulk theory. A necessary condition for a bulk field to be exactly marginal is that it has conformal weight $(1, 1)$, and that its three-point self-coupling vanishes [130, 30]. A brief introduction to this topic is given at the beginning of chapter 1.

The rest of chapter 1 then deals with the same analysis of the theory with boundary. The boundary condition and the gluing map describe the brane configuration, and the moduli of the branes, *i.e.* the open string moduli, are given by exactly marginal boundary operators. The presence of a boundary then changes the analysis presented above. Indeed, a bulk operator that is exactly marginal in the bulk theory may cease to be exactly marginal in the presence of a boundary. It then induces a flow on the boundary, which changes the boundary condition to a new configuration for which the perturbed theory is again conformal.

In chapter 2 symmetries of perturbed conformal field theories are analysed. As was mentioned before, CFTs can have additional symmetries. From a string theoretic point of view, such configurations are often much more accessible than generic, unsymmetric configurations. One can then start at a more symmetric point in the moduli space and investigate which symmetries are broken if one moves away from this point. It is also much easier to classify D-branes when they preserve a larger symmetry group of the theory. Again, one can then investigate what happens to such branes away from the point of enhanced symmetry.

From the CFT point of view, symmetries of the bulk theory are given by the chiral algebra, *i.e.* by fields of the theory which are purely left- or right-moving. The states of the theory then decompose into representations of this chiral algebra. The question is then which generators of the chiral algebras of the bulk theory survive a perturbation by an exactly marginal bulk field. In chapter 2 we find a simple criterion for this.

For theories with boundary, the situation is a bit more subtle. The symmetry algebra of the boundary is given by the set of local fields. It turns out that essentially the boundary continues to preserve as much symmetry as it possibly can, that is as much as is preserved in the bulk. The (somewhat technical) conclusions in this case can be found at the end of chapter 2.

Above, we described how the bulk theory could change the brane configuration. Chapter 3 then discusses the converse of this case, namely how a given brane can backreact on the bulk theory and so induce a bulk flow. The motivation for this analysis is to develop tools for calculating backreactions of branes, which are notoriously difficult to handle. We obtain the corresponding backreaction term in the RG equations and, in several examples, follow the resulting flow.

Note that this backreaction is no longer an effect of pure CFT, but appears only in the full string theoretic context via the so-called Fischler-Susskind mechanism. Let us thus briefly sketch the relationship of pure CFT to the full string theory. So far the CFTs we have considered have lived on the Riemann sphere (in the closed case) or on the disk or upper half plane (in the open case.) In principle, we can also define the theory on arbitrary Riemann surfaces with or without boundaries. The correlators then depend only on the conformal structure of the surface. For surfaces of genus g bigger than zero, the topology does not uniquely fix this structure; there exists a moduli space $\mathcal{M}$ of different surfaces.

String perturbation theory then instructs us to sum over all topologically different surfaces. Each diagram comes with a power of the string coupling constant g_s, the power given by the Euler character of the Riemann surface. Moreover, for each surface, we need to integrate over all moduli t_i. Note that these are worldsheet moduli which have nothing to do with the open or closed string moduli discussed before.

If there are massless fields in the theory, so-called tadpoles, then the integral over the t_i diverges. Regularising the divergence again introduces a length scale which can induce a renormalisation group flow. To put it another way, one has to introduce scale-dependent counterterms on worldsheet surfaces of lower genus. In chapter 3 we will see that, at least for the annulus, this backreaction can be incorporated in the RG equation.

The AdS/CFT-correspondence and dual CFTs

In the second part of this thesis we consider another application of two dimensional CFTs, namely as holographic duals to string theory on anti-de Sitter space AdS. We will first check the correspondence in one specific instance by explicitly constructing the dual CFT of a heterotic string setup. In a second part we discuss a special kind of dual CFTs, so-called extremal CFTs, and find strong evidence against their existence.

The AdS/CFT-correspondence is a special form of the so-called holographic principle. One seeks to describe the dynamics of a theory on a certain spacetime by constructing a dual theory which lives on the boundary of said spacetime. In its original form ([146], see *e.g.* [4] for a review) the correspondence describes the duality between a superstring theory living on the curved spacetime $AdS_5 \times S^5$ and four-dimensional $N = 4$ supersymmetric Yang-Mills-theory living on the boundary of the spacetime geometry. Here, S^5 is the five dimensional sphere, and AdS_5 is a maximally symmetric space of negative cosmological constant. $N = 4$ supersymmetric Yang-Mills-theory is the supersymmetric generalisation of gauge theory. Given the gauge group, supersymmetry fixes its field content completely. Moreover, its beta function vanishes to all orders, so that it is in fact a conformal theory.

The AdS/CFT-correspondence is a strong-weak duality, as it relates the strong-coupling limit of the CFT to the weak-coupling limit of the string theory. It can thus be used to obtain results in the strong-coupling regime of the gauge theory. This is of particular interest since many open questions in gauge theory such as quark confinement are strong-coupling effects not accessible to perturbation theory. On the other hand, because of the strong-weak nature of the duality, it is hard to perform test of the correspondence, as one tries to compare weak coupling quantities to strong coupling quantities. One way to circumvent this is to consider protected objects such as chiral fields whose behaviour does not change drastically when the strength of the coupling changes. Recently, there has also been much progress in this area by using integrable structures contained in $N = 4$ SYM [18] to obtain non-perturbative results. Another possibility is to consider a different instance of the correspondence where one can calculate exact correlators and is not restricted to the weak coupling limit.

It is for that reason that we are going to consider string theory on AdS_3, *i.e.* on three dimensional anti-de Sitter space. The dual CFT is then two dimensional. In this setup is then possible to calculate the exact correlation functions of the dual CFT using the methods of 2d CFT, and not just their perturbative expansion. Sometimes, one also knows the exact worldsheet theory, so that it is possible to perform precision tests of the AdS/CFT-correspondence.

In chapter 4 we perform such tests by investigating one specific instance of an AdS_3/CFT_2 correspondence, namely the heterotic three-charge model with $(0, 4)$ target space supersymmetry. We calculate in several different ways its central charge and check that they agree.

Let us briefly sketch the specific setup under investigation. Heterotic string theory is a theory in which the right-moving sector is supersymmetric, but the left-moving sector is not. Instead, it contains 10 bosons, which correspond to the usual spacetime coordinates, and 32 additional fermionic currents, which add up to the required central charge of 26. Consequently, the resulting dual CFT then has supersymmetry in the right-moving sector only. Three-charge model means that apart from the fundamental heterotic strings, which produce the AdS_3 geometry, we also have two more objects. In particular, there are NS5 branes, and so-called Kaluza-Klein monopoles. There are many dualities that relate this particular heterotic setup to equivalent setups in type I, type II and M-theory — see appendix B.1 for details. In our construction, we will make use of these dualities.

The worldsheet theory for heterotic strings for the near-horizon geometry we consider, namely $AdS_3 \times S^3/\mathbb{Z}_N \times T^4$, has been worked out in [137]. Moreover, for the 2+1 dimensional instance of the AdS/CFT correspondence, the (super) Virasoro algebra of the boundary theory can be worked out explicitly in terms of worldsheet vertex operators [111]. It is thus possible to directly identify operators of the two theories. In particular, one can determine the chiral primaries and the central charge of the boundary CFT.

We are interested in finding the boundary CFT that corresponds to this setup. We propose that it is given by a two-dimensional $(0, 4)$ sigma model arising on the Higgs branch of an orbifolded ADHM model. This generalises the models introduced in [184, 59]. To check this proposal, we compare the left- and right-moving central charges of the infrared conformal field theory to those predicted by the worldsheet model. More precisely, we need to flow the ADHM model, which we know in the UV, to its infrared fixed point. One can show however that there is no renormalisation group flow [138], so that one can simply count the massless degrees of freedom in the UV. Our counting is then in perfect agreement with the prediction from the worldsheet side.

Pure gravity and extremal CFTs

Chapters 5 and 6 deal with a very specific kind of dual CFT, so-called *extremal CFTs*. Their motivation comes from the analysis of Witten concerning pure gravity in AdS$_3$ [186]. In particular, he proposes the existence of extremal dual theories, which could serve as dual CFTs to pure gravity. We will give arguments that such extremal theories cannot exist.

Let us first introduce some concepts needed later on. By pure gravity we mean that the theory contains only the bare essentials necessary, and no additional matter or gauge fields. The dual CFT then also only contains a very limited number of states, which is why we call it extremal.

Our arguments will make use of modular properties of partitions functions. The *partition function* or *character* of a CFT is defined as

$$\chi_M(q) = \mathrm{Tr}_M(q^{L_0 - c/24}) ,$$

where the trace is taken over all states of the representation M of the theory. In particular, χ_M counts the number of states at each level. Physically, it corresponds to the zero-point function on the torus. The torus remains invariant under transformations of the modular group $SL(2, \mathbb{Z})$. All physical quantities that depend on τ should thus also be invariant under modular transformations. This places strong constraints on the partition function.

Witten then suggested that the boundary theories that correspond to pure gravity should be holomorphically factorising extremal bosonic conformal field theories of central charge $c = 24k$. The integer k is proportional to the AdS radius of the spacetime geometry. We are mainly interested in the case of large k where the curvature becomes weak.

There are only two kinds of excitations in pure gravity: perturbative excitations and black holes. The perturbative excitations are identified with Virasoro descendants of the vacuum following [23] while the BTZ black holes correspond to the new primaries. Since black holes are parametrically heavy there is a large gap from the vacuum to the first nontrivial primary. Extremality then means that the partition function of the boundary CFT is as close as possible to the Virasoro character of the vacuum, *i.e.* that it starts to differ from it only at a high level.

The above constraints specify the potential character of such meromorphic conformal field theories uniquely. For $k = 1$, the resulting character is the j invariant minus 744,

$$j(q) - 744 = \frac{1}{q} + 196884\,q + 21493760\,q^2 + \ldots\ .$$

To see that this theory is extremal note that the coefficient of q^0 is zero, so that there are no states at level 1. The corresponding conformal field theory is the famous Monster theory [83, 20]. For $k \geq 2$ one can write down similar partition functions, but an explicit realisation of these theories is so far not known. In chapter 5 we investigate if theories corresponding to these characters can exist at all. Following [91], we use null vectors and their associated modular differential equations to find consistency conditions for such theories which go beyond the existence of the partition function.

More specifically, if a null vector is inserted in the trace over any representation, one obtains a differential equation which annihilates the character. Moreover, the form of this differential equation is independent of the representation, *i.e.* the same equation annihilates all characters of a theory. The order of the equation depends on the level of the null vector. On the other hand, if one knows all the characters of a theory, one can by inspection find a modularly covariant differential equation which annihilates all of them.

In chapter 5 we show the converse, *i.e.* that each such equation must come from at least one null vector. It is possible, however, that the level of the null vectors is higher than the order of the differential equation, so that we cannot draw a direct conclusion on its conformal weight. On the other hand, the bigger the difference between the two, the more null vectors are needed. As long as the theory does not have an extremely high number of null vectors, the differential equation has approximately the same order as the level of the null vector.

For the proposed extremal theories this leads to a contradiction for $k \geq 42$, since the differential equation predicts null vectors at a level where they are excluded by extremality [91].

In chapter 6 we take a step back and perform the analogue of the analysis of [186], only this time for the $N = (2, 2)$ supersymmetric case. Again, we only allow $N = (2, 2)$ descendants of the vacuum and primary fields which correspond to BTZ black holes. BTZ black holes in this setup are characterised by their mass and charge, which we identify on the CFT side with their conformal weight and $U(1)$ charge. From classical supergravity, we know that there exists a so-called cosmic censorship bound: if the charge ℓ of a black hole is too big compared to its mass n, then the solution has a singularity which lies outside its event horizon. To avoid this, we allow only black holes for which

$$4mn - \ell^2 \geq 0\ ,$$

where m is the analogue of k in the bosonic case, *i.e.* the AdS radius. The cosmic censorship condition is the analogue of the requirement that the bosonic black holes be parametrically heavy, that is it restricts the kind of primary fields that can be introduced in the theory.

As was mentioned before, the $N = 2$ superconformal algebra has more structure than the Virasoro algebra; in particular, it contains a $U(1)$ current J. The zero mode J_0 is then an additional element of the Cartan subalgebra and can thus be inserted in the character much in the same way as L_0. This construction leads to the *elliptic genus* of the theory, the main object of interest in our analysis. The elliptic genus thus not only contains information on the weight of states, but also on their charges.

Mathematically, the elliptic genus is a so-called *weak Jacobi form.* Its Fourier expansion is given by

$$\chi(q,y) = \sum_{n\geq 0} \sum_{\ell\in\mathbb{Z}} c(n,\ell)\, q^n y^\ell \ ,$$

and it has nice transformation properties under modular transformations and spectral flow. The space of all possible weak Jacobi forms is a ring generated by four generators. The terms for which $4mn - \ell^2 < 0$ are called the polar part of χ. Note that this coincides with the cosmic censorship bound introduced above.

By this argument, we know that BTZ black holes cannot change the number of states below the cosmic censorship bound. This means that the coefficients of the polar terms of χ are fixed by the $N = 2$ vacuum character. The number of polar terms however is bigger than the number of allowed weak Jacobi forms, or, to put it another way, the system of equations so obtained is overdetermined. Generically, we thus do not expect an extremal elliptic genus to exist; using a rather technical analytical argument, one can show that for large AdS radii this is indeed true. The only genera that do exist are in fact $m = 1, 2, 3, 4, 5, 7, 8, 11, 13$.

On the other hand, one can show that an elliptic genus exists for all curvatures if one relaxes the cosmic censorship bound somewhat. This leads to the notion of near-extremal CFTs. In particular, since the original form of the bound is taken from a purely classical calculation, one cannot rule out that quantum corrections could modify the bound in just the right way to make the theory work. The $N = 4$ case however, for which one does not expect any quantum correction, does not seem to work much better. Our arguments thus give strong indications that pure supergravity theories on AdS_3 are inconsistent.

Chapters 1 and 2 are based on [79] and [80] written in collaboration with S. Fredenhagen and M. Gaberdiel. Chapter 3 is based on [134]. Chapter 4 is based on [121], written in collaboration with S. Hohenegger and I. Kirsch. Chapter 5 is based on [94] written in collaboration with M. Gaberdiel. Chapter 6 is based on [93] written in collaboration with M. Gaberdiel, S. Gukov, G. Moore, and H. Ooguri.

Part I

RG flows on the worldsheet

Chapter 1

Bulk-induced boundary perturbations

1.1 Overview

In the introduction we have seen that in CFT the closed string moduli space is described by the exactly marginal bulk perturbations. A necessary condition for a bulk field to be exactly marginal is that it has conformal weight $(1,1)$, and that its three-point self-coupling vanishes [130, 30]. This condition was derived for conformal field theories without boundary, but in the presence of a D-brane, the situation changes. Indeed, a marginal bulk operator that is exactly marginal in the bulk theory may cease to be exactly marginal in the presence of a boundary.

The simplest example where this phenomenon occurs, is the theory of a single free boson compactified on a circle. For this theory the full moduli space of conformal D-branes is known [97, 125] (see also [85, 169]). It depends in a very discontinuous manner on the radius of the circle, which is one of the bulk moduli. We always have the usual Dirichlet and Neumann branes, but if the radius is a rational multiple of the self-dual radius, the moduli space contains in addition a certain quotient of $SU(2)$. On the other hand, for an irrational multiple of the self-dual radius the additional part of the moduli space is just a line segment. The bulk operator that changes the radius is exactly marginal for the bulk theory, but in the presence of certain D-branes it is not. In particular, it ceases to be exactly marginal if we consider a rational multiple of the self-dual radius and a D-brane which is neither Dirichlet or Neumann, but which is associated to a generic group element g of $SU(2)$. If we change the radius infinitesimally, it is generically not a rational multiple of the self-dual radius any more, and thus the brane associated to g is no longer conformal.

In order to understand the response of the system to the bulk perturbation we set up the renormalisation group (RG) equations for bulk and boundary couplings. This can be done quite generally, and we find that whenever certain bulk-boundary coupling constants do not vanish, the exactly marginal bulk perturbation is not exactly marginal in the presence of a boundary, but rather induces a non-trivial RG flow on the boundary. In particular, this therefore gives a criterion for when an exactly marginal bulk deformation is also exactly marginal in the presence of a boundary.

For the above example of the free boson, the resulting RG flow equations can actually be studied in quite some detail. We find that upon changing the radius the resulting flow drives the brane associated to a generic group element g (that only exists at rational radii) to a superposition of pure Neumann or Dirichlet branes (that always exist). Whether the end-point is Dirichlet or Neumann depends on the sign of the perturbation, *i.e.* on whether the radius is increased or decreased. At the self-dual radius, the theory is equivalent to the $SU(2)$ WZW model at level 1, and the analysis can be done very elegantly. In this case we can actually give a closed formula for the boundary flow which is exact in the boundary coupling (at first order in the bulk coupling).

Some of these results can be easily generalised to arbitrary current-current deformations of WZW models at higher level and higher rank. While we cannot, in general, give an explicit description of the whole flow any more, we can still describe at least qualitatively the end-point of the boundary RG flow.

1.2 CFTs with boundaries

Let us give a very brief introduction to CFTs with boundaries, fleshing out some of the material mentioned in the introduction. More detailed introductions to conformal field theory are for example [48, 105]. As mentioned before, we can separate the energy-momentum tensor T into its holomorphic part $T(z)$ and its anti-holomorphic part $\bar{T}(\bar{z})$. We can then expand $T(z)$ in terms of its modes,

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-h} \ . \tag{1.2.1}$$

The Virasoro generators L_n satisfy the Virasoro algebra,

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n} \ . \tag{1.2.2}$$

Conformal symmetry requires that all states of the theory decompose into representations of the Virasoro generators L_n. So-called *primary* fields are fields which have good transformation properties under conformal transformations. In particular, a primary field $\phi(z,\bar{z})$ transforms as

$$\phi(z,\bar{z}) \mapsto \left(f'(z)\right)^h \left(\bar{f}'(\bar{z})\right)^{\bar{h}} \phi(f(z), \bar{f}(\bar{z})) \ , \tag{1.2.3}$$

where h and $\bar{h}$ are the (holomorphic and anti-holomporphic) conformal weight of ϕ. Further states, the so-called descendants, can be obtained by acting with Virasoro generators on primary fields. Using conformal symmetry, one can reduce correlation functions of descendants to correlation functions of primary fields.

Let us turn to CFTs with boundaries. It is often convenient to describe such theories as living on the upper half plane $\mathbb{H}_+$ of the complex plane; the boundary is then the real axis $z = \bar{z}$. Boundary conditions are imposed by relating left-moving fields to right-moving fields on said axis. If the theory is to remain conformal, it is in particular necessary to identify

$$T(z) = \bar{T}(\bar{z}) \qquad \text{for } z = \bar{z} \ . \tag{1.2.4}$$

It is then possible to analytically continue $T(z)$ to the lower half plane using the prescription

$$T(z) = \begin{cases} T(z) & \text{Im}z \geq 0 \\ \bar{T}(z) & \text{Im}z < 0 \end{cases} \ . \tag{1.2.5}$$

This means that we have essentially reduced the problem of a full CFT on the upper half plane to the left-moving sector of a CFT on the entire complex plane.

The above description of boundary conditions is the open string picture. It is often convenient to change to the closed string picture by performing a modular S transform on the worldsheet. In this case, for instance the 1-loop diagram of an open string whose endpoints lie on two D-branes becomes the diagram of a closed string propagating from an initial state to a final state. This state is the so-called boundary state $||B\rangle\rangle$. In this language, the condition for conformal invariance is then

$$(L_n - \bar{L}_{-n})||B\rangle\rangle = 0 \ . \tag{1.2.6}$$

Additional symmetries of the theory are described by its chiral algebra, *i.e.* by fields which only have a left- or right-moving component. In a theory with boundary such a symmetry is preserved if on the boundary the left- and right-moving chiral fields are related by an automorphism, the so-called gluing map. Many of these questions will be discussed in more detail in chapter 2.

1.3 The renormalisation group equation

In this section we shall analyse the RG flow involving bulk and boundary couplings. Bulk perturbations by relevant operators for conformal field theories with boundaries have been considered before in the context of integrable models starting from [37] and further developed in [175, 88, 102]. In particular, these flows have been studied using (an appropriate version of) the thermodynamic Bethe ansatz (see *e.g.* [141, 144, 56, 57, 54]), in terms of the truncated conformal space approach (see *e.g.* [56, 57, 55]), and recently by a form factor expansion [13, 34].

Let S^* be the action of a conformal field theory on the upper half plane. We denote the bulk fields by ϕ_i, and the boundary fields by ψ_j. Their operator product expansions are of the form

$$\begin{aligned} \phi_i(z)\phi_j(w) &= |z-w|^{h_k-h_i-h_j}\, C_{ijk}\, \phi_k(w) + \cdots \ , && (1.3.7) \\ \psi_i(x)\psi_j(y) &= (x-y)^{h_k-h_i-h_j}\, D_{ijk}\, \psi_k(y) + \cdots \ , && (1.3.8) \end{aligned}$$

where C_{ijk} and D_{ijk} are the bulk and boundary OPE coefficients, respectively. We are interested in the perturbation of this theory by bulk and boundary fields,

$$S = S^* + \sum_i \tilde{\lambda}_i \int \phi_i(z)\, d^2z + \sum_j \tilde{\mu}_j \int \psi_j(x)\, dx \ . \tag{1.3.9}$$

Introducing the length scale ℓ, we define dimensionless coupling constants λ_i and μ_j by

$$\tilde{\lambda}_i = \lambda_i \, \ell^{h_{\phi_i}-2} \ , \qquad \tilde{\mu}_j = \mu_j \, \ell^{h_{\psi_j}-1} \ . \tag{1.3.10}$$

Note that we do not assume here that ϕ_i and ψ_j are marginal operators.

Let $\langle \ldots \rangle$ denote the correlators in the unperturbed theory; the perturbed correlators are then defined as

$$\langle \phi_1(z_1,\bar{z}_1) \cdots \phi_n(z_n,\bar{z}_n) \rangle_\lambda = \frac{\langle \phi_1(z_1,\bar{z}_1) \cdots \phi_n(z_n,\bar{z}_n) e^{-\Delta \mathcal{S}} \rangle}{\langle e^{-\Delta \mathcal{S}} \rangle} \ , \tag{1.3.11}$$

and similarly for correlators involving boundary fields. The expression on the right hand side is divergent and has to be regularised. If we expand the exponential in powers of λ_i and μ_j, we get terms of the form

$$\begin{aligned} &\frac{\lambda_1^{l_1} \cdots \mu_1^{m_1} \cdots}{l_1! \cdots m_1! \cdots} \prod_i \ell^{(h_{\phi_i}-2)l_i} \prod_j \ell^{(h_{\psi_j}-1)m_j} \\ &\qquad \times \int \langle \phi_1(z_1^1) \phi_1(z_2^1) \cdots \phi_2(z_1^2) \cdots \psi_1(x_1^1) \cdots \rangle \prod d^2 z_k^i \prod dx_k^j \ . \end{aligned} \tag{1.3.12}$$

Since (1.3.12) is divergent, we need to specify a regularisation scheme. The most straightforward one is to cut out little disks around operator insertions by introducing an UV cutoff ℓ. More precisely, the prescription is

$$|z_k^i - z_{k'}^{i'}| > \ell \ , \qquad |x_k^j - x_{k'}^{j'}| > \ell \ , \qquad \mathrm{Im} z > \frac{\ell}{2} \ . \tag{1.3.13}$$

The parameter ℓ thus appears in (1.3.12) both explicitly as powers in h, and implicitly through the range of integration. In chapter 3 we will rederive the RG equations using a different regularisation scheme somewhat resembling dimensional regularisation; but for the moment, we shall use (1.3.13).

Following [30] we now consider a change of the scale ℓ, $\ell \to (1+\delta t)\ell$, and ask how the coupling constants have to be adjusted so as to leave the free energy unchanged. The explicit dependence of the expression (1.3.12) on ℓ leads to a change in λ_i and μ_j by

$$\begin{aligned} \lambda_i &\to (1 + (2 - h_{\phi_i})\, \delta t)\, \lambda_i \ , \\ \mu_j &\to (1 + (1 - h_{\psi_j})\, \delta t)\, \mu_j \ . \end{aligned} \tag{1.3.14}$$

The implicit dependence of (1.3.12) on ℓ through the UV prescription (1.3.13) gives rise to an additional change of the coupling constants. From the first inequality in (1.3.13), which controls the UV singularity in the bulk operator product expansion, we obtain the equation $\delta\lambda_k = \pi C_{ijk} \lambda_i \lambda_j \delta t$ [30]. A similar calculation gives $\delta\mu_k = D_{ijk}\mu_i\mu_j\delta t$ (see for example [3]) for the contribution from the boundary operator product expansion (the second inequality). Finally we have to consider the contribution from the third inequality which controls the singularity that arises when a bulk operator approaches the boundary. When we scale ℓ by $(1+\delta t)$ we change the integration region of a bulk operator by a strip parallel to the real axis of width $\ell\,\delta t/2$. This changes the expression (1.3.12) by terms of the form

$$-\lambda_i\,\ell^{h_{\phi_i}-2}\int dx\int_{\ell/2}^{\ell/2+\ell\delta t/2} dy\,\langle\cdots\phi_i(z)\cdots\rangle\ , \tag{1.3.15}$$

where we have written $z = x + iy$. In order to evaluate this contribution, we use the bulk-boundary operator product expansion

$$\phi_i(z,\bar{z}) = (2y)^{h_{\psi_j}-h_{\phi_i}}\,B_{ij}\,\psi_j(x) + \cdots\ , \tag{1.3.16}$$

where B_{ij} is the bulk-boundary OPE coefficient that depends on the boundary condition in question. The change of the free energy described by (1.3.15) is then

$$-\lambda_i\,\ell^{h_{\phi_i}-2}\int dx\,\frac{\ell\,\delta t}{2}\,B_{ij}\,\ell^{h_{\psi_j}-h_{\phi_i}}\,\langle\cdots\psi_j(x)\cdots\rangle = -\frac{1}{2}\,B_{ij}\,\ell^{h_{\psi_j}-1}\,\lambda_i\,\delta t\int dx\,\langle\cdots\psi_j(x)\cdots\rangle \tag{1.3.17}$$

which can be absorbed by a shift of $\delta\mu_j = \frac{1}{2}\,\lambda_i B_{ij}\delta t$. Collecting all terms, we thus obtain the RG equations to lowest order

$$\dot{\lambda}_k = (2-h_{\phi_k})\lambda_k + \pi C_{ijk}\,\lambda_i\lambda_j + \mathcal{O}(\lambda^3)\ , \tag{1.3.18}$$

$$\dot{\mu}_k = (1-h_{\psi_k})\mu_k + \frac{1}{2}\,B_{ik}\,\lambda_i + D_{ijk}\,\mu_i\mu_j + \mathcal{O}(\mu\lambda,\mu^3,\lambda^2)\ . \tag{1.3.19}$$

The flow of the bulk variables λ_k in (1.3.18) is independent of the boundary couplings μ_k on the disc. The RG flow in the bulk therefore does not depend on the boundary condition whereas the bulk has significant influence on the flow of the boundary couplings. Note that the terms we have written out explicitly are independent of the precise details of the UV cutoff (if the fields are marginal). Higher order corrections, on the other hand, will depend on the specific regularisation scheme.

Suppose now that ϕ_i is an exactly marginal bulk perturbation. The perturbation by ϕ_i is then exactly marginal in the presence of a boundary if the bulk boundary coupling constants B_{ik} vanish; this has to be the case for all boundary fields ψ_k (except for the vacuum) that are relevant or marginal, *i.e.* satisfy $h_{\psi_k} \leq 1$. Obviously, switching on the vacuum on the boundary just leads to a rescaling of the disc amplitude; for irrelevant operators, on the other hand, the flow is damped by the first term of (1.3.19), and thus the bulk perturbation only leads to a small correction of the boundary condition.

The above condition is the analogue of the usual statement about exact marginality: a necessary condition for a marginal bulk (boundary) operator to be exactly marginal is that the three point couplings C_{iik} (D_{iik}) vanish for all marginal or relevant fields ϕ_k (ψ_k), except for the identity (see for example [130, 30, 169]).

If the bulk boundary coefficient B_{ik} does not vanish for some relevant or marginal boundary operator ψ_k, the corresponding boundary coupling μ_k starts to run, and there is a non-trivial RG flow on the boundary. The bulk couplings λ_i are not affected by the flow ($\dot{\lambda}_i = 0$), and we can thus interpret it as a pure boundary flow in the marginally deformed bulk model. From that point of view it is then clear that the flow must respect the g-theorem [3, 86]. In particular, the g-function of the resulting brane is smaller than that of the initial brane. This is in fact readily verified for the examples we are about to study.

In chapter 2, we will rederive the above results using a somewhat different approach, namely by analysing symmetries of the theory. As a special case, one can then check under which conditions the conformal symmetry survives, and obtains the same result — see 2.3.1 for more details.

1.4 The free boson theory at the self-dual radius

As an application of these ideas, we now consider the example of the free boson theory at $c = 1$. We shall first consider the theory at the critical radius, where it is in fact equivalent to the WZW model of $su(2)$ at level 1. For this theory all conformal boundary states are known [98], and are labelled by group elements $g \in SU(2)$ (for earlier work see also [26, 167]).

Suppose that we are considering the boundary condition labelled by $g \in SU(2)$, where we write

$$g = \begin{pmatrix} a & b^* \\ -b & a^* \end{pmatrix} , \tag{1.4.1}$$

and a and b are complex numbers satisfying $|a|^2 + |b|^2 = 1$. (Geometrically, $SU(2)$ can be thought of as a product of two circles — see figure 1.1.) We shall choose the convention that the brane labelled by g satisfies the gluing condition[1]

$$\left(g\, J^\alpha_m \, g^{-1} + \bar{J}^\alpha_{-m} \right) \, \| g \rangle\!\rangle = 0 , \tag{1.4.2}$$

where J^α are the currents of the WZW model (the corresponding Lie algebra generators will be denoted by t^α). We shall furthermore use the identification that g diagonal ($b = 0$) describes a Dirichlet brane on the circle, whose position is given by the phase of a; conversely, if g is off-diagonal ($a = 0$), the brane is a Neumann brane, whose Wilson line on the dual circle is described by the phase of b.

1.4.1 Changing the radius

We want to consider the bulk perturbation by the field

$$\Phi = J^3 \bar{J}^3 , \qquad \text{where} \qquad t^3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{1.4.3}$$

This is an exactly marginal bulk perturbation that changes the radius of the underlying circle. With the above conventions, the perturbation $\lambda\Phi$ with $\lambda > 0$ increases the radius, while $\lambda < 0$ decreases it. At any rate, the perturbation by Φ breaks the $su(2)$ symmetry down to $u(1)$. However, in the presence of a boundary, the bulk perturbation is generically not exactly marginal any more. This is implicit in the results of [97, 125, 85] since the set of possible conformal boundary conditions is much smaller at generic (irrational) radius

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[1] Note that the labelling differs from the one used in [97].

relative to the self-dual case. Here we want to study in detail what happens to a generic boundary condition under this bulk deformation.

Even before studying the detailed RG equations that we derived in the previous section, it is not difficult to see that the above deformation is generically not exactly marginal. In particular, we can consider the perturbed one-point function of the field Φ in the presence of the boundary. To first order, this means evaluating the 2-point function

$$\lambda \int_{\mathbb{H}^+} d^2 z \langle (J^\alpha \bar{J}^\alpha)(z)\, (J^\alpha \bar{J}^\alpha)(w) \rangle \ , \tag{1.4.4}$$

where the label $\alpha = 3$ is not summed over. Using the usual doubling trick [28] this amplitude can be expressed as a chiral 4-point function, where we have the fields J^α at z and w, and the 'reflected' fields $J^\beta \equiv g J^\alpha g^{-1}$ at $\bar{z}$ and $\bar{w}$.
The chiral correlation functions of WZW models at level k can be calculated using the techniques of [84, 92]. Let t^α, $\alpha = 1, \ldots, \dim(g)$, be the Lie algebra generator (corresponding to J^α) in some representation; we choose the normalisation

$$\mathrm{Tr}(t^\alpha\, t^\beta) = k\, \delta^{\alpha\beta} \ . \tag{1.4.5}$$

To evaluate $\langle J^{\alpha_1}(z_1) \cdots J^{\alpha_n}(z_n) \rangle$, consider then all permutations $\rho \in S_n$ that have no fixed points; this subset of permutations is denoted by $\tilde{S}_n$. Each such ρ can be written as a product of disjoint cycles

$$\rho = \sigma_1 \sigma_2 \cdots \sigma_M \ . \tag{1.4.6}$$

To each cycle $\sigma = (i_1\, i_2 \cdots i_m)$ we assign the function

$$f_\sigma^{\alpha_{i_1} \cdots \alpha_{i_m}}(z_{i_1}, \ldots, z_{i_m}) = -\frac{\mathrm{Tr}(t^{\alpha_{i_1}} \cdots t^{\alpha_{i_m}})}{(z_{i_1} - z_{i_2})(z_{i_2} - z_{i_3}) \cdots (z_{i_m} - z_{i_1})} \ , \tag{1.4.7}$$

and to each ρ the product $f_{\sigma_1} \cdots f_{\sigma_M}$. The correlation function is then given by summing over all permutations without fixed points,

$$\langle J^{\alpha_1}(z_1) \cdots J^{\alpha_n}(z_n) \rangle = \sum_{\rho \in \tilde{S}_n} f_\rho \ . \tag{1.4.8}$$

In (1.4.4), ρ is either a 4-cycle or consists of two 2-cycles. In the latter case we get the terms

$$\frac{(\mathrm{Tr}(t^\alpha t^\beta))^2}{|z - \bar{z}|^2 |w - \bar{w}|^2} + \frac{(\mathrm{Tr}(t^\alpha t^\beta))^2}{|z - \bar{w}|^4} + \frac{\mathrm{Tr}(t^\alpha t^\alpha)\mathrm{Tr}(t^\beta t^\beta)}{|z - w|^4} \ . \tag{1.4.9}$$

Integration over the upper half plane gives (divergent) contributions proportional to $|w - \bar{w}|^{-2}$, which can be absorbed in the renormalisation of J^α. The six terms that come from the six different 4-cycles give a total contribution of

$$-\frac{\mathrm{Tr}([t^\alpha, t^\beta]^2)}{(z - \bar{z})(w - \bar{w})|z - \bar{w}|^2} \ . \tag{1.4.10}$$

Set $w = i|w|$ and $z = x + iy$. The resulting integral over the upper half plane is logarithmically divergent for $y \to 0$. Introducing an ultraviolet cutoff ϵ, we get

$$\int_{\mathbb{R}} dx \int_\epsilon^\infty dy\, \frac{1}{2iy 2i|w|} \frac{1}{x^2 + (y + |w|)^2} = \frac{\pi}{4|w|^2} \log \epsilon - \frac{\pi}{8|w|^2} \log |w|^2 + \mathcal{O}(\epsilon) \ . \tag{1.4.11}$$

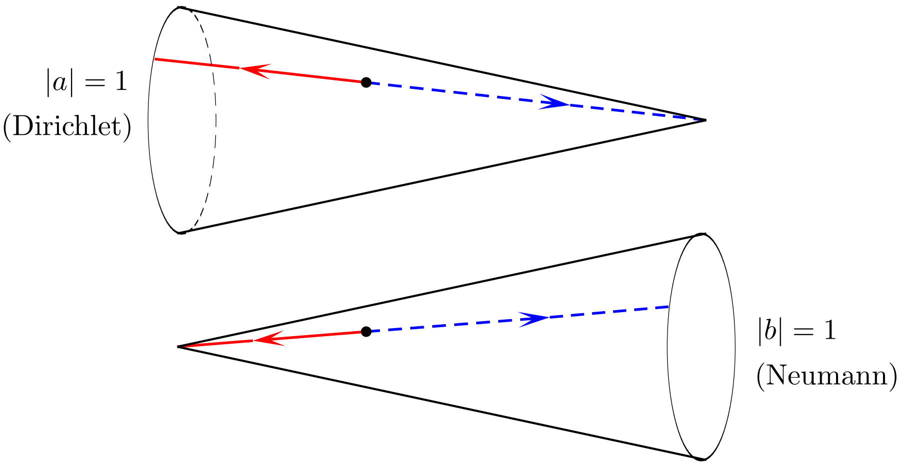

Figure 1.1: The moduli space of D-branes on the self-dual circle, $SU(2)$, can be described as a product of two circles S^1 (given by the phases of a and b in (1.4.1)) fibred over an interval where $|a|$ runs between 0 and 1, and $|a|^2+|b|^2 = 1$. The ends of the interval where one of the circles shrinks to zero describe Dirichlet and Neumann branes, respectively. If we start with a generic boundary condition and increase (decrease) the radius, the boundary condition will flow to a Dirichlet (Neumann) boundary condition.

The first term has the right w dependence to be absorbed by a suitable renormalisation of J^α. The second term, however, pushes the conformal weight away from $(1,1)$. Thus, if J^α is to be exactly marginal, the expression $\mathrm{Tr}([t^\alpha, t^\beta]^2)$ must vanish.

In the case above $\mathrm{Tr}([t^\alpha, t^\beta]^2)$ equals

$$\mathrm{Tr}([t^3, g\, t^3\, g^{-1}]^2) = -8|a|^2|b|^2 \ . \tag{1.4.12}$$

This only vanishes if either $|a| = 0$ or $|b| = 0$; the corresponding boundary conditions are therefore either pure Dirichlet or pure Neumann boundary conditions. This ties in with the expectations based on the analysis of the conformal boundary conditions since only pure Neumann or Dirichlet boundary conditions exist for all values of the radius.

The argument above can also be used in the general case to derive a necessary criterion for when a bulk deformation is exactly marginal in the presence of a boundary. It is not difficult to see that it leads to the same criterion as the one given in section 1.3.

1.4.2 The renormalisation group analysis

Now we want to analyse what happens if g does not describe a pure Neumann or pure Dirichlet boundary condition. In particular, we can use the results of section 2 to understand how the system reacts to the bulk perturbation by $\lambda\Phi$.

In order to see how the boundary theory is affected by the perturbation we have to compute the bulk boundary OPE of the perturbing field Φ. There are no relevant boundary fields (except the vacuum), and the marginal fields are all given by boundary currents J^γ. We can thus determine the bulk boundary OPE coefficient $B_{\Phi\gamma}$ from the two-point function

$$\langle J^\gamma(x)(J^3\bar{J}^3)(z)\rangle = B_{\Phi\gamma}\,|z-\bar{z}|^{-1}|x-z|^{-2}\ , \tag{1.4.13}$$

which – employing the general formula (1.4.8) – leads to

$$B_{\Phi\gamma} = -i\mathrm{Tr}(t^\gamma[t^3, g\,t^3 g^{-1}])\ . \tag{1.4.14}$$

We see that the only boundary field that is switched on by the bulk perturbation is the current J^γ whose (hermitian) Lie algebra generator t^γ is proportional to the commutator $[t^3, g\,t^3 g^{-1}]$. The normalised t^γ is given by

$$t^\gamma = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -e^{i\chi} \\ e^{-i\chi} & 0 \end{pmatrix} \qquad \text{with} \qquad a\,b^* = |ab|e^{i\chi}\ . \tag{1.4.15}$$

Its relation to the commutator is

$$-i[t^3, g\,t^3 g^{-1}] = -i\begin{pmatrix} 0 & -2ab^* \\ 2a^*b & 0 \end{pmatrix} = B\,t^\gamma\ , \tag{1.4.16}$$

where the bulk boundary coefficient $B = B_{\Phi\gamma}$ is given by

$$B = -2\sqrt{2}\,|a|\,|b|\ . \tag{1.4.17}$$

The boundary current proportional to t^γ modifies the boundary condition g by

$$\delta g = i\,t^\gamma\,g = \frac{1}{\sqrt{2}}\begin{pmatrix} -a\frac{|b|}{|a|} & b^*\frac{|a|}{|b|} \\ -b\frac{|a|}{|b|} & -a^*\frac{|b|}{|a|} \end{pmatrix}\ . \tag{1.4.18}$$

This leaves the phases of a and b unmodified, but decreases the modulus of a while increasing that of b.

Since the operators are marginal, the renormalisation group equation to lowest order in the coupling constants (1.3.19) is now

$$\dot{\mu} = \frac{1}{2}\,B\,\lambda + \mathcal{O}(\mu\lambda, \mu^2, \lambda^2)\ , \tag{1.4.19}$$

where μ is the boundary coupling constant of the field J^γ. Thus if the radius is increased ($\lambda > 0$), μ becomes negative, and the boundary condition flows to the boundary condition with $b = 0$ — the resulting brane is then a Dirichlet brane whose position is determined by the original phase of a. Conversely, if the radius is decreased ($\lambda < 0$), μ becomes positive, and the boundary condition flows to the boundary condition with $a = 0$. The resulting brane is then a Neumann brane whose Wilson line is determined by the original value of the phase of b (see figure 1.1). This is precisely what one should have expected since for radii larger than the self-dual radius, only the Dirichlet branes are stable, while for radii less then the self-dual radius, only Neumann branes are stable.

Actually, the renormalisation group flow can be studied in more detail. It follows from (1.4.18) that to lowest order in μ

$$a(\mu) = a_0 - \mu\, a_0 \frac{|b_0|}{\sqrt{2}|a_0|} + \mathcal{O}(\mu^2)\ , \tag{1.4.20}$$

where the initial values of a and b have been denoted by a_0 and b_0, respectively. Since a depends on the RG parameter only via μ, it thus follows that

$$\dot{a} = -\dot{\mu}\, a \frac{|b|}{\sqrt{2}|a|} = -\frac{B}{2\sqrt{2}} \frac{|b|}{|a|}\, a\, \lambda = |b|^2\, a\, \lambda = (1 - |a|^2)\, a\, \lambda\ . \tag{1.4.21}$$

If we write $|a| = \sin\psi$, this simplifies to

$$\dot{\psi} = \sin\psi\, \cos\psi\, \lambda\ . \tag{1.4.22}$$

Denoting the RG parameter by t, the solution to this differential equation is

$$\tan\psi(t) = \tan\psi(0)\, e^{\lambda t}\ . \tag{1.4.23}$$

Thus for $\lambda > 0$ this flows indeed to $|a_\infty| = 1$, while for $\lambda < 0$ we find $|a_\infty| = 0$, as expected.

Given the relation (1.4.20), we can deduce from the solution for $a(t)$ a differential equation for $\mu(t)$ which turns out to be

$$\dot{\mu} = -\sqrt{2}\, \dot{\psi}\ . \tag{1.4.24}$$

This can be integrated to

$$\mu(t) = -\sqrt{2}\, (\psi(t) - \psi(0))\ . \tag{1.4.25}$$

We can thus determine the path on the group manifold as

$$g(t) = e^{i\mu(t)\, t^\gamma}\, g\ . \tag{1.4.26}$$

As a consistency check one verifies that

$$\lim_{t\to\infty} g(t) = \begin{cases} \begin{pmatrix} \frac{a}{|a|} & 0 \\ 0 & \frac{a^*}{|a|} \end{pmatrix} & \text{if } \lambda > 0 \\ \begin{pmatrix} 0 & \frac{b^*}{|b|} \\ -\frac{b}{|b|} & 0 \end{pmatrix} & \text{if } \lambda < 0. \end{cases} \tag{1.4.27}$$

The path is actually a geodesic on $SU(2)$, relating the point g to the nearest diagonal or off-diagonal group element. In order to see this we write

$$g = \begin{pmatrix} \sin\psi\, e^{i\theta} & \cos\psi\, e^{-i\varphi} \\ -\cos\psi\, e^{i\varphi} & \sin\psi e^{-i\theta} \end{pmatrix}\ , \tag{1.4.28}$$

where $0 \le \psi \le \frac{\pi}{2}$ and $0 \le \theta, \varphi < 2\pi$. In these variables, the metric on $SU(2)$ is

$$ds^2 = d\psi^2 + \sin^2\psi\, d\theta^2 + \cos^2\psi\, d\varphi^2\ . \tag{1.4.29}$$

The above path in $SU(2)$ is the path with θ and φ constant. The variable μ (see eq. (1.4.25)) is simply proportional to $\psi - \psi_0$, which is the arc length parameter along the curve.

1.5 Generalisations

It is not difficult to generalise the above analysis in a number of different ways.

1.5.1 The free boson away from criticality

If the radius of the free boson is a rational multiple of the self-dual radius, $R = \frac{M}{N} R_{\rm sd}$, then a similar analysis applies. At this radius, the conformal boundary states are labelled by elements in the quotient space

$$g \in SU(2)/\mathbb{Z}_M \times \mathbb{Z}_N \ , \tag{1.5.1}$$

where $\mathbb{Z}_M$ and $\mathbb{Z}_N$ act by multiplication by roots of unity on a and b, respectively, leaving the absolute values unaffected [97] One way to arrive at this construction is to describe the theory at radius R as a freely-acting orbifold by $\mathbb{Z}_M \times \mathbb{Z}_N$ of the self-dual radius theory [179]. Under this orbifold action none of the generic $SU(2)$ branes are invariant, and thus the branes of the orbifold are simply the superpositions of MN branes of the $SU(2)$ level 1 theory.

In particular, it therefore follows that the bulk-boundary OPE coefficients that were relevant in the above analysis are (up to an MN dependent factor) unmodified. Therefore the same conclusions as above hold: if the radius is increased, a generic brane flows to M equally spaced Dirichlet branes (this is the interpretation of the branes with $b = 0$); if the radius is decreased, a generic brane flows to N Neumann branes whose Wilson lines are equally spaced on the dual circle (*i.e.* the branes with $a = 0$). Since the phases of a and b are unchanged along the flow, the flow is obviously compatible with the $\mathbb{Z}_M \times \mathbb{Z}_N$ orbifold operation that only acts on these phases.

1.5.2 The analysis at higher level

For $SU(2)$ at level k, the branes that preserve the affine symmetry (up to an inner automorphism by conjugation by a group element $g \in SU(2)$) are labelled by $\|j, g\rangle\!\rangle$, where $j = 0, \frac{1}{2}, 1, \ldots, \frac{k}{2}$ denotes the different representations of $\hat{su}(2)$ at level k (that label the different Cardy branes [29]), while g describes the automorphism

$$\left(g\, J^\alpha_m\, g^{-1} + \bar{J}^\alpha_{-m}\right) \|j, g\rangle\!\rangle = 0 \ . \tag{1.5.2}$$

In addition there is the identification,

$$\|j, g\rangle\!\rangle = \|\tfrac{k}{2} - j, -g\rangle\!\rangle \ , \tag{1.5.3}$$

where $-g \in SU(2)$ is minus the 2×2 matrix (1.4.1).

The field Φ is an exactly marginal bulk field for any level k [36, 120]. We can thus ask what happens to the boundary condition $\|j, g\rangle\!\rangle$ as we perturb the theory by Φ.

In fact, it is easy to see that the above analysis for level 1 still goes through — the only place where k enters is in the overall normalisation of the bulk-boundary OPE coefficient that is largely irrelevant for our analysis. Thus if we perturb the theory by the exactly marginal bulk perturbation $J^3\bar{J}^3$, the brane labelled by $\|j,g\rangle\!\rangle$ flows to $\|j,g_0\rangle\!\rangle$, where g_0 is either diagonal or off-diagonal (depending on the sign of the bulk coupling constant λ), and the relevant phase of a_0 or b_0 agrees with the original phase of a or b in g, respectively. In particular, this prescription therefore respects the identification (1.5.3). It is also worth noting that it does not mix different j, and therefore does not produce any additional flows that would reduce the K-theoretic charge group [5, 81].

The bulk perturbation breaks the $SU(2)$ symmetry down to $SU(2)/U(1)\times U(1)$, where the radius of the $U(1)$ factor is deformed away from the original value of $\sqrt{k}$ times the self-dual radius. The branes corresponding to g_0 (to which any brane will flow) describe factorisable boundary conditions that define a standard Dirichlet or Neumann boundary condition for the $U(1)$ factor. It is then clear that these branes exist for arbitrary radius of this $U(1)$ (this has been analysed previously in [76, 77]). The resulting picture is therefore again in agreement with expectations.

For large values of the level k we can give yet another geometric interpretation. The current-current deformation of the WZW model can be understood as deforming the metric, the B-field and the dilaton on the group. In particular, once the WZW model is deformed the dilaton ϕ is not constant any more, but has the dependence (see [120, 108, 76])

$$e^{-2\phi(\psi)} = \frac{1-(1-R^2)\cos^2\psi}{R} \ , \tag{1.5.4}$$

where R denotes the deformed radius of the embedded $U(1)$ ($R=1$ being the WZW case). If we start with a D0-brane on the group at position g, then after the deformation it will flow along the gradient of the dilaton to a maximum, such that its mass, which is proportional to $\frac{1}{g_s}\sim e^{-\phi}$, is minimal. Minimisation of (1.5.4) leads to the conditions

$$(1-R^2)\sin 2\psi = 0 \ , \qquad (1-R^2)\cos 2\psi > 0 \ . \tag{1.5.5}$$

When the radius is increased ($R>1$, corresponding to $\lambda>0$), we find $\psi=\frac{\pi}{2}$, *i.e.* $|a|=1$. For $R<1$ we obtain on the other hand $\psi=0$ ($|b|=1$). This is thus in nice agreement with our analysis of section 3.

1.5.3 Other bulk perturbations

So far we have only considered bulk perturbations by $J^3\bar{J}^3$, but it should be clear how to generalise this to the case where the perturbing bulk field is $J^\alpha\bar{J}^{\bar{\alpha}}$. In fact, if we write $t^\alpha = ht^3h^{-1}$ and $t^{\bar{\alpha}} = \bar{h}t^3\bar{h}^{-1}$, then the above analysis goes through provided we replace g by $\hat{g}=h^{-1}\,g\,\bar{h}$. Indeed, the relevant t^γ is in this case

$$it^\gamma \propto [t^\alpha, g\,t^{\bar{\alpha}}\,g^{-1}] = h[t^3,\hat{g}\,t^3\,\hat{g}^{-1}]h^{-1} \ , \tag{1.5.6}$$

and thus

$$\delta g = \delta(h\,\hat{g}\,\bar{h}^{-1}) = h\,\delta\hat{g}\,\bar{h}^{-1} \ . \tag{1.5.7}$$

At level 1, the perturbation by $J^\alpha\bar{J}^{\bar{\alpha}}$ can again be interpreted as changing the radius of a circle. Its embedding in $SU(2)$ is described as

$$\theta \mapsto he^{i\theta t^3}\bar{h}^{-1} \ . \tag{1.5.8}$$

1.5.4 Higher rank groups

Much of the discussion for $SU(2)$ carries over to Lie groups of higher rank, though in general it is not possible to give a closed expression for the integrated flow any more. For simplicity we shall restrict the following discussion to the Lie groups $G = SU(n)$.

Let us consider a D-brane that is characterised by the gluing condition (1.4.2) for a given $g \in SU(n)$. As in section 3.1, the perturbation $J^\alpha \bar{J}^\alpha$ with α fixed and $t^\alpha \in su(n)$ is exactly marginal in the bulk [36, 120], but leads to a flow of the gluing parameter g as

$$\dot{g} = \frac{\lambda}{2}\,[t^\alpha, t^\beta]\,g\ , \tag{1.5.9}$$

where $t^\beta = g\,t^\alpha\,g^{-1}$. This flow can be interpreted as a gradient flow,

$$\dot{g} = -\nabla V(g) \qquad \text{with potential} \qquad V(g) = -\frac{\lambda}{2}\mathrm{Tr}(t^\alpha\,g\,t^\alpha\,g^{-1})\ . \tag{1.5.10}$$

To see this, we first recall that the gradient is defined by

$$\frac{d}{ds}V(g+istg)\bigg|_{s=0} = -\mathrm{Tr}(\nabla V(g)\,g^{-1}\,it)\ , \tag{1.5.11}$$

where t is an arbitrary vector in the Lie algebra. Here the minus sign appears because the trace is negative definite on the Lie algebra; the factors of g map it to a tangent vector itg at g, and the tangent vector $\nabla V(g)$ to an element of the Lie algebra, $\nabla V(g)\,g^{-1}$. Evaluating the directional derivative we find

$$\begin{aligned}\frac{d}{ds}V(g+istg)\bigg|_{s=0} &= -\frac{\lambda}{2}\mathrm{Tr}\big(t^\alpha\,itg\,t^\alpha g^{-1} - t^\alpha g t^\alpha\,g^{-1}it\big)\\ &= \frac{\lambda}{2}\mathrm{Tr}\big([t^\alpha, gt^\alpha g^{-1}]\,g\,(g^{-1}it)\big)\ .\end{aligned} \tag{1.5.12}$$

Comparing this with (1.5.11) we deduce that

$$\nabla V(g) = -\frac{\lambda}{2}\,[t^\alpha, g\,t^\alpha\,g^{-1}]\,g\ , \tag{1.5.13}$$

which hence implies that (1.5.10) reproduces the flow equation (1.5.9).

In contradistinction to the $SU(2)$ case, however, this flow is generically not a geodesic flow. The change of the direction of the RG flow is

$$\frac{d}{dt}[t^\alpha, t^\beta] \propto [t^\alpha, [t^\beta, [t^\alpha, t^\beta]]] \tag{1.5.14}$$

which is in general not proportional to $[t^\alpha, t^\beta]$. Thus the tangent to the flow is not parallel to a fixed direction in the Lie algebra; this makes it hard to integrate the complete flow in the generic case.

We can nevertheless describe at least qualitatively the end point of the flow. To this end it is sufficient to understand the fixed points of the flow and their stability properties.

A boundary condition corresponding to the gluing condition g is a fixed point of the flow if $[t^\alpha, t^\beta] = 0$. This is only the case if the matrices t^α and t^β have common eigenspaces. Assume that t^α is generic, *i.e.* that all its eigenvalues τ_i are distinct and all eigenspaces $\mathbb{R}v_i$ are one-dimensional. Then $[t^\alpha, t^\beta] = 0$ if and only if g permutes the n eigenspaces and multiplies each one by a phase r_i. This means that there are $n!$ discrete choices for g, each coming with $n-1$ continuous degrees of freedom (note that $\det g = \pm \prod_i r_i = 1$).

This has a simple physical interpretation if the level of the WZW model is 1. Then the theory is equivalent to a compactification on a torus described by the momentum lattice

$$\{(p_L, p_R) \in \Lambda_W \oplus \Lambda_W \ , \ p_L - p_R \in \Lambda_R\} \ , \tag{1.5.15}$$

where Λ_W and Λ_R are the weight and root lattice of $su(n)$, respectively. Without loss of generality we may choose our Cartan subalgebra such that it contains t^α. A group element $g \in SU(n)$ that permutes the eigenvectors v_i acts by conjugation on the root lattice and hence corresponds to some element w_g of the Weyl group. The gluing condition (1.4.2) for the currents J^β then translates into the condition

$$w_g p_L = p_R \tag{1.5.16}$$

for the momenta. This is the gluing condition for the standard torus branes that couple to all momenta p_L (as $w_g p_L - p_L \in \Lambda_R$). The dimension of the brane is given by the number of eigenvalues of w_g that are not equal to 1 (this is the absolute length of w_g). The phases of g then correspond to the positions and Wilson lines of the brane.

These standard torus D-branes are the ones that are unaffected by a perturbation of the size of the torus and they correspond to the fixed points g of the flow equation (1.5.9).

