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Spectral properties of products of random matrices: theoretical results and data correlation analysis

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In this short review, some theoretical and applicative aspects of random matrix theory are discussed. First, the problem of determining the average eigenvalue density of a product of large random matrices with Gaussian entries will be addressed and solved, and the possible applications of this result in different scientific areas will be presented. After that, the random matrix analysis of financial correlation matrices will be discussed. Namely, the common knowledge, according to which most part of the eigenvalue spectrum of financial correlation matrices carries almost no information, will be challenged. This will be done by looking at some "microscopic" spectral properties of empirical eigenvalue spectra, analyzed by means of the relation, provided by random matrix theory, between a given correlation matrix and its empirically observed estimator.

1 Introduction

The very first notion of a random matrix ensemble dates back to a 1928 paper by J. Wishart [1]. After that, the topic remained almost undeveloped until the early fifties, when E. Wigner essentially threw the theoretical foundations for modern random matrix theory (RMT) and its applications in Physics [2], modeling the Hamiltonian of a heavy nucleus as a random matrix with statistically independent entries. Such a simple approach was found to be extremely effective in reproducing experimental data, such as energy level sequences, in the realm of nuclear Physics. Ever since then, RMT kept on developing through more than five decades, eventually reaching the status of one of the most relevant tools in the analysis of complex physical systems governed by very complicated, or even unknown, interaction laws. As a matter of fact, nowadays the applications of RMT in Physics cover very diverse fields, ranging from quantum gravity [3] and string theory [4, 5] to quantum chromodynamics [6], supersymmetric field theories [7], mesoscopic Physics [8, 9], entanglement Physics [10] and information theory [11].

In recent years, RMT also found several applications outside pure Physics. The mathematical result that gave rise to most of such applications was the one by V. A. Marčenko and L. A. Pastur [12], who derived a closed-form expression for the eigenvalue distribution of the correlation matrix of a large system of uncorrelated Gaussian random variables. This result established a benchmark in correlation analysis, showing how the eigenvalues of a completely noisy system are distributed, and paved the way for several statistical studies based upon the comparison of empirical correlation matrix spectra with the Marčenko-Pastur distribution. A much celebrated example can be found in the groundbreaking papers by L. Laloux *et al.* [13] and V. Plerou *et al.* [14] published in 1999, where a large portion of the correlation matrix eigenvalue spectrum of stocks traded on the New York Stock Exchange was shown to be well fitted by a Marčenko-Pastur distribution. This result also represents a beautiful example of the achievements that *Econophysics*, *i.e.* the application to Economics of ideas and techniques commonly used in Physics [15, 16], can reach.

The goal of this paper is to briefly sketch some recent results in RMT, focusing both on theoretical developments and on their possible applications.

2 Random matrix theory: introductory notions

One of the typical problems to be faced in RMT is the one of computing the average eigenvalue density of a given random matrix ensemble. So, suppose we are interested in studying the spectral properties of a given ensemble $\mathbf{H}$ of $N \times N$ matrices, possibly characterized by some symmetry properties (*e.g.* Hermiticity), whose matrix entries (whether real or complex) are independently drawn from some probability density. The very first ingredient we shall need is a probability measure $\mathrm{d}\mu(\mathbf{H})$ encoding all such properties. Quite intuitively, the ensemble average eigenvalue density can be defined as

$$\rho_{\mathbf{H}}(\lambda) = \frac{1}{N}\sum_{i=1}^{N}\langle\delta(\lambda-\lambda_i)\rangle, \tag{1}$$

where $\langle\ldots\rangle = \int(\ldots)\mathrm{d}\mu(\mathbf{H})$, and where the case of real eigenvalues (*i.e.* Hermitian matrices) has been considered. Now, if we suppose to remove the expectation from the right hand side of equation (2.1), then we are just looking at a single realization of the matrix ensemble $\mathbf{H}$ having eigenvalues $\lambda_1,\ldots,\lambda_N$: in the absence of any randomness, in this case the eigenvalue density is simply given by a sequence of Dirac deltas. On the other hand, when considering the average behavior of the eigenvalues, it is often convenient to introduce the following matrix function:

