## Effective potential in ultraviolet completions for composite Higgs models

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We consider a class of composite Higgs models based on asymptotically free SO(d) gauge theories with d odd, with fermions in two irreducible representations, and in which the Higgs field arises as a pseudo-Nambu-Goldstone boson and the top quark is partially composite. The Nambu-Goldstone coset containing the Higgs field, or Higgs coset, is either SU(4)/Sp(4) or SU(5)/SO(5), whereas the top partners live in two-index representations of the relevant flavor group [SU(4) or SU(5)]. In both cases, there is a large number of terms in the most general four-fermion Lagrangian describing the interaction of third-generation quarks with the top partners. We derive the top-induced effective potential for the Higgs coset together with the singlet pseudo-Nambu-Goldstone boson associated with the non-anomalous axial symmetry, to leading order in the couplings between the third-generation quarks and the composite sector. We obtain expressions for the lowenergy constants in terms of top-partner two-point functions. We revisit the effective potential of another composite Higgs model that we have studied previously, which is based on an SU(4) gauge theory and provides a different realization of the SU(5)/SO(5) coset. The top partners of this model live in the fundamental representation of SU(5), and, as a result, the effective potential of this model is qualitatively different from the SO(d) gauge theories. We also discuss the role of the isospin-triplet fields contained in the SU(5)/SO(5) coset, and show that, without further constraints on the four-fermion couplings, an expectation value for the Higgs field will trigger the subsequent condensation of an isospin-triplet field.

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#### I. INTRODUCTION

Among the mechanisms that have been proposed to keep the Higgs particle naturally light, the so-called composite Higgs paradigm [1,2] postulates the existence of a new strong sector, perhaps in the few TeV range, based on an asymptotically free gauge theory that we will call hypercolor. Spontaneous chiral symmetry breaking in the hypercolor theory produces a set of Nambu-Goldstone bosons (NGBs). When we couple the Standard Model and the hypercolor theory, this breaks explicitly the flavor symmetry group of the hypercolor theory to a smaller group, thereby generating an effective potential for the now pseudo-Nambu-Goldstone bosons (pNGBs). The Higgs doublet is composed of four of these pNGBs, and it is assumed that minimizing the effective potential triggers electroweak symmetry breaking. As the only fermion in the Standard Model with a mass comparable to the electroweak scale, the top quark is usually given a special role in composite-Higgs models. We will assume that the top quark couples linearly to baryons of the hypercolor sector, which we will refer to as hyperbaryons. The observed top-quark mass eigenstate is a linear superposition of the elementary top quark and the composite hyperbaryons, and this partial compositeness is responsible for its large mass [3].

The literature on composite Higgs leaves largely unspecified the details of the new dynamics, and focuses on its low-energy sector containing the pNGBs, which can be studied as a nonlinear sigma model (for reviews, see Refs. [4–6]). Nevertheless, a number of ultraviolet completions of composite Higgs models have been proposed [7,8]. All these models are asymptotically free gauge theories with fermionic matter, sharing some additional basic features. The models are free of gauge anomalies, both the hypercolor theory by itself, and the coupled system of hypercolor together with the Standard Model, including all their gauge interactions. While ultimately only a lattice calculation can settle it,<sup>1</sup> analytic considerations suggest

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<sup>&</sup>lt;sup>1</sup>For reviews of recent lattice work, see Refs. [9–12].

that all candidate models exist in a chirally broken phase, and are not infrared conformal.

As for the spectrum of the hypercolor theory, the Nambu-Goldstone coset must contain an  $SU(2)_L$  doublet that can be identified as the Higgs field. In addition, the hyperbaryon spectrum must contain top partners, states with the same Standard-Model quantum numbers as left-handed or right-handed quarks, that can couple linearly to thirdgeneration quarks. In this paper, we will consider mass generation for the top quark only,<sup>2</sup> and therefore we need hyperbaryons that can couple to  $q_L = (t_L, b_L)$  and to  $t_R$ , but not to  $b_R$ . From a "low-energy" point of view, the differences between the various models are mainly in the Nambu-Goldstone coset, and in which *irreps* of the flavor symmetry group of the hypercolor theory the top partners live [7,8]. As we will see, different hyperbaryon spectra can give rise to very different effective potentials even when the Nambu-Goldstone coset is the same.

A further assumption with important dynamical implications is that the Standard Model gauge symmetries are embedded into the unbroken flavor symmetry group of the hypercolor theory. This gives rise to the vacuum alignment phenomenon [14–16]. In particular, the effective potential induced by the coupling of electroweak gauge bosons to the hypercolor theory has its minimum at the origin for the Higgs field. As a result, the top-sector effective potential is instrumental in generating the nontrivial minimum for the Higgs field that will trigger electroweak symmetry breaking.

In this paper, we discuss mainly (but not only) composite Higgs models based on an SO(d) gauge group with d odd. Each model will contain fermions in the vector and in the spinor *irreps*. Since we will choose d to be odd, the spinor *irrep* is irreducible. The number of fermions of each *irrep* is just enough to accommodate the Standard Model's symmetries into the unbroken flavor symmetry group, while having pNGBs with the correct quantum numbers to be identified as the Higgs field. When the spinor *irrep* is pseudoreal, the symmetry breaking pattern is assumed to be [17–19]

$$\frac{G}{H} = \frac{SU(4)}{Sp(4)} \times \frac{SU(6)}{SO(6)} \times U(1)_A,$$
(1.1)

which corresponds to 6 Weyl (equivalently, Majorana) fermions in the (real) vector *irrep*, plus 4 Weyl fermions (or 2 Diracs) in the spinor *irrep*.  $U(1)_A$  is the nonanomalous axial symmetry whose generator is a linear combination of the axial charges of the two *irreps*. Demanding that the SO(d) theory will be asymptotically free allows us to choose d = 5 or d = 11 [7,8].<sup>3</sup> In the case that the spinor *irrep* is real, the symmetry breaking pattern is

$$\frac{G}{H} = \frac{SU(5)}{SO(5)} \times \frac{SU(6)}{SO(6)} \times U(1)_A,$$
 (1.2)

which requires 5 Majorana fermions in the spinor *irrep*. The asymptotically free cases are d = 7 and d = 9.

The Standard-Model symmetries are embedded into the unbroken subgroup H as follows. The OCD gauge symmetry  $SU(3)_c$  together with (ordinary) baryon number B are embedded into the unbroken SO(6), while  $SU(2)_L$  and  $SU(2)_R$  are embedded into the Sp(4) subgroup of SU(4), or into the SO(5) subgroup of SU(5). For all the fields of the hypercolor theory, as well as for the quark fields that will couple to it (namely,  $t_L$ ,  $b_L$  and  $t_R$ ), the usual Standard-Model hypercharge is given by  $Y = T_R^3 + 2B$ , where  $T_R^3$  is the third  $SU(2)_R$  generator, and baryon number has the usual normalization with B = 1/3 for a single quark. With these conventions, the electric charge is  $Q = T_L^3 + Y = T_L^3 + T_R^3 + 2B$ . The simplest hyperbaryons which can play the role of top partners are hypercolor singlet states made out of two SO(d) spinors and one SO(d) vector, which belong to two-index *irreps* of the SU(4) or SU(5) flavor group.

We comment that Sp(4) is the covering group of SO(5), and SU(4) of SO(6).<sup>4</sup> For the purpose of this paper it does not matter if the gauge group is Sp(4) or SO(5), and we opt for SO(5) just so that most of the gauge groups we deal with (except in Sec. V) will be SO(d) groups with dodd.<sup>5</sup> The symmetry-breaking cosets SU(4)/Sp(4) and SO(6)/SO(5) are isomorphic, and following Ref. [8] we opt for the former.

This paper is organized as follows. In Sec. II, we introduce our notation for the SO(d) gauge theories, and construct all the dimension-9/2 hyperbaryons that can serve as top partners. In Sec. III, we proceed to study the case of a pseudoreal spinor irrep. Of the 5 pNGBs in the SU(4)/Sp(4) coset, four make up the Higgs doublet, while the last one,  $\eta$ , is inert under all the Standard Model symmetries. We begin by listing all the possible embeddings of the quark fields  $q_L = (t_L, b_L)$  and  $t_R$  into spurions belonging to two-index *irreps* of SU(4). We write down the most general four-fermion Lagrangian describing the interaction of these spurions with the hyperbaryons, finding that it contains 15 independent terms. We then work out the resulting effective potential for the pNGBs. Thanks to the simplicity of the SU(4)/Sp(4) coset, this potential can be obtained in closed form. We also work out all the lowenergy constants, which can be expressed in terms of hyperbaryons two-point functions. A summary of our results for this coset is given in Sec. III F.

In Sec. IV, we deal with the case that the spinor *irrep* is real. The coset SU(5)/SO(5) contains 14 NGBs, five of

<sup>&</sup>lt;sup>2</sup>For a discussion of mass generation for the bottom quark, see Ref. [13]. For general remarks on mass generation for other Standard Model fermions, see Sec. VI.

<sup>&</sup>lt;sup>3</sup>See Sec. III for d = 13.

<sup>&</sup>lt;sup>4</sup>There exist isomorphisms  $SO(5) \simeq Sp(4)/Z_2$  and  $SO(6) \simeq SU_{\epsilon}(4)/Z_2$ .

For lattice work on the Sp(4) gauge group, see Refs. [20,21].

which are the same as before: a (2,2) of  $SU(2)_L \times SU(2)_R$ that constitutes the Higgs doublet, and the singlet  $\eta$ . The remaining nine NGBs fill up the (3,3) representation. Again there are 15 independent couplings in the four-fermion Lagrangian. The presence of the isospin-triplet pNGBs makes the analysis technically more involved, and we calculate the full effective potential only to third order in the pNGB fields. As before, we also discuss the low-energy constants.

We then turn to the following important issue (Sec. IV E). The SU(5)/SO(5) effective potential will in general contain cubic terms of the form  $\sim h^2 \varphi$ , where *h* is the physical Higgs, and  $\varphi$  is one of the nine new pNGBs.<sup>6</sup> The effective potential for  $\varphi$  takes the form

$$V(\varphi) = c_1 f h^2 \varphi + c_2 f^2 \varphi^2 + O(\varphi^3), \qquad (1.3)$$

where f represents the scale of the hypercolor theory, and the coefficients  $c_{1,2}$  are dimensionless. When the Higgs field h condenses, the cubic term [the first term on the right-hand side of Eq. (1.3) induces a linear term for  $\varphi$ . This, in turn, forces the subsequent condensation of the  $\varphi$  field [8]. Assuming  $c_2 > 0$  (and neglecting the  $O(\varphi^3)$  terms), the minimum of this potential is  $\varphi =$  $-(c_1/(2c_2))h^2/f$ . If the coefficients  $c_{1,2}$  have a similar magnitude, the  $\varphi$  expectation value will be suppressed by only one power of h/f relative to h itself. This is problematic, because  $\varphi$  transforms nontrivially under  $SU(2)_L \times SU(2)_R$ , and an expectation value for  $\varphi$  does not preserve the custodial symmetry. This diagonal subgroup of  $SU(2)_L \times SU(2)_R$  is needed in order to protect the  $\rho$ -parameter [22], for which there are tight experimental constraints. To shed more light on this issue we also calculate the full potential in the case that all the SU(5)/SO(5) pNGBs are turned off except for h and  $\varphi$ , and we discuss whether, and if so, how, those problematic cubic terms might be avoided.

In Sec. V, we revisit the SU(4) composite Higgs model that was previously studied by Ferretti in Ref. [13], and by us in Ref. [23]. In the latter work, we made rather restrictive assumptions that lead to a four-fermion Lagrangian containing just two terms, and to an effective potential that is quartic in the four-fermion couplings. Here we take essentially the opposite approach, and make only the most minimal assumptions, which lead to a four-fermion Lagrangian containing six terms. We find that, in general, an effective potential is then generated already at second order in the four-fermion couplings. However, as we explain in the concluding section (Sec. VI), this potential may suffer from a serious drawback. In addition, for the four-fermion Lagrangian we studied in Ref. [23] we find that the effective potential contains two more terms that we overlooked in Ref. [23].

Because of the length of this paper, we have collected the main phenomenological lessons that can be drawn from all our analyses in Sec. VI. The appendices cover various technical points.

#### II. SO(d) GAUGE THEORIES

The SO(d) gauge theories we study in this paper have fermions in the vector and spinor *irreps*. Since d is always odd, the spinor *irrep* is irreducible. The Higgs field is identified with pNGBs that arise from chiral symmetry breaking of the spinor-*irrep* fermions. We denote the Weyl fermions in the spinor *irrep* as  $\Upsilon_i$ , where i is the flavor index. There will be 4 of them when the spinor *irrep* is pseudoreal, and 5 when it is real. The flavor symmetry group is, correspondingly,  $G_{\Upsilon} = SU(4)$  or SU(5). We find it convenient to construct the hypercolor baryons in terms of 4-component fields

$$\chi_i = \begin{pmatrix} \Upsilon_i \\ \mathcal{C}\epsilon \tilde{\Upsilon}_i^T \end{pmatrix}, \qquad (2.1)$$

and

$$\bar{\chi}_i = \chi_i^T C \mathcal{C}. \tag{2.2}$$

Here *C* is the four-dimensional charge-conjugation matrix, and *C* is the charge-conjugation matrix in d = 2n + 1dimensions. For our notation, Dirac algebra conventions, and for the properties of the charge conjugation matrix in various dimensions, see Appendix A. (When the spinor *irrep* is real, as in Sec. IV below, the  $\chi_i$  are Majorana fermions.) For  $g \in G_{\Upsilon}$ , a flavor transformation acts as  $\Upsilon \rightarrow g\Upsilon, \tilde{\Upsilon} \rightarrow \tilde{\Upsilon}g^{\dagger}$ , or, in terms of the 4-component fields,

$$\chi \to (gP_R + g^*P_L)\chi, \qquad \bar{\chi} \to \bar{\chi}(g^T P_R + g^{\dagger}P_L).$$
 (2.3)

The infinitesimal form is

$$\delta \chi = i (P_R T_a - P_L T_a^T) \chi, \qquad \delta \bar{\chi} = i \bar{\chi} (P_R T_a^T - P_L T_a),$$
(2.4)

with  $T_a$  the hermitian generators. As we will discuss in the following sections, the  $SU(2)_L$  and  $SU(2)_R$  symmetries of the Standard Model are embedded into  $H_{\Upsilon}$ , the unbroken flavor symmetry group of the spinor-*irrep* fermions.

In addition, all models will contain 6 Majorana fermions in the real, vector *irrep*, with an assumed associated chiral symmetry breaking pattern  $SU(6) \rightarrow SO(6)$ . As already mentioned, the Standard Model symmetries  $SU(3)_c$  and  $U(1)_B$ , where *B* is ordinary baryon number, are both subgroups of the unbroken SO(6). We find it convenient to regroup the 6 Majorana fermions into 3 Dirac fermions,

<sup>&</sup>lt;sup>6</sup>The precise definitions are given in Appendix F.

<sup>&</sup>lt;sup>7</sup>If  $c_2 < 0$ , this is likely to lead to a larger expectation value for  $\varphi$ .

TABLE I. Top partners. The first column defines the *irrep* of the flavor group to which the hyperbaryon belongs, which can be a two-index *irrep*, or a singlet. The second column defines the SO(d) gauge theory. The next two columns give the hyperbaryon and antihyperbaryon operators. The chiral projector  $P_X$ , X = L, R, is always acting on the open Dirac index. The notation  $\{\bar{\chi}_i \cdots \chi_j\}$  for the adjoint *irrep* refers to the traceless part of the bilinear. Our choice of signs for the antihyperbaryons is explained in Sec. II B. The minus sign in  $\bar{B}_{ji,X}^r$  for the D and N *irreps* for d = 5, 11 arises because of the difference between the CP transformation rules (2.5c) and (2.5d).

irrep	d	$B_{ij,X}^r$	$ar{B}^r_{ji,X}$
$\overline{A, A^c}$	5,7 9,11	$\begin{array}{c} (\bar{\chi}_i P_{R,L} \Gamma_l \chi_j) P_X \psi_{Ia} \\ (\bar{\chi}_i P_{R,L} \Gamma_I \sigma_{\mu\nu} \chi_j) P_X \sigma_{\mu\nu} \psi_{Ia} \end{array}$	$\frac{(\bar{\chi}_i P_{L,R} \Gamma_l \chi_j) \bar{\psi}_{Ia} (1 - P_X)}{(\bar{\chi}_i P_{L,R} \Gamma_l \sigma_{\mu\nu} \chi_j) \bar{\psi}_{Ia} \sigma_{\mu\nu} (1 - P_X)}$
$S, S^c$	5,7 9,11	$\begin{array}{c} (\bar{\chi}_i P_{R,L} \Gamma_I \sigma_{\mu\nu} \chi_j) P_X \sigma_{\mu\nu} \psi_{Ia} \\ (\bar{\chi}_i P_{R,L} \Gamma_I \chi_j) P_X \psi_{Ia} \end{array}$	$\begin{array}{c} (\bar{\chi}_i P_{L,R} \Gamma_I \sigma_{\mu\nu} \chi_j) \bar{\psi}_{Ia} \sigma_{\mu\nu} (1 - P_X) \\ (\bar{\chi}_i P_{L,R} \Gamma_I \chi_j) \bar{\psi}_{Ia} (1 - P_X) \end{array}$
D	5,11 7,9	$ \begin{array}{l} \{ \bar{\chi}_i P_R \Gamma_I \gamma_\mu \chi_j \} P_X \gamma_\mu \psi_{Ia} \\ \{ \bar{\chi}_i P_R \Gamma_I \gamma_\mu \chi_j \} P_X \gamma_\mu \psi_{Ia} \end{array} $	$-\{\bar{\chi}_i P_L \Gamma_I \gamma_\mu \chi_j\} \bar{\psi}_{Ia} \gamma_\mu (1 - P_X) \{\bar{\chi}_i P_L \Gamma_I \gamma_\mu \chi_j\} \bar{\psi}_{Ia} \gamma_\mu (1 - P_X)$
N	5,11 7,9	$\delta_{ij}(\bar{\chi}_k P_R \Gamma_I \gamma_\mu \chi_k) P_X \gamma_\mu \psi_{Ia} \\ \delta_{ij}(\bar{\chi}_k P_R \Gamma_I \gamma_\mu \chi_k) P_X \gamma_\mu \psi_{Ia}$	$ \begin{array}{c} -\delta_{ij}(\bar{\chi}_{k}P_{L}\Gamma_{I}\gamma_{\mu}\chi_{k})\bar{\psi}_{Ia}\gamma_{\mu}(1-P_{X}) \\ \delta_{ij}(\bar{\chi}_{k}P_{L}\Gamma_{I}\gamma_{\mu}\chi_{k})\bar{\psi}_{Ia}\gamma_{\mu}(1-P_{X}) \end{array} $

 $\psi_{Ia}, \bar{\psi}_{Ia}$ , where I = 1, 2, ..., d is the SO(d) vector index, while a = 1, 2, 3 indexes ordinary color. Like quarks, the baryon number of these Dirac fermions is 1/3. The baryon number of the  $\chi$  fermions is zero.<sup>8</sup>

The embedding of the Standard Model symmetries is such that the pNGBs in the SU(6)/SO(6) coset carry ordinary color, but no  $SU(2)_L \times SU(2)_R$  quantum numbers. Since in this paper we are mainly interested in the Higgs potential, we will mostly ignore the SU(6)/SO(6) pNGBs.

### A. Top-partner hypercolor baryons

We will restrict the discussion to the simplest top partners, which are created by local 3-fermion operators constructed as follows. We first assemble two SO(d)spinor fermions into a bilinear transforming as an SO(d)vector, and then contract this bilinear with an SO(d)-vector fermion to form an SO(d)-singlet state. The resulting hyperbaryon and antihyperbaryon fields are tabulated in Table I. Unless it forms a singlet, the SO(d)-spinor bilinear belongs to one of the two-index *irreps* of the flavor group  $G_{\Upsilon}$ , which, we recall, can be SU(4) or SU(5). When a single four-dimensional Dirac matrix (aside from the chiral projector) is sandwiched between the two  $\chi$  fermions, we encounter the adjoint *irrep* (D), or a singlet (N). When the number of four-dimensional Dirac matrices is zero or two, the same chiral projector is applied to both of the  $\chi$ fermions, and the bilinear then has definite symmetry properties on its spin index. Taking into account also the symmetry properties on the SO(d) index (see the last column of Table V) fixes the symmetry on the flavor index. In view of Eq. (2.3), when the chiral projector is  $P_R$  we encounter the two-index symmetric (S) or two-index antisymmetric (A) representations, whereas for  $P_L$  we obtain the complex conjugate representations  $S^c$  and  $A^{c,9}$ 

We use the following notation. A generic hyperbaryon is denoted  $B_{ij,X}^r$ , where *i* and *j* are flavor indices, and the optional subscript X = L, *R* denotes the projector applied to the open Dirac index, which in turn is always carried by the  $\psi$  fermion. *r* labels the *irrep*, which can be one of *D*, *N*, *S*,  $S^c$ , *A* or  $A^c$ . Our notation is such that the antihyperbaryon of  $B_{ij}^r$  is denoted  $\bar{B}_{ji}^r$ , with the flavor indices flipped. This will prove convenient when using matrix notation in flavor space.

We comment in passing that the Ferretti-Karateev list of requirements is fairly restrictive [7,8]. Models that satisfy all the requirements and have a prescribed coset structure of the effective theory are so few, that in effect, knowing the coset structure essentially fixes the model, and thus, ultimately, also the top-partner content. However, by itself, the coset structure does not tell us what will be the *irreps* to which the top-partners belong. For example, the models of Sec. IV and Sec. V both share an SU(5)/SO(5) coset. But in Sec. IV the hyperbaryons belong to 2-index irreps of SU(5), whereas in Sec. V they belong to the (anti) fundamental *irrep*. Thus, the straightforward way to find the top partners of a given model is to explicitly construct the relevant gauge invariant operators. Of course, the explicit form of the hyperbaryon operators will also be needed for the derivation of the low-energy constants.

### B. CP symmetry

As a stand-alone theory, all the hypercolor theories we study in this paper are invariant under *C* and *P*. Because we couple the hypercolor theory to  $q_L = (t_L, b_L)$  and to  $t_R$ , but not to  $b_R$ , the four-fermion Lagrangian can be invariant

<sup>&</sup>lt;sup>8</sup>According to our naming conventions the roles of  $\chi$  and  $\psi$  are flipped relative to Refs. [7,8].

<sup>&</sup>lt;sup>9</sup>For the  $SO(5) \sim Sp(4)$  gauge theory, the N, D and A hyperbaryons were previously considered in Refs. [17,18].

only under the combined *CP* transformation.<sup>10</sup> The *CP* transformation acts on a gauge field as

$$A_{\mu}(x) \to \tilde{A}_{\mu}(\tilde{x}),$$
 (2.5a)

where  $\tilde{x}_{\mu} = x_{\mu}$  if  $\mu = 4$ , and  $\tilde{x}_{\mu} = -x_{\mu}$  if  $\mu = 1, 2, 3$ , with a similar definition for  $\tilde{A}_{\mu}$ . The SO(d) gauge field is invariant under charge conjugation, so that its transformation rule stems from parity only. The SO(d)-vector Dirac fermions transform as

$$\psi(x) \to i\gamma_4 C \bar{\psi}(\tilde{x})^T, \qquad \bar{\psi}(x) \to -i\psi(\tilde{x})^T C \gamma_4.$$
 (2.5b)

Except for the choice of phases, which is explained in Appendix B, this is the usual *CP* transformation rule of a Dirac fermion. The  $\chi$  fields transform according to

$$\chi_i(x) \to i\gamma_4\chi_i(\tilde{x}), \qquad \bar{\chi}_i(x) \to -i\bar{\chi}_i(\tilde{x})\gamma_4, \qquad (2.5c)$$

in the case that the spinor *irrep* is real ( $C = C^T$ ), whereas for the pseudoreal case ( $C = -C^T$ ) their transformation rule is

$$\chi_i(x) \to i\gamma_4\gamma_5\chi_i(\tilde{x}), \qquad \bar{\chi}_i(x) \to -i\bar{\chi}_i(\tilde{x})\gamma_5\gamma_4.$$
 (2.5d)

The induced transformation of the hyperbaryon fields is

$$B_{ij}^r(x) \to i\gamma_4 C \bar{B}_{ji}^r(\tilde{x})^T, \quad \bar{B}_{ij}^r(x) \to -i B_{ji}^r(\tilde{x})^T C \gamma_4.$$
 (2.6)

The sign choices we have made in Table I ensure that all hyperbaryons transform under *CP* like the  $\psi$  fermions. For more details, see Appendix B.

#### III. THE SU(4)/Sp(4) COSET

There are two models where the spinor *irrep* is pseudoreal, one based on an SO(5) gauge group and the other on SO(11). The SO(13) theory is asymptotically free as well, but according to analytic considerations it is probably inside the conformal window, and not chirally broken [8,24]. In any event, since all the relevant properties of the SO(d) theories are periodic in *d* modulo 8, the discussion of the SO(5)theory would carry over as is to the SO(13) case, if the latter were to be chirally broken. For previous work on the SU(4)/Sp(4) models, see Refs. [8,17–19,24].

The order parameter for the spontaneous breaking of the flavor symmetry  $G_{\Upsilon} = SU(4)$  is the expectation value of  $\bar{\chi}_i \chi_j$ . This order parameter is antisymmetric on its flavor indices. We will assume that  $\langle \bar{\chi}_i \chi_j \rangle \propto \epsilon_{0,ij}$ , where the 4 × 4 matrix  $\epsilon_0$  is defined in Eq. (C2). With this convention, we may take the order parameter to be  $\langle \bar{\chi} \epsilon_0 \chi \rangle$ . Applying an infinitesimal flavor transformation (2.4) to the order parameter we get

$$\delta_a(\bar{\chi}\epsilon_0\chi) = i\bar{\chi}(P_R(\epsilon_0T_a + T_a^I\epsilon_0) - P_L(T_a\epsilon_0 + \epsilon_0T_a^I))\chi.$$
(3.1)

Of the 15 generators of SU(4), there are 10 which leave the order parameter invariant [see Eq. (C5)]. They generate the unbroken group,  $H_{\Upsilon} = Sp(4)$ .

The remaining 5 generators belong to the coset  $G_{\Upsilon}/H_{\Upsilon} = SU(4)/Sp(4)$ . Taking  $T_a$  to be a coset generator, the variation of the order parameter gives rise to an interpolating field for one of the NGBs,

$$\delta_a(\bar{\chi}\epsilon_0\chi) = 2i\bar{\chi}(P_R\epsilon_0T_a - P_LT_a\epsilon_0)\chi. \qquad (3.2)$$

Equivalently, the full NGB field is

$$\hat{\Pi} = 2i\widehat{\mathrm{tr}}(P_R\chi\bar{\chi}\epsilon_0 - P_L\epsilon_0\chi\bar{\chi}), \qquad (3.3)$$

where the notation  $\hat{\mathbf{r}}$  indicates that the trace is over the Dirac and color indices, but not over the flavor indices. It readily follows that  $\epsilon_0 \hat{\Pi}$  (or  $\hat{\Pi} \epsilon_0$ ) is antisymmetric on its flavor indices. The flavor trace of  $\hat{\Pi}$  with a coset generator reproduces Eq. (3.2), while its trace with an Sp(4) generator vanishes identically, showing that  $\hat{\Pi}$  has the correct number of degrees of freedom. Using Eqs. (2.3) and (C1), we see that  $\hat{\Pi}$  transforms in the expected way under the unbroken group

$$\hat{\Pi} \to g\hat{\Pi}g^{\dagger}, \qquad g \in Sp(4).$$
 (3.4)

Under the CP transformation of the hypercolor theory, Eq. (2.5d), we have

$$\hat{\Pi}(x) \to -\hat{\Pi}^T(\tilde{x}),$$
 (3.5)

where we have used Eq. (2.2). Notice that (apart from the usual coordinates transformation) the *CP* transformation does not merely flip the sign of  $\hat{\Pi}$ . Related, when the coset generator  $T_a$  commutes with  $\epsilon_0$ , the NGB field (3.2) is a pseudoscalar, as in the familiar QCD case. But when  $T_a$  anticommutes with  $\epsilon_0$ , the NGB field is a scalar. We will discuss the phenomenological significance of this result shortly.

In the effective chiral theory, the NGBs of  $SU(4) \rightarrow Sp(4)$  symmetry breaking are represented by an antisymmetric unitary field  $\Sigma \in SU(4)$ ,  $\Sigma^T = -\Sigma$ . In addition, the effective theory depends on an SU(6)/SO(6) nonlinear field, which we will not discuss in this paper, and a field  $\Phi \in U(1)$  associated with the spontaneous breaking of the nonanomalous  $U(1)_A$  symmetry [7,17–19,23–26]. The axial transformations are

$$\delta_A \chi = \frac{i}{2} \gamma_5 \chi, \qquad \delta_A \bar{\chi} = \frac{i}{2} \bar{\chi} \gamma_5, \qquad (3.6a)$$

$$\delta_A \psi = i q \gamma_5 \psi, \qquad \delta_A \bar{\psi} = i q \bar{\psi} \gamma_5, \qquad (3.6b)$$

$$\delta_A \Phi = i\Phi. \tag{3.6c}$$

<sup>&</sup>lt;sup>10</sup>Provided that all the four-fermion coupling constants are real.