In order to understand where a generic brane flows to, it is furthermore important to understand the stability of the fixed points. Suppose we start with a boundary condition that is very close to one of the fixed points; if the brane is driven back to the fixed point it is *stable*, if it flows away (to some other fixed point) it is *unstable.*

To simplify the discussion we shall work in the eigenbasis $\{v_i\}$ of t^α. Using its spectral decomposition $t^\alpha = \sum \tau_i P_i$, we can rewrite (1.5.9) as

$$\dot{g} = \frac{\lambda}{2} \sum_{i,j} \tau_i \tau_j (P_i \, g \, P_j - g \, P_i \, g^{-1} P_j \, g) \ . \tag{1.5.17}$$

To check the stability of a fixed point $g = S$, consider the ansatz

$$g_{ij}(t) = S_{ij} + \epsilon \, h_{ij}(t) \ . \tag{1.5.18}$$

Here S is the matrix of a fixed point given by a permutation σ and phases r_i, *i.e.*

$$S : \ v_i \mapsto r_i \, v_{\sigma(i)} \ . \tag{1.5.19}$$

In particular, this means that

$$S \, P_i \, S^{-1} = P_{\sigma(i)} \ . \tag{1.5.20}$$

Evaluating (1.5.17) to first order yields

$$\dot{h}_{ij} = \frac{\lambda}{2} \left(\tau_i - \tau_{\sigma(j)}\right) \left(\tau_j - \tau_{\sigma^{-1}(i)}\right) h_{ij} \ . \tag{1.5.21}$$

We easily see that $\dot{h}_{ij} = 0$ for $i = \sigma(j)$; these are the $n-1$ flat directions we have identified before. In order for $g = S$ to be stable, all other components h_{lm} must have negative eigenvalues. Without loss of generality, we may assume that the τ_i are ordered,

$$\tau_1 < \tau_2 < \cdots < \tau_n \ . \tag{1.5.22}$$

Consider then the coefficient for $i = \sigma(p)$. If $\lambda > 0$ the condition is

$$j < p \Rightarrow \sigma(j) < \sigma(p) \ , \tag{1.5.23}$$

i.e. σ grows monotonically, which is only the case for $\sigma = \mathrm{id}$. For $\lambda < 0$, σ must be a decreasing function, *i.e.*

$$\sigma \, : i \mapsto n - i \ . \tag{1.5.24}$$

We thus obtain a very simple result: if $\lambda > 0$, g flows to the identity component; if $\lambda < 0$, the D-brane flows to the component where g inverts the order of the eigenvalues of t^α.

In the torus picture (for $k = 1$), the identity component corresponds to the D0-branes. This is what we expect: if the size increases ($\lambda > 0$) beyond the self-dual radius, the D0-branes are the lightest branes and a generic brane will flow to one of them. If the size decreases ($\lambda < 0$), the physical intuition is less clear, because there is a B-field on the torus which complicates things. The torus branes which are described by the inverse ordering of the eigenvectors correspond to the longest element w_0 in the Weyl group.[2] Its absolute length (minimal number of reflections, or minimal number of transpositions) is given by $\lfloor \frac{n-1}{2} \rfloor$ which gives us the dimension of the D-brane on the torus. In the example of $SU(3)$, the branes which are stable under a perturbation with $\lambda < 0$ are thus D1-branes.

So far we have restricted our discussion to a generic perturbation t^α. It is clear that there are special directions t^α for which the bulk perturbation breaks less symmetry. If two or more eigenvalues of t^α coincide, one observes from (1.5.21) that there are more directions h_{ij} which are unaffected by the flow ($\dot{h}_{ij} = 0$), *i.e.* the dimensions of the moduli spaces of fixed points can grow beyond $n-1$.

For other bulk perturbations $J^\alpha \bar{J}^{\bar{\alpha}}$ with $t^{\bar{\alpha}} \neq t^\alpha$, the discussion is very similar to the one above. Assume that $t^{\bar{\alpha}} = \bar{h}(\sum \bar{\tau}_i P_i)\bar{h}^{-1}$ with eigenvalues $\bar{\tau}_1 < \cdots < \bar{\tau}_n$. Then the arguments above apply if we replace g by $\hat{g} = g\bar{h}$. If the level is 1, we again have an interpretation in terms of a torus in $SU(n)$ which is obtained from the Cartan torus by translation by $\bar{h}^{-1}$ from the right.

For large values of the level k, we can – as in the $SU(2)$ case in section 4.2 – interpret the perturbation as a deformation of the metric, the B-field and the dilaton on the group (see [78]). One would then expect that the group values to which the branes flow are again characterised by the property that they maximise the dilaton; it would be interesting to check this directly.

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[2] Here 'long' refers to the standard length which is the minimal number of reflections at simple roots needed to write w_0, or, in terms of permutations, the minimal number of transpositions of neighbouring elements.

1.6 Outlook

It would be interesting to analyse the phenomena described in this chapter in a time-dependent string theory context. Suppose, for example, that we deform the bulk theory of some D-brane string background infinitesimally so that the D-brane is no longer conformal. One would then expect that the background evolves in a time dependent process towards a configuration in which the D-brane is again conformal. Neglecting closed string radiation, time dependence is essentially incorporated by substituting the first order derivatives in the RG equations by second order time derivatives (see *e.g.* [82, 113]). Since the models we considered are compact, unlike the situation studied in [113] there is no open string radiation that could escape to infinity. In particular, there is therefore no dissipation and the model will undergo eternal oscillations. It would be interesting to study the effects of closed string radiation in the examples we considered above. In particular, by suitably controlling the bulk deformation λ, the process can be made arbitrarily slow .

Our analysis was originally motivated by trying to understand the interpretation of the obstruction of [24]. There $N = 2$ supersymmetric B-type D-branes on the orbifold line $T^4/\mathbb{Z}_4$ of K3 were studied using matrix factorisation and conformal field theory techniques. It was found that a certain B-type brane (namely the brane that stretches diagonally across the two T^2s that make up the T^4) is obstructed against changing the relative radii of the two T^2s; this could be seen both from the matrix factorisation point of view, as well as in conformal field theory.

The analysis above suggests that upon changing the relative radii the brane simply readjusts its angle so that it continues to stretch diagonally across the two tori. From the point of view of conformal field theory, there is no obstruction in this. The obstruction that was observed in the matrix factorisation analysis only means that the resulting brane breaks the B-type supersymmetry, as could also be seen in conformal field theory [24]; for a discussion of such questions see chapter 2.

At least in this example the obstruction therefore does not 'lift' the corresponding bulk modulus. While we have only analysed the disc amplitude, we do not expect any higher order corrections since the brane remains supersymmetric (albeit not B-type supersymmetric). In general, however, one would expect that the backreaction of the brane on the background geometry could lift bulk moduli. We will discuss some aspects of this in chapter 3.

Chapter 2

Symmetries of perturbed CFTs

2.1 Overview

Chiral symmetry algebras play an important role in the construction of exactly solvable conformal field theories (see *e.g.* [157, 22, 89]). Often these symmetries are only present at specific points in the closed string moduli space and are partially broken when the theory is deformed away from these special points. An important example are Gepner models which are rational conformal field theories with $N = 2$ supersymmetry at a special point in the moduli space of Calabi-Yau compactifications of string theory. Starting from such a highly symmetric point, one can explore the moduli space by perturbing the original theory by some marginal operator (see *e.g.* [159, 160, 106, 173, 136]). In this context it is important to know which part of the symmetry algebra survives the deformation. In many situations of interest the string background also involves D-branes. It is then equally important to understand how much symmetry the branes continue to preserve under the bulk deformation.

The operators that describe changes of the closed string moduli are exactly marginal bulk operators. As we have seen in the previous chapter, in the presence of boundaries these operators may cease to be exactly marginal. If this is the case, the bulk perturbation breaks the conformal invariance of the boundary condition and induces a renormalisation group flow on the boundary. In general the resulting boundary condition is then 'far away' from the original boundary condition and it will be difficult to analyse the symmetries it continues to preserve. We shall therefore always assume that no such renormalisation group flow will be induced, *i.e.* that the boundary theory can be smoothly adjusted to the deformation of the bulk. This is equivalent to the statement that the boundary condition continues to be conformally invariant, *i.e.* that it continues to satisfy $T(z) = \bar{T}(\bar{z})$ on the boundary of the upper half plane.

But even if the conformal symmetry is maintained, other symmetries may be broken by the deformation. Let us assume that in the unperturbed theory the boundary preserves a chiral symmetry algebra $\mathcal{A}$. This means that the holomorphic fields $S \in \mathcal{A}$ are related to the anti-holomorphic fields $\bar{S} \in \bar{\mathcal{A}}$ on the boundary of the upper half plane by

$$\omega(S)(z) = \bar{S}(\bar{z}) \ . \tag{2.1.1}$$

Here ω is an automorphism of the symmetry algebra $\mathcal{A}$ that describes how the left- and

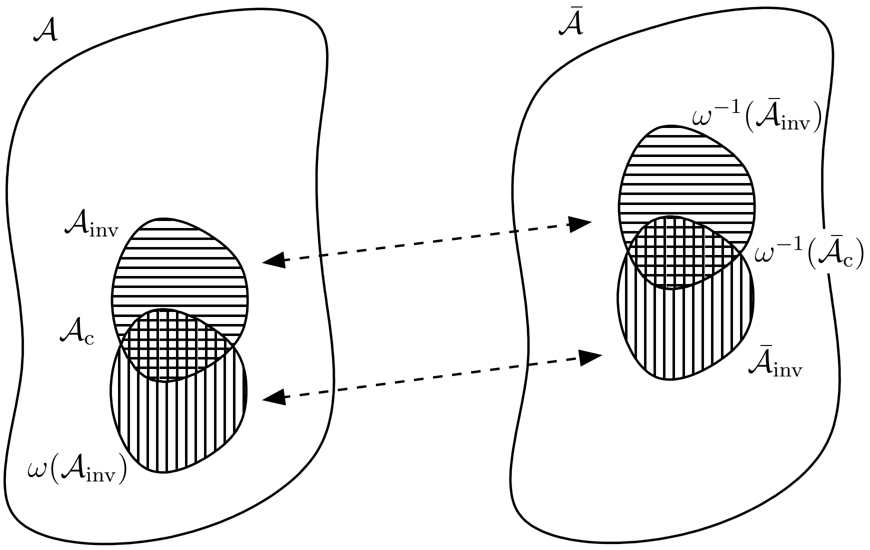

Figure 2.1: An illustration of the gluing of the left- and right-moving chiral algebras: By the gluing condition (2.1.1), the subalgebra $\bar{\mathcal{A}}_{\rm inv}$ is glued to $\omega(\mathcal{A}_{\rm inv})$, whereas $\mathcal{A}_{\rm inv}$ is glued to $\omega^{-1}(\bar{\mathcal{A}}_{\rm inv})$. After the deformation, only the fields in $\mathcal{A}_{\rm inv}$ and $\bar{\mathcal{A}}_{\rm inv}$ stay chiral, which means that it only makes sense to glue the fields in $\mathcal{A}_{\rm c}$ and $\omega^{-1}(\bar{\mathcal{A}}_{\rm c})$.

right-moving chiral algebras are glued together at the boundary; we have also assumed that the anti-chiral symmetry algebra $\bar{\mathcal{A}}$ is isomorphic to $\mathcal{A}$. If $\mathcal{A}$ continues to be a chiral symmetry upon perturbation we can ask whether the gluing condition (2.1.1) is violated or deformed by the perturbation.[1] For the case of a current-current deformation of the bulk we shall see that (2.1.1), with a suitably modified ω, will always continue to hold even after the perturbation. Thus boundary conditions always continue to preserve as much symmetry as they possibly can.

There is however an interesting subtlety that deserves a comment. If the bulk perturbation breaks $\mathcal{A}$ down to $\mathcal{A}_{\rm inv}$, it only makes sense (see figure 2.1) to require that (2.1.1) continues to hold for fields in[2]

$$\mathcal{A}_{\rm c} = \mathcal{A}_{\rm inv} \cap \omega(\mathcal{A}_{\rm inv}) \ . \tag{2.1.2}$$

The above argument then only implies that the boundary theory preserves $\mathcal{A}_{\rm c} \subseteq \mathcal{A}_{\rm inv}$. However, as we shall see, the actual symmetry of the boundary theory is in fact $\mathcal{A}_{\rm inv}$: the spectrum of boundary fields always contains an algebra of mutually local boundary fields associated to $\mathcal{A}_{\rm inv}$, and thus the full spectrum can be organised in (twisted) representations of $\mathcal{A}_{\rm inv}$. The subalgebra $\mathcal{A}_{\rm c} \subseteq \mathcal{A}_{\rm inv}$ also plays a special role: it consists precisely of those fields that are actually local with respect to all other boundary fields; with respect to $\mathcal{A}_{\rm c}$ the full boundary spectrum then forms a conventional (untwisted) representation.

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[1] For boundary perturbations a similar problem was studied in [26, 167, 169].

[2] Here we assume that the left- and right-moving algebras are broken in the same way, as is for example the case for a current-current deformation by $\Phi = J\bar{J}$.

We shall exemplify these findings (and subtleties) with a number of examples, most notably branes on a torus and in a product of $SU(2)$ WZW models. For these cases our results agree with the geometric intuition about the behaviour of branes under bulk deformations. We also consider complex structure deformations of B-type branes in Gepner models for which the perturbing field is not of current-current type. This problem can be conveniently studied using the language of matrix factorisations; the results we obtain are in nice agreement with the expectations based on our general analysis of current-current deformations.

Our setup and notations are the same as in chapter ch:bbp. We will concentrate mostly on a simple class of exactly marginal operators, the current-current deformations $\Phi = J\bar{J}$ with currents J and $\bar{J}$. In the presence of a boundary we shall also assume that $J = \omega(\bar{J})$ at the boundary. The perturbation Φ is then exactly marginal on worldsheets with boundary if the OPE of J and $\omega(J)$ does not contain a marginal or relevant field (see chapter 1). Some of our statements also generalise to more general current-current deformations of the form $\Phi = \sum J_i \bar{J}'_i$; this will be briefly discussed in section 2.5.

We shall assume that the spectrum of L_0 and $\bar{L}_0$ is bounded from below by zero, and that the state with $L_0 = \bar{L}_0 = 0$ is unique (the vacuum). Then it follows for example that the exactly marginal bulk field Φ is a primary field with respect to both Virasoro algebras.

2.2 Bulk symmetries

First we want to investigate under which conditions bulk symmetries are preserved by an exactly marginal bulk deformation. Let us assume that the field S belongs to the chiral symmetry algebra $\mathcal{A}$ before the perturbation. We want to ask whether the corresponding symmetry is preserved under the deformation, *i.e.* whether S remains a chiral field. Equivalently we can investigate whether the operator product expansion with the anti-holomorphic stress-energy tensor $\bar{T}$ remains trivial.

In order to study this question we consider the correlator $\langle \cdots S(z)\bar{T}(\bar{v})\rangle$ and look for singularities in $\bar{z} - \bar{v}$. To first order in the perturbation we find

$$\begin{aligned}
\lambda \int d^2w \, \langle \cdots S(z)\bar{T}(\bar{v})\Phi(w,\bar{w})\rangle & \\
\sim \; & \lambda \int d^2w \left(\frac{1}{(\bar{v}-\bar{w})^2} + \frac{1}{\bar{v}-\bar{w}}\partial_{\bar{w}}\right)\langle \cdots S(z)\Phi(w,\bar{w})\rangle \\
\sim \; & \lambda \int d^2w \, \partial_{\bar{w}}\Big(\frac{1}{\bar{v}-\bar{w}}\langle \cdots S(z)\Phi(w,\bar{w})\rangle\Big) \\
\sim \; & \frac{\lambda}{2i}\oint_{|z-w|=\ell} dw \; \frac{1}{\bar{v}-\bar{w}}\langle \cdots S(z)\Phi(w,\bar{w})\rangle \; . \qquad (2.2.1)
\end{aligned}$$

Here we have only kept the terms that could contribute to a singularity in $\bar{v} - \bar{z}$. From the last expression we see that — in the limit $\ell \to 0$ — we get a contribution precisely if there is a term proportional to $(w-z)^{-1}$ (expanded in w around z); in that case we find a singularity proportional to $(\bar{v}-\bar{z})^{-1}$. [One might have expected that there could also be a term $(\bar{v}-\bar{z})^{-2}$ which in the operator product expansion (OPE) would correspond to a non-vanishing right-moving conformal weight $\bar{h}$, but such a term can only arise at second order.]

We can therefore conclude that the chiral symmetry S is preserved to first order in the perturbation if

$$\lim_{\epsilon\to 0}\oint_{|w-z|=\epsilon} dw\ \Phi(w,\bar{w})S(z) = 0\ . \tag{2.2.2}$$

By assumption this condition must be satisfied if we take S to be the chiral component of the stress energy tensor $S=T$. Indeed, since Φ is marginal we have

$$\Phi(w,\bar{w})T(z) = \frac{1}{(z-w)^2}\,\Phi(w,\bar{w}) + \frac{1}{z-w}\partial_w\Phi(w,\bar{w}) + \mathcal{O}(1)\ , \tag{2.2.3}$$

from which (2.2.2) immediately follows.[3]

The condition (2.2.2) applies to general marginal deformations Φ. In the specific case where Φ is a current-current deformation, $\Phi(w,\bar{w}) = J(w)\bar{J}(\bar{w})$, the above condition simply amounts to the requirement that S does not carry any charge corresponding to J, so the symmetry algebra after deformation is $\mathcal{A}_{\text{inv}} = \{S\in\mathcal{A}, J_0S=0\}$. Similarly, the unbroken anti-chiral symmetry algebra after the deformation is $\bar{\mathcal{A}}_{\text{inv}} = \{\bar{S}\in\bar{\mathcal{A}}, \bar{J}_0\bar{S}=0\}$.

2.2.1 Higher order analysis

The above analysis has only been to first order in the perturbation, and the condition (2.2.2) is therefore only a necessary condition. In order to analyse whether it is also sufficient we have to check whether (2.2.2) remains true even after perturbation. If this is the case, then at every order in perturbation theory we can use the above argument to deduce that we do not obtain a non-trivial OPE between $\bar{T}$ and S.

Unfortunately, the analysis is quite complicated in the general case, but we can say something for the special case where the perturbation is a current-current deformation, $\Phi = J\bar{J}$. We normalise the currents so that

$$J(z)J(w) = \frac{1}{(z-w)^2} + \mathcal{O}(1)\ , \qquad \bar{J}(\bar{z})\bar{J}(\bar{w}) = \frac{1}{(\bar{z}-\bar{w})^2} + \mathcal{O}(1)\ . \tag{2.2.4}$$

The condition (2.2.2) implies that the OPE of J and S is of the form

$$J(w)S(z) = \sum_{n=1}^{h_S} \frac{1}{(w-z)^{n+1}} V(J_nS,z) + \mathcal{O}(1)\ , \tag{2.2.5}$$

i.e. that the $n=0$ term is not present. Here $V(\phi,z)$ denotes the field corresponding to the state ϕ. We want to check whether the property (2.2.2) remains true to next order in perturbation theory. To this end we consider the correlator

$$I = \langle\cdots\Phi(w,\bar{w})\,S(z)\int d^2v\,\Phi(v,\bar{v})\rangle\ , \tag{2.2.6}$$

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[3] In [174] the preservation of conformal symmetry was demonstrated by showing that the modes of the deformed energy-momentum tensor still form a Virasoro algebra with the same central charge.

and look for a term corresponding to a simple pole $(w-z)^{-1}$. The integrand can be calculated by expressing Φ in terms of the currents $\Phi = J\bar{J}$, and by using the operator product expansion of $\bar{J}(\bar{w})$ with the other fields; this leads to

$$I = \int d^2v \left\langle \cdots J(w)S(z)J(v)\frac{1}{(\bar{v}-\bar{w})^2}\right\rangle \tag{2.2.7}$$

$$+ \int d^2v \left\langle \Big(\sum_i \cdots \left[\bar{J}(\bar{w})\phi_i(z_i,\bar{z}_i)\right] \cdots \Big) J(w)S(z)J(v)\bar{J}(\bar{v}) \right\rangle \ , \tag{2.2.8}$$

where we have denoted the singular contribution of the OPE of $\bar{J}(\bar{w})$ with the field $\phi_i(z_i,\bar{z}_i)$ by a square bracket. By the same trick as in (2.2.1), the first integral can be written as a contour integral over dv which encircles the points w, z, as well as the other insertion points z_i,

$$\oint_{w,z,z_i} dv \, \langle \cdots J(w)S(z)J(v)\rangle \frac{1}{\bar{v}-\bar{w}} \ . \tag{2.2.9}$$

The contribution around w does not have any singularity in $w-z$, the contribution from z vanishes by the above assumption (2.2.5), and the contribution from the z_i cannot generate any new pole in $(w-z)$.

In the second integral (2.2.8), we replace $J(w)$ again by the singular terms of its operator product expansions. The contributions from the insertion points z_i and v do not have any singularity in $w-z$, and the contribution from z cannot produce a single pole $(w-z)^{-1}$ again because of (2.2.5).

Thus we can conclude that the condition (2.2.2) also holds in the deformed theory, and the chirality of S is not spoiled at second order. Assuming that J and $\bar{J}$ remain currents along the deformation, this shows that the chirality is preserved to all orders in perturbation theory.

2.2.2 An example: Gepner models

To illustrate the general condition (2.2.2) let us consider a Gepner model [101] corresponding to the Calabi-Yau 3-fold $W=0$ in weighted complex projective space, where

$$W = \sum_{i=1}^{5} x_i^{n_i} \ , \qquad \text{and} \qquad \sum_{i=1}^{5} \frac{1}{n_i} = 1 \ . \tag{2.2.10}$$

The relevant Gepner model is then (an orbifold of) the tensor product of five $N=2$ superconformal minimal models at level $k_i = n_i - 2$. The corresponding bulk conformal field theory possesses the diagonal $N=2$ superconformal symmetry at central charge $c=9$, but also the five individual $N=2$ superconformal symmetries at

$$c_i = \frac{3k_i}{k_i+2} \ , \qquad n_i = k_i + 2 \ . \tag{2.2.11}$$

Complex structure deformations of the Calabi-Yau manifold are described by polynomials in x_i that have the appropriate scaling behaviour in the weighted projective space (see *e.g.* the review [116]); in terms of conformal field theory, these deformations are described by the (cc) fields

$$\hat{\Phi} = \phi^1_{l_1 l_1 0}\, \phi^2_{l_2 l_2 0}\, \phi^3_{l_3 l_3 0}\, \phi^4_{l_4 l_4 0}\, \phi^5_{l_5 l_5 0}\ , \qquad \text{where} \qquad \sum_{i=1}^{5} \frac{l_i}{k_i+2} = 1\ . \tag{2.2.12}$$

Here ϕ^i_{lms} denotes the bulk field corresponding to the representation $(lms) \otimes \overline{(lms)}$ of the i^{th} $N=2$ factor, and (lms) are the usual coset labels of the $N=2$ superconformal algebra — see for example [116]. The field $\hat{\Phi}$ has total conformal weight equal to $h = 1/2$, and total $u(1)$ charge 1, $J_0 \hat{\Phi} = \hat{\Phi}$.

The actual perturbing field is obtained from $\hat{\Phi}$ by application of the $N=1$ supercharges $G = G^+ + G^-$ and $\bar{G} = \bar{G}^+ + \bar{G}^-$. Since $\hat{\Phi}$ is a cc field, $G^+_{-1/2}\hat{\Phi} = \bar{G}^+_{-1/2}\hat{\Phi} = 0$, and hence the perturbing field is $\Phi \equiv G^-_{-1/2}\bar{G}^-_{-1/2}\hat{\Phi}$. Since $\hat{\Phi}$ has total $u(1)$ charge equal to $q = \bar{q} = 1$, it follows that Φ has total $u(1)$ charge equal to zero,

$$J_0\, \Phi = 0\ , \tag{2.2.13}$$

and similarly $\bar{J}_0\, \Phi = 0$.

We can now apply our condition (2.2.2) to the different generators of the diagonal $N=2$ superconformal algebra. Since Φ is (exactly) marginal, the conformal symmetry is preserved (see the discussion following (2.2.2)). As for the total $u(1)$ current of the $N=2$ algebra, taking $S = J$, (2.2.2) vanishes because of (2.2.13). Finally, the operator product expansion of G^+ with Φ is a total derivative,

$$\begin{aligned} G^+(z)\Phi(w,\bar{w}) &= \frac{1}{(z-w)^2} V(G^+_{1/2} G^-_{-1/2}\hat{\Phi}, w) + \frac{1}{z-w} V(G^+_{-1/2} G^-_{-1/2}\hat{\Phi}, w) \\ &= \frac{1}{(z-w)^2} V((2L_0 + J_0)\hat{\Phi}, w) + \frac{1}{z-w} V(2L_{-1}\hat{\Phi}, w) = 2\,\partial_w \left(\frac{1}{z-w} V(\hat{\Phi}, w) \right)\ , \end{aligned} \tag{2.2.14}$$

so that the residue integral (2.2.2) is zero. On the other hand, the OPE with G^- vanishes directly. It thus follows that the diagonal $N=2$ superconformal algebra remains a symmetry under this perturbation. This is certainly what one expects since the perturbation preserves spacetime supersymmetry.

On the other hand, it follows from a similar reasoning that the i^{th} $N=2$ superconformal symmetry is not preserved unless $l_i = 0$. This is obviously also in line with general expectations.

2.2.3 Another example: WZW models and the free boson

As another example we consider a Wess-Zumino-Witten (WZW) model describing strings on a Lie group G. The symmetry algebra $\mathcal{A}$ is generated from the affine Lie algebra g corresponding to G. A current-current deformation $\Phi = J\bar{J}$ singles out a subgroup $U(1) \subset G$ (for a discussion of current-current deformations of WZW models see [120, 78]).

We can decompose the vacuum sector in terms of representations of the coset algebra $g/u(1)$ and the $u(1)$ theory,[4]

$$\mathcal{H}_0^g = \bigoplus_{m\in\mathbb{Z}} \mathcal{H}^{g/u(1)}_{(0,m)} \otimes \mathcal{H}^{u(1)}_m \ . \tag{2.2.15}$$

Projecting onto the states which are uncharged with respect to the $u(1)$ leaves us with the term with $m = 0$. Thus the subalgebra $\mathcal{A}_{\rm inv}$ that is preserved under the deformation is precisely the tensor product of the coset algebra and the $u(1)$ algebra.

A particular example is the $SU(2)$ WZW model at level $k = 1$. It describes the same theory as the free boson compactified on a circle at the selfdual radius. The marginal deformations have been investigated, and the complete connected moduli space is known [106, 51]. When the theory is infinitesimally deformed away from the selfdual radius, the symmetry is broken to the $u(1)$ algebra. When we continue to deform the theory, other points of enhanced symmetry will be reached: in fact at any rational value of the radius squared (when the radius is measured in units of the selfdual radius) there will be an enhanced symmetry. Similar considerations apply to the moduli space of $N = 1$ supersymmetric theories of a free boson and a free fermion on a circle [52].

2.3 Deformed gluing conditions

Up to now we have analysed whether symmetries of the bulk theory remain intact under perturbations by exactly marginal bulk operators. We have seen that a necessary condition for this is (2.2.2). For the case of current-current deformations we have furthermore shown that this condition guarantees that S remains chiral to arbitrary order in perturbation theory.

Now we want to analyse how symmetries of a boundary theory are affected by a bulk perturbation. To this end we introduce a boundary into our theory and consider the conformal field theory on the upper half plane $\mathbb{H}^+$. As we have seen in chapter 1, an exactly marginal bulk perturbation can break the conformal invariance at the boundary and induce a renormalisation group flow. If this is the case, it will be very difficult to make predictions about the symmetries of the fixed point theory (since the fixed point will be, in general, far away from the original boundary theory). We shall therefore restrict ourselves to bulk deformations which do not induce a non-trivial RG flow, and which therefore preserve the conformal invariance on the upper half plane. As has been shown in chapter 1, this will be the case provided that certain bulk-boundary OPE coefficients vanish. We shall give a second, independent proof of this result in the following subsection.

If this condition is satisfied, we can ask how the symmetries of a boundary theory (that come from bulk symmetries) behave under a bulk perturbation. We shall give arguments to suggest that the boundary condition always preserves those gluing conditions that continue to make sense in the bulk.

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[4] Here the $u(1)$ algebra is just the Heisenberg algebra of one current, without the inclusion of any charged fields.

2.3.1 Preserving the conformal invariance

Let us begin by analysing whether the boundary condition remains conformally invariant under a bulk perturbation. Conformal invariance of a boundary condition requires that the energy momentum tensor satisfies the gluing condition

$$T(z) = \bar{T}(\bar{z}) \quad \text{at } z = \bar{z} \ . \tag{2.3.1}$$

We want to study whether this condition remains true under the bulk perturbation, so we have to look at the limit of correlators

$$\lim_{y \to 0} \langle \cdots (T(z) - \bar{T}(\bar{z})) \rangle_\lambda \tag{2.3.2}$$

with $z = x + iy$. The first order correction to the gluing condition comes from

$$\Delta T = \lim_{y \to 0} \lambda \int_{\operatorname{Im} w > \ell/2} d^2 w \, \big(T(z) - \bar{T}(\bar{z})\big) \Phi(w, \bar{w}) \ , \tag{2.3.3}$$

where the expression is understood to be inserted into a correlator. Note that it is important that we first make a Laurent expansion in the regulator ℓ before taking the limit $y \longrightarrow 0$, as otherwise the expression vanishes trivially. Since Φ is primary, the singular part of the OPE in the presence of the boundary is

$$\begin{aligned} \big(T(z) - \bar{T}(\bar{z})\big) \Phi(w, \bar{w}) \sim{}& \frac{1}{(z-w)^2} \, \Phi(w, \bar{w}) + \frac{1}{(z-w)} \, \partial_w \Phi(w, \bar{w}) - ((z, w) \to (\bar{z}, \bar{w})) \\ &+ \frac{1}{(z-\bar{w})^2} \, \Phi(w, \bar{w}) + \frac{1}{(z-\bar{w})} \partial_{\bar{w}} \, \Phi(w, \bar{w}) - ((z, \bar{w}) \to (\bar{z}, w)) \ , \end{aligned}$$

where the second line arises from the mirror images that are required to guarantee that (2.3.1) holds in all correlators. As before in (2.2.1) we can rewrite the right hand side in terms of derivatives with respect to ∂_w and $\partial_{\bar{w}}$; we can thus write ΔT as

$$\begin{aligned} \Delta T = \lim_{y \to 0} \Bigg[& \frac{i\lambda}{2} \int_{\operatorname{Im} w = \ell/2} d\bar{w} \left\{ \frac{1}{z - w} \Phi(w, \bar{w}) - (z \to \bar{z}) \right\} \\ & - \frac{i\lambda}{2} \int_{\operatorname{Im} w = \ell/2} dw \left\{ \frac{1}{z - \bar{w}} \Phi(w, \bar{w}) - (z \to \bar{z}) \right\} \Bigg] \\ = \lim_{y \to 0} & \frac{i\lambda}{2} \int_{\mathbb{R}} du \left\{ \frac{1}{z - u - i\ell/2} - \frac{1}{z - u + i\ell/2} - (z \to \bar{z}) \right\} \Phi(u + i\ell/2; u - i\ell/2) \\ = \lim_{y \to 0} & \frac{i\lambda}{2} \int_{\mathbb{R}} du \left\{ \frac{i\ell}{(z-u)^2 + \ell^2/4} - (z \to \bar{z}) \right\} \Phi(u + i\ell/2, u - i\ell/2) \ . \end{aligned} \tag{2.3.4}$$

Here the minus sign in the second line arises because $d^2 w = \frac{i}{2} \, dw \wedge d\bar{w} = -\frac{i}{2} \, d\bar{w} \wedge dw$. For small ℓ we can now use the bulk-boundary operator product expansion to write

$$\Phi(u + i\ell/2, u - i\ell/2) \sim \sum_i \ell^{h_{\psi_i} - 2} \, B_i \, \psi_i(u) \ , \tag{2.3.5}$$

where ψ_i are boundary fields and the B_i are the bulk boundary coefficients. Since $h_\psi \geq 0$, the most singular term is proportional to ℓ^{-2}; thus we may drop the $\ell^2/4$ term in the denominator of the last line of (2.3.4). The limit $y \to 0$ of the bracket in (2.3.4) is of the form

$$\lim_{y\to 0}\left(\frac{1}{(x+iy-u)^n}-\frac{1}{(x-iy-u)^n}\right)=(-1)^n\frac{2\pi i}{(n-1)!}\,\delta^{(n-1)}(u-x) \qquad (2.3.6)$$

with $n = 2$, and hence leads to a derivative of a delta function. Altogether we therefore find

$$\Delta T = -\pi i\lambda \sum_i \ell^{h_{\psi_i}-1} B_i \partial_x \psi_i(x)\ . \qquad (2.3.7)$$

It follows that the gluing condition for T is violated if in the bulk-boundary operator product expansion of Φ there are relevant or marginal boundary fields ($h_{\psi_i} \leq 1$). The only exception is the vacuum ($h = 0$), since then there is no x-dependence and the derivative vanishes. This analysis reproduces precisely the condition that was found in chapter 1.

In the case of current-current deformations $\Phi = J\bar{J}$, for which $\bar{J} = \omega(J)$ at the boundary with $\omega(J)$ some chiral current, the above condition is simply the requirement that the OPE of J with $\omega(J)$ does not contain a simple pole. In this case the argument generalises to all orders: it is not difficult to show that the OPE of J and $\omega(J)$ will not acquire a pole term under the deformation, so that $\Delta T = 0$ also for finite λ. For a general perturbation, however, the first order criterion only provides a necessary, but not a sufficient condition for Φ to be exactly marginal.

2.3.2 Preserving a general symmetry

Let us now assume that the bulk deformation is exactly marginal on surfaces with boundary so that no relevant or marginal field is switched on at the boundary. In this case the boundary only adjusts infinitesimally to the bulk perturbation and we may hope to make statements about the symmetries it will continue to preserve.

In the following we shall only consider current-current deformations $\Phi = J\bar{J}$ for which $\bar{J} = \omega(J)$ at the boundary. Here ω is an automorphism of the chiral algebra that is preserved by the boundary. As we have just explained, in order for this perturbation to be exactly marginal in the presence of the boundary, we need to have that the OPE of J with $\omega(J)$ does not contain a simple pole.

Suppose now that the boundary condition preserves the symmetry associated to some chiral field S,

$$\omega(S)(z) = \bar{S}(\bar{z}) \quad \text{at } z = \bar{z}\ , \qquad (2.3.8)$$

where ω is an automorphism of the preserved chiral algebra $\mathcal{A}$. We want to ask whether after the perturbation by Φ, (2.3.8) still holds, possibly for some adjusted ω. Obviously for this to make sense we have to require that $\omega(S)$ continues to be a chiral field even after the perturbation ($\omega(S) \in \mathcal{A}_{\text{inv}}$), and similarly for $\bar{S}$; thus we want to assume that (2.2.2) holds for $\omega(S)$, and similarly for $\bar{S}$. Since $\bar{\mathcal{A}} \cong \mathcal{A}$ the latter condition is equivalent to the statement that J does not have a simple pole with S. Altogether we thus require that the OPEs of J with S and $\omega(S)$ do not have simple poles.

There is one subtle point that is worth mentioning. If $\omega(\mathcal{A}_{\text{inv}}) \not\subset \mathcal{A}_{\text{inv}}$, the field $\omega(S)$ can only be chosen from the intersection $\mathcal{A}_{\text{c}} = \mathcal{A}_{\text{inv}} \cap \omega(\mathcal{A}_{\text{inv}})$. The symmetry algebra on the boundary that can arise from gluing bulk fields is then smaller than the symmetry $\mathcal{A}_{\text{inv}}$ that is preserved in the bulk. As we shall explain in section 4, the boundary theory actually still preserves (in a certain sense) the full symmetry algebra $\mathcal{A}_{\text{inv}}$; for the time being, however, we concentrate on the symmetries $\mathcal{A}_{\text{c}}$ that can be understood in terms of gluing conditions.

Let us thus consider a field $\omega(S) \in \mathcal{A}_{\text{c}}$. As in the previous subsection we want to study the expression (this is again to be understood to be inserted into an arbitrary correlator)

$$\begin{aligned}\Delta S &= \lim_{y\to 0} \lambda \int d^2w \, \big(\omega(S)(z) - \bar{S}(\bar{z})\big)\Phi(w,\bar{w}) \\ &= \lim_{y\to 0} \lambda \int d^2w \, \Big(\sum_{n=1}^{h_S} \frac{1}{(w-z)^{n+1}} V(J_n\omega(S),z)\ \bar{J}(\bar{w}) \\ &\qquad\qquad + \sum_{n=1}^{h_S} \frac{1}{(\bar{w}-z)^{n+1}} V(\omega(J)_n\omega(S),z)\ J(w)\ - (z\to\bar{z})\Big) \qquad (2.3.9) \\ &= -\lim_{y\to 0} \frac{i\lambda}{2} \int_{\mathbb{R}} du \, \Big(\sum_{n=1}^{h_S} \frac{1}{n}\frac{1}{(u-z)^n} V(J_n\omega(S),z)\ \omega(J)(u) \\ &\qquad\qquad - \sum_{n=1}^{h_S} \frac{1}{n}\frac{1}{(u-z)^n} V(\omega(J)_n\omega(S),z)\ J(u) - (z\to\bar{z})\Big) \ . \qquad (2.3.10)\end{aligned}$$

We now want to close the contour in the upper half plane. The poles at insertion points of other fields in the correlator cancel in the expression in the limit $y \to 0$. The only pole that can give a contribution is at $u = z$. To determine its residue we use the full OPE of the fields,

$$\omega(J)(u)\ V(J_n\omega(S),z) = \sum_{m\le h_S-n} \frac{1}{(u-z)^{m+1}} V\Big(\omega(J)_m J_n\omega(S),z\Big) \ . \qquad (2.3.11)$$

The residue thus comes from the term with $m = -n$, so that we obtain

$$\Delta S = \pi\lambda \sum_{n=1}^{h_S} \frac{1}{n}\Big[V\Big(\omega(J)_{-n}J_n\omega(S),x\Big) - V\Big(J_{-n}\omega(J)_n\omega(S),x\Big)\Big] \ . \qquad (2.3.12)$$

Introducing the operator

$$K^\omega = -i\sum_{n>0}\frac{1}{n}\Big(\omega(J)_{-n}\,J_n - J_{-n}\,\omega(J)_n\Big) \ , \qquad (2.3.13)$$

we can rewrite the result as

$$\Delta S = \lim_{y\to 0}\lambda\int d^2w\,\big(\omega(S)(z) - \bar{S}(\bar{z})\big)\,\Phi(w,\bar{w}) = i\pi\lambda V(K^\omega\omega(S),x) \ . \qquad (2.3.14)$$

This suggests that we can absorb the change ΔS into a redefinition of the automorphism ω, *i.e.* that we have to first order in the perturbation

$$\omega_\lambda(S) = \omega(S) - i\pi\lambda V(K^\omega \omega(S)) \; . \tag{2.3.15}$$

We need to show that ω_λ (to first order in λ) still defines an automorphism. Suppose that we can write $J = i\partial X^1$ and $\omega(J) = i\partial X^2$, where X^1 and X^2 are free boson fields. Then K^ω is precisely the rotation generator in the $X^1 - X^2$ plane, except that in the definition of K^ω the zero modes are missing. Since by assumption $J_0\omega(S) = \omega(J)_0\omega(S) = 0$, these zero modes can be added to K^ω without modifying the action on $\omega(S)$. Thus we can think of the correction term in (2.3.15) as an infinitesimal rotation generator, implying that (2.3.15) defines indeed an automorphism of the chiral algebra.[5] More generally, we can prove without any further assumptions that (2.3.15) defines an automorphism if S is a current; this is explained in appendix A.1.

2.3.3 Example: Diagonal torus branes

As an example of the results above we consider a diagonal one-dimensional brane on a square torus with radii $R_1 = R_2 = R$, satisfying the gluing conditions

$$J^1(z) = \bar{J}^2(\bar{z}) \; , \qquad J^2(z) = \bar{J}^1(\bar{z}) \; , \qquad z = \bar{z} \; , \tag{2.3.16}$$

where $J^l = i\partial X^l$ is the $u(1)$ current corresponding to the l^{th} direction. We now deform the torus by changing the first of the two equal radii, setting $\Phi = J^1\bar{J}^1$.

Obviously this bulk perturbation preserves the chiral $u(1)$-symmetries, *i.e.* satisfies (2.2.2). Furthermore, if we take the bulk perturbation Φ to the boundary we do not switch on a marginal or relevant field since

$$\Phi(z,\bar{z})\big|_{y\to 0} \sim J^1(x+iy)\, J^2(x-iy) \sim \mathcal{O}(1) \; . \tag{2.3.17}$$

Thus the bulk perturbation is exactly marginal on the disk and we expect that the boundary condition continues to preserve the above $u(1)$-symmetries. The gluing condition, however, will get adjusted as detailed above. In fact, one can guess that the adjustment of the gluing conditions will simply describe the fact that the brane will continue to stretch diagonally across the torus. This motivates us to make the ansatz for the gluing conditions

$$\omega_\lambda(J^1) = -\cos\varphi_\lambda\, J^1 + \sin\varphi_\lambda\, J^2 \tag{2.3.18}$$
$$\omega_\lambda(J^2) = \sin\varphi_\lambda\, J^1 + \cos\varphi_\lambda\, J^2 \; , \tag{2.3.19}$$

where φ_λ is a λ-dependent angle with initial condition $\varphi_0 = \frac{\pi}{2}$. Let us fix some value for the parameter λ, and consider a small shift $\lambda \to \lambda + \delta\lambda$. The change in the gluing condition for J^1 is then given by

$$i\pi\,\delta\lambda\, V(K^{\omega_\lambda}\omega_\lambda(J^1)) = -\pi\,\delta\lambda(\sin^2\varphi_\lambda\, J^1 + \sin\varphi_\lambda\,\cos\varphi_\lambda\, J^2) \; , \tag{2.3.20}$$

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[5] As we shall see this is precisely what happens in the explicit example we are about to discuss.

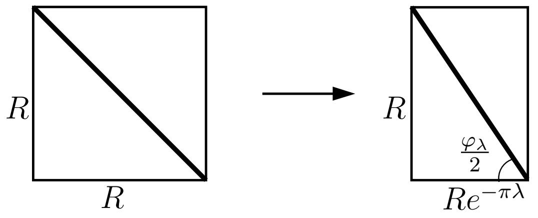

Figure 2.2: When one of the radii in the two-torus is deformed, the brane continues to stretch diagonally and its inclination changes.

and similarly for J^2. If our ansatz is correct, we must be able to absorb this shift into a redefinition of φ_λ. Thus we obtain the differential equations for the angle φ_λ,

$$\frac{d}{d\lambda}\cos\varphi_\lambda = -\pi\sin^2\varphi_\lambda \qquad \frac{d}{d\lambda}\sin\varphi_\lambda = \pi\sin\varphi_\lambda\,\cos\varphi_\lambda\ . \tag{2.3.21}$$

These are both equivalent to the differential equation for φ_λ

$$\frac{d}{d\lambda}\varphi_\lambda = \pi\sin\varphi_\lambda \qquad \text{or} \qquad \frac{d}{d\lambda}\xi(\lambda) = \pi\xi(\lambda)\ , \tag{2.3.22}$$

where $\xi(\lambda) = \tan\frac{\varphi_\lambda}{2}$. The solution is

$$\xi(\lambda) = e^{\pi\lambda} = \tan\frac{\varphi_\lambda}{2}\ . \tag{2.3.23}$$

The gluing condition ω_λ then corresponds to a brane at angle $\frac{\varphi_\lambda}{2}$ which sits on the diagonal of a torus with radii $R_1(\lambda) = Re^{-\pi\lambda}$ and $R_2 = R$, in precise agreement with the geometrical intuition (see figure 2.2).

This is in perfect agreement with the λ-dependence of the deformed radius $R_1(\lambda)$, as we shall now explain. The action for the free boson on the first circle of radius R_1 has the same form as the infinitesimal perturbation,

$$\mathcal{S} + \Delta\mathcal{S} = \frac{1}{2\pi}\int d^2w\,\Phi(w,\bar{w}) - \delta\lambda\int d^2w\,\Phi(w,\bar{w})\ , \tag{2.3.24}$$

with $\Phi = J_1\bar{J}_1$ as above. To get back to the standard normalisation of the action, we have to rescale J_1 to $J_1' = J_1(1-\pi\,\delta\lambda)$ and similarly for $\bar{J}_1$. Since $J = i\partial X_1$ and X_1 has periodicity $2\pi R_1$, the rescaled X_1' has periodicity $2\pi R_1'$, implying for the radius R_1 the differential equation

$$\frac{dR_1(\lambda)}{d\lambda} = -\pi R_1(\lambda)\ , \tag{2.3.25}$$

with the expected solution $R_1(\lambda) = Re^{-\pi\lambda}$.

2.4 Boundary symmetries

Up to now we have discussed the perturbed theory from the point of view of the bulk. In particular, we have analysed whether gluing conditions of the chiral bulk fields may be adjusted upon a bulk deformation. Obviously every bulk field for which we can find a gluing condition gives rise to a symmetry of the boundary theory. However, as we shall see, the converse is not strictly true.

In the following we shall therefore study the boundary symmetries directly from the point of view of the boundary theory. The symmetry algebra of the boundary theory is described by the set of 'holomorphic' fields S (with integer conformal weight) that are local with respect to all other boundary fields (in the sense that there are no branch cuts in the boundary OPEs). The full spectrum of boundary fields then forms a representation of this algebra. In a weaker sense, we can think of a boundary symmetry as an algebra of boundary fields that are only mutually local (but not necessarily local with respect to all the other boundary fields). This condition guarantees that these fields define a (conventional) vertex operator algebra. It is then clear that the full boundary spectrum also forms a representation of this vertex operator algebra, but in general the representations that appear in the spectrum will be twisted rather than the usual untwisted representations. (Twisted representations are characterised by the property that the monodromy of the operator in the vertex operator algebra, when taken around the field in question, may not be trivial; the general theory of twisted representations of vertex operator algebras has been developed in [53].)

To study the behaviour of the boundary symmetries under bulk deformations we thus need to determine changes in the boundary operator product expansion of two boundary fields. From this we will be able to read off changes in the conformal weights as well as in the locality properties of these boundary operators. For simplicity we shall only analyse current-current deformations $\Phi = J\bar{J}$ for which $\bar{J} = \omega(J)$ at the boundary. The results we shall obtain (see the following section) can then be summarised as follows:[6]

(i) A boundary field S retains its conformal weight if and only if $J_0\,\omega(J)_0 S = 0$.

(ii) A boundary field S retains its conformal weight and remains self-local if and only if $J_0 S = 0$ or $\omega(J)_0 S = 0$.

(iii) Two mutually local fields S_1, S_2 that satisfy (i) and (ii) remain mutually local to one another if and only if either $J_0 S_1 = J_0 S_2 = 0$ or $\omega(J)_0 S_1 = \omega(J)_0 S_2 = 0$.

(iv) A field S continues to be local with respect to all other boundary fields if $J_0 S = 0$ and $\omega(J)_0 S = 0$.

Note that $\omega(J)_0 S = 0$ is equivalent to $J_0\,\omega^{-1}(S) = 0$ since ω is an automorphism of the chiral algebra.

Case (iv) describes the strongest situation in which S remains a true symmetry of the boundary theory. If S arises from the gluing of bulk fields, then condition (iv) coincides with what we obtained in the last section, and the algebra of local fields is precisely the algebra $\mathcal{A}_{\rm c} = \mathcal{A}_{\rm inv} \cap \omega(\mathcal{A}_{\rm inv})$. [Note that $\mathcal{A}_{\rm c}$ consists of those fields S for which $J_0 S = 0$ and $J_0\,\omega^{-1}(S) = 0$.] On the other hand, we see that the condition to have self-local fields (condition (ii)) or an algebra of mutually local fields (condition (iii)) is

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[6] To prove that the conditions in (ii) and (iii) are necessary we have assumed that the perturbation is hermitian.

weaker. In particular the boundary has a symmetry algebra (in the above weak sense) that may be larger than what we get from gluing bulk symmetries. For example, if in the unperturbed theory the boundary preserves the full chiral algebra $\mathcal{A}$ of the bulk, then the set of boundary fields $\{S \in \mathcal{A}, J_0 S = 0\}$ forms an algebra of mutually local fields that is isomorphic to the algebra $\mathcal{A}_{\text{inv}}$ which is preserved in the bulk. In this sense, the boundary preserves the same symmetries as the bulk under the deformation. We shall illustrate the different conditions and their interpretation in an explicit example in section 2.4.3. We shall also see that these results have interesting implications for the structure of the open string moduli space; this will be explained in section 2.4.2.

2.4.1 The deformed boundary OPE

Let us now study the deformed OPE of two boundary fields $S_1(x_1)$ and $S_2(x_2)$. To this end we insert these two fields into arbitrary perturbed correlators and look for singularities in $x_1 - x_2$. To first order, the change in the OPE arises from the term

$$\int_{\mathbb{H}^+} d^2w \, S_1(x_1) S_2(x_2) \Phi(w, \bar{w}) \ , \tag{2.4.1}$$

where the integral is regulated by the prescription $\text{Im} w > \ell/2$. A change of the relative locality is indicated by a logarithmic term $\log(x_1 - x_2)$.

As before we assume that $\Phi = J\bar{J}$ with $\bar{J} = \omega(J)$ at the boundary. Furthermore, the OPE of J and $\omega(J)$ does not have a simple pole (since otherwise the perturbation will induce a non-trivial RG flow at the boundary); it is thus of the form

$$J(w)\, \omega(J)(\bar{w}) \sim \frac{C}{(w - \bar{w})^2} + \mathcal{O}(1) \ , \tag{2.4.2}$$

with some constant C.

By the usual recursive procedure, we can evaluate (2.4.1) by using the singular part of the OPE of $J(w)$ with the other fields. Since the singular term with $\bar{J}$ is independent of x_1 and x_2, it does not give rise to the term of interest. The other two OPEs on the other hand lead to

$$\int_{\mathbb{H}^+} d^2w \left[\sum_{m=0}^{h_{S_1}} \frac{1}{(w - x_1)^{m+1}} V(J_m S_1, x_1)\, S_2(x_2)\, \omega(J)(\bar{w}) \right. \\ \left. + \sum_{m=0}^{h_{S_2}} \frac{1}{(w - x_2)^{m+1}} S_1(x_1)\, V(J_m S_2, x_2)\, \omega(J)(\bar{w}) \right] \ . \tag{2.4.3}$$

We then apply the same recursive procedure for $\omega(J)$. For each of the above two terms there are in turn two terms, where the OPE of $\omega(J)$ with the fields at x_1 and x_2 is considered. Since we are only interested in a contribution proportional to $\log(x_1 - x_2)$, only the mixed terms can contribute

$$\int_{\mathbb{H}^+} d^2w \sum_{m=0}^{h_{S_1}} \sum_{n=0}^{h_{S_2}} \Bigg(V(J_m S_1, x_1)\; V(\omega(J)_n S_2, x_2) \frac{1}{(w-x_1)^{m+1}} \frac{1}{(\bar{w}-x_2)^{n+1}} \\ + V(\omega(J)_m S_1, x_1)\; V(J_n S_2, x_2) \frac{1}{(\bar{w}-x_1)^{m+1}} \frac{1}{(w-x_2)^{n+1}} \Bigg) \ . \tag{2.4.4}$$

To evaluate the integrals, let us first consider the terms with $n > 0$. By the familiar trick the integral can then be rewritten as an integral over the real axis,

$$\int_{\mathbb{H}^+} d^2w \, \frac{1}{(w-x_1)^{m+1}} \frac{1}{(\bar{w}-x_2)^{n+1}} = \frac{i}{2n} \int_{\mathbb{R}+i\ell/2} dw \, \frac{1}{(w-x_1)^{m+1}} \frac{1}{(\bar{w}-x_2)^{n}} \\ = \frac{i}{2n} \int_{\mathbb{R}} dx \, \frac{1}{(x-x_1+i\frac{\ell}{2})^{m+1}} \frac{1}{(x-x_2-i\frac{\ell}{2})^{n}} \ . \tag{2.4.5}$$

As $n > 0$, the integral falls off fast enough so that we can close the contour in the lower half plane. By the residue theorem, the result is then some inverse power of $x_1 - x_2$, but certainly not logarithmic. The same argument applies for $m > 0$, thus we can concentrate on $m = n = 0$.
In this case we have

$$\int_{\mathbb{H}^+} dxdy \, \frac{1}{x+iy-x_1} \frac{1}{x-iy-x_2} = \int_{\ell/2}^{\Lambda} dy \, \frac{2\pi i}{2iy + (x_2 - x_1)} \tag{2.4.6}$$

$$= \pi \log(2iy + (x_2 - x_1)) \Big|_{\ell/2}^{\Lambda} \ , \tag{2.4.7}$$

where we have introduced an infrared cut-off Λ. For $\Lambda \to \infty$, the corresponding term is independent of x_1, x_2 and thus harmless. The ℓ-term however produces a (real) logarithmic term in $x_1 - x_2$. As the second integral in (2.4.4) is just the complex conjugate of the first, the condition that S_1 and S_2 stay mutually local is therefore

$$V(J_0 S_1, x_1)\; V(\omega(J)_0 S_2, x_2) + V(\omega(J)_0 S_1, x_1)\; V(J_0 S_2, x_2) = 0 \ . \tag{2.4.8}$$

This should hold inside arbitrary correlators.

We can now use this result to derive the conditions (i) – (iv) from the beginning of this section. A boundary field $S = S_1$ will change its conformal weight if there exists a boundary field S_2 (its conjugate field) for which the two-point function picks up a logarithmic term. For this to be absent we therefore require that the vacuum expectation value of (2.4.8) vanishes. By a usual contour deformation argument we may move the zero mode acting on the field at x_2 to the field at x_1; since J_0 and $\omega(J)_0$ commute (because J and $\omega(J)$ do not have a simple pole), we then obtain the condition

$$2\langle V(J_0\, \omega(J)_0 S, x_1)\, V(S_2, x_2)\rangle = 0 \ . \tag{2.4.9}$$

Since the two-point function on the boundary is non-degenerate, this can only be the case for all S_2 if $J_0\, \omega(J)_0 S = 0$, thus proving (i).

A boundary field S will stay in addition self-local, if the OPE of S with itself does not have any logarithmic coefficients. This must be the case in arbitrary correlators. Thus the condition means that (2.4.8) with $S = S_1 = S_2$ must vanish as an operator identity. By considering the OPE with an arbitrary field $V(\phi, z)$ in the limit where $z \to x_1$ and $z \to x_2$, this can only be the case if either $J_0 S = 0$ or $\omega(J)_0 S = 0$, or if $J_0 S$ and $\omega(J)_0 S$ are proportional to one another (so that the two terms in (2.4.8) cancel). In the last case, using (i), it follows that $J_0 J_0 S = 0$ and $\omega(J)_0\, \omega(J)_0 S = 0$. If the perturbation is hermitian, *i.e.* if both J_0 and $\bar{J}_0$ are self-adjoint operators, we may diagonalise J_0. Then $J_0 J_0 S = 0$ implies that $J_0 S = 0$, and hence either $J_0 S = 0$ or $\omega(J)_0 S = 0$, thus proving (ii).

Now consider two mutually local fields S_1 and S_2 that satisfy (i) and (ii). The condition that they remain mutually local is again that (2.4.8) holds inside an arbitrary correlator, *i.e.* as an operator identity. Because of (ii), either $J_0 S_1 = 0$ or $\omega(J)_0 S_1 = 0$, and either $J_0 S_2 = 0$ or $\omega(J)_0 S_2 = 0$. It is then obvious that (2.4.8) only vanishes if either $J_0 S_1 = J_0 S_2 = 0$ or $\omega(J)_0 S_1 = \omega(J)_0 S_2 = 0$, thus giving (iii).

Finally if we want a field S to stay local relative to all fields S', (2.4.8) must hold as an operator identity for $S_1 = S$ and $S_2 = S'$ arbitrary. This is obviously the case if $J_0 S = \omega(J)_0 S = 0$. Thus we obtain (iv).

All the given arguments can be generalised to higher orders; in appendix A.2 this is explained for the analysis of (ii), but the line of arguments is similar in the other cases.

2.4.2 Open string moduli space

These considerations also have some interesting implications for the structure of the open string moduli space. The moduli space is spanned by the exactly marginal boundary fields. The fields S that keep conformal weight $h = 1$ and thus stay marginal under the bulk deformation satisfy $J_0 \omega(J)_0 S = 0$. This does not guarantee however that the fields remain *exactly* marginal. As was shown in [169] a sufficient criterion for exact marginality is that the marginal field is self-local. The criterion for self-locality (ii) thus provides a characterisation of at least some of the exactly marginal boundary fields.

On the other hand, at least to first order in perturbation theory, the condition for exact marginality of S is only that no non-trivial relevant or marginal fields appear in the OPE of S with itself. Thus the space of exactly marginal boundary fields (that parameterise the open string moduli space) could be bigger than just the self-local marginal fields. Actually, from the above analysis of the perturbed OPE it is clear that the only modification of the OPE of a marginal field S with itself appears in the form of terms containing a logarithm, which implies that the only effect is to change the conformal weights of the fields that appear in the OPE. If in the original theory the only fields that appear in the OPE of S with itself have $h > 1$, then this will continue to be so, at least for some finite range of λ. In particular, one should therefore expect that exactly marginal boundary fields that retain their conformal weight $h = 1$ under the deformation (*i.e.* that satisfy (i)) will continue to be exactly marginal for finite, but maybe small λ. In general there may thus be more exactly marginal fields than those characterised by (ii); we shall see an example of this phenomenon in section 2.4.3.