$$\mathbf{G}_{\mathbf{H}}(z) = \langle(\mathbf{Z}-\mathbf{H})^{-1}\rangle, \tag{2}$$

where $\mathbf{Z} = z\mathbf{1}_N$, $z \in \mathbb{C}$ and $\mathbf{1}_N$ is the $N \times N$ identity matrix. Taking the normalized trace of this expression we obtain the ensemble's Green's function:

$$g_{\mathbf{H}}(z) = \frac{1}{N}\mathbf{G}_{\mathbf{H}}(z) = \frac{1}{N}\sum_{i=1}^{N}\left\langle\frac{1}{z-\lambda_i}\right\rangle. \tag{3}$$

Suppose again that we remove the expectation map from the previous equation: then, $g_{\mathbf{H}}$ is a meromorphic function whose poles are on the real axis and correspond to the eigenvalues of the particular $\mathbf{H}$ matrix one is considering. Conversely, when the averaging operation is actually performed, and the $N \to \infty$ limit is taken, the poles of the Green's function start to merge into continuous intervals of the real line. In this limit the Green's function becomes a holomorphic function everywhere in the complex plane except for the aforementioned intervals. Remarkably, those are the intervals where the eigenvalue density (1) is actually defined. As a matter of fact, using Sokhotsky's formula $\lim_{\epsilon\to 0^+}(\lambda+\mathrm{i}\epsilon)^{-1} = \mathcal{P}(1/\lambda) - \mathrm{i}\pi\delta(\lambda)$, where $\mathcal{P}$ denotes the principal value, one can see from equations (1) and (3) that the following holds:

$$\rho_{\mathbf{H}}(\lambda) = -\frac{1}{\pi}\lim_{\epsilon\to 0^+}\mathrm{Im}\; g_{\mathbf{H}}(\lambda+\mathrm{i}\epsilon), \tag{4}$$

i.e. the Green's function, in the infinite matrix limit, is by all means equivalent to the eigenvalue density and encodes all of the spectral density of the matrix ensemble under study. Being holomorphic everywhere in the complex plane except for some cuts on the real line, the Green's function can be typically expanded into a power series around infinite z, whose coefficients can be shown to be given by the following expression:

$$g_{\mathbf{H}}(z) = \sum_{n=0}^{\infty}\frac{m_{\mathbf{H}}^{(n)}}{z^{n+1}}\,, \quad m_{\mathbf{H}}^{(n)} = \int \mathrm{d}\lambda\,\rho_{\mathbf{H}}(\lambda)\,\lambda^n. \tag{5}$$

The $m_{\mathbf{H}}^{(n)}$s are called matrix moments and are usually summed up in the M-transform, or moment generating function

$$m_{\mathbf{H}}(z) = z g_{\mathbf{H}}(z) - 1 = \sum_{n=1}^{\infty} \frac{m_{\mathbf{H}}^{(n)}}{z^n}. \tag{6}$$

3 Products of Gaussian random matrices

The formalism introduced in the previous section is suited to dealing with real eigenvalue spectra. When switching to non-Hermitian random matrix ensembles, hence to complex eigenvalues, the previous framework needs to be changed in a quite laborious way (see [17] for a review on this topic). This is essentially due to the structure of the eigenvalue density in the complex plane. Generalizing equation (1), when dealing with a non-Hermitian matrix ensemble $\mathbf{A}$ one can straightforwardly write

$$\rho_{\mathbf{A}}(\lambda, \lambda^*) = \frac{1}{N} \sum_{i=1} \left\langle \delta^{(2)}(\lambda - \lambda_i, \lambda^* - \lambda_i^*) \right\rangle, \tag{7}$$

where the two-dimensional Dirac delta function has been employed and the asterisk denotes complex conjugation. When generalizing the Green's function (3) to the complex domain a few problems arise, the most relevant one being the non-linear structure of the Green's function with respect to the ensemble $\mathbf{A}$. However, this drawback can be nicely circumvented by means of a linearization trick, first proposed in [18]. Also, such a trick can be further generalized in order to compute the spectral densities of products of random matrices.