Eq. (3.6a) gives the transformation rule of the spinor *irrep*, which sets the normalization of the nonanomalous axial transformation in the microscopic theory. Eq. (3.6b) is the transformation rule of the vector *irrep*, where  $q = -(1/3)T_{\chi}/T_{\psi}$ ,<sup>11</sup> and the group traces are  $T_{\chi} = 2^{(d-5)/2}$  and  $T_{\psi} = 2$ . Finally Eq. (3.6c) sets our normalization for the transformation rule of the corresponding effective field. The formal correspondence between the elementary and the effective fields is then

$$\Phi\Sigma \leftrightarrow \widehat{\operatorname{tr}}(P_R \chi \overline{\chi}), \qquad \Phi^* \Sigma^* \leftrightarrow \widehat{\operatorname{tr}}(P_L \chi \overline{\chi}).$$
 (3.7)

As already mentioned, we will assume that the vacuum is given by  $\langle \Sigma \rangle = \epsilon_0$  and  $\langle \Phi \rangle = 1$ , and parametrize the nonlinear field as

$$\Sigma = \exp(i\Pi/f)\epsilon_0 \exp(i\Pi^T/f) = \exp(2i\Pi/f)\epsilon_0, \quad (3.8)$$

where f is the decay constant. The effective NGB field  $\Pi$  is hermitian, traceless, and satisfies  $\epsilon_0 \Pi = \Pi^T \epsilon_0$ , just as  $\hat{\Pi}$ . Flavor transformations act on the nonlinear field as

$$\Sigma \to g\Sigma g^T, \qquad g \in SU(4).$$
 (3.9)

For  $g \in Sp(4)$ , it follows that the effective NBG field  $\Pi$  transforms in the same way as the NGB field of the microscopic theory, Eq. (3.4). The transformation rule of  $\Pi$  under *CP* is defined to be the same as in Eq. (3.5). The leading-order chiral Lagrangian is invariant under these transformations.

The embedding of  $SU(2)_L$  and  $SU(2)_R$  in Sp(4) is given in Eq. (C6), and the parametrization of the effective field  $\Pi$ is given in Eq. (C7). Four of the NGBs are identified with the Higgs doublet,  $H = (H_+, H_0)$ , whereas the fifth,  $\eta$ , is a singlet under  $SU(2)_L \times SU(2)_R$ . Using the parametrization (C7), a *CP* transformation acts as

$$H_0 \to H_0^*, \qquad H_+ \to H_+^*, \qquad \eta \to -\eta.$$
 (3.10)

This correctly reproduces the *CP* transformation of the Higgs field in the Standard Model.

The rest of this section is organized as follows. In Sec. III A, we obtain all the spurion embeddings of the quark fields. In Sec. III B, we write down the four-fermion Lagrangian  $\mathcal{L}_{EHC}$ , and in Sec. III C we list all the effective top Yukawa couplings allowed by it. In Sec. III D, we begin the discussion of the effective potential of the pNGBs,  $V_{eff}$ . We group the various contributions into twelve "template" forms, and then work out all the contributions to  $V_{eff}$  in closed form. In Sec. III E, we derive the low-energy constants. We summarize our findings in Sec. III F, which

also contains a simple example of a phenomenologically viable potential. Finally, we discuss spontaneous *CP* breaking in Sec. III G.

### A. Spurions

Much like in technicolor theories, the coupling of the Higgs field to the gauge bosons of the Standard Model arises naturally when the relevant global symmetries of the hypercolor theory are gauged; but a more elaborate setup is needed to generate masses for fermions. Here we postulate the existence of yet another gauge symmetry, dubbed "extended hypercolor" (EHC). We assume that the EHC gauge symmetry breaks spontaneously at some scale  $\Lambda_{\text{EHC}}$ which is large relative to the scale of the hypercolor theory,  $\Lambda_{\rm HC}$ . The remnant of the EHC interactions at the hypercolor scale is a set of four-fermion interactions, and we assume that these four-fermion interactions couple the third generation quark fields  $q_L = (t_L, b_L)$  and  $t_R$  to the hyperbaryon fields constructed in Sec. II A. The EHC theory will thus generate a mass for the top quark through the mechanism of partial compositeness. We comment that this setup does not necessarily generate a mass for any other Standard Model's fermion. Their masses may have to involve some other dynamics (see Sec. VI).

Unlike the hyperbaryon fields, quark fields fit into *irreps* of the smaller, Standard-Model symmetry. They do not fill up any *irreps* of the global symmetry group of the hypercolor theory. The coupling of quark and hyperbaryon fields therefore explicitly breaks the flavor symmetry of the hypercolor theory. This will induce a potential  $V_{\rm eff}$  for the NGBs.

While  $V_{\text{eff}}$  is invariant only under Standard-Model symmetries, it depends on low-energy constants that can be expressed in terms of correlation functions of the standalone hypercolor theory. When we derive expressions for these low-energy constants, we may benefit from the full global symmetry of the hypercolor theory, including in particular  $G_{\Upsilon}$ . The way to do this is to promote the quark fields to spurion fields transforming in *irreps* of  $G_{\Upsilon}$ .

In the rest of this subsection, we construct the spurions explicitly. Each embedding of  $q_L$  is defined by

$$X_L(x) = t_L(x)\hat{X}_{t_L} + b_L(x)\hat{X}_{b_L},$$
  
$$\bar{X}_L(x) = \bar{t}_L(x)\hat{X}_{t_L} + \bar{b}_L(x)\hat{X}_{b_L},$$
 (3.11)

and similarly for  $t_R$ ,

$$X_R(x) = t_R(x)\hat{X}_{t_R}, \qquad \bar{X}_R(x) = \bar{t}_R(x)\hat{X}_{t_R}, \qquad (3.12)$$

where the hatted objects are constant  $4 \times 4$  matrices. Because the EHC theory is not known, we will allow for the most general four-fermion Lagrangian which is compatible with the (spurionized) symmetries of the hypercolor theory, and with *CP*.

<sup>&</sup>lt;sup>11</sup>In the case of the SU(5)/SO(5) models of Sec. IV,  $q = -(5/12)T_{\chi}/T_{\psi}$ . For more details see, e.g., Refs. [24,26,27].

In order to build the four-fermion Lagrangian, we have to allow for all embeddings of the quark fields into spurions belonging to two-index *irreps* of  $G_{\Upsilon} = SU(4)$  (or to a singlet), which are consistent with the embedding of  $SU(2)_L$  and  $SU(2)_R$  into SU(4). We begin with the spurion embeddings of  $q_L$ . For the adjoint *irrep* of SU(4) there are two options,

Remembering that  $q_L = (t_L, b_L)$  is an  $SU(2)_L$  doublet with  $T_R^3 = -1/2$ , one can check that these spurions are consistent with the Standard-Model transformation properties of  $q_L$ . To this end we use that the adjoint spurions transform as  $D_L^i \rightarrow g D_L^i g^{\dagger}$  under  $g \in SU(4)$ , and the embedding (C6) of  $SU(2)_L$  and  $SU(2)_R$  into SU(4). For the two-index antisymmetric *irrep* we have one embedding,

$$A_L = \begin{pmatrix} 0 & 0 & 0 & t_L \\ 0 & 0 & 0 & b_L \\ 0 & 0 & 0 & 0 \\ -t_L & -b_L & 0 & 0 \end{pmatrix},$$
(3.15)

and likewise for the two-index symmetric irrep,

$$S_L = \begin{pmatrix} 0 & 0 & 0 & t_L \\ 0 & 0 & 0 & b_L \\ 0 & 0 & 0 & 0 \\ t_L & b_L & 0 & 0 \end{pmatrix}.$$
 (3.16)

The  $A_L$  and  $S_L$  spurions transforms as  $X_L \rightarrow gX_Lg^T$ ,  $X \in \{A, S\}$ , under  $g \in SU(4)$ , and again one can verify consistency with Standard-Model quantum numbers. The embeddings for the complex conjugate *irreps*  $A^c$  and  $S^c$  may be obtained using the rule

$$X^c = -\epsilon_0 X \epsilon_0, \tag{3.17}$$

where again  $X \in \{A, S\}$ . Let us explain this rule. We first observe that  $X^c$  spurions transform under  $g \in SU(4)$  as  $X^c \to g^* X^c g^{\dagger}$ . Restricting to  $g \in Sp(4)$ , and using Eqs. (3.17) and (C1), we have

$$g^* X^c g^{\dagger} = -g^* \epsilon_0 X \epsilon_0 g^{\dagger} = -\epsilon_0 g X g^T \epsilon_0.$$
 (3.18)

The rightmost expression involves the transformation rule of a field in the *A* or *S irreps*, and we have already verified that this correctly reproduces the Standard-Model transformation rules for the  $A_L$  and  $S_L$  spurions. Since  $SU(2)_L \times SU(2)_R$  is a subgroup of Sp(4), it follows that the spurion  $X^c$  defined by Eq. (3.17) will again reproduce the correct Standard-Model transformation rules. Applying Eq. (3.17) we find the explicit forms

$$A_{L}^{c} = \begin{pmatrix} 0 & 0 & -b_{L} & 0\\ 0 & 0 & t_{L} & 0\\ b_{L} & -t_{L} & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad (3.19)$$
$$S_{L}^{c} = \begin{pmatrix} 0 & 0 & -b_{L} & 0\\ 0 & 0 & t_{L} & 0\\ -b_{L} & t_{L} & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}. \qquad (3.20)$$

Let us move on to  $t_R$ , which is a singlet of  $SU(2)_L$  with  $T_R^3 = 0$  (note that  $t_R$  is not required to be invariant under the full  $SU(2)_R$ , but only under rotations generated by  $T_R^3$ ). In this case, we have more options, starting with the singlet

$$N_R = t_R \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
 (3.21)

There are two linearly independent options for the adjoint *irrep*,

another two for the antisymmetric irrep,

and one for the symmetric *irrep*,

The spurion embeddings for the  $A^c$  and  $S^c$  *irreps* again follow using Eq. (3.17). Explicitly,

$$A_R^{c1} = A_R^1,$$
  
 $A_R^{c2} = A_R^2,$   
 $S_R^c = -S_R.$  (3.27)

It remains to construct the antispurion embeddings. Referring to the decompositions (3.11) and (3.12), we define the *c*-number coefficients of the antispurion fields via

$$\hat{\bar{X}} \equiv \hat{X}^{\dagger} = \hat{X}^T. \tag{3.28}$$

The last equality follows because we have chosen all the *c*-number spurions  $\hat{X}$  to be real.

# B. $\mathcal{L}_{EHC}$

With the top-partner hyperbaryons and the spurions at hand, the most general four-fermion Lagrangian that couples them is given by

$$\mathcal{L}_{\rm EHC} = \mathcal{L}_{\rm EHC,1} + \mathcal{L}_{\rm EHC,2}, \qquad (3.29a)$$

$$\mathcal{L}_{\text{EHC},1} = \text{tr}(\lambda_1 \bar{A}_L B_R^A + \lambda_2 \bar{A}_L^c B_R^{A^c} + \lambda_3 \bar{S}_L B_R^S + \lambda_4 \bar{S}_L^c B_R^{S^c} + (\lambda_5 \bar{D}_R^1 + \lambda_6 \bar{D}_R^2) B_L^D + \lambda_7 \bar{N}_R B_L^N + \text{H.c.}),$$
(3.29b)

$$\begin{aligned} \mathcal{L}_{\text{EHC},2} &= \text{tr}((\tilde{\lambda}_1 \bar{A}_R^1 + \tilde{\lambda}_2 \bar{A}_R^2) B_L^A + (\tilde{\lambda}_3 \bar{A}_R^{c1} + \tilde{\lambda}_4 \bar{A}_R^{c2}) B_L^{A^c} \\ &+ \tilde{\lambda}_5 \bar{S}_R B_L^S + \tilde{\lambda}_6 \bar{S}_R^c B_L^{S^c} \\ &+ (\tilde{\lambda}_7 \bar{D}_L^1 + \tilde{\lambda}_8 \bar{D}_L^2) B_R^D + \text{H.c.}), \end{aligned}$$
(3.29c)

where the trace is over SU(4) indices.  $\lambda_1, ..., \lambda_7$  and  $\tilde{\lambda}_1, ..., \tilde{\lambda}_8$  are (dimensionful) coupling constants. We have grouped in  $\mathcal{L}_{\text{EHC},1}$  those terms where  $B_L$  belongs to D or N, while  $B_R$  belongs to A,  $A^c$ , S or  $S^c$ , and the other way around for  $\mathcal{L}_{\text{EHC},2}$ . The four-fermion Lagrangian is invariant under the spurionized SU(4) symmetry. In addition, it is

truly invariant under the Standard-Model gauge symmetries  $SU(3)_c$ ,  $SU(2)_L$ , and  $U(1)_Y$ , and it conserves baryon number, or, which is equivalent, the  $T_R^3$  charge.

Assuming that all the coupling constants are real, the four-fermion Lagrangian is also invariant under the combined *CP* transformation of the hypercolor theory and the Standard Model, in which the *c*-number spurions are inert. How *CP* works is best illustrated through an example. The *CP* rules of Sec. II B imply in particular that  $\bar{t}_R B_{L,ij} \leftrightarrow \bar{B}_{L,ji} t_R$ . Remembering that *c*-number spurions don't transform, we have

$$\operatorname{tr}(\bar{X}_R B_L) = \bar{t}_R \operatorname{tr}(\hat{\bar{X}}_R B_L) \leftrightarrow \operatorname{tr}(\hat{\bar{X}}_R \bar{B}_L^T) t_R$$
$$= \operatorname{tr}(\hat{X}_R^T \bar{B}_L^T) t_R = \operatorname{tr}(\bar{B}_L X_R), \qquad (3.30)$$

where again the trace and transpose operations are applied to the flavor indices. In order to establish the CP-invariance of  $\mathcal{L}_{EHC}$ , we have used Eq. (3.28), which in turn relies on the fact that all the *c*-number spurions are real. That such a choice can be made, is a special feature of the SU(4)/Sp(4) coset. (As we will see in Sec. IV, things are slightly more involved for the SU(5)/SO(5) case.) Of course, we could have chosen to multiply some *c*-number spurions by arbitrary phases. This would invalidate Eq. (3.28) for those *c*-number spurions, and, as a result, there would be fewer terms in  $\mathcal{L}_{EHC}$  if we wish to maintain *CP* invariance. However, opting to do this is arbitrary. Once again, the point is that apart from some very general assumptions, we do not know the EHC theory. Therefore, we must consider the most general four-fermion Lagrangian consistent with those general assumptions. When all the four-fermion couplings are taken to be real, this requires choosing all the *c*-number spurions to be real as well.

As already noted, in this paper we do not study the SU(6)/SO(6) pNGBs associated with the vector-*irrep* fermions, and therefore we only gave the  $SU(3)_c$  quantum numbers of the hyperbaryons. Requiring full SU(6) invariance will give rise to the same four-fermion Lagrangian once the spurions assume their Standard Model values. Indeed, each term in Eq. (3.29) can be trivially "lifted" to an SU(6)-invariant form, as we illustrate through the following examples. For definiteness, we will refer to the hyperbaryons of the SO(5) gauge theory.

We begin with the first term on the right-hand side of Eq. (3.29b),  $\bar{A}_L B_R^A$ . Since the  $\chi$  fermions play little role, for brevity we express the hyperbaryon operator as  $B_{R,a}^A = f(\chi)_I^A \psi_{R,Ia} = f(\chi)_I^A \Psi_{Ia}$ , where in the last equality we have used that the 3 Dirac fermions introduced earlier are composed of 6 right-handed vector-*irrep* Weyl fermions  $\Psi_1, \ldots, \Psi_6$  according to

$$\psi_a = \begin{pmatrix} \Psi_a \\ \epsilon \bar{\Psi}_{3+a}^T \end{pmatrix}, \quad \bar{\psi}_a = \begin{pmatrix} -\Psi_{3+a}^T \epsilon & \bar{\Psi}_a \end{pmatrix}, \quad a = 1, 2, 3.$$
(3.31)

A complete SU(6) irrep is now obtained by simply replacing the index a = 1, 2, 3, with a new index  $\bar{a} = 1, ..., 6$ , explicitly,  $\tilde{B}^{A}_{R,\bar{a}} = f(\chi)^{A}_{I}\Psi_{I\bar{a}}$ , where we are using a tilde to refer to SU(6)irreps. The SU(6)-invariant interaction is thus  $\tilde{A}_{L,\bar{a}}\tilde{B}^{A}_{R,\bar{a}}$ . In order to ensure equality between the  $SU(3)_{c}$  and SU(6)versions, we simply embed the  $SU(3)_{c}$  spurion into the SU(6)spurion, namely, we define  $\tilde{A}_{L,\bar{a}} = \bar{A}_{L,a}$  for  $\bar{a} = a = 1, 2, 3$ , and  $\tilde{A}_{L,\bar{a}} = 0$  for  $\bar{a} = 4, 5, 6$ .

At this point we have not made use of the last three components of the SU(6) multiplet,  $\tilde{B}^A_{R,\bar{a}}$ ,  $\bar{a} = 4$ , 5, 6. These components occur in a difference place in  $\mathcal{L}_{EHC}$ , in the term that involve  $\bar{B}^A_L$ , and, thus, contains  $\bar{\psi}_{L,a} \sim \Psi_{a+3}$ , as follows from Eq. (3.31). This time, we "lift" the spurions to SU(6) by letting  $\tilde{A}^i_{R,\bar{a}} = 0$  for  $\bar{a} = 1$ , 2, 3, and  $\tilde{A}^i_{R,\bar{a}} = A^i_{R,\bar{a}-3}$  for  $\bar{a} = 4$ , 5, 6 and i = 1, 2.

These examples demonstrate that there is one-to-one correspondence between the  $SU(3)_c$ -invariant and SU(6)-invariant forms of  $\mathcal{L}_{EHC}$ . The underlying reason is that the "expectation values" of the spurions are only constrained by SM symmetries.

### C. Top Yukawa couplings

Effective top-Yukawa couplings are generated by integrating out all the states of the hypercolor theory except for the pNGBs. These effective interactions are organized in a weak-coupling expansion in the four-fermion couplings, as well as according to the usual power counting of the chiral Lagrangian. To second order in the four-fermion couplings, and to leading order in the chiral expansion, we find effective interactions that are either linear or bilinear in  $\Sigma$  or  $\Sigma^*$ . Any effective interaction which is cubic or higher in the nonlinear field must contain additional derivatives and/or mass insertions,<sup>12</sup> and therefore belongs to a higher order in the chiral expansion.

We begin with effective interactions that are linear in  $\Sigma$  or  $\Sigma^*$ . Each effective interaction contains one spurion and one antispurion, one of which must be left-handed and the other right-handed. The effective Yukawa interactions have the same symmetries as  $\mathcal{L}_{EHC}$ . In order to form an SU(4)singlet, the spurion must belong to  $A, A^c, S$  or  $S^c$  and the antispurion to D or N, or the other way around, because the effective interaction has to contain a  $\Sigma$  or a  $\Sigma^*$ . It follows that the spurion and the antispurion must both come from  $\mathcal{L}_{EHC,1}$ , or both from  $\mathcal{L}_{EHC,2}$ , which explains why we have grouped the four-fermion interactions this way. The list of possible top-Yukawa effective interactions is thus

TABLE II. Axial charges of the hyperbaryons.

$B^A_{R,L}$	$B^{A^c}_{R,L}$	$B^S_{R,L}$	$B^{S^c}_{R,L}$	$B^D_{R,L}$	$B^N_{R,L}$
$1 \pm q$	$-1 \pm q$	$1 \pm q$	$-1 \pm q$	$\mp q$	$\mp q$

where the hermitian conjugate is to be added to each operator.  $X_L$  can be  $A_L$  or  $S_L$ , and  $X_R$  can be  $A_R^{1,2}$  or  $S_R$ . The explicit form of each effective interaction can be worked out by assigning to each spurion its Standard-Model value from Sec. III A, and using Eqs. (C7) and (C9) for the  $\Sigma$  field. Out of a total of 22 possible contributions to the top-Yukawa coupling, we find that 8 of the possibilities vanish identically, while the other 14 generate a nonzero top-Yukawa coupling.

Each effective interaction in Eq. (3.32) is (formally) invariant under SU(4) and  $U(1)_A$ .<sup>13</sup> The power of  $\Phi$  is fixed by the axial charges of the spurions, which, in turn, are determined by the axial charges of the hyperbaryons, and the requirement that the four-fermion Lagrangian (3.29) will be invariant. For example, the power of  $\Phi$  in the first effective interaction matches the axial charge of the product  $B_R^A \bar{B}_L^D$  (or  $B_R^S \bar{B}_L^D$ ). See Eq. (3.6) for the axial transformations, and Table I for the field content of the hyperbaryons. The axial charges of the hyperbaryons are listed in Table II. Notice that the dependence on the axial charge q of the vector *irrep* always cancels out in the effective Yukawa interactions.

Similar considerations give rise to the list of effective interactions which are bilinear in  $\Sigma$  or  $\Sigma^{\dagger}$ , given by

$$\begin{aligned} &\operatorname{tr}(\bar{A}_{L}\Sigma)\operatorname{tr}(A_{R}\Sigma^{*}), & \operatorname{tr}(A_{L}^{c}\Sigma)\operatorname{tr}(\bar{A}_{R}^{c}\Sigma^{*}), & \operatorname{tr}(\bar{D}_{L}^{T}\Sigma^{*}D_{R}\Sigma), \\ &\Phi^{2}\operatorname{tr}(A_{L,R}^{c}\Sigma)\operatorname{tr}(\bar{A}_{R,L}\Sigma), & \Phi^{2}\operatorname{tr}(\bar{S}_{R,L}\Sigma S_{L,R}^{c}\Sigma), \\ &\Phi^{2}\operatorname{tr}(\bar{A}_{R,L}\Sigma A_{L,R}^{c}\Sigma), \end{aligned}$$
(3.33)

where again the hermitian conjugate is to be added to each operator. This amounts to 18 additional possibilities, none of which vanish.

The coupling constant that multiplies a given effective top-Yukawa interaction term is obtained using the procedure that we have discussed in detail in Ref. [23]. As an example, let us consider the term  $\Phi tr(\bar{A}_L \Sigma N_R)$ . Denoting by  $y_{A_L,N_R}$  the coupling constant that multiplies this term in the effective theory, and using  $\langle \Phi \rangle = 1$ , we have

$$\frac{\partial}{\partial N_{Raa}(y)} \frac{\partial}{\partial \bar{A}_{Lijb\beta}(x)} \log Z_{\text{eff}}$$

$$= -y_{A_L,N_R} \langle \Sigma_{ji} \rangle \delta_{ab} \delta_{a\beta} \delta(x-y) + \cdots$$

$$= y_{A_L,N_R} \epsilon_{0,ij} \delta_{ab} \delta_{a\beta} \delta(x-y) + \cdots, \qquad (3.34)$$

<sup>&</sup>lt;sup>12</sup>See Sec. III F below for a discussion of explicit mass terms for the fermions of the hypercolor theory.

<sup>&</sup>lt;sup>13</sup>To maintain the invariance under SU(6) we would have to reintroduce the corresponding nonlinear field.

where *a*, *b* are SU(3)-color indices,  $\alpha$ ,  $\beta$  are Dirac indices, and we have treated the singlet *N* as an SU(4) scalar with no flavor indices. In the microscopic theory,

$$\frac{\partial}{\partial N_{Ra\alpha}(y)} \frac{\partial}{\partial \bar{A}_{Lijb\beta}(x)} \log Z = \lambda_1 \lambda_7 \langle B^A_{Rjib\beta}(x) \bar{B}^N_{La\alpha}(y) \rangle.$$
(3.35)

Demanding equality between the effective and microscopic theories, and using that the right-hand side of Eq. (3.34) is the leading term in a derivative expansion, we obtain

$$y_{A_L,N_R} = \frac{\lambda_1 \lambda_7}{48} \epsilon_{0,ij} \delta_{ab} \delta_{\alpha\beta} S_{ijab\alpha\beta}(0), \qquad (3.36)$$

where

$$\langle B^A_{Rija\alpha}(x)\bar{B}^N_{Lb\beta}(y)\rangle = \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} S_{ijab\alpha\beta}(p). \quad (3.37)$$

In the absence of spontaneous symmetry breaking, this two-point function would evidently vanish, because  $B^A$  and  $B^N$  belong to two different *irreps* of SU(4). But the antisymmetric *irrep* of SU(4) contains an Sp(4) singlet (see Appendix C), and so this two-point function is nonzero after symmetry breaking. In terms of the elementary fermions of the hypercolor theory, we have, using Table I,

$$\langle B_{Rija}^{A}(x)\bar{B}_{Lb}^{N}(y)\rangle$$

$$= -\int \mathcal{D}A\mu(A)P_{R}\langle\psi_{Ia}(x)\bar{\psi}_{Jb}(y)\gamma_{\nu}\rangle P_{R}$$

$$\times \langle (\bar{\chi}_{i}(x)P_{R}\Gamma_{I}\chi_{j}(x))(\bar{\chi}_{k}(y)P_{L}\gamma_{\nu}\Gamma_{J}\chi_{k}(y))\rangle, \quad (3.38)$$

where  $\mathcal{D}A$  denotes the Haar measure for the gauge field, and  $\mu(A)$  is the Boltzmann weight. Inside the gauge-field integral, the expectation values denote correlation functions of the elementary fermions in a fixed gauge-field background. There are three different ways to contract the four  $\chi$ fermions into a product of two  $\langle \chi \bar{\chi} \rangle$  propagators. In every case, we will have a  $P_R$  applied to both sides of one  $\langle \chi \bar{\chi} \rangle$ propagator, which projects out an order parameter for the  $SU(4) \rightarrow Sp(4)$  symmetry breaking. Expressions for all other contributions to the top-Yukawa coupling can be worked out in a similar way. It is clear that the experimental value of the top-Yukawa coupling in the Standard Model provides only one constraint on the many couplings present in  $\mathcal{L}_{EHC}$ .

### **D.** $V_{\rm eff}$

The effective potential for the pNGBs is generated by integrating out all other states of the hypercolor theory, and, in addition, the Standard-Model gauge and fermion fields. Here we will calculate the effective potential  $V_{\text{eff}}$  for the SU(4)/Sp(4) and  $U(1)_A$  pNGBs (we keep disregarding

the SU(6)/SO(6) pNGBs), which is obtained by integrating out the third-generation quarks.<sup>14</sup> To leading order in the four-fermion couplings, the effective potential arises from correlation functions of two four-fermion vertices, where every correlation function is a convolution of a hyperbaryon two-point function with a single massless quark propagator. As a result, every term in the effective potential will be quadratic in the spurions, and both spurions will have the same handedness.

There is a large number of ways to generate an effective potential, which we organize into twelve "templates,"

$$\begin{aligned} \mathcal{T}_{1} &= \Phi^{1-2q} \mathrm{tr}(\bar{A}\Sigma N) + \mathrm{H.c.}, \\ \mathcal{T}_{2} &= \Phi^{-1-2q} \mathrm{tr}(\bar{A}^{c}\Sigma^{*}N) + \mathrm{H.c.}, \\ \mathcal{T}_{3} &= \Phi^{1\mp 2q} \mathrm{tr}(\bar{A}\Sigma D^{T}) + \mathrm{H.c.}, \\ \mathcal{T}_{4} &= \Phi^{1\mp 2q} \mathrm{tr}(\bar{S}\Sigma D^{T}) + \mathrm{H.c.}, \\ \mathcal{T}_{5} &= \Phi^{-1\mp 2q} \mathrm{tr}(\bar{A}^{c}\Sigma^{*}D) + \mathrm{H.c.}, \\ \mathcal{T}_{6} &= \Phi^{-1\mp 2q} \mathrm{tr}(\bar{S}^{c}\Sigma^{*}D) + \mathrm{H.c.}, \\ \mathcal{T}_{7} &= \mathrm{tr}(\bar{A}\Sigma) \mathrm{tr}(A\Sigma^{*}), \\ \mathcal{T}_{8} &= \mathrm{tr}(A^{c}\Sigma) \mathrm{tr}(\bar{A}^{c}\Sigma^{*}), \\ \mathcal{T}_{9} &= \Phi^{2} \mathrm{tr}(A^{c}\Sigma) \mathrm{tr}(\bar{A}\Sigma) + \mathrm{H.c.}, \\ \mathcal{T}_{10} &= \Phi^{2} \mathrm{tr}(S^{c}\Sigma\bar{S}\Sigma) + \mathrm{H.c.}, \\ \mathcal{T}_{11} &= \Phi^{2} \mathrm{tr}(A^{c}\Sigma\bar{A}\Sigma) + \mathrm{H.c.}, \\ \mathcal{T}_{12} &= \mathrm{tr}(D\Sigma\bar{D}^{T}\Sigma^{*}). \end{aligned}$$
(3.39)

As in Sec. III C, the power of  $\Phi$  in each template matches the axial charge of the associated product of hyperbaryons in the microscopic theory. The axial charge vanishes for templates  $\mathcal{T}_7$ ,  $\mathcal{T}_8$  and  $\mathcal{T}_{12}$ . For the other templates it doesn't. We have normalized the axial charge such that the  $\chi$ 's give rise to an integer power of  $\Phi$ . Templates  $\mathcal{T}_1$ through  $\mathcal{T}_6$  are sensitive also to q, the axial charge of the  $\psi$ 's. In templates  $\mathcal{T}_3$  through  $\mathcal{T}_6$ ,  $\Phi^{-2q}$  ( $\Phi^{+2q}$ ) corresponds to right-handed (left-handed) spurions. For  $\mathcal{T}_1$  and  $\mathcal{T}_2$  we always obtain  $\Phi^{-2q}$  from the  $\psi$ 's, because only  $t_R$  can be embedded into a neutral spurion (see below).