One can also arrive at this conclusion from a different point of view. Suppose that S is an exactly marginal boundary field before the deformation. Since we are only considering bulk deformations Φ that are exactly marginal (in the presence of the boundary), we know that no marginal or relevant boundary fields appear in the bulk-boundary OPE of Φ. If the direction in the open string moduli space corresponding to S survives the bulk deformation, then Φ must also remain exactly marginal with respect to the perturbed boundary condition. To analyse the bulk-boundary OPE of Φ in the deformed boundary theory, we look at the deformed correlators containing $\Phi(w, \bar{w})$ and look for singularities in $w - \bar{w}$. The first order contribution is

$$\Phi(w, \bar{w}) \int_{\mathbb{R}} dx \, S(x) \ , \tag{2.4.10}$$

where, as usual, the expression is understood as being inserted into correlators. Writing $\Phi = J\bar{J}$, and using the OPE of J and $\omega(J)$ with S, we see that the only terms that could change the singular terms in $w - \bar{w}$ are

$$\int_{\mathbb{R}} dx \sum_{\substack{m,n \geq 0 \\ m+n \leq 1}} V(\omega(J)_n J_m S, x) \frac{1}{(w-x)^{m+1}(\bar{w}-x)^{n+1}} \ . \tag{2.4.11}$$

Each summand gives a contribution $\sim (w - \bar{w})^{-m-n-1}$, so the only problematic term is the one with $m = n = 0$. For this term to be absent, we need that $\omega(J)_0 J_0 S = 0$. This coincides with the condition (i) that S does not change its conformal weight under the bulk deformation Φ.

2.4.3 Example: Deformed $SU(2) \times SU(2)$ permutation branes

Let us illustrate our analysis in an example. We consider the product of two $SU(2)$ WZW models at level k. On the upper half plane we impose permutation gluing conditions on the currents (see [72, 168]) corresponding to the automorphism

$$\omega(J^{(1)}) = g \, J^{(2)} \, g^{-1} \ , \qquad \omega(J^{(2)}) = h \, J^{(1)} \, h^{-1} \ , \tag{2.4.12}$$

where g and h are group elements in $SU(2)$. Since this gluing condition preserves the full $su(2)_k \times su(2)_k$ symmetry, the boundary spectrum of each permutation brane forms a (conventional) representation of $su(2)_k \times su(2)_k$. In the simplest case (the brane associated to the identity representation with g and h being arbitrary) the spectrum takes the form

$$\mathcal{H} = \bigoplus_{l=0}^{k} \mathcal{H}_l^{su(2)_k} \otimes \mathcal{H}_l^{su(2)_k} \ . \tag{2.4.13}$$

Now we perturb the theory by the operator $\Phi = J_3^{(1)} \bar{J}_3^{(1)}$, *i.e.* we deform[7] the $U(1)$ sitting inside the first $SU(2)$. The symmetry in the bulk is broken to

$$\mathcal{A}_{\text{inv}} = su(2)_k / u(1)_{2k} \times u(1) \times su(2)_k \ . \tag{2.4.14}$$

---

[7] The deformation of untwisted D-branes in a single copy of $SU(2)$ has been analysed in [76, 77, 79].

The chiral bulk symmetry thus only contains four fields of conformal weight one after the perturbation; these are $J_3^{(1)}$ and $J_a^{(2)}$. We also note that the bulk deformation is exactly marginal in the presence of the permutation boundary, because the OPE of $J_3^{(1)}$ and $\omega(J_3^{(1)}) = gJ_3^{(2)}g^{-1}$ is non-singular.

As we have just seen, the chiral and anti-chiral fields $J_\pm^{(1)}$ and $\bar{J}_\pm^{(1)}$ do not remain chiral under the deformation. After the deformation we therefore cannot glue $J_\pm^{(1)}$ to $h^{-1}\bar{J}_\pm^{(2)}h$ and $\bar{J}_\pm^{(1)}$ to $gJ_\pm^{(2)}g^{-1}$ any more. The chiral algebra that can still be preserved by gluing bulk fields is therefore only

$$\mathcal{A}_{\rm c} = \mathcal{A}_{\rm inv} \cap \omega(\mathcal{A}_{\rm inv}) = \frac{su(2)_k}{u(1)_{2k}} \times u(1) \times \frac{su(2)_k}{u(1)_{2k}} \times u(1) \ . \qquad (2.4.15)$$

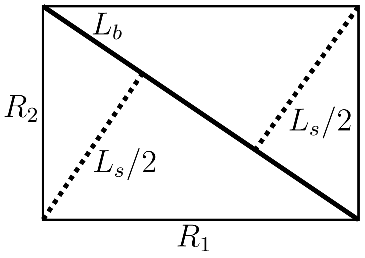
Thus we are in the situation where $\mathcal{A}_{\rm c} \subsetneq \mathcal{A}_{\rm inv}$. Note that $\mathcal{A}_{\rm c}$ has only two fields of conformal weight one.

Now we turn to the boundary description of the system. The boundary fields that belong to the $SU(2)$ currents $J_\pm^{(1)}$ and $gJ_\pm^{(2)}g^{-1}$ keep their conformal weight: they satisfy criterion (i) of section 2.4, because they are either annihilated by the zero mode of $J_3^{(1)}$ or by the zero mode of $\omega(J_3^{(1)}) = gJ_3^{(2)}g^{-1}$. The boundary theory therefore continues to have six marginal fields. Furthermore, since they are exactly marginal in the original theory, they remain exactly marginal, at least for some finite perturbation. (As we shall see momentarily, they will actually remain exactly marginal for arbitrary finite perturbations.) This is in agreement with the arguments of section 2.4.2; the six-dimensional moduli space of permutation branes should survive the perturbation, because the bulk deformation Φ is exactly marginal for arbitrary g and h. In the perturbed theory, these six degrees of freedom can be described as follows: two parameterise the choice of gluing $J_3^{(1)}$ to any current of the second, undeformed $su(2)$, similarly two come from gluing $\bar{J}_3^{(1)}$, and the remaining two come from the two $u(1)$s that are conserved by the brane.

Not all of these fields are however mutually local, and therefore arbitrary linear combinations will not be self-local. Given the analysis of section 2.4 we expect that a subalgebra of fields isomorphic to $\mathcal{A}_{\rm inv}$ remains mutually local. For example we can take the mutually local fields to be those that are annihilated by the zero mode of $J_3^{(1)}$. This subalgebra contains then four fields of $h = 1$, namely $J_3^{(1)}$ as well as the three fields $J_a^{(2)}$ from the second $su(2)_k$.

Finally, the fields that are local with respect to all boundary fields are those that are annihilated by both the zero mode of $J_3^{(1)}$ and $J_3^{(2)}$; this algebra is then precisely $\mathcal{A}_{\rm c}$ and contains only two fields of weight one, namely $J_3^{(1)}$ and $J_3^{(2)}$.

We can check these assertions by computing the boundary spectrum. The deformed theory is a $\mathbb{Z}_k \times \mathbb{Z}_k$ orbifold of the product of two parafermion theories $su(2)_k/u(1)_{2k}$ and a square torus, with permutation gluing conditions in the coset part and a diagonal one-dimensional brane on the torus (similar to the situation in section 2.3.3). The permutation boundary state on the parafermions is not affected by the perturbation, and it is straightforward to determine the boundary states for the diagonal branes on the deformed torus. It is then not hard to obtain the boundary states in the orbifolded theory and from there the boundary spectra.



$$L_b = \sqrt{R_1^2 + R_2^2}$$

$$L_s = \frac{R_1 R_2}{L_b}$$

Figure 2.3: The diagonal brane with length L_b and a string with length L_s that winds perpendicular to the brane around the torus.

We shall take a shorter route here which uses geometric arguments. First we decompose the boundary spectrum (2.4.13) of the unperturbed theory into representations of $su(2)_k/u(1)_{2k}$ and $u(1)_{2k}$,

$$\mathcal{H} = \bigoplus_{l,m_1,m_2} \mathcal{H}_{(l,m_1)} \otimes \mathcal{H}_{(l,m_2)} \otimes \mathcal{H}^{u(1)_{2k}}_{m_1} \otimes \mathcal{H}^{u(1)_{2k}}_{m_2} \ , \tag{2.4.16}$$

where m_1, m_2 run from $-k+1$ up to k (with the condition $l + m_i$ even). The product of the $u(1)_{2k}$ representations can be explicitly expressed in terms of momentum and winding modes. On the level of characters we have ($m_1 + m_2$ is even)

$$\chi_{m_1}(q)\chi_{m_2}(q) = \frac{1}{\eta^2(q)} \sum_{n\in\mathbb{Z}} \sum_{m=\frac{m_1+m_2}{2}+nk \mod 2k} q^{\frac{m^2}{2k}+(n-\frac{m_1-m_2}{2k})^2\frac{k}{2}} \ . \tag{2.4.17}$$

Here, m corresponds to the momentum modes of the open strings, and n corresponds to the winding modes. When we deform the radius of the first $U(1)$, $R_1 \to \kappa R_1$, the contribution of momentum and winding modes change. The length of the brane is changed to $\sqrt{R_1^2 + R_2^2}$, which means that the conformal weight of a momentum mode is changed by the factor $2/(1+\kappa^2)$. A string winding perpendicular to the brane has length $R_1 R_2/\sqrt{R_1^2 + R_2^2}$, so the conformal weight of a winding mode is changed by a factor of $2\kappa^2/(1+\kappa^2)$ (see figure 2.3). Hence the boundary partition function of the permutation brane in the perturbed theory is

$$Z(q) = \sum_{l,m_1,m_2} \sum_{n} \sum_{m=\frac{m_1+m_2}{2}+nk \mod 2k} \chi_{(l,m_1)}(q)\chi_{(l,m_2)}(q) \frac{1}{\eta^2(q)} q^{\frac{m^2}{k(1+\kappa^2)}+(n-\frac{m_1-m_2}{2k})^2\frac{k\kappa^2}{(1+\kappa^2)}} \ . \tag{2.4.18}$$

In particular, the partition function can be written in terms of untwisted representations of $\mathcal{A}_{\rm c}$, as anticipated. Given our analysis we expect, however, that we can also write the partition function in terms of (twisted) representations of $\mathcal{A}_{\rm inv}$. To see that this is indeed possible we rewrite the partition function as

$$Z(q) = \sum_{l,m_1,m_2} \sum_{N,M} \chi_{(l,m_1)}(q)\chi_{(l,m_2)}(q) \frac{1}{\eta^2(q)} q^{\left[(\frac{m_1+m_2}{2k}+M+N)^2+(N-\frac{m_1-m_2}{2k}-M)^2\kappa^2\right]\frac{k}{1+\kappa^2}} \ , \tag{2.4.19}$$

where we introduced new summation variables M, N which replace the variables m, n in (2.4.18) by $m = \frac{m_1+m_2}{2} + (N+M)k$, $n = N - M$. A simple transformation of the exponent of q yields now

$$Z(q) = \sum_{l,m_1,m_2} \sum_{N,M} \chi_{(l,m_1)}(q) \chi_{(l,m_2)}(q) \frac{1}{\eta^2(q)} q^{\frac{(m_2+2Nk)^2}{4k} + \frac{(m_1+2Mk)^2}{4k} + \frac{1}{2k}(m_2+2Nk)(m_1+2Mk)\frac{1-\kappa^2}{1+\kappa^2}} . \tag{2.4.20}$$

The only effect of the deformation $\kappa \neq 1$ is the term that is a product of the two $u(1)$ charges. Introducing twisted $su(2)$ characters,

$$\hat{\chi}_{l,\theta}(q) = \sum_{m \in \mathbb{Z}} \chi_{(l,m)}(q) \frac{1}{\eta(q)} q^{\frac{m^2}{4k} + \theta m} , \tag{2.4.21}$$

we can rewrite the partition function as

$$Z(q) = \sum_{l,m_1} \sum_{M} \chi_{(l,m_1)}(q)\, \hat{\chi}_{l,(M+\frac{m_1}{2k})\frac{1-\kappa^2}{1+\kappa^2}}(q)\, \frac{1}{\eta(q)}\, q^{\frac{(m_1+2Mk)^2}{4k}} . \tag{2.4.22}$$

Thus the partition function can indeed be written in terms of twisted $su(2)$ characters, coset characters and $u(1)$ characters, giving strong support to our claim that the boundary spectrum forms a (twisted) representation of $\mathcal{A}_{\text{inv}}$.

Finally, we want to check our assertion that the two sets of $SU(2)$ currents remain marginal (and in fact exactly marginal). For concreteness let us consider the $SU(2)$ current $J_+^{(1)}$ that appears in the representation with $l = 0$, $m_1 = 2$, $m_2 = 0$, $n = 0$, $m = 1$. We can easily evaluate the conformal weight of this mode in the perturbed theory,

$$h = h_{(0,2)} + h_{(0,0)} + \frac{1}{k(1+\kappa^2)} + \frac{\kappa^2}{k(1+\kappa^2)} = 1 , \tag{2.4.23}$$

where $h_{(0,2)} = 1 - \frac{1}{k}$ and $h_{(0,0)} = 0$. The analysis is similar for $J_-^{(1)}$ and $J_\pm^{(2)}$. Thus it follows that for all values of κ, there are six marginal fields which come from the $SU(2) \times SU(2)$ symmetry of the undeformed theory. As we have explained before they describe the six exactly marginal boundary fields that generate the six-dimensional moduli space of the brane.

2.4.4 Matrix factorisation examples

As a final example we consider theories for which the perturbation is not a current-current deformation. While we cannot study these cases in general, there is an interesting class of examples for which we can test the above ideas. These are B-type branes in Gepner models of Calabi-Yau 3-folds discussed in section 2.2.2. As was explained there, under complex structure deformations, the diagonal $N = 2$ algebra continues to be a symmetry of the bulk conformal field theory. Suppose we consider a B-type D-brane in this background. Then we can ask whether it will continue to be a B-type $N = 2$ brane under these complex structure deformations. This problem is very accessible since B-type branes in these theories can be described in terms of matrix factorisations (see [131, 25, 132] for some early references and [123] for a review) of the associated Landau-Ginzburg model.

Our arguments above now suggest that the brane will remain B-type provided that the bulk perturbation is exactly marginal in the presence of the brane, *i.e.* that no relevant or marginal boundary field is induced by the bulk perturbation. As in the situation discussed in [17], a non-trivial RG flow will be induced if and only if the bulk-boundary operator product coefficient $B_{\Phi\psi} \neq 0$ for some boundary fermion ψ of $u(1)$ charge 1. Thus we want to show that the matrix factorisation is obstructed against the bulk perturbation by Φ if and only if $B_{\Phi\psi} \neq 0$ for some boundary fermion ψ of $u(1)$ charge 1.

From a matrix factorisation point of view a D-brane remains B-type if we can adjust the matrix factorisation Q as $Q(\lambda) = Q + \lambda Q_1 + \cdots$ so that $Q(\lambda)$ is a matrix factorisation of $W(\lambda) = W + \lambda\Phi$. To first order in λ a necessary and sufficient condition for this is that Φ is exact with respect to Q [124]. If Φ is exact, then the bulk-boundary correlator $B_{\Phi\psi}$ vanishes for all boundary fields ψ, as follows by standard Landau-Ginzburg arguments. Thus if the matrix factorisation can be adjusted as above, the bulk-boundary correlator $B_{\Phi\psi}$ vanishes, and no marginal or relevant boundary field is turned on.

To show the converse direction we need to prove that if Φ is not exact with respect to Q_0 (so that the matrix factorisation adjustment is obstructed), then there is a boundary field ψ of $u(1)$ charge 1 such that $B_{\Phi\psi} \neq 0$. If Φ is not exact, this means that Φ taken to the boundary induces a chiral primary field on the boundary. From conformal field theory we then know that there is a non-trivial bulk boundary correlator involving these two fields; since both are chiral primary fields, this amplitude is then also non-trivial in the topologically twisted theory, and hence $B_{\Phi\psi} \neq 0$ for some boundary field ψ. Furthermore, it follows from charge conservation considerations in the Landau-Ginzburg theory that the bulk-boundary OPE can only be non-trivial if ψ is a fermion of charge 1. This then proves our claim.

As an aside we note that the above analysis only applies to the Calabi-Yau 3-fold case. For the case of K3 the charge conservation analysis implies that the only boundary field that can have a non-trivial bulk-boundary correlator with the complex structure deformation Φ is a boson of charge 0. Such a field does *not* correspond to an exactly marginal boundary field. However, this does not invalidate our claim concerning the symmetries of the boundary since in the K3 case there is in fact $N = 4$ supersymmetry, and a brane need not remain B-type with respect to any $N = 2$ subalgebra even if it continues to preserve the full $N = 4$ superconformal algebra — see [24] for an example. Finally, we also note that the above point of view suggests how the criterion of [24] for obstructions of matrix factorisations on K3 against complex structure deformations can be sharpened: the matrix factorisation will be obstructed to first order if and only if $B_{\Phi\psi} \neq 0$ for some boundary boson of charge 0. One easily checks that the examples of [24] are in agreement with this criterion.

2.5 Summary

As the results of our analysis are somewhat subtle, we will briefly summarise the main points. For a pure bulk theory, if the perturbation is a current-current field $\Phi = J\bar{J}$, then the surviving symmetry algebra $\mathcal{A}_{\text{inv}}$ is formed by the fields $S \in \mathcal{A}$ for which $J_0 S = 0$. If there is a boundary, we have seen that if the boundary condition preserves $\mathcal{A}_{\text{inv}}$ before deforming the bulk, it always does so after the deformation. Thus the boundary never destroys any additional symmetries!

As we have seen the statement about the boundary symmetries is actually quite subtle, and two cases need to be distinguished. In the first case the boundary condition originally preserves the algebra $\mathcal{A}_{\text{inv}}$ in the sense that the fields in $\mathcal{A}_{\text{inv}}$ and $\bar{\mathcal{A}}_{\text{inv}}$ are glued at the boundary via some automorphism ω (that is an automorphism of $\mathcal{A}_{\text{inv}}$). Then the symmetries of the boundary theory actually arise from gluing preserved bulk symmetries (see section 2.3.2). The prime example for this phenomenon is the diagonal brane on a torus when the torus is deformed (see section 2.3.3).

In the second case, the automorphism ω of $\mathcal{A}$ actually does not define an automorphism of the preserved subalgebra $\mathcal{A}_{\text{inv}}$. Then only fields in $\mathcal{A}_{\text{c}} = \mathcal{A}_{\text{inv}} \cap \omega(\mathcal{A}_{\text{inv}})$ can be glued after the deformation. From the point of view of the gluing conditions of the bulk the boundary symmetry thus appears to be reduced to $\mathcal{A}_{\text{c}}$. However, as discussed in section 4, the boundary theory still has mutually local boundary fields associated to all elements of $\mathcal{A}_{\text{inv}}$, and the spectrum of the boundary theory can be decomposed into (twisted) representations of $\mathcal{A}_{\text{inv}}$. The prime example for this phenomenon is the permutation brane in the product of two SU(2) WZW models (see section 2.4.3).

These results were obtained for current-current deformations, but we believe that the general observation — the boundary symmetry is not further reduced than the bulk symmetry — is also true for a larger class of deformations. We gave one example in section 2.4.4 using techniques from matrix factorisations in Landau-Ginzburg models.

We have only considered current-current deformations of the form $\Phi = J\bar{J}$ with $\omega(J) = \bar{J}$ at the boundary. It is straightforward to generalise our analysis to the more general case where $\Phi = J\bar{J}'$ with $\omega(J') = \bar{J}'$ at the boundary. We then have to distinguish $\mathcal{A}_{\text{inv}} = \{S; J_0 S = 0\}$ and $\bar{\mathcal{A}}'_{\text{inv}} = \{\bar{S}; \bar{J}'_0 \bar{S} = 0\}$. The gluing automorphism can be deformed for fields in $\mathcal{A}_{\text{inv}} \cap \omega(\mathcal{A}'_{\text{inv}})$, and $\mathcal{A}_{\text{inv}}$ still plays the role of an algebra of mutually local boundary fields.

Many results also carry over to the most general current-current deformation where $\Phi = \sum_i J_i \bar{J}'_i$. This deformation is exactly marginal in the bulk if the currents J_i do not have simple poles among themselves, and similarly for the $\bar{J}'_i$ [36]. Using the same arguments as in section 2.2 one easily shows that the symmetry algebra that is preserved by such a deformation is $\mathcal{A}_{\text{inv}} = \{S; J_{i,0} S = 0 \text{ for all } i\}$ and similarly $\bar{\mathcal{A}}'_{\text{inv}} = \{\bar{S}; \bar{J}'_{i,0} \bar{S} = 0 \text{ for all } i\}$. In the presence of a boundary, the deformation Φ is exactly marginal if $\bar{J}'_i = \omega(J'_i)$ at the boundary and if the J_i do not have simple poles with the $\omega(J'_j)$. In this case one can then also generalise straightforwardly our analysis of deformed gluing conditions and boundary algebras.

Chapter 3

Brane backreactions and the Fischler-Susskind mechanism

3.1 Overview

In this chapter we want to extend extend the RG equations derived in chapter 1 to include the backreaction of branes on the bulk theory. Whereas so far the entire analysis took place in the framework of pure CFT, we will now need to consider effects arising from the fact that we consider a complete string theory.

The idea for the mechanism underlying the backreaction goes back to [40, 38, 73, 74, 27]: in string theory, to calculate amplitudes one considers not only the disk diagram, but also diagrams of higher genus. The total amplitude is obtained by summing over all topologies and integrating over the moduli of the conformal structure of the diagrams. This integration can lead to new divergences at the boundary of the moduli space $\mathcal{M}$, *i.e.* when the surface degenerates. More precisely, the spectrum of the theory may contain tadpoles, *i.e.* massless modes, which give logarithmic divergences when integrated over $\mathcal{M}$. According to [73, 74], these can be absorbed by a suitable shift of the coupling constants in lower genus diagrams, thus contributing to the RG flow of the bulk couplings. Since the nature of the tadpoles depends on the boundary condition that is imposed, this describes the backreaction of the brane on the bulk.

It turns out that this prescription works for the annulus diagram, *i.e.* that the tadpole divergences can be compensated by local counterterms on the disk diagram, leading to additional terms in the bulk RG equations of chapter 1. The brane backreaction can thus be incorporated quite naturally in the language of renormalisation group flows.

We will then use the RG equations so obtained to study various examples. In many cases, we know already from geometric considerations how the brane should deform the bulk theory, so that we can compare our results. For instance, we expect that a D1-brane wrapping a circle should shrink its radius. This is confirmed by the RG analysis. In other, more complicated examples we also find agreement between the RG analysis and geometric expectations or supergravity calculations.

3.2 Renormalisation group equations

3.2.1 Dimensional regularisation on the disk

We will first rederive the renormalisation group equations (1.3.18) using a different regularisation scheme. The divergent expression was given by

$$\frac{\lambda_1^{l_1} \cdots \mu_1^{m_1} \cdots}{l_1! \cdots m_1! \cdots} \prod_i \ell^{(h_{\phi_i}-2)l_i} \prod_j \ell^{(h_{\psi_j}-1)m_j}$$
$$\times \int \langle \phi_1(z_1^1)\phi_1(z_2^1) \cdots \phi_2(z_1^2) \cdots \psi_1(x_1^1) \cdots \rangle \prod d^2 z_k^i \prod dx_k^j \ . \quad (3.2.1)$$

Note that the disk has the conformal symmetry group $SU(1,1)$. The integration measure $d\mu$ must transform with conformal weight $(-1,-1)$ under such transformations, so that integrals of marginal $(1,1)$ fields $\int d\mu \phi_{(1,1)}$ are invariant. Clearly, d^2z satisfies this property. Since we can use $SU(1,1)$ to map any point to 0, it follows that up to a constant factor this is the only possible measure. Already because of the symmetry group, the integrals in (3.2.1) are infinite. To render them finite, we use $SU(1,1)$ to fix the position of one bulk and one boundary insertion. Alternatively, we can (formally) divide by the volume of $SU(1,1)$. This will turn out to be important later on.

To regularise (3.2.1), we will not cut disks, but use a scheme which resembles dimensional regularisation. To evaluate diverging integrals, we change the conformal dimension of the fields involved to such values that the integral converges, and evaluate the original integral by analytic continuation. One motivation for using this scheme comes from the spacetime interpretation of the divergences that will show up in the modular integrals: they can be interpreted as infrared divergences due to massless modes, so that a natural regularisation is to introduce a small mass term. In the worldsheet theory, this corresponds to a shift of the conformal dimension of the field. From a more technical point of view, it is favourable to keep conformal covariance of all expressions, which is destroyed if we cut out small disks.

Let us shift the conformal weight of boundary fields as $h_\psi \mapsto h_\psi - \epsilon$, and that of bulk fields as $h_\phi \mapsto h_\phi - 2\epsilon$.[1] As an example for how the scheme works, consider two marginal bulk fields ϕ_i, ϕ_j that come close to each other to produce another marginal field ϕ_k,

$$\lambda_i \ell^{-2\epsilon} \lambda_j \ell^{-2\epsilon} \phi_i(z)\phi_j(0) \sim \lambda_i \lambda_j \ell^{-4\epsilon} \frac{\phi_k(0) C_{ijk}}{|z|^{h_i+h_j-h_k}} = \lambda_i \lambda_j \ell^{-4\epsilon} \phi_k(0) C_{ijk} |z|^{-2+2\epsilon} \ . \quad (3.2.2)$$

For simplicity, we have fixed the position of ϕ_j to 0. We perform the d^2z integral up to some IR cutoff L to obtain

$$\lambda_i \lambda_j \ell^{-2\epsilon} \phi_k(0) \, 2\pi C_{ijk} \frac{\ell^{-2\epsilon}}{2\epsilon} L^{2\epsilon} \ . \quad (3.2.3)$$

We have pulled out a factor $\ell^{-2\epsilon}$ which will be absorbed in the shift of λ_k (see (3.2.5)). The second factor $\ell^{-2\epsilon}$ gives

$$\frac{\ell^{-2\epsilon}}{2\epsilon} L^{2\epsilon} = \frac{1}{2\epsilon} - \log \ell + \log L + \mathcal{O}(\epsilon) \ . \quad (3.2.4)$$

---

[1] Note that for bulk fields in a theory with boundary $h = h_L + h_R$.

In the limit $\epsilon \to \infty$, only the second term gives a dependence on ℓ which contributes to the RG flow. We see that the regularisation scheme has introduced an implicit dependence of the integral on ℓ. As $\langle e^{-S} \rangle$ must be independent of ℓ, we must compensate a shift in $\log \ell$ by shifting λ_i and μ_j. A combinatorial analysis shows that the shift needed is

$$\lambda_k \ell^{-2\epsilon} \mapsto \lambda_k \ell^{-2\epsilon} + \lambda_i \lambda_j \ell^{-2\epsilon} \pi C_{ijk} \cdot \log \ell \ . \tag{3.2.5}$$

In a similar way, we treat the other types of divergences. The resulting renormalisation group equations are then the same as (1.3.18), (1.3.19)

$$\begin{aligned} \dot{\lambda}_k &= (2 - h_{\phi_k})\lambda_k + \pi C_{ijk}\, \lambda_i \lambda_j + \mathcal{O}(\lambda^3) \ , & (3.2.6) \\ \dot{\mu}_k &= (1 - h_{\psi_k})\mu_k + \frac{1}{2}\, B_{ik}\, \lambda_i + D_{ijk}\, \mu_i \mu_j + \mathcal{O}(\mu\lambda, \mu^3, \lambda^2) \ , & (3.2.7) \end{aligned}$$

where the dot indicates a derivative with respect to the flow parameter $t = \log \ell$.

Note that although the both the result and the flow parameter turn out to be same for both schemes, the divergences themselves are quite different: in the first analysis, we obtained divergences in ℓ itself, whereas here the divergence was in ϵ, the shift of the conformal weight.

3.2.2 Higher genus: general strategy

To calculate amplitudes in string theory, we have to take into account higher genus diagrams as well. For simplicity assume that there is only one type of field ϕ in our theory. As before, a string amplitude F can be expanded in powers of λ, $F = \sum_n \lambda^n F_n$. Each term F_n itself contains contributions from all topologically different diagrams with n insertions of ϕ. Moreover, for a given topology we must integrate over all conformal structures, parametrised by modular parameters t_i. In full,

$$F_n = \sum_k g_s^{\chi_k} \int_{\mathcal{M}_k} dt_i F_n^k(t_i) \ , \tag{3.2.8}$$

where g_s is the string coupling constant and χ_k is the Euler characteristic of the diagram F^k. Integration over the moduli space $\mathcal{M}_k$ leads to new divergences due to marginal and relevant modes in the spectrum of the theory. The divergences have to be regularised, and we must try to compensate for them by introducing counterterms on diagrams of lower genus. These ℓ-dependent terms then give the the backreaction terms in the bulk RG equations.

3.2.3 The annulus diagram

We will now calculate the backreaction terms caused by the annulus diagram $A_n = F_n^1$. The annulus has a single real modular parameter q, its inner radius. The integral over q produces a divergence for $q \to 0$. In this case there is an intuitive way to see how the counterterm on the disk arises, as shown in figure 3.1: the divergent part of the annulus diagram with n integrated insertions corresponds to a disk diagram with an additional

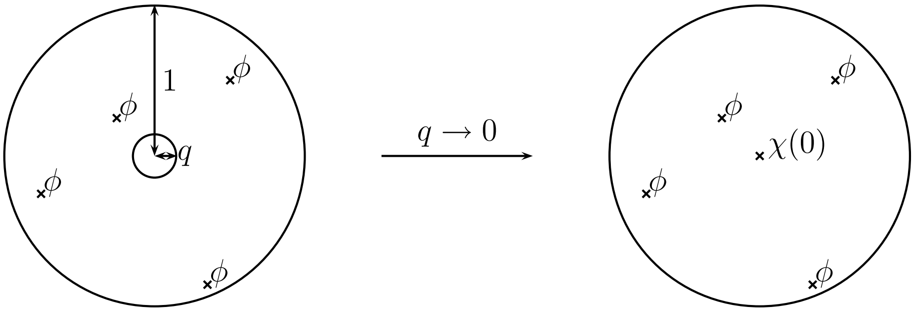


Figure 3.1: Divergences of the annulus diagram

field $\chi(0)$ inserted. A shift $\lambda \mapsto \lambda + \delta\lambda$ on the disk diagram $D_{n+1} = F^0_{n+1}$ can thus compensate the divergence. The corresponding term is of order g_s.

Although we will only calculate the term of order $g_s\lambda^0$, some comments on terms of higher order in λ are necessary. The analysis on the disk showed that λ^2 terms are produced by two fields approaching each other, and that higher order terms appear when n fields come close together. In the situation here, higher order corrections arise when additional fields move close to the new field produced on the disk or to the boundary of the annulus. If for instance a single ϕ moves close to the centre of the annulus A_n, the divergence can be compensated by the disk diagram D_n, which produces a contribution of order $g_s\lambda$. As we are only interested in the lowest order correction, we can thus subtract divergences which arise from fields moving close to each other or to the boundary.

Note that the symmetry group of the annulus is only $U(1)$ — we can fix the position of one boundary insertion, or alternatively we can divide the amplitude by 2π. This also means that unlike on the disk, the conformal symmetry no longer uniquely fixes the integration measure. Nevertheless, the correct measure is still d^2z, see *e.g.* [115].

For a given radius q, the integrated n-point amplitude of the annulus is given by

$$A_n(q) = \frac{1}{\pi}\prod_{i=1}^{n}\int_1^q d^2z_i \langle\langle B||\phi(z_1)\ldots\phi(z_n)q^{L_0+\bar{L}_0-2}||B\rangle\rangle \ . \tag{3.2.9}$$

For simplicity, we have only included one type of marginal field ϕ. As usual, $\langle\langle B||$ is the boundary state at the outer radius 1. To obtain the boundary state at the inner radius, we transport $||B\rangle\rangle$ to the inner radius q using the propagator $\pi^{-1}q^{L_0+\bar{L}_0-2}$, whose normalisation is fixed by the construction of the boundary states. By inserting a complete set of states, we expand the boundary state in a sum of fields inserted at the point 0. The action of the propagator then gives

$$\pi^{-1}q^{L_0+\bar{L}_0-2}||B\rangle\rangle = \pi^{-1}\sum_i q^{h_i+\bar{h}_i-2}|\phi_i\rangle\langle\phi_i||B\rangle\rangle \ . \tag{3.2.10}$$

Here $\langle\phi_i||B\rangle\rangle$ is the disk one-point function with ϕ_i sitting at the point 0. Integrating (3.2.9) over its moduli space using the measure $q^{-1}dq$, we see from (3.2.10) that divergences arise for $q \to 0$ for all fields with $h_i = \bar{h}_i \leq 1$. In a supersymmetric setup, we expect no relevant, *i.e.* tachyonic fields. In the bosonic theories we will consider, the only such field is usually the vacuum $h = \bar{h} = 0$. The vacuum only changes overall normalisations, so that we will ignore it in what follows. The only divergences are then due to marginal fields $h_i = \bar{h}_i = 1 - \epsilon$. Their contribution is

$$||B(q)\rangle\rangle \simeq \frac{q^{-2\epsilon}}{\pi} \sum_i \langle\phi_i||B\rangle\rangle\phi_i(0) \ . \tag{3.2.11}$$

For the moment, let us assume that there are no integrated bulk insertions. The integral of (3.2.11) over moduli space converges if $\epsilon < 0$, and we will use its analytic continuation,

$$\int_0^1 q^{-1}dq\, ||B(q)\rangle\rangle = -\frac{1}{\pi}\frac{1}{2\epsilon} \sum_i \langle\phi_i||B\rangle\rangle\phi_i(0) \ . \tag{3.2.12}$$

The pole in ϵ will then contribute to the RG equations as in (3.2.3).
If the diagram contains integrated bulk insertions, the comparison is a bit more subtle: in the disk diagram, the additional bulk insertions are integrated over the entire disk, whereas on the annulus they are only integrated up to the inner radius q. The divergent contribution of the tadpole, however, comes from the limit $q \to 0$. We can thus concentrate on annulus diagrams where $q < |\epsilon|$. Indeed,

$$\int_{|\epsilon|}^1 dq q^{-1-2\epsilon} = -\frac{1}{2\epsilon}(1 - e^{-2\epsilon \ln|\epsilon|}) = \mathcal{O}(\ln|\epsilon|) \tag{3.2.13}$$

is only a subleading contribution compared to (3.2.12). We claim then that to lowest order in λ we can rewrite the annular integral as

$$\int_0^{|\epsilon|} dq \int_q^1 d^2z_i \langle\ldots\rangle = \int_{|\epsilon|}^1 d^2z_i \int_0^{|\epsilon|} dq \langle\ldots\rangle + \mathcal{O}(\epsilon^2) \ . \tag{3.2.14}$$

This holds because we can estimate the contribution of the fields ϕ integrated over the small disk of radius $|\epsilon|$: since we only calculate the lowest order term in λ, we subtract all singular terms in ϕ. The remaining expression is then bounded by some constant B, and we can estimate its contribution as $\leq \pi\epsilon^2 B$. A similar argument shows that we can cut out the same small disk in the disk diagram without changing the result. This shows that we can compare annulus diagrams with disk diagrams even if they contain integrated insertions.
So far, the fields ϕ_i introduced by the tadpoles are inserted at the point $z = 0$. In order to be able to compensate them with a disk diagram, we need to rewrite them as integrated insertions. To do this, we use the fact that the disk has a larger symmetry group than the annulus. Consider the disk diagram with n integrated fields $\phi(z_i)$ and one additional field $\chi(z)$, each of them marginal. We can use part of the symmetry group $SU(1,1)$ to fix the position of χ to 0. In particular, for each z choose $f_z \in SU(1,1)$ such that $f_z(z) = 0$. Defining $\hat{z}_i = f_z(z_i)$, conformal covariance tells us that the z_i integral changes as

$$\int d^2z_i\phi(z_i) \to \int d^2\hat{z}_i \left|\frac{\partial z_i}{\partial \hat{z}_i}\right|^{-2\epsilon} \phi(\hat{z}_i) = \int d^2\hat{z}_i\phi(\hat{z}_i) + \mathcal{O}(\epsilon) \ . \tag{3.2.15}$$

Up to terms of order ϵ, the resulting integral is thus independent of z, and the additional field $\chi(z)$ is fixed at the position $z = 0$. Formally, we can write this manipulation as

$$\frac{1}{|SU(1,1)|}\int d^2z \int d^2z_i \langle \chi(z)\phi(z_1)\ldots\rangle = \frac{1}{|U(1)|}\int d^2\hat{z}_i\langle\chi(0)\phi(\hat{z}_1)\ldots\rangle + \mathcal{O}(\epsilon) \ , \quad (3.2.16)$$

where $|G|$ denotes the volume of the respective symmetry group. On the right hand side of (3.2.16), we divide by $|U(1)|$ because we still have not fixed the entire symmetry: after choosing f_z, we can always rotate the disk around its centre. This remaining $U(1)$ symmetry however is exactly the symmetry group of the annulus, so that the right hand side of (3.2.16) is the standard annulus diagram with one fixed insertion.

The upshot of this analysis is that the divergent part of A_n has the same form as D_{n+1}, so that it can be compensated by introducing a counterterm on the disk diagram. As before, we need to split off a factor $\ell^{-2\epsilon}$ to be included in λ. The annulus contribution to the disk diagram is thus

$$-\ell^{-2\epsilon}\frac{g_s}{\pi}\frac{\ell^{2\epsilon}}{2\epsilon}\langle\phi_i||B\rangle\rangle\int d^2z\,\phi_i(z) = -\ell^{-2\epsilon}\frac{g_s}{\pi}\left(\frac{1}{2\epsilon}+\log\ell+\mathcal{O}(\epsilon)\right)\langle\phi_i||B\rangle\rangle\int d^2z\,\phi_i(z) \quad (3.2.17)$$

for each marginal field ϕ_i. The usual combinatorial analysis shows that this can be compensated by shifting the coupling constant λ_i.

Putting everything together we obtain the modified bulk RG equations

$$\dot{\lambda}_k = (2-h_{\phi_k})\lambda_k + \frac{g_s}{\pi}\langle\phi_k||B\rangle\rangle + \pi C_{ijk}\,\lambda_i\lambda_j + \mathcal{O}(g_s\lambda, \lambda^3, g_s^2) \ . \quad (3.2.18)$$

3.3 WZW models and the free boson

We now apply equation (3.2.18) to some examples. First we consider the free boson compactified on a circle, subject to Neumann or Dirichlet boundary conditions. Then we turn to Wess-Zumino-Witten models based on compact Lie groups. These models and their boundary states are very well understood and can be interpreted geometrically. We can thus check RG flow results against geometric expectations.

3.3.1 The free boson on a circle

Let $X(z,\bar{z})$ be the free boson compactified on a circle of radius R, $X \sim X + 2\pi R$. Its action is given by

$$S = \frac{1}{2\pi}\int d^2z\,\partial X\bar{\partial}X \ . \quad (3.3.1)$$

Neumann and Dirichlet boundary conditions are given by identifying on the real axis $z = \bar{z}$

$$\partial X = \bar{\partial}X \text{ (Neumann)} \qquad \text{and} \qquad \partial X = -\bar{\partial}X \text{ (Dirichlet).}$$

As usual, we can switch to the closed string picture by mapping the upper half-plane to the disk. The boundary condition is then described by the boundary states $||N\rangle\rangle$ and $||D\rangle\rangle$, respectively.

The ground states of the system are parametrised by momentum and winding numbers $n, w \in \mathbb{Z}$ such that

$$(p_L, p_R) = \left(\frac{n}{2R} + wR, \frac{n}{2R} - wR\right) \ , \tag{3.3.2}$$

with conformal weight given by $(\frac{1}{2}p_L^2, \frac{1}{2}p_R^2)$. At a generic radius R, the only marginal operator is $\partial X \bar{\partial} X$. Its one-point function is given by

$$\langle \partial X \bar{\partial} X || N \rangle\rangle = 1 \qquad \text{and} \qquad \langle \partial X \bar{\partial} X || D \rangle\rangle = -1 \ . \tag{3.3.3}$$

We will also have to deal with the relevant fields that are present in theory.

Let us analyse the Neumann case first. The one-point function vanishes unless $p_L = -p_R$, *i.e.* $n = 0$, so that only pure winding modes couple. If we take R big enough, (3.3.2) shows that all these modes become irrelevant. It is thus sufficient to only consider the perturbation by $\partial X \bar{\partial} X$,

$$S = \frac{1}{2\pi} \int d^2z \, \partial X \bar{\partial} X \ - \lambda \int d^2 z \, \partial X \bar{\partial} X \ . \tag{3.3.4}$$

We see that (3.2.18) yields $\dot{\lambda} = g_s/\pi > 0$. An increase in λ means that the circle shrinks, as can be seen from (3.3.4): to maintain the correct normalisation of the action, we have to introduce rescaled fields $X' = \sqrt{1 - 2\pi\lambda} X$, which satisfy $X' \sim X' + 2\pi R' = X' + 2\pi R \sqrt{1 - 2\pi\lambda}$.

This shows that a Neumann brane that wraps the circle shrinks its radius. Similar reasoning shows that the D0 brane given by $||D\rangle\rangle$ increases the radius of the circle.

When R becomes of the order of the self-dual radius $R_0 = 1/\sqrt{2}$, new relevant and marginal fields appear, and the above analysis breaks down. To analyse this case, we will use the fact that the free boson at the self-dual radius is equivalent to the $SU(2)$ Wess-Zumino-Witten-model at level 1. We therefore turn our attention to WZW-models.

3.3.2 Renormalisation group flows in general WZW models

Wess-Zumino-Witten models are often described as σ-models on a group manifold of a Lie group G [181]. A different, more algebraic approach is to define them via their operator content and correlation functions. For the moment, we will use this more abstract formulation, before changing to a more geometric picture in the next section.

The currents of the WZW model of a Lie group G at level k correspond to elements of the Lie algebra $\mathfrak{g}$ of G and satisfy the operator product expansion

$$J^a(z) J^b(w) \sim \frac{k\delta^{ab}}{(z-w)^2} + i f^{ab}{}_c \frac{J^c(w)}{(z-w)} \ , \tag{3.3.5}$$

where $f^{ab}{}_c$ are the structure constants of $\mathfrak{g}$. The marginal fields of the theory are given by all possible combinations $J^a \bar{J}^b$ of left-moving and right-moving currents. We consider branes that preserve the affine symmetry up to conjugation by $g \in G$ [98, 26, 167]. In the closed string picture this means that the boundary state $||B\rangle\rangle$ has to satisfy the gluing condition

$$(g J^a_m g^{-1} + \bar{J}^a_{-m}) ||B\rangle\rangle = 0 \ , \tag{3.3.6}$$

whereas in the open string picture the left and right moving currents are identified at the boundary as

$$gJ^a(z)g^{-1} = \bar{J}^a(\bar{z}) \quad \text{for } z = \bar{z} \ . \tag{3.3.7}$$

The one-point function is best evaluated in the open string picture and gives [84, 92]

$$\langle (J^a \bar{J}^b)(u) \rangle_B = k \frac{\mathrm{tr}\,(J^a g J^b g^{-1})}{(u-\bar{u})^2} = -k \frac{\mathrm{tr}\,(J^a g J^b g^{-1})}{|u-\bar{u}|^2} \ , \tag{3.3.8}$$

so that $\langle J^a \bar{J}^b || B \rangle\rangle = -k \mathrm{tr}\,(J^a g J^b g^{-1})$. Note that the currents are normalised such that $\mathrm{tr}\,(J^a J^b) = \delta^{ab}$. The orthonormal marginal fields are thus

$$\phi_{ab}(z) = k^{-1} J^a \bar{J}^b \ . \tag{3.3.9}$$

Let us start from the model which is initially unperturbed. To lowest order, (3.2.18) gives then

$$\dot{\lambda}_{ab} = -\frac{g_s}{\pi} \mathrm{tr}\,(J^a g J^b g^{-1}) \ . \tag{3.3.10}$$

Higher order contributions in the bulk come from evaluating connected n-point functions. They are given [84, 92] by the product of traces $k\mathrm{tr}\,(J^{a_1} \dots J^{a_n}) k \mathrm{tr}\,(\bar{J}^{b_1} \dots \bar{J}^{b_n})$, so that in the normalisation (3.3.9) they go as k^{2-n}. In the limit $k \to \infty$ they only give subleading contributions.

Let us make a side remark. We can choose an orthogonal basis $J^a, a = 1, \dots, r$ for the left moving currents, and a corresponding basis $\bar{J}^b := g^{-1} J^b g, b = 1, \dots, r$ for the right moving currents. (3.3.10) then shows that only the fields ϕ_{aa} are switched on. Note that these fields leave the boundary conditions unchanged, as

$$[t^a, g\bar{t}^a g^{-1}] = [t^a, t^a] = 0 \ , \tag{3.3.11}$$

which means that all B_{ik} in (3.2.7) vanish, so that no boundary fields are switched on [79]. The brane changes the bulk without inducing a backreaction on itself.

3.3.3 Geometric interpretation of $SU(2)_k$

To get a geometric picture of the brane backreaction, we switch to a more geometric description of WZW models. We will concentrate on $G = SU(2)$. We can write this theory as a σ-model on the group manifold, using the parametrisation [120]

$$g = e^{i(\theta+\tilde{\theta})\sigma_2/2} e^{i\phi\sigma_1/2} e^{-i(\theta-\tilde{\theta})\sigma_2/2} \ , \tag{3.3.12}$$

or explicitly

$$g = \begin{pmatrix} \cos\frac{\phi}{2}\cos\tilde{\theta} + i\sin\frac{\phi}{2}\sin\theta & \cos\frac{\phi}{2}\sin\tilde{\theta} + i\sin\frac{\phi}{2}\cos\theta \\ -\cos\frac{\phi}{2}\sin\tilde{\theta} + i\sin\frac{\phi}{2}\cos\theta & \cos\frac{\phi}{2}\cos\tilde{\theta} - i\sin\frac{\phi}{2}\sin\theta \end{pmatrix} \ . \tag{3.3.13}$$

At level k the action then becomes

$$S_0(\phi, \theta, \tilde{\theta}) = \frac{k}{2\pi} \int d^2z \left(\frac{1}{4}\, \bar{\partial}\phi\partial\phi + \sin^2\frac{\phi}{2}\, \bar{\partial}\theta\partial\theta + \cos^2\frac{\phi}{2}\, \bar{\partial}\tilde{\theta}\partial\tilde{\theta} + \cos^2\frac{\phi}{2} (\bar{\partial}\theta\partial\tilde{\theta} - \bar{\partial}\tilde{\theta}\partial\theta) \right) \ . \tag{3.3.14}$$

For later use, we also derive explicit expressions for the currents $J = -k\partial g\ g^{-1}$ and $\bar{J} = kg^{-1}\bar{\partial}g$,

$$\begin{aligned} J^1 &= -k\frac{i}{\sqrt{2}}(\partial\phi\cos(\tilde{\theta}+\theta) - \partial\theta\sin\phi\sin(\tilde{\theta}+\theta) + \partial\tilde{\theta}\sin\phi\sin(\tilde{\theta}+\theta) \\ J^2 &= -k\frac{i}{\sqrt{2}}(\partial\theta(1-\cos\phi) + \partial\tilde{\theta}(1+\cos\phi)) \\ J^3 &= -k\frac{i}{\sqrt{2}}(\partial\phi\sin(\tilde{\theta}+\theta) + \partial\theta\sin\phi\cos(\tilde{\theta}+\theta) - \partial\tilde{\theta}\sin\phi\cos(\tilde{\theta}+\theta) \end{aligned}$$

and

$$\begin{aligned} \bar{J}^1 &= k\frac{i}{\sqrt{2}}(\bar{\partial}\phi\cos(\tilde{\theta}-\theta) + \bar{\partial}\theta\sin\phi\sin(\tilde{\theta}-\theta) + \bar{\partial}\tilde{\theta}\sin\phi\sin(\tilde{\theta}-\theta)) \\ \bar{J}^2 &= k\frac{i}{\sqrt{2}}(\bar{\partial}\theta(-1+\cos\phi) + \bar{\partial}\tilde{\theta}(1+\cos\phi)) \\ \bar{J}^3 &= k\frac{i}{\sqrt{2}}(-\bar{\partial}\phi\sin(\tilde{\theta}-\theta) + \bar{\partial}\theta\sin\phi\cos(\tilde{\theta}-\theta) + \bar{\partial}\tilde{\theta}\sin\phi\cos(\tilde{\theta}-\theta))\ . \end{aligned}$$

The boundary states are given by $||j,g\rangle\rangle$. For each gluing condition g there are $k+1$ possible branes, labelled by $j = 0, \frac{1}{2}, \ldots, \frac{k}{2}$. [6] gives a geometric interpretation for these branes in terms of conjugacy classes: if g is the identity e, then $||j,e\rangle\rangle$ is the S^2 that wraps the conjugacy class given by

$$h\begin{pmatrix} e^{2\pi ij/k} & 0 \\ 0 & e^{-2\pi ij/k} \end{pmatrix}h^{-1}\ . \tag{3.3.15}$$

In particular, for $j=0$ and $j=\frac{k}{2}$, the conjugacy class collapses to a point and the brane describes a D0 brane sitting at the point e and $-e$, respectively. If the gluing map is given by a general g, the position of the brane shifts accordingly.
To go to the geometric limit, we fix j and let $k \to \infty$. Independent of j the brane thus becomes a D0 brane sitting at the point g. Also, (3.3.10) shows that the flow induced depends only on g. We can therefore suppress the index j and parametrise the brane only by $g = g(\Phi, \Theta, \tilde{\Theta})$. Note that we denote its position by capital letters $\Phi, \Theta, \tilde{\Theta}$, as opposed to small letters for the coordinates of the manifold.
In the geometric limit the $SU(2)_k$ model corresponds to a non-linear σ-model on S^3 with radius $r \sim \sqrt{k}$. We can read off the target space metric G and the field B from the coefficients of the action. In the unperturbed case (3.3.14) this gives

$$G_0 = \begin{pmatrix} k/4 & 0 & 0 \\ 0 & k\sin^2\frac{\phi}{2} & 0 \\ 0 & 0 & k\cos^2\frac{\phi}{2} \end{pmatrix}, \qquad B_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & k\cos^2\frac{\phi}{2} \\ 0 & -k\cos^2\frac{\phi}{2} & 0 \end{pmatrix}. \tag{3.3.16}$$

3.3.4 Minimising the brane mass

Let us now calculate the RG flow and try to interpret it. (3.3.10) shows that the marginal fields $J^i\bar{J}^j$ are turned on with the respective strength

$$\dot{\lambda}_{ij}(\Phi,\Theta,\tilde{\Theta}) = -\frac{g_s}{\pi}\mathrm{tr}\,(J^igJ^jg^{-1}) =: -\frac{g_s}{\pi}K_{ij}(\Phi,\Theta,\tilde{\Theta})\ . \tag{3.3.17}$$

The coefficients K_{ij} depend on the position of the brane and are given by

$$K_{ij} = 2 \begin{pmatrix} \cos 2\tilde{\Theta} \cos^2 \frac{\Phi}{2} + \cos 2\Theta \sin^2 \frac{\Phi}{2} & \sin(\Theta + \tilde{\Theta}) \sin \Phi & \sin 2\Theta \sin^2 \frac{\Phi}{2} - \sin 2\tilde{\Theta} \cos^2 \frac{\Phi}{2} \\ - \sin(\Theta - \tilde{\Theta}) \sin \Phi & \cos \Phi & \cos(\Theta - \tilde{\Theta}) \sin \Phi \\ \sin 2\Theta \sin^2 \frac{\Phi}{2} + \sin 2\tilde{\Theta} \cos^2 \frac{\Phi}{2} & - \cos(\Theta + \tilde{\Theta}) \sin \Phi & \cos 2\tilde{\Theta} \cos^2 \frac{\Phi}{2} - \cos 2\Theta \sin^2 \frac{\Phi}{2} \end{pmatrix} . \tag{3.3.18}$$

This flow has a nice geometric interpretation. The mass of a brane is given by the value of the dilaton φ. Perturbing the metric of S^3 induces a non-constant dilaton and so changes the mass of the brane. [79] showed that in the case of an induced boundary flow, the brane deformed in such a way as to minimise its mass. We will show that a similar thing happens here: this time, the brane remains at the same place, but it deforms the geometry in such a way that its mass is minimised.

To show this, let us first find the change in geometry that decreases the mass of the brane as much as possible. The most general current-current deformation of the original theory is

$$S = S_0 - \alpha \int d^2z \sum_{i,j} a_{ij} J^i(z) \bar{J}^j(\bar{z}) \ , \tag{3.3.19}$$

where the a_{ij} are real coefficients. This gives a new metric $G'(\phi, \theta, \tilde{\theta}) = G_0 - \alpha G_1$ and a new B-field. The new, nontrivial dilaton φ can be calculated by [120, 76]

$$e^{-2\varphi_0} \sqrt{\det G_0} = e^{-2\varphi(\phi,\theta,\tilde{\theta})} \sqrt{\det G'(\phi, \theta, \tilde{\theta})} \ . \tag{3.3.20}$$

The mass of the brane at $g = g(\Phi, \Theta, \tilde{\Theta})$ is proportional to $g_s^{-1} \sim e^{-\varphi(\Phi,\Theta,\tilde{\Theta})}$. We thus want to maximise the increase of $\det G_0^{-1} G'$ at the point $(\Phi, \Theta, \tilde{\Theta})$. Its derivative is given by

$$\partial_\alpha \det(1 - \alpha G_0^{-1} G_1)|_0 = -\mathrm{tr}\, G_0^{-1} G_1 \ . \tag{3.3.21}$$

A straightforward calculation then shows

$$\mathrm{tr}\, G_0^{-1} G_1(\Phi, \Theta, \tilde{\Theta}) = k \sum_{i,j} a_{ij} K_{ij}(\Phi, \Theta, \tilde{\Theta}) \ , \tag{3.3.22}$$

where K_{ij} is the same expression as in (3.3.18). Introducing a Lagrange multiplier term $\nu \sum_{i,j} a_{ij}^2$ shows that the expression is extremised by $a_{ij} = -K_{ij}(\Phi, \Theta, \tilde{\Theta})$. Comparing to (3.3.17) we find perfect agreement.

We can try to follow the flow further and describe the geometry of the deformed manifold. By the symmetry of the problem, it is sufficient to consider the brane sitting at $\theta = 0, \tilde{\theta} = 0$[2] so that

$$g = \begin{pmatrix} e^{i\Phi/2} & 0 \\ 0 & e^{-i\Phi/2} \end{pmatrix} . \tag{3.3.23}$$

(3.3.18) then turns on the fields

$$\lambda_{ij} = -2 \, \frac{g_s}{\pi} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \Phi & \sin \Phi \\ 0 & - \sin \Phi & \cos \Phi \end{pmatrix} . \tag{3.3.24}$$

---

[2] We could simply restrict to $g = e$, but e is a coordinate singularity in the parametrisation chosen.

They change the metric G_0 by some expression $2\,\frac{g_s}{\pi}G_1^\Phi(\phi,\theta,\tilde{\theta})$. At the point of the brane, G_1^Φ simplifies:

$$G_1^\Phi(\Phi,0,0) = \begin{pmatrix} k^2/2 & 0 & 0 \\ 0 & 2k^2\sin^2\frac{\Phi}{2} & 0 \\ 0 & 0 & 2k^2\cos^2\frac{\Phi}{2} \end{pmatrix} = 2kG_0(\Phi,0,0)\ . \tag{3.3.25}$$

The effect of the backreaction is simply to rescale the original metric. We can continue to use our original reasoning even away from the point $t = 0$ to obtain the differential equation

$$\dot{G}^\Phi(\Phi,0,0) \sim G_0(\Phi,0,0)\ . \tag{3.3.26}$$

The geometric analysis only gives the direction of the flow, so that we are free to choose the actual flow parameter. Writing

$$G^\Phi_{\mu\nu}(t) = G_{0\mu\nu} + 4\frac{g_s}{\pi}ktG^\Phi_{1\mu\nu} \tag{3.3.27}$$

we fix t so that it agrees with the conformal field theory flow parameter at $t = 0$.
Note that this analysis agrees with the observation in section 3.3.2, where we argued that in the limit $k \to \infty$, only the zero order term is important, and that thus no new bulk fields are turned on. This translates to the statement that (3.3.26) remains valid away from the starting point.
We can now try to understand how the geometry of the S^3 changes as we start to flow, and we can also try to estimate how far we should trust our analysis. Define a new flow parameter $t' = 4g_st/\pi$. Take the metric $G^\Phi_{\mu\nu}(t')$ and calculate the associated Ricci scalar $R(t')$. At the point g it is given by

$$R(t') = \frac{6 + 84kt'}{k(1+2kt')^2}\ . \tag{3.3.28}$$

The curvature thus increases at first, in agreement with the intuition that the brane warps the space around it.
The geometric picture breaks down as soon as the curvature becomes too big. In fact, if one considers $R(t')$ on all of S^3, it turns out that at $kt' = \frac{1}{2}$ the curvature becomes singular at some points. The geometric approximation thus becomes unreliable as soon as $kt' \sim 1$. In particular, one should not trust (3.3.28) for values $kt' \sim \frac{5}{14}$, where $R(t')$ seemingly starts to decrease.