In the past few years, products of random matrices have been increasingly employed in very different scientific areas, ranging from statistical Physics [19] and Yang-Mills theories [20, 21] to wireless telecommunications [11]. In a recent paper [22], the average eigenvalue spectrum of a product of matrices of the type

$$\mathbf{P}_L = \mathbf{A}_1 \mathbf{A}_2 \dots \mathbf{A}_L, \tag{8}$$

where each $\mathbf{A}_l$ is a $N \times N$ matrix with independent complex Gaussian entries, was derived (taking the infinite matrix size limit $N \to \infty$) in closed-form for an arbitrary number of matrices L in the product. Such a density was shown to be rotationally symmetric and bounded within a circular domain in the complex plane. These results have recently been generalized to the case in which each $\mathbf{A}_l$ in the product (8) is a $N_l \times N_{l+1}$ rectangular matrix [23] (assuming $N_1 = N_{L+1}$ in order for the product matrix to be square and have eigenvalues). More precisely, in [23] each $\mathbf{A}_L$ matrix was assumed to be a Ginibre matrix [24], *i.e.* a random matrix with complex entries whose real and imaginary parts are independently drawn from a Gaussian distribution with the following normalizations:

$$\begin{aligned} \langle \mathrm{Re}[\mathbf{A}_l]_{ij} \rangle &= \langle \mathrm{Im}[\mathbf{A}_l]_{ij} \rangle = 0 \\ \langle (\mathrm{Re}[\mathbf{A}_l]_{ij})^2 \rangle &= \langle (\mathrm{Im}[\mathbf{A}_l]_{ij})^2 \rangle = \frac{\sigma_l^2}{2\sqrt{N_l N_{l+1}}} \quad \Longrightarrow \quad \langle |[\mathbf{A}_l]_{ij}|^2 \rangle = \frac{\sigma_l^2}{\sqrt{N_l N_{l+1}}} \end{aligned} \tag{9}$$

where $i = 1, \dots, N_l$, $j = 1, \dots, N_{l+1}$ and $\sigma_l > 0$ is a real parameter. The infinite matrix size limit was implemented in the form of a "thermodynamic" limit for matrices:

$$N_l \to \infty \,, \quad \text{with} \quad R_l = \frac{N_l}{N_{L+1}} = \text{constant} \,, \quad l = 1, \dots, L. \tag{10}$$

The spectral problem was then solved in the form of a polynomial equation for the M-transform $m_{\mathbf{P}_L}$ of the $\mathbf{P}_L$ matrix ensemble:

$$\prod_{l=1}^{L}\left(1+\frac{m_{\mathbf{P}_L}(z,z^*)}{R_l}\right)=\frac{|z|^2}{\sigma^2}, \tag{11}$$

where $\sigma=\sigma_1\sigma_2\ldots\sigma_L$. Once this equation has been solved (whether analytically or numerically, depending on the value of L), the Green's function can be immediately recovered (as $g_{\mathbf{P}_L}(z,z^*)=(m_{\mathbf{P}_L}(z,z^*)+1)/z$, see equation (6)), then the corresponding eigenvalue density $\rho_{\mathbf{P}_L}$ can be derived. In [23] this result was thoroughly tested against numerically obtained eigenvalue densities of product matrices, always finding an excellent agreement. However, a few things about the density $\rho_{\mathbf{P}_L}$ can be inferred from equation (11) without actually solving it. Namely, it can be shown that $\rho_{\mathbf{P}_L}$ is rotationally symmetric and bounded within a circle of radius σ in the complex plane, and both these facts generalize the square matrix case solved in [22]. Moreover, it can also be shown from equation (11) that, close to the origin of the complex plane, one has $\rho_{\mathbf{P}_L}(\lambda,\lambda^*)\sim|\lambda|^{-2(1-1/s)}$, where s is the number of rectangularity ratios R_l being equal to one.