The alert reader will have noticed the similarity between templates  $\mathcal{T}_1$  through  $\mathcal{T}_6$  and the effective Yukawa interactions in Eq. (3.32), and likewise, between templates  $\mathcal{T}_7$  through  $\mathcal{T}_{12}$  and the effective Yukawa interactions in Eq. (3.33). The underlying reason is the similar group theoretic structure, as well as the power counting, which again allows for a maximum of two nonlinear fields ( $\Sigma$  or  $\Sigma^*$ ) in the leading-order effective potential. While we will shortly explain in detail how the templates encode the effective potential, already at this stage we point out several important differences. First, in the effective Yukawa

<sup>&</sup>lt;sup>14</sup>For the gauge boson's contribution to the effective potential, see Sec. III F below.

interactions the quark fields are present, whereas in the effective potential they have been integrated out. Second, the two spurions in the effective Yukawa interactions are one right-handed and one left-handed, whereas here both of them have the same handedness. As a result, the pattern of axial charges in the effective Yukawa interactions and in the effective potential is different as well.

Every template from Eq. (3.39) will expand out to several terms in  $V_{\rm eff}$ .<sup>15</sup> We illustrate this using the example of  $\mathcal{T}_1$ . In this case, the two spurions must be right-handed, because  $q_L$  cannot be embedded into a singlet of SU(4). As for  $t_R$ , it can be embedded into an antisymmetric spurion in two different ways. Template  $\mathcal{T}_1$  thus gives rise to the following two terms

$$C_{1R}(\lambda_7 \tilde{\lambda}_1 \langle \Phi tr(\bar{A}_R^1 \Sigma N_R) + H.c. \rangle + \lambda_7 \tilde{\lambda}_2 \langle \Phi tr(\bar{A}_R^2 \Sigma N_R) + H.c. \rangle).$$
(3.40)

Each term consists of the product of three elements: a lowenergy constant, a pair of coupling constants from  $\mathcal{L}_{EHC}$ , and an expression of the form  $\langle \mathcal{T}_1 \rangle$ , where we have made a particular choice for the spurions in the template  $\mathcal{T}_1$ . The meaning of the notation  $\langle \cdot \rangle$  here is the following. For the right-handed case, this is the outcome of integrating out the  $t_R$  field, and the hyperbaryon fields to which it couples in Eq. (3.29). In practice, denoting the spurion and antispurion fields generically as  $X_R$  and  $\bar{X}_R$ , they are traded inside the  $\langle \cdot \rangle$  symbol with the corresponding constant spurion matrices  $\hat{X}_{t_R}$  and  $\hat{X}_{t_R}$ , see Eq. (3.12). In the lefthanded case, we in addition sum over the contributions of  $t_L$  and  $b_L$  [Eq. (3.11)].

As mentioned above, each correlation function that contributes to the leading-order effective potential is built from two vertices from  $\mathcal{L}_{EHC}$ , and so it contains a hyperbaryon two-point function together with a single quark propagator, which, in this approximation, is a free massless propagator of a given chirality. The spurion and the antispurion in each template must therefore have the same chirality. In the example of Eq. (3.40), only  $t_R$  can be embedded into a spurion belonging to the singlet *irrep*. Since there are two independent options for the embedding of  $t_R$  into the antisymmetric *irrep*,  $A_R^1$  and  $A_R^2$ , the template expands out to two terms in  $V_{\text{eff}}$ .

The four-fermion coupling constants together with the low-energy constant are inferred from a matching procedure that we have discussed in detail in Ref. [23], and which is similar to the one used in the previous subsection for the case of the effective top-Yukawa couplings. The four-fermion coupling constants are the two coupling constants from  $\mathcal{L}_{EHC}$  associated with the spurion and the antispurion that occur inside the  $\langle \cdot \rangle$  symbol. The remaining

low-energy constant is expressed in terms of a correlation function of the stand-alone hypercolor theory, which does

not depend on the particular embedding of the quark fields into the spurion. Therefore, for each template  $T_i$  we have just two low-energy constants  $C_{iL}$  and  $C_{iR}$ , one for each chirality. For the first two templates we only need  $C_{iR}$ , because only  $t_R$  can be embedded into an N spurion.

Since both the hypercolor theory and the four-fermion Lagrangian are *CP* invariant, so will be the effective potential  $V_{\text{eff}}$ . Using the assumed reality of the four-fermion coupling constants, one can also verify directly the *CP* invariance of Eq. (3.40), and of the corresponding expressions for all other templates. Because  $V_{\text{eff}}$  is always real, it follows as a corollary that all the low-energy constants that multiply the effective Yukawa couplings discussed in the previous subsection.

We comment in passing that *CP* is only an approximate symmetry of the Standard Model, whose breaking is encoded in the Yukawa couplings. In a similar spirit, one may relax the assumption that the coupling constants in  $\mathcal{L}_{EHC}$  are all real, and assume, instead, that any imaginary parts of these coupling constants are parametrically small. How the EHC theory would induce this small amount of *CP* violation goes beyond the scope of this paper. To avoid confusion, we stress that since we have defined the lowenergy constants to be independent of the four-fermion Lagrangian, their reality is true regardless of whether or not the coupling constants of  $\mathcal{L}_{EHC}$  are real.

In the rest of this subsection, we list all the contributions to  $V_{\text{eff}}$  for the twelve templates. As explained above, the four-fermion couplings that multiply each expression are easily read off from  $\mathcal{L}_{\text{EHC}}$ . The low-energy constants will be derived in the next subsection. Thanks to the simplicity of the SU(4)/Sp(4) coset, it is possible to obtain the potential in closed form. Because some of the templates depend on the U(1) field  $\Phi$ , in general an effective potential will be generated for the  $U(1)_A$  pNGB as well.

We begin with  $T_1$ , which gives rise to the two terms in Eq. (3.40). Using Eqs. (C7) and (C9), we have

$$\begin{split} \langle \Phi^{1-2q} \mathrm{tr}(\bar{A}_{R}^{1}\Sigma N_{R}) + \mathrm{H.c.} \rangle \\ &= 4\cos(\alpha)\cos((1-2q)\zeta) \\ &- \frac{2\sqrt{2}\eta\sin(\alpha)\sin((1-2q)\zeta)}{\alpha f}, \end{split}$$
(3.41a)

where  $\alpha$  is given by (cf. Appendix C)

<sup>&</sup>lt;sup>15</sup>For an alternative, but ultimately equivalent, spurion technique, see for example Ref. [6].

$$\alpha = \frac{1}{f} \left( \frac{1}{2} \eta^2 + H^{\dagger} H \right)^{1/2}, \qquad (3.42)$$

and we wrote

$$\Phi = e^{i\zeta}.\tag{3.43}$$

The field  $\zeta$  is dimensionless, and is introduced here for the sake of brevity. For the chiral expansion, it is more natural to use instead the expansion  $\Phi = \exp(i\zeta/(\sqrt{2}f_{\zeta}))$ , where

the NGB field has the appropriate canonical dimension, and  $f_{\zeta}$  is the decay constant of the  $U(1)_A$  NGB [26].

Because  $t_R$  is embedded into the  $A_R$  and  $A_R^c$  spurions in the same way, each result for  $\mathcal{T}_2$  may be obtained from the corresponding result for  $\mathcal{T}_1$  by flipping the signs of the SU(4)/Sp(4) pNGBs, and multiplying  $\zeta$  by -1-2qinstead of 1-2q. The outcome is the same as just replacing 1-2q by 1+2q everywhere.

Considering next templates  $\mathcal{T}_3$  through  $\mathcal{T}_6$ , which also have a single nonlinear field, but a *D* spurion instead of the *N* spurion, we find for  $\mathcal{T}_3$ 

$$\langle \Phi^{1-2q} \operatorname{tr}(\bar{A}_{R}^{1}\Sigma D_{R}^{1T}) + \operatorname{H.c.} \rangle = 4\cos(\alpha)\cos((1-2q)\zeta) - \frac{2\sqrt{2}\eta\sin(\alpha)\sin((1-2q)\zeta)}{\alpha f},$$

$$\langle \Phi^{1-2q} \operatorname{tr}(\bar{A}_{R}^{2}\Sigma D_{R}^{2T}) + \operatorname{H.c.} \rangle = 0,$$

$$\langle \Phi^{1-2q} \operatorname{tr}(\bar{A}_{R}^{2}\Sigma D_{R}^{1T}) + \operatorname{H.c.} \rangle = 4\cos(\alpha)\cos((1-2q)\zeta) + \frac{2\sqrt{2}\eta\sin(\alpha)\sin((1-2q)\zeta)}{\alpha f},$$

$$\langle \Phi^{1-2q} \operatorname{tr}(\bar{A}_{R}^{1}\Sigma D_{R}^{2T}) + \operatorname{H.c.} \rangle = 0,$$

$$\langle \Phi^{1+2q} \operatorname{tr}(\bar{A}_{L}\Sigma D_{L}^{1T}) + \operatorname{H.c.} \rangle = -4\cos(\alpha)\cos((1+2q)\zeta) - \frac{2\sqrt{2}\eta\sin(\alpha)\sin((1+2q)\zeta)}{\alpha f},$$

$$\langle \Phi^{1+2q} \operatorname{tr}(\bar{A}_{L}\Sigma D_{L}^{1T}) + \operatorname{H.c.} \rangle = 0.$$

$$(3.44)$$

For  $\mathcal{T}_4$  we have

$$\begin{split} \langle \Phi^{1-2q} \mathrm{tr}(\bar{S}_R \Sigma D_R^{1T}) + \mathrm{H.c.} \rangle &= 0 \\ \langle \Phi^{1-2q} \mathrm{tr}(\bar{S}_R \Sigma D_R^{2T}) + \mathrm{H.c.} \rangle &= 4 \cos(\alpha) \cos((1-2q)\zeta) + \frac{2\sqrt{2\eta} \sin(\alpha) \sin((1-2q)\zeta)}{\alpha f}. \\ \langle \Phi^{1+2q} \mathrm{tr}(\bar{S}_L \Sigma D_L^{1T}) + \mathrm{H.c.} \rangle &= 4 \cos(\alpha) \cos((1+2q)\zeta) + \frac{2\sqrt{2\eta} \sin(\alpha) \sin((1+2q)\zeta)}{\alpha f}, \\ \langle \Phi^{1+2q} \mathrm{tr}(\bar{S}_L \Sigma D_L^{2T}) + \mathrm{H.c.} \rangle &= 0, \end{split}$$
(3.45)

for  $\mathcal{T}_5$ ,

$$\begin{split} \langle \Phi^{-1-2q} \mathrm{tr}(\bar{A}_{R}^{c1} \Sigma^{*} D_{R}^{1}) + \mathrm{H.c.} \rangle &= 4 \cos(\alpha) \cos((1+2q)\zeta) - \frac{2\sqrt{2\eta} \sin(\alpha) \sin((1+2q)\zeta)}{\alpha f}, \\ \langle \Phi^{-1-2q} \mathrm{tr}(\bar{A}_{R}^{c2} \Sigma^{*} D_{R}^{2}) + \mathrm{H.c.} \rangle &= 0, \\ \langle \Phi^{-1-2q} \mathrm{tr}(\bar{A}_{R}^{c2} \Sigma^{*} D_{R}^{1}) + \mathrm{H.c.} \rangle &= 4 \cos(\alpha) \cos((1+2q)\zeta) + \frac{2\sqrt{2\eta} \sin(\alpha) \sin((1+2q)\zeta)}{\alpha f}, \\ \langle \Phi^{-1-2q} \mathrm{tr}(\bar{A}_{R}^{c1} \Sigma^{*} D_{R}^{2}) + \mathrm{H.c.} \rangle &= 0, \\ \langle \Phi^{-1+2q} \mathrm{tr}(\bar{A}_{L}^{c} \Sigma^{*} D_{L}^{1}) + \mathrm{H.c.} \rangle &= -4 \cos(\alpha) \cos((1-2q)\zeta) + \frac{2\sqrt{2\eta} \sin(\alpha) \sin((1-2q)\zeta)}{\alpha f}, \\ \langle \Phi^{-1+2q} \mathrm{tr}(\bar{A}_{L}^{c} \Sigma^{*} D_{L}^{2}) + \mathrm{H.c.} \rangle &= 0, \end{split}$$
(3.46)

and for  $\mathcal{T}_6$ ,

$$\langle \Phi^{-1-2q} \operatorname{tr}(\bar{S}_{R}^{c} \Sigma^{*} D_{R}^{1}) + \operatorname{H.c.} \rangle = 0,$$

$$\langle \Phi^{-1-2q} \operatorname{tr}(\bar{S}_{R}^{c} \Sigma^{*} D_{R}^{2}) + \operatorname{H.c.} \rangle = -4 \cos(\alpha) \cos((1+2q)\zeta) - \frac{2\sqrt{2}\eta \sin(\alpha) \sin((1+2q)\zeta)}{\alpha f},$$

$$\langle \Phi^{-1+2q} \operatorname{tr}(\bar{S}_{L}^{c} \Sigma^{*} D_{L}^{1}) + \operatorname{H.c.} \rangle = -4 \cos(\alpha) \cos((1-2q)\zeta) + \frac{2\sqrt{2}\eta \sin(\alpha) \sin((1-2q)\zeta)}{\alpha f},$$

$$\langle \Phi^{-1+2q} \operatorname{tr}(\bar{S}_{L}^{c} \Sigma^{*} D_{L}^{2}) + \operatorname{H.c.} \rangle = 0.$$

$$(3.47)$$

Turning to the templates with two nonlinear fields, for  $\mathcal{T}_7$  we have

$$\langle \operatorname{tr}(\bar{A}_{R}^{1}\Sigma)\operatorname{tr}(A_{R}^{1}\Sigma^{*})\rangle = -4 + \frac{4\sin^{2}\alpha}{\alpha^{2}}\frac{H^{\dagger}H}{f^{2}}, \qquad (3.48a)$$

$$\langle \operatorname{tr}(\bar{A}_R^2 \Sigma) \operatorname{tr}(A_R^2 \Sigma^*) \rangle = -4 + \frac{4 \sin^2 \alpha}{\alpha^2} \frac{H^{\dagger} H}{f^2}, \qquad (3.48b)$$

$$\langle \operatorname{tr}(\bar{A}_{R}^{1}\Sigma)\operatorname{tr}(A_{R}^{2}\Sigma^{*}) + \operatorname{H.c.} \rangle = 8 - \frac{8\sin^{2}\alpha}{\alpha^{2}} \frac{(\eta^{2} + H^{\dagger}H)}{f^{2}}, \qquad (3.48c)$$

$$\langle \operatorname{tr}(\bar{A}_L \Sigma) \operatorname{tr}(A_L \Sigma^*) \rangle = -\frac{\sin^2 \alpha}{\alpha^2} \frac{4H^{\dagger}H}{f^2}.$$
 (3.48d)

Notice that  $\langle \operatorname{tr}(\bar{A}_R^2\Sigma)\operatorname{tr}(A_R^1\Sigma^*)\rangle$  is the hermitian conjugate of  $\langle \operatorname{tr}(\bar{A}_R^1\Sigma)\operatorname{tr}(A_R^2\Sigma^*)\rangle$ . The results for  $\mathcal{T}_8$  are the same as for the corresponding results for  $\mathcal{T}_7$ . The last double-trace template is  $\mathcal{T}_9$ , for which we obtain

$$\langle \Phi^2 \operatorname{tr}(\bar{A}_R^1 \Sigma) \operatorname{tr}(A_R^{c1} \Sigma) + \operatorname{H.c.} \rangle = 8 \cos(2\zeta) \left( -1 + \frac{\sin^2 \alpha}{\alpha^2} \frac{(\eta^2 + H^{\dagger} H)}{f^2} \right) + \frac{4\sqrt{2}}{\alpha f} \eta \sin(2\zeta) \sin(2\alpha), \quad (3.49a)$$

$$\langle \Phi^2 \operatorname{tr}(\bar{A}_R^2 \Sigma) \operatorname{tr}(A_R^{c2} \Sigma) + \mathrm{H.c.} \rangle = 8\cos(2\zeta) \left( -1 + \frac{\sin^2 \alpha}{\alpha^2} \frac{(\eta^2 + H^{\dagger}H)}{f^2} \right) - \frac{4\sqrt{2}}{\alpha f} \eta \sin(2\zeta) \sin(2\alpha), \qquad (3.49b)$$

$$\langle \Phi^2 \operatorname{tr}(\bar{A}_R^1 \Sigma) \operatorname{tr}(A_R^{c2} \Sigma) + \mathrm{H.c.} \rangle = 8 \cos(2\zeta) \left( 1 - \frac{\sin^2 \alpha}{\alpha^2} \frac{H^{\dagger} H}{f^2} \right),$$
(3.49c)

$$\langle \Phi^2 \operatorname{tr}(\bar{A}_R^2 \Sigma) \operatorname{tr}(A_R^{c1} \Sigma) + \operatorname{H.c.} \rangle = 8 \cos(2\zeta) \left( 1 - \frac{\sin^2 \alpha}{\alpha^2} \frac{H^{\dagger} H}{f^2} \right),$$
(3.49d)

$$\langle \Phi^2 \operatorname{tr}(\bar{A}_L \Sigma) \operatorname{tr}(A_L^c \Sigma) + \mathrm{H.c.} \rangle = 8 \cos(2\zeta) \frac{\sin^2 \alpha}{\alpha^2} \frac{H^{\dagger} H}{f^2}.$$
(3.49e)

Moving on to the single-trace templates, for  ${\cal T}_{10}$  we find

$$\langle \operatorname{tr}(\bar{S}_R \Sigma S_R^c \Sigma) + \operatorname{H.c.} \rangle = 4 \cos(2\zeta) \left( -1 + \frac{\sin^2 \alpha}{\alpha^2} \frac{(\eta^2 + H^{\dagger} H)}{f^2} \right) - \frac{2\sqrt{2\eta} \sin(2\zeta) \sin(2\alpha)}{\alpha f}, \quad (3.50a)$$

$$\langle \operatorname{tr}(\bar{S}_L \Sigma S_L^c \Sigma) + \operatorname{H.c.} \rangle = 4 \cos(2\zeta) \left( -2 + \frac{3 \sin^2 \alpha}{\alpha^2} \frac{H^{\dagger} H}{f^2} \right),$$
(3.50b)

and for  $\mathcal{T}_{11}$ ,

$$\langle \operatorname{tr}(\bar{A}_{R}^{1}\Sigma A_{R}^{c1}\Sigma) + \operatorname{H.c.}\rangle = 4\cos(2\zeta) \left( -1 + \frac{\sin^{2}\alpha}{\alpha^{2}} \frac{\eta^{2} + H^{\dagger}H}{f^{2}} \right) + \frac{2\sqrt{2}\eta\sin(2\zeta)\sin(2\alpha)}{\alpha f}, \quad (3.51a)$$

$$\langle \operatorname{tr}(\bar{A}_{R}^{2}\Sigma A_{R}^{c2}\Sigma) + \operatorname{H.c.} \rangle = 4\cos(2\zeta) \left( -1 + \frac{\sin^{2}\alpha}{\alpha^{2}} \frac{\eta^{2} + H^{\dagger}H}{f^{2}} \right) - \frac{2\sqrt{2}\eta\sin(2\zeta)\sin(2\alpha)}{\alpha f}, \quad (3.51b)$$

$$\langle \operatorname{tr}(\bar{A}_{R}^{1}\Sigma A_{R}^{c2}\Sigma) + \operatorname{H.c.}\rangle = -4\cos(2\zeta)\frac{\sin^{2}\alpha}{\alpha^{2}}\frac{H^{\dagger}H}{f^{2}}, \quad (3.51c)$$

$$\langle \operatorname{tr}(\bar{A}_R^2 \Sigma A_R^{c1} \Sigma) + \operatorname{H.c.} \rangle = -4 \cos(2\zeta) \frac{\sin^2 \alpha}{\alpha^2} \frac{H^{\dagger} H}{f^2}, \quad (3.51d)$$

$$\langle \operatorname{tr}(\bar{A}_L \Sigma A_L^c \Sigma) + \operatorname{H.c.} \rangle = 4 \cos(2\zeta) \left( -2 + \frac{\sin^2 \alpha}{\alpha^2} \frac{H^{\dagger} H}{f^2} \right).$$
  
(3.51e)

Finally, for  $\mathcal{T}_{12}$ , the nonzero results are

$$\langle \operatorname{tr}((\bar{D}_R^1)^T \Sigma^* D_R^1 \Sigma) \rangle = -4 + 8 \frac{\sin^2 \alpha}{\alpha^2} \frac{H^{\dagger} H}{f^2}, \qquad (3.52a)$$

$$\langle \operatorname{tr}((\bar{D}_R^2)^T \Sigma^* D_R^2 \Sigma) \rangle = 2 - 2 \frac{\sin^2 \alpha}{\alpha^2} \frac{H^{\dagger} H}{f^2}, \qquad (3.52b)$$

$$\langle \operatorname{tr}((\bar{D}_L^1)^T \Sigma^* D_L^1 \Sigma) \rangle = \frac{\sin^2 \alpha}{\alpha^2} \frac{H^{\dagger} H}{f^2},$$
 (3.52c)

$$\langle \operatorname{tr}((\bar{D}_L^2)^T \Sigma^* D_L^2 \Sigma) \rangle = \frac{\sin^2 \alpha}{\alpha^2} \frac{H^{\dagger} H}{f^2}.$$
 (3.52d)

#### E. Low-energy constants

To complete the construction of the effective potential, we need the low-energy constants. In order to fully benefit from the SU(4) symmetry of the hypercolor theory, we now expand each spurion as

$$X_{L,R}(x) = \eta_{L,R}(x)\hat{X}_{L,R}, \qquad \bar{X}_{L,R}(x) = \bar{\eta}_{L,R}(x)\hat{\bar{X}}_{L,R},$$
(3.53)

where  $\eta_{L,R}(x)$  is a free massless Weyl field. Let us compare this with Eqs. (3.11) and (3.12). In the latter case, the (hatted) matrices that carry the SU(4) indices are assigned a fixed numerical value that defines a particular embedding of a quark field. By contrast, we now treat  $\hat{X}_{L,R}$  and  $\bar{\hat{X}}_{L,R}$  as global spurions that do not have any particular value, but, instead, transform in an *irrep* of SU(4). As a final preparatory step, we eliminate from  $\mathcal{L}_{EHC}$  the information about any specific embedding of the quark fields while keeping only the information about the SU(4) *irreps*, by writing, e.g.,  $(\lambda_5 D_R^1(x) + \lambda_6 D_R^2(x)) B_L^D(x) = \mathcal{D}_R \eta_R(x) B_L^D(x)$ , where  $\mathcal{D}_R$  is a global spurion in the adjoint *irrep*. In this process, we also deliberately suppress the information about the four-fermion coupling constants. As discussed above, this information can easily be read off from the original definition (3.29). We end up re-expressing  $\mathcal{L}_{EHC}$  in terms of the hyperbaryon fields, the  $\eta_{L,R}(x)$  field, and a pair of global spurions for each *irrep*: singlet  $\mathcal{N}_{L,R}$ , adjoint  $\mathcal{D}_{L,R}$ , two-index antisymmetric  $\mathcal{A}_{L,R}$ , two-index symmetric  $\mathcal{S}_{L,R}$ , and their complex conjugates  $\mathcal{A}_{L,R}^c$  and  $\mathcal{S}_{L,R}^c$ .

In the (templates for the) effective potential, Eq. (3.39), we simply trade every spurion field with the corresponding global spurion. Each low-energy constant will be obtained by taking ordinary derivatives with respect to the global spurions, and matching the results between the microscopic and the effective theories. This matching procedure will allow us to replace the  $\Sigma$  field in the effective theory by its expectation value. This, in turn, simplifies considerably the calculation of the low-energy constants. Indeed, by making use of the global symmetry, we are able to extract the lowenergy constants from correlation functions of the microscopic theory that do not involve any NGB asymptotic states.

We start with  $T_1$ , whose contribution to  $V_{\text{eff}}$  now reads

$$C_{1R}(\Phi^{1-2q}\mathrm{tr}(\bar{\mathcal{A}}_R\Sigma\mathcal{N}_R) + \mathrm{H.c.}). \tag{3.54}$$

We recall that we only need the right-handed low-energy constant  $C_{1R}$ , because the left-handed quarks cannot be embedded into the singlet *irrep*. In the effective theory,

$$\epsilon_{0,ij} \frac{\partial}{\partial \mathcal{N}_R} \frac{\partial}{\partial \bar{\mathcal{A}}_{Rij}} \frac{\log Z_{\text{eff}}}{V} = -C_{1R} \epsilon_{0,ij} \langle \Sigma_{ji} \rangle$$
$$= -C_{1R} \epsilon_{0,ij} \epsilon_{0,ji} = 4C_{1R}, \quad (3.55)$$

where we have used that  $\langle \Sigma \rangle = \epsilon_0$  and  $\langle \Phi \rangle = 1$ . In the microscopic theory, we have

$$\begin{aligned} \epsilon_{0,ij} \frac{\partial}{\partial \mathcal{N}_R} \frac{\partial}{\partial \bar{\mathcal{A}}_{Rij}} \log Z \\ &= \epsilon_{0,ij} \int d^4 x \int d^4 y \langle \bar{B}_L^N(y) \eta_R(y) \bar{\eta}_R(x) B^A_{Lji}(x) \rangle \\ &= i \epsilon_{0,ij} \int d^4 x \int d^4 y \int \frac{d^4 p}{(2\pi)^4} \frac{p_\mu}{p^2} e^{i p(y-x)} \langle \bar{B}_L^N(y) \gamma_\mu B^A_{Lji}(x) \rangle. \end{aligned}$$
(3.56)

Hence

$$C_{1R} = \frac{i\epsilon_{0,ij}}{4} \int d^4x \int \frac{d^4p}{(2\pi)^4} \frac{p_{\mu}}{p^2} e^{-ipx} \langle \bar{B}_L^N(0) \gamma_{\mu} B_{Lji}^A(x) \rangle.$$
(3.57)

As in Sec. III C we may express the hyperbaryon two-point function in terms of the elementary fermions. As can be seen from Table I, while in the case of the D and N *irreps* 

the hyperbaryon fields have the same form for the SO(5) and SO(11) gauge theories, their forms for the other *irreps* are different in the two theories. For definiteness, we will assume in this subsection that the microscopic theory is the SO(5) gauge theory,<sup>16</sup> obtaining

$$\langle \bar{B}_{L}^{N}(y)\gamma_{\mu}B_{Lji}^{A}(x)\rangle = \frac{1}{2}\int \mathcal{D}A\mu(A)\langle \bar{\psi}_{J}(y)\gamma_{\nu}\gamma_{\mu}P_{L}\psi_{I}(x)\rangle \\ \times \langle \bar{\chi}_{k}(y)\gamma_{\nu}\Gamma_{J}\chi_{k}(y)\rangle \langle \bar{\chi}_{j}(x)P_{R}\Gamma_{I}\chi_{i}(x)\rangle \\ + \cdots,$$
(3.58)

where we have used that  $\bar{\chi}_k \Gamma_J \gamma_5 \gamma_\nu \chi_k = 0$ , and the ellipses stand for a term that vanishes when contracted with  $\epsilon_{0,ij}$  in Eq. (3.57). As expected, the expectation value of  $\bar{\chi}_j P_R \Gamma_l \chi_i$ provides for an order parameter for  $SU(4) \rightarrow Sp(4)$ symmetry breaking. Unlike the basic local order parameter [Eq. (3.7)], because of the presence of the SO(d) matrices  $\Gamma_I$  and  $\Gamma_J$  inside of the  $\chi$  bilinears, only the two-point function as a whole is a gauge invariant (nonlocal) order parameter. In addition, the factor  $\langle \bar{\psi}_J(y) \gamma_\nu \gamma_\mu P_L \psi_I(x) \rangle$ does not vanish because of the symmetry breaking  $SU(6) \rightarrow SO(6)$ , so the nonvanishing of the correlator  $\langle \bar{B}_L^N(y) \gamma_\mu B_{Lji}^A(x) \rangle$  requires both SU(4) and SU(6) to be spontaneously broken.

For template  $\mathcal{T}_2$ , the only difference in the calculation of  $C_{2R}$  is that the hyperbaryon  $B_L^A$  is replaced by  $B_L^{A^c}$ . This has the effect of replacing the  $P_L$  projector inside the  $\bar{\chi}_j(x)P_L\Gamma_L\chi_i(x)$  bilinear in Eq. (3.58) by a  $P_R$ . For  $\mathcal{T}_3$ , we need an adjoint hyperbaryon instead of the neutral one. In this case, both chiralities are needed, and by similar arguments we find

$$C_{3R,L} = -\frac{i\epsilon_{0,jk}}{4} \int d^4x \int \frac{d^4p}{(2\pi)^4} \frac{p_{\mu}}{p^2} \\ \times e^{-ipx} \langle \bar{B}^D_{ki}(0) \gamma_{\mu} P_{L,R} B^A_{ji}(x) \rangle.$$
(3.59)

The low-energy constants for templates  $T_4$ ,  $T_5$  and  $T_6$  can be similarly obtained.