3.4 Flat space

The last example we consider is the bosonic string in flat space in the presence of a Dp-brane. In this case, one can consider the low-energy supergravity limit of the theory. The D-brane is then given by a p-brane, a solution of the corresponding supergravity equation. [49] performed a boundary state calculation and found agreement with the supergravity results. We will reproduce their results using the extended RG equations.

3.4.1 The boundary state

The conformal field theory is described by 26 free bosons with ladder operators $a^\mu_n, \bar{a}^\nu_n$. A Dp-brane located at y is described by the boundary state [49]

$$||Dp; y\rangle\rangle = \frac{T_p}{2} \int \frac{d^{d_\perp} k_\perp}{(2\pi)^{d_\perp}} e^{ik_\perp y} \exp\left[-\sum_{n=1}^{\infty} a^\mu_{-n} \mathcal{S}_{\mu\nu} \bar{a}^\nu_{-n} \right] |0; k_\| = 0, k_\perp\rangle \quad . \tag{3.4.1}$$

The diagonal matrix $\mathcal{S}_{\mu\nu}$ is given by

$$\mathcal{S}_{\mu\nu} = (\eta_{\alpha\beta}, -\delta_{ij}) \ , \tag{3.4.2}$$

where α, β run over the $d_\| = p + 1$ dimensions parallel to the brane, and i, j over the $d_\perp = 26 - p - 1$ transverse dimension. Its tension is

$$T_p = \frac{\sqrt{\pi}}{2^{(d-10)/4}} (4\pi^2 \alpha')^{(d-2p-4)/4} \ . \tag{3.4.3}$$

Again, we will ignore the tachyon and concentrate on marginal fields. The corresponding states are of the form

$$a^\mu_{-1} \bar{a}^\nu_{-1} |0; k\rangle \ . \tag{3.4.4}$$

Here $|0; k\rangle$ is the ground state of momentum k, normalised as $\langle k|k'\rangle = 2\pi\delta(k - k')$, with $(2\pi)^d \delta^{(d)}(0) = V$. The conformal weight of (3.4.4) is $(1 + \alpha' k^2/4, 1 + \alpha' k^2/4)$, and it couples to the Dp-brane centred at $y = 0$ as

$$A^{\mu\nu}_k := \langle 0; k| a^\mu_1 \bar{a}^\nu_1 ||Dp; 0\rangle\rangle = -\frac{T_p}{2} \delta^{(d_\|)}(k_\|) \mathcal{S}^{\mu\nu} \ . \tag{3.4.5}$$

We see that only states with $k_\| = 0$ couple to the brane. It is thus necessary to consider states with non-vanishing transverse momentum, which means $k_\perp^2 > 0$, such that $k^2 > 0$. This poses a problem, as in string theory vertex operators have to be marginal, so that $k^2 = 0$.

This analysis indicates that we need to go off-shell to find states that couple to the brane. From the CFT point of view such this means that we need to consider states (3.4.4) that are almost marginal.

3.4.2 Applying the RG equations

We would like to apply (3.2.18) and find the fixed point to which the theory flows. Although we derived (3.2.18) only for marginal fields, the argument also works for almost marginal fields with $h = 1 + \delta h$. δh then takes the role of ϵ, and the counterterm needed is $\sim \ell^{\delta h} (\delta h)^{-1}$. The contribution to (3.2.18) is again $\frac{g_s}{\pi} \langle \phi_k ||B\rangle\rangle$. It is clear however that several steps of the derivation depended on taking $\epsilon \to 0$ in the end. We should therefore trust (3.2.18) only for almost marginal fields with $\delta h \ll 1$.

A fixed point of (3.2.18) is given by

$$0 = \dot{\lambda}^{\mu\nu} = (2 - h)\lambda^{\mu\nu} + \frac{g_s}{\pi} A^{\mu\nu}_k + \mathcal{O}(g_s \lambda, \lambda^3, g_s^2) \ , \tag{3.4.6}$$

so that to lowest order

$$\lambda^{\mu\nu} = \frac{2g_s}{\pi\alpha'}\frac{A_k^{\mu\nu}}{k_\perp^2} = \frac{g_s T_p (2\pi)^{p+1} V_{p+1}}{\pi\alpha'}\frac{\mathcal{S}^{\mu\nu}}{k_\perp^2} \ . \tag{3.4.7}$$

To compare to the metric in the supergravity solution, we calculate the expectation value of the graviton, *i.e.* its one-point function. Assuming that the fields $\phi_{\mu\nu}(k)$ were orthonormal in the original theory, the perturbed one-point function of $a_{-1}^{\mu}\bar{a}_{-1}^{\nu}|0;k\rangle$ is

$$\langle\phi^{\mu\nu}(k)\rangle_\lambda = \lambda^{\sigma\rho}\langle\phi_{\sigma\rho}(k)\phi^{\mu\nu}(k)\rangle_0 + O(\lambda^2) = \lambda^{\mu\nu} + O(\lambda^2) \ . \tag{3.4.8}$$

To obtain the expectation value of the graviton, we have to extract the symmetric traceless part of (3.4.7), as has been done in [49]. (3.4.7) then agrees with their results, up to a constant factor due to different normalisations.
Our analysis is only valid if $\alpha' k_\perp^2 \ll 1$, since else δh is too big. Moreover $\frac{g_s T_p}{\alpha' k_\perp^2} \ll 1$ is needed, since otherwise higher order terms will become important. Geometrically this means that we can only consider weakly curved configurations, and only probe the long distance limit. Our analysis is thus valid in the same range of parameters as the supergravity calculation.

Part II

Dual CFTs

Chapter 4

Heterotic AdS_3/CFT_2 duality with $(0, 4)$ spacetime supersymmetry

4.1 Introduction

In this chapter we study a specific instance of the AdS$_3$/CFT$_2$ duality. Several authors [109, 176, 42, 127, 140, 135, 8] have studied the possibility of an AdS$_3$/CFT$_2$ duality for the fundamental *heterotic* string. Heterotic strings are dual to type I D1-branes whose low-energy effective field theory is expected to be conformally invariant. The dual near-horizon geometry of the heterotic string should therefore contain an AdS_3 factor. This was confirmed in [31, 32] (see also [33]) in which an $AdS_3 \times S^2$ factor was found in an $\mathcal{N} = 2$, $d = 5$ R^2-corrected supergravity solution corresponding to heterotic strings in five dimensions.

In general, heterotic string setups may contain additional charged objects such as NS5-branes and Kaluza-Klein monopoles. Such setups generically have $(0, 4)$ target space supersymmetry. It has been found however that in the absense of some or all of these additional charges the target space supersymmetry is enhanced to $(0, 8)$ [176, 127, 140, 135] (see also [7]). Such theories are expected to be very different from those with only $(0, 4)$ supersymmetry. For one thing, there are no linear superconformal algebras with more than four supercurrents. Indeed, it has been argued in [140, 135] that the global supergroup of the boundary CFT is $Osp(4^*|4)$, whose affine extension is given by a *nonlinear* $\mathcal{N} = 8$, $d = 2$ superconformal algebra. For another, it is not clear if these theories possess unitary representations.

In this chapter we we will address the construction of a heterotic AdS/CFT duality with only $(0, 4)$ target space supersymmetry. For this we revisit a heterotic three-charge model previously studied in [137]; this setup consists of p fundamental strings embedded in the worldvolume of N' NS5 branes and N Kaluza-Klein (KK) monopoles. The worldsheet theory for string theory on the corresponding near-horizon geometry $AdS_3 \times S^3/\mathbb{Z}_N \times T^4$ turns out to be essentially the product of an $SL(2)$ WZW model and a "twisted" $SU(2)$ WZW model corresponding to the asymmetric orbifold $S^3/\mathbb{Z}_N$. In contrast, not much is known about the dual conformal field theory on the boundary of the AdS_3 space.

In this chapter we construct the dual two-dimensional boundary conformal field theory. To achieve this, we first apply heterotic/type I duality to map the three-charge configuration to an intersection of p D1-branes and N' D5-branes plus N KK monopoles in type I string theory. In the absence of any KK monopoles the low-energy effective theory corresponds to Witten's ADHM sigma model of Yang-Mills instantons [184], as shown by Douglas in [59]. To also include KK monopoles, which have a $\mathcal{C}^2/\mathbb{Z}_N$ near-core geometry, it is natural to construct a $\mathbb{Z}_N$ orbifold theory of the massive ADHM sigma model. (Refs. [177, 161] also use an orbifold construction to obtain the boundary CFT dual to *type II* string theory on $AdS_3 \times S^3/\mathbb{Z}_N \times T^4$.)

Our proposal is then that the sought-after boundary conformal field theory arises on the Higgs branch of the orbifolded ADHM model, which corresponds to the bound state phase of the D-brane setup. We will perform a consistency check for the proposal by the following line of reasoning. Lambert has shown in [138] that, even though the ADHM model is classically not conformal, it is ultraviolet finite to all orders in perturbation theory. There is no renormalisation group flow, and anomalous conformal dimensions are absent [138]. The conformal Higgs branch theory can therefore be obtained by integrating out the massive degrees of freedom in the ADHM model [139]. Moreover, the central charges of the Higgs branch theory can be determined by counting the massless degrees of freedom of the ultraviolet theory. In other words, they are given by the dimension of the instanton moduli space of the ADHM model. Repeating these steps for the orbifold version of the ADHM model, we determine the central charges of the low-energy theory of the three-charge model and match them to those predicted by the worldsheet theory.

4.2 Heterotic AdS_3/CFT_2 duality

4.2.1 Three-charge model for heterotic strings

We consider heterotic string theory compactified on $S^1 \times T^4$ which we take along the directions $\{x^5\}$ and $\{x^6, x^7, x^8, x^9\}$ respectively. In particular, following [137], we study the following brane setup:

- p fundamental strings F1 infinitely stretched in the x^1 direction,
- N' NS5-branes wrapped around the T^4 and infinitely stretched along x^1,
- N KK monopoles wrapped around T^4 and extended in x^1.

We can depict this configuration schematically in the following table:

		0	1	2	3	4	5	6	7	8	9
p	F1	●	●								
N'	NS5	●	●					●	●	●	●
N	KKM	●	●					●	●	●	●

From a 5-dimensional spacetime point of view this configuration looks like an infinitely stretched string in the x^1 direction, which preserves $(0,4)$ supersymmetry, *i.e.* it is non-supersymmetric in the left sector and contains four supercharges in the right sector. Let us recall the classical solution as given in [137]. The metric is given by

$$\begin{aligned} ds^2 =& F^{-1}(-dt^2+dx_1^2)+H_5\big[H_K^{-1}(dx_5+P_K(1-\cos\theta)d\varphi)^2 \\ &+H_K(dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2))\big]+\sum_{i=6}^{9}dx_i^2, \end{aligned} \tag{4.2.1}$$

with the following harmonic functions

$$H_5=1+\frac{P_5}{r}\,, \qquad H_K=1+\frac{P_K}{r}\,, \qquad F=1+\frac{Q}{r}\,. \tag{4.2.2}$$

Here we use spherical coordinates (r,θ,φ) for the directions (x^2,x^3,x^4). The corresponding gauge fields and the dilaton read

$$B_{t1}=F\,, \qquad B_{\varphi 5}=P_5(1-\cos\theta)\,, \qquad e^{-2[\Phi_{10}(r)-\Phi_{10}(\infty)]}=\frac{F}{H_5}\,. \tag{4.2.3}$$

The quantities P_5, P_K, Q are related to N', N, p by

$$P_5=\frac{\alpha'}{2R}N'\,, \qquad P_K=\frac{R}{2}N\,, \qquad Q=\frac{\alpha'^3e^{2\Phi_{10}(\infty)}}{2RV}p\,, \tag{4.2.4}$$

where R is the asymptotic radius of the S^1, V the volume of the torus and $\Phi_{10}(\infty)$ the asymptotic value of the dilaton.

In the near-horizon limit $r\to 0$, the metric (4.2.1) reduces to

$$\begin{aligned} ds^2 =& \frac{r'^2}{4P_5P_K}(-dt^2+dx_1^2)+\frac{P_5}{P_K}(dx_5+P_K(1-\cos\theta)d\varphi)^2 \\ &+P_5P_K\Big(4dr'^2+(d\theta^2+\sin^2\theta d\varphi^2)\Big)+\sum_{i=6}^{9}dx_i^2\,, \end{aligned} \tag{4.2.5}$$

where we have defined r' by

$$r=\frac{4P_5P_Kr'^2}{Q}\,. \tag{4.2.6}$$

In [137] this metric was interpreted as describing the space

$$AdS_3\times S^3/\mathbb{Z}_N\times T^4\,, \tag{4.2.7}$$

with AdS radius and six-dimensional string coupling

$$R^2_{AdS,\,\mathrm{uncorr}}=\alpha' NN'\,, \qquad g_6^2=e^{2\Phi_6^{\mathrm{hor}}}=\frac{N'}{p}\,. \tag{4.2.8}$$

Obviously, string theory on this background is weakly-coupled for $N'\ll p$. Note that so far we have only discussed an *uncorrected* supergravity solution, *i.e.* a solution to an action at the two derivative level.

4.2.2 Lift to M-theory

In order to understand why the supergravity solution (4.2.5) is expected to receive α' corrections, we now determine the central charges of the boundary CFT. We begin by mapping the heterotic setup to M-theory compactified on $CY_3 = K3 \times T^2$. For this, we first dualize to type IIA theory, from where (after additional S and T dualities) we may lift to M-theory — see appendix B.1 for details. We obtain the following setup of M5 branes:

		0	1	2	3	4	5	6	7	8	9	10
p	M5	●	●					●	●	●	●	
N'	M5	●	●				●	●	●			●
N	M5	●	●				●			●	●	●

Our convention will be that the internal T^2 is spanned by the directions $\{x^5, x^{10}\}$ while the $K3$ resides in $\{x^6, x^7, x^8, x^9\}$.

A general method for determining the central charges of the low-energy effective theory on M5-branes wrapping a 4-cycle in a Calabi-Yau three-fold CY_3 is given in [148]. The low-energy effective field theory is given by a two-dimensional (heterotic) sigma model with the M5-brane moduli space as target space. The left- and right-moving central charges $c_{L,R}$ of this sigma model are given by

$$\begin{aligned} c_L = 6D + c_2 \cdot p \,, \qquad & c_R = 6D + \frac{1}{2} c_2 \cdot p \,, \\ D = \frac{1}{6} c_{IJK} & p^I p^J p^K \,, \end{aligned} \tag{4.2.9}$$

where c_{IJK} are the intersection numbers of CY_3, and p^I is the (magnetic) charge of the M5-brane wrapping the Ith 4-cycle [148]. The product $c_2 \cdot p$ contains the second Chern class of CY_3.[1]

Let us apply these formulae to the present case[2] and identify

$$p^1 = p \,, \qquad\qquad p^2 = N \,, \qquad\qquad p^3 = N' \,. \tag{4.2.10}$$

Denoting the single modulus of the T^2 by p^1, the only non-vanishing intersection numbers are $c_{1ij} = c_{ij}$, where c_{ij} is the intersection matrix for $K3$. For p M5-branes wrapping $K3$, $c_2 \cdot p = c_2(K3)p = 24p$ [31, 32], and (4.2.9) provides the central charges

$$\begin{aligned} c_L &= 6NN'p + 24p \,, \\ c_R &= 6NN'p + 12p \,. \end{aligned} \tag{4.2.11}$$

Since $D \neq 0$, this three-charge model preserves only $(0, 4)$ supersymmetry [148]. For $N = N' = 0$, we have $D = 0$ and $(c_L, c_R) = (24p, 12p)$. These are the central charges of

---

[1] For an exact definition of the product $c_2 \cdot p$ see [148].

[2] In contrast to what is assumed in [148] for the four-cycle inside the CY_3, $K3$ is not a very ample divisor in $K3 \times T^2$. Nevertheless, we may still use (4.2.9), since $b_1(K3) = 0$, even though $b_1(K3 \times T^2) \neq 0$.

the $(0,8)$ low-energy effective field theory describing a stack of p heterotic strings.

Let us compare the central charges $c_{L,R}$ with that obtained from the supergravity solution by applying the Brown-Henneaux formula [23],

$$c = \frac{3R_{AdS}}{2G_N^{(3)}}\,, \tag{4.2.12}$$

where $G_N^{(3)}$ is Newton's constant in three dimensions. Substituting the AdS radius (4.2.8) of the uncorrected supergravity solution into (4.2.12), we get

$$c = 6NN'p\,, \tag{4.2.13}$$

as was already found in [137]. We notice that (4.2.13) agrees with (4.2.11) only to leading order in the charges. The reason for the absence of the subleading term in (4.2.13) is the fact that it is computed from an uncorrected supergravity solution. Taking into account higher derivative terms in the action as well, one recovers the full expression (4.2.11), as was recently shown for a dual setup [31, 32]. This result has also been reproduced with somewhat different methods in [121].

4.2.3 $\mathcal{N} = (0,2)$ worldsheet theory

We now discuss heterotic string theory on the $AdS_3 \times S^3/\mathbb{Z}_N \times T^4$ near-horizon geometry of the F1-NS5-KKM three-charge model introduced in section 4.2.1. The corresponding worldsheet theory has been constructed in [137], and we will only review some of its features relevant for the construction of the boundary conformal field theory.

The worldsheet theory is expected to be the product of a heterotic $SL(2)$ WZW model, a conformal field theory on $S^3/\mathbb{Z}_N$ and a free $U(1)^4$ CFT on the four-torus T^4. As a heterotic model, the product theory is bosonic in the left-moving sector and supersymmetric in the right-moving sector. The heterotic $SL(2)$ WZW model therefore has a bosonic affine $SL(2)$ algebra of level k_b in the left-moving sector and a supersymmetric one of level $k_s = k_b - 2$ in the right-moving sector. Accordingly, the right-moving sector is generated by three bosonic and three fermionic currents, $\bar{J}^A$ and $\bar{\psi}^A$ $(A = 1,2,3)$, while the left-moving sector contains only J^A. Similarly, the right-moving CFT on T^4 is constructed from four bosonic fields $\bar{Y}^i$ and four fermions $\bar{\lambda}_i$ $(i = 1,2,3,4)$. The left-moving sector contains only the bosonic currents Y^i.

In the unorbifolded case, the S^3 factor of the geometry would be described by an $SU(2)$ WZW model with levels k'_b and $k'_s = k'_b + 2$ in the left- and right-moving sector, respectively. The right-moving sector of the $SU(2)$ model contains three bosonic currents, $\bar{K}^a$, and three fermions $\bar{\chi}^a$ $(a = 1,2,3)$. The left-moving sector has the same bosonic currents K^a $(a = 1,2,3)$, but again no fermions.

Let us now implement the $\mathbb{Z}_N$ orbifold. We start from the $SU(2)$ WZW model in which we parameterise the $SU(2)$ group manifold in terms of the Euler angles

$$0 \leq \theta \leq \pi\,, \qquad 0 \leq \phi \leq 2\pi\,, \qquad 0 \leq \xi \leq 4\pi\,, \tag{4.2.14}$$

where ξ parameterises the fibre, and θ, ϕ are the base coordinates. As in [137], we consider an $SU(2)$ model at level

$$k'_b = NN' \tag{4.2.15}$$

and identify

$$\xi \sim \xi + \frac{4\pi}{N} . \tag{4.2.16}$$

The orbifold acts asymmetrically in the near-horizon geometry. We therefore turn the $SU(2)$ WZW model into a coset model of the type

$$\frac{SU(2)_L \times SU(2)_R}{(\mathbb{Z}_N)_L} , \tag{4.2.17}$$

where the orbifold is embedded in $SU(2)_L$: $\mathbb{Z}_N$ acts on the currents as

$$\begin{aligned} K^{\pm} &\to e^{\pm\frac{4\pi i}{N}} K^{\pm} , \qquad & K^3 &\to K^3 , \\ \bar{K}^{\pm,3} &\to \bar{K}^{\pm,3} , \qquad & \chi^{\pm,3} &\to \chi^{\pm,3} . \end{aligned} \tag{4.2.18}$$

For $N > 2$, the effect of the asymmetric orbifold is to break the $SU(2)$ of the left-moving sector down to $U(1)$, whose current K^3 is invariant under the orbifold action.[3]

The consistency of the theory requires that the worldsheet central charges are $(c_L^{\text{ws}}, c_R^{\text{ws}}) = (26, 15)$. The central charges in the right-moving sector are

$$c_R^{\text{ws}}(AdS_3) = \frac{3}{2} + \frac{3k_b}{k_b - 2} , \qquad c_R^{\text{ws}}(S^3/\mathbb{Z}_N) = \frac{3}{2} + \frac{3k'_b}{k'_b + 2} , \qquad c_R^{\text{ws}}(T^4) = 6 , \tag{4.2.19}$$

which adds up to $c_R^{\text{ws}} = 15$ provided that

$$k_b = k'_b + 4 . \tag{4.2.20}$$

Similarly, for the left-moving sector we have

$$c_L^{\text{ws}}(AdS_3) = \frac{3k_b}{k_b - 2} , \qquad c_L^{\text{ws}}(S^3/\mathbb{Z}_N) = \frac{3k'_b}{k'_b + 2} , \qquad c_L^{\text{ws}}(T^4) = 4 , \tag{4.2.21}$$

which adds up to ten. Heterotic string theory also contains 32 left-moving current algebra fermions, *i.e.* 16 for each E_8. We thus get $c_L^{\text{ws}} = 10 + 16 = 26$, as required.

The worldsheet theory also provides some information on the boundary conformal field theory. As shown in [111], the left- and right-moving (super)Virasoro algebras of the boundary CFT can be constructed from the worldsheet affine $SL(2)$ Lie algebra. Their central charges are

$$(c_L, c_R) = (6k_b p, 6k_s p) , \tag{4.2.22}$$

---

[3] In the related S^2 theory of [103] the orbifold is embedded in the supersymmetric (right) sector, and $(0, 2)$ worldsheet supersymmetry relates N and N'. In the present case N and N' are independent since the orbifold is embedded in the non-supersymmetric (left) sector.

where, as before, k_b and $k_s = k_b - 2$ are the levels of left- and right-moving $SL(2)$ algebras, and p is the number of heterotic strings. Substituting (4.2.15) and (4.2.20) in (4.2.22), we find the central charges

$$(c_L, c_R) = (24p + 6NN'p, 12p + 6NN'p) \tag{4.2.23}$$

which agree with (4.2.11) and satisfy the constraint $c_L - c_R = 12p$ as also found in [137, 135].

Let us finally consider the amount of worldsheet and target space supersymmetry. From the geometry we expect that the worldsheet model preserves a $(0,4)$ target space supersymmetry. Since T^4 is Kähler, the heterotic worldsheet CFT on T^4 has $(0,2)$ supersymmetry. The Kähler structure also ensures that the $(0,2)$ worldsheet supersymmetry leads to $(0,4)$ spacetime supersymmetry. The heterotic $SL(2)$ model and the "twisted" $SU(2)$ model separately preserve only $(0,1)$ supersymmetry. Only the product of both models has a chance to have $(0,2)$ worldsheet supersymmetry. In order to enhance $\mathcal{N} = 1$ to $\mathcal{N} = 2$ supersymmetry in the right sector, one must find a $U(1)_R$ current $J_{\mathcal{N}=2}$, which is part of the $\mathcal{N} = 2$ algebra. The existence of such a current is guaranteed by the fact that the orbifold is embedded in $SU(2)_L$ such that the right sector remains unaffected by it. The $\mathcal{N} = 2$ $U(1)_R$ current therefore has the same structure as in the (unorbifolded) type II case, see [111].

4.3 Two-dimensional boundary sigma model

4.3.1 General remarks

In this section we discuss the two-dimensional $(0,4)$ conformal field theory living on the boundary of the AdS_3 space. Our starting point is the heterotic brane setup introduced in the previous section. We first T-dualize in x^5 to go from $E_8 \times E_8$ to $SO(32)$ heterotic string theory and then use heterotic/type I duality in order to obtain the following type I brane configuration:

		0	1	2	3	4	5	6	7	8	9
p	D1	●	●								
N'	D5	●	●					●	●	●	●
N	KKM	●	●					●	●	●	●
32	D9	●	●	●	●	●	●	●	●	●	●

Let us consider the type I setup in detail. Since the heterotic/type I duality involves a strong-coupling transition, the heterotic F1 and NS5-branes naturally map to D1 and D5-branes. Moreover, since we are dealing with a type I string theory we are also required to introduce 32 D9-branes and perform an orientifold projection. In order to understand the contribution of the KK monopoles, we recall that the approximation of the near-core region of N KK monopoles is a $\mathcal{C}^2/\mathbb{Z}_N$ orbifold. This instructs us to study a $\mathbb{Z}_N$ orbifold in the directions $x^{2,3,4,5}$ of the D1-D5-D9-brane theory.

In order to set up our notation we remark that the D1-D5-D9 brane configuration breaks ten-dimensional Lorentz symmetry to $SO(1,1) \times SO(4)_E \times SO(4)_I$, where $SO(4)_E$ and $SO(4)_I$ rotate $x^{2,3,4,5}$ and $x^{6,7,8,9}$, respectively. We will use the standard decomposition

$$SO(4)_E \times SO(4)_I \simeq SU(2)_A \times SU(2)_Y \times SU(2)_{A'} \times SU(2)_{\tilde{A}'}$$

to label the appearing representations in terms of doublet representations with $(A', \tilde{A}', A, Y = \pm)$. The orbifold is embedded in $SU(2)_Y$.

We will start out our construction by reviewing the low-energy effective theory of the type I D1-D5-D9 intersection. In the absence of any KK monopoles this theory was shown in [59] (for $p = 1$) to be equivalent to Witten's ADHM model of Yang-Mills instantons. In section 4.3.2 we will review the model for $p > 1$ as constructed in [145]. In section 4.3.3 we will include the effect of the KK monopoles by orbifolding the ADHM model. Subsequently, in section 4.3.4 we discuss its instanton moduli space and determine the central charges of the Higgs branch theory.

4.3.2 Spectrum of D1-D5-D9 and the ADHM model

Let us briefly recall some basic facts. Spacetime fermions arise in the Ramond sector, and spacetime bosons in the Neveu-Schwarz sector. If the boundary conditions on both ends of the string are the same, then the worldsheet fermions of the R sector have integral modes, and those in the NS sector half-integers. If the boundary conditions are different, the additional signs introduced exchange the moddings, which also changes the ground state energy of the sector. In particular, the NS ground state energy in the case of N_{DN} mixed boundary conditions is given by

$$-\frac{1}{2} + \frac{N_{DN}}{8} \ , \tag{4.3.1}$$

whereas the ground state energy in the R sector is always zero.

Let us now discuss the strings stretching between the various types of branes.

1-1 strings

In the NS-sector, the massless modes form a ten-dimensional vector A^μ_{ab}, the Chan-Paton indices running over $a, b = 1, \ldots, p$. Considered as an object on the D1, it splits into a 2d vector A^μ_{ab} and 8 scalars b^i_{ab}. The orientifold projection Ω maps $A^\mu_{ab} \mapsto -A^\mu_{ba}$. We are thus left with the gauge bosons $A^\mu_{[ab]}$ in the adjoint of the gauge group $SO(p)$. On the other hand, the vertex operator of b picks up no sign under Ω, as it contains no derivative along the boundary. This leaves 8 bosons $b^i_{(ab)}$ in the symmetric representation of $SO(p)$ which we group in a pair of 4 bosons, $b^{AY}_{(ab)}$ and $b^{A'\tilde{A}'}_{(ab)}$.

In the R-sector, the GSO projection restricts to modes which are invariant under $\bar{\Gamma} := \Gamma^0 \ldots \Gamma^9$, where Γ^μ denotes the fermionic zero modes. To obtain the action of Ω, note that the fermionic modes $\psi^2, \ldots, \psi^9$ reflect from the boundary with an extra minus sign, so that they pick up an additional minus sign under exchange of right and left movers. Ω thus acts on massless fermions as $\Omega = -\Gamma^2\Gamma^3 \dots \Gamma^9$. The massless spinors thus must satisfy the two conditions

$$\psi_{ab} = \bar{\Gamma}\psi_{ab} = -\Gamma^2 \dots \Gamma^9 \psi_{ba} \ . \tag{4.3.2}$$

The first condition simply states that ψ is in the $\mathbf{16}$ of $SO(1,9)$. To obtain the worldsheet behaviour of ψ, we need to decompose $\mathbf{16}$ into representations of $SO(1,1) \times SO(8)$, which gives $\mathbf{16} = \mathbf{8}'_+ \oplus \mathbf{8}''_-$, where $\mathbf{8}'$, $\mathbf{8}''$ are the two spinor representations of $SO(8)$ and $\pm$ denotes the chirality with respect to $SO(1,1)$. The second condition in (4.3.2) then states that $\psi_{(ab)}$ transforms as $\mathbf{8}''_-$, and $\psi_{[ab]}$ as $\mathbf{8}'_+$. The $\psi_{(ab)}$ are the right-moving superpartners of the $b_{(ab)}$. Due to the D5-branes each $\mathbf{8}$ decomposes into a pair of $\mathbf{4}$'s of the $SO(4)$'s. Following [59], these will be denoted by $\psi^{A'Y}_{-\,(ab)}$,$\psi^{A\tilde{A}'}_{-\,(ab)}$ and $\psi^{A'A}_{+\,[ab]}$, $\psi^{Y\tilde{A}'}_{+\,[ab]}$. The left-moving fermions $\psi_{+\,[ab]}$ are antisymmetric and therefore do not appear in the case of a single D1-brane.

1-5 strings

The analysis of this sector has been performed in [58]. Since $N_{DN} = 4$, the ground state energy is also zero in the NS-sector, so that there appear both bosons and fermions. In total, we obtain bosons $\phi^{A'm}_a$ in the $(p, 2N', 1)$ of $SO(p) \times Sp(2N') \times SO(32)$, and their right- and left-moving fermionic superpartners $\chi^{Am}_{-\,a}$ and $\chi^{Ym}_{+\,a}$. The index m runs over $m = 1, ..., 2N'$.

1-9 strings

Since $N_{DN} = 8$, the ground state energy of the NS-sector is strictly positive, so that there are no bosons. In the R-sector there are two massless modes Γ^0, Γ^1. The GSO projection eliminates one of them, leaving only the left moving mode. We thus obtain $32p$ left-moving fermions λ^M_{+a}, where $M = 1, ..., 32$ is the Chan-Paton index of $SO(32)$.

5-5 strings, 5-9 strings

The analysis of the remaining sectors has been performed in [59]. Since their field content is not very important in what follows, we only cite the results. The 5-brane fields form a $Sp(2N')$ gauge theory, a hypermultiplet in the antisymmetric representation with scalar component $X^{AY}_{[mn]}$, and "half-hypermultiplets" in $(1, 2N', 32)$ with scalar component h^{Am}_M.

We summarise the results by listing the relevant fields in the following table (see also [145]):

We have not listed fields coming from 5-5 and 5-9 strings, since here we are only interested in the case of vanishing instanton size which corresponds to setting the 5-9 fields to zero (see [185, 139]). Moreover, the 5-5 fields X^{AY}_{mn} denote the position of the D5-branes in the

strings	bosons	fermions	$SO(p)$ rep.
1-1	$A^{\mu}_{[ab]}$	$\psi^{A'A}_{+\,[ab]}$, $\psi^{Y\tilde{A}'}_{+\,[ab]}$	adj.=anti-sym.
	$b^{AY}_{(ab)}$	$\psi^{A'Y}_{-\,(ab)}$	sym.
	$b^{A'\tilde{A}'}_{(ab)}$	$\psi^{A\tilde{A}'}_{-\,(ab)}$	sym.
1-5	$\phi^{A'm}_{a}$	$\chi^{Am}_{-\,a}$	fund.
		$\chi^{Ym}_{+\,a}$	fund.
1-9		$\lambda^{M}_{+\,a}$	fund.

Table 4.1: Summary of fields in the ADHM model.

transversal space, which we treat as parameters of the low energy theory.[4]

The Lagrangian describing the low-energy physics of the type I D1-D5-D9 intersection can now be written in terms of the fields of table 4.1. For $p \geq 1$, it is convenient to divide the Lagrangian into three parts,

$$\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{pot}} + \mathcal{L}_{\text{int}}\,, \tag{4.3.3}$$

where $\mathcal{L}_{\text{kin}}$ contains the kinetic terms for all fields in table 4.1, and $\mathcal{L}_{\text{pot}}$ describes their potential. In general, $\mathcal{L}_{\text{pot}}$ contains Yukawa couplings of the type $b\psi_+\psi_-$ and D-terms for the scalars b. For details, see ref. [145].

The Lagrangian $\mathcal{L}_{\text{int}}$ describes the interaction of 1-1 with 1-5 string modes and is given by [59, 145]

$$\begin{aligned}\mathcal{L}_{\text{int}} = \text{Tr}\,\Bigg(&\frac{im}{2}\big(\psi_-^{A'Y}\chi_{+\,Ym} + \psi_+^{AA'}\chi_{-\,Am}\big)\phi_{A'}{}^{m} + \frac{im}{2}\chi_{+\,Ym}(X^{AY}_{mn} - b^{AY}\delta_{mn})\chi_-^{An} \\ &+\frac{m^2}{8}(X^{AY}_{mn} - b^{AY}\delta_{mn})^2\phi_{A'm}\phi^{A'n}\Bigg) + c.c.\,,\end{aligned} \tag{4.3.4}$$

where the trace is taken over the $SO(p)$ indices. As first found in [59] for $p = 1$, this Lagrangian corresponds to Witten's ADHM model [184] describing an $Sp(2N')$ instanton with instanton number one. It is believed that for $p \geq 1$ the ADHM model describes the moduli space of $Sp(2N')$ instantons with instanton number p.

4.3.3 ADHM orbifold theory

Field content of the orbifold theory

Let us now include the effect of the KK monopoles in the ADHM model. This requires us to consider the D1-D5-D9 intersection at the origin of a $\mathcal{C}^2/\mathbb{Z}_N$ orbifold acting along

---

[4] As we will explain in more detail when discussing the orbifolded theory in section 4.3.3, the D5-branes will all be clustered at the orbifold fixed point ($x^2 = x^3 = x^4 = x^5 = 0$), which instructs us to set $X^{AY}_{mn} = 0$.

$x^{2,3,4,5}$. Following Refs. [61, 128, 126], we start with pN D1-branes intersecting $2N'N$ D5-branes and $32N$ D9-branes in flat space and take the corresponding ADHM Lagrangian with gauge group $U(Np)\times U(2NN')\times U(32N)$ as the parent theory.[5] The ADHM orbifold theory is then obtained by projecting out the degrees of freedom which are not invariant under the $\mathbb{Z}_N$ orbifold group.

The $\mathcal{C}^2/\mathbb{Z}_N$ orbifold is realized as follows. Denote the matrix b^{AY} by

$$b = (b^{AY}) = \begin{pmatrix} b^1 & -\bar{b}^2 \\ b^2 & \bar{b}^1 \end{pmatrix} , \tag{4.3.5}$$

where $b^1 = x^2 + ix^3$ and $b^2 = x^4 + ix^5$. Then the action of $(g_A, g_Y) \in SO(4)_E = SU(2)_A \times SU(2)_Y$ along $x^{2,3,4,5}$ is realized by

$$b \mapsto g_Y b g_A \, . \tag{4.3.6}$$

We now embed the $\mathbb{Z}_N$ action in $SU(2)_Y$ by choosing $g_Y = \text{diag}(\omega, \omega^{-1})$ with $\omega = e^{2\pi i/N}$. Then,

$$b^1 \mapsto \omega b^1 \, , \quad b^2 \mapsto \omega^{-1} b^2 \, , \tag{4.3.7}$$

or, alternatively, $b^{AY} \mapsto \omega^Y b^{AY}$. The scalars $b^{A'\tilde{A}'}$ (along $x^{6,7,8,9}$) remain unaffected by the orbifold. The origin of $x^{2,3,4,5}$ is the only fixed point of the orbifold.

The orbifold therefore acts on the fields of the ADHM model as follows (gauge indices suppressed):

$$\begin{aligned}
1-1: \quad & b^{AY} \to \omega^Y g_1(\omega) b^{AY} g_1^\dagger(\omega) \, , && \psi_-^{A'Y} \to \omega^Y g_1(\omega) \psi_-^{A'Y} g_1^\dagger(\omega) \, , \\
& && \psi_+^{Y\tilde{A}'} \to \omega^Y g_1(\omega) \psi_+^{Y\tilde{A}'} g_1^\dagger(\omega) \, , \\
& b^{A'\tilde{A}'} \to g_1(\omega) b^{A'\tilde{A}'} g_1^\dagger(\omega) \, , && \psi_-^{A\tilde{A}'} \to g_1(\omega) \psi_-^{A\tilde{A}'} g_1^\dagger(\omega) \, , \\
& && \psi_+^{A'A} \to g_1(\omega) \psi_+^{A'A} g_1^\dagger(\omega) \\
\\
1-5: \quad & \phi^{A'} \to g_1(\omega) \phi^{A'} g_5^\dagger(\omega) \, , && \chi_-^{A} \to g_1(\omega) \chi_-^{A} g_5^\dagger(\omega) \, , \\
& \chi_+^{Y} \to \omega^Y g_1(\omega) \chi_+^{Y} g_5^\dagger(\omega) \\
\\
1-9: \quad & \lambda_+ \to g_1(\omega) \lambda_+ g_9^\dagger(\omega) \, .
\end{aligned} \tag{4.3.8}$$

Here $g_1(\omega)$, $g_5(\omega)$, $g_9(\omega)$ denote the usual embeddings of the $\mathbb{Z}_N$ orbifold group in the gauge groups $U(Np)$, $U(2NN')$ and $U(32N)$, respectively. We choose a basis such that the embedding matrices have the block-diagonal form $g_i(\omega) = \text{diag}(\mathbf{1}, \omega\mathbf{1}, \omega^2\mathbf{1}, ..., \omega^{N-1}\mathbf{1})$, where $\mathbf{1}$ denotes a $p \times p$, $2N' \times 2N'$ and 32×32 unit matrix for $i = 1, 5, 9$, respectively. The fields thus decompose into N orbifold sectors which we denote by $j, j' = 0, \ldots, N-1$.

---

[5] Formally, we begin with the type IIB version of the ADHM model [59] and perform the orientifold projection in the next subsection. The overall factor 2 in $U(2NN')$ reflects the pairing of the D5-branes for invariance under Ω.

We observe that all fields carrying an index Y transform non-trivially under the orbifold group, *i.e.* the transformation law contains an additional factor ω^Y.

Substituting the embeddings $g_i(\omega)$ into (4.3.8), we get the following transformation behaviour in component form:

$$\begin{array}{lll}
1-1: & b^{AY}_{j,j'} \mapsto \omega^{Y+j-j'} b^{AY}_{j,j'}\,, & \psi^{A'Y}_{-\,j,j'} \mapsto \omega^{Y+j-j'} \psi^{A'Y}_{-\,j,j'}\,, \\
 & & \psi^{Y\tilde{A}'}_{+\,j,j'} \mapsto \omega^{Y+j-j'} \psi^{Y\tilde{A}'}_{+\,j,j'} \\
 & b^{A'\tilde{A}'}_{j,j'} \mapsto \omega^{j-j'} b^{A'\tilde{A}'}\,, & \psi^{A\tilde{A}'}_{-\,j,j'} \mapsto \omega^{j-j'} \psi^{A\tilde{A}'}_{-\,j,j'}\,, \\
 & & \psi^{A'A}_{+\,j,j'} \mapsto \omega^{j-j'} \psi^{A'A}_{+\,j,j'} \\
 & & \\
1-5: & \phi^{A'm}_{j,j'} \mapsto \omega^{j-j'} \phi^{A'm}_{j,j'}\,, & \chi^{Am}_{-\,j,j'} \mapsto \omega^{j-j'} \chi^{Am}_{-\,j,j'}\,, \\
 & \chi^{Ym}_{+\,j,j'} \mapsto \omega^{Y+j-j'} \chi^{Ym}_{+\,j,j'} & \\
 & & \\
1-9: & \lambda^{M}_{+\,j,j'} \mapsto \omega^{j-j'} \lambda^{M}_{+\,j,j'} &
\end{array} \tag{4.3.9}$$

where $Y = \pm 1$.

The fields invariant under the orbifold action (4.3.9) are thus

- 1-1: $(b^{A'\tilde{A}'}_{j,j}, \psi^{A\tilde{A}'}_{-\,j,j}, \psi^{A'A}_{+\,j,j})$ and $(b^{AY}_{j,j+Y}, \psi^{A'Y}_{-\,j,j+Y}, \psi^{Y\tilde{A}'}_{+\,j,j+Y})$
- 1-5: $(\phi^{A'm}_{j,j}, \chi^{Am}_{-\,j,j}, \chi^{Ym}_{+\,j,j+Y})$
- 1-9: $\lambda^{M}_{+\,j,j}$

Another important question concerns the gauge groups and representations under which these fields transform. Due to the Ω-projection of type I string theory, this issue is more intricate than in type II theories and will now be discussed at length.

Type I effective action

The type I effective theory is obtained by imposing, in addition to the $\mathbb{Z}_N$ orbifold projection, the orientifold Ω [61]. Let us denote the embedding of Ω into the gauge groups $U(Np)$, $U(2NN')$ and $U(32N)$ by $g_1(\Omega)$, $g_5(\Omega)$ and $g_9(\Omega)$, respectively. A generic (scalar) field y then transforms under worldsheet parity according to

$$y \mapsto g(\Omega) y^t g(\Omega)^{-1}\ , \tag{4.3.10}$$

while an element U of one of the above gauge groups satisfies

$$U g(\Omega) U^t g(\Omega)^{-1} = 1\ . \tag{4.3.11}$$

Here t denotes the transpose and g is one of the embeddings g_1, g_5, g_9.

To determine g, we have to solve various consistency conditions [61]. The first condition is

$$g(\Omega)_{ij} = \chi(\omega, \Omega)\omega^{i+j} g(\Omega)_{ij}\,. \tag{4.3.12}$$

We choose the phase $\chi(\omega,\Omega)=1$ which then implies that only $g(\Omega)_{i,N-i}$ is non-vanishing.[6] A second condition requires

$$g(\Omega)_{i,N-i}=\chi(\Omega)g(\Omega)^t_{N-i,i}\ , \tag{4.3.13}$$

with some phase factor $\chi(\Omega)=\pm1$. To reproduce the standard type I action, which has an $SO(32)$ gauge group for the D9-branes, we choose the phases $\chi(\Omega)=+1,-1,+1$ for $g=g_1,g_5,g_9$, respectively.[7] The solutions of (4.3.13) can be brought into the form

$$\begin{aligned}
g_{1,9}(\Omega)_{0,0}&=\mathbf{1}\,, & g_5(\Omega)_{0,0}&=\epsilon\,, & &\\
g_{1,9}(\Omega)_{i,N-i}&=\mathbf{1}\,, & g_5(\Omega)_{i,N-i}&=\mathbf{1}\,, & &0<i<N/2\,,\\
g_{1,9}(\Omega)_{N-i,i}&=\mathbf{1}\,, & g_5(\Omega)_{N-i,i}&=-\mathbf{1}\,, & &N/2<i<N\,,
\end{aligned} \tag{4.3.14}$$

where $\mathbf{1}$ is the corresponding $p\times p$, $2N'\times 2N'$ or 32×32 unit matrix. For even orbifolds, we have in addition

$$g_{1,9}(\Omega)_{N/2,N/2}=\mathbf{1}\,,\qquad g_5(\Omega)_{N/2,N/2}=\epsilon\,.$$

Let us now determine the unbroken gauge groups from (4.3.11). We distinguish between even and odd orbifolds:

- **even** N
  For $g=g_1$, the gauge group of the D1-branes is

  $$\begin{aligned} G^1_{\text{even}}&=\{(U_0,U_1,...,U_{N-1}):U_iU^t_{N-i}=1,0\le i\le N\}\\ &=SO(p)\times U(p)^{N/2-1}\times SO(p)\,, \end{aligned} \tag{4.3.15}$$

  while for the D5-branes, it is

  $$\begin{aligned} G^5_{\text{even}}&=\{(U_0,U_1,...,U_{N-1}):U_iU^t_{N-i}=1,0\le i\le N-1,i\neq N/2\}\\ &=Sp(2N')\times U(2N')^{N/2-1}\times Sp(2N')\,. \end{aligned} \tag{4.3.16}$$

- **odd** N
  For $g=g_1$, we get the gauge group

  $$\begin{aligned} G^1_{\text{odd}}&=\{(U_0,U_1,...,U_{N-1}):U_iU^t_{N-i}=1,\ 1\le i\le N-1\}\\ &=SO(p)\times U(p)^{\frac{N-1}{2}}\,, \end{aligned} \tag{4.3.17}$$

  while for $g=g_5$, it is

  $$\begin{aligned} G^5_{\text{odd}}&=\{(U_0,U_1,...,U_{N-1}):U_iU^t_{N-i}=1,\ 1\le i\le N-1\}\\ &=Sp(2N')\times U(2N')^{\frac{N-1}{2}}\,. \end{aligned} \tag{4.3.18}$$

---

[6] For even orbifolds one could also choose $\chi(\omega,\Omega)=\omega$, which would not invalidate our final conclusion. We will therefore not consider this case here.

[7] In fact, once we have set $\chi_9(\Omega)=+1$, which is necessary to get a consistent $SO(32)$ type I string theory, the other values follow (see [104]).

The effect on the matter fields is as follows. For the b^{AY}, equation (4.3.10) reads

$$(b^{AY}_{N-i-Y,N-i})^t = b^{AY}_{i,i+Y} \ . \tag{4.3.19}$$

For N even, this relates one half of the fields to the other half, but gives no additional constraints. The same holds true for the fermions $\psi_+^{Y\tilde{A}'}$ and $\psi_-^{A'Y}$. If N is odd, there is the additional condition

$$(b^{AY}_{(N-Y)/2,(N+Y)/2})^t = b^{AY}_{(N-Y)/2,(N+Y)/2} \ , \tag{4.3.20}$$

so that these particular b transform in the symmetric instead of the bifundamental. The situation is analogous to the analysis in section 4.3.2, so that their fermionic partners $\psi^{A'Y}_{-\,(N-Y)/2,(N+Y)/2}$ and $\psi^{Y\tilde{A}'}_{+\,(N-Y)/2,(N+Y)/2}$ transform in the symmetric and antisymmetric, respectively.

The $b^{A'\tilde{A}'}$ are subject to

$$(b^{A'\tilde{A}'}_{i,i})^t = b^{A'\tilde{A}'}_{N-i,N-i} \tag{4.3.21}$$

for all $i = 0, \ldots, (N-1)/2$ for N odd and $i = 0, \ldots, N/2$ for N even. Note that the fields $b^{A'\tilde{A}'}_{00}$ (and also $b^{A'\tilde{A}'}_{N/2,N/2}$ if N is even) are symmetric. Again, the situation is exactly as described above such that the corresponding fermionic modes, $\psi^{A\tilde{A}'}_{-\,0,0}$ and $\psi^{A'A}_{+\,0,0}$ (and $\psi^{A\tilde{A}'}_{-\,N/2,N/2}$ and $\psi^{A'A}_{+\,N/2,N/2}$ for N even), are in the symmetric and anti-symmetric representation, respectively.

We omit the corresponding relations for the $\phi^{A'm}_{i,i}$, as they again only relate half of the fields to the other half [61].

Quiver theory

So far we have determined the spectrum of fields that survive the orientifold projection along with the gauge groups of the world-volume theories of the various branes. It remains to determine the representations under which the matter fields transform. In fact they are given by

$b^{A'\tilde{A}'}_{j,j}, \psi^{A\tilde{A}'}_{-\,j,j}, \psi^{A'A}_{+\,j,j}$	adjoint rep. if $G^1_j = U(p)$, rep. as in table 4.1 if $G^1_j = SO(p)$,
$b^{AY}_{j,j+Y}, \psi^{A'Y}_{-\,j,j+Y}, \psi^{Y\tilde{A}'}_{+\,j,j+Y}$	bifundamentals of $G^1_j \times G^1_{j+Y}$,
$\phi^{A'm}_{j,j}, \chi^{Am}_{-\,j,j}$	bifundamentals of $G^1_j \times G^5_j$,
$\chi^{Ym}_{+\,j,j+Y}$	bifundamentals of $G^1_j \times G^5_{j+Y}$.

The gauge groups and matter content of the theory can now be encoded in a quiver diagram, see figure 4.1 for examples.[8]

Each node in the inner circle corresponds to a gauge group G^1_j (D1-branes), while an outer node represents a gauge group G^5_j (D5-branes). In principle, there are also nodes

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[8] A similar quiver diagram was also found in [39] for the $(0,4)$ quiver theory located on a D3/D3$'$ intersection at a $\mathcal{C}^2/\mathbb{Z}_N$ orbifold.

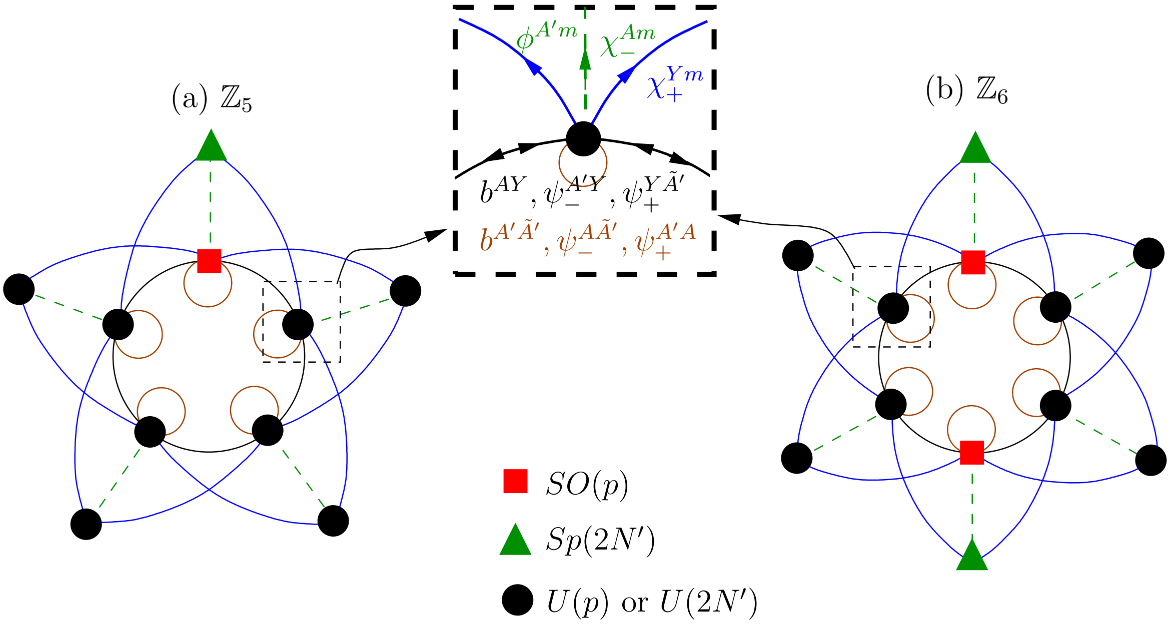

Figure 4.1: Quiver diagrams for odd ($\mathbb{Z}_5$) and even ($\mathbb{Z}_6$) N. The detail view in the centre shows the notation for the fields. For simplicity, we have not included the fields λ_+^M.

corresponding to $SO(32)$ gauge groups (D9-branes). The latter are not needed for the interaction Lagrangian and are therefore not shown in figure 4.1. The fields $b^{A'\tilde{A}'}_{j,j}$, $\psi^{A\tilde{A}'}_{-\,j,j}$, $\psi^{A'A}_{+\,j,j}$ transform under a single gauge group and are represented as brown circles. The bifundamentals $b^{AY}_{j,j+Y}$, $\psi^{A'Y}_{-\,j,j+Y}$, $\psi^{Y\tilde{A}'}_{+\,j,j+Y}$ (shown as black lines), $\phi^{A'm}_{j,j}$, $\chi^{Am}_{-\,j,j}$ (green lines), and $\chi^{Ym}_{+\,j,j+Y}$ (blue lines) connect different nodes. We have omitted bifundamentals connecting the outer nodes. These are generated by 5-5 strings which decouple at low-energies, as already discussed earlier.

We may now write down the corresponding quiver Lagrangian which descends from the ADHM Lagrangian in flat space, Eq. (4.3.3). Upon projecting out the degrees of freedom which are not invariant under the orbifold, we obtain

$$\mathcal{L} = \mathcal{L}_{\text{kin, quiv}} + \mathcal{L}_{\text{pot, quiv}} + \mathcal{L}_{\text{int, quiv}} \tag{4.3.22}$$

with the quiver interaction

$$\begin{aligned}
&\mathcal{L}_{\text{int, quiv}} = \\
&\quad \mathrm{Tr}\left(\frac{im}{2}(\chi_{+\,Ym})_{j,j+Y}(\phi_{A'}{}^{m})_{j+Y,j+Y}(\psi_-^{A'Y})_{j+Y,j} + \frac{im}{2}(\psi_+^{AA'})_{j,j}(\chi_{-\,Am})_{j,j}(\phi_{A'}{}^{m})_{j,j}\right.\\
&\quad \left. +\frac{im}{2}(\chi_{+\,Ym})_{j,j+Y}(\chi_-^{Am})_{j+Y,j+Y}(b_A{}^{Y})_{j+Y,j} + \frac{m^2}{8}(b^{AY}b_{AY})_{j,j}(\phi^{A'm}\phi_{A'm})_{j,j}\right) + c.c.\,,
\end{aligned}$$

and, similarly, $\mathcal{L}_{\text{kin, quiv}}$ and $\mathcal{L}_{\text{pot, quiv}}$ are the projections of $\mathcal{L}_{\text{kin}}$ and $\mathcal{L}_{\text{pot}}$ in (4.3.3), respectively. The range of summation over j and Y is restricted by the Ω projection. For instance, for N even, consider again the quiver diagram shown in figure 4.1. Each Yukawa coupling corresponds to a triangle in the quiver diagram. The field identifications of the previous section introduce a kind of reflection axis, which vertically divides the quiver in two parts. The $SO(p)$ gauge groups at $j = 0, N/2$ lie on the $\mathbb{Z}_2$ reflection axis. Due to constraints such as (4.3.19), each field on the right hand side of the axis is identified with one on the left hand side. In (4.3.22) we therefore sum only over $j = 0, ..., N/2$ and set $Y = +1$ at $j = 0$ and $Y = -1$ at $j = N/2$, $Y = \pm 1$ otherwise. The gauge groups are chosen as in (4.3.15) and (4.3.16). For N odd, the Lagrangian is constructed in a similar way.

4.3.4 Higgs branch theory and instanton moduli space

Higgs branch theory

In this section we investigate the infrared fixed point theory of the ADHM quiver model (4.3.22). This theory will be interpreted as the boundary conformal field theory dual to the worldsheet theory described in section 4.2.3. For its construction, we first have to choose a vacuum solution which sets the potential of (4.3.22) to zero. Inspecting the term $m^2 b^2 \phi^2$ in (4.3.22), we find two different possibilities for the scalars b^{AY} and $\phi^{A'm}$ and their vacuum expectation values $\langle b^{AY} \rangle$ and $\langle \phi^{A'm} \rangle$ [184]:[9]

- Coulomb branch: $\langle b^{AY} \rangle \neq 0$ and $\langle \phi^{A'm} \rangle = 0$
  On the Coulomb branch the D1-branes are transversely displaced from the D5-branes with $\langle b^{AY} \rangle$ proportional to the distance. In this case the $\phi^{A'm}$ become massive.