Before concluding this section, it is important to mention that the aforementioned generalization of the spectral problem from square to rectangular matrices is far from being a mere mathematical exercise, being actually quite relevant from the viewpoint of practical applications. As a matter of fact, products of rectangular Gaussian matrices are very important for the information theoretical description of multiple antenna channels known as MIMO (multiple-input / multiple-output) links [11, 25]. Also, the previously illustrated results recently found interesting applications to the problem of composing quantum operations [26] and generating random density matrices [27]. Moreover, as it will be discussed in the next section, when the product of two matrices (*i.e.* $L=2$ in equation (8)) the eigenvalue density can actually be derived in closed-form, and this result has quite relevant applications in multivariate statistics.

4 Random matrix analysis of financial data

As already mentioned in the introduction, in two well known papers [13, 14] it was shown that most of the eigenvalue spectrum of a financial correlation matrix can be well fitted by a Marčenko-Pastur distribution (see Figure 1), and this fact is robust with respect to changes in the data time scale, ranging from intra-day to daily data. Moreover, in the two aforementioned papers it was also shown that typically one has a very large ($\mathcal{O}(N)$, N being the number of stocks) and isolated eigenvalue plus a few other large eigenvalues "leaking out" of such bulks. Performing a principal component analysis (see for example [28] for an introduction to this topic) easily shows that the former can be interpreted as a "market eigenvalue", emerging as a consequence of global fluctuations involving the market as a whole. On the other hand, the remaining large eigenvalues can be interpreted as "sector eigenvalues", *i.e.* a reflection of the presence of clusters of correlated stocks belonging to the same market sector (*e.g.* financial, pharmaceutical). Thus, according to this picture, the large majority of the observed eigenvalue spectrum of a financial correlation matrix, being close to the eigenvalue density of a completely noisy system, carries no information at all, whereas all the information about correlations amongst stocks is encoded within a few large eigenvalues. Let us now analyze these observation from a more formal point of view.

When studying the correlations in a set of N random variables $r_1,\ldots,r_N$, one can reasonably assume that a "true" correlation C_{ij} exists between the i-th and j-th variable. The best one can do in order to obtain an empirical estimate of C_{ij} is then to collect T equally spaced (in time) observations of the random variables under study, and to compute Pearson's estimator for the correlation coefficient, which for standardized data (*i.e.* zero mean and unit variance) reads

$$c_{ij}=\frac{1}{T}\sum_{t=1}^{N}r_{it}r_{jt}, \tag{12}$$

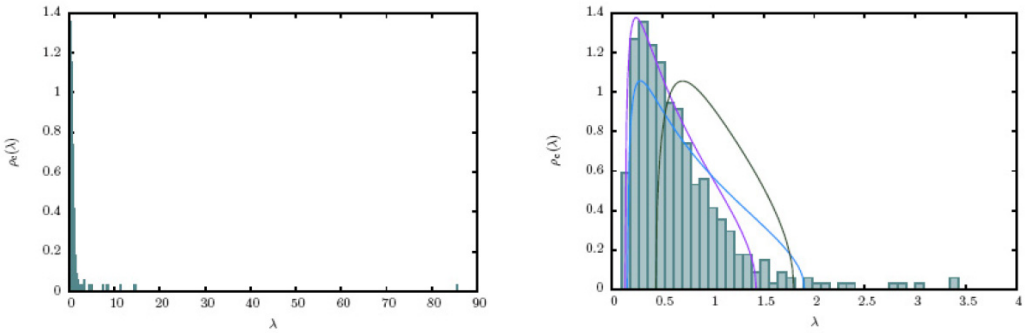

Fig. 1 Left: eigenvalue spectrum of the covariance matrix for $T = 3400$ daily returns of $N = 396$ assets belonging the S&P500 index over the years 1996-2009. Right: the main eigenvalue bulk of the S&P data used in the left plot is compared with three different Marčenko-Pastur densities: the one obtained with with the "natural" values of the parameters ($q = N/T = 0.116$, $\sigma = 1$, green line), the one obtained by the subtracting the market eigenvalue's contribution to σ and fitting q ($q = 0.313$, $\sigma = 0.885$, blue line) and the one obtained by fitting both q and σ ($q = 0.287$, $\sigma = 0.778$, purple line).