In the case of template  $\mathcal{T}_7$ , we need to do a little more work, because one can construct from the  $\mathcal{A}$  and  $\overline{\mathcal{A}}$  spurions also a symmetry-preserving term that does not depend on the  $\Sigma$  field, tr( $\overline{\mathcal{A}}\mathcal{A}$ ). Considering the left-handed case for definiteness, the relevant terms are

$$C_{7L} \operatorname{tr}(\bar{\mathcal{A}}_L \Sigma) \operatorname{tr}(\mathcal{A}_L \Sigma^*) + C'_{7L} \operatorname{tr}(\bar{\mathcal{A}}_L \mathcal{A}_L), \quad (3.60)$$

and so

$$\frac{\partial}{\partial \mathcal{A}_{Lij}} \frac{\partial}{\partial \bar{\mathcal{A}}_{Lk\ell}} \frac{\log Z_{\text{eff}}}{V} = -C_{7L} \epsilon_{0,ij} \epsilon_{0,k\ell} - C_{7L}' (\delta_{jk} \delta_{i\ell} - \delta_{ik} \delta_{j\ell}). \quad (3.61)$$

<sup>16</sup>The reader can easily work out the minor changes for the SO(11) case.

We may now extract  $C_{7L}$  by contracting this result with the fully antisymmetric four-dimensional tensor  $\epsilon_{ijk\ell}$ . By applying the same differentiations to the microscopic theory, and comparing the results, we find

$$C_{7L} = \frac{i\epsilon_{ijk\ell}}{8} \int d^4x \int \frac{d^4p}{(2\pi)^4} \frac{p_{\mu}}{p^2} e^{-ipx} \langle \bar{B}^A_{ji}(0) \gamma_{\mu} P_R B^A_{\ell k}(x) \rangle.$$
(3.62)

For  $C_{7R}$ , the chiral projector inside the hyperbaryon twopoint function is  $P_L$ . In terms of the elementary fermions,

$$\begin{split} \langle \bar{B}_{ji}^{A}(y) \gamma_{\mu} P_{R,L} B_{\ell k}^{A}(x) \rangle \\ &= \int \mathcal{D}A\mu(A) \langle \bar{\psi}_{I}(y) \gamma_{\mu} P_{R,L} \psi_{J}(x) \rangle \\ &\times \langle \bar{\chi}_{i}(y) P_{L} \Gamma_{I} \chi_{j}(y) \rangle \langle \bar{\chi}_{\ell}(x) P_{R} \Gamma_{J} \chi_{k}(x) \rangle + \cdots, \end{split}$$
(3.63)

where again the ellipses denote terms that vanish when contracted with  $\epsilon_{ijk\ell}$  in Eq. (3.62). We see that from each  $\chi$  propagator we pick up the part proportional to  $\epsilon_0$  in flavor space, which is nonzero in the broken phase.

For template  $\mathcal{T}_8$ , the  $\mathcal{A}$  and  $\mathcal{A}$  spurions are replaced by  $\overline{\mathcal{A}}^c$  and  $\mathcal{A}^c$  spurions, respectively. The result is similar, except that, in Eq. (3.63), the chiral projectors inside the  $\chi$  bilinears get flipped.

For templates  $\mathcal{T}_9$ ,  $\mathcal{T}_{10}$  and  $\mathcal{T}_{11}$  there are no  $\Sigma$  independent terms. For  $\mathcal{T}_9$  we find

$$C_{9L,R} = \frac{i\epsilon_{ijk\ell}}{8} \int d^4x \int \frac{d^4p}{(2\pi)^4} \frac{p_{\mu}}{p^2} \\ \times e^{-ipx} \langle \bar{B}_{ji}^{A^c}(0)\gamma_{\mu}P_{R,L}B^A_{\ell k}(x) \rangle, \quad (3.64)$$

where

$$\begin{split} \langle \bar{B}_{ji}^{A^{c}}(y)\gamma_{\mu}P_{R,L}B_{\ell k}^{A}(x)\rangle \\ &= \int \mathcal{D}A\mu(A)\langle \bar{\psi}_{I}(y)\gamma_{\mu}P_{R,L}\psi_{J}(x)\rangle \\ &\times \langle (\bar{\chi}_{i}(y)P_{R}\Gamma_{I}\chi_{j}(y))(\bar{\chi}_{\ell}(x)P_{R}\Gamma_{J}\chi_{k}(x))\rangle. \end{split}$$
(3.65)

This time, the three possible contractions of the  $\chi$ 's are all nonzero in the broken phase, and contribute to the low-energy constants. For template  $T_{10}$ ,

$$C_{10L,R} = -\frac{i\epsilon_{ijk\ell}}{8} \int d^4x \int \frac{d^4p}{(2\pi)^4} \frac{p_{\mu}}{p^2} \\ \times e^{-ipx} \langle \bar{B}_{ji}^{Sc}(0)\gamma_{\mu}P_{R,L}B_{\ell k}^S(x) \rangle, \quad (3.66)$$

where

$$\begin{split} \langle \bar{B}_{ji}^{S^{c}}(y) \gamma_{\mu} P_{R,L} B_{\ell k}^{S}(x) \rangle \\ &= \int \mathcal{D}A\mu(A) \langle \bar{\psi}_{I}(y) \sigma_{\sigma \rho} \gamma_{\mu} P_{R,L} \sigma_{\kappa \lambda} \psi_{J}(x) \rangle \\ &\times \langle (\bar{\chi}_{i}(y) P_{R} \sigma_{\sigma \rho} \Gamma_{I} \chi_{j}(y)) (\bar{\chi}_{\ell}(x) P_{R} \sigma_{\kappa \lambda} \Gamma_{J} \chi_{k}(x)) \rangle. \end{split}$$
(3.67)

For template  $\mathcal{T}_{11}$  we find  $C_{11L,R} = -C_{9L,R}$ .

Finally, in the case of template  $\mathcal{T}_{12}$  we once more have a symmetry preserving term,  $C'_{12L,R} \text{tr}(\bar{\mathcal{D}}_{L,R} \mathcal{D}_{L,R})$ , that we need to separate out.<sup>17</sup> Expanding the adjoint fields on the basis of SU(4) generators  $T_a$  we have in the effective theory (omitting the chirality label)

$$\frac{\partial}{\partial \mathcal{D}_a} \frac{\partial}{\partial \bar{\mathcal{D}}_b} \frac{\log Z_{\text{eff}}}{V} = -C_{12} \text{tr} (T_b^T \epsilon_0 T_a \epsilon_0) - C_{12}' \text{tr} (T_b T_a).$$
(3.68)

The right-hand side is proportional to  $(\pm C_{12} + C'_{12})\delta_{ab}$ when  $T_a$  is an unbroken, respectively, broken generator. By considering both cases we may extract the low-energy constant. In the microscopic theory (considering the lefthanded spurions for definiteness),

$$\frac{\partial}{\partial \mathcal{D}_{La}} \frac{\partial}{\partial \bar{\mathcal{D}}_{Lb}} \frac{\log Z}{V} 
= i \int d^4x \int \frac{d^4p}{(2\pi)^4} \frac{p_{\mu}}{p^2} e^{-ipx} \langle \bar{B}^D_a(0) \gamma_{\mu} P_R B^D_b(x) \rangle,$$
(3.69)

where

$$\begin{split} \langle \bar{B}^{D}_{a}(y) \gamma_{\mu} P_{L,R} B^{D}_{b}(x) \rangle \\ &= \int \mathcal{D}A\mu(A) \mathcal{F}_{IJ\nu\rho ab}(x,y) \langle \bar{\psi}_{J}(y) \gamma_{\rho} \gamma_{\mu} \gamma_{\nu} P_{L,R} \psi_{I}(x) \rangle, \end{split}$$
(3.70)

and

$$\begin{aligned} \mathcal{F}_{IJ\nu\rho ab}(x,y) \\ &= \langle (\bar{\chi}(x)P_R\gamma_{\nu}\Gamma_{I}T_b\chi(x))(\bar{\chi}(y)P_R\gamma_{\rho}\Gamma_{J}T_a\chi(y))\rangle \\ &= -\mathrm{tr}(T_b\langle\chi(x)\bar{\chi}(y)\rangle P_R\gamma_{\rho}\Gamma_{J}T_a\langle\chi(y)\bar{\chi}(x)\rangle P_R\gamma_{\nu}\Gamma_{I}) \\ &- \mathrm{tr}(T_b\langle\chi(x)\bar{\chi}(y)\rangle\gamma_{\rho}P_R\Gamma_{J}T_a^T\langle\chi(y)\bar{\chi}(x)\rangle P_R\gamma_{\nu}\Gamma_{I}). \end{aligned}$$
(3.71)

The first term on the right-hand side of the second equality picks up the kinetic part of the  $\chi$  propagator, which is symmetry preserving and proportional to  $\delta_{ij}$  in flavor space. The flavor trace therefore collapses to tr( $T_bT_a$ ), which corresponds to the  $C'_{12}$  term in Eq. (3.68). The last

term picks up the symmetry breaking part of the  $\chi$  propagator, which is proportional to  $\epsilon_{0,ij}$ . This precisely corresponds to the flavor trace multiplying the  $C_{12}$  term in Eq. (3.68), and therefore the low-energy constants  $C_{12L,R}$  are obtained by substituting this term into Eq. (3.70). This completes the derivation of the low-energy constants for this theory.

#### F. Summary

Collecting everything, we see that the effective potential arising from integrating out the third-generation quarks takes the form

$$V_{\rm eff} = c_0 + \sum_{i=1}^{9} c_i f_i, \qquad (3.72)$$

with the following nine functions

$$f_{1,2} = \cos(\alpha) \cos((1 \pm 2q)\zeta),$$

$$f_{3,4} = \frac{\eta \sin(\alpha) \sin((1 \pm 2q)\zeta)}{\alpha f},$$

$$f_5 = \frac{\eta \sin(2\alpha) \sin(2\zeta)}{\alpha f},$$

$$f_6 = \frac{\sin^2 \alpha}{\alpha^2 f^2} H^{\dagger} H, \qquad f_7 = \cos(2\zeta) \frac{\sin^2 \alpha}{\alpha^2 f^2} H^{\dagger} H,$$

$$f_8 = \frac{\sin^2 \alpha}{\alpha^2 f^2} \eta^2, \qquad f_9 = \cos(2\zeta) \frac{\sin^2 \alpha}{\alpha^2 f^2} \eta^2, \qquad (3.73)$$

and where  $\alpha$  is given by Eq. (3.42). An interesting feature of this result is that, in general, a potential is generated not only for the Higgs doublet and for  $\eta$ , which are the NGBs of the SU(4)/Sp(4) coset, but also for the singlet NGB  $\zeta$ . (We recall that in this paper we disregard the NGBs of the SU(6)/SO(6) coset.) The  $c_i$ 's of Eq. (3.72) can be expressed in terms of the coupling constants of  $\mathcal{L}_{EHC}$ and the low-energy constants that we have derived in the previous subsection. The low-energy constants can be determined from a lattice calculation, which would then allow for a study of the experimental constraints on the four-fermion coupling constants. We note that experimental constraints on the effective potential alone can, of course, be studied directly in terms of the  $c_i$ 's. However, if one wants to incorporate the top Yukawa coupling into this analysis, then it has to be done in terms of the four-fermion couplings, and thus, it depends on the knowledge of the low-energy constants.

For completeness, we also give the gauge-boson contribution to the effective potential, which is

$$V_{\rm EW} = -C_w {\rm tr}(\Sigma Q_a \Sigma^* Q_a^*), \qquad (3.74)$$

where  $Q_a$  is to be summed over  $gT_L^i$  and  $g'Y = g'T_R^3$ , and where  $C_w > 0$  [15]. The expression for the low-energy

<sup>&</sup>lt;sup>17</sup>See Ref. [16] for a similar calculation.

constant  $C_w$  may be found in Ref. [16] for the case of a real *irrep*. The case of a pseudoreal *irrep* defers only by the overall sign. However, relative to the definition of  $V_{\rm EW}$  given in Ref. [16], in Eq. (3.74) we have introduced an extra a minus sign on the right-hand side. This cancels out against the sign that is encountered in the derivation, so that now  $C_w$  comes out positive in the pseudoreal case as well. With this, we find

$$V_{\rm EW} = -\frac{C_w}{2}(3g^2 + g'^2)(1 - f_6), \qquad (3.75)$$

where  $f_6$  is defined in Eq. (3.73). The gauge bosons contribution will therefore add up to the coefficient  $c_6$ . As usual, taken by itself this contribution prefers the trivial vacuum  $\langle H \rangle = 0$ , a phenomenon that goes under the name of vacuum alignment [14]. But considering  $V_{\rm eff}$  as a whole, there is ample room for a nontrivial minimum of the Higgs field.

A final contribution to the effective potential might come from mass terms for the  $\chi$  fermions. One can write down two mass terms which are invariant under the Standard Model symmetries [28]. Introducing  $\epsilon_0^{\pm} = \pm (i/2)(1 \pm \tau_3) \times \tau_2$  (where we are using the notation of Appendix C), these mass terms are

$$V_{m}^{\pm} = Bm^{\pm} \operatorname{tr}(\Phi\Sigma\epsilon_{0}^{\pm} + \text{H.c.})$$
  
=  $Bm^{\pm} \left(-4\cos(\zeta)\cos(\alpha) \pm \frac{2\sqrt{2}\sin(\zeta)\sin(\alpha)}{\alpha f}\right),$   
(3.76)

where we have used Eqs. (3.7), (3.43) and (C9), and *B* is a low-energy constant. For  $m^+ = m^- = m$ , the mass term simplifies to

$$V_m = Bm tr(\Phi \Sigma \epsilon_0 + H.c.)$$
  
= -8Bm cos( $\zeta$ ) cos( $\alpha$ ). (3.77)

The mass term (3.77) breaks the global SU(4) symmetry explicitly to Sp(4), and the individual mass terms (3.76) further break it explicitly to the Standard Model symmetry  $SU(2)_L \times SU(2)_R$ . From the point of view of the standalone hypercolor theory it may be more natural to avoid any mass terms, since this keeps the full SU(4) global symmetry intact. Having said this, we observe that explicit breaking of the flavor symmetry of the hypercolor theory, encoded in the four-fermion Lagrangian (3.29), must originate from the EHC theory. Since we do not know the details of this EHC theory, we cannot rule out that it might also induce some of the mass terms discussed above. Similar statements apply to a Dirac mass term  $\propto \bar{\psi}\psi$  for the vector-*irrep* fermions, which breaks the SU(6) symmetry explicitly to  $SU(3)_c$ . The structure of the total potential is complicated. Its minimum will depend on the values of the low-energy constants, which can be determined within the hypercolor theory, and on the four-fermion couplings  $\lambda_i$  and  $\tilde{\lambda}_i$ , which arise from integrating out heavy degrees of freedom of the EHC theory. In addition, the potential depends on the electroweak couplings through Eq. (3.75), and possibly, on the mass term (3.76) or (3.77). Here we will be content with an example of a phenomenologically viable potential obtained by setting to zero by hand most of the four-fermion couplings.

Our example consists of turning on the following couplings:  $\lambda_2$ ,  $\lambda_7$ , and  $\tilde{\lambda}_1 = -\tilde{\lambda}_2$ , setting to zero the rest of the four-fermion couplings and the mass terms. Notice that  $\tilde{\lambda}_1$  and  $\tilde{\lambda}_2$  involve the same hyperbaryon,  $B_L^A$ , hence the notion of a fixed ratio  $\tilde{\lambda}_1/\tilde{\lambda}_2$  is invariant under renormalization-group evolution. Also,  $\tilde{\lambda}_1 = -\tilde{\lambda}_2$  implies that the spurions  $A_R^1$  and  $A_R^2$  always occur as the linear combination  $A_R^1 - A_R^2 \propto \epsilon_0$ .

With this choice, the only contribution that depends on  $\zeta$  arises from template  $\mathcal{T}_1$  (see Eq. (3.41)), and is given by

$$8C_{1R}\lambda_7\tilde{\lambda}_1\cos(\alpha)\cos((1-2q)\zeta). \tag{3.78}$$

We will demand that the minimum of the potential occurs for  $|\alpha| < \pi/2$ , as is required for a phenomenologically viable solution. Further assuming that

$$C_{1R}\lambda_7\tilde{\lambda}_1 < 0, \tag{3.79}$$

then implies that  $\langle \zeta \rangle = 0$  at the minimum of the potential. (Alternatively, we may set  $\lambda_7 = 0$  and achieve a similar result by turning on the mass term (3.77) with m > 0.) Setting  $\zeta = 0$ , the complete potential is then give by

$$V(H,\eta) = -a_1 \cos \alpha + a_2 \sin^2 \alpha + a_3 \frac{\sin^2 \alpha}{\alpha^2 f^2} H^{\dagger} H, \quad (3.80)$$

where  $a_1 = -8C_{1R}\lambda_7\tilde{\lambda}_1$ ,  $a_2 = 16C_{3R}\tilde{\lambda}_1^2$ , and

$$a_3 = C_w (3g^2 + g'^2)/2 - 4C_{8L}\lambda_2^2.$$
(3.81)

The  $a_2$  term arises from the contributions of right-handed spurions to  $\mathcal{T}_7$ , while the  $a_3$  term arises from the gaugebosons contribution as well as from the left-handed spurions in  $\mathcal{T}_7$ . The  $a_1$  and  $a_2$  terms have full Sp(4)invariance since they depend on H and  $\eta$  only through  $\alpha$ . It follows that, if the minimum of the potential occurs for nonzero  $\alpha$ , it will point in the H direction (i.e.,  $\langle H \rangle \neq 0$  and  $\langle \eta \rangle = 0$ ) when  $a_3 < 0$ , and in the  $\eta$  direction when  $a_3 > 0$ . This conclusion is confirmed by studying the saddle-point equations. Thus, to be phenomenologically viable, the topsector contribution to  $a_3$  must be (negative and) large enough to overcome the positive contribution of the gauge bosons. A sufficient set of conditions to ensure a vacuum with  $\langle \zeta \rangle = \langle \eta \rangle = 0$  and  $\langle H \rangle \neq 0$  is  $a_1 > 0$ ,  $a_3 < 0$ , and

$$-C_{1R}\lambda_{7}\tilde{\lambda}_{1} + 4C_{7R}\tilde{\lambda}_{1}^{2} - C_{8L}\lambda_{2}^{2} + C_{w}(3g^{2} + g'^{2})/8 < 0,$$
(3.82)

where this last condition implies that the curvature in the *H* direction is negative at the origin, and thus that the minimum of the potential cannot occur for  $\alpha = 0$ . Once  $\langle \zeta \rangle = \langle \eta \rangle = 0$ , the potential further simplifies. We defer further discussion of the resulting potential to the concluding section.

Returning momentarily to the EHC theory, we observe that if the four-fermion couplings arise from integrating out heavy gauge bosons, then each four-fermion term must take the form of a current-current interaction (possibly up to a Fierz rearrangement). Checking Table I shows that this condition is satisfied for all the four-fermion couplings that contribute to our example potential. Some other fourfermion couplings, such as, for example, the  $\lambda_1$  term, cannot be brought to the form of a current-current interaction, and would thus vanish. However, if the heavy EHC degrees of freedom that have been integrated out include not only gauge bosons but also fermions (whose mass could have either an explicit or a dynamical origin), or scalars, then none of the four-fermion operators in Eq. (3.29) is ruled out. In that case we could, for example, turn off  $\lambda_2$  and turn on  $\lambda_1$  instead. The only change in the potential would be that  $C_{8L}\lambda_2^2$  gets replaced by  $C_{7L}\lambda_1^2$ .

### G. Spontaneous CP breaking

The Standard-Model neutral fields  $\eta$  and  $\zeta$  are pseudoscalars, and so, at face value, their expectation values break *CP* spontaneously. (We are assuming that all the fourfermion couplings are real, so that *CP* is not broken explicitly.) Recently, it has been pointed out in Ref. [29] that this is not necessarily true, because it might be possible to shift the expectation value to zero through field redefinitions.<sup>18</sup> Here we address this question, first for  $\langle \eta \rangle$ , and then for  $\langle \zeta \rangle$ .

Assume that at the minimum of the effective potential,  $\langle \eta \rangle = \eta_0 \neq 0$ . In order to "rotate away" this expectation value, we need to apply to the  $\chi$  fields of the hypercolor theory the SU(4) transformation  $U_0 = \exp(-i\tilde{\eta}_0 X/2)$ , where we have introduced the dimensionless quantity  $\tilde{\eta}_0 = \eta_0/(\sqrt{2}f)$ , and  $X = \tau_3 \times 1$  is the generator associated with  $\eta$  [see Eq. (C7)]. Indeed, if  $\langle \Sigma \rangle = \exp(i\tilde{\eta}_0 X)\epsilon_0$ , then  $U_0 \langle \Sigma \rangle U_0^T = \epsilon_0$ . If initially both  $\eta$  and H have nonzero expectation values, then the  $U_0$  transformation will set  $\langle \eta \rangle = 0$  while in general changing the expectation value of H as well.

TABLE III. Values n = -1, 0, +1 of the phase transformation  $\exp(-in\tilde{\eta}_0)$ , which is to be applied to a Standard Model field, together with the spurion embeddings of  $q_L$  (2nd column) and  $t_R$  (3rd column) for which, for this *n*, the corresponding term in  $\mathcal{L}_{\text{EHC}}$  remains invariant when the SU(4) transformation  $U_0$  is applied to the  $\chi$  fields.

+1	$D_L^1$	$A_R^1, A_R^{c1}$
0	$A_L, A_L^c, S_L, S_L^c$	$N_R, D_R^1, D_R^2$
-1	$D_L^2$	$A_R^2, A_R^{c2}, S_R, S_R^c$

The question now is whether we can find a matching transformation of the Standard Model fields  $q_L$  and  $t_R$ , such that, together with the transformation  $\chi \to U_0 \chi$ , the total Lagrangian  $\mathcal{L}_{\text{HC}} + \mathcal{L}_{\text{EHC}}$  will be invariant. If the answer is Yes, then we have achieved  $\langle \eta \rangle = 0$  via the field redefinitions, which implies that  $\langle \eta \rangle$  was indeed unphysical.

In order to keep a particular term in  $\mathcal{L}_{EHC}$  invariant, the transformation needed for a given Standard Model field depends on its spurion embedding. Using the SU(4) transformation rules of the spurions, and applying the transformation to each spurion embedding in turn, we find that this transformation can always be realized via the multiplication of the Standard Model field by a U(1) phase  $\exp(-in\tilde{\eta}_0)$ , where the possible values of n are -1, 0, +1. We list the values of n for all spurion embeddings of  $q_L$  and  $t_R$  in Table III.

The answer to the question is now clear. Consider the set of nonzero couplings in  $\mathcal{L}_{EHC}$ . If all of the spurion embeddings of  $q_L$  belong to the same row of Table III, and the same is true also for the embeddings of  $t_R$ , then invariance of  $\mathcal{L}_{EHC}$  will be achieved by applying the corresponding phase transformations to  $q_L$  and to  $t_R$ . In this case, the expectation value of  $\eta$  can indeed be rotated away, and is thus unphysical. But if the spurion embeddings of  $q_L$  and/or  $t_R$  belong to more than one row of the table, then it is not possible to maintain the invariance of  $\mathcal{L}_{EHC}$ . In this case,  $\langle \eta \rangle$  is physical, and  $\langle \eta \rangle \neq 0$  signifies the spontaneous breaking of *CP* (for an exception, see below).

A similar argument applies to  $\langle \zeta \rangle$ . The phase transformation of a Standard Model field that we now need for a particular term in  $\mathcal{L}_{EHC}$  is determined by the axial charge of the hyperbaryon to which it couples (see Table II). Once again, in order to be able to rotate  $\langle \zeta \rangle$  away, the necessary and sufficient condition is that  $q_L$  couples to hyperbaryons that all have the same axial charge, and that the same is true for  $t_R$ .

For the example potential discussed in the previous subsection we have turned on the couplings  $\lambda_2$ ,  $\lambda_7$ ,  $\tilde{\lambda}_1$  and  $\tilde{\lambda}_2$ . Only the  $\lambda_2$  term is a spurion embedding of  $q_L$ , so this poses no difficulty. However, the three spurion embeddings of  $t_R$  associated with the remaining three couplings

<sup>&</sup>lt;sup>18</sup>However, in our opinion the discussion of Ref. [29] is incomplete.

populate all three lines of Table III. Therefore, the invariance of  $\mathcal{L}_{EHC}$  under the field redefinition  $\chi \to U_0\chi$  cannot be maintained, which implies that  $\langle \eta \rangle$  is physical. The same is true for  $\langle \zeta \rangle$  since the axial charges of the relevant hyperbaryons are all different from each other. As a result, for  $\langle \eta \rangle \neq 0$  and/or  $\langle \zeta \rangle \neq 0$ , *CP* is broken spontaneously.

An exception is the special case  $\langle \Sigma \rangle = (\tau_3 \times 1)\epsilon_0$ , which corresponds to specific nonzero values of both  $\langle \eta \rangle$  and  $\langle \zeta \rangle$ . Even if both expectation values are physical, in this special case *CP* is not broken spontaneously, because  $\langle \Sigma \rangle$  is real, and so it remains invariant under the combined sign flip of  $\eta$  and  $\zeta$ .

Finally, we comment that an advantage of the SU(4)/Sp(4) coset is that it does not contain any isospin-triplet fields, and, as a result, the difficulties with triplet expectation values and their potential influence on the  $\rho$ -parameter do not arise.

### IV. THE SU(5)/SO(5) COSET

The list of Ref. [8] includes two models in which the spinor *irrep* is real, based on the gauge groups SO(7) and SO(9). These models are the subject of this section. While the vector-*irrep* fermions  $\psi_{Ia}$  are the same as before,  $\chi_i$  will now denote five Majorana fermions in the real spinor *irrep* [the relation between  $\chi_i$  and  $\bar{\chi}_i$  is still given by Eq. (2.2)]. In comparison with the SU(4)/Sp(4) coset we have studied in the previous section, the SU(5)/SO(5) coset is larger. Apart from the Higgs field and the singlet  $\eta$ , it contains nine additional NGBs that fill up the (3,3) representation of  $SU(2)_L \times SU(2)_R$ . For the basic features of the SU(5)/SO(5) coset, and the embedding of the 14 NGBs into the pion field, see Appendix D.

The order parameter  $\langle \bar{\chi}_i \chi_j \rangle$  is symmetric on its indices for a real *irrep*. We will assume that the vacuum state has  $\langle \bar{\chi}_i \chi_j \rangle \propto \delta_{ij}$ . Applying the infinitesimal flavor transformation, Eq. (2.4), we see that the NGB fields are all pseudoscalars,

$$\delta_a(\bar{\chi}\chi) = i\bar{\chi}\gamma_5(T_a + T_a^T)\chi. \tag{4.1}$$

The NGBs correspond to the 14 real symmetric generators of SU(5). For the 10 antisymmetric, imaginary generators of SU(5), we have  $\delta_a(\bar{\chi}\chi) = 0$ , showing that the unbroken group is SO(5).

These features of the NGBs resembles QCD, and are different from what we saw in the previous section for the case of a pseudoreal *irrep*. As in QCD, it is easy to check that all the NGB fields flip sign under the *CP* transformation of the hypercolor theory, Eq. (2.5c). This creates a phenomenological problem concerning the Higgs field. The Standard Model's *CP* transformation, which we will denote as  $\widehat{CP}$ , must be different from the original *CP* transformation of the hypercolor theory, because the real

components of  $H_0$  and  $H_+$  are even under  $\widehat{CP}$ , but, like all NGBs, they are odd under the *CP* transformation of the hypercolor theory. As it turns out,  $\widehat{CP}$  may be obtained as the product of the original *CP* and a diagonal SO(5) transformation.<sup>19</sup> Explicitly,

$$\widehat{CP} = \mathcal{Q} \circ CP, \qquad \mathcal{Q} = \operatorname{diag}(1, -1, 1, -1, 1). \quad (4.2)$$

The formal correspondence of the effective fields with the microscopic theory takes a similar form to Eq. (3.7), except that now the nonlinear coset field  $\Sigma$  is a symmetric unitary 5 × 5 matrix. The pion field  $\Pi$  is real symmetric, and traceless [see Eq. (D2)]. Using the embedding of the Higgs field into the pion field, given in Appendix D, it is straightforward to check that Eq. (4.2) correctly reproduces the Standard Model transformation rules of all components of the Higgs field.

The organization of this section is as follows. Since the methodology is the same as in the previous section, we will be brief, and focus on those features of the SU(5)/SO(5)coset that are different from the SU(4)/Sp(4) coset. As before we begin with the spurions in Sec. IVA, and write down the four-fermion Lagrangian in Sec. IV B, which is then followed by the list of top Yukawa effective couplings in Sec. IV C. Turning to the effective potential for the pNGBs, we begin in Sec. IV D with the templates, which are followed by the list of low-energy constants. Because of the complexity of the SU(5)/SO(5) coset we were unable to obtain the effective potential in closed form. The expansion of  $V_{\rm eff}$  to second order in the pNGB fields is relegated to Appendix E, while in Sec. IVE and Appendix F we focus on the third order terms and their phenomenological role.