- Higgs branch: $\langle b^{AY} \rangle = 0$ and $\langle \phi^{A'm} \rangle \neq 0$
  On the Higgs branch the D1-branes and D5-branes form a bound state with $\langle \phi^{A'm} \rangle$ proportional to the binding strength between the two. In this case the b^{AY} become massive.

In the following we are interested in the situation where all branes form stacks located at the orbifold fixed point. We will therefore consider the Higgs branch of the theory.

In principle, we could now proceed as in [139] and integrate out all massive modes of the quiver theory. As in [139], this would lead to a $(0, 4)$ sigma model whose target space is the instanton moduli space $\mathcal{M}$ of the ultraviolet theory. The actual construction would be along the lines of [139] and involves a non-trivial gauge field $F^{pq}_{mn\,jj}$ which is defined in terms of the bifundamentals $\phi^{A'm}_{jj}$. Although straightforward, we will not do this explicitly here. Instead we only determine the left- and right-moving central charges of the infrared theory and compare them to those expected from the dual worldsheet model.

As outlined in the introduction, our strategy to find these charges is as follows. The ADHM quiver model is classically not conformally invariant, but ultraviolet finite such that there is no renormalisation group flow. This follows from the fact that the one-loop diagrams cancel, and all higher loop diagrams are finite [138]. The massless fields of the

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[9] We will not discuss the rather delicate case $\langle b^{AY} \rangle = \langle \phi^{A'm} \rangle = 0$.

quiver model therefore do not acquire anomalous conformal dimensions and contribute to the central charges of the infrared conformal field theory. This allows us to determine the left- and right-moving central charges of the infrared conformal field theory from the number of massless modes in the ultraviolet quiver theory.

Number of massless modes for N even

We begin by counting the massless degrees of freedom in the case of even orbifolds: First, there are the bifundamental fields $(\phi_a^{A'm})_{j,j}$ descending from 1-5 strings and their left- and right-moving fermionic partners $(\chi_{-\,a}^{Am})_{j,j}$ and $(\chi_{+\,a}^{Ym})_{j,j+1}$. These fields are not constrained by any D-term relations and thus contribute $2 \cdot N \cdot 2N' \cdot p = 4NN'p$ scalars and an equal number of left- and right-moving fermions. The 5-1 string modes are related to the 1-5 modes by the Ω reflection and therefore do not contribute any additional massless modes.

Second, consider the bosons $(b_{ab}^{AY})_{j,j+Y}$ which are massive on the Higgs branch. Since the theory has $(0,4)$-supersymmetry, we know immediately that an equal number of right-moving fermions $(\psi_{-\,ab}^{A'Y})_{j,j+Y}$ has to obtain mass. However, since only non-chiral fermions can be massive, it follows that also all left-moving $(\psi_{+\,ab}^{AA'})_{j,j}$ become massive. The mass terms for the latter arise due to couplings of the type $\psi_+\chi_-\phi$ in (4.3.22). This sector thus has no massless modes.

Third, consider the scalars $(b_{ab}^{A'\tilde{A}'})_{j,j}$. Those fields $(b_{ab}^{A'\tilde{A}'})_{j,j}$ which are adjoints of a $U(p)$ gauge group do not contribute to the counting: The $4p^2$ degrees of freedom of $(b_{ab}^{A'\tilde{A}'})_{j,j}$ (for fixed $j \neq 0, N/2$) are removed by $3p^2+p^2$ conditions coming from the vanishing of the corresponding D-term and $U(p)$ gauge equivalence. By supersymmetry, the same number of $(\psi_{-\,ab}^{A\tilde{A}'})_{j,j}$ are removed, and by the same pairing mechanism as described above also all of the $(\psi_{+\,ab}^{Y\tilde{A}'})_{j,j+Y}$. These fields thus give no contribution.

For $j = 0$ and $j = N/2$, however, the gauge group is $SO(p)$, and the counting is similar as in the unorbifolded case [58, 16]: the fields $(b_{(ab)}^{A'\tilde{A}'})_{0,0}$ and $(b_{(ab)}^{A'\tilde{A}'})_{N/2,N/2}$ are in the symmetric representation of $SO(p)$ and contribute $4p(p+1)/2$ real scalars each. However, there are also $4p(p-1)/2$ constraints due to D-term relations and gauge equivalences. In total, $(b_{(ab)}^{A'\tilde{A}'})_{0,0}$ and $(b_{(ab)}^{A'\tilde{A}'})_{N/2,N/2}$ thus contribute $2(4p(p+1)/2-4p(p-1)/2) = 8p$ massless bosons. Supersymmetry then dictates that of the $8p(p+1)/2$ right-moving fermions $(\psi_{-(ab)}^{A'\tilde{A}'})_{0,0}$ and $(\psi_{-(ab)}^{A'\tilde{A}'})_{N/2,N/2}$ only $8p$ survive. To eliminate the remaining $8p(p-1)/2$, we need to pair up all of the $8p(p-1)/2$ left-moving fermions $(\psi_{+[ab]}^{A'\tilde{A}'})_{0,0}$ and $(\psi_{+[ab]}^{A'\tilde{A}'})_{N/2,N/2}$. This leaves us with no left-moving massless fermions.

Number of massless modes for N odd

Much of the above analysis carries over to odd orbifolds. The fields $(\phi_a^{A'm})_{j,j}$ again contribute $4NN'p$ massless bosonic degrees of freedom and an equal number of left- and right-moving fermions. For $j \neq 0$, the $(b_{ab}^{A'\tilde{A}'})_{j,j}$ of the $U(p)$ gauge groups are eliminated by D-terms, and for $j = 0$ $(b_{(ab)}^{A'\tilde{A}'})_{0,0}$ give $4p$ degrees of freedom. Note that we only have one $SO(p)$ gauge group and we therefore get only half as many massless degrees of freedom from these fields as required.

Since we are on the Higgs branch, all the $(b^{AY}_{ab})_{j,j+Y}$ become massive, except for the fields $(b^{A+}_{ab})_{(N-1)/2,(N+1)/2}$ and $(b^{A-}_{ab})_{(N+1)/2,(N-1)/2}$ shown by red arrows in figure 4.2. These fields are special and essentially take on the role played by the second $SO(p)$ gauge group in the even case. By (4.3.20) these particular b^{AY} fields and their superpartners ψ_- are symmetric fields with $4p(p+1)/2$ components each, while the corresponding left-moving fermions ψ_+ are antisymmetric fields with $4p(p-1)/2$ components. From the type II theory we know that the only other left-moving fermions, the χ_+, remain massless. We can thus only form $4p(p-1)/2$ Yukawa terms so that of the ψ_-, $4p(p+1)/2 - 4p(p-1)/2 = 4p$ remain. By supersymmetry, the same number of bosons b must remain massless. The total number of bosonic degrees of freedom is thus again $4NN'p + 8p$ (for $N > 1$), the same as in the even case.

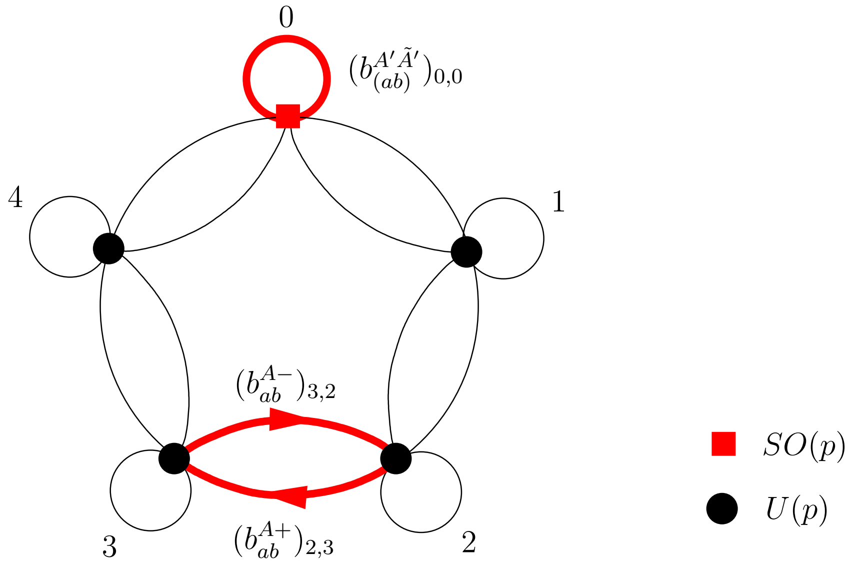

Figure 4.2: Inner circle of the quiver diagram for an odd type I orbifold ($\mathbb{Z}_5$). The special fields that contribute to the counting are denoted by red arrows.

In the degenerate case $N = 1$ there is one $SO(p)$ gauge group, but no bifundamentals b^{AY} of the type described above. We therefore get only $4NN'p + 4p$ bosonic massless degrees of freedom, in agreement with the unorbifolded ADHM model.

Central charges of the Higgs branch theory

From the above counting of massless degrees of freedom, we find that the moduli space of the ultraviolet theory is spanned by the $4NN'p$ fields $(\phi^{A'm}_a)_{j,j}$ and the $8p$ independent degrees of freedom provided by $(b^{A'\tilde{A}'}_{(ab)})_{j,j}$ $(j = 0, N/2)$. Its dimension is therefore given by

$$\dim \mathcal{M} = 4NN'p + 8p\,. \qquad (4.3.23)$$

Recalling that the target space of the conformal sigma model on the Higgs branch is the instanton moduli space of the ADHM quiver model, we may now also determine the central charges of the infrared theory. For $N \geq 2$ we find

$$(c_L, c_R) = (6NN'p + 24p, 6NN'p + 12p) \tag{4.3.24}$$

in agreement with (4.2.11) and (4.2.22). The leading term, $6NN'p$, is given by the ADHM instanton fields $\phi_{jj}^{A'm}$ and their fermionic partners (1-5 strings). The subleading term in the right sector, $12p$, is given by the conformal charges of the $8p$ independent degrees of freedom of the scalars $b_{jj}^{A'\tilde{A}'}$ and their fermionic superpartners (1-1 strings). One contribution to the term $24p$ in the left-moving sector is given by the $8p$ bosonic fields descending from the $b_{jj}^{A'\tilde{A}'}$. The remaining $16p$ are given by the 32 fermions λ_{+a}^{M} (1-9 strings).

In conclusion, we propose the $(0,4)$ sigma model on the Higgs branch of the type I quiver model (4.3.22) as the appropriate candidate for the boundary conformal field theory of heterotic string theory on $AdS_3 \times S^3/\mathbb{Z}_N \times T^4$ $(N \geq 2)$.

4.4 Heterotic two-charge models

In view of a possible heterotic string duality with $(0,8)$ spacetime supersymmetry [140, 135], it is an interesting question whether we can systematically switch off charges in the present $(0,4)$ duality. Clearly, the worldsheet theory for strings on $AdS_3 \times S^3/\mathbb{Z}_N \times T^4$ requires at least one KK monopole and is not applicable for vanishing KK monopole charge. Since the KK monopoles break supersymmetry down to $(0,4)$ there seems to be no obvious way to generalise the model to $(0,8)$. Nevertheless, it is interesting to consider models with less charges such as the F1-KKM and the NS5-KKM intersection.

4.4.1 F1-KKM intersection $N' = 0$

We shall first consider a heterotic two-charge model consisting of a stack of p fundamental strings in the background of a KK monopole with charge proportional to $N \geq 2$. The setup is the same as in section 2.1, but now $N' = 0$ (no D5-branes). From (4.2.11), we find the central charges of the boundary conformal field theory to be $(c_L, c_R) = (24p, 12p)$. Remarkably, the central charges do not depend on the charge of the KK monopole since the leading term cubic in the charges ($\propto NN'p$) is absent. This has some interesting consequences.

Let us first have a look at the supergravity solution. Classically, the solution has a horizon of zero area leaving a naked curvature singularity at the origin. This corresponds to a vanishing Bekenstein-Hawking entropy on the classical level. It is however believed that higher-derivative corrections to the supergravity solution resolve the classical singularity leading to a finite entropy. The corrected supergravity solution presented in the previous section is valid for large NN' and thus cannot be applied to this case.

The heterotic worldsheet theory for this case has some peculiar features. The left sector of the CFT on the $S^3/\mathbb{Z}_N$ has collapsed to a trivial theory with bosonic level $k_b' = c_L^{\mathrm{ws}}(S^3/\mathbb{Z}_N) = 0$. The supersymmetric level corresponding to the right sector is $k_s' = k_b' + 2 = 2$, and we have $c_R^{\mathrm{ws}}(S^3/\mathbb{Z}_N) = \frac{3}{2}$. We are thus left with a trivial theory in the left sector and three fermions $\bar{\chi}^a$ $(a = 1, 2, 3)$ in the right sector. The AdS$_3$ part of the geometry is described by a heterotic $SL(2)$ WZW model with levels $k_b = 4$ and $k_s = 2$. The full (supersymmetric part of the) background is thus

$$SL(2,\mathbb{R})_2 \times \{\bar{\chi}^1, \bar{\chi}^2, \bar{\chi}^3\} \times T^4 \,, \tag{4.4.1}$$

and the central charges of the worldsheet model are:

$$\begin{aligned} c_L^{\mathrm{ws}}(SL(2)) &= 6\,, \qquad c_L^{\mathrm{ws}}(S^3/\mathbb{Z}_N) = 0\,, \qquad c_L^{\mathrm{ws}}(T^4) = 4\,, \\ c_R^{\mathrm{ws}}(SL(2)) &= 15/2\,, \qquad c_R^{\mathrm{ws}}(S^3/\mathbb{Z}_N) = 3/2\,, \qquad c_R^{\mathrm{ws}}(T^4) = 6\,, \end{aligned} \tag{4.4.2}$$

ensuring criticality, $(c_L^{\mathrm{ws}}, c_R^{\mathrm{ws}}) = (26, 15)$, given that $c_L^{\mathrm{ws}}(E_8 \times E_8) = 16$. The worldsheet model also gives the correct central charges for the boundary CFT, cf. Eq. (4.2.23). Related heterotic models involving three fermions can be found in [109, 42].

We conclude with some comments on the dual boundary conformal field theory. Removing the D5 branes in the quiver ADHM theory corresponds to the removal of the outer circle and the spikes in the quiver diagram in figure 4.1. The ADHM part of the quiver action disappears, leaving only that part of the action which corresponds to the inner circle of the quiver diagram. Nevertheless, the counting of the massless degrees of freedom in the remaining quiver theory seems to yield the correct central charges, $(c_L, c_R) = (24p, 12p)$ (for $N \geq 2$). It is interesting to observe that the independence of $c_{L,R}$ on N is reflected by fact that varying N changes only the number of sites in the quiver diagram corresponding to $U(p)$ gauge groups. Recall, however, that the fields of the $U(p)$ gauge groups do not contribute to the central charges of the infrared conformal field theory. Certainly, it would be interesting to study this field theory in more detail.

4.4.2 Heterotic NS5-KKM intersection $p = 0$

For completeness, we also consider the NS5-KKM intersection which can be obtained from the three-charge model of section 4.2.1 by setting $p = 0$.

Let us approach this setup from a slightly different point of view. In [140] Lapan, Simons and Strominger suggested to start from a four-dimensional monopole black hole with near-horizon geometry

$$\mathbb{R}^t \times \mathbb{R}^\phi \times S^2 \times T^6 \,, \tag{4.4.3}$$

where $\mathbb{R}^t$ denotes time and $\mathbb{R}^\phi$ a real line labelled by ϕ with linear dilaton. Decompactifying one of the compact directions, *i.e.* replacing $\mathbb{R}^t \times S^1$ by a two-dimensional Minkowski space $\mathbb{R}^{1,1}$ leads to the geometry

$$\mathbb{R}^{1,1} \times \mathbb{R}^\phi \times S^2 \times T^5 \,. \tag{4.4.4}$$

The CFT on (4.4.4) is then expected to describe a monopole string in five dimensions [140]. Ref. [140] also suggested that the S^2 factor could be described by the coset model of [103].

Here, however, we deviate from the proposal of [140] and include a KK monopole charge by replacing $S^2 \times T^5$ by $S^3/\mathbb{Z}_N \times T^4$. Of course, we thereby break half of the target space supersymmetry. Heterotic string theory in the background of a five-dimensional monopole string with additional KK monopole charge is then expected to be given by the CFT on

$$\mathbb{R}^{1,1} \times \mathbb{R}^{\phi} \times S^3/\mathbb{Z}_N \times T^4 \,. \tag{4.4.5}$$

In fact, the thus derived background is nothing but the near-horizon geometry of the F1-NS5-KKM set-up for *vanishing* electrical F1 charge, $p = 0$. This can be seen by setting $F = 1$ in (4.2.1) and taking the limit $r \rightarrow 0$.

Heterotic string theory on the background (4.4.5) can be described by a linear dilaton theory with central charges

$$c_L^{\mathrm{ws}}(\mathbb{R}^{1,1} \times \mathbb{R}^{\phi}) = 2 + (1 + 3Q_D^2)\,, \qquad c_R^{\mathrm{ws}}(\mathbb{R}^{1,1} \times \mathbb{R}^{\phi}) = 3 + (\tfrac{3}{2} + 3Q_D^2)\,, \tag{4.4.6}$$

and dilaton charge Q_D. The internal part of the geometry, $S^3/\mathbb{Z}_N$ and T^4, will be described as before, see section 2.3. By criticality, the linear dilaton charge Q_D is related to the bosonic level k_b' of the $S^3/\mathbb{Z}_N$ theory as

$$Q_D^2 = \frac{2}{k_b' + 2}\,, \tag{4.4.7}$$

where $k_b' = k_s' - 2 = NN'$, if we assume $k_s' = NN' + 2$.

Finally, as explained in [110], there is a simple relation between linear dilaton and $SL(2)$ models. Adding p D1-branes along the $R^{1,1}$ and taking the near-horizon limit amounts to replacing the factor $\mathbb{R}^{1,1} \times \mathbb{R}^{\phi}$ by AdS_3. The level of $SL(2)$ is related to the dilaton charge by $k_s = 2/Q_D^2$ ($k_b = k_s + 2$). This leads back to $AdS_3 \times S^3/\mathbb{Z}_N \times T^4$, as expected.

In summary, the proposed heterotic duality obviously requires further investigation. The evidence given here is based on the counting of the massless degrees of freedom of the ultraviolet orbifold theory. These modes are not renormalised and therefore also constitute the Higgs branch theory. Its actual construction is expected to be straightforward along the lines of [139] by integrating out the massive modes in the UV theory. This procedure will be made more complicated by the fact that the Higgs branch metric will receive α' corrections and seems to be divergent at the origin [139].

Another interesting check of our proposal would then be to work out the dictionary between the chiral primaries of the boundary CFT and those of the worldsheet model [137]. The primaries of the boundary CFT will be composite operators of the massless fields of the ultraviolet ADHM quiver model. A comparison of the corresponding n-point functions should then provide further evidence for the duality. Such tests have been performed in the type II AdS$_3$/CFT$_2$ duality in [95, 43, 163, 178, 107].

Chapter 5

Modular differential equations and null vectors

5.1 Overview

Every rational conformal field theory possesses a modular differential equation. This is to say, the different characters of the finitely many irreducible highest weight representations satisfy a common differential equation in the modular parameter. This fact was first observed, using the transformation properties of the characters under the modular group, in [62, 11, 151, 152]; later developments of these ideas are described in [67, 66, 69, 68, 75, 15]. Following the work of Zhu [187], the modular transformation properties of the characters were derived from first principles (see also [158]). Zhu's derivation suggests that the modular differential equation is a consequence of a null-vector relation in the vacuum Verma module [91], see also [75]. The main result of this chapter is that this idea is indeed correct.

The recent interest in this problem arose from the analysis of Witten concerning pure gravity in AdS_3 [186]. He suggested that the corresponding boundary theories should be holomorphically factorising bosonic conformal field theories at $c = 24k$ with $k = 1, 2, \ldots$, where $k \to \infty$ describes the classical limit of the AdS_3 theory. Furthermore, the corresponding chiral theories should be extremal, meaning that up to level $k + 1$ above the vacuum, the theory only consists of Virasoro descendants of the vacuum state. For $k = 1$, the resulting conformal field theory is the famous Monster theory [83, 20], but for $k \geq 2$ an explicit realisation of these theories is so far not known. The above constraints, however, specify the character of these meromorphic conformal field theories uniquely [122, 186].

It was proposed in [91] that the analysis of their modular differential equations could answer the question whether such theories exist for $k \geq 2$. Since these theories are self-dual, they only have a single highest weight representation, and thus only a single character. One can then obtain an estimate for the order s of the differential equation that annihilates this character; this is proportional, for large k, to $s \sim \sqrt{k}$. On the other hand, if there is a direct relation between modular differential equations and null-vectors in the vacuum Verma module, a modular differential equation at order s should imply that the vacuum Verma module has a null-vector at conformal weight $2s$. This would then lead to a contradiction for $k \geq 42$ since the extremal theories do not have any null-vectors at such low levels [91].

In [91] a specific conjecture was made (and supported by some evidence). It was suggested that if a conformal field theory satisfies an order s modular differential equation, then $L_{-2}^{s}\Omega \in O_{[2]}$. (In particular, this conjecture implies the weaker statement that the vacuum Verma module has a null-vector at level $2s$.) The original form of the conjecture has turned out to be incorrect: the example of Gaiotto [99] involving tensor products of the Monster theory demonstrates this fact. This example is however not in conflict with the weaker statement that the vacuum Verma module possesses a null-vector at level $2s$ — albeit one that is of a somewhat different form. In fact, the tensor products of the Monster theory have many null-vectors at levels that are even below the one suggested by the order of the modular differential equation!

In this chapter we show that every modular differential equation comes from a null-vector in the vacuum Verma module (see (5.3.13)). We shall explain under which conditions this leads to a relation of the form $L_{-2}^{s}\Omega \in O_{[2]}$, thus giving in particular a null-vector at level $2s$. We shall also explain in detail how the counterexample of Gaiotto avoids this conclusion; as we shall see, this is intimately related to the fact that the tensor product of two (or more) Monster theories has many other null-vectors. We also comment on the fact that the existence of these additional null-vectors can be seen from an analysis of the Monster theory character; the same is true for Witten's theory at $k = 2$, but, at least from the point of view of the character, there are no indications that the theories with $k \geq 3$ should have sufficiently many null vectors to avoid a contradiction along these lines.

5.2 The modular differential equation

Let us begin by explaining the structure of torus amplitudes in a rational conformal field theory. It is usually believed (and it follows in fact from the analysis of Zhu [187]) that the torus amplitudes can be described in terms of the characters of the highest weight representations of the conformal field theory. These characters satisfy a modular differential equation [151, 152] (for earlier work see [62, 11]). In this section we want to explain the origin of this differential equation from the point of view of Zhu [187].

Let V be a meromorphic conformal field theory (or vertex operator algebra). For each state $a \in V$ we have a vertex operator $V(a, z)$, whose modes we denote by a_n (using the usual physicists' conventions). The zero mode of a plays a special role, and we denote it by $o(a) \equiv a_0$. On the torus, it is more advantageous to use different coordinates; the associated modes are then denoted by $a_{[n]}$. All of this is explained in more detail in appendix C.1.

It follows from an elementary (but somewhat tedious) calculation due to Zhu (Proposition 4.3.5 of [187] — we sketch an outline of the argument in appendix C.2) that

$$\mathrm{Tr}_{\mathcal{H}}\Big(o(a_{[-h_a]}b)\, q^{L_0}\Big) = \mathrm{Tr}_{\mathcal{H}}\Big(o(a)\, o(b)\, q^{L_0}\Big) + \sum_{k=1}^{\infty} G_{2k}(q)\mathrm{Tr}_{\mathcal{H}}\Big(o(a_{[2k-h_a]}b)\, q^{L_0}\Big) \,. \qquad (5.2.1)$$

Here the trace is taken in any highest weight representation $\mathcal{H}$ of the chiral algebra, and $G_n(q)$ denotes the n^{th} Eisenstein series; our conventions for the Eisenstein series (as well as their main properties) are summarised in appendix C.3. Next we apply (5.2.1) with a replaced by $L_{[-1]}a$, and use that $(L_{[-1]}a)_{[n]} = -(h_a + n)a_{[n]}$ (as follows from (C.1.4) upon taking a derivative), as well as $o(L_{[-1]}a) = (2\pi i)\, o(L_{-1}a + L_0 a) = 0$, which is a consequence of (C.1.7); this leads to (see Proposition 4.3.6 of [187])

$$\mathrm{Tr}_{\mathcal{H}}\Big(o(a_{[-h_a-1]}b)\, q^{L_0}\Big) + \sum_{k\geq 1}(2k-1)G_{2k}(q)\, \mathrm{Tr}_{\mathcal{H}}\Big(o(a_{[2k-h_a-1]}b)\, q^{L_0}\Big) = 0 \ . \tag{5.2.2}$$

The term with $k = 1$ does not contribute here since the trace of $o(a_{[-h_a+1]}b)$ vanishes — as follows from (C.2.11) in the appendix, it is a commutator and hence vanishes in the trace.

Equation (5.2.2) motivates now the following definition. Let $V[G_4(q), G_6(q)]$ be the space of polynomials in the Eisenstein series with coefficients in V. This is a module over the ring $R = \mathbb{C}[G_4(q), G_6(q)]$ which carries a natural grading given by the modular weight of each monomial; since G_4 and G_6 generate all modular forms, we have in particular that $G_{2k}(q) \in R$ for $k \geq 2$. Then we define $O_q(V)$ to be the submodule of $V[G_4(q), G_6(q)]$ generated by states of the form

$$O_q(V): \qquad a_{[-h_a-1]}b + \sum_{k\geq 2}(2k-1)\, G_{2k}(q)\, a_{[2k-h_a-1]}\, b \ , \tag{5.2.3}$$

where $a, b \in V$. Here the sum is finite, as $a_{[n]}$ annihilates b for sufficiently large n. By (5.2.2), it is now clear that

$$\mathrm{Tr}_{\mathcal{H}}\Big(o(v)\, q^{L_0}\Big) = 0 \qquad \text{if } v \in O_q(V). \tag{5.2.4}$$

This is true for every character, *i.e.* independent of the highest weight representation $\mathcal{H}$ that is being considered. For later convenience we also note that

$$a_{[-h_a-n]}b - (-1)^n \sum_{2k\geq n+1} \binom{2k-1}{n} G_{2k}(q)\, a_{[2k-h_a-n]}b \in O_q(V) \ , \quad \forall n \geq 1 \ , \tag{5.2.5}$$

as can be seen by evaluating the above identity repeatedly with a being replaced by $L_{[-1]}a$.

Suppose now that for a conformal field theory we can find an integer s and modular forms $g_r(q)$ of weight $2(s-r)$ such that [1]

$$\big(L_{[-2]}\big)^s\Omega + \sum_{r=0}^{s-2} g_r(q)\, \big(L_{[-2]}\big)^r\Omega \in O_q(V) \ . \tag{5.2.6}$$

---

[1] As is explained in [187], the existence of such a vector follows for example from the C_2 condition that is believed to hold for every rational conformal field theory — see also appendix A.2.

We then claim that all the characters $\chi_{\mathcal{H}}(q) := \mathrm{Tr}_{\mathcal{H}}\big(q^{L_0-\frac{c}{24}}\big)$ of the conformal field theory satisfy a common modular covariant differential equation, *i.e.* an equation of the form

$$\left[D^s + \sum_{r=0}^{s-2} f_r(q) D^r\right] \chi_M(q) = 0 \ . \tag{5.2.7}$$

Here D^s is the order s differential operator (see appendix C.3)

$$D^s = D_{2s-2}\, D_{2s-4} \cdots D_2\, D_0 \ , \qquad \text{with} \qquad D_r = q\frac{d}{dq} - \frac{r}{4\pi^2} G_2(q) = q\frac{d}{dq} - \frac{r}{12} E_2(q) \ , \tag{5.2.8}$$

and $f_r(q)$ is a modular form of weight $2(s-r)$.[2]

To show this, note that because of the defining property of $O_q(V)$ (5.2.4), we know that the character of the zero mode of the left hand side of (5.2.6) vanishes. On the other hand, each term in this expression can be turned into a differential operator

$$\mathrm{Tr}_{\mathcal{H}}\Big(o\big((L_{[-2]})^r\Omega\big)\, q^{L_0-\frac{c}{24}}\Big) = P_r(D)\mathrm{Tr}_{\mathcal{H}}\Big(q^{L_0-\frac{c}{24}}\Big) \ , \tag{5.2.9}$$

where $P_r(D)$ is a modular covariant differential operator of order r with modular weight $2r$. To see (5.2.9) we note that for $r=1$ we obtain directly

$$\mathrm{Tr}_{\mathcal{H}}\Big(o(L_{[-2]}\Omega)\, q^{L_0-\frac{c}{24}}\Big) = (2\pi i)^2 \mathrm{Tr}_{\mathcal{H}}\Big(\big(L_0 - \frac{c}{24}\big)\, q^{L_0-\frac{c}{24}}\Big) = (2\pi i)^2 \Big(q\frac{d}{dq}\Big)\mathrm{Tr}_{\mathcal{H}}\Big(q^{L_0-\frac{c}{24}}\Big) \ , \tag{5.2.10}$$

which is modular covariant since the character has modular weight 0. The case of general r follows by applying (5.2.1) (which clearly still works if we replace q^{L_0} by $q^{L_0-c/24}$)

$$\begin{aligned}
\mathrm{Tr}_{\mathcal{H}}\Big(o(L_{[-2]}(L_{[-2]})^r\Omega)\, q^{L_0-\frac{c}{24}}\Big) \ &= \ (2\pi i)^2 q\frac{d}{dq}\mathrm{Tr}_{\mathcal{H}}\Big(o((L_{[-2]})^r\Omega)\, q^{L_0-\frac{c}{24}}\Big) \\
&\quad + 2rG_2(q)\, \mathrm{Tr}_{\mathcal{H}}\Big(o((L_{[-2]})^r\Omega)\, q^{L_0-\frac{c}{24}}\Big) \\
&\quad + \sum_{k\geq 2} G_{2k}(q)\, \mathrm{Tr}_{\mathcal{H}}\Big(o(L_{[2k-2]}(L_{[-2]})^r\Omega)\, q^{L_0-\frac{c}{24}}\Big) \ .
\end{aligned} \tag{5.2.11}$$

In the last line we commute the positive $L_{[2k-2]}$ modes to the right, using the Virasoro commutation relations. The final result is a vector of the form $(L_{[-2]})^{r+1-k}\Omega$, which leads to a differential operator of lower order, multiplied by the modular form of appropriate weight. The first two terms, on the other hand, just produce the covariant derivative D_{2r} for a form of weight $2r$. Collecting all terms, we get the desired operator $P_r(D)$. Note that the leading term of $P_r(D)$ is proportional to D^r; for the first few values of r, the explicit formula for $P_r(D)$ is given in appendix B.1. This completes the derivation of the modular differential equation.

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[2] We shall use two different conventions for the Eisenstein series, namely $G_n(q)$ and $E_n(q)$, in this chapter; the two functions only differ by an overall normalisation constant, see appendix C.

5.2.1 A simple example

Let us illustrate this construction with a simple example, the Yang-Lee minimal model at $c = -\frac{22}{5}$. This is the 'simplest' minimal model since it only has two highest weight representations, the vacuum representation at $h = 0$ as well as the representation at $h = -\frac{1}{5}$. The vacuum representation has a null-vector at level 4,

$$\mathcal{N} = \left(L_{[-4]} - \frac{5}{3} L_{[-2]}^2 \right) \Omega \ . \tag{5.2.12}$$

We want to use $\mathcal{N}$ to obtain an expression of the form (5.2.6). To this end we observe that $L_{[-4]}\Omega$ is already in $O_q(V)$ since (5.2.5) implies that

$$O_q(V) \ni L_{[-4]}\Omega - \sum_{k\geq 2} \binom{2k-1}{2} G_{2k}(q)\, L_{[2k-4]}\Omega = L_{[-4]}\Omega \ . \tag{5.2.13}$$

Since $\mathcal{N}$ is a null-vector, the sought-after relation is then simply

$$L_{[-2]}^2 \Omega \in O_q(V) \ . \tag{5.2.14}$$

Using the explicit expression for (5.2.9) derived in appendix C.2.1, we obtain the differential equation

$$0 = \mathrm{Tr}_{\mathcal{H}}\Big(o(L_{[-2]}L_{[-2]}\Omega)\, q^{L_0 - \frac{c}{24}} \Big) = (2\pi i)^4 \left[D^2 - \frac{11}{3600}\, E_4(q) \right] \chi_{\mathcal{H}}(q) \ . \tag{5.2.15}$$

The two characters of the Yang-Lee model are explicitly given as (see for example [48])

$$\chi_0(q) = \frac{1}{\eta(q)} \sum_{n\in\mathbb{Z}} \left(q^{\frac{(20n-3)^2}{40}} - q^{\frac{(20n+7)^2}{40}} \right) \tag{5.2.16}$$

$$\chi_{-1/5}(q) = \frac{1}{\eta(q)} \sum_{n\in\mathbb{Z}} \left(q^{\frac{(20n-1)^2}{40}} - q^{\frac{(20n+9)^2}{40}} \right) \ , \tag{5.2.17}$$

where $\eta(q)$ is the usual Dedekind eta function

$$\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \ . \tag{5.2.18}$$

One easily checks (using for example Mathematica) that the two characters are indeed the two solutions of this second order differential equation. We have also performed the analogeous analysis for the Ising model.

5.2.2 Relation to the null-vector

In the above example, the vector of the form (5.2.6) in $O_q(V)$ was a direct consequence of a null-vector relation in the vacuum representation, see (5.2.12). This is actually generally true: a vector of the form (5.2.6) in $O_q(V)$ can only exist if the vacuum representation has a null-vector at level $2s$. To see this we recall that $V[G_4(q), G_6(q)]$ carries two grades: the *conformal weight* of the vectors in V (with respect to $L_{[0]}$), and the *modular weight* of the coefficient functions (that are polynomials in G_4 and G_6). Furthermore, the relations that define $O_q(V)$ are homogeneous with respect to the grade that is the sum of these two grades, as is manifest from (5.2.3).

Since the relation (5.2.6) is a relation in $V[G_4(q), G_6(q)]$ it must hold separately for every conformal weight and every modular weight. If we consider the component at conformal weight $2s$ and modular weight zero, we therefore get a relation of the form

$$\big(L_{[-2]}\big)^s\Omega + \sum_j a^j_{[-h(a^j)-1]}\, b^j = 0 \ , \tag{5.2.19}$$

where $h(a^j)$ is the conformal weight (with respect to $L_{[0]}$) of a^j. This is necessarily a non-trivial relation in the Verma module since $L_{[-2]}$ is not of the form $a_{[-h_a-1]}$ for any a. Such a non-trivial relation is usually called a null-vector. We mention in passing that it implies that $\big(L_{[-2]}\big)^s\Omega$ vanishes in the C_2 quotient space of Zhu (that is briefly discussed in appendix A.2), as was already mentioned in [91].

5.3 Reconstructing the null-vector

As we have seen above, a vector of the form (5.2.6) in $O_q(V)$ implies that the characters of the theory satisfy a common order s modular differential equation. We have also shown that such a relation in $O_q(V)$ can only exist if the vacuum representation has a null-vector at conformal weight $2s$, see (5.2.19).

We would now like to show a partial converse to these statements, namely that every modular differential equation implies that the vacuum Verma module has a null-vector. We shall assume that Zhu's algebra is semisimple, as is known to be the case for rational conformal field theories (in the mathematical sense) [187]. In particular, this is the case for the self-dual theories, for which Zhu's algebra is one-dimensional, consisting only of the identity.

5.3.1 The underlying vector

Suppose now that we have a modular covariant differential equation of the form (5.2.7) that annihilates all characters of the conformal field theory. Using the arguments of section 2 in reverse order, it is easy to see that there is then a vector $K(q)$ of the form

$$K(q) \equiv \big(L_{[-2]}\big)^s\Omega + \sum_{r=0}^{s-2} g_r(q)\, \big(L_{[-2]}\big)^r\Omega \tag{5.3.1}$$

that has the property that

$$\mathrm{Tr}_{\mathcal{H}}\Big(o(K(q))\, q^{L_0-\frac{c}{24}}\Big) = 0 \tag{5.3.2}$$

for all characters of the conformal field theory. Let us consider the limit

$$\lim_{q\to 0} q^{\frac{c}{24}-h}\mathrm{Tr}_{\mathcal{H}}\Big(o(K(q))\, q^{L_0-\frac{c}{24}}\Big) = 0 \ , \tag{5.3.3}$$

where h is the conformal weight of the highest weight state in $\mathcal{H}$. In this limit only the highest weight states $\mathcal{H}^0$ in $\mathcal{H}$ contribute, and we conclude that

$$\mathrm{Tr}_{\mathcal{H}^0}\Big(o(K(0))\Big) = 0 \ . \tag{5.3.4}$$

5.3.2 Using Zhu's Theorem

The above argument has shown that $K(0)$ acts trivially in the trace of an arbitrary highest weight representation. The action of the elements of V on highest weight states is captured by Zhu's algebra (for a brief introduction see appendix C.1.1). If Zhu's algebra is semisimple (as we shall assume) then the fact that $K(0)$ is trivial in all traces implies that $K(0)$ must equal a commutator in Zhu's algebra. This follows for example from a standard theorem of associative algebras, the Wedderburn structure theorem [71]. It states that every semisimple associative algebra is isomorphic to the product of algebras of $n \times n$ matrices over $\mathbb{C}$,

$$A(V) \cong \prod_{i=1}^{N} \mathcal{M}_{n_i}(\mathbb{C}) \ , \tag{5.3.5}$$

where n_i is the dimension of the i^{th} irreducible representation M_i of $A(V)$. Assume we are given $a \in A(V)$ such that $\mathrm{Tr}_{M_i}(a) = 0$ for all irreducible representations M_i. By (5.3.5), a is isomorphic to a blockdiagonal matrix whose blocks all have vanishing trace. It is then a straightforward exercise to show that each such matrix can be written as a sum of commutators, *i.e.* that up to elements in $O_{[1,1]}$

$$2\pi i \cdot a = 2\pi i \sum_l (d^l * e^l - e^l * d^l) = \sum_l d^l_{[-h(d^l)+1]}\, e^l \ . \tag{5.3.6}$$

In the last equation we have used the identity (C.1.12).

This argument thus implies that up to commutator terms (5.3.6), $K(0)$ lies in $O_{[1,1]}$, the subspace by which we quotient to obtain Zhu's algebra $A(V)$. On the other hand, $O_{[1,1]}$ is closely related to $O_q(V)$: for any state in $O_q(V)$,

$$a_{[-h_a-1]}b + \sum_{k\geq 2}(2k-1)G_{2k}(q)a_{[2k-h_a-1]}b \ , \tag{5.3.7}$$

we can formally take the limit $q \longrightarrow 0$, *i.e.* we can consider its constant part only. Then we obtain (see [187], Lemma 5.3.2)

$$\frac{\pi i}{6}a_{[-h_a+1]}b + 2\pi i \oint dz \left(V(a,z)\frac{(1+z)^{h_a}}{z^2}b \right) \ , \tag{5.3.8}$$

i.e. up to a commutator term, the limit is in $O_{[1,1]}$. In fact, it is obvious that every element in $O_{[1,1]}$ can be obtained in this manner.

Taking these two statements together they now imply that $K(0)$ equals the sum of commutator terms (5.3.6) and the $q \to 0$ limit of elements in $O_q(V)$,

$$K(0) - \sum_l d^l_{[-h(d^l)+1]}\, e^l - \sum_j H^j(0) = 0 \ , \tag{5.3.9}$$

where

$$H^j(q) = a^j_{[-h(a^j)-1]}b^j + \sum_{k\geq 2}(2k-1)G_{2k}(q)\, a^j_{[2k-h(a^j)-1]}b^j \in O_q(V)\ . \tag{5.3.10}$$

This relation now holds in the full vacuum representation V. Next we define the element $N(q) \in V[G_4, G_6]$ by

$$N(q) \equiv K(q) - \sum_l d^l_{[-h(d^l)+1]}\, e^l - \sum_j H^j(q)\ . \tag{5.3.11}$$

By construction, $N(0) = 0$, and hence $N(q)$ is proportional to q. We can then divide by q, and repeat the above argument. Recursively this allows us to prove that

$$K(q) - \sum_l f_l(q)\, d^l_{[-h(d^l)+1]}\, e^l - \sum_j h_j(q)\, H^j(q) = 0 \tag{5.3.12}$$

in $V[q]$, where $f_l(q)$ and $h_j(q)$ are power series in q. (Here $V[q]$ consists of vectors in V with coefficients that are formal power series in q.) If we assume that the theory is C_2-finite we can furthermore show that only finitely many terms appear and that the power series have a non-trivial radius of convergence; however, this will not be important in the following.

Putting everything together, we can now use (5.3.12) as well as (5.3.1) and (5.3.10) to arrive at the identity

$$\begin{aligned}&(L_{[-2]})^s\Omega + \sum_{i=0}^{s-1} g_i(q)(L_{[-2]})^i\Omega \qquad\qquad (5.3.13)\\ &\quad = \sum_l f_l(q) d^l_{[-h(d^l)+1]}e^l + \sum_j h_j(q)\Big(a^j_{[-h(a^j)-1]}b^j + \sum_{k\geq 2}(2k-1)G_{2k}(q)\, a^j_{[2k-h(a^j)-1]}b^j\Big)\ .\end{aligned}$$

This defines the sought after 'null-vector' relation in the vacuum Verma module. Obviously, the full expression is not homogeneous with respect to conformal weight, and therefore each component (*i.e.* the terms of each fixed conformal weight) must vanish separately (and indeed for any value of q). Some of these relations may be trivial in the Verma module, but not all of them can if the original modular differential equation from which we started was non-trivial.

5.3.3 Consequences

We have thus shown that every modular differential equation comes from a null-vector in the vacuum Verma module. We would now like to obtain more detailed information from (5.3.13). For the application to the extremal self-dual conformal field theories, it is for instance also important to determine the conformal weights of the constituent null-vectors. In particular, one may expect that the term of highest conformal weight on the left-hand-side — this is the vector $(L_{-2})^s\Omega$ — should be part of a non-trivial null-vector relation.

In order to motivate this proposal we observe that the coefficients of the vectors of the left-hand-side of (5.3.13) are all analytic functions in q on the unit disc, $|q| < 1$. Therefore the same has to be true for the coefficients on the right-hand-side. Generically, one should then expect that the functions $f_l(q)$ and $h_j(q)$ will also be analytic functions on $|q| < 1$; as we shall discuss later on, there are however situations where this is not the case.

Now we recall that $V[G_4(q), G_6(q)]$ has two gradings, namely the ones given by conformal weight and modular weight. By construction $(L_{[-2]})^s\Omega$ has modular weight 0 and conformal weight $2s$. If $f_l(q)$ and $h_j(q)$ are indeed analytic, then the only terms of modular weight 0 on the right hand side of (5.3.13) have constant coefficients. Moreover, comparing the conformal weights, only terms of $L_{[0]}$-weight $2s$ can contribute. Thus we can conclude that we have an identity of the form

$$(L_{[-2]})^s\Omega = \sum_j{}' a^j_{[-h(a^j)-1]}b^j + \sum_l{}' d^l_{[-h(d^l)+1]}e^l \ , \tag{5.3.14}$$

where the prime over the sum indicates that we only include states of $L_{[0]}$-weight $2s$, *i.e.* terms with $h(a^j) + h(b^j) + 1 = 2s$ and $h(d^l) + h(e^l) - 1 = 2s$. Because of the 'commutator terms', *i.e.* the first sum in (5.3.14), this identity does not quite imply that $L^s_{[-2]}\Omega \in O_{[2]}$. However, for the case of the extremal self-dual theories at $c = 24k$ we can show (see section 4 below) that this is so, and hence that (5.3.14) defines indeed a non-trivial null-vector relation.

In the above argument we have used that there are no holomorphic functions of negative modular weight; in particular, this implied that $h_j(q)G_{2k}(q)$ had modular weight greater or equal to $2k$, and hence could not contribute to the identity (5.3.14). However, as soon as we allow h_j to be meromorphic, we can no longer guarantee this. For example, we can then construct other contributions to (5.3.14) from terms with $k \neq 0$ by choosing $h_j(q) = G_{2k}(q)^{-1}$. We will now discuss an example of such a situation.

5.3.4 A counterexample

It was observed in [99] that for the tensor product of two (or more) Monster theories, there exist modular differential equations that do not come from relations of the type (5.3.14). As we shall explain in the following, this 'counterexample' to (5.3.14) can be traced back to the failure of h_j to be holomorphic. We shall also see that this is only compatible with the holomorphicity of (5.3.13) because the Monster theory (and indeed the tensor products of the Monster theory) has many other null-vectors at low levels. These null-vectors are necessary to guarantee that the apparent non-holomorphic terms on the right-hand-side of (5.3.13) in fact vanish in the vacuum representation. Thus it seems that (5.3.14) can only be avoided if the theory has other non-trivial null-vectors at low levels.

The Monster theory

To set up the notation we first recall a few facts about the case of a single Monster theory; for an introduction to these matters see for example [100]. The Monster theory has no fields of conformal dimension one, and 196884 fields of conformal dimension 2. The latter consist of the stress-energy tensor whose modes L_n satisfy a Virasoro algebra at central charge $c = 24$,

$$[L_m, L_n] = (m-n)L_{m+n} + 2m(m^2-1)\delta_{m,-n} \ . \tag{5.3.15}$$

The remaining 196883 fields W^i transform in an irreducible representation of the Monster group and satisfy the commutation relations

$$\begin{aligned} [L_m, W^i_n] &= (m-n)W^i_{m+n} \\ [W^i_m, W^j_n] &= \frac{1}{6}\delta^{ij} m(m^2-1)\delta_{m,-n} + \frac{1}{12}\delta^{ij}(m-n)L_{m+n} \\ &\quad + h^{ij}_k (m-n) W^k_{m+n} + f^{ij}_\alpha V^\alpha_{m+n} \ , \end{aligned} \tag{5.3.16}$$

where V^α_l are the modes of the primary fields at conformal weight three that transform in the 21296876-dimensional irreducible representation of the Monster group. The coefficients h^{ij}_k are totally symmetric in all three indices, and define the structure constants of the so-called Griess algebra. In our conventions, the metric on the space of the W^i fields is orthonormal, so we can raise and lower the i, j, k indices freely.

The Monster conformal field theory has many non-trivial relations; the first non-trivial relation already occurs at level four since we have the identity (see for example [153])[3]

$$\mathcal{N}_4 = L^2_{-2}\Omega + \frac{36}{11}L_{-4}\Omega - \frac{12}{30503}\sum_i W^i_{-2}W^i_{-2}\Omega = 0 \ . \tag{5.3.17}$$

This null-relation does, however, not directly lead to a differential equation since it is not of the form (5.3.14). As was already explained in [91], the character of the Monster theory $\chi_M(q)$ satisfies only a third order differential equation

$$\left[D^3 + \frac{16}{31}E_6(q) - \frac{290}{279}E_4(q)\,D\right]\chi_M(q) = 0 \ . \tag{5.3.18}$$

This differential equation can be obtained from the null-vector at level six (see again [153])

$$\mathcal{N}_6 = L^3_{-2}\Omega + \frac{41}{8}L^2_{-3}\Omega + \frac{15623}{1488}L_{-4}L_{-2}\Omega + \frac{873}{31}L_{-6}\Omega - \frac{1}{124}\sum_i W^i_{-4}W^i_{-2}\Omega = 0 \ . \tag{5.3.19}$$

In fact, it is easy to see that evaluating the trace of $V_0(\mathcal{N}_6)$ as in section 2 (where in the definition of $\mathcal{N}_6$ we replace the L_{-n} modes by $L_{[-n]}$ modes) leads to the above modular differential equation. (In order to do this calculation, one also needs to use the commutation relations of the W^i-modes.)

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[3] This follows from the equation after (2.9) in [153] upon rewriting his modes x^i with $i = 1, \ldots, 196884$ in terms of the W^i and L.

There is an independent null-vector at level eight, which is of the form[4]

$$\begin{aligned}\mathcal{N}_8 \;=\;& h_{ijk}W^i_{-4}W^j_{-2}W^k_{-2}\Omega - H_0\Big[\frac{503352}{8072203}\,L_{-8}\Omega + \frac{81048}{8072203}\,L_{-6}L_{-2}\Omega \\ & +\frac{34565}{8072203}\,L_{-5}L_{-3}\Omega + \frac{26403}{16144406}\,L_{-4}L_{-4}\Omega + \frac{110221}{96866436}\,L_{-4}L_{-2}L_{-2}\Omega \\ & +\frac{3193}{16144406}\,L_{-3}L_{-3}L_{-2}\Omega + \frac{5210}{24216609}\,L_{-2}L_{-2}L_{-2}L_{-2}\Omega\Big]\;, \qquad (5.3.20)\end{aligned}$$

where $H_0 = h_{ijk}h^{ijk}$, which equals in our conventions $H_0 = 196883\,\frac{6929}{6} = \frac{1364202307}{6}$. By the same token as above (and with somewhat more effort — in particular, we now also have to use the null-vector $\mathcal{N}_4$ in order to express the term $h_{ijk}W^i_{[0]}W^j_{[-2]}W^k_{[-2]}\Omega$ that appears in the course of this calculation in terms of Virasoro generators) it leads to the fourth order modular differential equation

$$\Big[D^4 - \frac{73421}{93780}\,E_4(q)\,D^2 + \frac{527029}{562680}\,E_6(q)\,D - \frac{1259}{2605}E_4^2(q)\Big]\,\chi_M(q) = 0\;. \qquad (5.3.21)$$

This differential equation is actually linearly independent from the other fourth order modular differential equation of the Monster theory, namely the one coming from the null-vector $L_{-2}\mathcal{N}_6$. The latter differential equation equals

$$\Big[D^4 - \frac{290}{279}\,E_4(q)\,D^2 + \frac{722}{837}\,E_6(q)\,D - \frac{8}{31}E_4^2(q)\Big]\,\chi_M(q) = 0\;, \qquad (5.3.22)$$

which is in fact simply equal to the D-derivative of (5.3.18). Taking the difference of (5.3.21) and (5.3.22) the Monster theory therefore also satisfies a modular differential equation of order two,

$$\Big[E_4(q)D^2 + \frac{71}{246}\,E_6(q)\,D - \frac{36}{41}E_4^2(q)\Big]\,\chi_M(q) = 0\;. \qquad (5.3.23)$$

[Another way of saying this, is that this is the modular differential equation that comes from the nullvector

$$\mathcal{M}_8 = (2\pi i)^{-8}\frac{16151}{2982996}\Big(\frac{24216609}{5210\,H_0}\mathcal{N}_8 - L_{-2}\,\mathcal{N}_6\Big)\;.\Big] \qquad (5.3.24)$$

Note that the existence of this second order modular differential equation is not in conflict with what was said above (or in [91]), since (5.3.23) is not holomorphic in the above sense: if we divide by $E_4(q)$ to obtain a differential equation whose leading term is D^2, the coefficient of the term proportional to D is not holomorphic but only meromorphic. If we allow for meromorphic coefficients, every self-dual conformal field theory obviously also satisfies a first order modular differential equation (see also [15]).

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[4] Such a null-vector must exist since, up to level 10, all states that are Monster invariant can be expressed in terms of Virasoro descendants of the vacuum. The coefficients can then be fixed by evaluating the inner products with all Virasoro descendants.

Tensor products of Monster theories

Now let us turn to the case of the tensor product of two Monster theories. (As we shall see momentarily, the answer for the tensor product of an arbitrary number of Monster theories can be understood once we have done so for the two-fold tensor product.) It is not difficult to show that if (5.3.14) was true, an order s modular differential equation for the tensor product of the two Monster theories would imply that

$$\left(L^{(1)}_{-2} + L^{(2)}_{-2}\right)^s \Omega \in O_{[2]} \ . \tag{5.3.25}$$

Given the arguments of [91, 99] it is easy to see that (5.3.25) can only hold for $s \geq 5$. On the other hand, one finds that the tensor product of two Monster theories actually satisfies a fourth order differential equation [99], namely

$$\left[D^4 - \frac{175117}{45756} E_4(q)\, D^2 + \frac{47539165}{11255976}\, E_6(q)\, D - \frac{12838}{52111}\, E_4^2(q)\right] \chi_M^2(q) = 0 \ . \tag{5.3.26}$$

We now want to explain how to obtain this differential equation from a null vector in the vacuum Verma module. First we observe that the leading term D^4 in (5.3.26) comes from the vector

$$\begin{aligned} \left(L^{(1)}_{[-2]} + L^{(2)}_{[-2]}\right)^4 \Omega \ &= \ \Big[\left(L^{(1)}_{[-2]}\right)^4 + 4\left(L^{(1)}_{[-2]}\right)^3 L^{(2)}_{[-2]} \\ &\qquad + 6\left(L^{(1)}_{[-2]}\right)^2 \left(L^{(2)}_{[-2]}\right)^2 + 4 L^{(1)}_{[-2]} \left(L^{(2)}_{[-2]}\right)^3 + \left(L^{(2)}_{[-2]}\right)^4\Big]\Omega \ . \end{aligned} \tag{5.3.27}$$

In the following we want to show how this vector can be expressed, up to terms of lower conformal weight, in terms of elements in $O_q(V)$. The terms in $O_q(V)$ vanish inside any trace, and the terms of lower conformal weight can be expressed in terms of Virasoro generators, and hence give rise to the lower coefficients in (5.3.26).[5]

The various terms in (5.3.27) can now be rewritten as follows. First of all, we observe that every element in $O_q(V)$ is of the form

$$O_q(V): \quad v + \sum_{n\geq 2} G_n(q) v_n \ , \qquad \text{where} \quad v \in O_{[2]} \ , \tag{5.3.28}$$

and that for any $v \in O_{[2]}$, there is such an element in $O_q(V)$. We call v the 'head', and the remaining terms the 'tail'. Note that the conformal weights of the terms in the tail are always strictly smaller than that of v.

Now we can use the null vector $\mathcal{N}_8$ (or $L_{[-2]}\mathcal{N}_6$) to express $(L^{(i)}_{[-2]})^4\Omega$, where $i = 1, 2$, in terms of a vector in $O_{[2]}$. This can be taken to form the head of an element in $O_q(V)$, and hence we can rewrite $(L^{(i)}_{[-2]})^4\Omega$, up to elements of lower conformal weight that come from the tail, as an element of $O_q(V)$. Similarly, we can reduce $(L^{(1)}_{[-2]})^3 L^{(2)}_{[-2]}\Omega$ by using

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[5] Strictly speaking we also have to guarantee that the resulting terms of lower conformal weight can be expressed in terms of powers of $(L^{(1)}_{[-2]} + L^{(2)}_{[-2]})$, but this can indeed be arranged — this is again a consequence of the fact that there are two independent null-vectors at level eight.

the null-vector $\mathcal{N}_6^{(1)} \otimes L^{(2)}_{[-2]}\Omega$, and likewise for the term $L^{(1)}_{[-2]}(L^{(2)}_{[-2]})^3\Omega$. The only difficult term is $(L^{(1)}_{[-2]})^2\,(L^{(2)}_{[-2]})^2\Omega$ for which this is not possible — in fact, this is the reason why (5.3.25) with $s = 4$ does not hold. We now want to explain how this can be circumvented by making use of the null vector $\mathcal{M}_8$.

As we have seen above, the single Monster theory has a null-vector at level 8, $\mathcal{M}_8$, that lies entirely inside $O_{[2]}$, $\mathcal{M}_8 \in O_{[2]}$. Let us denote by $O_{\mathcal{M}_8}$ its tail, so that $\mathcal{M}_8 + O_{\mathcal{M}_8} \cong O_{\mathcal{M}_8} \in O_q(V)$; this is explicitly given (up to an overall normalisation) as

$$O_{\mathcal{M}_8} = \Big[G_4(q)L^2_{[-2]} - \frac{497}{41}G_6(q)L_{[-2]} - \frac{26412}{41}G_4(q)^2\Big]\Omega\ , \qquad (5.3.29)$$

where we have made use of the null vector $\mathcal{N}_4$ at level four to rewrite the term $W^i_{[-2]}W^i_{[-2]}\Omega$ that appeared in the course of this calculation in terms of $L^2_{[-2]}\Omega$.