where r_{it} represents the time t observation of the i-th variable. The two sets of numbers (C_{ij} and c_{ij}) can then be collected into $N \times N$ symmetric and positive definite correlation matrices $\mathbf{C}$ and $\mathbf{c}$. So, in this framework, the Marčenko-Pastur distribution can be seen as the empirically observed, *i.e.* noise-dressed, version of the correlation matrix $\mathbf{C} = \mathbf{1}_N$ of an N-dimensional set of uncorrelated Gaussian random variables. The Marčenko-Pastur distribution reads

$$\rho_{\mathbf{c}}(\lambda) = \frac{1}{2\pi q\sigma^2} \frac{\sqrt{(\lambda_+ - \lambda)(\lambda - \lambda_-)}}{\lambda}, \quad \lambda_\pm = \sigma^2(1 \pm \sqrt{q})^2, \tag{13}$$

where $q = N/T$ is the ratio between the number of variables and the depth of the time series employed to compute the estimator $\mathbf{c}$, while σ is the standard deviation of the variables in use. So, as one can immediately realize, whenever some dataset has been chosen there is actually no flexibility in the definition of the Marčenko-Pastur distribution, whose parameters q and σ are fixed once and for all by the dimensions and normalizations of the data being analyzed. However, as already mentioned, the Marčenko-Pastur distribution usually needs to be fitted on the empirically observed eigenvalue bulks of financial correlation matrices, allowing q and σ to differ from their "natural" values (see Figure 1). This fact seems to suggest that the aforementioned simple picture (*i.e.* one "noisy" eigenvalue bulk plus a few eigenvalues carrying all relevant information on correlations) is actually quite simplistic. As a matter of fact, a few authors recently questioned and critically revised such a framework (see for example [29, 30, 31]).

In a recent paper [32], the Marčenko-Pastur framework was challenged from a "microscopic" point of view. The starting point of that work was the random matrix description of the relation between a "true" correlation matrix $\mathbf{C}$ and its estimator (12). In a nutshell, it can be proven that the moment generating functions of the two matrices, which we denote as $\mu_{\mathbf{C}}$ and $m_{\mathbf{c}}$ respectively, are related as follows [33, 34, 35]:

$$m_{\mathbf{c}}(z) = \mu_{\mathbf{C}}(Z) \quad \text{where} \quad Z = \frac{z}{1 + q m_{\mathbf{c}}(z)}, \tag{14}$$

where $q = N/T$ as before[1]. As a matter of fact, the Marčenko-Pastur eigenvalue density (13) can be derived from equation (14) when posing $\mathbf{C} = \mathbf{1}_N$. Equation (14) allows in principle to compute, at least

[1] It is worth remarking that equation (14) is derived under the thermodynamic limit $N, T \to \infty$ keeping their ratio q fixed.