### A. Spurions

As usual, we assume that the third-generation quark fields couple linearly to three-constituent baryons of the hypercolor theory, via four-fermion interactions that originate from an extended hypercolor theory which is operative at an as yet much higher energy scale. In view of our ignorance of the EHC theory, we must allow for the most general form of the four-fermion Lagrangian which is compatible with the symmetries of the Standard Model: the continuous symmetries  $SU(3)_c$ ,  $SU(2)_L$ ,  $T_R^3$  and B, and the discrete symmetry  $\hat{CP}$ . Analogous to Sec. III, we do this by looking for all the embeddings of  $q_L$  and  $t_R$  into SU(5) spurions. Demanding consistency with the assignment of Standard-Model quantum numbers then yields the most general coupling between the third-generation quarks and the hyperbaryons.

We begin with the left-handed doublet  $q_L = (t_L, b_L)$ . Introducing the 5 × 5 matrices,

<sup>&</sup>lt;sup>19</sup>For a similar situation, see Ref. [30].

$$\Theta_q = -i \begin{pmatrix} 0 & 0 & 0 & 0 & ib_L \\ 0 & 0 & 0 & 0 & b_L \\ 0 & 0 & 0 & 0 & it_L \\ 0 & 0 & 0 & 0 & -t_L \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

all the spurion embeddings of  $q_L$  may be constructed using  $\Theta_q$  and  $\Theta_q^T$ , and all the embeddings of  $\bar{q}_L$  may be constructed using  $\bar{\Theta}_q$  and  $\bar{\Theta}_q^T$ . For the adjoint *irrep* we have two independent embeddings,  $D_L^1 = \Theta_q$  and  $D_L^2 = \Theta_q^T$ . For the symmetric *irrep* there is only one embedding  $S_L = S_L^c = \Theta_q + \Theta_q^T$ , and similarly for the antisymmetric *irrep* we have  $A_L = A_L^c = \Theta_q - \Theta_q^T$ . Notice that while the quark content of the spurions  $S_L$ and  $S_L^c$  is the same (and similarly for  $A_L$  and  $A_L^c$ ), they are nevertheless different spurions, because their SU(5) transformation rules are different. For  $g \in SO(5)$ , the transformation rules of all the two-index SU(5) irreps collapse to the common rule  $X \to qXq^T$ . The relative phases of different entries of  $\Theta_q$  and  $\overline{\Theta}_q$  are fixed by the embedding of  $SU(2)_L$ and  $SU(2)_R$  as subgroups of SO(5) [see Eq. (D1)]. Our choice of the overall phase of  $\Theta_q$  will be explained shortly.

Being an  $SU(2)_L$  singlet with  $T_R^3 = 0$ , the right-handed quark field  $t_R$  can be embedded into a (1,1) or into a (1,3) of  $SU(2)_L \times SU(2)_R$ <sup>20</sup> The simplest possibility is the SU(5) singlet  $N_R = \text{diag}(1, 1, 1, 1, 1)t_R$ . For the adjoint *irrep* we again have two embeddings,  $D_R^1 = T_R^3 t_R$  and  $D_R^2 = \text{diag}(1, 1, 1, 1, -4)t_R$ , which correspond to the (1,3) and (1,1) cases, respectively. There are two more possibilities for the symmetric *irrep*,  $S_R^1 = S_R^{1c} = \text{diag}(1, 1, 1, 1, 0)t_R$ and  $S_R^2 = S_R^{2c} = \text{diag}(0, 0, 0, 0, 1)t_R$ , both of which correspond to the (1,1) case. Finally, there is a single embedding for antisymmetric *irrep*,  $A_R = A_R^c = T_R^3 t_R$ , which belongs to (1,3). As for  $q_L$ , we sometimes encounter the same embedding of  $t_R$  for different SU(5) irreps. For example, in each of the spurions  $D_R^1$ ,  $A_R$  and  $A_R^c$ , the quark field  $t_R$  is multiplied by the same constant matrix,  $T_R^3$ . Again, these are nevertheless different spurions, because of their different SU(5) transformation properties.

The *c*-number matrices that define the antispurion embeddings [recall Eqs. (3.11) and (3.12)] are always given by

$$\hat{\bar{X}} \equiv \hat{X}^{\dagger} = \mathcal{Q}\hat{X}^{T}\mathcal{Q}, \qquad (4.4)$$

where the SO(5) matrix Q is defined in Eq. (4.2). The last equality, which can be verified on a case-by-case basis, depends on the fact that all the right-handed spurion

matrices  $\hat{X}_R$  are real, and all the left-handed spurion matrices  $\hat{X}_L$  were constructed using  $\Theta_q$  (and its transpose), which implies that  $\hat{X}_{L,ij}$  is always real for even i + j, and imaginary for odd i + j. Of course, choosing to multiply any spurion matrix by some arbitrary phase would spoil these features. As already explained in Sec. III B, we refrain from doing this because we are after the most general fourfermion Lagrangian which is consistent with the Standard Model's symmetries, including, in particular,  $\hat{CP}$ .

#### **B.** $\mathcal{L}_{\text{EHC}}$

With all the spurion embeddings at hand, the four-fermion Lagrangian is

$$\mathcal{L}_{EHC} = \mathcal{L}_{EHC,1} + \mathcal{L}_{EHC,2}, \qquad (4.5a)$$

$$\mathcal{L}_{EHC,1} = \operatorname{tr}(\lambda_1 \bar{S}_L B_R^S + \lambda_2 \bar{S}_L^c B_R^{S^c} + \lambda_3 \bar{A}_L B_R^A + \lambda_4 \bar{A}_L^c B_R^{A^c} + (\lambda_5 \bar{D}_R^1 + \lambda_6 \bar{D}_R^2) B_L^D + \lambda_7 \bar{N}_R B_L^N + \text{H.c.}), \qquad (4.5b)$$

$$\begin{aligned} \mathcal{L}_{\text{EHC},2} &= \text{tr}((\tilde{\lambda}_1 \tilde{S}_R^1 + \tilde{\lambda}_2 \tilde{S}_R^2) B_L^S + (\tilde{\lambda}_3 \tilde{S}_R^{1c} + \tilde{\lambda}_4 \tilde{S}_R^{2c}) B_L^{Sc} \\ &+ \tilde{\lambda}_5 \bar{A}_R B_L^A + \tilde{\lambda}_6 \bar{A}_R^c B_L^{Ac} \\ &+ (\tilde{\lambda}_7 \bar{D}_L^1 + \tilde{\lambda}_8 \bar{D}_L^2) B_R^D + \text{H.c.}), \end{aligned}$$
(4.5c)

where now the trace is over SU(5) indices. As usual, the invariance of  $\mathcal{L}_{EHC}$  under Standard-Model continuous symmetries follows from the consistency of the spurion embeddings with those symmetries. Assuming again that all the coupling constants are real, and using that all the *c*-number spurion matrices satisfy the algebraic property (4.4), one can verify that  $\mathcal{L}_{EHC}$  is also invariant under  $\widehat{CP}$ . As discussed above, our spurion construction ensures that  $\mathcal{L}_{EHC}$  is in fact the most general four-fermion Lagrangian that enjoys these symmetries. As in Sec. III, one can then infer that all the low-energy constants occurring in the effective top Yukawa interactions and in the effective Higgs potential are real.

### C. Top Yukawa couplings

As in Sec. III C, the leading effective top Yukawa couplings are either linear or bilinear in  $\Sigma$  and  $\Sigma^*$ . For the same reason as before, those interactions that are linear in  $\Sigma$  or  $\Sigma^*$  must involve a spurion and an antispurion that

<sup>&</sup>lt;sup>20</sup>The basis elements that span the (3,1) and (1,3) *irreps* are the generators  $T_L^i$ , respectively  $T_R^i$ , themselves.

both come from  $\mathcal{L}_{EHC,1}$  or both from  $\mathcal{L}_{EHC,2}$ . In the former case, we obtain 10 effective interactions

where i = 1, 2, and in the latter case we obtain 12 more,

$$\begin{aligned} \Phi \mathrm{tr}(\bar{D}_{L}^{i}\Sigma S_{R}^{jc}), & \Phi \mathrm{tr}(\bar{D}_{L}^{i}\Sigma A_{R}^{c}), \\ \Phi^{*}\mathrm{tr}(\bar{D}_{L}^{iT}\Sigma^{*}S_{R}^{j}), & \Phi^{*}\mathrm{tr}(\bar{D}_{L}^{iT}\Sigma^{*}A_{R}), \end{aligned}$$
(4.7)

where i, j = 1, 2. The extraction of the associated lowenergy constants can be done following the example we have given in Sec. III C.

The effective Yukawa couplings that are bilinear in  $\Sigma$  and  $\Sigma^*$  may be read off from templates  $\mathcal{T}_7$  thru  $\mathcal{T}_{12}$  in Eq. (4.8) below, in the same way that the effective interactions in Eq. (3.33) are related to templates  $\mathcal{T}_7$  through  $\mathcal{T}_{12}$  of Eq. (3.39).

#### **D.** Low-energy constants

We now move on to the effective potential for the pNGBs, and begin by listing the templates for  $V_{\text{eff}}$ . This time, they are given by

$$\begin{split} \mathcal{T}_{1} &= \Phi^{1-2q} \mathrm{tr}(\bar{S}\Sigma N) + \mathrm{H.c.}, \\ \mathcal{T}_{2} &= \Phi^{-1-2q} \mathrm{tr}(\bar{S}^{c}\Sigma^{*}N) + \mathrm{H.c.}, \\ \mathcal{T}_{3} &= \Phi^{1\mp 2q} \mathrm{tr}(\bar{A}\Sigma D^{T}) + \mathrm{H.c.}, \\ \mathcal{T}_{4} &= \Phi^{1\mp 2q} \mathrm{tr}(\bar{S}\Sigma D^{T}) + \mathrm{H.c.}, \\ \mathcal{T}_{5} &= \Phi^{-1\mp 2q} \mathrm{tr}(\bar{A}^{c}\Sigma^{*}D) + \mathrm{H.c.}, \\ \mathcal{T}_{6} &= \Phi^{-1\mp 2q} \mathrm{tr}(\bar{S}^{c}\Sigma^{*}D) + \mathrm{H.c.}, \\ \mathcal{T}_{7} &= \mathrm{tr}(\bar{S}\Sigma) \mathrm{tr}(\bar{S}\Sigma^{*}), \\ \mathcal{T}_{8} &= \mathrm{tr}(S^{c}\Sigma) \mathrm{tr}(\bar{S}\Sigma^{*}), \\ \mathcal{T}_{9} &= \Phi^{2} \mathrm{tr}(S^{c}\Sigma) \mathrm{tr}(\bar{S}\Sigma) + \mathrm{H.c.}, \\ \mathcal{T}_{10} &= \Phi^{2} \mathrm{tr}(S^{c}\Sigma\bar{S}\Sigma) + \mathrm{H.c.}, \\ \mathcal{T}_{11} &= \Phi^{2} \mathrm{tr}(A^{c}\Sigma\bar{A}\Sigma) + \mathrm{H.c.}, \\ \mathcal{T}_{12} &= \mathrm{tr}(D\Sigma\bar{D}^{T}\Sigma^{*}). \end{split}$$
(4.8)

The main difference compared to the previous case [Eq. (3.39)] is that the roles of the *A* and *S* irreps have been interchanged, because  $\Sigma$  is now symmetric instead of antisymmetric.

For completeness, we note that one can write down two mass terms which are invariant under the Standard model symmetries, given by  $Btr((m_1M_1 + m_2M_2)\Sigma + H.c.)$ , where the mass matrices are  $M_1 = diag(1, 1, 1, 1, 0)$  and  $M_2 = diag(0, 0, 0, 0, 1)$ . For  $m_1 = m_2$ , the mass term is

invariant under SO(5). Because of the similarity between the mass matrices  $M_{1,2}$  and the symmetric right-handed spurions  $S_R^{1,2}$ , the explicit form of the mass terms bears resemblance to the effective potential for template  $T_1$ . We leave the details to the reader.

The derivation of the low-energy constants is very similar to the previous section, and so we will only give the results. Also, except for  $T_{12}$ , we leave it to the reader to work out the explicit expressions for the hyperbaryon two-point functions, using Table I. In all cases, it can be verified that SU(5) must break spontaneously to SO(5) for the relevant two-point function not to vanish. In some cases, SU(6) must be broken to SO(6) as well.

As in the previous section, for  $\mathcal{T}_1$  we only need the righthanded low-energy constant,

$$C_{1R} = -\frac{i}{5} \int d^4x \int \frac{d^4p}{(2\pi)^4} \frac{p_{\mu}}{p^2} e^{-ipx} \langle \bar{B}^N(0) \gamma_{\mu} P_L B^S_{ii}(x) \rangle.$$
(4.9)

For  $\mathcal{T}_2$ ,  $B^S$  gets replaced by  $B^{S^c}$ . For  $\mathcal{T}_3$  both chiralities occur in  $V_{\text{eff}}$ , and

$$C_{3R,L} = -\frac{i}{5} \int d^4x \int \frac{d^4p}{(2\pi)^4} \frac{p_{\mu}}{p^2} e^{-ipx} \langle \bar{B}_{ji}^D(0) \gamma_{\mu} P_{L,R} B_{ji}^A(x) \rangle.$$
(4.10)

Again the low-energy constants for templates  $T_4$ ,  $T_5$  and  $T_6$  can be similarly obtained. For  $T_7$  we find

$$C_{7L,R} = -i \int d^4x \int \frac{d^4p}{(2\pi)^4} \frac{p_{\mu}}{p^2} \\ \times e^{-ipx} \langle \bar{B}^S_{k\ell}(0) \gamma_{\mu} P_{R,L} B^S_{ij}(x) \rangle |_{i=j \neq k=\ell'}.$$
(4.11)

The special choice of flavor indices we have made separates out the coefficient of  $tr(\bar{S}\Sigma)tr(S\Sigma^*)$ , which is what we need for  $V_{eff}$ , from the coefficient of  $tr(\bar{S}S)$ , which is a  $\Sigma$ -independent effective term (for the spurion notation we use here, see Sec. III E). For  $\mathcal{T}_8$ , we replace  $B^S$  by  $B^{S^c}$  and  $\bar{B}^S$  by  $\bar{B}^{S^c}$  in Eq. (4.11). Next, the low-energy constants for  $\mathcal{T}_9$  and  $\mathcal{T}_{10}$  are obtained from the same hyperbaryon twopoint function,

$$-i\int d^4x \int \frac{d^4p}{(2\pi)^4} \frac{p_{\mu}}{p^2} e^{-ipx} \langle \bar{B}^{S^c}_{k\ell}(0)\gamma_{\mu}P_{R,L}B^S_{ij}(x)\rangle, \quad (4.12)$$

and differ only by the choice of flavor indices needed to project them out. For  $C_{9L,R}$  we set  $i = j \neq k = \ell$  in Eq. (4.12), whereas for  $C_{10L,R}$  we set  $j = k \neq \ell = i$ . For  $\mathcal{T}_{11}$  we use the same choice of flavor indices as for  $\mathcal{T}_{10}$ , so that

$$C_{11L,R} = -i \int d^4x \int \frac{d^4p}{(2\pi)^4} \frac{p_{\mu}}{p^2} \\ \times e^{-ipx} \langle \bar{B}^{A^c}_{k\ell}(0) \gamma_{\mu} P_{R,L} B^A_{ij}(x) \rangle |_{j=k \neq \ell=i}.$$
(4.13)

We finally consider  $\mathcal{T}_{12}$ , where, just like in Sec. III E, we need to separate out the low-energy constant of interest from the  $\Sigma$ -independent effective term  $C'_{12L,R} \text{tr}(\bar{\mathcal{D}}_{L,R} \mathcal{D}_{L,R})$ . Instead of Eq. (3.68), in the effective theory we now have (again omitting the common chirality index)

$$\frac{\partial}{\partial \mathcal{D}_a} \frac{\partial}{\partial \bar{\mathcal{D}}_b} \frac{\log Z_{\text{eff}}}{V} = -C_{12} \text{tr}(T_b^T T_a) - C_{12}' \text{tr}(T_b T_a). \quad (4.14)$$

In the microscopic theory, the hyperbaryon two-point function is given by Eqs. (3.69) through (3.71) as before. But the symmetry-breaking part of  $\langle \chi_i \bar{\chi}_j \rangle$  is now proportional to  $\delta_{ij}$ , instead of to  $\epsilon_{0,ij}$ , as it was in Sec. III. The upshot is that  $C_{12L,R}$  can be expressed in terms of the contraction on the last line of Eq. (3.71) in the same way as in the previous section.

## E. $V_{eff}$

With its nine additional NGBs, the structure of the SU(5)/SO(5) coset is richer than that of SU(4)/Sp(4), and the calculation of  $V_{\text{eff}}$  is more difficult. We have not been able to obtain  $V_{\text{eff}}$  in closed form. As a first step, we have worked it out to second order in the pNGBs. The results may be found in Appendix E.

One way to understand the extra complexity of the SU(5)/SO(5) coset is to consider the invariants of the Standard Model symmetries  $SU(2)_L$  and  $T_R^3$  that can be constructed from the pNGB fields. If, in addition, such an operator (possibly together with its hermitian conjugate) is invariant also under CP, it can occur as a separate term in the effective potential. In the case of the SU(4)/Sp(4)coset, the simplest invariants that can occur in  $V_{\rm eff}$  were the bilinear  $H^{\dagger}H$  and powers of the inert pNGBs  $\eta$  and  $\zeta$ . Moreover, the SU(4)/Sp(4) nonlinear field  $\Sigma$  can be expressed as a linear function of the pion field  $\Pi$ , with coefficients that depend on the bilinears  $\eta^2$  and  $H^{\dagger}H$  [see Eq. (C9)]. This has enabled us to obtain the effective potential in closed form. By contrast, in the case of the SU(5)/SO(5) coset we also have a (3,3)-plet of  $SU(2)_L \times$  $SU(2)_R$  at our disposal. There are two new invariant bilinears, given by  $tr(\hat{\Phi}_0^2)$  and  $tr(\hat{\Phi}_+\hat{\Phi}_-)$  in the notation of Appendix D. At third order there are new invariants that depend only on the triplet fields: tr( $\hat{\Phi}_0^3$ ) and tr( $\hat{\Phi}_0 \hat{\Phi}_+ \hat{\Phi}_-$ ), as well as mixed invariants that depend on both the Higgs and the triplet fields:  $H^{\dagger}\hat{\Phi}_{0}H$ ,  $H^{T}\epsilon\hat{\Phi}_{-}H$  and  $H^{\dagger}\hat{\Phi}_{+}\epsilon H^{*}$ .

The mixed invariants are particularly important for phenomenology. This is best illustrated through an example. We consider the contribution of  $q_L$  to template  $T_7$ , whose third-order term is (see Sec. III D for the  $\langle \cdot \rangle$  notation)

$$\langle \operatorname{tr}(S_L \Sigma) \operatorname{tr}(S_L \Sigma^*) \rangle |_{\operatorname{3rd order}}$$
  
=  $\frac{32}{f^3} (H^T \epsilon \hat{\Phi}_- H + \mathrm{H.c.})$   
=  $\frac{32}{f^3} (2H_0 H_+ \phi_-^0 - i\sqrt{2}H_+^2 \phi_-^- + i\sqrt{2}H_0^2 \phi_+^- + \mathrm{H.c.}).$   
(4.15)

We see that once the Higgs field acquires an expectation value,  $\langle H_0 \rangle = h/\sqrt{2} \neq 0$ , this induces a linear potential for  $\mathrm{Im}\phi_+^-$  [see Eq. (1.3)]. As a result, the expectation value  $\langle \mathrm{Im}\phi_+^- \rangle = \varphi/\sqrt{2}$  will necessarily move away from zero [8], while the expectation values of all the remaining components of the (3,3)-plet remain zero at this order. As explained in the introduction, this is undesirable, because  $\langle \mathrm{Im}\phi_+^\pm \rangle$  does not preserve the diagonal subgroup of  $SU(2)_L \times SU(2)_R$  (the custodial symmetry) [22]. Therefore, this expectation value will drive the  $\rho$ -parameter away from unity.

Let us investigate this issue in more detail. While we have not been able to obtain the effective potential in closed form for arbitrary values of the pNGB fields, this can be done when only h and  $\varphi$  are turned on. The results may be found in Appendix F. Examining these results, we see that odd-order terms, and, in particular, the cubic term  $h^2\varphi$ , are present in several cases. These include the contribution of  $q_L$  to templates  $\mathcal{T}_7$  [Eq. (F9)] and  $\mathcal{T}_8$  [Eq. (F10)]. Similar terms are obtained for template  $\mathcal{T}_{12}$ , see Eqs. (F14a), (F14b), (F14g) and (F14h).

The question arises whether these undesirable contributions can be avoided. A simple observation is that oddorder terms would be absent if one could show that the effective potential is invariant under an "intrinsic parity" transformation that takes  $\Sigma \rightarrow \Sigma^*$  and  $\Phi \rightarrow \Phi^*$ , while leaving the Standard-Model quark fields unchanged. The obvious reason is that this transformation flips the sign of all the pNGB fields.<sup>21</sup> A case-by-case check, using the explicit forms of the spurions [and assuming the general form of the pion field, Eq. (D3)], reveals that the individual contributions to  $V_{\rm eff}$  are each invariant under the intrinsic parity transformation, except for the six cases we have listed above, where the cubic term  $h^2\varphi$  is actually present.

Individual odd-order contributions can be avoided by imposing suitable constraints on the coupling constants of  $\mathcal{L}_{EHC}$ . For example, the contributions of Eqs. (F14a) and (F14b) cancel each other if  $\tilde{\lambda}_7 = \pm \tilde{\lambda}_8$  [8]. The contributions from Eqs. (F14g) and (F14h) are absent if  $\lambda_5$  and/ or  $\lambda_6$  vanish. Similarly, Eq. (F9a) is absent when  $\lambda_1$ vanishes, and Eq. (F10) when  $\lambda_2$  does. Interestingly, for

<sup>&</sup>lt;sup>21</sup>The transformation  $\Sigma \to \Sigma^*$  is physically equivalent to the transformation  $P_{\pi}$  considered in Ref. [8], because the difference between them is an SO(5) transformation.

the parametrization (F2) all the odd-order contributions happen to involve the same function of h and  $\varphi$ . In  $V_{\text{eff}}$ , every term from Appendix F comes multiplied by two coupling constants from  $\mathcal{L}_{EHC}$ , and a low-energy constant [see Eq. (3.40)]. Therefore, mathematically, the minimal requirement that would eliminate all the odd-order terms for the parametrization (F2) is a single constraint, which is bilinear in the coupling constants of  $\mathcal{L}_{EHC}$ , and linear in the low-energy constants.

Physically, the four-fermion couplings and the lowenergy constants have an entirely different origin. The former arise from integrating out heavy gauge bosons of the EHC theory, whereas the latter only depend on correlation functions of the hypercolor theory. Therefore, it is unlikely that they will satisfy a constraint of the kind described above. Intuitively, what makes more sense is that the oddorder terms in  $V_{\rm eff}$  might vanish thanks to the vanishing of sufficiently many four-fermion couplings. Some new constraint in the EHC theory would have to set the proper linear combinations of the couplings  $\lambda_i$  and  $\tilde{\lambda}_i$  equal to zero. One way this might happen is if the intrinsic parity symmetry discussed above would arise from some discrete symmetry of the EHC theory. Unfortunately, we have not been able to identify such a symmetry. Having said this, it remains a possibility that integrating out the heavy gauge bosons of the EHC theory would give rise to a small set of four-fermion couplings, that happens to satisfy the needed constraints on the couplings of  $\mathcal{L}_{EHC}$ , at least when the heavy gauge bosons exchange is considered at tree level.

## V. REVISITING THE SU(4)COMPOSITE-HIGGS MODEL

Another composite Higgs model whose low-energy sector yields the SU(5)/SO(5) coset was first studied in detail by Ferretti in Ref. [13], and later by us in Ref. [23]. In this section, we revisit the effective potential induced by the coupling to third generation quarks in this model. We begin with a brief summary. The model is an SU(4) gauge theory. The matter content includes 5 Majorana fermions  $\chi_i$  in the 2-index antisymmetric (sextet) irrep, together with 3 Dirac fermions  $\psi_a$  in the fundamental *irrep*.<sup>22</sup> The global symmetry is<sup>23</sup>

$$G = SU(5) \times SU(3) \times SU(3)' \times U(1)_X \times U(1)_A, \quad (5.1)$$

where  $\chi_R$  transforms as **5** of SU(5),  $\psi_R$  as  $\bar{\mathbf{3}}$  of SU(3), and  $\psi_L$  as  $\bar{\mathbf{3}}$  of SU(3)'. The embedding of  $SU(2)_L \times$  $SU(2)_R \subset SO(5) \subset SU(5)$  is the same as in Sec. IV (see Appendix D), while  $SU(3)_c$  is the vector subgroup of  $SU(3) \times SU(3)'$ .  $U(1)_X$  is the conserved fermion number of the  $\psi$ 's. If we take the  $U(1)_X$  charge of  $\psi$  to be 1/6<sup>24</sup> it will coincide with ordinary baryon number.  $U(1)_{4}$  is the conserved axial current. As in Ref. [23], we take the axial charge of  $\chi_R$  to be -1. The axial charges are then 5/3 for  $\psi_R$  and -5/3 for  $\psi_L$ .

In Ref. [23], we studied the top-induced effective potential. Making rather restrictive assumptions, we found that the potential is quartic in the spurions (equivalently, in the four-fermion couplings), and we discussed it in some detail. In this section, as in the rest of this paper, we will instead make minimal assumptions about the four-fermion Lagrangian. We begin by reconsidering the dimension-9/2 hyperbaryons that can serve as top partners, finding two more operators that can play this role, in addition to the four operators already considered in Ref. [23]. The most general fourfermion Lagrangian thus contain six independent couplings. Using this Lagrangian, we find that, in general, an effective potential is induced already at second order in the four-fermion couplings. We also reconsider the potential that is induced by the same four-fermion Lagrangian as in Ref. [23], and find that it contains two additional terms that we overlooked. We conclude with a short discussion of the phenomenological implications of our findings.

The top-partners we consider are limited to threefermion operators of the minimal dimension, 9/2. They must transform as **3** under  $SU(3)_c$ , and can belong to **5** or  $\overline{\mathbf{5}}$  of SU(5). Since  $SU(3)_c$  is the diagonal subgroup of  $SU(3) \times SU(3)'$ , this allows for several possibilities for the  $SU(3) \times SU(3)'$  quantum numbers of the hyperbaryons. Altogether, we can construct 3 right-handed and 3 left-handed hyperbaryons that satisfy the requirements. We list them in Table IV. The first four were already introduced in Ref. [23]. The last two are given by

$$B_{R,bc}^{(\bar{\mathbf{5}},\bar{\mathbf{3}},\bar{\mathbf{3}})} = \epsilon_{ABCD}(\gamma_{\mu}\chi_{L,AB})(\psi_{L,Cb}^{T}C\gamma_{\mu}\psi_{R,Dc}),$$
  

$$B_{L,bc}^{(\bar{\mathbf{5}},\bar{\mathbf{3}},\bar{\mathbf{3}})} = \epsilon_{ABCD}(\gamma_{\mu}\chi_{R,AB})(\psi_{L,Cb}^{T}C\gamma_{\mu}\psi_{R,Dc}), \qquad (5.2)$$

where the subscripts A, B, ..., are SU(4)-hypercolor indices. In this section, we label the hyperbaryons by a superscript that specifies the  $SU(5) \times SU(3) \times$ SU(3)' quantum numbers.<sup>25</sup> Under  $SU(3)_c$ , the operators in Eq. (5.2) describe a **3** and a  $\overline{6}$ , but only the **3** will couple to Standard-Model fields.

The most general four-fermion Lagrangian that we can construct using these hyperbaryons is

<sup>&</sup>lt;sup>22</sup>For a lattice study of a closely related SU(4) gauge theory, see Ref. [31]. <sup>23</sup>In Ref. [23],  $U(1)_A$  is denoted U(1)'.

<sup>&</sup>lt;sup>24</sup>This normalization is different by a factor two from that of Ref. [23].

We label a hyperbaryon and its antihyperbaryon by the same superscript.

TABLE IV. Hyperbaryon operators. The first four lines correspond to Table I of Ref. [23] (omitting the antihyperbaryons), and the last two lines to Eq. (5.2). The left column is the name of the hyperbaryon in the notation used in this section. When relevant, we give for comparison the name we used for the same operator in Ref. [23] in the second column. The remaining columns list the quantum numbers. The (ordinary) baryon number of all these hyperbaryons is 1/3.