The same argument also applies to the null vector $\mathcal{M}_{10} := L_{[-2]}\mathcal{M}_8$. Up to an overall constant, its tail is

$$O_{\mathcal{M}_{10}} = \Big(G_4(q)L^3_{[-2]} + \lambda_1 G_6(q)L^2_{[-2]} + \lambda_2 G_4^2(q)L_{[-2]} + \lambda_3 G_4(q)G_6(q)\Big)\Omega\ , \qquad (5.3.30)$$

where

$$\lambda_1 = \frac{2334255}{1158254}\ , \qquad \lambda_2 = -\frac{451255338}{579127}\ , \qquad \lambda_3 = -\frac{10493019690}{579127}\ , \qquad (5.3.31)$$

and we have used the null-vector relation $\widehat{\mathcal{N}}_6 = 0$ with

$$\begin{aligned}\widehat{\mathcal{N}}_6 \;=\; & h_{ijk}W^i_{-2}W^j_{-2}W^k_{-2}\Omega - H_0\Big[\frac{20403}{196883}\,L_{-6}\Omega + \frac{5607}{393766}\,L_{-4}L_{-2}\Omega \qquad (5.3.32)\\ & + \frac{279}{196883}\,L_{-3}L_{-3}\Omega + \frac{8837}{2362596}\,L_{-2}L_{-2}L_{-2}\Omega\Big]\ .\end{aligned}$$

Now we can combine these null-vectors to write

$$\begin{aligned}&\Big[\Big(L^{(1)}_{[-2]}\Big)^2\Big(L^{(2)}_{[-2]}\Big)^2 + \frac{497}{41\,\lambda_1}\Big(L^{(1)}_{[-2]}\Big)^3\Big(L^{(2)}_{[-2]}\Big)\Big]\Omega + \text{terms of lower conformal weight}\\ &\quad = \frac{1}{G_4(q)}\Big\{\Big(L^{(1)}_{[-2]}\Big)^2\Omega^{(1)}\otimes O^{(2)}_{\mathcal{M}_8} + \frac{497}{41\,\lambda_1}\,O^{(1)}_{\mathcal{M}_{10}}\otimes\Big(L^{(2)}_{[-2]}\Big)\Omega^{(2)}\Big\} \in O(q)\ . \qquad (5.3.33)\end{aligned}$$

Generically, such an identity will involve coefficients that are not holomorphic in q, since the terms in the bracket on the right-hand-side will not automatically be divisible by $G_4(q)$. However, for the specific linear combination that we have chosen — *i.e.* for the relative coefficient $\frac{497}{41\lambda_1}$ — the expression is actually holomorphic. To see this we observe that the the coefficients that appear in the bracket are proportional to Eisenstein series G_n with $n = 6, 8, 10$. Except for G_6, these Eisenstein series are automatically divisible by G_4. Thus we only need to guarantee that the coefficient of G_6 vanishes, and this is precisely achieved by the above linear combination.

It should be clear from this analysis that in order to avoid the coefficients f_l and h_j to be holomorphic, one needs a sufficiently large number of null-vectors to guarantee that all non-holomorphic terms in (5.3.13) are actually zero. (In the above case we had to use, for both theories, the null-vector at level four, the two null-vectors at level six, and the null vector at level eight.) For larger conformal weight the situation becomes even more constraining since then the tail will generically also involve Eisenstein series G_n with $n > 14$, none of which are divisible by G_4. Thus there will be even more coefficients that will need to be cancelled!

Finally, let us comment on the question of how this analysis generalises to higher tensor powers of the Monster theory. It is clear from the above analysis that for the k-fold tensor product we can always construct a modular differential equation of order $k+2$. To see this we expand out

$$\left(\sum_{i=1}^{k} L_{-2}^{(i)}\right)^{k+2} \Omega \; . \tag{5.3.34}$$

Then each term will either be proportional to $(L_{-2}^{(i)})^3\Omega$ for some i — such terms lie in $O_{[2]}$ by virtue of the null-vector $\mathcal{N}_6$ — or to terms of the form $(L_{-2}^{(i)})^2(L_{-2}^{(j)})^2\Omega$ which can be dealt with as explained above. Thus using the above methods we can construct a modular differential equation at order $k+2$. On the other hand, this seems to be the minimal order for which such a differential equation exists [99]. Thus there do not seem to be any additional cancellations beyond what is already visible for the case of the tensor product of two Monster theories. Finally, we should stress that the k-fold tensor product has a plethora of low-lying null-vectors: there are at least k linearly independent null-vectors at level 4, $2k$ at level 6, k additional ones at level 8, *etc*, that are relevant for this analysis.

5.4 Application to extremal self-dual CFTs

In this final section we want to comment on the implications of these considerations for the existence of the extremal self-dual conformal field theories at $c = 24k$ that were proposed by Witten [186]. As was shown in [91], these theories satisfy a modular differential equation of degree s where, for large k, $s \sim \sqrt{k}$.

As we have shown in section 3 above, every modular differential equation comes from a null-vector in the vacuum Verma module, see (5.3.13). Provided that f_l and h_j are holomorphic for $|q| < 1$, the null-vector relation (5.3.13) implies that (5.3.14) holds. We now want to show that (5.3.14) leads to a contradiction for $k \geq 42$. Thus the extremal conformal field theories can only be consistent for large k, provided that the assumption about the analyticity of f_l and h_j is not satisfied; we shall comment on this possibility further below.

Suppose then the the extremal conformal field theories have a 'null-vector-relation' of the form (5.3.14) at conformal weight $2s$. For $k \geq 42$ this relation is at $L_{[0]}$-weight $2s \leq k$, and thus arises at a weight where the proposed conformal field theory only possesses Virasoro descendants of the vacuum. This then leads to a contradiction: by the above argument, the right hand side can only contain Virasoro operators, which we may bring to the standard Poincaré-Birkhoff-Witt basis. We now claim that no term

$(L_{[-2]})^s\Omega$ can arise in the process. Consider first the terms $a_{[-h(a)-1]}b$. Since b can only be a Virasoro descendant of the vacuum, we can write it as a sum of terms

$$L_{[-n_1]} \cdots L_{[-n_N]}\Omega \ , \tag{5.4.1}$$

where all $n_l \geq 2$. Since the level of b is $h(b)$, we have necessarily that $N \leq \lfloor \frac{h(b)}{2} \rfloor$, where $\lfloor \cdot \rfloor$ denotes the truncated part. Similar statements also hold for a. We now have to the evaluate the $(-h(a)-1)$-th mode of a and apply it to b. The crucial point is that this mode contains at most as many $L_{[-n]}$ as a, see *e.g.* [96]. $a_{[-h(a)-1]}b$ thus has at most $\lfloor \frac{h(b)+h(a)}{2} \rfloor = \lfloor s - \frac{1}{2} \rfloor = s-1$ $L_{[-n]}$. Since going to the standard basis only decreases their number, it is clear that we cannot obtain $(L_{[-2]})^s\Omega$ from this term.

If we apply the same argument to $d_{[-h(d)+1]}e$, it seems that we could obtain s Virasoro operators. Note however that $d_{[-h(d)+1]}$ annihilates the vacuum and must therefore contain at least one $L_{[-n]}$ with $n \leq 1$. Bringing this operator to the right, commuting through the modes of e, we decrease the number of Virasoro operators at least by one, so that we are again left with at most $s-1$ Virasoro generators.

It therefore follows that the right hand side of (5.3.14) does not contain the term $(L_{[-2]})^s\Omega$. To satisfy the equality the theory must therefore have a non-trivial null-vector. At $c > 1$, however, we know that the pure Virasoro theory does not have any non-trivial null-vectors. This then leads to the desired contradiction.

5.4.1 A way out?

This leaves us with the possibility that (5.3.13) does not imply (5.3.14), *i.e.* that f_l and h_j are not holomorphic for $|q| < 1$. As we have seen in section 3.3, this can only be the case if the theory has many additional null-vectors (that guarantee that all coefficients of the meromorphic functions that would generically appear are actually zero). It is certainly conceivable that this can be achieved with only null-relations at $h > k$,[6] and we do not have any hard argument against this possibility. There is however a curious observation that seems to throw some doubt on this scenario.

As we have explained above, the extremal theory at $k = 1$, the Monster theory, has many low-lying null-vectors. This property is something one can actually read off from the character. To explain this, let us recall that the partition function of the Monster theory is

$$Z_M(q) = q^{-1} + 196884\,q + 21493760\,q^2 + 864299970\,q^3 + \cdots \ . \tag{5.4.2}$$

We can read off from this formula that there are $N_1 = 196884$ states at level two; these consist of the stress energy tensor L, as well as the fields W^i we have introduced before. Now consider the $N_1^2 = 38\,763\,309\,456$ states

$$L_{-2}L_{-2}\Omega \ , \qquad L_{-2}W^i_{-2}\Omega \ , \qquad W^i_{-2}L_{-2}\Omega \ , \qquad W^i_{-2}W^j_{-2}\Omega \ . \tag{5.4.3}$$

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[6] This is not, though, what happened in the example of the tensor products of the Monster theories: there the null-vectors that are responsible for this cancellation appear at or below the level suggested by the order of the differential equation.

These states appear at level four. On the other hand, we know from the partition function (5.4.2) that the total number of states at conformal weight four (above the vacuum) is

$$M_1 = 864\,299\,970 \ll 38\,763\,309\,456 = N_1^2 \ . \tag{5.4.4}$$

Thus it follows from this simple counting argument that there must be many 'null'-relations among the states (5.4.3); one of them is for example the null-vector relation (5.3.17).

One may ask how this counting argument works for the other extremal self-dual theories. For general k we define N_k and D_k by

$$Z_k(q) = q^{-k} + \cdots + N_k\, q + \cdots + M_k\, q^{k+2} \ , \tag{5.4.5}$$

where $Z_k(q)$ is the extremal partition function. By the same token as above, the theory will have many null-vectors if $M_k - N_k^2 < 0$. For the first few values of k we find the following numbers:

k	N_k	M_k	$M_k - N_k^2$
k=1	196884	864299970	-37899009486
k=2	42987520	802981794805760	-1044945080944640
k=3	2593096794	378428749730548169825	371704598747495091389
k=4	81026609428	141229814494885904705260482	141223249183450507046773298

Table 5.1: The coefficients N_k and M_k for the extremal self-dual theories at $c = 24k$.

We have checked these numbers for up to $k = 150$, and the pattern seems to continue — in fact it appears that $N_k^2 \leq d_1\, e^{-d_2 k}\, M_k$ for some constants d_1 and d_2. Thus this counting argument explains why the Monster theory has many low-lying null-vectors. It also predicts that the same is true for the theory with $k = 2$, but at least from this point of view, there are no indications that the theories with $k \geq 3$ should have many low-lying null-vectors. We regard this as evidence against the possibility that the extremal theories avoid the above contradiction.

5.5 Summary

To sum up, we have shown that every modular differential equation of a rational conformal field theory comes from a non-trivial null vector in the Verma module — see (5.3.13). Generically, the functions f_l and h_j that appear in this identity are analytic in $|q| < 1$, and then (5.3.13) implies that there is a relation of the form (5.3.14). At least for the extremal self-dual theories at $c = 24k$ this relation is a non-trivial null relation. This then implies, following the arguments of [91], that these theories are inconsistent for $k \geq 42$.

This analysis is however not completely conclusive since it *is* possible that the functions f_l and h_j appearing in (5.3.13) are non-holomorphic — indeed, this is what happens for the example of Gaiotto [99] concerning tensor products of the Monster theory. However, this then requires that the non-holomorphic terms that appear on the right-hand side of (5.3.13) must actually vanish, thus indicating that there are many other null vector relations (albeit none of the form (5.3.14)). This is indeed what happens for the case of the tensor product of the Monster theories.

Finally, we have seen from the analysis of the partition functions, that the theories at $k = 1, 2$ must have many non-trivial null-vector relations, but that there are no indications (from this point of view) that this should be the case for $k \geq 3$. Taken together we regard this as suggestive evidence for the assertion that the extremal self-dual theories at $c = 24k$ are inconsistent for $k \geq 42$.

Chapter 6

Extremal $N = (2,2)$ 2D CFTs and constraints of modularity

6.1 Overview and summary of the main results

In the last chapter we have refined the original analysis of pure bosonic gravity in [186]. In this chapter we take a step back and address questions for pure quantum gravity with extended $\mathcal{N} = 2$ supersymmetry. Our main tool will be the elliptic genus of an $\mathcal{N} = 2$ superconformal field theory. As we recall below, this is a weak Jacobi form, and its modular properties impose tight constraints on the partition function. The advantage of this approach is that, unlike the bosonic case, we do not have to assume the complete holomorphic factorization of the partition function. The holomorphy and modularity of the elliptic genus holds for any conformal field theory with $\mathcal{N} = 2$ supersymmetry. Thus, we can test the hypothetical existence of a theory of pure AdS_3 supergravity without relying on the additional assumption of holomorphic factorization. It turns out that there is some tension between these modular properties and the notion of extremality.

A brief summary of our results is the following:

1. In section 6.3.1 we give a definition of an extremal $(2,2)$ superconformal field theory which, one might expect would constitute a holographic dual to "pure $(2,2)$ AdS_3 supergravity." In any case, it is a natural generalization of the notion of extremality to $(2,2)$ supersymmetry. We will restrict attention to theories with integral $U(1)_R$ charges for the left- and right-moving $\mathcal{N} = 2$ algebras.

2. In section 6.4 we give numerical evidence that only a finite number of "sporadic" examples of extremal $(2,2)$ theories can exist. Then in section 6.5 we give an analytic proof that this is indeed the case.

3. In section 6.6 we then introduce the notion of a "nearly extremal $(2,2)$ superconformal theory," whose spectrum only approximates that of pure $(2,2)$ supergravity. We show that if the degree of approximation is relaxed then candidate elliptic genera do indeed exist.

4. By quantifying the degree of approximation required to produce candidate elliptic genera we are able to constrain the spectrum as follows. Consider states (in the NSNS sector) which are right-chiral-primary and left $\mathcal{N} = 2$ primary with (L_0, J_0) eigenvalue (h, ℓ). Suppose the central charge is $c = 6m$. In section 6.6.1, equation (6.6.128) we show that for m large any theory with modular elliptic genus must have some such state with

$$h < \frac{m}{4} + \frac{\ell^2}{4m} - \frac{1}{8} + \mathcal{O}(m^{-1/2}) \ . \tag{6.1.1}$$

   This result is conjectural. It is supported by numerical evidence described in section 6.6. Finding a rigorous justification of (6.6.128) (or a counterexample) is an interesting open problem.

5. On the other hand, in section 6.7 we show that it is possible to construct an elliptic genus which is compatible with the spectrum of an extremal $(2, 2)$ superconformal theory for conformal weights $h \leq \frac{m}{4}$.

6. In section 6.9 we comment on a partial generalization of our results to $\mathcal{N} = 4$ theories.

In the remainder of the chapter we discuss some implications of the above results. First, in section 6.8 we discuss the implications for the existence of pure $(2, 2)$ AdS_3 supergravity. While our results cast some doubt on the existence of such theories, they are not conclusive. It is conceivable that quantum corrections to the cosmic censorship bound for the existence of black holes imply that one should identify a near-extremal rather than an extremal $(2, 2)$ CFT as a holographic dual of pure supergravity.

A second motivation for the present work is that constraints on conformal field theory spectra implied by modular invariance might have interesting applications to flux compactifications of string theory and M-theory. This is briefly explained in section 6.10. Again, the development of this idea is left to future work.

6.2 Polar states and the elliptic genus

We will focus on theories with $\mathcal{N} = (2, 2)$ two-dimensional superconformal symmetry. It will be convenient to parametrize the (left = right) central charge as $c = 6m$. A simple example of such a theory that the reader might wish to keep in mind is an $\mathcal{N} = (2, 2)$ sigma-model based on a Calabi-Yau target space of complex dimension $2m$. In the present paper we only consider integer values of m, and thus the relevant Calabi-Yau manifolds have even complex dimension.[1] In particular, the smallest non-trivial value of m corresponds to a Calabi-Yau 2-fold, that is a torus T^4 or a $K3$ surface.

We assume that the Hilbert space of our theory is a direct sum of unitary highest weight representations of the $\mathcal{N} = 2$ algebra. This allows us to define the RR-sector

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[1] A generalization to half-integer values of m should be possible, but we will not attempt it in the present paper.

partition function

$$Z_{RR}(\tau,z;\bar{\tau},\bar{z}) := \mathrm{Tr}_{\mathcal{H}_{RR}} q^{L_0-c/24} e^{2\pi i z J_0} \bar{q}^{\tilde{L}_0-c/24} e^{2\pi i \bar{z}\tilde{J}_0} e^{i\pi(J_0-\tilde{J}_0)} \tag{6.2.2}$$

which has good modular properties under the $SL(2,\mathbb{Z})$ action $(\tau,z)\to(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d})$. Here, as usual, $q=e^{2\pi i\tau}$ and $y=e^{2\pi iz}$, and similarly for $\bar{q}$ and $\bar{y}$.

In these conventions, the elliptic genus of an $\mathcal{N}=(2,2)$ superconformal field theory $\mathcal{C}$ is defined to be

$$\chi(\tau,z;\mathcal{C}) := Z_{RR}(\tau,z;\bar{\tau},0)\ . \tag{6.2.3}$$

It is holomorphic in (τ,z) by the standard properties of the Witten index. For references on the elliptic genus see [10, 9, 63, 133, 142, 143, 162, 164, 165, 171, 170, 180, 182].

$\mathcal{N}=2$ algebras have the crucial spectral flow isomorphism [172], which allows us to relate the NS and R-sector partition functions. Recall that spectral flow SF_θ for $\theta\in\frac{1}{2}\mathbb{Z}$ is an isomorphism of $\mathcal{N}=2$ superconformal algebras which maps eigenvalues

$$\begin{aligned} L_0 &\to L_0+\theta J_0+\theta^2 m && (6.2.4)\\ J_0 &\to J_0+2\theta m\ . && (6.2.5)\end{aligned}$$

The spectral flow operators act on $Z=Z_{RR}$ as:

$$(SF_\theta\widetilde{SF}_{\tilde{\theta}}Z) = e\left(m\theta^2\tau+2m\theta(z+\frac{1}{2})\right)\cdot e\left(m\tilde{\theta}^2\bar{\tau}+2m\tilde{\theta}(\bar{z}-\frac{1}{2})\right)Z(\tau,z+\tau\theta;\bar{\tau},\bar{z}+\tilde{\theta}\bar{\tau}), \tag{6.2.6}$$

where $e(x):=e^{2\pi ix}$. For simplicity we restrict our attention to theories with integral spectrum of left- and right-moving $U(1)$ charges $J_0,\tilde{J}_0$. Again, it should be possible, and would be interesting, to relax this assumption. Spectral-flow invariant theories with integral $U(1)$ charges satisfy

$$\begin{aligned} Z_{RR} &= (SF_\theta\widetilde{SF}_{\tilde{\theta}})Z_{RR} \qquad \theta,\tilde{\theta}\in\mathbb{Z} && (6.2.7)\\ Z_{NSNS} &= (SF_\theta\widetilde{SF}_{\tilde{\theta}})Z_{RR} \qquad \theta,\tilde{\theta}\in\mathbb{Z}+\frac{1}{2}\ . && (6.2.8)\end{aligned}$$

As is well-known [133], the modularity properties of Z_{RR} together with spectral flow invariance and unitarity imply that the elliptic genus is a *weak Jacobi form* of index m [70]. A weak Jacobi form $\phi(\tau,z)$ of weight w and index $m\in\mathbb{Z}$, with $(\tau,z)\in\mathbb{H}\times\mathbb{C}$, satisfies the transformation laws

$$\phi(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}) = (c\tau+d)^w e^{2\pi im\frac{cz^2}{c\tau+d}}\phi(\tau,z) \qquad \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2,\mathbb{Z})\ , \tag{6.2.9}$$

$$\phi(\tau,z+\ell\tau+\ell') = e^{-2\pi im(\ell^2\tau+2\ell z)}\phi(\tau,z) \qquad \ell,\ell'\in\mathbb{Z}\ , \tag{6.2.10}$$

and has a Fourier expansion

$$\phi(\tau,z) = \sum_{n\geq 0,\ell\in\mathbb{Z}} c(n,\ell)q^n y^\ell \tag{6.2.11}$$

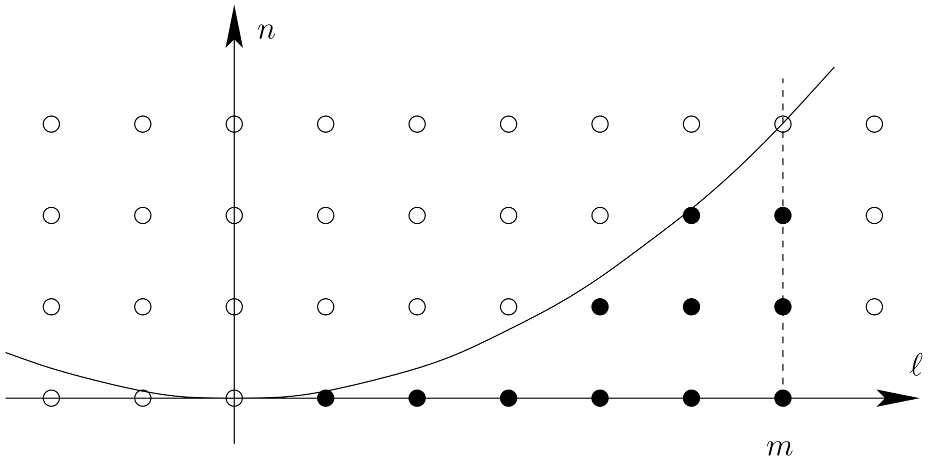

Figure 6.1: A cartoon showing polar states (represented by "•") in the region $\mathcal{P}^{(m)}$. Spectral flow by $\theta = \frac{1}{2}$ relates these to states in the NS sector of an $\mathcal{N} = 2$ superconformal field theory which are holographically dual to particle states in AdS_3.

with $c(n, \ell) = (-1)^w c(n, -\ell)$. It follows from the spectral flow identity that $c(n, \ell) = 0$ for $4mn - \ell^2 < -m^2$. Following [70], we denote by $\tilde{J}_{w,m}$ the vector space of weak Jacobi forms of weight w and index m. A Jacobi form is then a weak Jacobi form whose polar part vanishes (see below).

Suppose we are given an integer $m \in \mathbb{Z}_+$. If $(\ell, n) \in \mathbb{Z}^2$ is a lattice point we refer to its *polarity* as $p = 4mn - \ell^2$. If $\phi \in \tilde{J}_{0,m}$ let us define the *polar part* of ϕ, denoted ϕ^-, to be the sum of the terms in the Fourier expansion corresponding to lattice points of negative polarity. By spectral flow one can always relate the degeneracies to those in the fundamental domain with $|\ell| \le m$. If we impose the modular transformation (6.2.9) with $-\mathbf{1} \in SL(2, \mathbb{Z})$, which implements charge conjugation, then $c(n, \ell) = c(n, -\ell)$ and therefore the polar coefficients which cannot be related to each other by spectral flow and charge conjugation are $c(n, \ell)$ where (ℓ, n) is valued in the *polar region* $\mathcal{P}$ (of index m), defined to be

$$\mathcal{P}^{(m)} := \{(\ell, n) : 1 \le \ell \le m, \quad 0 \le n, \quad p = 4mn - \ell^2 < 0\} . \qquad (6.2.12)$$

For an example, see figure 6.1.

Given any Fourier expansion

$$\psi(\tau, z) = \sum_{\ell, n \in \mathbb{Z}} \hat{\psi}(n, \ell) q^n y^\ell \qquad (6.2.13)$$

we define its *polar polynomial* (of index m) to be the sum restricted to the polar region $\mathcal{P}^{(m)}$:

$$\mathrm{Pol}(\psi) := \sum_{(\ell, n) \in \mathcal{P}^{(m)}} \hat{\psi}(n, \ell) q^n y^\ell . \qquad (6.2.14)$$

Let us moreover denote by V_m the space of polar polynomials, *i.e.* the vector space generated by the monomials $q^n y^\ell$ with $(\ell, n) \in \mathcal{P}^{(m)}$.

The key mathematical fact we need is that one can reconstruct a weak Jacobi form of weight zero from its polar polynomial, as explained in [50, 156, 150]. The results of [150] imply that there is a sequence

$$0 \to \tilde{J}_{0,m} \overset{\mathrm{Pol}}{\to} V_m \overset{\mathrm{Per}}{\to} S_{5/2} \tag{6.2.15}$$

exact at V_m, where Per is a “period map” to a certain space of vector-valued cusp forms of weight 5/2. A nonzero image in the space of cusp forms means that the polar polynomial cannot be realized by a true Jacobi form. In the next two sections we will show that there can indeed be nontrivial obstructions by computing the dimensions of $\tilde{J}_{0,m}$ and V_m.

Returning to the conformal field theory $\mathcal{C}$, an eigenstate of L_0, J_0 is called a *polar state* if it has negative polarity:

$$p = 4mL_0 - J_0^2 - m^2 = 4m(L_0 - \frac{c}{24}) - J_0^2 < 0 \ . \tag{6.2.16}$$

One checks that $4mL_0 - J_0^2$ is spectral flow invariant, so we can speak of polar states in both the R and NS sector. Using the mathematical results explained above we see that the significance of polar states is that the polar degeneracies of the elliptic genus determine all the other Fourier coefficients of the elliptic genus.

6.2.1 Counting weight zero weak Jacobi forms

Let $\tilde{J}_{ev,*} = \oplus_{w\in 2\mathbb{Z}, m\in\mathbb{Z}} \tilde{J}_{w,m}$ denote the bigraded ring of weak Jacobi forms of even weight. According to [70], Theorem 9.3, $\tilde{J}_{ev,*}$ is a polynomial algebra on four generators of degree

$$(w,m) \ = \ (4,0), \ \ (6,0), \ \ (-2,1), \ \ (0,1) \ . \tag{6.2.17}$$

The first two generators correspond to the Eisenstein series E_4 and E_6 as defined in appendix C.3. A generalization of Eisenstein series to Jacobi forms is described in [70]:

$$E_{k,m}(\tau,z) = \frac{1}{2} \sum_{c,d\in\mathbb{Z},(c,d)=1} \sum_{\ell\in\mathbb{Z}} (c\tau+d)^{-k} e^{2\pi i m\left(\ell^2 \frac{a\tau+b}{c\tau+d} + 2\ell \frac{z}{c\tau+d} - \frac{cz^2}{c\tau+d}\right)} \ . \tag{6.2.18}$$

In terms of these generalized Eisenstein series one can write the remaining two generators in (6.2.17) as

$$\tilde{\phi}_{-2,1} = \frac{\phi_{10,1}}{\Delta} \in \tilde{J}_{-2,1} \qquad \tilde{\phi}_{0,1} = \frac{\phi_{12,1}}{\Delta} \in \tilde{J}_{0,1} \ , \tag{6.2.19}$$

where the first subscript on $\tilde{\phi}$ denotes the weight and the second denotes the index. Here, $\Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24}$ and

$$\begin{aligned} \phi_{10,1} &= \frac{1}{144}(E_6 E_{4,1} - E_4 E_{6,1}) \\ &= (y - 2 + y^{-1})q + (-2y^2 - 16y + 36 - 16y^{-1} - 2y^{-2})q^2 + \dots \ , \\ \phi_{12,1} &= \frac{1}{144}(E_4^2 E_{4,1} - E_6 E_{6,1}) \\ &= (y + 10 + y^{-1})q + (10y^2 - 88y - 132 - 88y^{-1} + 10y^{-2})q^2 + \dots \ . \end{aligned} \tag{6.2.20}$$

Thus the two weak Jacobi forms $\tilde{\phi}_{-2,1}$ and $\tilde{\phi}_{0,1}$ have the series expansion

$$\begin{aligned} \tilde{\phi}_{-2,1} &= \left(y - 2 + y^{-1}\right) + \left(-2y^2 + 8y - 12 + 8y^{-1} - 2y^{-2}\right) q + \dots \ , \\ \tilde{\phi}_{0,1} &= (y + 10 + y^{-1}) + \left(10y^2 - 64y + 108 - 64y^{-1} + 10y^{-2}\right) q + \dots \ . \end{aligned} \tag{6.2.21}$$

Much useful information about Jacobi forms can be found in [70].

To summarize, a natural vector space basis of $\tilde{J}_{0,m}$ is given by

$$(\tilde{\phi}_{-2,1})^a (\tilde{\phi}_{0,1})^b E_4^c E_6^d \ , \tag{6.2.22}$$

where a, b, c, d are nonnegative integers such that $a + b = m$, and $a = 2c + 3d$. It is straightforward to compute the number of solutions to these constraints and thereby show that

$$j(m) := \dim \tilde{J}_{0,m} = \frac{m^2}{12} + \frac{m}{2} + \left(\delta_{s,0} + \frac{s}{2} - \frac{s^2}{12}\right) \ , \tag{6.2.23}$$

where $m = 6\rho + s$ with $\rho \geq 0$ and $0 \leq s \leq 5$. Specifically,

$$j(m) = \begin{cases} m^2/12 + m/2 + 1 & m = 0 \bmod 6 \\ m^2/12 + m/2 + 5/12 & m = 1, 5 \bmod 6 \\ m^2/12 + m/2 + 2/3 & m = 2, 4 \bmod 6 \\ m^2/12 + m/2 + 3/4 & m = 3 \bmod 6 \ . \end{cases} \tag{6.2.24}$$

6.2.2 Counting polar monomials

Let us now compute the dimension of the space V_m, and compare it to $j(m)$. In other words, we wish to count the number of integer points in the (ℓ, n) plane bounded (on one side) by the parabola $4mn - \ell^2 = 0$, as shown in figure 6.1. We have

$$P(m) := \dim V_m = \sum_{\ell=1}^{m} \lceil \frac{\ell^2}{4m} \rceil \ . \tag{6.2.25}$$

Note that we want the *ceiling* function and not the floor function, as we include $n = 0$ up to the largest n with $n < \ell^2/(4m)$ for each $\ell = 1, \dots, m$.

To compute this we follow [70] and write our sum as a sum of three terms.

$$\sum_{\ell=1}^{m} \lceil \frac{\ell^2}{4m} \rceil = \sum_{\ell=1}^{m} \frac{\ell^2}{4m} - \sum_{\ell=1}^{m} ((\frac{\ell^2}{4m})) + \frac{1}{2} \sum_{\ell=1}^{m} \left(\lceil \frac{\ell^2}{4m} \rceil - \lfloor \frac{\ell^2}{4m} \rfloor \right) \ , \tag{6.2.26}$$

where

$$((x)) := x - \frac{1}{2}(\lceil x \rceil + \lfloor x \rfloor) = \begin{cases} 0 & x \in \mathbb{Z} \\ \alpha - \frac{1}{2} & x = n + \alpha, 0 < \alpha < 1 \ . \end{cases}$$

Note that $((x))$ is the sawtooth function. It is periodic of period 1.

Now we evaluate the three terms. The main term comes from the elementary formula

$$\sum_{\ell=1}^{m} \frac{\ell^2}{4m} = \frac{m^2}{12} + \frac{m}{8} + \frac{1}{24} \ . \tag{6.2.27}$$

Next, note that the number of integers ℓ with $1 \leq \ell \leq m$ with $\ell^2 = 0 \bmod 4m$ is $\lfloor \frac{b}{2} \rfloor$ where b is the largest integer with $b^2|m$. This follows from the prime factorization of m. Thus, we obtain:

$$\sum_{\ell=1}^{m} \lceil \frac{\ell^2}{4m} \rceil - \sum_{\ell=1}^{m} \lfloor \frac{\ell^2}{4m} \rfloor = m - \lfloor \frac{b}{2} \rfloor \ . \tag{6.2.28}$$

Finally we come to the most subtle term $\sum_{\ell=1}^{m} ((\frac{\ell^2}{4m}))$. The numbers $((\frac{\ell^2}{4m}))$ are, very roughly speaking, randomly distributed between $-1/2$ and $+1/2$, Therefore, the average will go to zero. In fact, they roughly make a random walk so we expect a quantity on the order of $m^{1/2}$. To be more precise the discussion of [70], pp. 122-124 shows that

$$\sum_{\ell=1}^{m} ((\frac{\ell^2}{4m})) = -\frac{1}{4} \sum_{d|4m} h'(-d) + \frac{1}{2}((\frac{m}{4})) \ ,$$

where $h'(-d)$ is the class number of a quadratic imaginary field of discriminant $-d$.

Putting the three terms together we obtain:

$$P(m) = \frac{m^2}{12} + \frac{5m}{8} + A(m) \tag{6.2.29}$$

where $A(m)$ is the arithmetic function

$$A(m) = \frac{1}{4} \sum_{d|4m} h'(-d) - \frac{1}{2} \lfloor \frac{b}{2} \rfloor - \frac{1}{2}((\frac{m}{4})) + \frac{1}{24} \ . \tag{6.2.30}$$

The first few values of $P(m)$ and $j(m)$ are:

m	$\dim \tilde{J}_{0,m}$	$\dim V_m$
$m=0$	1	0
$m=1$	1	1
$m=2$	2	2
$m=3$	3	3
$m=4$	4	4
$m=5$	5	6
$m=6$	7	8
$m=7$	8	9
$m=8$	10	11
$m=9$	12	13
$m=10$	14	16
$m=11$	16	18
$m=12$	19	21

(6.2.31)

Note that $P(m) > j(m)$ for $m \geq 5$. Roughly speaking,[2] $A(m)$ grows like $\mathcal{O}(m^{1/2})$ so for large m we have

$$P(m) - j(m) = \frac{m}{8} + \mathcal{O}(m^{1/2}) \ . \tag{6.2.32}$$

The important conclusion that we draw is that for large m there are on the order of $\frac{m}{8}$ linear constraints on the polar coefficients of the elliptic genus expressing modularity.

Remarks

1. The action of charge conjugation together with spectral flow defines an action of D_∞ on the (ℓ, n) plane which preserves the space $\mathcal{Q}$ of polar values $-m^2 \leq 4mn - \ell^2 < 0$. A fundamental domain is given by the polar region $\mathcal{P}^{(m)}$, but the quotient $\mathcal{Q}/D_\infty$ has fixed points: for $\ell = -m$ the spectral flow to $\ell = +m$ can be undone by charge conjugation. Therefore, if we compute the *orbifold* Euler character of $\mathcal{Q}/D_\infty$ the line of states (ℓ, h) with $\ell = m$ should be counted with weight $\frac{1}{2}$. There are precisely $m/4$ states on this line and hence $\chi_{\rm orb}(\mathcal{Q}/D_\infty) = P(m) - m/8$, which is a much closer approximation to $j(m)$.

2. Recently, J. Manschot [149] has reproduced the formula for $P(m) - j(m)$ by directly computing the dimension of the image of the period map Per in (6.2.15).

6.3 Extremal $\mathcal{N} = (2,2)$ conformal field theories

6.3.1 Definition

In [186] Witten suggested that the holographic dual of pure 2+1 dimensional quantum gravity should be an "extremal conformal field theory." The latter is defined to be a conformal field theory whose modular invariant partition function is "as close as possible" to the Virasoro character of the vacuum. When $c = 24k$ the vacuum character is

$$\chi^{(k)}_{Vac}(\tau) = q^{-k} \prod_{n=2}^{\infty} \frac{1}{1-q^n} \ . \tag{6.3.33}$$

The partition function $Z_k(\tau)$ has weight zero. Unlike the elliptic genus case, there is no obstruction to completing an arbitrary polynomial in q^{-1} to a modular function by adding nonpolar terms. Therefore, Witten defines $Z_k(\tau)$ to be the unique modular function with no singularities for $\tau \in \mathbb{H}$ such that the expansion around the cusp at infinity satisfies

$$Z_k(\tau) := \left[q^{-k} \prod_{n=2}^{\infty} \frac{1}{1-q^n} \right]_{q^{\leq 0}} + \mathcal{O}(q) \ . \tag{6.3.34}$$

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[2] A theorem of Siegel states that $\lim_{d\to\infty} \frac{\log h'(-d)}{\log d} = \frac{1}{2}$ as d runs through discriminants of quadratic imaginary fields, but $h'(-d)$ itself does not have a simple asymptotic expansion. This follows from its relation to the Dirichlet Series $L_d(s)$ at $s = 1$. For a discussion of these and related matters, together with their possible applications to black holes and with references to the math literature see [154].

Following [50], Witten interprets the first Virasoro primary above the vacuum representation to be a state corresponding to the lightest possible BTZ black hole in AdS_3.

Following Witten [186] let us consider "pure $\mathcal{N} = (2, 2)$ supergravity" with negative cosmological constant. This is the hypothetical quantum theory whose classical action is a supersymmetric completion of the Einstein-Hilbert action,

$$I_{sugra} = \frac{1}{16\pi G} \int d^3x \sqrt{g} \left(\mathcal{R}(g) + \frac{2}{R^2} + \ldots \right) \ . \tag{6.3.35}$$

Here, R is the AdS length scale and the ellipses denote contributions of other fields in the $\mathcal{N} = 2$ supergravity multiplet. Specifically, apart from the metric, these fields include real spin-$\frac{3}{2}$ gravitino fields, ψ^i_L and ψ^i_R, $i = 1, 2$ as well as two abelian gauge fields, a_L and a_R. In general, if we were interested in $\mathcal{N} = (p, q)$ supergravity theory, the corresponding gauge group would be $SO(p) \times SO(q)$. Thus, in the present context of $\mathcal{N} = (2, 2)$ theory we have $SO(2) \times SO(2)$ gauge fields.

In fact, by enlarging the gauge group one can write the entire supergravity action (6.3.35) as the Chern-Simons action [1, 2]:

$$\begin{aligned} I_{CS} =& \frac{k_L}{4\pi} \int \mathrm{tr} \left(A_L \wedge dA_L + \frac{2}{3} A_L \wedge A_L \wedge A_L \right) \\ &- \frac{k_R}{4\pi} \int \mathrm{tr} \left(A_R \wedge dA_R + \frac{2}{3} A_R \wedge A_R \wedge A_R \right) \end{aligned} \tag{6.3.36}$$

where the gauge fields A_L and A_R take values in the Lie algebra of the supergroup

$$G = G_L \times G_R = OSp(2|2)_L \times OSp(2|2)_R \ . \tag{6.3.37}$$

Since the bosonic part of the supergroup $OSp(2|2)$ is $SO(2) \times SL(2, \mathbb{R})$, the gauge group (6.3.37) contains the classical symmetry[3] group, $SL(2, \mathbb{R})_L \times SL(2, \mathbb{R})_R$, of the three-dimensional AdS space. In the simple case $k_L = k_R$, which will be of interest to us in the present paper, one finds the following relation between the parameters:

$$k_L = k_R = \frac{R}{16G} \ . \tag{6.3.38}$$

Combining this with the Brown-Henneaux formula $c_L = c_R = \frac{3R}{2G}$ and using our expression for the central charge $c_L = c_R = 6m$, we can conveniently write (6.3.38) as

$$k_L = k_R = \frac{m}{4} \ . \tag{6.3.39}$$

Since we take m to be integer, it follows that k_L and k_R take values in $\frac{1}{4}\mathbb{Z}$. This is consistent with the fact that the bosonic part of our supergroup $OSp(2|2)$ contains $SL(2, \mathbb{R})$, which is a double cover of $SO(2, 1)$; see section 2.1 of [186] for further details on the allowed values of k_L and k_R.

The equivalence of $\mathcal{N} = (2, 2)$ supergravity and Chern-Simons theory based on the supergroup (6.3.37) is valid not only classically, but to all orders in perturbation theory, as

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[3] This symmetry group is the gauge group of the analogous formulation of $\mathcal{N} = 0$ gravity theory.

long as the perturbative expansion starts with a non-degenerate classical solution. This way of formulating perturbative $\mathcal{N} = (2,2)$ supergravity will be useful to us in what follows, in particular, in section 6.8 where we discuss quantum corrections.

The $\mathcal{N} = (2,2)$ case is similar to the $\mathcal{N} = 0$ case of Chern-Simons gravity: There are no local degrees of freedom, but the Chern-Simons theory does give rise to "edge states." These are $\mathcal{N} = 2$ descendants of the vacuum representation, that is, the irreducible highest weight representation defined by $(h = 0, q = 0)$.

The natural generalization of Witten's proposal to $(2,2)$ supergravity in $2+1$ dimensions is that the holographic dual should be an "extremal $(2,2)$ superconformal field theory," where we define the latter to be a theory whose partition function is "as close as possible" to the vacuum character of the $\mathcal{N} = 2$ algebra. The vacuum character of the $\mathcal{N} = 2$ algebra is [21]

$$\begin{aligned}\chi^{(m)}_{vac}(\tau, z) &:= \mathrm{Tr}_{V_{0,0}} q^{L_0 - c/24} e^{2\pi i(z+\frac{1}{2})J_0} \\ &= q^{-m/4}(1-q)\prod_{n=1}^{\infty}\frac{(1-yq^{n+1/2})(1-y^{-1}q^{n+1/2})}{(1-q^n)^2}\ .\end{aligned} \tag{6.3.40}$$

We have shifted z by $1/2$ relative to the standard definition for later convenience. The expression in (6.3.40) is neither spectral flow invariant, nor modular invariant.

It has been observed that in $\mathcal{N} = (2,2)$ supergravity the cosmic censorship bound for charged black holes is given by $p, \tilde{p} \geq 0$, where p and $\tilde{p}$ refer to the polarity of the left- and right-moving states (*i.e.*, $p = 4mn - \ell^2$) [41, 50]. It is thus natural to only allow new states with non-negative polarity. By explicitly enforcing spectral flow invariance, we take our definition of an $\mathcal{N} = (2,2)$ extremal conformal field theory to be:

Definition: An $\mathcal{N} = (2,2)$ *extremal conformal field theory of level* m ("$\mathcal{N} = 2$ ECFT" for short) is a hypothetical theory whose partition function is of the form:

$$\begin{aligned}Z_{NSNS}(\tau, z; \bar{\tau}, \bar{z}) &:= \mathrm{Tr}_{\mathcal{H}_{NSNS}} q^{L_0 - c/24} e^{2\pi i z J_0} \bar{q}^{\tilde{L}_0 - c/24} e^{2\pi i \bar{z}\tilde{J}_0} e^{i\pi(J_0 - \tilde{J}_0)} \\ &= \sum_{s,\bar{s}\in\mathbb{Z}} SF_s \chi^{(m)}_{\text{vac}}(\tau, z) SF_{\bar{s}} \bar{\chi}^{(m)}_{\text{vac}}(\bar{\tau}, \bar{z}) \\ &\quad + \sum_{s\in\mathbb{Z}} SF_s \chi^{(m)}_{\text{vac}}(\tau, z) \bar{f}(\bar{\tau}, \bar{z}) + \sum_{\bar{s}\in\mathbb{Z}} f(\tau, z) SF_{\bar{s}} \bar{\chi}^{(m)}_{\text{vac}}(\bar{\tau}, \bar{z}) \\ &\quad + \sum_{p,\tilde{p}\geq 0} a(n, \ell; \tilde{n}, \tilde{\ell}) q^n y^\ell \bar{q}^{\tilde{n}} \bar{y}^{\tilde{\ell}}\ .\end{aligned} \tag{6.3.41}$$

Here the coefficients $a(n, \ell; \tilde{n}, \tilde{\ell})$ are integers, and the sum over nonpolar states in the last line means that *both* the left and right polarity of the state is non-negative. The functions $f(\tau, z)$ and $\bar{f}(\bar{\tau}, \bar{z})$ describe the contribution of terms with non-negative polarity with respect to the left and right polarity, respectively. We need to include such terms since states with either $p \geq 0$ or $\tilde{p} \geq 0$ are not polar and are allowed by the extremality condition.

Using spectral flow (6.2.3) we can compute $Z_{RR}(\tau, z; \bar{\tau}, \bar{z})$ for an $\mathcal{N} = 2$ ECFT from (6.3.41). The elliptic genus is then obtained upon setting $\bar{z} = 0$. In this limit only those terms contribute that have $\bar{q}^0$. All of these terms are polar, with the exception of the $\bar{q}^0\bar{y}^0$

term that has polarity zero. Thus the elliptic genus of an $\mathcal{N}=2$ ECFT of level m is of the form

$$(2(-1)^m + k) \sum_{\theta \in \mathbb{Z}+\frac{1}{2}} SF_\theta \chi_{vac}^{(m)} + \text{Nonpolar} \; , \tag{6.3.42}$$

where k is the coefficient of the $\bar{q}^0 \bar{y}^0$ term coming from $\bar{f}(\bar{\tau}, \bar{z})$. The factor $2(-1)^m$ is the limit $\bar{z} \to 0$ of the first term in (6.3.41), as we will see momentarily. Using (6.5.92) below one can determine the constant to be $k = 12m - 2$. For convenience we drop the overall constant factor from the right-movers and define:

$$\chi_{\text{ext}}^{(m)}(\tau, z) := \sum_{\theta \in \mathbb{Z}+\frac{1}{2}} SF_\theta \chi_{vac}^{(m)} + \text{Nonpolar} \; . \tag{6.3.43}$$

We will call a weak Jacobi form that satisfies (6.3.43) an *extremal elliptic genus*. Because the only unknown terms in (6.3.43) are nonpolar terms we can compute the polar polynomial of such an extremal elliptic genus. We will give an explicit formula for it in section 6.3.2. Then, in section 6.4 we investigate whether such a polar polynomial is consistent with modularity.

6.3.2 The extremal polar polynomial

Let us compute the polar polynomial of a would-be extremal elliptic genus. We begin by demonstrating the following useful fact:

$$\text{Pol}\Big(\sum_{\theta \in \mathbb{Z}+\frac{1}{2}} SF_\theta \chi_{vac}^{(m)} \Big) = \text{Pol}(SF_{1/2} \chi_{vac}^{(m)}) \; . \tag{6.3.44}$$

Indeed, if we apply the spectral flow by $\theta = l + \frac{1}{2}$ to the vacuum character (6.3.40) we obtain an expression of the form

$$(-1)^m \, q^{l(l+1)m} y^{(2l+1)m} (1-q) \prod_{n=1}^{\infty} \frac{(1 - yq^{n+l+1})(1 - y^{-1} q^{n-l})}{(1-q^n)^2} \; . \tag{6.3.45}$$

We wish to show that this expression contains no polar terms in the fundamental domain (6.2.12) for $l \neq 0$. Without loss of generality, we can assume $l > 0$. Note that it is not true that (6.3.45) has no polar terms. In fact, already the first term $q^{l(l+1)m} y^{(2l+1)m}$ is polar for every l; it has polarity $p = -m^2$. However, it does not belong the polar region $\mathcal{P}^{(m)}$ since the power of y is not in the allowed range $1 \leq \ell \leq m$.

On the other hand, there are terms in (6.3.45) with $1 \leq \ell \leq m$ but, as we show momentarily, these terms are not polar. We can simplify the problem a little bit and omit the denominator in (6.3.45) and the factor $(1-q)$ which can only increase the polarity. Then, our goal is to show that

$$q^{l(l+1)m} y^{(2l+1)m} \prod_{n=1}^{\infty} (1 - yq^{n+l+1})(1 - y^{-1} q^{n-l}) \tag{6.3.46}$$

has no polar terms in the range $1 \leq \ell \leq m$. From the above discussion, we already know that the term $q^{l(l+1)m}y^{(2l+1)m}$ is polar. We can combine it with the terms from factors $(1-yq^{n+l+1})$ and $(1-y^{-1}q^{n-l})$ for various n to bring the power of y to the desired range. Since l is assumed to be positive, it is easy to see that the terms coming from factors $(1-yq^{n+l+1})$ can be ignored, while from $\prod_{n=1}^{\infty}(1-y^{-1}q^{n-l})$ we need to collect at least $2lm$ factors of y^{-1} to bring the overall power of y to the desired range. The most economical way to do this (which yields the minimal increase in polarity) is to collect the factors in the infinite product with the smallest powers of q. These are the terms with $n = 1, \ldots, 2lm$:

$$q^{l(l+1)m}y^{(2l+1)m}\prod_{n=1}^{2lm} y^{-1}q^{n-l} = q^{(2lm-l+2)lm}y^m \ . \tag{6.3.47}$$

The resulting term has polarity $p = 4(2lm-l+2)lm^2 - m^2$ which satisfies $p > 0$ for any $l, m \geq 1$. It is easy to see that including other factors from the infinite product in (6.3.46) only increases the polarity further.

Having proven (6.3.44) we now define

$$p_{\text{ext}}^{(m)} := (-1)^m \, \text{Pol} SF_{1/2}\chi_{vac}^{(m)} \ . \tag{6.3.48}$$

On the other hand, setting $l = 0$ in (6.3.45) one finds

$$(-1)^m SF_{1/2}\chi_{vac}^{(m)} = (1-q)y^m \prod_{n=1}^{\infty} \frac{(1-yq^{n+1})(1-y^{-1}q^n)}{(1-q^n)^2} \ . \tag{6.3.49}$$

The Fourier expansion of (6.3.49) begins:

$$y^m + q(y^m - y^{m-1}) + q^2(-2y^{-1+m} + 3y^m - y^{1+m}) + \cdots \ . \tag{6.3.50}$$

The first few polar polynomials follow easily from (6.3.50) since the polar terms for index m have $n \leq \lfloor \frac{m}{4} \rfloor$. In this way we find that the first few polar polynomials are:

$$p_{\text{ext}}^1 = y \tag{6.3.51}$$
$$p_{\text{ext}}^2 = y^2 \tag{6.3.52}$$
$$p_{\text{ext}}^3 = y^3 \tag{6.3.53}$$
$$p_{\text{ext}}^4 = y^4 \tag{6.3.54}$$
$$p_{\text{ext}}^5 = (1+q)y^5 \tag{6.3.55}$$
$$p_{\text{ext}}^6 = (1+q)y^6 - qy^5 \tag{6.3.56}$$
$$p_{\text{ext}}^7 = (1+q)y^7 - qy^6 \tag{6.3.57}$$
$$p_{\text{ext}}^8 = (1+q)y^8 - qy^7 \tag{6.3.58}$$
$$p_{\text{ext}}^9 = (1+q+3q^2)y^9 - qy^8 \tag{6.3.59}$$
$$p_{\text{ext}}^{10} = (1+q+3q^2)y^{10} - (q+2q^2)y^9 \tag{6.3.60}$$
$$p_{\text{ext}}^{11} = (1+q+3q^2)y^{11} - (q+2q^2)y^{10} \tag{6.3.61}$$
$$p_{\text{ext}}^{12} = (1+q+3q^2)y^{12} - (q+2q^2)y^{11} \ . \tag{6.3.62}$$

6.4 Experimental search for the extremal elliptic genus

Since $P(m) > j(m)$ for $m \geq 5$, and since eq. (6.3.49) does not have any obvious modular properties, it is far from obvious that (6.3.44) is the polar polynomial of a true weak Jacobi form. In this section we describe numerical results suggesting that in fact, for all but finitely many m it is not in the image of Pol applied to $\tilde{J}_{0,m}$. We will find that there are actually some "sporadic" cases where it is in the image for $m \geq 5$. In section 6.5 we will show analytically that there can only be a finite number of such sporadic cases. That might seem to obviate the need for the present section, but the methods we employ here will prove very useful when we come to describe nearly extremal theories in section 6.6.

Choose a basis ϕ_i, $i = 1, \ldots, j(m)$ for $\tilde{J}_{0,m}$. We are searching for real numbers x_i such that

$$\sum_{i=1}^{j(m)} x_i \mathrm{Pol}(\phi_i) = p_{\mathrm{ext}}^{(m)} \ . \tag{6.4.63}$$

A useful way of trying to solve this equation is the following. We choose a polarity-ordered basis of monomials $q^n y^\ell$ for V_m, that is the basis monomials $q^{n(a)} y^{l(a)}$ where $a = 1, \ldots, \dim V_m = P(m)$ so that polarity increases as a increases, and terms with the same polarity are ordered in increasing powers of y. For example for $a = 1$ the most polar term is y^m. A polarity-ordered basis for V_5 would be

$$y^5, y^4, y^3, qy^5, y^2, y^1 \tag{6.4.64}$$

with $a = 1, \ldots, 6$. The polarity-ordered basis will be very useful for our discussion of β-extremal $\mathcal{N} = 2$ conformal field theories in section 6.6.

Having chosen these two bases we can define a matrix N_{ia} of dimensions $j(m) \times P(m)$ from the expansion

$$\mathrm{Pol}(\phi_i) = \sum_{a=1}^{P(m)} N_{ia} q^{n(a)} y^{\ell(a)}. \tag{6.4.65}$$

Similarly, we can define coefficients d_a by

$$p_{\mathrm{ext}}^{(m)} = \sum_{a=1}^{P(m)} d_a q^{n(a)} y^{\ell(a)}. \tag{6.4.66}$$

Thus, we are trying to solve the linear equations

$$\sum_{i=1}^{j(m)} x_i N_{ia} = d_a, \qquad\qquad a = 1, \ldots, P(m). \tag{6.4.67}$$

It should be stressed that even if we can find a solution x_i to (6.4.67) we are far from establishing the existence of an $\mathcal{N} = 2$ extremal theory. If a solution exists then the next test we should apply is to see whether the resulting form $\sum x_i \phi_i$ has *integral* Fourier coefficients. Integrality is clearly a necessary condition for any candidate elliptic genus since it arises in conformal field theory from the trace on a Hilbert space.

Using a computer (and the explicit basis (6.2.22) above) we have examined equation (6.4.67) for $1 \leq m \leq 36$. We have found that there is a solution x_i in rational numbers for $1 \leq m \leq 5$ and for $m = 7, 8, 11, 13$, but there is no solution for $m = 6, 9, 10$ and $14 \leq m \leq 36$.[4] Moreover, remarkably, for those values of m which give a solution, the Fourier coefficients we have explicitly evaluated turn out to be integral.

The simplest example is the case $m = 1$, in which case $\chi^{(1)}_{ext} = \tilde{\phi}_{0,1}$. The next simplest case, $m = 2$ yields

$$\chi^{(2)}_{ext} = \frac{1}{6}(\tilde{\phi}_{0,1})^2 + \frac{5}{6}(\tilde{\phi}_{-2,1})^2 E_4 \ . \tag{6.4.68}$$

Although it is not obvious, one can prove that the Fourier coefficients are all integral. Indeed, the claim that this expression has integer Fourier coefficients is equivalent to the statement

$$(\tilde{\phi}_{0,1})^2 + 5(\tilde{\phi}_{-2,1})^2 E_4 = 0 \mod 6 \ . \tag{6.4.69}$$

In order to prove this, it is convenient to note (see (C.3.15) and (C.3.16)) that:

$$E_4 = 1 \text{ mod } 6 \ , \quad E_6 = 1 \text{ mod } 6 \ .$$

Moreover, from (6.2.20) it also follows that $\phi_{10,1} = \phi_{12,1}$ mod 6, which in turn implies $\tilde{\phi}_{-2,1} = \tilde{\phi}_{0,1}$ mod 6, *cf.* (6.2.21). Substituting this into $(\tilde{\phi}_{0,1})^2 + 5(\tilde{\phi}_{-2,1})^2$ and using the fact that $\tilde{\phi}_{0,1}$ and $\tilde{\phi}_{-2,1}$ have integer Fourier coefficients we therefore demonstrate (6.4.69).

When we use the basis (6.2.22) the solutions x_i are rational numbers with increasingly large denominators as m increases. For example, already the next case, $m = 3$, looks like

$$\chi^{(3)}_{ext} = \frac{1}{48}(\tilde{\phi}_{0,1})^3 + \frac{7}{16}\tilde{\phi}_{0,1}(\tilde{\phi}_{-2,1})^2 E_4 + \frac{13}{24}(\tilde{\phi}_{-2,1})^3 E_6 \ . \tag{6.4.70}$$

Even though the coefficients x_i of every monomial $(\tilde{\phi}_{-2,1})^a(\tilde{\phi}_{0,1})^b E_4^c E_6^d$ are rational numbers, the Fourier coefficients $c(n, \ell)$ are integers. In order to show this, as in the previous example, we express this as the following statement

$$(\tilde{\phi}_{0,1})^3 + 21\tilde{\phi}_{0,1}(\tilde{\phi}_{-2,1})^2 E_4 + 26(\tilde{\phi}_{-2,1})^3 E_6 = 0 \mod 48 \ . \tag{6.4.71}$$

Then, using (C.3.15) we note that $E_4 = 1$ mod 48, so we can ignore E_4 in this computation. It is not true, however, that $E_6 = 1$ mod 48. Instead, from (C.3.16) we find that $E_6^2 = 1$ mod 48. According to (6.2.19) and (6.2.20), this implies the following identity,

$$\tilde{\phi}_{-2,1} = \tilde{\phi}_{0,1} E_6 \mod 48 \ ,$$

which, after substituting in the LHS of (6.4.71), proves the desired result.