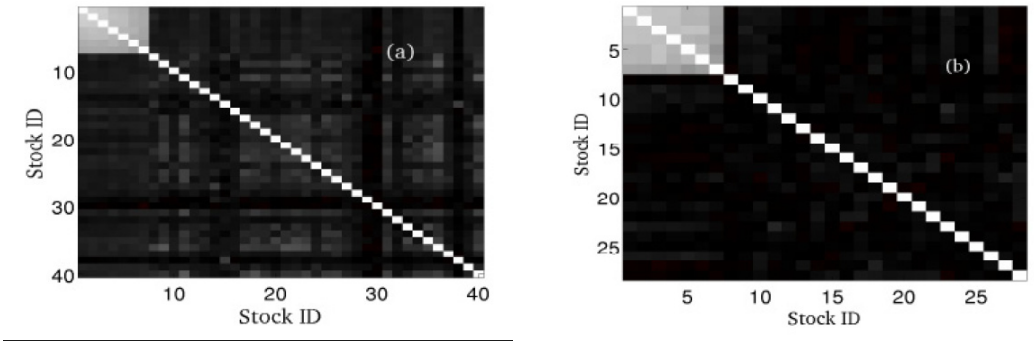

Fig. 2 Filtered correlation matrices for the S&P500 (left) and FTSE350 (right) datasets (see [32] for more details on the filtering technique employed).

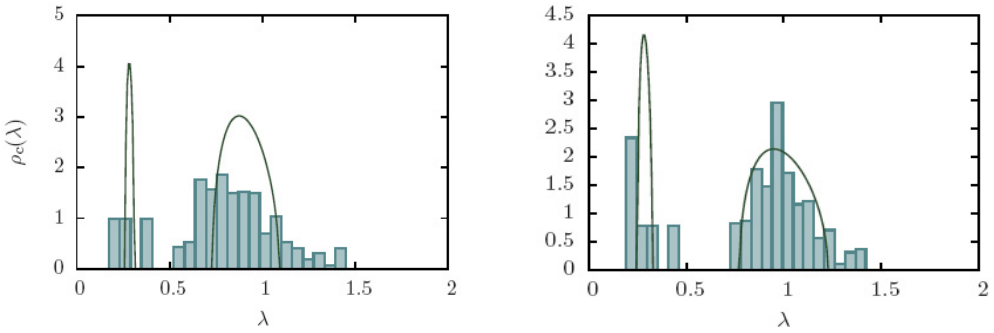

Fig. 3 Comparison between the empirical eigenvalue spectra of the filtered correlation matrices shown in Figure 2 and the eigenvalue densities obtained from the noise dressing, via equation (14) of the spectrum in equation (15).

numerically, the average eigenvalue density $\rho_{\mathbf{c}}$ of the noise-dressed estimator of a given correlation matrix. In [32] such a framework was employed to study in detail the correlation matrices of two financial datasets made of stocks belonging to the S&P500 and FTSE350 indices, respectively. More specifically, such correlation matrices were filtered in order to end up with the block-diagonal matrices shown in Figure 2, displaying a very clear structure with one cluster of strongly correlated stocks, plus additional stocks very weakly correlated with the cluster and amongst themselves. Such a structure can be approximated with a block-diagonal matrix: a first block in which elements belonging to the correlated cluster are all mutually correlated via the same coefficient ρ, computed as the average correlation in the cluster, and a second block simply assumed to be equal to the identity matrix. Now, the eigenvalue spectrum of such a block-diagonal matrix can be easily computed, and it reads

$$\det(\mathbf{C} - \lambda \mathbf{1}_N) = [\lambda - (1-\rho)]^{\bar{N}-1}(\lambda - 1)^{N-\bar{N}}[\lambda - (\bar{N}\rho + (1-\rho))] = 0, \tag{15}$$

where $\bar{N}$ is the number of stocks belonging to the correlated cluster. So, apart from one large eigenvalue ($\sim \bar{N}\rho$), one has two groups of degenerate eigenvalues. The moment generating function $\mu_{\mathbf{C}}$ of this type of spectra can be computed very easily, and plugging it into equation (14) yields the spectrum of the corresponding estimator. In Figure 3 the eigenvalue densities obtained in such a way are compared with the empirical ones obtained by diagonalizing the correlation matrices shown in Figure 2. A few considerations are in order. First, the two degenerate eigenvalues in equation (15), equal to $1-\rho$ and 1, are observed as