		SU(5)	$SU(3) \times SU(3)'$	$SU(3)_c$	$U(1)_A$
$B_{p}^{(5,3,1)}$	$B_R$	5	$(\bar{3},1) \times (\bar{3},1) \to (3,1)$	3	7/3
$B_{I}^{(\bar{5},3,1)}$	$B_L$	<b>5</b>	$(\bar{3},1) \times (\bar{3},1) \rightarrow (3,1)$	3	13/3
$B_{R}^{(5,1,3)}$	$B'_R$	5	$(1,\mathbf{\bar{3}}) \times (1,\mathbf{\bar{3}}) \rightarrow (1,3)$	3	-13/3
$B_{L}^{(\bar{5},1,3)}$	$B_L'$	<b>5</b>	$(1,\bar{3})\times(1,\bar{3})\to(1,3)$	3	-7/3
$B_R^{(\bar{5},\bar{3},\bar{3})}$		5	$(\bar{3},1) imes(1,\bar{3})=(\bar{3},\bar{3})$	$3+\mathbf{ar{6}}$	1
$B_L^{(5,\mathbf{ar{3}},\mathbf{ar{3}})}$		5	$(\bar{3},1)\times(1,\bar{3})=(\bar{3},\bar{3})$	$3+\bar{6}$	-1

$$\mathcal{L}_{\rm EHC} = \lambda_1 \bar{T}_L^{(5,3,1)} B_R^{(5,3,1)} + \lambda_2 \bar{T}_R^{(5,3,1)} B_L^{(5,3,1)} + \lambda_3 \bar{T}_L^{(5,1,3)} B_R^{(5,1,3)} + \lambda_4 \bar{T}_R^{(\bar{5},1,3)} B_L^{(\bar{5},1,3)} + \lambda_5 \bar{T}_L^{(\bar{5},\bar{3},\bar{3})} B_R^{(\bar{5},\bar{3},\bar{3})} + \lambda_6 \bar{T}_R^{(5,\bar{3},\bar{3})} B_L^{(5,\bar{3},\bar{3})} + \text{H.c..}$$
(5.3)

The embeddings of the Standard Model fields into the spurions are  $T_{L,a}^{(5,3,1)} = T_{L,a}^{(5,1,3)} = T_{L,a}$ ,  $T_{R,a}^{(\bar{5},3,1)} = T_{R,a}^{(\bar{5},1,3)} = T_{R,a}$ ,  $T_{L,bc}^{(\bar{5},\bar{3},\bar{3})} = \epsilon_{abc}T_{L,a}$  and  $T_{R,bc}^{(\bar{5},\bar{3},\bar{3})} = \epsilon_{abc}T_{R,a}$ . Here

$$T_{L,a}(x) = t_{L,a}(x)\hat{t}_L + b_{L,a}(x)\hat{b}_L, \qquad (5.4)$$

$$T_{R,a}(x) = t_{R,a}(x)\hat{t}_R,$$
 (5.5)

where the constant 5-vectors are

$$\hat{t}_{L} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0\\0\\i\\-1\\0 \end{pmatrix}, \qquad \hat{b}_{L} = \frac{1}{\sqrt{2}} \begin{pmatrix} i\\1\\0\\0\\0 \\0 \end{pmatrix}, \qquad \hat{t}_{R} = \begin{pmatrix} 0\\0\\0\\0\\1 \end{pmatrix}.$$
(5.6)

In Ref. [23], the effective potential was  $O(\lambda^4)$ , i.e., it was quartic in the coupling constants of  $\mathcal{L}_{EHC}$ . Correspondingly, the low-energy constant discussed in Ref. [23] was determined in terms of a hyperbaryon 4-point function. The additional terms proportional to  $\lambda_5$ and  $\lambda_6$  present in Eq. (5.3) allow for the generation of an effective potential already at  $O(\lambda^2)$ , with low-energy constants that depend on hyperbaryon two-point functions. The  $O(\lambda^2)$  potential is given by

$$V_{\text{eff}}^{\text{top}} = \frac{1}{2} \lambda_1 \lambda_5 C_L \epsilon_{abc} \Omega_{cd}^* \Phi^{4/3} v_{L,a}^{(\mathbf{5},\mathbf{3},1)^{\dagger}} \Sigma v_{L,bd}^{(\mathbf{5},\mathbf{3},\mathbf{3})} + \frac{1}{2} \lambda_3 \lambda_5 C'_L \epsilon_{abc} \Omega_{dc} \Phi^{-16/3} v_{L,a}^{(\mathbf{5},\mathbf{1},\mathbf{3})^{\dagger}} \Sigma v_{L,db}^{(\mathbf{5},\mathbf{3},\mathbf{3})} + \frac{1}{2} \lambda_2 \lambda_6 C_R \epsilon_{abc} \Omega_{cd}^* \Phi^{16/3} v_{R,a}^{(\mathbf{5},\mathbf{3},\mathbf{1})^{\dagger}} \Sigma^* v_{R,bd}^{(\mathbf{5},\mathbf{3},\mathbf{3})} + \frac{1}{2} \lambda_4 \lambda_6 C'_R \epsilon_{abc} \Omega_{dc} \Phi^{-4/3} v_{R,a}^{(\mathbf{5},\mathbf{1},\mathbf{3})^{\dagger}} \Sigma^* v_{R,db}^{(\mathbf{5},\mathbf{3},\mathbf{3})} + \text{H.c..}$$
(5.7)

As in the previous sections, the global spurions (the v's) result from integrating over the quark fields  $q_L$  and  $t_R$ . The dependence of each spurion field on the relevant global spurions is similar to Eqs. (3.11) and (3.12). It follows that in the right-handed case we simply need to substitute  $\hat{t}_R$  for  $v_R$ . For the left-handed case, we have to sum over  $\hat{t}_L$  and  $\hat{b}_L$ , paying attention to the possible presence of the  $SU(3)_c$  invariant tensor  $\epsilon_{abc}$  in the embedding of the Standard-Model fields into the spurions.

Since in this section we keep track of the  $SU(3) \times SU(3)'$  symmetry, we show in Eq. (5.7) the dependence of the potential on  $\Omega$ , the nonlinear field for  $SU(3) \times SU(3)' \rightarrow SU(3)_c$  symmetry breaking.  $\Omega$  transforms as  $\Omega \rightarrow g\Omega g'^{\dagger}$ , with  $g \in SU(3)$  and  $g' \in SU(3)'$ , i.e., it belongs to  $(\mathbf{1}, \mathbf{3}, \mathbf{3})$ . As for the dependence on the  $U(1)_A$  nonlinear field  $\Phi$ , its power in each term is given by the axial charge of the hyperbaryon two-point function occurring in the calculation of the low-energy constant. (The actual calculation of the low-energy constants is similar to Ref. [23], and is left for the reader.)

In order to proceed, we will for simplicity set  $\Omega_{ab} = \delta_{ab}$ . This means that, as in the previous sections, we do not calculate the effective potential for the colored pNGBs. The result is

$$V_{\text{eff}}^{\text{top}} = \lambda_5 (\lambda_1 C_L \Phi^{4/3} + \lambda_3 C'_L \Phi^{-16/3}) \text{tr}(\Sigma P_1) + \lambda_6 (\lambda_2 C_R \Phi^{16/3} + \lambda_4 C'_R \Phi^{-4/3}) \text{tr}(\Sigma^* P_2) + \text{H.c..}$$
(5.8)

Here we introduced the orthogonal projectors

$$P_1 = \sum_{v_L = \hat{\imath}_L, \hat{b}_L} v_L \times v_L^{\dagger}, \qquad (5.9a)$$

$$P_2 = v_R \times v_R^{\dagger}, \tag{5.9b}$$

$$P_3 = \sum_{v_L = \hat{\imath}_L, \hat{b}_L} v_L^* \times v_L^T, \qquad (5.9c)$$

whose sum  $P_1 + P_2 + P_3$  is equal to the 5 × 5 identity matrix.

As in the previous section, we were unable to work out the dependence of  $V_{\text{eff}}^{\text{top}}$  on all the pNGBs in closed form. But, as before, we can obtain the potential in some special cases. First, expanding the potential to second order in all the pNGBs gives

$$V_{\text{eff}}^{\text{top}} = -(\lambda_1 \lambda_5 C_L + \lambda_3 \lambda_5 C'_L) \frac{(2/5)\eta^2 + 4H^{\dagger}H + 4\Phi_0^2 + 8\Phi_+ \Phi_-}{f^2} - (\lambda_2 \lambda_6 C_R + \lambda_4 \lambda_6 C'_R) \frac{(16/5)\eta^2 + 8H^{\dagger}H}{f^2} + (-\lambda_1 \lambda_5 C_L + 4\lambda_3 \lambda_5 C'_L - 8\lambda_2 \lambda_6 C_R + 2\lambda_4 \lambda_6 C'_R) \frac{16\eta\zeta}{3\sqrt{5}f} - (2\lambda_1 \lambda_5 C_L + 32\lambda_3 \lambda_5 C'_L + 16\lambda_2 \lambda_6 C_R + \lambda_4 \lambda_6 C'_R) \frac{16\zeta^2}{9} + \cdots.$$
(5.10)

If we use the parametrization (F2), i.e., we retain only the h and  $\varphi$  fields of Eq. (F1), the potential is given by

$$V_{\text{eff}}^{\text{top}} = 4(\lambda_1 \lambda_5 C_L \cos(4\zeta/3) + \lambda_3 \lambda_5 C'_L \cos(16\zeta/3)) \left(1 - \hat{c} \frac{2\varphi^2 + h^2}{f^2}\right) + 2(\lambda_2 \lambda_6 C_R \cos(16\zeta/3) + \lambda_4 \lambda_6 C'_R \cos(4\zeta/3)) \left(1 - 4\hat{c} \frac{h^2}{f^2}\right).$$
(5.11)

We observe that there are no odd-order terms. Indeed, it is easy to check that the potential (5.8) is invariant under the intrinsic parity transformation of Sec. IV E. The gauge bosons contribution for this parametrization is the same as in Sec. IV, see Eq. (F15).

In this section, we have allowed for spurions with all possible  $SU(3) \times SU(3)'$  quantum numbers, resulting in the four-fermion Lagrangian (5.3). By contrast, in Ref. [23] we only considered top spurions with particular  $SU(3) \times SU(3)'$  quantum numbers. This corresponds to retaining only the  $\lambda_1$  and  $\lambda_2$  terms in Eq. (5.3), while setting  $\lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 0$ . In this case, the  $O(\lambda^2)$  potential vanishes, and the leading potential is  $O(\lambda^4)$ . Explicitly,

$$\begin{aligned} V_{\text{eff}}^{\text{top}} &= \lambda_{1}^{2} \lambda_{2}^{2} C_{LR}^{\text{top}} \sum_{v_{L} = \hat{t}_{L}, \hat{b}_{L}} |v_{L}^{\dagger} \Sigma \hat{t}_{R}|^{2} + \lambda_{2}^{4} C_{RR}^{\text{top}} |\hat{t}_{R}^{T} \Sigma \hat{t}_{R}|^{2} \\ &+ \lambda_{1}^{4} C_{LL}^{\text{top}} \sum_{v_{L}, u_{L} = \hat{t}_{L}, \hat{b}_{L}} |u_{L}^{\dagger} \Sigma v_{L}^{*}|^{2} \\ &= \lambda_{1}^{2} \lambda_{2}^{2} C_{LR}^{\text{top}} \text{tr}(P_{1} \Sigma P_{2} \Sigma^{*}) + \lambda_{2}^{4} C_{RR}^{\text{top}} \text{tr}(P_{2} \Sigma P_{2} \Sigma^{*}) \\ &+ \lambda_{1}^{4} C_{LL}^{\text{top}} \text{tr}(P_{1} \Sigma P_{3} \Sigma^{*}), \end{aligned}$$
(5.12)

where we have used Eq. (5.9). The  $C_{LR}^{top}$  term was discussed in Ref. [23], whereas the other two terms were overlooked.<sup>26</sup> As in the rest of this paper, the low-energy constants introduced in this section are always determined by the stand-alone hypercolor theory. Expanding this potential to second order in the pNGB fields gives

$$V_{\text{eff}}^{\text{top}} = \left( \left( 4\lambda_1^2 \lambda_2^2 C_{LR}^{\text{top}} - 8\lambda_2^4 C_{RR}^{\text{top}} \right) H^{\dagger} H + 8\lambda_1^4 C_{LL}^{\text{top}} \Phi_+ \Phi_- \right) + \cdots,$$
(5.13)

while for the parametrization (F2) we obtain

$$V_{\text{eff}}^{\text{top}} = \lambda_1^2 \lambda_2^2 C_{LR}^{\text{top}} \frac{2h^2}{f^2} \left( \hat{s} - \frac{2\hat{c}\varphi}{f} \right)^2 + \lambda_2^4 C_{RR}^{\text{top}} \left( 1 - \frac{4\hat{c}h^2}{f^2} \right)^2 + 4\lambda_1^4 C_{LL}^{\text{top}} \left( \frac{\hat{c}h^2}{f^2} + \frac{\hat{s}\varphi}{f} \right)^2.$$
(5.14)

This result shows that there are no odd-order terms associated with  $C_{RR}^{\text{top}}$ , consistent with the invariance of the corresponding term in Eq. (5.12) under the intrinsic parity transformation of Sec. IV E. Cubic terms arise from the contributions associated with  $C_{LR}^{\text{top}}$  and  $C_{LL}^{\text{top}}$ . These contributions will be absent if  $\lambda_1 = 0$ .<sup>27</sup> If both  $\lambda_1$  and  $\lambda_2$  are nonzero, then the cubic terms will be present except in the (unlikely, because arbitrarily fine-tuned) case that  $\lambda_2^2 C_{LR}^{\text{top}} = \lambda_1^2 C_{LL}^{\text{top}}$ . In this case, the sum of the two terms is proportional to

<sup>&</sup>lt;sup>26</sup>In the conventions of Ref. [23],  $\lambda_1^2 \lambda_2^2 C_{LR}^{top}$  corresponds to  $y^2 C_{top}$ . In Ref. [23] we argued that  $C_{LR}^{top}$  dominates over the gauge bosons contribution in a certain large-*N* framework. Unfortunately, it is not possible to incorporate  $C_{RR}^{top}$  and  $C_{LL}^{top}$  into the same large-*N* framework in a meaningful way.

<sup>&</sup>lt;sup>27</sup>Notice, however, that in order to generate a mass for the top quark, at least two four-fermion couplings must be nonzero, e.g.,  $\lambda_1$  and  $\lambda_2$  [23].

$$\operatorname{tr}(P_1 \Sigma P_2 \Sigma^*) + \operatorname{tr}(P_1 \Sigma P_3 \Sigma^*) = \operatorname{tr}(P_1 \Sigma (1 - P_1) \Sigma^*),$$
(5.15)

which is again invariant under the intrinsic parity transformation.

The main phenomenological implications of the results of this section are discussed in the concluding section.

### VI. DISCUSSION AND CONCLUSIONS

The composite Higgs approach is often discussed taking the low-energy, nonlinear sigma model as a starting point. In this paper, we studied in detail several concrete realizations (ultraviolet completions) of this approach as an asymptotically free gauge theory with fermionic matter. In this concluding section, we discuss the lessons that can be drawn from our findings.

We begin with a simple technical observation about the Higgs potential. It is a generic feature of composite Higgs models that, if we turn off all the pNGBs except for  $h = \sqrt{2} \operatorname{Re} H_0$ , then the coset field  $\Sigma$  describes a rotation matrix by an angle  $\alpha \propto h$  in some generalized space. In other words, the nonzero entries of  $\Sigma$  depend linearly on  $\cos(\alpha)$  or  $\sin(\alpha)$ . This is true in particular for the two cosets discussed in this paper.<sup>28</sup> For an effective potential that is at most quadratic in  $\Sigma$  and/or  $\Sigma^*$ , it follows that the effective potential is then a second-order polynomial in  $\cos(\alpha)$  and  $\sin(\alpha)$ . Furthermore,  $SU(2)_{I}$  invariance requires that, when all triplet fields are turned off, the potential must be an even function of the Higgs field H, and this remains true when we retain  $h = \sqrt{2} \operatorname{Re} H_0$ only. The form of the resulting effective potential is very restricted. It depends on just two trigonometric functions of  $\alpha$ , and we may take it to be [6,23,32]

$$V_{\text{eff}} = \text{const} - A\cos(\alpha) + B\cos^2(\alpha).$$
 (6.1)

The solutions of the saddle-point equation are  $sin(\alpha) = 0$  or

$$\frac{A}{2B} = \cos(\alpha), \tag{6.2}$$

which is the symmetry-breaking solution of interest.<sup>29</sup> We may rewrite this solution as

$$1 - \frac{A}{2B} = \frac{\alpha^2}{2} + O(\alpha^4).$$
 (6.3)

Current experimental constraints suggest  $h^2/f^2 \lesssim 0.1$  as a figure of merit [5,6,33].<sup>30</sup> Thus, for the right-hand side of

Eq. (6.3) to be small, an "irreducible fine-tuning" at a similar level of the coefficients *A* and *B* is needed.

The effective potential receives contributions from two different sources. First, there are  $O(q^2, q'^2)$  terms, arising from the interaction between the electroweak gauge bosons and the pNGBs. The form of these terms is constrained by gauge invariance, and they depend on a single low-energy constant  $C_{LR}$ . The other source of an effective potential arises from integrating out the third generation quark fields. This is the prime focus of this paper. In order to explain the four-fermion Lagrangian that couples the quark fields to three-fermion states of the hypercolor theory, we have to postulate the existence of an "extended hypercolor" theory. This new dynamics is operative at a yet higher energy scale,  $\Lambda_{\rm EHC}$ , and requires the existence of new heavy gauge bosons that can transform an ordinary quark into one of the fermion species of the hypercolor theory.<sup>31</sup> The leading contributions to the effective potential from this sector are  $O(\lambda^2)$ , where we use  $\lambda$  as a generic name for a four-fermion coupling. In the case of the model of Sec. V, for reasons that we explain below, we are also interested in  $O(\lambda^4)$ contributions.

Having a minimum of the effective potential with  $h^2/f^2 \lesssim 0.1$  thus requires balancing between  $O(g^2, g'^2)$ effects, which depend on the gauge couplings of the Standard Model, and  $O(\lambda^2)$  effects (or, in special circumstances,  $O(\lambda^4)$  effects), which depend on the dynamics of the EHC theory, and can generically arise from several distinct four-fermion couplings. If the effects of the thirdgeneration quarks dominates over the gauge bosons, then the balancing has to happen between the contributions coming from different four-fermion couplings. We have studied an example potential in Sec. III F. However, it remains an open question how the four-fermion couplings originating from the EHC theory can be arranged to give the desired result. We note that we did not make any *ad hoc* assumptions about the EHC sector. It turns out that, in all cases considered here, the most general form that the fourfermion Lagrangian may take is quite complicated, leading to many possibilities for the low-energy effective theory (both the induced Higgs potential and the Yukawa couplings). New ideas will be needed to simplify the situation, but those would necessarily address the specific form of the EHC sector, and are beyond the scope of this paper.

Let us briefly touch on another basic difficulty, which is the inherent tension between fermion masses and flavor constraints. Traditionally, fermion masses are generated in technicolor models via four-fermion couplings that are induced by an extended technicolor (ETC) dynamics, of which our extended hypercolor (EHC) dynamics is a close cousin. The main difference is the following. If we generically use  $\psi$  to denote a Standard Model fermion

<sup>&</sup>lt;sup>28</sup>For the SU(4)/Sp(4) coset  $\alpha = h/(\sqrt{2}f)$  by Eq. (3.42). For the SU(5)/SO(5) coset  $\alpha = 2h/f$ , see Appendix F.

<sup>&</sup>lt;sup>29</sup>For Eq. (6.2) to describe a global minimum at small  $\alpha$  we must have 0 < A < 2B.

<sup>&</sup>lt;sup>30</sup>We may expect this bound to become tighter in the future if no new particles are found at the LHC.

<sup>&</sup>lt;sup>31</sup>Here we assume that also the EHC theory, which otherwise remains unspecified, is a renormalizable gauge theory.

field, and  $\Psi$  for a fermion of the new strong dynamics (be it technicolor or hypercolor), then ETC requires four-fermion interactions of the generic form  $\psi\psi\Psi\Psi\Psi$ , whereas the EHC interactions are assumed to have the form  $\psi \Psi \Psi \Psi$ . The ETC four-fermion interactions induce a fermion mass term,  $\psi\psi\langle\Psi\Psi\rangle$ , once the operator  $\Psi\Psi$  acquires an expectation value. By contrast, the EHC four-fermion interaction  $\psi\Psi\Psi\Psi$  allows for a linear coupling of a Standard Model fermion to a hyperbaryon, thereby giving rise to a partially composite state.<sup>32</sup>

The basic problem is that the same ETC or EHC dynamics that gives rise to the desired four-fermion interactions can, generically, also give rise to four-fermion interactions  $\sim \psi \psi \psi \psi$ , namely interactions that involve four Standard Model fermions. These interactions will trigger flavor-changing processes that, if too strong, will be in conflict with experiment. According to naive power counting, fermion masses in ETC are suppressed relative to the technicolor scale  $\Lambda_{TC}$  by  $z_{TC}^2$ , where  $z_{TC} = \Lambda_{TC} / \Lambda_{ETC}$ , with  $\Lambda_{\text{ETC}}$  being the ETC scale. Because of the flavor constraints  $\Lambda_{\text{ETC}}$  must be quite large, making the ratio  $z_{\text{TC}}$  small. The resulting fermion masses, of order  $\Lambda_{TC} z_{TC}^2$ , are then too small in many cases.

A partial solution may be provided by walking technicolor, where the technicolor dynamics is assumed to be nearly conformal. Taking quantum effects into account, the induced fermion mass in walking technicolor is  $\sim \Lambda_{TC} z_{TC}^{2-\gamma_m}$ , where  $\gamma_m$  is the (approximately constant) mass anomalous dimension of the technifermion  $\Psi$ . Ideally, a very large anomalous dimension  $\gamma_m \lesssim 2$  would wipe out entirely the suppression factor  $z_{TC}^{2-\gamma_m}$ . But various theoretical considerations suggest that such large values of  $\gamma_m$  are unlikely [4–6]. Lattice calculations in various models find that  $\gamma_m$ does not exceed 1 (see the review articles [9,11]). If indeed  $\gamma_m \lesssim 1$  then the induced fermion mass can only be as large as  $\Lambda_{TC} z_{TC}$ , i.e., still suppressed by one power of  $z_{TC}$ . Thus, while near-conformality together a large  $\gamma_m$  help in generating larger fermion masses, it remains very difficult to generate a mass as large as that of the top quark. As an illustration, according to Ref. [6],  $\Lambda_{\text{ETC}}$  cannot be smaller than about 10<sup>5</sup> TeV,<sup>33</sup> so that  $z_{TC}$  cannot be larger than ~10<sup>-4</sup>. With  $\gamma_m \sim 1$  this might have allowed for generating the  $\sim$ 1 GeV mass of the charm quark, but certainly not the top-quark mass.

If, instead, the top quark receives its mass via the partial compositeness mechanism, this mass will be naively of order  $\Lambda_{\rm HC} z_{\rm HC}^4$ , where  $z_{\rm HC} = \Lambda_{\rm HC} / \Lambda_{\rm EHC}$ , because, when measured in units of the hypercolor theory, each four-fermion coupling is naively of order  $z_{HC}^2$ , and two

four-fermion couplings are needed to generate a mass for the top: the top must transform into a hyperbaryon, and then back into a top. At tree level, the case for partial compositeness is thus worse than traditional ETC. Of course, one has to take into account quantum effects. If again the theory is nearly conformal, the induced top mass is of order  $\Lambda_{\rm HC} z_{\rm HC}^{4-2\gamma'}$ , where  $\gamma'$  is the (again, approximately constant) anomalous dimension of the relevant fourfermion operators. Once again, the suppression factor  $z_{\rm HC}^{4-2\gamma'}$  would be wiped out when  $\gamma' \lesssim 2$ . The popularity of partial compositeness stems from the fact that there are no theoretical considerations against such large values of  $\gamma'$ . Thus, at least in principle, one could end up with a suppression by a very small power of  $z_{\rm HC}$  [4–6].<sup>34</sup>

We stress that in order to achieve a large enhancement, be it in the context of extended technicolor or in the context of a partially composite top, the anomalous dimension must be approximately constant, and large, over many energy decades. This requires the dynamics to be nearly conformal. In contrast, if the gauge dynamics is QCD-like, then this mechanism is unlikely to be effective. The reason is that as we increase the energy scale, the gauge coupling quickly becomes perturbative. Existing perturbative calculations of the anomalous dimension of various four-fermion operators always find small values [27,34]. It remains an open question whether a realistic top-quark mass can be achieved by invoking a strong near-conformal dynamics. Lattice calculations of  $\gamma'$  in candidate hypercolor theories could help shed light on this important issue.

An alternative approach would be to assume that, while the top quark receives its mass through partial compositeness from an extended hypercolor dynamics, yet some other dynamics (or, more generally, some additional high scales), are involved in mass generation of all other Standard Model fermions.<sup>35</sup> This approach is, obviously, less economic, but eventually it might be forced upon us by the tension between flavor-changing processes and quark masses. In a way, in this paper we are following this approach, because we study the interaction between the third-generation quarks and the hypercolor theory, while disregarding the rest of the fermions of the Standard Model. In particular, we are in effect allowing for the extended hypercolor scale  $\Lambda_{EHC}$  to be close enough to the hypercolor scale  $\Lambda_{HC}$ , so that the four-fermion couplings will be large enough to generate phenomenologically viable mass for the top quark and effective potential for the pNGBs.

In this paper, we studied two SO(d) gauge theories with d = 5, 11, where chiral symmetry breaking gives rise to pNGBs in the SU(4)/Sp(4) coset (Sec. III); and three models where the coset is SU(5)/SO(5), two are again based on an SO(d) gauge theory with d = 7, 9, and have a

<sup>&</sup>lt;sup>32</sup>In principle, a given EHC theory may induce both  $\psi\psi\Psi\Psi\Psi$ and  $\psi \Psi \Psi \Psi$  type four-fermion interactions, in which case both mechanisms for fermion mass generation will be operative (see, e.g., Ref. [13]).  $^{33}$  In the notation of Ref. [6],  $\Lambda_{ETC}$  is  $\Lambda_{UV}.$ 

 $<sup>^{34}</sup>$  For a calculation of  $\gamma_m$  and  $\gamma'$  in a gauged Nambu–Jona-Lasinio model, see Ref. [17].

See, for example, Refs. [35-37].

similar set of top partners (Sec. IV), while the third is an SU(4) gauge theory with a rather different set of top partners (Sec. V). Each model contains fermions in two different *irreps*, leading to a nonanomalous Abelian axial symmetry,  $U(1)_A$ , with an associated pNGB,  $\zeta$ , which is inert under all the Standard-Model gauge interactions. For each theory we first listed all the dimension-9/2 hyperbaryons that can serve as top partners, and wrote down the most general four-fermion Lagrangian that couples them to  $t_L$ ,  $b_L$  and  $t_R$ . We then worked out the resulting effective potential for the multiplet of pNGBs containing the Higgs field together with the  $U(1)_A$  pNGB.

We started with the SU(4)/Sp(4) coset. Its structure is simpler in that, besides Higgs doublet H, this coset contains only one additional pNGB,  $\eta$ , which is inert under the Standard Model gauge interactions, like the  $U(1)_A$  pNGB  $\zeta$ . We worked out the  $O(\lambda^2)$  potential in closed form. We found that it consists of a linear superposition of nine functions of the variables  $H^{\dagger}H$ ,  $\eta$  and  $\zeta$  (cf. Sec. III F). Thus, in general, a potential is generated for all the pNGBs, including the  $U(1)_A$  pNGB. Each coefficient  $c_i$  consists of a sum of terms, where each term is the product of a lowenergy constant and two four-fermion couplings. The effective potential generated by the electroweak gauge bosons also depends on one of these functions,  $f_6$ , and so it contributes only to its coefficient  $c_6$ . Finally, there is an additional contribution to the effective potential if a mass term for the  $\chi$  fermions is turned on in the hypercolor theory. By itself, experimental constraints on the effective potential can be studied directly in terms of the  $c_i$ 's. But, if one wants to incorporate also the top Yukawa coupling into this analysis, then it has to be done in terms of the fourfermion couplings, and requires knowledge of the lowenergy constants. The latter can, in principle, be calculated on the lattice.

Studying the minima of the full effective potential as a function of all the relevant parameters is challenging. Generically, minimizing the potential might give rise to the condensation of not just the Higgs field, but also the "inert" fields  $\zeta$  and  $\eta$ . Since these fields are pseudoscalars, the expectation values  $\langle \eta \rangle$  or  $\langle \zeta \rangle$  break *CP* spontaneously, and will thus be constrained by experiment. We have discussed the conditions that these expectation values are physical, and cannot be rotated away (Sec. III G). Our discussion of the effective potential was limited to a simple example, in which most of the four-fermion couplings are turned off by hand (Sec. III F). For this potential  $\langle \eta \rangle$  and  $\langle \zeta \rangle$  are both physical. We wrote down the conditions needed to have  $\langle \eta \rangle = \langle \zeta \rangle = 0$ , at which point the potential reduces to the familiar form of Eq. (6.1).<sup>36</sup>

The low-energy constants of the SO(d) models depend on two-point functions of the hyperbaryons. While, as we have explained above, one can sometimes by-pass the calculation of the low-energy constants by studying directly the  $c_i$  coefficients in Eq. (3.72), the correct form of the effective potential cannot be determined without the knowledge of the dimension-9/2 hyperbaryons. In other words, if one starts directly from the nonlinear sigma model it is just not possible to determine the correct effective potential. One can, of course, determine the structure of the effective potential for a given set of spurion fields. But the spurions must match the top-partner hyperbaryons. An *ad-hoc* list of spurions could amount to arbitrarily setting some of the four-fermion couplings to zero, in a manner that cannot be reproduced by any extended hyper-color theory.