Using the basis of weak Jacobi forms described in section 6.7 below one can check that for the "miraculous" values $m = 5, 7, 8, 11, 13$ the solution does indeed have the property that all the Fourier coefficients $c(n, \ell)$ are integers.

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[4] The arguments of section 5 demonstrate that there can only be finitely many solutions. Using the constraints of that section it is easy to check that there are no further solutions up to $m \leq 400$. This suggests that the above list is in fact complete.

6.5 The extremal elliptic genus does not exist for m sufficiently large

In this section we give an analytic proof that there is no weak Jacobi form in $\tilde{J}_{0,m}$ satisfying (6.3.43) for m sufficiently large. Since this section is rather long and technical let us summarize the main idea here. Using the spectral flow symmetry one can determine the NS sector character (without an insertion of y^{J_0} or $(-1)^F$) from the elliptic genus. This character is a modular form for a congruence subgroup Γ_θ of the modular group. It is therefore highly constrained, and as in the case discussed in [186], determined by the coefficients of the negative powers of q, which in turn are fixed by the polar terms of the original elliptic genus. On the other hand, given the full NS sector character, we can also determine from it, by a suitable modular transformation, the R-sector character (without an insertion of $(-1)^F$), and thus, in particular, its leading term in the q-expansion. This coefficient is however also directly determined by the extremal hypothesis and a sum rule (6.5.91) for Fourier coefficients. The two ways of evaluating the same coefficient lead to a non-trivial constraint on m, equation (6.5.90). Using properties of modular forms one can show that this constraint is violated for sufficiently large m. The argument must be broken up into cases: m odd, $m = 2 \bmod 4$ and $m = 0 \bmod 4$, of which the last case is technically the most difficult. In this section we will give the main line of argument, whereas the technical details can be found in appendix D.1.

6.5.1 NS-sector elliptic genus

Suppose $\chi(\tau, z)$ is the elliptic genus of a CFT with $\chi \in \tilde{J}_{0,m}$. By spectral flow we define the "NS sector elliptic genus" to be[5]

$$\chi_{NS}(\tau, z) := e\left[m\left(\frac{\tau}{4} + z + \frac{1}{2}\right)\right] \chi\left(\tau, z + \frac{\tau}{2} + \frac{1}{2}\right) . \tag{6.5.72}$$

Using the transformation properties of a Jacobi form it follows easily that

$$\begin{aligned} \chi_{NS}(-1/\tau, z/\tau) &= (-1)^m e\left(\frac{mz^2}{\tau}\right) \chi_{NS}(\tau, z) \\ \chi_{NS}(\tau + 2, z) &= (-1)^m \chi_{NS}(\tau, z) . \end{aligned} \tag{6.5.73}$$

If we put $z = 0$ we thus obtain simple transformation laws for $\chi_{NS}(\tau) := \chi_{NS}(\tau, 0)$ under the congruence subgroup $\Gamma_\theta = \langle T^2, S\rangle$. For m even we have a strict modular function and for m odd we have a function with multiplier system given by -1 on the two generators.

To begin, let us sketch a few mathematical facts. The group Γ_θ is a genus zero subgroup of Γ. It has modular domain $\mathcal{F}_\theta = \mathcal{F} \cup T \cdot \mathcal{F} \cup TS \cdot \mathcal{F}$ shown in figure 6.2. Note there are two cusps, equivalent to $\tau = i\infty$ and $\tau = 1$.

Since $\mathbb{H}/\Gamma_\theta$ is genus zero the function field has a generator $\hat{K}(\tau)$ which can be uniquely specified (up to an additive and multiplicative constant) by demanding that $\hat{K}$ takes $i\infty$

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[5] Note that unlike the NS vacuum character (6.3.40), $\chi_{NS}(\tau, z)$ does not involve the shift of z by 1/2.

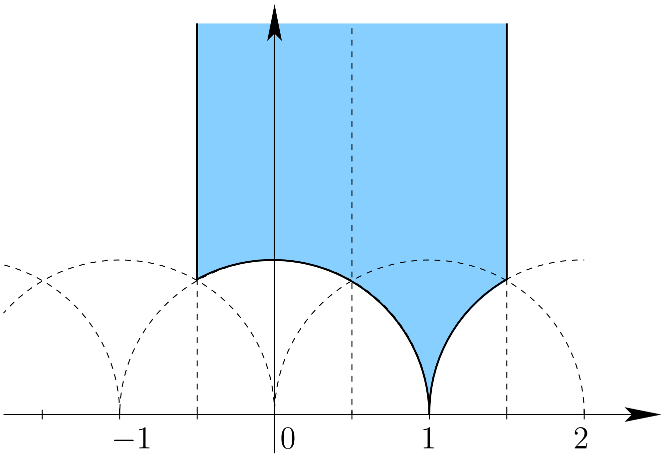

Figure 6.2: The fundamental domain $\mathcal{F}_\theta$ of the genus zero subgroup Γ_θ of Γ.

to ∞.[6] An explicit choice is:

$$\hat{K} := \frac{\vartheta_3^{12}}{\eta^{12}} = \frac{\Delta^2(\tau)}{\Delta(2\tau)\Delta(\tau/2)} = q^{-1/2} + 24 + 276q^{1/2} + \cdots \ . \tag{6.5.74}$$

The expansion of $\hat{K}$ around the cusp at $\tau = 1$ is obtained by writing $\tau = 1 - \frac{1}{\tau_r}$ and observing that

$$\hat{K}(\tau) := \tilde{K}(\tau_r) = -\frac{\vartheta_2^{12}}{\eta^{12}}(\tau_r) = -2^{12}q_r + \cdots \ , \tag{6.5.75}$$

where $q_r = e(\tau_r)$.

In order to work with the case of m odd it will be useful to consider the index two subgroup[7] $\tilde{\Gamma}_\theta := \langle T^4, ST^2\rangle$ such that $\Gamma_\theta = \tilde{\Gamma}_\theta \cup S\cdot\tilde{\Gamma}_\theta$. This is again a genus zero subgroup, and its Hauptmodul is the NS-sector character of $\tilde{\phi}_{0,1}$ (*i.e.* the elliptic genus for $K3$ divided by two). Applying (6.5.72) to $\tilde{\phi}_{0,1}$ one finds

$$\kappa(\tau) := \left(\frac{2\vartheta_4}{\vartheta_2}\right)^2 - \left(\frac{2\vartheta_2}{\vartheta_4}\right)^2 = q^{-1/4}(1 - 20q^{1/2} + \cdots) \ . \tag{6.5.76}$$

This function satisfies $\kappa|_S = -\kappa$ and $\kappa|_{T^2} = -\kappa$, and is thus odd under the Deck transformation $\mathbb{H}/\tilde{\Gamma}_\theta \to \mathbb{H}/\Gamma_\theta$. Indeed,

$$\kappa^2(\tau) = \hat{K}(\tau) - 64 \ , \tag{6.5.77}$$

giving the explicit double cover. Near the Ramond cusp κ has the expansion

$$\kappa(1 - 1/\tau_r) := \tilde{\kappa}(\tau_r) = -4i\left[\left(\frac{\vartheta_3}{\vartheta_4}\right)^2 + \left(\frac{\vartheta_4}{\vartheta_3}\right)^2\right](\tau_r) = -8i\Big(1 + 32q_r + \mathcal{O}(q_r^2)\Big) \ . \tag{6.5.78}$$

---

[6] Such a function for a genus zero congruence subgroup is often referred to as a "Hauptmodul."

[7] To prove the subgroup is index two note that for all $n \in \mathbb{Z}$, T^{4n}, ST^{4n+2}, $T^{4n+2}S$ and $ST^{4n}S$ are in $\tilde{\Gamma}_\theta$. Then use induction on the length of the word in S, T^2.

Now, χ_{NS} has no singularities for $\tau \in \mathbb{H}$, and, moreover, using again the transformation laws of a Jacobi form

$$\chi_{NS}(1-1/\tau_r) = e\left(-\frac{m}{4}\right)\chi(\tau_r,\frac{1}{2}) = e\left(-\frac{m}{4}\right)\sum_{n,\ell} c(n,\ell)\,(-1)^{\ell}\,q_r^n \ . \tag{6.5.79}$$

By unitarity the sum is over $n \geq 0$ and hence $\chi_{NS}(\tau)$ must be a polynomial in $\kappa(\tau)$. This polynomial will be even for m even and odd for m odd. Moreover, the polynomial is fixed by the coefficients of the nonpositive powers of q. Those coefficients in turn are related to the polar contributions to $\chi(\tau,z)$. To demonstrate the relationship note that

$$\chi_{NS}(\tau) = \sum_{n,\ell} c(n,\ell)\,(-1)^{m+\ell}\,q^{\frac{m}{4}+n+\frac{\ell}{2}} \ . \tag{6.5.80}$$

Now write:

$$4mn-\ell^2 = 4m\left(\frac{m}{4}+n+\frac{\ell}{2}\right)-(m+\ell)^2 \ . \tag{6.5.81}$$

The nonpolar terms in $\chi(\tau,z)$ have $4mn-\ell^2 \geq 0$ and therefore from (6.5.81) contribute only nonnegative powers of q in (6.5.80). In fact, they always contribute positive powers with precisely one exception: when $4mn-\ell^2=0$ and $\ell=-m$. In that case $n=m/4$. Note that this cannot happen if $m \neq 0 \bmod 4$ because n is integral.

6.5.2 A nontrivial constraint

In this subsection we assume $m \neq 0 \bmod 4$. We return to a discussion of the case $m = 0 \bmod 4$ in subsection 6.5.3 below.

Our conclusion thus far is that for $m \neq 0 \bmod 4$, $\chi_{NS}(\tau)$ is a modular function for $\tilde{\Gamma}_\theta$ such that

$$\chi_{NS}(\tau) = \sum_{\theta\in\mathbb{Z}} SF_\theta \chi^{(m)}_{vac}(\tau,z)|_{z=\frac{1}{2}} + \mathcal{O}(q^{1/4}) \ . \tag{6.5.82}$$

One easily finds that only $\theta=0$ can contribute to negative powers of q and hence we can simplify this equation to

$$\chi_{NS}(\tau) = q^{-m/4}(1-q)\prod_{n=1}^{\infty}\frac{(1+q^{n+1/2})^2}{(1-q^n)^2} + \mathcal{O}(q^{1/4}) \ . \tag{6.5.83}$$

This has expansion

$$q^{-m/4}\left(1+q+2q^{3/2}+3q^2+4q^{5/2}+6q^3+\cdots\right) \ . \tag{6.5.84}$$

While the expression on the RHS of (6.5.83) is not modular, it can be written as:

$$\chi_{NS}(\tau) = q^{-m/4}\frac{1-q^{1/2}}{1+q^{1/2}}q^{1/8}\frac{\vartheta_3}{\eta^3} + \mathcal{O}(q^{1/4}) \ . \tag{6.5.85}$$

Now we can write an explicit formula for $\chi_{NS}(\tau)$. Define expansion coefficients:

$$q^{-m/4}q^{1/8}\frac{1-q^{1/2}}{1+q^{1/2}}\frac{\vartheta_3}{\eta^3} = \sum_{\alpha=-m/4}^{\infty}\tilde{h}(\alpha)q^{\alpha} \ . \tag{6.5.86}$$

Note that $\tilde{h}(\alpha)$ is only nonzero for $\alpha \in \frac{1}{2}\mathbb{Z}$, for m even and $\frac{1}{4}+\frac{1}{2}\mathbb{Z}$ for m odd. For $\alpha \in \frac{1}{4}\mathbb{Z}_+$ let $\wp_\alpha$ be the unique polynomial of degree 4α such that

$$\wp_\alpha(\kappa) = q^{-\alpha} + \mathcal{O}(q^{1/4}) \ , \qquad \alpha \in \frac{1}{4}\mathbb{Z}_+ \ . \tag{6.5.87}$$

Then for $m \neq 0 \bmod 4$

$$\chi_{NS} = \sum_{\alpha=-m/4}^{0} \tilde{h}(\alpha)\, \wp_{-\alpha}(\kappa) \ . \tag{6.5.88}$$

On the other hand, if we expand around the cusp $\tau = 1$ then, by (6.5.79)

$$\sum_{\alpha=-m/4}^{0} \tilde{h}(\alpha)\, \wp_{-\alpha}(\tilde{\kappa}(\tau_r)) = e^{-i\pi m/2} \sum_{n,\ell\in\mathbb{Z}} c(n,\ell)\,(-1)^\ell\, q_r^n \ . \tag{6.5.89}$$

In particular, if we take $\tau_r \to i\infty$, then we arrive at the key constraint:

$$L := \sum_{\alpha=-m/4}^{0} \tilde{h}(\alpha)\, \wp_{-\alpha}(-8i) = e^{-i\pi m/2} \sum_{\ell} c(0,\ell)\,(-1)^\ell \ . \tag{6.5.90}$$

The argument for the non-existence of the extremal elliptic genus is based on showing that, for large m, the left-hand side and right-hand side of (6.5.90) have different growth rates. As we shall see momentarily, the right-hand side is always an affine linear function of m, while the left-hand side grows exponentially for $m = 2 \bmod 4$; for m odd, the left-hand side grows also linearly in m, but the coefficient is different.

Let us first establish the growth property of the right-hand side. By the ansatz for pure supergravity we know that the only nonzero polar coefficients $c(0,\ell)$ occur for $\ell = \pm m$ and are given by 1. The coefficient $c(0,0)$ is *not* polar. Fortunately, Gritsenko has proven a useful identity for the Fourier coefficients of weak Jacobi forms of index m [117]: [8]

$$m \sum_{\ell} c(0,\ell) = 6 \sum_{\ell} \ell^2 c(0,\ell) \ . \tag{6.5.91}$$

Using (6.5.91) and (6.3.49) we can solve for $c(0,0)$ to get $c(0,0) = 12m - 2$, and therefore

$$\sum_{\ell} c(0,\ell)(-1)^\ell = 12m - 2 + 2(-1)^m = \begin{cases} 12m & m \ \text{ even} \\ 12m - 4 & m \ \text{ odd.} \end{cases} \tag{6.5.92}$$

In particular, the right-hand side of (6.5.90) grows linearly with m.

Now let us turn to the left-hand side of (6.5.90). Observe that this is the q^0 term in the q-expansion of

$$\left(\sum_{\alpha \geq -m/4} \tilde{h}(\alpha) q^\alpha \right) \left(\sum_{n\geq 0} q^{n/4} \wp_{n/4}(-8i) \right) \ . \tag{6.5.93}$$

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[8] The proof is very simple: $\exp[-8\pi^2 m G_2(\tau) z^2]\chi(\tau,z)$ transforms as a weight zero modular form. Therefore the coefficients of z^{2n} in the Taylor series around $z=0$ transform like forms of weight $2n$. In particular the coefficient of z^2 must vanish, since there are no modular forms of weight two.

On the other hand, using the fact that κ is a Hauptmodul one can show, that [9]

$$\sum_{n=0}^{\infty} q^{n/4} \wp_{n/4}(z) = \frac{4q\frac{d}{dq}\kappa}{z-\kappa} \ . \tag{6.5.94}$$

In order to apply this to our problem we use the identities [10]

$$\begin{aligned} 24q\frac{d}{dq}\log\vartheta_4 &= E_2 - (\vartheta_2^4+\vartheta_3^4) \\ 24q\frac{d}{dq}\log\vartheta_3 &= E_2 + (\vartheta_2^4-\vartheta_4^4) \\ 24q\frac{d}{dq}\log\vartheta_2 &= E_2 + (\vartheta_3^4+\vartheta_4^4) \end{aligned} \tag{6.5.95}$$

to compute $4q\frac{d}{dq}\kappa = -4\vartheta_3^8/(\vartheta_2^2\vartheta_4^2)$. Using the "abstruse identity" $\vartheta_3^4 = \vartheta_2^4 + \vartheta_4^4$ it follows that

$$\sum_{n=0}^{\infty} q^{n/4} \wp_{n/4}(-8i) = (\vartheta_4^2 - i\vartheta_2^2)^2 \ . \tag{6.5.96}$$

Thus, we need to estimate the large m behavior of

$$L := \left[q^{-\frac{m}{4}+\frac{1}{8}} \frac{1-q^{1/2}}{1+q^{1/2}} \frac{\vartheta_3}{\eta^3} (\vartheta_4^2 - i\vartheta_2^2)^2 \right]_{q^0} \ . \tag{6.5.97}$$

We estimate the growth behavior of L in appendix D.1, and it turns out to be quite different for even and odd.

For m odd, $e^{i\pi m/2}L$ is positive, and is bounded below by

$$e^{i\pi m/2} L \geq 4\pi m - 8\pi\sqrt{m-\frac{5}{2}} - 6\pi \ . \tag{6.5.98}$$

Since $4\pi > 12$, this will asymptotically (*i.e.* for $m \geq 2000$) grow more quickly than (6.5.92). We have checked that among the first 2000 terms, the two numbers only agree

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[9] Write $\wp_\alpha(z) = \oint_C \frac{\wp_\alpha(\ell)}{\ell - z}\frac{d\ell}{2\pi i}$ where the contour is on a large circle C in the ℓ plane. Now make the change of variables $\ell \to \ell(r) := r^{-1} - 20r + \cdots$ so that $\ell(q^{1/4}) = \kappa$. This gives a one-one map of C to a small circle C' around the origin. Using $\wp_\alpha(\ell(r)) = r^{-4\alpha} + \mathcal{O}(r)$, and taking the circle to be small we see $\wp_\alpha(z) = -\oint_{C'} \frac{\ell'(r) r^{-4\alpha}}{\ell(r)-z}\frac{dr}{2\pi i}$. It is now straightforward to sum the series and apply Cauchy's theorem to arrive at (6.5.94). We thank Terry Gannon for pointing out this crucial identity to us.

[10] To prove these identities note that $(24q\frac{d}{dq} - E_2)\vartheta_2$ must be a weight 5/2 modular form for $\Gamma(2)$ and hence is a polynomial of degree 5 in $\vartheta_2, \vartheta_3, \vartheta_4$. Moreover, the q expansion has only coefficients $q^{\frac{1}{8}+n}$ with n integer. Together with the transformation property under $\tau \to \tau + 1$ this fixes it to be of the form $\vartheta_2(a(\vartheta_3^4+\vartheta_4^4) + b\vartheta_3\vartheta_4(\vartheta_3^2+\vartheta_4^2) + c\vartheta_3^2\vartheta_4^2)$ for some constants a, b, c. Now, matching the first 3 coefficients of the q expansion on the left and right hand sides we find $a = 1, b = c = 0$. The other two equations now follow by modular transformations. These identities also have nice interpretations in terms of massless free fermions on a two-dimensional torus. One can compute the expectation value of their energy either by differentiating their partition function or by evaluating the energy-momentum tensor using the fermion two-point function. Requiring that these two methods produce the same answer implies these identities [62].

for $m = 1, 3, 5, 7, 11, 13, 19, 31, 41$. For $m = 1, 3, 5, 7, 11, 13$ there exists indeed a sugra elliptic genus, while for $m = 19, 31, 41$ there does not, as we have verified explicitly. (Note that the fact that the two numbers agree does not imply that there must exist a sugra elliptic genus!)

For $m = 2 \bmod 4$, L turns out to grow exponentially, so that (6.5.90) cannot be satisfied for m large enough. For details of the calculation, see again appendix D.1.

6.5.3 A constraint for $m = 0 \bmod 4$

We now turn to the case $m = 0 \bmod 4$. As we have pointed out above, in this case non-polar terms contribute to the constant term of χ_{NS}. We thus need to make the more general ansatz

$$\chi_{NS}(\tau, z) = q^{-\frac{m}{4}+\frac{1}{8}} \frac{1-q}{(1+yq^{1/2})(1+y^{-1}q^{1/2})} \frac{\vartheta_3(\tau, z)}{\eta^3} + d + \mathcal{O}(q^{1/2}) \ . \tag{6.5.99}$$

Instead of (6.5.88) we obtain

$$\chi_{NS} = \sum_{\alpha=-m/4}^{0} \tilde{h}(\alpha) \wp_{-\alpha}(\kappa) + d \ . \tag{6.5.100}$$

The argument of section 6.5.2 can then be used to fix the value of d:

$$d = 12m - \left[q^{-\frac{m}{4}+\frac{1}{8}} \frac{1-q^{1/2}}{1+q^{1/2}} \frac{\vartheta_3(\tau)}{\eta^3} (\vartheta_4^4 - \vartheta_2^4) \right]_{q^0} \ . \tag{6.5.101}$$

We obtain an additional constraint on the theory in the following way: Let

$$\hat{D} := \left(y \frac{d}{dy} \right)^2 - \frac{m}{6} E_2 \ . \tag{6.5.102}$$

Then $\hat{\chi}_{NS}(\tau) := \hat{D}(\chi_{NS}(\tau, z))|_{z=0}$ is a weight two weakly holomorphic modular form for Γ_θ which moreover satisfies

$$\hat{\chi}_{NS}(1 - 1/\tau_r) = \tau_r^2 \hat{D}(\chi(\tau, z))|_{z=1/2} \ . \tag{6.5.103}$$

The $q_r \longrightarrow 0$ limit of the coefficient of τ_r^2 of the right-hand-side of (6.5.103) is

$$\sum_\ell c(0, \ell)(-1)^\ell \ell^2 - \frac{m}{6} \sum c(0, \ell)(-1)^\ell = 2m^2 - \frac{m}{6} 12m = 0 \ . \tag{6.5.104}$$

On the other hand, weakly holomorphic modular forms of weight two for Γ_θ are of the form

$$(\vartheta_2^4 - \vartheta_4^4) \times L(\hat{K}) \ , \tag{6.5.105}$$

where $L(\hat{K})$ is a Laurent series in $\hat{K}$. By examining the Ramond cusp we see that $L(\hat{K})$ must be a polynomial in $\hat{K}$. Define polynomials $P_a(\hat{K}) = q^{-a/2} + \mathcal{O}(q^{1/2})$ for $a \geq 0$ and

$$\tilde{P}_a(\hat{K})(\vartheta_2^4 - \vartheta_4^4) = \begin{cases} 1 + \mathcal{O}(q^{1/2}) & a = 0 \\ aq^{-a/2} + \mathcal{O}(q^{1/2}) & a > 0 \ . \end{cases} \tag{6.5.106}$$

Using (6.5.95) we find

$$2q\frac{d}{dq}\hat{K}=\hat{K}(\vartheta_2^4-\vartheta_4^4)\ , \tag{6.5.107}$$

from which we deduce

$$\tilde{P}_a(z)=\begin{cases}-1 & a=0\\ -zP_a'(z) & a>0\ .\end{cases} \tag{6.5.108}$$

Define expansion coefficients

$$\hat{\chi}_{NS}(\tau)=\sum_{\alpha=-m/4}(-2\alpha)x(\alpha)q^\alpha+X(0)\ . \tag{6.5.109}$$

If the extremal elliptic genus exists then

$$\hat{\chi}_{NS}(\tau)=\sum_{\alpha<0}x(\alpha)\tilde{P}_{-\alpha}(\hat{K})(\vartheta_2^4-\vartheta_4^4)-X(0)(\vartheta_2^4-\vartheta_4^4)\ . \tag{6.5.110}$$

Evaluating at the Ramond cusp we have

$$\tau_r^2\left(X(0)(\vartheta_4^4+\vartheta_3^4)-\sum_{\alpha<0}x(\alpha)\tilde{P}_{-\alpha}(\tilde{K})(\vartheta_4^4+\vartheta_3^4)\right)\ , \tag{6.5.111}$$

and evaluating at $q_r\to 0$ the coefficient of τ_r^2 becomes simply $2X(0)$ since $\tilde{P}_\alpha(0)=0$ for $\alpha>0$. Therefore, $X(0)=0$.

On the other hand, we can deduce the coefficient $X(0)$ directly from the q^0 term of $\hat{D}\chi_{NS}$. Expressing χ_{NS} by (6.5.99) and (6.5.101) and then using

$$\begin{aligned}(y\partial_y)^2\frac{1}{(1+yq^{1/2})(1+y^{-1}q^{1/2})}\bigg|_{y=1} &= -\frac{2q^{1/2}}{(1+q^{1/2})^4}\ , && (6.5.112)\\ y\partial_y\vartheta_3\Big|_{y=1} &= 0\ , && (6.5.113)\\ (y\partial_y)^2\vartheta_3\Big|_{y=1} &= 2q\partial_q\vartheta_3\ , && (6.5.114)\end{aligned}$$

and (6.5.95), we obtain the constraint

$$\begin{aligned}0 &= \left[\hat{D}\chi_{NS}\right]_{q^0}=\left[(y\partial_y)^2\chi_{NS}-\frac{m}{6}E_2\,\chi_{NS}\right]_{q^0}\\ &= -2m^2+\left[q^{-m/4+1/8}\frac{1-q^{1/2}}{1+q^{1/2}}\frac{-2q^{1/2}}{(1+q^{1/2})^2}\frac{\vartheta_3}{\eta^3}\right]_{q^0}\\ &\qquad -(4m-2)\left[q^{-m/4+1/8}\frac{1-q^{1/2}}{1+q^{1/2}}\frac{q\partial_q\vartheta_3}{\eta^3}\right]_{q^0}\\ &= -2m^2-R_1-(4m-2)\,R_2\ ,\end{aligned} \tag{6.5.115}$$

where R_1 and R_2 are defined as

$$\begin{aligned}R_1 &= \left[2\,q^{1/2}\,\frac{(1-q^{1/2})^4}{(1-q)^3}\frac{\vartheta_3}{\eta^3}\right]_{q^{\frac{m}{4}-\frac{1}{8}}} && (6.5.116)\\ R_2 &= \left[\frac{(1-q^{1/2})^2}{1-q}\frac{q\partial_q\vartheta_3}{\eta^3}\right]_{q^{\frac{m}{4}-\frac{1}{8}}}\ . && (6.5.117)\end{aligned}$$

In appendix D.1 we show that for large enough m both R_1 and R_2 are positive. It is then clear that (6.5.115) cannot be satisfied.

6.6 Near-extremal $\mathcal{N} = 2$ conformal field theories

In section 6.5 we showed that $\mathcal{N} = 2$ ECFT's, as we have defined them, at best exist only for a finite number of sporadic values of m. One might object that our definition is too narrow, and that we should simply modify the definition of an extremal theory.

In this section we consider one way of modifying the notion of an extremal theory, by demanding only that some "significant" fraction of the polar degeneracies $c(n, \ell)$ coincide with those predicted from the vacuum character.

Returning to the system of equations (6.4.67), for fixed m define $k(m)$ to be the largest integer such that

$$\sum_{i=1}^{j(m)} x_i N_{ia} = d_a \ , \qquad\qquad a = 1, \dots, k(m) \tag{6.6.118}$$

admits a solution x_i for which the elliptic genus $\sum x_i \phi_i$ has an integral Fourier expansion. We would like to show that we can choose $k(m)$ to be "close" to $P(m)$.

Turning again to a numerical analysis, we studied the truncation of (6.6.118) to the first $j(m)$ equations: $1 \le a \le j(m)$ where we ordered the polar terms via their polarity. We found that in all cases $1 \le m \le 36$ there is indeed a solution x_i in rational numbers. Moreover, for all m *except* $m = 17$ the Fourier expansion coefficients are integral — in so far as we have tested them. This indicates that $k(m) = j(m) + \mathcal{O}(1)$.[11] We conjecture that this is the case in general, and in section 6.6.1, assuming this conjecture to be true, we derive an interesting constraint on the spectrum of $\mathcal{N} = 2$ CFTs.

For the analysis in section 6.6.1 it turns out to be more convenient to define a "β-extremal $\mathcal{N} = 2$ CFT" by imposing the less restrictive condition of only requiring that polar degeneracies are predicted from the vacuum character in the β-truncated polar region:

$$\mathcal{P}_\beta := \{(\ell, n) : 1 \le \ell \le m, n \ge 0, 4mn - \ell^2 \le -\beta\} \ . \tag{6.6.119}$$

We know that for suitable β candidate elliptic genera exist. For example, if we take $\beta = m^2$ then we can always construct a candidate elliptic genus. We get a better approximation to an extremal theory by lowering the value of β. Therefore, let $P_\beta(m)$ be the number of independent polar monomials of polarity $\le -\beta$, and let β_* be the *smallest* integer β such that

$$\sum_{i=1}^{j(m)} x_i N_{ia} = d_a \ , \qquad\qquad a = 1, \dots, P_\beta(m) \tag{6.6.120}$$

admits a solution x_i for which $\sum x_i \phi_i$ has *integral* coefficients in its Fourier expansion. According to our conjecture $P_{\beta_*}(m) \cong j(m)$. We would therefore like to estimate the value of β for which $P_\beta(m) = j(m) + \mathcal{O}(m^{1/2})$ for large m. The computation follows closely the analysis of section 6.2.2.

---

[11] Note that at least for the sporadic solutions $m = 7, 8, 11, 13$ we have $k(m) > j(m)$.

We now have

$$P_\beta(m) = \sum_{r=r_0}^{m} \lceil \frac{r^2-\beta}{4m} \rceil \ , \tag{6.6.121}$$

where $r_0 := \lceil\sqrt{\beta}\rceil$. As before, we write this as a sum of three terms,

$$P_\beta(m) = \sum_{r=r_0}^{m} \frac{r^2-\beta}{4m} - \sum_{r=r_0}^{m} ((\frac{r^2-\beta}{4m})) + \frac{1}{2}\sum_{r=r_0}^{m} \left(\lceil \frac{r^2-\beta}{4m} \rceil - \lfloor \frac{r^2-\beta}{4m} \rfloor\right) \ . \tag{6.6.122}$$

The first term is

$$\sum_{r=r_0}^{m} \frac{r^2-\beta}{4m} = \frac{m^2}{12} + \frac{m}{8} + \frac{1}{24} - \frac{r_0(2r_0-1)(r_0-1)}{24m} - \beta\frac{(m-r_0+1)}{4m} \ . \tag{6.6.123}$$

Denote the number of integers r such that $r_0 \leq r \leq m$ with $r^2 = \beta \bmod 4m$ by $\nu(m,\beta)$. Unlike the case $\beta = 0$ we cannot write down an exact formula, but it is clear that asymptotically $\nu(m,\beta) \sim m^{1/2}$. The second term is

$$\sum_{r=r_0}^{m} \lceil \frac{r^2-\beta}{4m} \rceil - \sum_{r=r_0}^{m} \lfloor \frac{r^2-\beta}{4m} \rfloor = m + 1 - r_0 - \nu(m,\beta) \ . \tag{6.6.124}$$

For the third term we again use the argument that the numbers $((\frac{r^2-\beta}{4m}))$ are randomly distributed. We thus have a random walk between $-1/2$ and $+1/2$ and the sum is expected to be of order $m^{1/2}$.

To conclude, note that for $\beta = \alpha m$ with α a constant $0 < \alpha < 1$ we have $r_0 \sim m^{1/2}$, so the large m asymptotics are

$$P_\beta(m) = \frac{m^2}{12} + \left(\frac{5}{8} - \frac{\alpha}{4}\right) m + \mathcal{O}(m^{1/2}) \ . \tag{6.6.125}$$

Comparing to (6.2.24) we see that for large m the reduction of polarity to obtain the truncated supergravity elliptic genus is given by $\beta = \frac{1}{2}m + \mathcal{O}(m^{1/2})$.

6.6.1 A constraint on the spectrum of $\mathcal{N} = 2$ theories with integral $U(1)$ charges

In the previous sections we have found strong evidence that we must have $P_{\beta_*}(m) \cong j(m)$, and hence by (6.6.125)

$$\beta_* \geq \frac{m}{2} + \mathcal{O}(m^{1/2}) \tag{6.6.126}$$

for large m.

Now a monomial $q^n y^\ell$ of polarity β corresponds by spectral flow to a state in the NS sector that contributes as $q^{h-\frac{m}{4}} y^\ell$ with

$$h = \frac{m}{4} + \frac{\ell^2}{4m} - \frac{\beta}{4m} \ . \tag{6.6.127}$$

Therefore, if we accept (6.6.126) then we can obtain an interesting constraint on the spectrum of a $(2,2)$ AdS_3 supergravity with a holographically dual CFT: It must contain at least one state which is a left-moving $\mathcal{N}=2$ primary (not necessarily chiral primary) tensored with a right-moving chiral primary such that

$$h < \frac{m}{4} + \frac{\ell^2}{4m} - \frac{1}{8} + \mathcal{O}(m^{-1/2}) \ . \tag{6.6.128}$$

It would be interesting and useful to sharpen this bound. However, we will show in section 6.7 below that it *is* possible to construct elliptic genera, which, after spectral flow, do match the spectrum of the vacuum character for all conformal weights with $h \leq \frac{m}{4}$. There is no contradiction between this result and (6.6.128) because under 1/2 unit of spectral flow $0 \leq |\ell| \leq 2m$ and hence $\frac{\ell^2}{4m}$ could be as large as m, and thus the bound can be as large as $\frac{5m}{4} - \frac{1}{8} + \mathcal{O}(m^{-1/2})$.

6.7 Construction of nearly extremal elliptic genera

In this section we consider an alternative basis for the weak Jacobi forms which has a "triangular" nature, allowing us to replace the polar region $\mathcal{P}^{(m)}$ by an alternative region S. We will see that for large m, S "approximates" $\mathcal{P}^{(m)}$. In the next section we discuss the possible physical significance of this fact.

It is shown in [117] that there is an *integral* basis of the ring of weak Jacobi forms of weight zero with integral coefficients

$$\tilde{J}^{\mathbb{Z}}_{0,*} = \mathbb{Z}[\phi_{0,1}, \phi_{0,2}, \phi_{0,3}, \phi_{0,4}]/I \ , \tag{6.7.129}$$

where I is the ideal generated by the relation

$$\phi_{0,1}\phi_{0,3} = 4\phi_{0,4} + \phi_{0,2}^2 \ . \tag{6.7.130}$$

The generators are elliptic genera of Calabi-Yau manifolds, and explicit formulae are given in [117]. In the basis (6.2.22) they can be expressed as[12]

$$\begin{aligned}
\phi_{0,1} &= \tilde{\phi}_{0,1} && (6.7.131)\\
\phi_{0,2} &= \frac{1}{24}\tilde{\phi}_{0,1}^2 - \frac{1}{24}\tilde{\phi}_{-2,1}^2 E_4 && (6.7.132)\\
\phi_{0,3} &= \frac{1}{432}\tilde{\phi}_{0,1}^3 - \frac{1}{144}\tilde{\phi}_{0,1}\tilde{\phi}_{-2,1}^2 E_4 + \frac{1}{216}\tilde{\phi}_{-2,1}^3 E_6 && (6.7.133)\\
\phi_{0,4} &= \frac{1}{6912}\tilde{\phi}_{0,1}^4 - \frac{1}{1152}\tilde{\phi}_{0,1}^2\tilde{\phi}_{-2,1}^2 E_4 + \frac{1}{864}\tilde{\phi}_{0,1}\tilde{\phi}_{-2,1}^3 E_6 - \frac{1}{2304}\tilde{\phi}_{-2,1}^4 E_4^2 \ . && (6.7.134)
\end{aligned}$$

To make the triangular nature of this basis manifest it is useful to consider the NS sector images of the generators,

$$\hat{\phi}_{0,m} = (-1)^m q^{m/4} y^m \phi_{0,m}(\tau, z + \frac{\tau}{2} + \frac{1}{2}) \ . \tag{6.7.135}$$

---

[12] We have redefined $\phi_{0,4}$ in [117] by a factor of -1.

We now consider ordering the q, y expansion by the leading power of q and, for each power of q by the *largest* positive power of y. (Recall that $\chi_{NS}(\tau, z)$ is an even function of z, so the positive powers of y determine the negative powers of y.) With this ordering of terms we have

$$\begin{aligned} \hat{\phi}_{0,1} &= q^{-1/4} + \mathcal{O}(q^{1/4}) \\ \hat{\phi}_{0,2} &= (y + y^{-1}) + \mathcal{O}(q^{1/2}) \\ \hat{\phi}_{0,3} &= q^{1/4}(y - y^{-1})^2 + \mathcal{O}(q^{3/4}) \\ \hat{\phi}_{0,4} &= 1 + \mathcal{O}(q^{1/2}) \ . \end{aligned} \tag{6.7.136}$$

By (6.7.129) an overcomplete linear basis of $\tilde{J}_{0,m}$ is given by

$$(\hat{\phi}_{0,1})^i (\hat{\phi}_{0,2})^j (\hat{\phi}_{0,3})^k (\hat{\phi}_{0,4})^l \tag{6.7.137}$$

with $i + 2j + 3k + 4l = m$, $i, j, k, l \geq 0$. In order to obtain a set of linearly independent basis vectors we distinguish the monomials in (6.7.137) according to whether $i > k$ or $i \leq k$ and then use identity (6.7.130) to eliminate $\hat{\phi}_{0,3}$ or $\hat{\phi}_{0,1}$, respectively. The result is that there exists a vector space basis for $\tilde{J}_{0,m}$ which is a disjoint union of two sets $A \amalg B$ with

$$A := \{(\hat{\phi}_{0,1})^i (\hat{\phi}_{0,2})^j (\hat{\phi}_{0,4})^l | \quad i > 0, j \geq 0, l \geq 0 \qquad i + 2j + 4l = m\} \ , \tag{6.7.138}$$

$$B := \{(\hat{\phi}_{0,2})^j (\hat{\phi}_{0,3})^k (\hat{\phi}_{0,4})^l | \quad j \geq 0, k \geq 0, l \geq 0 \qquad 2j + 3k + 4l = m\} \ . \tag{6.7.139}$$

A tedious but elementary counting argument shows that

$$|A| = \begin{cases} \frac{m^2}{16} + \frac{3m}{8} - \frac{s^2}{16} + \frac{s}{8} + \frac{1}{2} & m = s \bmod 4, s = 1,3 \\ \frac{m^2}{16} + \frac{m}{4} - \frac{s^2}{16} + \frac{s}{4} & m = s \bmod 4, s = 0,2 \end{cases} \tag{6.7.140}$$

and $|A| + |B| = j(m)$.

Now note that the leading expression in the q, y expansion of an element in the set A is $q^{-i/4} y^j$, while that in the set B is $q^{k/4} y^{j+2k}$. It thus follows that *an (NS-sector) Jacobi form of weight zero and index* m *with integral Fourier coefficients is uniquely determined by the coefficients of* $q^n y^\ell$ *where* (ℓ, n) *run over the set:*

$$S = S_A \amalg S_B \tag{6.7.141}$$

where

$$S_A = \Big\{(\ell, n) | n < 0, \quad 0 \leq \ell, \quad n + \frac{m}{4} \geq \frac{\ell}{2}\Big\} \tag{6.7.142}$$

and

$$S_B = \Big\{(\ell, n) | 0 \leq n, \quad 8n \leq \ell, \quad n + \frac{m}{4} \geq \frac{\ell}{2}\Big\} \ . \tag{6.7.143}$$

In both S_A and S_B the (ℓ, n) are in the lattice $(\ell, n) \in \mathbb{Z} \times \frac{1}{4}\mathbb{Z}$, subject to the quantization condition

$$\Big(n + \frac{m}{4}\Big) - \frac{\ell}{2} = 0 \mod 1 \ . \tag{6.7.144}$$

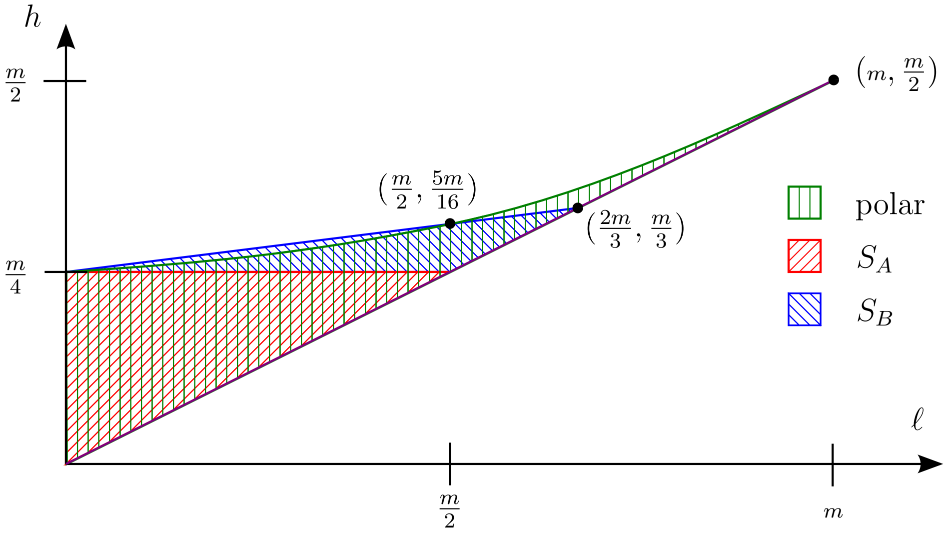

Figure 6.3: A comparison of the polar region $\mathcal{P}^{(m)}$ and the region S. The NS sector polar region is bounded by $\ell \geq 0, h \geq \ell/2, h \leq \frac{m}{4} + \frac{\ell^2}{4m}$. The region S is the triangular region, $\ell \geq 0, h \geq \frac{\ell}{2}, h - \frac{m}{4} \leq \frac{\ell}{8}$, which itself is a union of two triangular regions S_A and S_B, where S_A is the subregion of S with $h < \frac{m}{4}$. The polar region contains S_A, while S_B is an "approximation" to the remainder.

(This quantization is equivalent to the statement that in the Ramond sector the elliptic genus has a Fourier expansion in q, y with integral powers of q, y.) The regions S_A and S_B in the (ℓ, h) plane are triangles and their union is a triangle. The full region S can serve as a surrogate for the polar region $\mathcal{P}^{(m)}$, as explained in figure 6.3.

Recall that n, the power of q in the NS sector character, is related to h as $n = h - \frac{m}{4}$. It then follows from (6.7.142) that S_A contains all possible points with $h < m/4$ that occur in the NS vacuum character (6.3.40). Thus it *is* possible to construct a weak Jacobi form with integral coefficients whose q-expansion agrees with that of an extremal theory for all NS-sector Virasoro weights up to $h = m/4$ (for m even) and $h = (m-1)/4$ (for m odd). This fits in very nicely with the bound (6.6.128), which puts an upper bound on the range of h where all states can be descendants of the vacuum.

6.8 Discussion: quantum corrections to the cosmic censorship bound

If the pure $\mathcal{N} = (2,2)$ supergravity is a consistent quantum theory, its Hilbert space should be spanned by states which can be identified as excitations of the supergravity fields. One class of such states are perturbative and normalizable excitations of the supergravity fields in AdS_3, which generate the vacuum representation in the boundary CFT [23]. It is expected that these are the only states up to the cosmic censorship bound. We define this bound to be the boundary of the region in the space of energy and charges in which states corresponding semiclassically to black hole solutions can exist. In the classical limit the cosmic censorship bound is the condition on mass and charges of a black hole such that there is a regular horizon.

It turns out that the classical cosmic censorship bound is exactly equal to the upper bound of the polar part of the CFT spectrum [50]. This was the motivation for the definition of $\mathcal{N} = (2,2)$ extremal conformal field theory in section 6.3.1. On the other hand, in section 6.5, we proved that such a conformal field theory does not exist for sufficiently large m. This result, however, does not immediately rule out the conjectured existence of pure $\mathcal{N} = (2,2)$ supergravity since the cosmic censorship bound might receive quantum corrections. That is, there might be quantum corrections to the relation between the values of the mass and charges of those quantum states whose semiclassical manifestation are black holes. There are two potential sources for such corrections, and we will discuss each of them below.

As far as perturbative effects are concerned, the pure supergravity theory can be treated as the Chern-Simons gauge theory with the gauge group (6.3.37). Since the classical equations of motion of the Chern-Simons theory imply vanishing of the gauge field strength and since any perturbative corrections to the equations of motion can be expressed as a polynomial of the field strength and its covariant derivatives, black hole solutions are not corrected to all orders in the perturbative (*i.e.* $1/m$) expansion. However, values of the mass and charge of a given black hole solution can receive corrections since computing them requires knowing the action as well as the equations of motion. In particular, the "level" m, whose inverse appears in front of the action, can be corrected. The leading discrepancy between the dimension of the space of polar polynomials, $P(m)$, and the dimension of the space of weak Jacobi forms, $j(m)$,

$$P(m) - j(m) = \frac{m}{8} + \mathcal{O}(m^{1/2}) \ , \tag{6.8.145}$$

can be explained if m is shifted by an appropriate constant by quantum effects. Such a shift is known to occur in perturbative Chern-Simons gauge theory [183], where the level k is shifted at one loop by the dual Coxeter number of the gauge group, $C_2(G)$. For the supergroup $OSp(2|2)$, we have $C_2 = -2$, so that in the present case both k_L and k_R are shifted as[13]

$$k_L \rightarrow k_L - 2 \ . \tag{6.8.146}$$

Combining this with equation (6.3.39), we can express this as the shift of m,

$$m \rightarrow m - 8 \ , \tag{6.8.147}$$

which, unfortunately, does not account for the difference in (6.8.145). Furthermore, it seems difficult to attribute sub-leading terms in $(P(m) - j(m))$ to higher order perturbative effects since sub-leading terms in $P(m)$ contains the arithmetic function $A(m)$, which does not have a nice $1/m$ expansion (see footnote 2).

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[13] One way to think about this shift is as follows. The supergroup $OSp(2|2)$ is the superconformal group of AdS_2, and its dual Coxeter number, C_2, can be thought of as the beta-function of the world-sheet sigma-model defining AdS_2 space-time. If instead of AdS_2 we consider a positive curvature space, that is a 2-sphere S^2, the contribution to the beta-function of the world-sheet theory should have opposite sign and, hence, the opposite shift of k. In particular, for S^2, which has the isometry group $SU(2)$, the shift $k \rightarrow k + 2$ is familiar in the study of $SU(2)$ Chern-Simons theory [183]. In the case of $OSp(2|2)$ Chern-Simons theory the shift should have opposite sign, therefore justifying (6.8.146).

There is another source of corrections which are non-perturbative in nature. To see this, we note that conformal weights h for states counted by the elliptic genus are integers, as required by modular invariance. This granularity, which is smeared out in any perturbative analysis, gives rise to an intrinsic ambiguity in the cosmic censorship bound of $O(1)$ in h. Since the boundary of the polar region in the (L_0, J_0) plane has a length of order m, it is possible that the discrepancy of $P(m)$ and $j(m)$ mentioned in the above is entirely attributed to this granularity. For example, the bound on h for a new primary state found in (6.6.128) is within $O(1)$ of the cosmic censorship bound.

It is possible that a combination of these two effects resolves the apparent contradiction between the conjectured existence of pure $\mathcal{N} = (2, 2)$ supergravity and the properties of the elliptic genus we found in this paper. It is even conceivable that the fully quantum corrected cosmic censorship bound is in fact the region $S_A \cup S_B$ identified in section 6.7. One way to falsify this conjecture would be to exhibit a quantum state which is in the region S_B, but not in the polar region, and which is described semiclassically as a black hole.

6.9 Extremal $\mathcal{N} = 4$ theories

The analysis for the case of the pure $\mathcal{N} = (2, 2)$ supergravity theories is somewhat inconclusive since we cannot rule out that there are quantum corrections to the classical supergravity ansatz. The situation is sharper for the case with $\mathcal{N} = (4, 4)$ superconformal symmetry since the possible quantum corrections of these theories are well constrained [44]. Therefore, in this section we shall begin to address whether modular invariance allows for a pure $\mathcal{N} = (4, 4)$ supergravity theory. Unfortunately, our results are somewhat incomplete.

Following the earlier definition we define an extremal $\mathcal{N} = (4, 4)$ theory to be a theory whose partition function is of the form (6.3.41) where $\chi^{(m)}_{\text{vac}}$ is now the vacuum character of the $\mathcal{N} = 4$ algebra [64, 65]:

$$\chi^{(m)}_{\text{vac}} = q^{-m/4} \prod_{n=1}^{\infty} \frac{(1 - yq^{n-1/2})^2(1 - y^{-1}q^{n-1/2})^2}{(1 - q^n)} \chi(q, y) \ , \tag{6.9.148}$$

with

$$\begin{aligned} \chi(q, y) \ &= \ \prod_{n=1}^{\infty} \frac{1}{(1 - q^n)(1 - y^2q^n)(1 - y^{-2}q^{n-1})} \\ &\qquad \times \sum_{j \in \mathbb{Z}} q^{(m+1)j^2+j} \left(\frac{y^{2(m+1)j}}{(1 - yq^{j+1/2})^2} - \frac{y^{-2(m+1)j-2}}{(1 - y^{-1}q^{j+1/2})^2} \right) . \end{aligned} \tag{6.9.149}$$

As in the case of the $\mathcal{N} = 2$ vacuum character, we have evaluated this expression at $z + \frac{1}{2}$. To get rid of the negative powers of q in the denominator, we can rewrite it as two separate sums over positive j,

$$\begin{aligned}\chi(q,y) &= \prod_{n=1}^{\infty}\frac{1}{(1-q^n)(1-y^2q^n)(1-y^{-2}q^{n-1})}\times\\ &\quad\Big[\sum_{j\geq 0} q^{(k+1)j^2+j}\Big(\frac{y^{2(m+1)j}}{(1-yq^{j+1/2})^2}-\frac{y^{-2(m+1)j-2}}{(1-y^{-1}q^{j+1/2})^2}\Big)\\ &\qquad+\sum_{j\geq 1} q^{(m+1)j^2+j-1}\Big(\frac{y^{-2(m+1)j-2}}{(1-y^{-1}q^{j-1/2})^2}-\frac{y^{2(m+1)j}}{(1-yq^{j-1/2})^2}\Big)\Big]\end{aligned} \tag{6.9.150}$$

It is straightforward to read off the polar polynomial from this expression.

Using the same methods as in section 6.4, we have analyzed whether this polar polynomial can be completed to a weak Jacobi form. We have performed the analysis for $1 \leq m \leq 20$, and we have found that the only cases where this is possible are $m = 1, 2, 3, 4, 5$. (Note that for $1 \leq m \leq 4$ this is automatic since $P(m) = j(m)$.) Thus, apart from a few low level exceptions, we expect that the pure $\mathcal{N} = (4, 4)$ sugra ansatz is incompatible with modular invariance. It might be possible to prove this assertion by suitably modifying the methods of section 6.5, but the expressions appear to be challenging and we have not attempted to do so.

An important loophole in our argument is the possibility that there are zero-modes making the elliptic genus vanish. This might happen when there is an extension of the chiral algebra and m is odd. In order to demonstrate this write the character expansion of the RR sector partition function as

$$Z_{RR} = \sum_{1\leq \ell,\tilde{\ell}\leq m} c_{\ell\tilde{\ell}}\chi_\ell\overline{\chi_{\tilde{\ell}}} + c_{00}\chi_0\overline{\chi_0} + \sum_{1\leq\ell\leq m} c_{\ell 0}\chi_\ell\overline{\chi_0} + \sum_{1\leq\tilde{\ell}\leq m} c_{0\tilde{\ell}}\chi_0\overline{\chi_{\tilde{\ell}}} + \cdots \tag{6.9.151}$$

Here χ_ℓ denote the characters of the unitary massless representations, with $0 \leq \ell \leq m$ denoting twice the spin of the highest weight vector, and $+\cdots$ refers to terms with a massive representation on the left or the right. The reason for separating out the $\ell = 0$ spin as special is that its highest weight vector is not a polar state, whereas the highest weight vectors of all the other massless representations are polar states. An extremal theory must have an expansion of the form

$$Z_{RR} = \chi_m\overline{\chi_{\tilde{m}}} + c_{00}\chi_0\overline{\chi_0} + \sum_{1\leq\ell\leq m} c_{\ell 0}\chi_\ell\overline{\chi_0} + \sum_{1\leq\tilde{\ell}\leq m} c_{0\tilde{\ell}}\chi_0\overline{\chi_{\tilde{\ell}}} + \cdots \tag{6.9.152}$$

since χ_m is the spectral flow image of the NS vacuum. Now, the elliptic genus of χ_ℓ is $(-1)^\ell(\ell+1)$, while that of the massive representations is zero. Thus, if the elliptic genus vanishes then, comparing the coefficient of the left-moving vacuum character χ_m we see that

$$c_{m0} = (-1)^{m+1}(m+1)\ . \tag{6.9.153}$$

Note that a non vanishing coefficient c_{m0} implies that the right-moving chiral algebra is enhanced, as claimed. Also, since c_{m0} is a positive integer this can only happen when m is odd. Moreover, by comparing the coefficients of the other left-moving characters we find the constraints $c_{\ell 0} = 0$ for $1 \leq \ell \leq m-1$ and $\sum_{\tilde{\ell}=0}^{m} c_{0\tilde{\ell}}(-1)^{\tilde{\ell}}(\tilde{\ell}+1) = 0$. Since our no-go theorem would apply if either the holomorphic or anti-holomorphic elliptic genus is nonvanishing we might as well assume the anti-holomorphic elliptic genus also vanishes. In this case we find that $c_{0\ell} = 0$ for $1 \leq \ell \leq m-1$ and hence $c_{00} = (m+1)^2$, so that $Z_{RR} = |\chi_m + (m+1)\chi_0|^2 + \cdots$. Thus, for extremal theories of this type our arguments fail, and further investigation is necessary.

It should be noted that a vanishing elliptic genus does indeed occur in some important examples. One example arises in $AdS_3 \times S^3 \times T^4$ compactifications [147]. A second example is in the MSW conformal field theory with $(0,4)$ supersymmetry, which is dual to an $AdS_3 \times S^2 \times X$ compactification, where X is Calabi-Yau [148, 155]. In all these cases there is an extended chiral algebra due to singleton modes. In such a case one must take derivatives with respect to $\bar{z}$ and set $\bar{z} = 0$ [35, 147]. The resulting modular object is a non-holomorphic generalization of a Jacobi theta function [45, 47]. A similar phenomenon happens in the analog of the elliptic genus for the *large* $\mathcal{N} = 4$ superconformal algebra [118]. Of course, the examples we have just cited are not extremal theories. However, these examples do suggest that it would be useful to extend the investigation of extremal theories to the cases of vanishing elliptic genera, or $(0,4)$ supersymmetry, or large $\mathcal{N} = 4$ supersymmetry.

6.10 Applications to flux compactifications

Flux compactifications of M-theory and string theory have been a very popular subject of investigation in recent years [60, 46]. Unfortunately, these compactifications are in general very complicated and it is difficult to be sure that they are valid solutions of string theory within a controlled approximation scheme. The demonstration of holographically dual conformal field theories would definitively settle such difficulties, at least for flux compactifications to anti-de Sitter spacetimes. The considerations and techniques of this paper might put interesting constraints on the allowed spectra of some classes of flux compactifications, namely compactifications to AdS_3 with a holographically dual $(2,2)$ conformal field theory. One could imagine, for example, flux compactifications of M-theory on a suitable Calabi-Yau 4-fold, where one includes $M5$ instanton effects, in order to exclude no-scale compactifications.

The compactifications of greatest interest are those with a small cosmological constant and a large gap from the ground state to the Kaluza-Klein scale. These simple aspects of the spectrum already have implications for the conformal field theory. If the cosmological constant is small then the Brown-Henneaux central charge $c = \frac{3}{2}RM_{pl}^{(3)}$ is large. This implies that the level

$$m = \frac{RM_{pl}^{(3)}}{4} \qquad (6.10.154)$$

is large.

Now let us consider the spectrum of the theory. The supergravity multiplet corresponds to the super-Virasoro descendants. Next, if V_8 is the volume of the Calabi-Yau

4-fold in 11-dimensional Planck units then

$$[V_8(M_{pl}^{(11)})^8]M_{pl}^{(11)} = M_{pl}^{(3)} \tag{6.10.155}$$

and therefore, $M_{pl}^{(11)} \sim M_{pl}^{(3)}$ unless V_8 is unnaturally large, and hence in AdS units, the KK scale is of order m. Thus, we naturally expect a large gap to the primary fields corresponding to the KK modes.

In addition to the supergravity multiplet and the KK modes there will typically be other primary fields, for example the moduli fields, many of which might have acquired masses in the compactification scheme. Our conjectured bound (6.6.128) can be viewed as putting constraints on the masses which the moduli acquire. For scalar fields in AdS_3 the relation between the conformal weight h and the mass M of a corresponding particle of spin zero is

$$(MR)^2 = h(h-3) \ , \tag{6.10.156}$$

with similar formulae for particles of other spins. (See section 3.3.1 of [4].) For a large gap we have $MR \sim h$ for all spins, and hence we estimate

$$h \sim M_i R = 4m\frac{M_i}{M_{pl}^{(3)}} \ .$$

Let M_* be the lightest conformal primary other than the identity. If m is very large we clearly should have

$$\frac{M_*}{M_{pl}^{(3)}} < \frac{5}{16} - \frac{1}{32m} + \mathcal{O}(m^{-3/2})$$

or we will rule out the compactification. This can be interpreted as putting an upper bound on the gap to the lightest stabilized moduli field.