separate bulks (since strong correlations in the cluster ensure a small value of $1 - \rho$). This fact represents a clear phenomenological evidence that the mere assumption of a Marčenko-Pastur (which represents the noise-dressing of one single eigenvalue) is not enough to provide a realistic description. However, as one can immediately see, the agreement between the empirical densities and the theoretically predicted ones for the approximated block-diagonal correlation matrices is rather poor: as shown in [32], this is due to unresolved correlation structures in the filtered correlation matrices, the "gray shadings" in Figure 2. However, despite the poor agreement, Figure 3 suggests that the noise-dressing mechanism for degenerate eigenvalues described by the map (14) is actually acting on real financial data. All in all, in the light of these considerations, one is quite tempted to conjecture that the empirically observed eigenvalue bulks emerge from the superposition and subtle interplay between different cluster structures like the ones in Figure 2. Some further evidence in this respect can be found in [32], where it was shown, by means of Monte Carlo simulations, that eigenvalues bulks which are very well fitted by Marčenko-Pastur distributions can actually emerge from properly defined sets of correlated random variables.

As a conclusion to this section, let us mention a possible generalization of this type of analysis. In the previous section, an equation for the M-transform of the product of L rectangular Gaussian random matrices was derived. As already mentioned, such equation can be solved analytically when the product of two matrices is considered, eventually obtaining an equation for the corresponding eigenvalue density. So, if we consider the product $\mathbf{k} = \mathbf{A}_1\mathbf{A}_2^{\mathrm{T}}$, where $\mathbf{A}_1$ and $\mathbf{A}_2$ are two $N \times T$ matrices, we are essentially introducing a generalized correlation matrix between two systems. Therefore, the eigenvalue density computed in [22] for this case when $\mathbf{A}_1$ and $\mathbf{A}_2$ are both random Gaussian matrices represents the generalization of the Marčenko-Pastur distribution to two systems having no self-correlations of their own and no correlations between them.

5 Conclusions and outlook

In this review, a few different aspects of RMT were presented, both from a theoretical and an applicative viewpoint. The main theoretical contribution that was discussed is the solution of the spectral problem for products of an arbitrary number of large rectangular Gaussian random matrices. An equation for the moment generating function of the product matrix ensemble $\mathbf{P}_L = \mathbf{A}_1\mathbf{A}_2 \ldots \mathbf{A}_L$ was introduced. The solution of this equations, to be found numerically except for one notable exception, leads to the average eigenvalue density of the $\mathbf{P}_L$ matrix. From the viewpoint of possible applications, the spectral analysis of products of Gaussian matrices is quite relevant to the information theoretical approach to multiple antenna channels in wireless telecommunications. Moreover, the eigenvalue density for the product of two Gaussian random matrices can be derived in closed-form, and this result is quite important to correlation analysis, since it represent the generalization of the Marčenko- Pastur density as a pure noise benchmark for two separate systems.

The possible applications of RMT to financial data analysis were also discussed. The simplified picture, according to which most of the eigenvalue spectrum of a financial correlation matrix essentially carries no information, was critically revised. As a matter of fact, the main evidence against such a viewpoint is the fact that the empirically observed eigenvalue bulks, representing the large majority of eigenvalue spectra, need to be fitted by a Marčenko-Pastur distribution whose parameters differ from the "natural" ones fixed by the data under study. The relation (14) between a "true" correlation matrix and its empirically observed estimator was employed to study some properly filtered correlation matrices of financial datasets displaying a very clear cluster structure. This analysis revealed some of the "microscopic" structure of the observed eigenvalue bulks, confirming the Marčenko-Pastur framework not to be suited to a realistic description of the data.

The ultimate application of RMT to financial data analysis would be the one of inspiring and proposing cleaning recipes for empirical correlation matrices, something which would be of huge usefulness for practical applications in portfolio management. In this respect, a lot of work still needs to be done. As

a matter of fact, the current knowledge in RMT only leads to correlation matrix cleaning procedures that only slightly outperform more traditional ones. However, the analytical and technical difficulties of RMT would still reasonably lead one to prefer the latter. Thus, a more application-oriented program of random matrix financial data analysis seems to be needed in years to come.

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