Next let us discuss the models that yield the SU(5)/SO(5) coset. In addition to the pNGBs that are present in the SU(4)/Sp(4) case, there are nine additional pNGBs that fill a (3,3)-plet of  $SU(2)_L \times SU(2)_R$ . Because of this more complicated structure we were not able to obtain the full potential in closed form. Instead, we studied the potential in various simplified cases. First, we obtained the potential to second order in all the pNGBs. Some useful constraints can already be obtained from this result because, ideally, we would like the curvature at the origin to be negative in the direction of the Higgs field, and to be positive in the direction of the triplet fields, to prevent any triplet from condensing.<sup>37</sup>

We also considered third order terms. These terms arise because one can construct invariants of both  $SU(2)_L$  and the U(1) generated by  $T_R^3$  from a pair of Higgs fields and one triplet field. For a concrete example, see Eq. (4.15). As we explained in the introduction, and in more detail in Sec. IV E, these terms are especially dangerous for phenomenology. If the potential contains cubic terms, then, once the Higgs field acquires an expectation value, this induces a term linear in the triplet field. This, in turn, will necessarily drive the expectation value of the triplet field away from zero. The resulting triplet expectation value is different from the one that preserves the custodial symmetry [22], and so it will drive the  $\rho$ -parameter away from unity. The magnitude of this triplet expectation value is thus tightly constrained by experiment.

Studying this issue further, we have worked out the full effective potential in the case that only  $h = \sqrt{2} \operatorname{Re} H_0$  and  $\varphi = \sqrt{2} \operatorname{Im} \phi_+^-$  are turned on. We checked which "templates" for the effective potential can give rise to odd-order terms, and, in particular, to the cubic term  $h^2\varphi$ , finding that such contributions are possible in all the SU(5)/SO(5) models.<sup>38</sup> We then raised the question

<sup>&</sup>lt;sup>36</sup>For further discussion of the role of  $\eta$ , see, for example, Refs. [24,38,39].

<sup>&</sup>lt;sup>37</sup>In order to obtain the complete second-order potential, one should add the contribution of the electroweak gauge bosons, calculated in Ref. [23].

<sup>&</sup>lt;sup>38</sup>No odd-order terms arise from the gauge bosons contribution in this case, see Eq. (F15).

how likely it is that all cubic terms (or, more generally, all odd-order terms) will be absent from the effective potential thanks to cancellations.

As we explained in Sec. IV E, if all the four-fermion couplings are nonzero, the vanishing of the coefficient of a particular (cubic) term in the effective potential requires a "conspiracy" between the four-fermion couplings and the low-energy constants. What might be more natural is that the cubic terms will vanish thanks to the vanishing of suitable (linear combinations of) four-fermion couplings. The intrinsic parity transformation introduced in Sec. IV E is a convenient device to determine which linear combinations of the four-fermion couplings should vanish. Unfortunately, we were unable to conceive of any obvious symmetry at the level of the EHC theory that would induce the intrinsic parity symmetry at the level of the low-energy effective theory. Still, one should remember that the fourfermion couplings must be induced by integrating out the heavy degrees of freedom of an EHC theory, and a good candidate EHC theory will conceivably induce only a small number of four-fermion couplings.

The SU(4) model of Sec. V was already studied in detail previously [13,23]. We found that if we allow for the most general four-fermion Lagrangian, an effective potential is induced already at  $O(\lambda^2)$ . While this potential contains no cubic terms, it does have another serious phenomenological drawback. If we set to zero all the pNGB fields except for h, then the contribution from the  $O(\lambda^2)$  potential is proportional to  $\cos(\alpha)$ . Because the gauge bosons also contribute to the same term, we would end up with the situation that  $A \neq 0$  but B = 0 in Eq. (6.1). This appears to be incompatible with the requirement of having small h/f. A possible way out that we have discussed above is that, when turning on also the inert pNGBs  $\eta$  and  $\zeta$ , this would reveal new minima of the potential.

An alternative is that only a smaller subset of the fourfermion couplings is actually induced by the EHC, and, as a result, the  $O(\lambda^2)$  potential vanishes. We rederived the potential in the case that only the two four-fermion couplings we considered in Ref. [23] are nonzero, finding two more terms that we overlooked in Ref. [23]. Like the other SU(5)/SO(5) models, this  $O(\lambda^4)$  potential will generically have the undesired cubic terms  $\propto h^2\varphi$ , so that, as explained above, further constraints must be satisfied in order to achieve a phenomenologically viable minimum. In this paper, we discussed the nonlinear field  $\Sigma$  associated with the SU(4)/Sp(4) or SU(5)/SO(5) coset, and the field  $\Phi$  that describes the pNGB of the nonanomalous  $U(1)_A$  symmetry. We did not discuss the other nonlinear field containing the colored pNGBs, which is associated with the SU(6)/SO(6) coset in the case of the SO(d) theories of Sec. III and Sec. IV, or with  $SU(3) \times SU(3)'/SU(3)_c$  in the case of the SU(4) model of Sec. V. While our results and conclusions are valid by themselves, a more complete analysis that includes the potential for the remaining nonlinear effective field would allow for a more detailed study of the phenomenological consequences. The obvious additional constraint on the complete potential is that the colored pNGBs are not allowed to condense.

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### APPENDIX A: CHARGE CONJUGATION MATRIX IN *d* DIMENSIONS

Here we review some properties of the charge conjugation matrix  $C_d$  in euclidean space. These properties are periodic in the dimensionality d modulo 8, as shown in Table V. The basic relation satisfied by the charge conjugation matrix  $C_d$  in d = 2n dimensions is

$$C_d \Gamma_I = -\Gamma_I^T C_d, \qquad I = 1, 2, \dots, d.$$
(A1)

In addition,  $C_d$  always satisfies  $C_d^T = C_d^{\dagger} = C_d^{-1}$ . For the symmetry properties of  $C_d$ , see the second column of Table V. The chirality matrix is defined by  $\Gamma_{2n+1} = e^{i\eta_n}\Gamma_1\cdots\Gamma_{2n}$ , where the phase  $e^{i\eta_n}$  is chosen such that  $\Gamma_{2n+1}^2 = 1$ . One has  $C_{2n}\Gamma_{2n+1} = \pm\Gamma_{2n+1}^T C_{2n}$ , where the sign is given in the third column of Table V. When this sign is negative, Eq. (A1) generalizes to include  $\Gamma_{2n+1}$ , and we define  $C_{2n+1} = C_{2n}$ . When this sign is positive we define, instead,  $C_{2n+1} = C_{2n}\Gamma_{2n+1}$ , which implies

TABLE V. Some properties of the charge conjugation matrix. These properties are periodic in the dimensionality d modulo 8, shown in the left column. See Appendix A for an explanation on the other columns.

2	$C_d^T = -C_d$	$C_d \Gamma_{d+1} = -\Gamma_{d+1}^T C_d$	$C_{d+1} = C_d$	$(C_{d+1}\Gamma_I)^T = +C_{d+1}\Gamma_I$
4	$C_d^{\tilde{T}} = -C_d$	$C_d\Gamma_{d+1} = +\Gamma_{d+1}^T C_d$	$C_{d+1} = C_d \Gamma_{d+1}$	$(C_{d+1}\Gamma_I)^T = -C_{d+1}\Gamma_I$
6	$C_d^T = +C_d$	$C_d \Gamma_{d+1} = -\Gamma_{d+1}^T C_d$	$C_{d+1} = C_d$	$(C_{d+1}\Gamma_I)^T = -C_{d+1}\Gamma_I$
0	$C_d^T = +C_d$	$C_d \Gamma_{d+1} = + \Gamma_{d+1}^T C_d$	$C_{d+1} = C_d \Gamma_{d+1}$	$(C_{d+1}\Gamma_I)^T = +C_{d+1}\Gamma_I$

$$C_{2n+1}\Gamma_I = +\Gamma_I^T C_{2n+1}, \qquad I = 1, 2, ..., 2n+1.$$
 (A2)

In all cases,  $C_{2n+1}$  has the same symmetry properties as  $C_{2n}$ . It follows that for all *d*, the generators of SO(d) rotations on spinors,  $\Sigma_{II} = \frac{1}{4} [\Gamma_I, \Gamma_J]$ , satisfy

$$C_d \Sigma_{IJ} = -\Sigma_{IJ}^T C_d, \qquad I, J = 1, 2, ..., d.$$
 (A3)

For *d* odd, the spinor *irrep* is irreducible, and from the symmetry properties of  $C_d$  it follows that the spinor *irrep* is real for d = 1, 7 mod 8, and pseudoreal for d = 3, 5 mod 8. For brevity, we will use the notation *C* for the 4-dimensional charge conjugation matrix, and *C* for the charge conjugation matrix in a given odd dimension.

The construction of the 4-component spinor  $\chi$  in Eq. (2.1) assumes the chiral representation of the Dirac matrices,

$$\gamma_{\mu} = \begin{pmatrix} 0 & \sigma_{\mu}^{\dagger} \\ \sigma_{\mu} & 0 \end{pmatrix}, \tag{A4}$$

where  $\sigma_4 = 1$ , and  $\sigma_{\mu}$  is equal to  $-i\sigma_k$  for  $\mu = k = 1, 2, 3$ , where  $\sigma_k$  are the Pauli matrices. Also,  $\gamma_5 = \text{diag}(1, 1, -1, -1)$ , and, as usual,  $P_R = (1 + \gamma_5)/2$  and  $P_L = (1 - \gamma_5)/2$ . The charge conjugation matrix is then

$$C = \gamma_4 \gamma_2 = \begin{pmatrix} -\epsilon & 0\\ 0 & \epsilon \end{pmatrix}, \tag{A5}$$

where  $\epsilon = i\sigma_2$ .

### **APPENDIX B: DISCRETE SYMMETRIES**

Here we discuss the discrete symmetries C, P and CP in SO(d) gauge theories. We first recall the familiar case of an SU(N) gauge theory with Dirac fermions in the fundamental *irrep*. Charge-conjugation symmetry acts as

$$\psi \to C \bar{\psi}^T, \qquad \bar{\psi} \to \psi^T C,$$
 (B1a)

$$A_{\mu} \rightarrow -A_{\mu}^{T}$$
. (B1b)

Writing  $A_{\mu} = A_{\mu a}T_a$  we infer the transformation rule of the individual components, which is  $A_{\mu a} \rightarrow \mp A_{\mu a}$  if  $T_a^T = \pm T_a$ . Because all SU(N) irreps may be constructed from tensor products of the fundamental irrep, these transformation rules remain valid for Dirac fermions in any irrep.

We take parity to act as

$$\psi(x) \to i\gamma_4 \psi(\tilde{x}), \qquad \bar{\psi}(x) \to -i\bar{\psi}(\tilde{x})\gamma_4, \qquad (B2a)$$

$$A_{\mu}(x) \to \tilde{A}_{\mu}(\tilde{x}).$$
 (B2b)

Here  $\tilde{x}_{\mu} = x_{\mu}$  if  $\mu = 4$ , while  $\tilde{x}_{\mu} = -x_{\mu}$  if  $\mu = 1, 2, 3$ . A similar definition applies to  $\tilde{A}_{\mu}$ . The *C* and *P* fermion transformation rules, Eqs. (B1a) and (B2a), both involve a choice of phase. The reason for the particular choices we have made is that we want the transformation rules to take

the same form for Majorana fermions. If we replace the Dirac fermion  $\psi$  by a Majorana fermion  $\chi$ , and  $\bar{\psi}$  by  $\bar{\chi}$ , then  $\bar{\chi}$  is not an independent field, but rather, it is related to  $\chi$  via Eq. (2.2). With the phases we have chosen in Eqs. (B1a) and (B2a), these transformation rules are consistent with Eq. (2.2).

Moving on to SO(d) gauge theories, charge conjugation is still given by Eq. (B1a) for Dirac fermions in the fundamental, vector *irrep*. Because the generators in the vector *irrep* are all antisymmetric, the rule (B1b) implies that the SO(d) gauge field is charge-conjugation invariant. The rule for a Dirac fermion in a spinor *irrep* is

$$\eta \to C \mathcal{C}^T \bar{\eta}^T, \qquad \bar{\eta} \to \eta^T C \mathcal{C},$$
 (B3)

where the presence of C in Eq. (B3) compensates for the fact that the SO(d) gauge field is invariant [note Eq. (A3)]. In the case of a real *irrep*, the same rules [Eqs. (B1b) or (B3)] may be applied to Majorana fermions, because again our choice of phases is consistent with Eq. (2.2). In fact, using Eq. (2.2) it immediately follows that Majorana fermions of SO(d) gauge theories are charge-conjugation invariant. (For the case of a Majorana fermion in the vector *irrep*, the matrix C in Eq. (2.2) is replaced by the identity matrix.)

We define CP by first applying P and then C. The resulting transformation rules are given in Sec. II B. The rules for the gauge field, and for the Dirac and Majorana fermions that we will encounter, follow from the transformation rules we have already discussed above.

In the case of the SU(4)/Sp(4) coset, we have 4 Weyl fermions in the pseudoreal spinor *irrep*. The discrete symmetries can be approached in two ways. First, we may assemble the four Weyl fermions into two Dirac fermions. In this case, P acts in the usual way, while C acts as described above. However, the Dirac formulation has the disadvantage that it obscures the SU(4) flavor symmetry of the pseudoreal Weyl fermions.<sup>39</sup> The alternative we choose in this paper is to work in terms of the 4-component fields  $\chi_i$  and  $\bar{\chi}_i$  introduced in Eqs. (2.1) and (2.2), also for the pseudoreal case. The advantage is that the flavor symmetry is manifest. The separate P and C transformations will look more complicated in terms of  $\chi_i$  and  $\bar{\chi}_i$ , but, because of the properties of the four-fermion Lagrangian (Sec. II B), we only need the explicit form of the combined CP transformation, which we can derive as follows. We start from the observation that the Weyl action

$$S = \int d^4 x \bar{\Upsilon} \sigma_\mu D_\mu \Upsilon, \qquad (B4)$$

is invariant under CP symmetry where the SO(d) gauge field transforms as described above, and

<sup>&</sup>lt;sup>39</sup>A similar situation is discussed in Ref. [40].

$$\Upsilon_i(x) \to i\mathcal{C}\epsilon \bar{\Upsilon}_i^T(\tilde{x}), \qquad \bar{\Upsilon}_i(x) \to i\Upsilon_i^T(\tilde{x})\epsilon\mathcal{C}^T.$$
(B5)

In terms of the four-component fields  $\chi_i$  and  $\bar{\chi}_i$ , the transformation (B5) takes the form of Eq. (2.5d) when the fermions belong to a pseudoreal *irrep*. For a real *irrep*, we recover Eq. (2.5c).

To avoid confusion, we recall that in the case of a real *irrep*, the action (B4) may be rewritten as

$$S = \frac{1}{2} \int d^4 x \bar{\chi}_i \not D \chi_i, \qquad (B6)$$

where the Majorana fermions are defined by Eqs. (2.1) and (2.2). But if we keep using the same 4-component fields for a pseudoreal *irrep*, then the right-hand side of Eq. (B6) will vanish identically. Of course, for both real and pseudoreal *irreps* we may recover Eq. (B4) from Eq. (B6) by inserting  $2P_L$  between  $\not D$  and  $\chi_i$ .

### APPENDIX C: THE SU(4)/Sp(4) COSET

The Sp(4) subgroup of SU(4) is defined as the set of elements satisfying

$$g^T \epsilon_0 g = \epsilon_0, \tag{C1}$$

where

$$\epsilon_0 = \tau_3 \times i\tau_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$
(C2)

(If  $g \in Sp(4)$ , then so is  $g^T$ .) The 15 generators of SU(4) split into 10 generators of Sp(4),

$$1 \times \tau_i, \qquad \tau_1 \times 1, \qquad \tau_2 \times \tau_i, \qquad \tau_3 \times \tau_i, \qquad (C3)$$

and 5 generators for the coset SU(4)/Sp(4),

$$au_1 imes au_i, au_2 imes 1, au_3 imes 1, au_{C4}$$

where  $\tau_i$  are the Pauli matrices, and 1 stands for the 2 × 2 identity matrix. These generators satisfy

$$\epsilon_0 T_a = \begin{cases} -T_a^T \epsilon_0, & Sp(4) \text{ generators,} \\ +T_a^T \epsilon_0, & SU(4)/Sp(4) \text{ generators.} \end{cases}$$
(C5)

The tensor product of two fundamental SU(4) *irreps* contains the six-dimensional antisymmetric, and the tendimensional symmetric *irreps*. Under the reduction  $SU(4) \rightarrow Sp(4)$ , the **10** remains irreducible, whereas the **6** reduces to a **5** and a singlet. If  $A_{ij} = -A_{ji}$  transforms in the **6** of SU(4), the singlet is tr( $\epsilon_0 A$ ), and the **5** is formed by  $A + \frac{1}{4}\epsilon_0 \operatorname{tr}(\epsilon_0 A)$ . The effective NGB field  $\Pi$  introduced in Eq. (3.8) transforms in the **5** of Sp(4). Following Ref. [8], the Standard Model's  $SU(2)_L$  and  $SU(2)_R$  symmetries are identified with the subgroups of Sp(4) with generators

$$T_{L}^{i} = \frac{1}{2}(1+\tau_{3}) \times \frac{1}{2}\tau_{i},$$
  

$$T_{R}^{i} = \frac{1}{2}(1-\tau_{3}) \times \frac{1}{2}\tau_{i}.$$
 (C6)

Correspondingly, the NGB field is parametrized as

$$2\Pi = -(\mathrm{Im}H_{+})\tau_{1} \times \tau_{1} - (\mathrm{Re}H_{+})\tau_{1} \times \tau_{2}$$

$$+ (\mathrm{Im}H_{0})\tau_{1} \times \tau_{3} - (\mathrm{Re}H_{0})\tau_{2} \times 1 + \frac{1}{\sqrt{2}}\eta\tau_{3} \times 1$$

$$= \begin{pmatrix} \eta/\sqrt{2} & 0 & iH_{0}^{*} & iH_{+} \\ 0 & \eta/\sqrt{2} & -iH_{+}^{*} & iH_{0} \\ -iH_{0} & iH_{+} & -\eta/\sqrt{2} & 0 \\ -iH_{0}^{*} & -iH_{0}^{*} & 0 & -n/\sqrt{2} \end{pmatrix}. \quad (C7)$$

The coset generators (C4) satisfy the 5-dimensional Dirac algebra. (This property is closely related to the existence of the isomorphisms  $SU(4)/Z_2 \simeq SO(6)$  and  $Sp(4)/Z_2 \simeq SO(5)$ .) Using

$$\Pi^2 = (\eta^2 + 2H^{\dagger}H)/8, \tag{C8}$$

one can express  $\Sigma$  in closed form,

$$\Sigma = \left(\cos(\alpha) + \frac{2i}{\alpha f}\sin(\alpha)\Pi\right)\epsilon_0,$$
 (C9)

where

$$\alpha^2 = ((1/2)\eta^2 + H^{\dagger}H)/f^2.$$
 (C10)

#### APPENDIX D: THE SU(5)/SO(5) COSET

The unbroken SO(5) subgroup is generated by the 10 antisymmetric, purely imaginary, generators of SU(5). We embed the generators of  $SU(2)_L \times SU(2)_R$ , which is isomorphic to SO(4), in the upper-left  $4 \times 4$  block. They are given explicitly by the following tensor products of the Pauli matrices [8]

$$2T_L^1 = \tau_2 \times \tau_1,$$
  

$$2T_L^2 = -\tau_2 \times \tau_3,$$
  

$$2T_L^3 = 1 \times \tau_2,$$
  

$$2T_R^1 = \tau_1 \times \tau_2,$$
  

$$2T_R^2 = \tau_2 \times 1,$$
  

$$2T_R^3 = \tau_3 \times \tau_2.$$
 (D1)

The nonlinear field  $\Sigma \in SU(5)/SO(5)$  is expanded as

$$\Sigma = \exp(i\Pi/f)\Sigma_0 \exp(i\Pi/f)^T = \exp(2i\Pi/f), \quad (D2)$$

where in the last equality we have set  $\Sigma_0 = 1$ . The pion field  $\Pi$  is expanded in terms of the 14 real symmetric generators of SU(5). Its  $SU(2)_L \times SU(2)_R$  content is

$$\Pi = \Theta + \Theta^{\dagger} + \tilde{\Phi}_0 + \tilde{\Phi}_+ + \tilde{\Phi}_+^{\dagger} + \tilde{\eta}, \qquad (D3)$$

where

<

$$\Theta = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 & -iH_{+} \\ 0 & 0 & 0 & 0 & H_{+} \\ 0 & 0 & 0 & 0 & iH_{0} \\ 0 & 0 & 0 & 0 & H_{0} \\ -iH_{+} & H_{+} & iH_{0} & H_{0} & 0 \end{pmatrix}, \quad (D4)$$

$$\tilde{\Phi}_{0} = \begin{pmatrix} \phi_{0}^{0}/\sqrt{2} & 0 & a & b & 0 \\ 0 & \phi_{0}^{0}/\sqrt{2} & b & -a & 0 \\ a & b & -\phi_{0}^{0}/\sqrt{2} & 0 & 0 \\ b & -a & 0 & -\phi_{0}^{0}/\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (D5)$$

with  $a = (i/2)(\phi_0^- - \phi_0^+)$  and  $b = (1/2)(\phi_0^- + \phi_0^+)$ ,

$$\tilde{\Phi}_{+} = \begin{pmatrix} \phi_{+}^{+}/\sqrt{2} & i\phi_{+}^{+}/\sqrt{2} & i\phi_{+}^{0}/2 & 0\\ i\phi_{+}^{+}/\sqrt{2} & -\phi_{+}^{+}/\sqrt{2} & -\phi_{+}^{0}/2 & 0\\ i\phi_{+}^{0}/2 & -\phi_{+}^{0}/2 & \phi_{-}^{-}/\sqrt{2} & 0\\ \phi_{+}^{0}/2 & i\phi_{+}^{0}/2 & -i\phi_{-}^{-}/\sqrt{2} & 0\\ 0 & 0 & 0 & 0 \end{pmatrix},$$
(D6)

and  $\tilde{\eta} = \eta \text{diag}(1, 1, 1, 1, -4)/\sqrt{20}$ . These conventions are the same as in Ref. [8], except for a slightly different normalization of the  $\eta$  field.

The  $\eta$  and H fields, which we have already encountered in the SU(4)/Sp(4) case, constitute the (1,1), respectively (2,2), representations of  $SU(2)_L \times SU(2)_R$ . The SU(5)/SO(5) coset contains nine additional NGBs, the  $\phi$ 's, that belong to (3,3). Their superscript and subscript label their  $SU(2)_L$ , respectively  $SU(2)_R$ , quantum numbers. The electric charge is  $T_L^3 + T_R^3$  for the coset fields, hence the electric charge of each  $\phi$  field is the sum of its superscript and subscript. Complex conjugation works on the  $\phi$ 's by interchanging  $+ \leftrightarrow -$  for both the superscript and the subscript. The  $SU(2)_L$  triplets are  $\Phi_0 = \{\phi_0^+, \phi_0^0, \phi_0^-\}$ (where  $\phi_0^0$  is real and  $\phi_0^- = (\phi_0^+)^*$ ), with  $T_R^3 = 0$ ,  $\Phi_+ = \{\phi_+^+, \phi_+^0, \phi_+^-\}, \text{ with } T_R^3 = +1, \text{ and } \Phi_- = (\Phi_+)^* =$  $\{\phi_{-}^{-}, \phi_{-}^{0}, \phi_{-}^{+}\}$ , with  $T_{R}^{3} = -1$ . The invariant bilinears are  $\Phi_0^2 \equiv (\phi_0^0)^2 + 2\phi_0^+\phi_0^-$  and  $\Phi_+\Phi_- \equiv \phi_+^+\phi_-^- + \phi_+^0\phi_-^0 + \phi_-^-\phi_+^-$ . We also introduce  $2 \times 2$  matrix formats,  $\hat{\Phi}_0 = \phi_0^0 \tau_3 +$  $2^{1/2}(\phi_0^+\tau_++\phi_0^-\tau_-)$  and  $\hat{\Phi}_{\pm}=\phi_{\pm}^0\tau_3+2^{1/2}(-i\phi_{\pm}^+\tau_++\phi_0^-\tau_-)$  $i\phi_{\pm}^{-}\tau_{-})$ , where  $\tau_{\pm} = (\tau_1 \pm i\tau_2)/2$ , which satisfy  $\operatorname{tr}(\overline{\Phi}_0^2) =$  $2\Phi_0^2$  and tr $(\hat{\Phi}_+\hat{\Phi}_-) = 2\Phi_+\Phi_-$ .

## APPENDIX E: $V_{eff}$ AT SECOND ORDER FOR SU(5)/SO(5)

In this appendix, we list all the contributions to  $V_{\rm eff}$ , truncated to second order in the pNGB fields. We use the expansion of the coset field  $\Sigma$  given in Appendix D, and the expansion of the singlet NGB field  $\Phi$  given in Eq. (3.43). The  $\langle \cdot \rangle$  notation is explained in Sec. III D, and the list of templates for  $V_{\rm eff}$  may be found in Eq. (4.8).