It would clearly be of interest to make these considerations more precise, and moreover to extend them to theories with holographic duals with only $(1,1)$ supersymmetry. We believe the techniques used in section 6.5 can be usefully applied to this question, but we leave that for future work.

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Part III

Appendices

Appendix A

A.1 Automorphism for current algebras

In the case of current algebras, let us explicitly check that the modified map ω_λ (2.3.15) is in fact an automorphism. Let $S^{(1)}$, $S^{(2)}$ be two currents such that $\omega(S^{(i)})$ commute with J and $\omega(J)$, so that there is no single pole in their OPE. The effect of ω_λ on the modes $S^{(i)}_m$ is to first order

$$S^{(i)}_m \mapsto \omega(S^{(i)}_m) - \pi\lambda\big(J, \omega(S)^{(i)}\big)\omega(J)_m + \pi\lambda\big(\omega(J), \omega(S^{(i)})\big) J_m \ . \tag{A.1.1}$$

Here, (J, S) denotes the inner product of the generators corresponding to J and S. Now let us check that the commutation relations of the current algebra are preserved under the map ω_λ, so we have to check whether

$$[\omega_\lambda(S^{(1)}_m), \omega_\lambda(S^{(2)}_n)] = \omega_\lambda([S^{(1)}_m, S^{(2)}_n]) \ . \tag{A.1.2}$$

Evaluating the left hand side, we find

$$[\omega_\lambda(S^{(1)}_m), \omega_\lambda(S^{(2)}_n)] = \omega([S^{(1)}_m, S^{(2)}_n]) \ . \tag{A.1.3}$$

We used here that in the commutators of $\omega(S^{(i)})$ with $J, \omega(J)$ only central terms appear; the total central term is then easily seen to vanish.

The commutator on the right hand side is just the one that we would have got with the unchanged automorphism ω. Therefore ω_λ is an automorphism if it acts in the same way as ω on the commutator $[S^{(1)}_m, S^{(2)}_n]$. By definition

$$\begin{aligned} \omega_\lambda([S^{(1)}_m, S^{(2)}_n]) = \omega([S^{(1)}_m, S^{(2)}_n]) &- \pi\lambda\big(\omega(J), [\omega(S^{(1)}), \omega(S^{(2)})]\big) J_{m+n} \\ &+ \pi\lambda\big(J, [\omega(S^{(1)}), \omega(S^{(2)})]\big)\omega(J)_{m+n} \ . \end{aligned} \tag{A.1.4}$$

Here, $[S^{(1)}, S^{(2)}]$ denotes the Lie algebra bracket corresponding to the currents $S^{(i)}$. Since $(J, [\omega(S^{(1)}), \omega(S^{(2)})]) = ([J, \omega(S^{(1)})], \omega(S^{(2)})) = 0$ and similarly for the other term, the two additional terms in (A.1.4) cancel, and we have shown that ω_λ is an automorphism.

A.2 Higher order analysis of boundary locality

In this appendix we shall analyse under which conditions the self-locality of a boundary field S remains unaffected by higher order perturbations. This continues the analysis of (ii) in section 4.1 to higher orders.

Suppose that $J(w)\bar{J}(\bar{w})$ is exactly marginal on the disk, *i.e.* that

$$J(w)J(z) = \frac{1}{(w-z)^2} + \mathcal{O}(1) \ , \qquad \bar{J}(\bar{w})\bar{J}(\bar{z}) = \frac{1}{(\bar{w}-\bar{z})^2} + \mathcal{O}(1) \ , \tag{A.2.1}$$

and

$$J(w)\, \omega(J)(\bar{z}) = \frac{C}{(w-\bar{z})^2} + \mathcal{O}(1) \ , \tag{A.2.2}$$

where C is some constant (that may be zero). Moreover we assume that S does not change its conformal weight to first order in the perturbation. The analysis in section 4.1 then tells us that either $J_0 S = 0$ or $\omega(J)_0 S = 0$. For definiteness let us assume in the following that the first case holds, $J_0 S = 0$.
The contribution to the deformed OPE at order n is given by

$$\lambda^n S(x_1) S(x_2) \prod_{i=1}^{n} \int d^2 w_i \, J(w_i)\, \omega(J)(\bar{w}_i) \ , \tag{A.2.3}$$

which as usual is understood to be inserted into arbitrary correlators. By the usual arguments this can be evaluated by summing the singular terms that arise from the OPE of the currents with the other fields. Note that the integral in (A.2.3) is regularised by $|w_i - w_j| > \ell$ (and of course $\mathrm{Im}(w_i) > \ell/2$). The regularisation is obviously only important if there actually is a pole in $w_i - w_j$; otherwise it will only lead to terms of order ℓ that we can neglect.

Let us now discuss the different pole terms that arise. In the following we will not distinguish between J and $\omega(J)$, and w and $\bar{w}$, unless stated otherwise. We will denote the singular contribution from the operator product expansion between operators at u and v by the symbol $u \to v$; with these conventions there are the following terms to consider:

(1) $w_k \to \bar{w}_k$: The pole $(w_k - \bar{w}_k)^{-2}$ only gives a (divergent) overall renormalisation.

(2) $w_k \to \bar{w}_j$ and $\bar{w}_k \to w_j$: Again the w_k and w_j integrals are independent of all other variables and thus only give an overall normalisation factor. The same applies for $w_k \to w_j$ and $\bar{w}_k \to \bar{w}_j$.

(3) $w_k \to w_j$ but $\bar{w}_k \to w_l$: Evaluate

$$\int d^2 w_k \, (w_k - w_j)^{-2} (\bar{w}_k - w_l)^{-2} = \int_{\partial \mathbb{H}^+, w_i} dw_k \, (w_k - w_j)^{-2} (\bar{w}_k - w_l)^{-1} \ , \tag{A.2.4}$$

where the integral is taken along the real axis, as well as around little circles surrounding the points w_i. The contour integral around w_i with $i \neq j, l$ is zero since the function is regular at that point. Similarly, the integral around w_j and w_l gives zero. The integral along the real axis can be evaluated by closing the contour. Depending on whether w_j and w_l are in the upper or in the lower half plane (or more precisely, whether we consider w or $\bar{w}$), it gives either zero or a term $\sim (w_l - w_j)^{-2}$. Effectively we have thus reduced the problem to the one where $J(w_k)$ is absent, and we are considering the pole contribution from $w_l \to w_j$.

This deals with all the poles between the different currents J and $\omega(J)$. It remains to analyse the poles that involve at least one S. There are two more cases to consider:

(4) $w_k \to x_i$ and $\bar{w}_k \to x_j$: This is the situation discussed in section 4.1. Note that we can still use translation invariance, as the integrand is regular at all points $w_k = w_i$. A logarithmic term in $x_1 - x_2$ could only be produced if both zero modes J_0 and $\omega(J)_0$ act on the fields S, but this gives zero, because we assumed $J_0 S = 0$. We thus obtain meromorphic terms in x_1, x_2, and new operators of the form $V(J_m S; x_i)$ and $V(\omega(J)_n S; x_j)$, $m > 0, n \geq 0$. Obviously $[J_0, J_n] = 0$, and because of (A.2.2) $[J_0, \omega(J)_n] = 0$. This means that the new fields do not have any simple pole with the current J, so that we can continue to use the same arguments as before.

(5) $w_k \to x_1$, but $\bar{w}_k \to \bar{w}_l$: By the same arguments as in (3) above we do not get anything new if $w_l \to w_j$, so we may assume that $w_l \to x_i$. If $w_l \to x_1$, then the w_k and w_l integrals only depend on x_1 and no other variables. By translation invariance this is just a constant factor.

The only remaining cases are thus $w_k \to x_1, \bar{w}_k \to \bar{w}_l$, $w_l \to x_2$, and similar situations with w and $\bar{w}$ exchanged. Let us consider first the contribution

$$\int d^2 w_l \, d^2 w_k \, [J(w_k) S(x_1)][J(w_l) S(x_2)] \frac{1}{(\bar{w}_k - \bar{w}_l)^2} \; , \tag{A.2.5}$$

where we denote the singular part of the OPE by square brackets. The domain of integration is restricted by $|w_l - w_k| > \ell$ and $\mathrm{Im} w_l, \mathrm{Im} w_k > \frac{\ell}{2}$. Write $w_l = u_l + i v_l$ and redefine $\hat{w}_k = w_k - u_l$. The variable u_l is integrated over the real axis without any restrictions. The integrand has poles in u_l at $u_l = x_2 - i v_l$ and at $u_l = x_1 - \hat{w}_k$ which both lie on the lower half plane. By closing the contour in the upper half plane we thus see that this contribution vanishes.

The same argument applies for the case $\bar{w}_k \to x_1$, $w_k \to w_l$, $\bar{w}_l \to x_2$, so the only remaining contribution that we need to consider comes from $\bar{w}_k \to x_1$, $w_k \to \bar{w}_l$, $w_l \to x_2$, which is of the form

$$\int d^2 w_l \, d^2 w_k \, \frac{1}{(w_k - \bar{w}_l)^2} \sum_{m=0}^{h_S} \sum_{n=1}^{h_S} \frac{1}{(\bar{w}_k - x_1)^{m+1}} \frac{1}{(w_l - x_2)^{n+1}} V(\omega(J)_m S, x_1) V(J_n S, x_2) \; . \tag{A.2.6}$$

Note that there is no term with $n = 0$. It is not hard to see that the integral for each summand produces a contribution $\sim (x_1 - x_2)^{-(m+n)}$, so that a logarithmic term could only appear for $m = n = 0$.

In summary, we have thus shown that there will be no logarithmic terms to any order in perturbation theory if they do not arise at first order.

Appendix B

B.1 Web of Dualities

In this appendix we display the various dualities leading from the heterotic theory on T^5 (along $x^{5,6,7,8,9}$) to M-theory on $K3 \times T^2$ (along $x^{6,7,8,9}$ and $x^{5,10}$). For a review of string dualities see e.g. [12]. In order to facilitate keeping track of the various steps, we have depicted a schematic overview in the following web

$$\begin{tikzcd}
 & \boxed{\begin{array}{c}\textbf{M-theory}\\ \hline \begin{array}{lll} p & \text{M5} & 01\ \ \ 6789 \\ N' & \text{M5} & 01\ 567\ \ \ 10 \\ N & \text{M5} & 01\ 5\ \ \ 8910 \end{array}\end{array}} \arrow[d, leftrightarrow, "\text{lift}"] & \\
\boxed{\begin{array}{c}\textbf{Type IIB}\\ \hline \begin{array}{lll} p & \text{KK} & 01\ \ \ 6789 \\ N' & \text{D1} & 01 \\ N & \text{D5} & 01\ \ \ 6789 \end{array}\end{array}} \arrow[r, leftrightarrow, "T_{567}"] \arrow[d, leftrightarrow, "S"] & \boxed{\begin{array}{c}\textbf{Type IIA}\\ \hline \begin{array}{lll} p & \text{NS5} & 01\ \ \ 6789 \\ N' & \text{D4} & 01\ 567 \\ N & \text{D4} & 01\ 5\ \ \ 89 \end{array}\end{array}} & \boxed{\begin{array}{c}\textbf{Type I}\\ \hline \begin{array}{lll} p & \text{D1} & 01 \\ N' & \text{D5} & 01\ \ \ 6789 \\ N & \text{KK} & 01\ \ \ 6789 \end{array}\end{array}} \arrow[d, leftrightarrow, "\text{het/type I}"] \\
\boxed{\begin{array}{c}\textbf{Type IIB}\\ \hline \begin{array}{lll} p & \text{KK} & 01\ \ \ 6789 \\ N' & \text{F1} & 01 \\ N & \text{NS5} & 01\ \ \ 6789 \end{array}\end{array}} \arrow[r, leftrightarrow, "T_5"] & \boxed{\begin{array}{c}\textbf{Type IIA}\\ \hline \begin{array}{lll} p & \text{NS5} & 01\ \ \ 6789 \\ N' & \text{F1} & 01 \\ N & \text{KK} & 01\ \ \ 6789 \end{array}\end{array}} \arrow[r, leftrightarrow] & \boxed{\begin{array}{c}\textbf{heterotic}\\ \hline \begin{array}{lll} p & \text{F1} & 01 \\ N' & \text{NS5} & 01\ \ \ 6789 \\ N & \text{KK} & 01\ \ \ 6789 \end{array}\end{array}}
\end{tikzcd}$$

We start in the lower right corner with the heterotic theory as described in section 4.2.1. Following the first arrow to the left[1], heterotic-type IIA duality takes us to a setup with NS5-branes, fundamental strings and KK monopoles as described in the corresponding

---

[1] The arrow pointing upwards is just included for completeness and represents the heterotic-type I duality which we exploit in section 4.3.

box. Going further to the left (using the arrow labelled T_5), we perform a T-duality along the isometry direction of the KK monopoles (direction x^5), which exchanges the KK monopoles and the NS5-branes but leaves the F1 untouched. Since we have performed the T-duality only along a single direction, the setup is now in the type IIB theory. Following the next arrow upwards (labelled by S), we perform S-duality in the type IIB framework, which turns the NS5-branes and F1 into D5- and D1-branes, respectively. Next we follow the arrow labelled T_{567} to the right, which represents T-duality transformations along $x^{5,6,7}$. Since again the isometry direction of the KK monopoles is affected, they are transformed to NS5-branes, while the D1 and D5-branes are mapped to D4-branes. Since we have performed the duality transformation in an odd number of dimensions, we are back to the type IIA framework. The final arrow pointing upwards is the M-theory lift, which takes us to the setup of three stacks of M5-branes described in section 4.2.2.

Appendix C

C.1 Vertex operator algebras and Zhu's algebra

The vacuum representation of a (chiral) conformal field theory describes a meromorphic conformal field theory [112]. In mathematics, this structure is usually called a vertex operator algebra (see for example [83, 129] for a more detailed introduction). A vertex operator algebra is a vector space $V = \bigoplus_{n=0}^{\infty} V_n$ of states, graded by the conformal weight. Each element in V of grade h defines a linear map on V via

$$a \mapsto V(a,z) = \sum_{n\in\mathbb{Z}} a_n \, z^{-n-h} \qquad (a_n \in \text{End } V) \ . \tag{C.1.1}$$

In this paper we follow the usual physicists' convention for the numberings of the modes; this differs by a shift by $h-1$ from the standard mathematical convention that is also, for example, used in [187]. We also use sometimes (as in [187]) the symbol

$$o(a) = a_0 \ . \tag{C.1.2}$$

Every meromorphic conformal field theory contains an energy-momentum tensor L with modes

$$V(L,z) = \sum_n L_n \, z^{-n-2} \ . \tag{C.1.3}$$

The modes L_n satisfy the Virasoro algebra.

Since much of our analysis is concerned with torus amplitudes it will be convenient to work with the modes that naturally appear on the torus; they can be obtained via a conformal transformation from the modes on the sphere. More specifically, we define (see section 4.2 of [187])

$$V[a,z] = e^{2\pi i z h_a} \, V(a, e^{2\pi i z} - 1) = \sum_n a_{[n]} \, z^{-n-h} \ . \tag{C.1.4}$$

The explicit relation is then

$$a_{[m]} = (2\pi i)^{-m-h_a} \sum_{j\geq m} c(h_a, j+h-1, m+h-1) \, a_j \ , \tag{C.1.5}$$

where

$$(\log(1+z))^m (1+z)^{h_a-1} = \sum_{j\geq m} c(h_a, j, m) \, z^j \ . \tag{C.1.6}$$

This defines a new vertex operator algebra with a new Virasoro tensor whose modes $L_{[n]}$ are given by

$$L_{[n]} = (2\pi i)^{-n} \sum_{j \geq n+1} c(2, j, n+1) L_{j-1} - (2\pi i)^2 \frac{c}{24} \delta_{n,-2} \ . \tag{C.1.7}$$

The appearance of the correction term for $n = -2$ is due to the fact that L is only quasiprimary, rather than primary. Since the two descriptions are related by a conformal transformation to one another, the new modes $S_{[n]}$ satisfy the same commutation relations as the original modes S_n. In particular, the modes $L_{[n]}$ satisfy a Virasoro algebra with the same central charge as the modes L_n.

C.1.1 Zhu's algebra

One of the key results of Zhu [187] is his characterisation of the highest weight representations of a vertex operator algebra in terms of representations of an associative algebra $A_{[1,1]}$, usually now called Zhu's algebra. This algebra is defined as the quotient space of V by the subspace $O_{[1,1]}$, where $O_{[1,1]}$ is spanned by elements of the form

$$\oint dz \left(V(a, z) \frac{(z+1)^{h_a}}{z^2} b \right) \ . \tag{C.1.8}$$

This definition is motivated by the observation (see for example [92, 90] for a more detailed exposition) that

$$\left\langle \phi_1 \middle| \phi_2(1) \oint dz \left(V(a, z) \frac{(z+1)^{h_a}}{z^2} b \right) \right\rangle = 0 \tag{C.1.9}$$

provided only that ϕ_1 and ϕ_2 are highest weight states, *i.e.* are annihilated by all a_n with $n > 0$. Thus any combination of two highest weight states defines an element in the dual space of $A_{[1,1]}$. Zhu showed that also the converse is true; more specifically he proved that $A_{[1,1]}$ carries the structure of an associative algebra with product

$$a * b = \oint dz \left(V(a, z) \frac{(1+z)^{h_a}}{z} b \right) \ , \tag{C.1.10}$$

and that the representations of this associative algebra are in one-to-one correspondence with the highest weight representations of the vertex operator algebra. The product structure (C.1.10) describes the multiplication of the zero modes on highest weight states; in particular, if ψ is a highest weight state, then

$$o(a) o(b) \, \psi = o(a * b) \, \psi \ . \tag{C.1.11}$$

For future reference we also note that in Zhu's algebra (see [187], p.296)

$$a * b - b * a = \frac{1}{2\pi i} a_{[-h_a+1]} \, b \ . \tag{C.1.12}$$

Finally, if V is a rational vertex operator algebra, $A_{[1,1]}$ is a semisimple algebra.

C.1.2 The C_2 space

The states of the form (C.1.8) are not homogeneous with respect to the L_0 grading, even if a and b are. The 'leading term', *i.e.* the term with the highest conformal weight is the term of the form $a_{-h_a-1}b$. Let us denote the subspace that is generated by states of this form by $O_{[2]}$. (We are using here the same conventions as in [96].) A vertex operator algebra is said to satisfy the C_2 criterion if the quotient space $A_{[2]} = V/O_{[2]}$ is finite dimensional. It is easy to see (and proven in [187]) that the C_2 condition implies that Zhu's algebra is finite dimensional. In fact, the dimension of $A_{[2]}$ provides an upper bound on the dimension of Zhu's algebra. Actually, in many cases these two dimensions agree, but this is not always the case: in particular, the dimension of the C_2 space is always at least two [91], while Zhu's algebra is for example one-dimensional for self-dual theories.

Similarly, the leading term of the generating vectors of $O_q(V)$ in (5.2.3) is $a_{[-h_a-1]}b$; since the vertex operator algebras defined by a_n and $a_{[n]}$ are isomorphic, the C_2 condition (formulated for either $a_{-h_a-1}b$ or $a_{[-h_a-1]}b$) then also implies that the $\mathbb{C}[G_4(q), G_6(q)]$-ideal of $O_q(V)$ in $V[G_4(q), G_6(q)]$ has finite codimension. From this it follows that there is a relation of the type (5.2.6).

C.2 Torus recursion relations

In this appendix we briefly sketch the derivation of the recursion relation (5.2.1); for the detailed argument see [187]. Let us introduce the notation

$$F_{\mathcal{H}}\Big((a^1, z_1), \ldots, (a^n, z_n); q\Big) = z_1^{h_1} \ldots z_n^{h_n} \mathrm{Tr}_{\mathcal{H}}\Big(V(a^1, z_1) \cdots V(a^n, z_n)\, q^{L_0}\Big) \,. \qquad \text{(C.2.1)}$$

The derivation of (5.2.1) consists of several steps. We first need the following proposition:

$$\begin{aligned} F_{\mathcal{H}} & \Big((a^1, z_1), (a, w), (a^2, z_2), \ldots, (a^n, z_n); q\Big) \\ = \;& z_1^{-h_1} \ldots z_n^{-h_n} \mathrm{Tr}_{\mathcal{H}}\Big(o(a) V(a^1, z_1) \ldots V(a^n, z_n)\, q^{L_0}\Big) \\ & + \sum_{m \in \mathbb{N}_0} \mathcal{P}_{m+1}\left(\frac{z_1}{w}, q\right) \times F_{\mathcal{H}}\Big((a_{[m-h_a+1]}a^1, z_1), (a^2, z_2), \ldots, (a^n, z_n); q\Big) \qquad \text{(C.2.2)} \\ & + \sum_{j=2}^{n} \sum_{m \in \mathbb{N}_0} \mathcal{P}_{m+1}\left(\frac{z_j}{w}, q\right) \times F_{\mathcal{H}}\Big((a^1, z_1), (a^2, z_2), \ldots, (a_{[m-h_a+1]}a^j, z_j), \ldots, (a^n, z_n); q\Big) \,. \end{aligned}$$

Note that there is actually no difference between the terms in the third line and the fourth line — we have only distinguished between them to clarify the derivation below. In fact, it is easy to show that $F_{\mathcal{H}}$ is actually independent of the order in which the (a^j, z^j) (or (a, w)) appear, as must be the case.

Sketch of proof: The proof is in principle simple: expand out $V(a, w)$ in modes as in (C.1.1). Commute the zero mode $o(a)$ to the left to get the second line in (C.2.2); the commutator will eventually be absorbed into the $\mathcal{P}_1(\frac{z_1}{w}, q)$ of the third line, using (C.3.3). For the other terms in the mode expansion of $V(a, w)$ we commute each mode a_k through the other fields, using

$$[a_k, V(a^j, z_j)] = \sum_{m\in\mathbb{N}} \begin{pmatrix} h+k-1 \\ m \end{pmatrix} V(a_{m-h_a+1}a^j, z_j)\, z_j^{h+k-1-m} \ . \tag{C.2.3}$$

As a_k is taken past q^{L_0}, we pick up

$$a_k\, q^{L_0} = q^k\, q^{L_0} a_k \ . \tag{C.2.4}$$

Thus when a_k comes back to its original position, it is multiplied by q^k. We can therefore solve for the original expression to get

$$\begin{aligned}
&\mathrm{Tr}_{\mathcal{H}}\Big(V(a^1,z_1)\, a_k \cdots V(a^n,z_n)\, q^{L_0}\Big) \\
&\quad = \frac{1}{1-q^k} \sum_{j=2}^{n} \sum_{l\in\mathbb{N}} \binom{h_a-1+k}{l} z_j^{h_a-1+k-l} \\
&\qquad\qquad \times \mathrm{Tr}_{\mathcal{H}}\Big(V(a^1,z_1)\cdots V(a_{l-h_a+1}a^j,z_j)\cdots V(a^n,z_n)\, q^{L_0}\Big) \\
&\qquad + \frac{q^k}{1-q^k} \sum_{l\in\mathbb{N}} \binom{h_a-1+k}{l} z_1^{h_a-1+k-l} \mathrm{Tr}_{\mathcal{H}}\Big(V(a_{l-h_a+1}a^1,z_1)\cdots V(a^n,z_n)\, q^{L_0}\Big) \ .
\end{aligned} \tag{C.2.5}$$

We can then plug this into the original expansion and use the identity

$$\begin{aligned}
&\sum_{l\in\mathbb{N}} \sum_{k=1}^{\infty} \left(\binom{h_a-1+k}{l} \frac{1}{1-q^k} x^k + \binom{h_a-1-k}{l} \frac{1}{1-q^{-k}} x^{-k} \right) a_{l-h_a+1}a^j \\
&\qquad\qquad = \sum_{m\in\mathbb{N}} \mathcal{P}_{m+1}(x,q)\, a_{[m-h_a+1]}a^j \ ,
\end{aligned} \tag{C.2.6}$$

where $\mathcal{P}_{m+1}(x,q)$ is the Weierstrass function, see appendix C. For the terms with $j \neq 1$, $x = z_j/w$, so that we obtain directly the last line of (C.2.2). For $j = 1$, $x = qz_1/w$, and we apply (C.3.3) to get the third line. Note that for $m = 0$ the shift by $2\pi i$ is exactly compensated by the commutator term that comes from the second line. □

We will now use (C.2.2) to calculate the action of $a_{[-h_a]}$ on one of the inserted operators. We claim that

$$\begin{aligned}
&F_{\mathcal{H}}\Big((a_{[-h_a]}a^1,z_1),(a^2,z_2),\ldots,(a^n,z_n);q\Big) \\
&\quad = z_1^{h_1}\ldots z_n^{h_n} \mathrm{Tr}_{\mathcal{H}}\Big(o(a)V(a^1,z_1)\cdots V(a^n,z_n)\, q^{L_0}\Big) \\
&\qquad -\pi i\, F_{\mathcal{H}}\Big((a_{[-h_a+1]}a^1,z_1),(a^2,z_2),\ldots,(a^n,z_n);q\Big) \\
&\qquad + \sum_{k=1}^{\infty} G_{2k}(q)\, F_{\mathcal{H}}\Big((a_{[2k-h_a]}a^1,z_1),(a^2,z_2),\ldots,(a^n,z_n);q\Big) \\
&\qquad + \sum_{j=2}^{n} \sum_{m\in\mathbb{N}_0} \mathcal{P}_{m+1}\left(\frac{z_j}{z_1},q\right) F_{\mathcal{H}}\Big((a^1,z_1),\ldots,(a_{[m-h_a+1]}a^j,z_j),\ldots,(a^n,z_n);q\Big) \ .
\end{aligned} \tag{C.2.7}$$

Proof: We can write the first line of (C.2.7) as

$$\int_C w^{-1}\left(\log\left(\frac{w}{z_1}\right)\right)^{-1} F_{\mathcal{H}}\Big((a,w),(a^1,z_1),\ldots,(a^n,z_n);q\Big)dw\ . \tag{C.2.8}$$

This can be seen by rewriting $a_{[-h_a]}$ in terms of the original modes, using

$$V(a_l\,a^1,z_1) = \oint dw(w-z_1)^{-l-h_a}\,V(a,w)\,V(a^1,z_1) \tag{C.2.9}$$

and by the definition of the $c(h,j,m)$,

$$\sum_{j\geq -1} c(h,j,-1)\,(w-z_1)^j z_1^{h-1-j} w^{-h} = w^{-1}\left(\log\left(\frac{w}{z_1}\right)\right)^{-1}\ . \tag{C.2.10}$$

We then use (C.2.2) to evaluate $F_{\mathcal{H}}$. From (C.2.8) we see that in the terms that are regular in $w = z_1$, we simply need to replace w by z_1. To evaluate the third line of (C.2.2) we substitute $z_1 = \exp(2\pi i z_1')$, $w = \exp(2\pi i w')$, which shows that we obtain the constant term in the w' expansion of $\mathcal{P}_{m+1}(e^{2\pi i w'})$, which can be read off directly from (C.3.6). □

To get (5.2.1), we specialise (C.2.7) to the case $n = 1$. Furthermore we use that (see [187])

$$[o(a), V(b,z)] = V(a_{[-h_a+1]}b, z)\ , \tag{C.2.11}$$

implying that $F_{\mathcal{H}}((a_{[-h_a+1]}b,z);q) = 0$. If we consider the terms of (C.2.7) of power z^0, we thus obtain

$$\mathrm{Tr}_{\mathcal{H}}\Big(o(a_{[-h_a]}b)\,q^{L_0}\Big) = \mathrm{Tr}_{\mathcal{H}}\Big(o(a)\,o(b)\,q^{L_0}\Big) + \sum_{k=1}^{\infty} G_{2k}(q,y)\mathrm{Tr}_{\mathcal{H}}\Big(o(a_{[2k-h_a]}b)\,q^{L_0}\Big)\ . \tag{C.2.12}$$

C.2.1 Differential operators

For the determination of the modular differential equation, one of the key steps is the calculation of the differential operators $P_s(D)$, see (5.2.9). In the following, we give explicit formulae for them for the first few values of s

$$\begin{aligned}
P_2(D) &= (2\pi i)^2 D && \text{(C.2.13)}\\
P_4(D) &= (2\pi i)^4 D^2 + \frac{c}{2}\,G_4(q) && \text{(C.2.14)}\\
P_6(D) &= (2\pi i)^6 D^3 + \left(8+\frac{3c}{2}\right) G_4(q)\,(2\pi i)^2 D + 10c\,G_6(q) && \text{(C.2.15)}\\
P_8(D) &= (2\pi i)^8 D^4 + (32+3c)\,G_4(q)\,(2\pi i)^4 D^2 + (160+40c)\,G_6(q)\,(2\pi i)^2 D\\
&\quad + \left(108c + \frac{3}{4}c^2\right) G_4(q)^2\ . && \text{(C.2.16)}
\end{aligned}$$

Here c is the central charge of the corresponding conformal field theory.

C.3 Weierstrass functions and Eisenstein series

Let us define the function

$$\mathcal{P}_k(q_z,q) = \frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\left(\frac{n^{k-1}q_z^n}{1-q^n} + \frac{(-1)^k n^{k-1}q_z^{-n}q^n}{1-q^n}\right) \ , \tag{C.3.1}$$

which converges for $|q| < |q_z| < 1$. Since $q_z\frac{d}{dq_z}\mathcal{P}_k(q_z,q) = \frac{k}{2\pi i}\mathcal{P}_{k+1}(q_z,q)$, we will concentrate on $\mathcal{P}_1(q_z,q)$. In what follows, we shall be interested in the behaviour around $q_z = 1$. $\mathcal{P}_1^Q(q_z,q,y)$ has a simple pole at $q_z = 1$, but we can find a meromorphic continuation on $|q| < |q_z| < |q|^{-1}$ by rewriting

$$\mathcal{P}_1(q_z,q) = \frac{2\pi i}{1-q_z} - 2\pi i + 2\pi i\sum_{n=1}^{\infty}\left(\frac{q_z^n q^n}{1-q^n} - \frac{q_z^{-n}q^n}{1-q^n}\right) \ . \tag{C.3.2}$$

A straightforward calculation then shows the identity

$$\mathcal{P}_1(qq_z,q) = \mathcal{P}_1(q_z,q) + 2\pi i \ . \tag{C.3.3}$$

Introducing the new variable z by $q_z = e^{2\pi i z}$, we want to calculate the Laurent expansion in z around 0. The crucial point is that the coefficients of this Laurent expansion are essentially the Eisenstein series $G_{2k}(q)$ that will eventually appear in (5.2.1). In fact, expanding q^z in z and using the definition of the Bernoulli numbers,

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty}\frac{B_n}{n!}x^n \ , \tag{C.3.4}$$

along with the identity

$$B_{2n} = \frac{(-1)^{n-1}2(2n)!}{(2\pi)^{2n}}\zeta(2n) \ , \tag{C.3.5}$$

we obtain

$$\mathcal{P}_1(q_z,q) = -\frac{1}{z} - \pi i + \sum_{k=1}^{\infty}G_{2k}(q)z^{2k-1} \ , \tag{C.3.6}$$

where the Eisenstein series are defined by

$$G_{2k}(q) = 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty}\frac{n^{2k-1}q^n}{1-q^n} \ . \tag{C.3.7}$$

The Laurent expansions of the higher $\mathcal{P}_k(q_z,q)$ functions can be directly obtained by

$$\partial_z\mathcal{P}_k(q_z,q) = k\mathcal{P}_{k+1}(q_z,q) \ . \tag{C.3.8}$$

C.3.1 The Eisenstein series

The Eisenstein series $G_{2k}(\tau)$ can also be alternatively defined by

$$\begin{aligned} G_{2k}(\tau) &= \sum_{(m,n)\neq(0,0)} \frac{1}{(m\tau+n)^{2k}} \qquad k \geq 2 \ , && \text{(C.3.9)} \\ G_2(\tau) &= \frac{\pi^2}{3} + \sum_{m\in\mathbb{Z}-\{0\}} \sum_{n\in\mathbb{Z}} \frac{1}{(m\tau+n)^2} \ . && \text{(C.3.10)} \end{aligned}$$

For $k \geq 2$, $G_{2k}(\tau)$ is a modular form of weight $2k$, *i.e.*

$$G_{2k}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{2k} G_{2k}(\tau) \ , \tag{C.3.11}$$

whereas $G_2(\tau)$ transforms with a modular anomaly

$$G_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 G_2(\tau) - 2\pi i c(c\tau+d) \ . \tag{C.3.12}$$

We can use G_2 to define a modular covariant derivative: If $f(q)$ is a modular form of weight s, then $D_s f(q)$ is a modular form of weight $s+2$, where

$$D_s = q\frac{d}{dq} - \frac{s}{4\pi^2} G_2(q) \ . \tag{C.3.13}$$

The space of modular covariant functions is given by the ring $\mathbb{C}[G_4(q), G_6(q)]$ that is freely generated by $G_4(q)$ and $G_6(q)$. In particular, all higher $G_{2k}(q)$ can be written as polynomials in $G_4(q), G_6(q)$.

It is also sometimes convenient to work with a different normalisation for the Eisenstein series, so that the constant term is 1; the corresponding series will be noted by $E_n(q)$. For the first few values of n, they are explicitly given as

$$\begin{aligned} E_2(q) &= 1 - 24\,q - 72\,q^2 - 96\,q^3 - 168\,q^4 - 144\,q^5 - \cdots \ , && \text{(C.3.14)} \\ E_4(q) &= 1 + 240\,q + 2160\,q^2 + 6720\,q^3 + 17520\,q^4 + 30240\,q^5 + \cdots \ , && \text{(C.3.15)} \\ E_6(q) &= 1 - 504\,q - 16632\,q^2 - 122976\,q^3 - 532728\,q^4 - 1575504\,q^5 - \cdots && \text{(C.3.16)} \end{aligned}$$

The relation between the $G_n(q)$ and $E_n(q)$ is simply $G_n(q) = 2\zeta(n)\,E_n(q)$; for the first few values of n, we have explicitly

$$G_2(q) = -\frac{(2\pi i)^2}{12}\,E_2(q) \ , \quad G_4(q) = \frac{(2\pi i)^4}{720}\,E_4(q) \ , \quad G_6(q) = -\frac{(2\pi i)^6}{30240}\,E_6(q) \ . \tag{C.3.17}$$

Finally, we mention that we have the identities

$$DE_4 = -\frac{1}{3}E_6 \ , \qquad DE_6 = -\frac{1}{2}E_4^2 \ , \qquad DE_4^2 = -\frac{2}{3}E_4E_6 \ . \tag{C.3.18}$$

Appendix D

D.1 Growth properties

D.1.1 Analysis of the constraint for m odd

For m odd we have

$$L = \left[-2iq^{-\frac{m}{4}+\frac{1}{8}} \frac{1-q^{1/2}}{1+q^{1/2}} \frac{\vartheta_3(\tau)}{\eta^3} \vartheta_2^2 \vartheta_4^2 \right]_{q^0} . \tag{D.1.1}$$

where L was defined in (6.5.97). We can simplify this significantly using the triple product identity $\vartheta_2\vartheta_3\vartheta_4 = 2\eta^3$. Next, shifting $\tau \to \tau + 1$ (which cannot change the q^0 coefficient) we obtain:

$$L = 4e^{-i\pi m/2} \left[q^{-\frac{m}{4}+\frac{1}{8}} \frac{1+q^{1/2}}{1-q^{1/2}} \vartheta_2\vartheta_3 \right]_{q^0} . \tag{D.1.2}$$

Now use the usual sum formula for ϑ_2 and ϑ_3 to obtain

$$\vartheta_2\vartheta_3 = \sum_{r,s\in\mathbb{Z}} q^{(r-1/2)^2/2+s^2/2} = \sum_{r,s\in\mathbb{Z}} q^{(2r-1)^2/8+(2s)^2/8} = \sum_{n\in\mathbb{N}_0} B(n) q^{n/8} , \tag{D.1.3}$$

where $B(n)$ is the number of ways of writing n as a sum of an even and an odd integer squared, *i.e.* $n = (2r-1)^2 + (2s)^2$ with both r and s integer. We also observe that the series expansion of the other factor is

$$\frac{1+q^{1/2}}{1-q^{1/2}} = 1 + 2\sum_{\ell=1}^{\infty} q^{\ell/2} . \tag{D.1.4}$$

Thus the exact result for (6.5.97) is

$$L = 4e^{-i\pi m/2} \left[B\left(2m-1\right) + 2\sum_{\ell=1}^{\frac{2m-1}{4}} B\left(2m-1-4\ell\right) \right] . \tag{D.1.5}$$

The dominant contribution comes from the second term. This sum is precisely equal to all combinations of an odd and an even integer whose square sum up to a number less or equal to $2m-5$. Now draw a rectangular lattice whose unit cell is a square with length 2, where we shift the lattice by one unit in the x^1-direction say, so that the centers of the

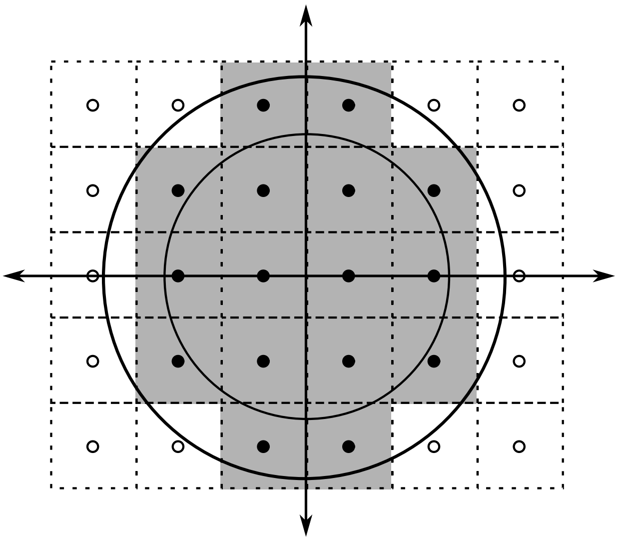

Figure D.1: The grey area is given by those boxes whose centers lie within the outer circle of radius $\sqrt{2m-5}$. The inner circle has radius $\sqrt{2m-5}-\sqrt{2}$ and is completely contained in the grey area.

cells are at $(x^1, x^2) = (2r-1, 2s)$. Consider the area of all those unit cells for which the corresponding center point $(2r-1, 2s)$ has the property that $(2r-1)^2 + (2s)^2 \leq 2m-5$. It follows from elementary geometry that this area is bigger than the area of the disk with radius $\sqrt{2m-5}-\sqrt{2}$ (see figure D.1). Since each unit cell has area 4, it follows that

$$\sum_{\ell=1}^{\frac{2m-1}{4}} B\left(2m-1-4\ell\right) \geq \frac{1}{4}\pi\left(\sqrt{2m-5}-\sqrt{2}\right)^2 = \frac{\pi}{2}m - \pi\sqrt{m-\frac{5}{2}} - \frac{3}{4}\pi \ . \tag{D.1.6}$$

Thus it follows that $e^{i\pi m/2}L$, which is positive, is bounded below by

$$e^{i\pi m/2}L \geq 4\pi m - 8\pi\sqrt{m-\frac{5}{2}} - 6\pi \ . \tag{D.1.7}$$

D.1.2 Analysis of the constraint for $m = 2 \bmod 4$

In the case of m odd we saw that L only grew linearly. Since the original expression contained exponentially growing function such as η^{-3}, this means that there had to occur cancellations. We will now show that for $m = 2 \bmod 4$ such cancellation do not occur, *i.e.* that

$$L = \left[q^{-\frac{m}{4}+\frac{1}{8}}\frac{1-q^{1/2}}{1+q^{1/2}}\frac{\vartheta_3}{\eta^3}(\vartheta_4^4 - \vartheta_2^4)\right]_{q^0} \tag{D.1.8}$$

grows exponentially with m. To this end, use (6.5.95) to write

$$\left[q^{-\frac{1}{2}+\frac{1}{8}}\frac{(1-q^{1/2})^2}{1-q}\left(-24\frac{q\partial_q\vartheta_3}{\eta^3} + \frac{E_2\vartheta_3}{\eta^3}\right)\right]_{q^N} \ , \tag{D.1.9}$$

where $N = m/4 - 1/2$. The following form of E_2 will be useful:

$$E_2(\tau) = 1 - 24\sum_{k=1}^{\infty} \sigma_1(k) q^k \ , \tag{D.1.10}$$

where $\sigma_1(k)$ is the divisor function.

Let us first consider the second term of (D.1.9). We will show that this is negative and grows exponentially fast with N. We introduce the expansion coefficients of ϑ_3/η^3,

$$\frac{\vartheta_3}{\eta^3} = q^{-1/8} \sum_{n\geq 0} (F_1(n) q^n + F_2(n) q^{n+1/2}) \ . \tag{D.1.11}$$

From these we obtain the discrete derivative $(1-q^{1/2})^2 \vartheta_3/\eta^3$,

$$q^{-\frac{1}{2}+\frac{1}{8}} (1-q^{1/2})^2 \frac{\vartheta_3}{\eta^3} = \sum_{n\geq 0} \left(K(n) q^n + K'(n) q^{n-1/2} \right) \tag{D.1.12}$$

with $K(n) = F_2(n) - 2F_1(n) + F_2(n-1)$, and, including E_2,

$$q^{-\frac{1}{2}+\frac{1}{8}} E_2 (1-q^{1/2})^2 \frac{\vartheta_3}{\eta^3} = \sum_{n\geq 0} \left(\hat{K}(n) q^n + \hat{K}'(n) q^{n-1/2} \right) \tag{D.1.13}$$

with

$$\hat{K}(n) = K(n) - 24 \sum_{s=1}^{n} \sigma(s) K(n-s) \ . \tag{D.1.14}$$

Finally, the desired second term of (D.1.9) is $\sum^N \hat{K}(n)$. It will therefore suffice to show that $\hat{K}(n)$ grows exponentially and is negative for large n.

To examine the large n behavior we begin with the Rademacher expansions for $F_1(n)$ and $F_2(n)$. These are summarized in appendix D.2 with the result that

$$\begin{aligned} F_1(n) &= (8n)^{-5/4} e^{\pi\sqrt{2n}} \left(1 - \frac{15+\pi^2}{8\sqrt{2}\pi} n^{-1/2} + \frac{105 + 10\pi^2 + \pi^4}{256\pi^2} n^{-1} + \mathcal{O}(n^{-3/2}) \right) \ , \\ F_2(n) &= (8n)^{-5/4} e^{\pi\sqrt{2n}} \left(1 + \frac{3\,(\pi^2-5)}{8\sqrt{2}\pi} n^{-1/2} + \frac{3\,(35 - 10\pi^2 + 3\pi^4)}{256\pi^2} n^{-1} + \mathcal{O}(n^{-3/2}) \right) \ . \end{aligned}$$

From this we compute the discrete derivative:

$$K(n) = \pi^2 (8n)^{-9/4} e^{\pi\sqrt{2n}} (1 + O(n^{-1/2})) \ . \tag{D.1.15}$$

Note the exponential growth with n. Now write

$$\hat{K}(n) = K(n) - 24K(n-1) - 24S \tag{D.1.16}$$

with $S := \sum_{s=2}^{n} \sigma(s) K(n-s)$. It is straightforward to see that the sum S is positive definite for large n: first note that because of (D.1.15) $K(n)$ is negative for at most finitely many n. Since $K(n)$ grows exponentially and $\sigma(s)$ only grows like $\sigma(s) \sim e^{\gamma} s \ln\ln s$, where γ is the Euler-Mascheroni constant [119], it follows that the first terms of the sum dominate the (potentially negative) terms at its tail. The first two terms on the RHS of (D.1.16) clearly grow like $-23\pi^2(8n)^{-9/4}e^{\pi\sqrt{2n}}$, hence $\hat{K}(n)$ is negative and exponentially growing for large n. Therefore the same is true for $\sum^N \hat{K}(n)$.

In the analysis of the case $m = 0 \bmod 4$ below we will show that the first term of (D.1.9) is negative, so that there can be no cancellations between the two. We thus conclude that (D.1.9) grows exponentially.

D.1.3 Analysis of the constraint for $m = 0$ mod 4

Define

$$R_1 = \left[2\,q^{1/2}\,\frac{(1-q^{1/2})^4}{(1-q)^3}\frac{\vartheta_3}{\eta^3}\right]_{q^{\frac{m}{4}-\frac{1}{8}}} \tag{D.1.17}$$

$$R_2 = \left[\frac{(1-q^{1/2})^2}{1-q}\frac{q\partial_q\vartheta_3}{\eta^3}\right]_{q^{\frac{m}{4}-\frac{1}{8}}} . \tag{D.1.18}$$

We shall show that for large enough m both R_1 and R_2 are positive. Consider first R_2. Note that the only negative coefficients that can appear are due to the factor $(1-q^{1/2})^2$. It will suffice to show that the coefficients

$$\left[\frac{(1-q^{1/2})^2}{(1-q)^3(1-q^2)^3}\,q\partial_q\vartheta_3\right]_{q^N} \tag{D.1.19}$$

are positive for N large enough. We have dropped the factor of $(1-q)^{-1}$ and included only the first two factors of η^3, which will turn out to be sufficient to ensure positivity. Defining

$$\frac{1}{(1-q)^3(1-q^2)^3} = \sum_{n=0}^{\infty} b(n)q^n \ , \tag{D.1.20}$$

it is straightforward to calculate

$$b(n) = \begin{cases} \frac{1}{1920}(2+n)(4+n)(6+n)(8+n)(5+2n) & n \text{ even} \\ \frac{1}{1920}(1+n)(3+n)(5+n)(7+n)(13+2n) & n \text{ odd.} \end{cases} \tag{D.1.21}$$

Note in particular that

$$b(n) = \frac{n^5}{960} + \frac{3n^4}{128} + \frac{19n^3}{96} + \mathcal{O}(n^2) \ . \tag{D.1.22}$$

We now want to calculate the coefficients $p(N)$ of

$$\frac{1}{(1-q)^3(1-q^2)^3}\,q\partial_q\vartheta_3 = \sum_{N\in\frac{1}{2}\mathbb{N}} p(N)q^N \ . \tag{D.1.23}$$

We need to distinguish the cases $N \in \mathbb{N}$ and $N \in \mathbb{N} + \frac{1}{2}$:

$$N \in \mathbb{N} \quad : \quad p(N,K) = \sum_{s=0}^{K} b(N - 2s^2)\, 4s^2 \tag{D.1.24}$$

$$N \in \mathbb{N} + \frac{1}{2} \quad : \quad p(N,K) = \sum_{s=0}^{K} b(N - (2s+1)^2/2)\, (2s+1)^2 \tag{D.1.25}$$

In principle, the upper bound K is given by the requirement that the argument of b be non-negative, and its explicit expression will involve some floor function of a square root of N. For the moment, we will leave K as an auxiliary integer parameter. One can then evaluate the sums explicitly to obtain polynomials in both N and K, again distinguishing the cases N odd and N even. As the resulting expressions are rather lengthy, we refrain from writing them down explicitly. To determine the Nth coefficient of (D.1.19), we then need to evaluate

$$p(N, K_1) - 2p(N - 1/2, K_2) + p(N-1, K_3)\ . \tag{D.1.26}$$

In principle, we would now have to determine the exact values of K_i, which are complicated step functions of $N^{1/2}$. For our purposes however it is enough to know their leading behavior. In particular, we know that $K_i = \sqrt{\frac{N}{2}} - \epsilon_i$, where $0 \leq \epsilon_i < 2$, so that ϵ_i is of order one. We then obtain for (D.1.26) the expression

$$\frac{N^{9/2}}{1890\sqrt{2}} + \mathcal{O}(N^{7/2})\ . \tag{D.1.27}$$

Note that this holds for all $N \in \frac{1}{2}\mathbb{N}$. (Hence, our estimates can also be applied to the analysis of section D.1.2.) This shows that the leading term has a positive coefficient and that it is independent of the ϵ_i, which only appear in the subleading terms. This then shows that (D.1.19) has positive coefficients for N large enough.

Note that for low values of N the coefficients of (D.1.19) can still be negative. To complete the argument, we thus have to show that after convolution with the remaining factors in (D.1.18) the potentially negative coefficients for $N < N_0$ cannot render negative the coefficients at arbitrarily large N. To see this, note that it follows from the Rademacher expansion that, for any set of positive integers $a_1, \ldots, a_k$, the Fourier coefficients of

$$(1-q)^{a_1}(1-q^2)^{a_2} \cdots (1-q^k)^{a_k} \eta^{-3} \tag{D.1.28}$$

will have the asymptotic behavior $\sim n^p e^{\pi\sqrt{2n}}$. For example in the appendix we show that for the case of interest, $(1-q)^3(1-q^2)^3\eta^{-3}$ the leading asymptotics is given by

$$\frac{\pi^6}{8\sqrt{2}} n^{-9/2} e^{\pi\sqrt{2n}}\ . \tag{D.1.29}$$

We approximate the convolution sum as the integral

$$\int^{N} ds\, s^{9/2} (N-s)^{-9/2}\, e^{\pi\sqrt{2(N-s)}}\ . \tag{D.1.30}$$

The position of the saddle point of this integral grows as

$$s_0 \sim N^{1/2} \ . \tag{D.1.31}$$

This means that for N large enough the contribution of the negative coefficients around $s \sim 1$ will be negligible, so that the total coefficient is positive.

Turning to R_1, we need to consider

$$(1-q)^{-3}(1-q^2)^{-3}(1-q^3)^{-3}(1-q^4)^{-3} = \sum_{n=0}^{\infty} \tilde{b}(n) q^n \ . \tag{D.1.32}$$

A straightforward, but somewhat tedious calculation then gives expressions similar to (D.1.21) whose explicit forms depend on n mod 12. Again, the leading terms are independent of this, so that we can write

$$\tilde{b}(n) = \frac{n^{11}}{551809843200} + \frac{n^{10}}{3344302080} + \frac{29\ n^9}{1337720832} + \frac{5\ n^8}{5505024} + \frac{16949\ n^7}{696729600} + \mathcal{O}(n^6) \ . \tag{D.1.33}$$

We can now define $\tilde{p}(N,K)$ analogously to (D.1.24), (D.1.25) and evaluate

$$\tilde{p}(N,K_1) - 4\tilde{p}(N-1/2,K_2) + 6\tilde{p}(N-1,K_3) - 4\tilde{p}(N-3/2,K_4) + \tilde{p}(N-2,K_5) \ , \tag{D.1.34}$$

which leads to

$$\frac{N^{15/2}}{1751349600\sqrt{2}} + \mathcal{O}(N^{13/2}) \ . \tag{D.1.35}$$

Since sums of terms of order n^6 give contributions of at most N^7, this also shows that it was sufficient to consider (D.1.33) only up to n^6. The coefficients of the truncated η^{-3} expansion grow as in (D.2.56), and the rest of the argument is then completely analogous to the case of R_2.

D.2 Rademacher expansions

The proofs in appendix D.1 require some asymptotic expansions for coefficients of some modular forms. We collect these here.

First, we apply the expansion to the modular vector

$$f_1(\tau) = \frac{1}{2}\frac{\vartheta_3+\vartheta_4}{\eta^3} = q^{-1/8}\sum_{n=0}^{\infty} F_1(n) q^n \tag{D.2.36}$$

$$f_2(\tau) = \frac{1}{2}\frac{\vartheta_3-\vartheta_4}{\eta^3} = q^{3/8}\sum_{n=0}^{\infty} F_2(n) q^n \tag{D.2.37}$$

$$f_3(\tau) = \frac{\vartheta_2}{\eta^3} = \sum_{j=0}^{\infty} F_3(n) q^n \ . \tag{D.2.38}$$

We have weight $w = -1$, the representation is manifest for T, and for S it is computed from

$$
\begin{aligned}
f_1(-1/\tau) &= (-i\tau)^{-1}\frac{1}{2}(f_1 + f_2 + f_3) && \text{(D.2.39)}\\
f_2(-1/\tau) &= (-i\tau)^{-1}\frac{1}{2}(f_1 + f_2 - f_3) && \text{(D.2.40)}\\
f_3(-1/\tau) &= (-i\tau)^{-1}(f_1 - f_2)\ . && \text{(D.2.41)}\\
& && \text{(D.2.42)}
\end{aligned}
$$

We now have convergent expansions

$$
\begin{aligned}
F_1(n) &= \frac{\pi}{8}(n - 1/8)^{-1} I_2(4\pi\sqrt{\frac{1}{8}(n - \frac{1}{8})}) + \mathcal{O}(e^{2\pi\sqrt{n/8}}) && \text{(D.2.43)}\\
F_2(n) &= \frac{\pi}{8}(n + 3/8)^{-1} I_2(4\pi\sqrt{\frac{1}{8}(n + \frac{3}{8})}) + \mathcal{O}(e^{2\pi\sqrt{n/8}}) && \text{(D.2.44)}\\
F_3(n) &= \frac{\pi}{8}(n)^{-1} I_2(4\pi\sqrt{\frac{1}{8}n}) + \mathcal{O}(e^{2\pi\sqrt{n/8}})\ . && \text{(D.2.45)}
\end{aligned}
$$

Now use

$$I_\nu(x) \sim \frac{1}{\sqrt{2\pi x}} e^x \left(1 - \frac{4\nu^2 - 1}{8x} + \frac{(4\nu^2 - 1)(4\nu^2 - 9)}{128x^2} + \cdots\right) \tag{D.2.46}$$

for $x \to +\infty$ to get

$$F_1(n) = (8n)^{-5/4} e^{4\pi\sqrt{\frac{n}{8}}} \left(1 - \frac{\pi^2 + 15}{8\sqrt{2}\pi}\frac{1}{n^{1/2}} + \frac{\pi^4 + 70\pi^2 + 105}{256\pi^2}\frac{1}{n} + \cdots\right) \tag{D.2.47}$$

$$F_2(n) = (8n)^{-5/4} e^{4\pi\sqrt{\frac{n}{8}}} \left(1 + \frac{3(\pi^2 - 5)}{8\sqrt{2}\pi}\frac{1}{n^{1/2}} + \frac{3(3\pi^4 - 70\pi^2 + 35)}{256\pi^2}\frac{1}{n} + \cdots\right)\ . \tag{D.2.48}$$

We also need the asymptotic expansion of functions that are obtained from η^{-3} by dropping the first few factors in the product formula. Defining

$$\eta^{-3} = q^{-1/8}\sum_n p_3(n) q^n \tag{D.2.49}$$

(with $p_3(n) = 0$ for $n < 0$), we have the Rademacher formula

$$p_3(n) = 2\pi(8n - 1)^{-5/4} I_{3/2}(\pi\sqrt{2(n - 1/8)}) + \mathcal{O}(e^{\pi\sqrt{n/2}})\ . \tag{D.2.50}$$

Note that the Bessel function is elementary

$$I_{3/2}(x) = \frac{2}{\sqrt{2\pi x}}(\cosh x - \frac{\sinh x}{x})\ . \tag{D.2.51}$$

Define

$$(1-q)^3(1-q^2)^3\eta^{-3} = q^{-1/8}\sum_n \hat{p}_3(n)q^n \ , \tag{D.2.52}$$

which is a kind of sixth-order discrete derivative:

$$\begin{aligned}\hat{p}_3(n) = p_3(n) - 3p_3(n-1) + 8p_3(n-3) \\ - 6p_3(n-4) - 6p_3(n-5) + 8p_3(n-6) - 3p_3(n-8) + p_3(n-9) \ .\end{aligned} \tag{D.2.53}$$

Substituting the asymptotic expansion (D.2.50) one finds after some algebraic manipulations

$$\hat{p}_3(n) = \left(\frac{\pi^6}{8\sqrt{2}}n^{-9/2} + \mathcal{O}(n^{-5})\right) e^{\pi\sqrt{2n}} \ . \tag{D.2.54}$$

Similarly, the coefficients

$$(1-q)^3(1-q^2)^3(1-q^3)^3(1-q^4)^3\eta^{-3} = q^{-1/8}\sum_n \tilde{p}_3(n)q^n \tag{D.2.55}$$

have leading asymptotics

$$\tilde{p}_3(n) \sim \left(\frac{27\pi^{12}}{\sqrt{2}}n^{-15/2} + \mathcal{O}(n^{-8})\right) e^{\pi\sqrt{2n}} \ . \tag{D.2.56}$$

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