For template  $\mathcal{T}_1$  the spurions must be right-handed. We get

$$\langle \Phi^{1-2q} \mathrm{tr}(\bar{S}_R^1 \Sigma N_R) + \mathrm{H.c.} \rangle = 8 - \frac{4}{f^2} \left( (1/5)\eta^2 + 2H^{\dagger}H + \Phi_0^2 + 2\Phi_+\Phi_- \right) - \frac{8(1-2q)\zeta\eta}{\sqrt{5}f} - 4(1-2q)^2\zeta^2, \qquad (E1a)$$

$$\langle \Phi^{1-2q} \text{tr}(\bar{S}_R^2 \Sigma N_R) + \text{H.c.} \rangle = 2 - \frac{8}{f^2} ((2/5)\eta^2 + H^{\dagger}H) + \frac{8(1-2q)\zeta\eta}{\sqrt{5}f} - (1-2q)^2 \zeta^2.$$
(E1b)

Since  $t_R$  is embedded into the spurions  $S_R$  and  $S_R^c$  in the same way, the result for template  $\mathcal{T}_2$  is obtained from the corresponding result for  $\mathcal{T}_1$  by replacing 1 - 2q with 1 + 2q. The results for template  $\mathcal{T}_3$  are

$$\langle \Phi^{1-2q} \text{tr}(\bar{A}_R \Sigma D_R^{1T}) + \text{H.c.} \rangle = -2 + \frac{1}{f^2} ((1/5)\eta^2 + 2H^{\dagger}H + 2\Phi_0^2 + 4\Phi_+\Phi_-) + \frac{2(1-2q)}{\sqrt{5}} \frac{\eta\zeta}{f} + (1-2q)^2 \zeta^2, \quad (\text{E2a})$$

$$\langle \Phi^{1-2q} \text{tr}(\bar{A}_R \Sigma D_R^{2T}) + \text{H.c.} \rangle = 0.$$
(E2b)

$$\langle \Phi^{1+2q} \text{tr}(\bar{A}_L \Sigma D_L^{1T}) + \text{H.c.} \rangle = -8 + \frac{32}{f^2} ((2/5)\eta^2 + H^{\dagger}H) - \frac{32(1+2q)}{\sqrt{5}} \frac{\eta\zeta}{f} + 4(1+2q)^2 \zeta^2, \tag{E2c}$$

$$\langle \Phi^{1+2q} \text{tr}(\bar{A}_L \Sigma D_L^{2T}) + \text{H.c.} \rangle = 8 - \frac{4}{f^2} ((1/5)\eta^2 + 2H^{\dagger}H + 2\Phi_0^2 + 4\Phi_+\Phi_-) - \frac{8(1+2q)}{\sqrt{5}}\frac{\eta\zeta}{f} - 4(1+2q)^2\zeta^2, \quad \text{(E2d)}$$

for template  $T_4$ ,

$$\langle \Phi^{1-2q} \operatorname{tr}(\bar{S}_R^1 \Sigma D_R^{1T}) + \mathrm{H.c.} \rangle = 0, \tag{E3a}$$

$$\langle \Phi^{1-2q} \text{tr}(\bar{S}_R^1 \Sigma D_R^{2T}) + \text{H.c.} \rangle = 8 - \frac{4}{f^2} ((1/5)\eta^2 + 2H^{\dagger}H + 2\Phi_0^2 + 4\Phi_+\Phi_-) - \frac{8(1-2q)\zeta\eta}{\sqrt{5}f} - 4(1-2q)^2\zeta^2, \quad (E3b)$$

$$\langle \Phi^{1-2q} \operatorname{tr}(\bar{S}_R^2 \Sigma D_R^{1T}) + \mathrm{H.c.} \rangle = 0, \tag{E3c}$$

$$\langle \Phi^{1-2q} \text{tr}(\bar{S}_R^2 \Sigma D_R^{2T}) + \text{H.c.} \rangle = -8 + \frac{32}{f^2} ((2/5)\eta^2 + H^{\dagger}H) - \frac{32(1-2q)\zeta\eta}{\sqrt{5}f} + 4(1-2q)^2 \zeta^2.$$
(E3d)

$$\langle \Phi^{1+2q} \text{tr}(\bar{S}_L \Sigma D_L^{1T}) + \text{H.c.} \rangle = 8 - \frac{32}{f^2} ((2/5)\eta^2 + H^{\dagger}H) + \frac{32(1+2q)\eta\zeta}{\sqrt{5}f} - 4(1+2q)^2 \zeta^2.$$
(E3e)

$$\langle \Phi^{1+2q} \text{tr}(\bar{S}_L \Sigma D_L^{2T}) + \text{H.c.} \rangle = 8 - \frac{4}{f^2} ((1/5)\eta^2 + 2H^{\dagger}H + 2\Phi_0^2 + 4\Phi_+\Phi_-) - \frac{8(1+2q)\eta\zeta}{\sqrt{5}f} - 4(1+2q)^2\zeta^2, \quad \text{(E3f)}$$

for template  $T_5$ ,

$$\langle \Phi^{-1-2q} \operatorname{tr}(\bar{A}_R^c \Sigma^* D_R^1) + \operatorname{H.c.}\rangle = 2 - \frac{1}{f^2} ((1/5)\eta^2 + 2H^{\dagger}H + 2\Phi_0^2 + 4\Phi_+\Phi_-) - \frac{2(1+2q)}{\sqrt{5}}\frac{\eta\zeta}{f} - (1+2q)^2\zeta^2, \quad (E4a)$$

$$\langle \Phi^{-1-2q} \operatorname{tr}(\bar{A}_R^c \Sigma^* D_R^2) + \mathrm{H.c.} \rangle = 0.$$
(E4b)

$$\langle \Phi^{-1+2q} \text{tr}(\bar{A}_L^c \Sigma^* D_L^1) + \text{H.c.} \rangle = 8 - \frac{4}{f^2} ((1/5)\eta^2 + 2H^{\dagger}H + 2\Phi_0^2 + 4\Phi_+\Phi_-) - \frac{8(1-2q)\eta\zeta}{\sqrt{5}f} - 4(1-2q)^2\zeta^2, \quad \text{(E4c)}$$

$$\langle \Phi^{-1+2q} \text{tr}(\bar{A}_L^c \Sigma^* D_L^2) + \text{H.c.} \rangle = -8 + \frac{32}{f^2} ((2/5)\eta^2 + H^{\dagger}H) - \frac{32(1-2q)}{\sqrt{5}} \frac{\eta\zeta}{f} + 4(1-2q)^2 \zeta^2, \tag{E4d}$$

and for template  $T_6$ ,

$$\langle \Phi^{-1-2q} \operatorname{tr}(\bar{S}_R^{1c} \Sigma^* D_R^1) + \mathrm{H.c.} \rangle = 0, \tag{E5a}$$

$$\langle \Phi^{-1-2q} \text{tr}(\bar{S}_R^{1c} \Sigma^* D_R^2) + \text{H.c.} \rangle = 8 - \frac{4}{f^2} ((1/5)\eta^2 + 2H^{\dagger}H + 2\Phi_0^2 + 4\Phi_+\Phi_-) - \frac{8(1+2q)}{\sqrt{5}} \frac{\zeta\eta}{f} - 4(1+2q)^2 \zeta^2, \quad \text{(E5b)}$$

$$\langle \Phi^{-1-2q} \operatorname{tr}(\bar{S}_R^{2c} \Sigma^* D_R^1) + \mathrm{H.c.} \rangle = 0, \tag{E5c}$$

$$\langle \Phi^{-1-2q} \operatorname{tr}(\bar{S}_R^{2c} \Sigma^* D_R^2) + \mathrm{H.c.} \rangle = -8 + \frac{32}{f^2} ((2/5)\eta^2 + H^{\dagger} H) - \frac{32(1+2q)\zeta\eta}{\sqrt{5}} \frac{\zeta\eta}{f} + 4(1+2q)^2 \zeta^2.$$
(E5d)

$$\langle \Phi^{-1+2q} \text{tr}(\bar{S}_L^c \Sigma^* D_L^1) + \text{H.c.} \rangle = 8 - \frac{4}{f^2} ((1/5)\eta^2 + 2H^{\dagger}H + 2\Phi_0^2 + 4\Phi_+\Phi_-) - \frac{8(1-2q)}{\sqrt{5}}\frac{\eta\zeta}{f} - 4(1-2q)^2\zeta^2, \quad \text{(E5e)}$$

$$\langle \Phi^{-1+2q} \text{tr}(\bar{S}_L^c \Sigma^* D_L^2) + \text{H.c.} \rangle = 8 - \frac{32}{f^2} ((2/5)\eta^2 + H^{\dagger} H) + \frac{32(1-2q)}{\sqrt{5}} \frac{\eta\zeta}{f} - 4(1-2q)^2 \zeta^2.$$
(E5f)

For  $\mathcal{T}_7$  we obtain

$$\langle \operatorname{tr}(\bar{S}_L \Sigma) \operatorname{tr}(S_L \Sigma^*) \rangle = \frac{32}{f^2} H^{\dagger} H,$$
 (E6a)

$$\langle \operatorname{tr}(\bar{S}_{R}^{1}\Sigma)\operatorname{tr}(S_{R}^{1}\Sigma^{*})\rangle = 16 - \frac{32}{f^{2}}(H^{\dagger}H + \Phi_{0}^{2} + 2\Phi_{+}\Phi_{-}),$$
 (E6b)

$$\langle \operatorname{tr}(\bar{S}_R^2 \Sigma) \operatorname{tr}(S_R^2 \Sigma^*) \rangle = 1 - \frac{8}{f^2} H^{\dagger} H, \tag{E6c}$$

$$\langle \operatorname{tr}(\bar{S}_{R}^{1}\Sigma)\operatorname{tr}(S_{R}^{2}\Sigma^{*}) + \operatorname{H.c.}\rangle = 8 - \frac{4}{f^{2}}(5\eta^{2} + 10H^{\dagger}H + 2\Phi_{0}^{2} + 4\Phi_{+}\Phi_{-}).$$
 (E6d)

Again the second-order results for template  $\mathcal{T}_8$  can be obtained by replacing *S* with  $S^c$  and  $\Sigma$  with  $\Sigma^*$  on the left-hand sides, while keeping the right-hand sides unchanged. The results for template  $\mathcal{T}_9$  are

$$\langle \Phi^2 \operatorname{tr}(S_L^c \Sigma) \operatorname{tr}(\bar{S}_L \Sigma) + \mathrm{H.c.} \rangle = -\frac{64}{f^2} H^{\dagger} H, \tag{E7a}$$

$$\langle \Phi^{2} \mathrm{tr}(S_{R}^{1c} \Sigma) \mathrm{tr}(\bar{S}_{R}^{1} \Sigma) + \mathrm{H.c.} \rangle = -\frac{64}{f^{2}} ((1/5)\eta^{2} + H^{\dagger}H + \Phi_{0}^{2} + 2\Phi_{+}\Phi_{-}) - \frac{128}{\sqrt{5}} \frac{\zeta\eta}{f} - 64\zeta^{2}, \tag{E7b}$$

$$\langle \Phi^{2} \text{tr}(S_{R}^{2c} \Sigma) \text{tr}(\bar{S}_{R}^{2} \Sigma) + \text{H.c.} \rangle = -\frac{16}{f^{2}} ((4/5)\eta^{2} + H^{\dagger}H) + \frac{32}{\sqrt{5}} \frac{\zeta\eta}{f} - 4\zeta^{2}, \tag{E7c}$$

$$\langle \Phi^2 \text{tr}(S_R^{1c}\Sigma)\text{tr}(\bar{S}_R^2\Sigma) + \text{H.c.} \rangle = -\frac{8}{f^2} ((9/10)\eta^2 + 5H^{\dagger}H + \Phi_0^2 + 2\Phi_+\Phi_-) + \frac{48}{\sqrt{5}}\frac{\zeta\eta}{f} - 16\zeta^2, \tag{E7d}$$

$$\langle \Phi^2 \text{tr}(S_R^{2c}\Sigma)\text{tr}(\bar{S}_R^1\Sigma) + \text{H.c.} \rangle = -\frac{8}{f^2} ((9/10)\eta^2 + 5H^{\dagger}H + \Phi_0^2 + 2\Phi_+\Phi_-) + \frac{48}{\sqrt{5}}\frac{\zeta\eta}{f} - 16\zeta^2, \quad (E7e)$$

for template  $T_{10}$ ,

$$\langle \Phi^2 \text{tr}(S_L^c \Sigma \bar{S}_L \Sigma) + \text{H.c.} \rangle = -\frac{16}{f^2} ((9/10)\eta^2 + 7H^{\dagger}H + \Phi_0^2 + 2\Phi_-\Phi_+) + \frac{96}{\sqrt{5}} \frac{\zeta \eta}{f} - 32\zeta^2,$$
(E8a)

$$\langle \Phi^{2} \text{tr}(S_{R}^{1c} \Sigma \bar{S}_{R}^{1} \Sigma) + \text{H.c.} \rangle = -\frac{16}{f^{2}} \left( (1/5)\eta^{2} + H^{\dagger}H + 2\Phi_{0}^{2} + 4\Phi_{-}\Phi_{+} \right) - \frac{32}{\sqrt{5}} \frac{\zeta \eta}{f} - 16\zeta^{2}, \tag{E8b}$$

$$\langle \Phi^2 \text{tr}(S_R^{2c} \Sigma \bar{S}_R^2 \Sigma) + \text{H.c.} \rangle = -\frac{1}{f^2} ((64/5)\eta^2 + 16H^{\dagger}H) + \frac{32}{\sqrt{5}} \frac{\zeta \eta}{f} - 4\zeta^2,$$
(E8c)

$$\langle \Phi^2 \text{tr}(S_R^{1c} \Sigma \bar{S}_R^2 \Sigma) + \text{H.c.} \rangle = -\frac{16}{f^2} H^{\dagger} H, \tag{E8d}$$

$$\langle \Phi^2 \text{tr}(S_R^{2c} \Sigma \bar{S}_R^1 \Sigma) + \text{H.c.} \rangle = -\frac{16}{f^2} H^{\dagger} H, \tag{E8e}$$

for template  $T_{11}$ ,

$$\langle \Phi^2 \text{tr}(A_L^c \Sigma \bar{A}_L \Sigma) + \text{H.c.} \rangle = -\frac{16}{f^2} ((9/10)\eta^2 + 3H^{\dagger}H + \Phi_0^2 + 2\Phi_+ \Phi_-) + \frac{96}{\sqrt{5}} \frac{\zeta \eta}{f} - 32\zeta^2, \tag{E9a}$$

$$\langle \Phi^2 \text{tr}(A_R^c \Sigma \bar{A}_R \Sigma) + \text{H.c.} \rangle = -\frac{4}{f^2} \left( (1/5)\eta^2 + H^{\dagger} H + 2\Phi_0^2 \right) - \frac{8}{\sqrt{5}} \frac{\zeta \eta}{f} - 4\zeta^2, \tag{E9b}$$

and for template  ${\cal T}_{12},$ 

$$\langle \operatorname{tr}(D_L^1 \Sigma \bar{D}_L^{1T} \Sigma^*) \rangle = \frac{8}{f^2} H^{\dagger} H,$$
 (E10a)

$$\langle \operatorname{tr}(D_L^2 \Sigma \bar{D}_L^{2T} \Sigma^*) \rangle = \frac{8}{f^2} H^{\dagger} H,$$
 (E10b)

$$\begin{split} \langle \mathrm{tr}(D_L^1 \Sigma \bar{D}_L^{2T} \Sigma^*) + \mathrm{H.c.} \rangle &= -\frac{8}{f^2} ((5/2) \eta^2 + 5 H^{\dagger} H \\ &+ \Phi_0^2 + 2 \Phi_+ \Phi_-), \end{split} \tag{E10c}$$

$$\langle \operatorname{tr}(D_R^1 \Sigma \bar{D}_R^{1T} \Sigma^*) \rangle = -\frac{2}{f^2} (H^{\dagger} H + 4\Phi_+ \Phi_-), \quad (\text{E10d})$$

$$\langle \operatorname{tr}(D_R^2 \Sigma \bar{D}_R^{2T} \Sigma^*) \rangle = -\frac{200}{f^2} H^{\dagger} H,$$
 (E10e)

$$\langle \operatorname{tr}(D_R^1 \Sigma \overline{D}_R^{2T} \Sigma^*) + \operatorname{H.c.} \rangle = 0.$$
 (E10f)

Notice that  $\langle \operatorname{tr}(D_L^2 \Sigma \overline{D}_L^{1T} \Sigma^*) + \operatorname{H.c.} \rangle = \langle \operatorname{tr}(D_L^1 \Sigma \overline{D}_L^{2T} \Sigma^*) + \operatorname{H.c.} \rangle.$ 

# APPENDIX F: $V_{eff}$ FOR THE *h* AND $\varphi$ FIELDS

In this appendix, we obtain the exact form of the effective potential, assuming that the SU(5)/SO(5) pion field is given by  $\Pi = V$ , where

A comparison to Eq. (D3) shows that this corresponds to keeping  $h = \sqrt{2} \operatorname{Re} H_0$  and  $\varphi = \sqrt{2} \operatorname{Im} \phi_+^-$  arbitrary, while turning off the other 12 pNGBs. (Like  $H_0$ , also  $\phi_+^-$  is electrically neutral.) The coset field is then given by

$$\Sigma = 1 + \frac{2iV}{f} \frac{\sin(\alpha)}{\alpha} - \frac{4V^2}{f^2} \frac{1 - \cos(\alpha)}{\alpha^2}$$
  
= 1 +  $\frac{2iV}{f} \hat{s} - \frac{4V^2}{f^2} \hat{c}$ , (F2)

where

 $\alpha^2 = 4(h^2 + \varphi^2)/f^2$ , and we have introduced the short-hands  $\hat{s} = \frac{\sin(\alpha)}{\alpha}$  and  $\hat{c} = \frac{1 - \cos(\alpha)}{\alpha^2}$ .

The contributions to the effective potential from template  $\ensuremath{\mathcal{T}}_1$  are

$$\langle \Phi^{1-2q} \text{tr}(\bar{S}_R^1 \Sigma N_R) + \text{H.c.} \rangle = 8\cos((1-2q)\zeta) \left(1 - \hat{c}\frac{2\varphi^2 + h^2}{f^2}\right),$$
 (F4a)

$$\langle \Phi^{1-2q} \operatorname{tr}(\bar{S}_R^2 \Sigma N_R) + \mathrm{H.c.} \rangle = \cos((1-2q)\zeta) \left(2 - 8\hat{c}\frac{h^2}{f^2}\right) \qquad . \tag{F4b}$$

The contributions of template  $T_2$  are again obtained from those of  $T_1$  by replacing 1 - 2q with 1 + 2q. For  $T_3$  we have

$$\langle \Phi^{1-2q} \text{tr}(\bar{A}_R \Sigma D_R^{1T}) + \text{H.c.} \rangle = -2\cos((1-2q)\zeta) \left(1 - \hat{c} \frac{2\varphi^2 + h^2}{f^2}\right),$$
 (F5a)

$$\langle \Phi^{1-2q} \operatorname{tr}(\bar{A}_R \Sigma D_R^{2T}) + \mathrm{H.c.} \rangle = 0, \tag{F5b}$$

$$\langle \Phi^{1+2q} \text{tr}(\bar{A}_L \Sigma D_L^{1T}) + \text{H.c.} \rangle = -8\cos((1+2q)\zeta) \left(1 - 4\hat{c}\frac{h^2}{f^2}\right),$$
 (F5c)

$$\langle \Phi^{1+2q} \text{tr}(\bar{A}_L \Sigma D_L^{2T}) + \text{H.c.} \rangle = 8\cos((1+2q)\zeta) \left(1 - \hat{c}\frac{2\varphi^2 + h^2}{f^2}\right) ,$$
 (F5d)

for  $\mathcal{T}_4$ ,

$$\langle \Phi^{1-2q} \operatorname{tr}(\bar{S}_R^1 \Sigma D_R^{1T}) + \operatorname{H.c.} \rangle = 0, \tag{F6a}$$

$$\langle \Phi^{1-2q} \operatorname{tr}(\bar{S}_R^1 \Sigma D_R^{2T}) + \mathrm{H.c.} \rangle = 8 \cos((1-2q)\zeta) \left(1 - \hat{c} \frac{2\varphi^2 + h^2}{f^2}\right),$$
 (F6b)

$$\langle \Phi^{1-2q} \mathrm{tr}(\bar{S}_R^2 \Sigma D_R^{1T}) + \mathrm{H.c.} \rangle = 0, \tag{F6c}$$

$$\langle \Phi^{1-2q} \operatorname{tr}(\bar{S}_R^2 \Sigma D_R^{2T}) + \operatorname{H.c.} \rangle = -8 \cos((1-2q)\zeta) \left(1 - 4\hat{c} \frac{h^2}{f^2}\right),$$
(F6d)

$$\langle \Phi^{1+2q} \text{tr}(\bar{S}_L \Sigma D_L^{1T}) + \text{H.c.} \rangle = 8\cos((1-2q)\zeta) \left(1 - 4\hat{c}\frac{h^2}{f^2}\right),$$
 (F6e)

$$\langle \Phi^{1+2q} \operatorname{tr}(\bar{S}_L \Sigma D_L^{2T}) + \mathrm{H.c.} \rangle = 8 \cos((1-2q)\zeta) \left(1 - \hat{c} \frac{2\varphi^2 + h^2}{f^2}\right),$$
 (F6f)

for  $\mathcal{T}_5$ 

$$\langle \Phi^{-1-2q} \operatorname{tr}(\bar{A}_R^c \Sigma^* D_R^1) + \operatorname{H.c.}\rangle = 2\cos((1+2q)\zeta) \left(1 - \hat{c} \frac{2\varphi^2 + h^2}{f^2}\right),$$
 (F7a)

$$\langle \Phi^{-1-2q} \mathrm{tr}(\bar{A}_R^c \Sigma^* D_R^2) + \mathrm{H.c.} \rangle = 0, \tag{F7b}$$

$$\langle \Phi^{-1+2q} \operatorname{tr}(\bar{A}_L^c \Sigma^* D_L^1) + \mathrm{H.c.} \rangle = 8 \cos((1-2q)\zeta) \left(1 - \hat{c} \frac{2\varphi^2 + h^2}{f^2}\right),$$
 (F7c)

$$\langle \Phi^{-1+2q} \operatorname{tr}(\bar{A}_L^c \Sigma^* D_L^2) + \mathrm{H.c.} \rangle = -8 \cos((1-2q)\zeta) \left(1 - 4\hat{c}\frac{h^2}{f^2}\right),$$
 (F7d)

and for  $\mathcal{T}_6$ ,

$$\langle \Phi^{-1-2q} \operatorname{tr}(\bar{S}_R^{1c} \Sigma^* D_R^1) + \operatorname{H.c.} \rangle = 0, \tag{F8a}$$

$$\langle \Phi^{-1-2q} \operatorname{tr}(\bar{S}_R^{1c} \Sigma^* D_R^2) + \operatorname{H.c.} \rangle = 8 \cos((1+2q)\zeta) \left(1 - \hat{c} \frac{2\varphi^2 + h^2}{f^2}\right),$$
 (F8b)

$$\langle \Phi^{-1-2q} \operatorname{tr}(\bar{S}_R^{2c} \Sigma^* D_R^1) + \mathrm{H.c.} \rangle = 0, \tag{F8c}$$

$$\langle \Phi^{-1-2q} \operatorname{tr}(\bar{S}_R^{2c} \Sigma^* D_R^2) + \mathrm{H.c.} \rangle = -8 \cos((1+2q)\zeta) \left(1 - 4\hat{c} \frac{h^2}{f^2}\right),$$
 (F8d)

$$\langle \Phi^{-1+2q} \operatorname{tr}(\bar{S}_L^c \Sigma^* D_L^1) + \operatorname{H.c.} \rangle = 8 \cos((1-2q)\zeta) \left(1 - \hat{c} \frac{2\varphi^2 + h^2}{f^2}\right),$$
 (F8e)

$$\langle \Phi^{-1+2q} \operatorname{tr}(\bar{S}_L^c \Sigma^* D_L^2) + \mathrm{H.c.} \rangle = 8 \cos((1-2q)\zeta) \left(1 - 4\hat{c} \frac{h^2}{f^2}\right).$$
(F8f)

The results for template  $T_7$  are

$$\langle \operatorname{tr}(\bar{S}_L \Sigma) \operatorname{tr}(S_L \Sigma^*) \rangle = 16 \frac{h^2}{f^2} \left( \hat{s} - 2\hat{c} \frac{\varphi}{f} \right)^2,$$
 (F9a)

$$\langle \operatorname{tr}(\bar{S}_R^1 \Sigma) \operatorname{tr}(S_R^1 \Sigma^*) \rangle = 16 \left( -1 + \hat{c} \frac{2\varphi^2 + h^2}{f^2} \right)^2, \qquad (F9b)$$

$$\langle \operatorname{tr}(\bar{S}_R^2 \Sigma) \operatorname{tr}(S_R^2 \Sigma^*) \rangle = \left( -1 + 4\hat{c} \frac{h^2}{f^2} \right)^2, \tag{F9c}$$

$$\langle \operatorname{tr}(\bar{S}_{R}^{1}\Sigma)\operatorname{tr}(S_{R}^{2}\Sigma^{*})\rangle = 4\left(-1 + 4\hat{c}\frac{h^{2}}{f^{2}}\right)\left(-1 + \hat{c}\frac{2\varphi^{2} + h^{2}}{f^{2}}\right).$$
(F9d)

The expansion of Eq. (F9a) contains the cubic term  $h^2\varphi$  [compare Eq. (4.15)].

In the case of template  $\mathcal{T}_8$ , the results for the righthanded spurions may again be obtained from those for  $\mathcal{T}_7$  as in Appendix E. For the contribution of  $q_L$  we now find

$$\langle \operatorname{tr}(S_L^c \Sigma) \operatorname{tr}(\bar{S}_L^c \Sigma^*) \rangle = 16 \frac{h^2}{f^2} \left( \hat{s} + 2\hat{c} \frac{\varphi}{f} \right)^2.$$
 (F10)

This differs from Eq. (F9a) by the relative sign inside the square, and again contains a cubic term. We observe that a cancellation of the cubic terms between the contributions of Eqs. (F9a) and (F10), while technically possible, is unlikely. In  $V_{\rm eff}$ , each contribution gets multiplied by two coupling constants from  $\mathcal{L}_{\rm EHC}$ , and by a low-energy constant. In order for this cancellation to happen, the ratio of the relevant low-energy constants, which is a feature of the stand-alone hypercolor theory, would have to be equal to the ratio of the coupling constants squared, which are features of the EHC theory.

The results for template  $T_9$  are

$$\langle \Phi^2 \operatorname{tr}(S_L^c \Sigma) \operatorname{tr}(\bar{S}_L \Sigma) + \mathrm{H.c.} \rangle = 32 \cos(2\zeta) \frac{h^2}{f^2} \left( 4\hat{c}^2 \frac{\varphi^2}{f^2} - \hat{s}^2 \right),$$
(F11a)

$$\langle \Phi^2 \operatorname{tr}(S_R^{1c}\Sigma) \operatorname{tr}(\bar{S}_R^1\Sigma) + \mathrm{H.c.} \rangle = 32\cos(2\zeta) \left( -1 + \hat{c} \frac{2\varphi^2 + h^2}{f^2} \right)^2,$$
(F11b)

$$\langle \Phi^2 \operatorname{tr}(S_R^{2c}\Sigma) \operatorname{tr}(\bar{S}_R^2\Sigma) + \mathrm{H.c.} \rangle = 2\cos(2\zeta) \left(-1 + 4\hat{c}\frac{h^2}{f^2}\right)^2,\tag{F11c}$$

$$\langle \Phi^2 \operatorname{tr}(S_R^{1c}\Sigma) \operatorname{tr}(\bar{S}_R^2\Sigma) + \mathrm{H.c.} \rangle = 8\cos(2\zeta) \left( -1 + 4\hat{c}\frac{h^2}{f^2} \right) \left( -1 + \hat{c}\frac{2\varphi^2 + h^2}{f^2} \right),$$
(F11d)

for template  $T_{10}$ ,

$$\langle \Phi^2 \text{tr}(S_L^c \Sigma \bar{S}_L \Sigma) + \text{H.c.} \rangle = 16 \cos(2\zeta) \left( 1 - \frac{(\hat{s}^2 + 5\hat{c})h^2 + 2\hat{c}\varphi^2}{f^2} + 4\hat{c}^2 \frac{h^2(h^2 + 3\varphi^2)}{f^4} \right), \tag{F12a}$$

$$\langle \Phi^2 \operatorname{tr}(S_R^{1c} \Sigma \bar{S}_R^1 \Sigma) + \mathrm{H.c.} \rangle = 8 \cos(2\zeta) \left( 1 - \frac{2\hat{c}h^2 + (4\hat{c} + 2\hat{s}^2)\varphi^2}{f^2} + 4\hat{c}^2 \frac{2\varphi^4 + 2\varphi^2 h^2 + h^4}{f^4} \right),$$
(F12b)

$$\langle \Phi^2 \text{tr}(S_R^{2c}\Sigma\bar{S}_R^2\Sigma) + \text{H.c.} \rangle = 2\cos(2\zeta) \left(-1 + 4\hat{c}\frac{h^2}{f^2}\right)^2,\tag{F12c}$$

$$\langle \Phi^2 \operatorname{tr}(S_R^{1c} \Sigma \bar{S}_R^2 \Sigma) + \mathrm{H.c.} \rangle = 8 \cos(2\zeta) \frac{h^2}{f^2} \left( -\hat{s}^2 + 4\hat{c}^2 \frac{\varphi^2}{f^2} \right),$$
(F12d)

for template  $\mathcal{T}_{11}$ ,

$$\langle \Phi^2 \text{tr}(A_L^c \Sigma \bar{A}_L \Sigma) + \text{H.c.} \rangle = 16 \cos(2\zeta) \left( 1 + \frac{(\hat{s}^2 - 5\hat{c})h^2 - 2\hat{c}\varphi^2}{f^2} + 4\hat{c}^2 \frac{h^2(\varphi^2 + h^2)}{f^4} \right), \tag{F13a}$$

$$\langle \Phi^{2} \text{tr}(A_{R}^{c} \Sigma \bar{A}_{R} \Sigma) + \text{H.c.} \rangle = 2\cos(2\zeta) \left( 1 + 2\frac{(\hat{s}^{2} - 2\hat{c})\varphi^{2} - \hat{c}h^{2}}{f^{2}} + 8\hat{c}^{2}\frac{\varphi^{2}(\varphi^{2} + h^{2})}{f^{4}} \right), \tag{F13b}$$

and finally, for template  $T_{12}$ ,

$$\langle \operatorname{tr}(D_L^1 \Sigma \bar{D}_L^{1T} \Sigma^*) \rangle = 4 \frac{h^2}{f^2} \left( \hat{s} - 2\hat{c} \frac{\varphi}{f} \right)^2, \tag{F14a}$$

$$\langle \operatorname{tr}(D_L^2 \Sigma \bar{D}_L^{2T} \Sigma^*) \rangle = 4 \frac{h^2}{f^2} \left( \hat{s} + 2\hat{c} \frac{\varphi}{f} \right)^2, \tag{F14b}$$

$$\langle \operatorname{tr}(D_L^2 \Sigma \bar{D}_L^{1T} \Sigma^*) \rangle = 4 \left( -1 + 4\hat{c} \frac{h^2}{f^2} \right) \left( -1 + \hat{c} \frac{2\varphi^2 + h^2}{f^2} \right),$$
 (F14c)

$$\langle \operatorname{tr}(D_L^1 \Sigma \bar{D}_L^{2T} \Sigma^*) \rangle = 4 \left( -1 + 4\hat{c} \frac{h^2}{f^2} \right) \left( -1 + \hat{c} \frac{2\varphi^2 + h^2}{f^2} \right),$$
 (F14d)

$$\langle \operatorname{tr}(D_R^1 \Sigma \bar{D}_R^{1T} \Sigma^*) \rangle = -1 + (2\hat{s}^2 + 4\hat{c}) \frac{\varphi^2}{f^2} + 2\hat{c} \frac{h^2}{f^2} - 8\hat{c}^2 \frac{\varphi^2(\varphi^2 + h^2)}{f^4},$$
(F14e)

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$$\langle \operatorname{tr}(D_R^2 \Sigma \bar{D}_R^{2T} \Sigma^*) \rangle = 20 + 8(\hat{s}^2 - 2\hat{c}) \frac{\varphi^2}{f^2} - 8(4\hat{s}^2 + 17\hat{c}) \frac{h^2}{f^2} + 16\hat{c}^2 \frac{2\varphi^4 - 6\varphi^2 h^2 + 17h^4}{f^4},$$
(F14f)

$$\langle \operatorname{tr}(D_R^1 \Sigma \bar{D}_R^{2T} \Sigma^*) \rangle = -40\hat{c} \,\hat{s} \frac{h^2 \varphi}{f^3},\tag{F14g}$$

$$\langle \operatorname{tr}(D_R^2 \Sigma \bar{D}_R^{1T} \Sigma^*) \rangle = -40\hat{c} \,\hat{s} \, \frac{h^2 \varphi}{f^3}. \tag{F14h}$$

An interesting property is that, for the parametrization (F2), only one odd function of the pNGB fields occurs in the potential, namely,  $\hat{c} \hat{s} h^2 \varphi$  (see Sec. IV E for further discussion). By contrast, the effective potential depends on a large number of even functions of the pNGBs. For completeness, the gauge sector's contribution for the parametrization (F2) is<sup>40</sup>

$$V_{\text{eff}}^{\text{gauge}} = -g^2 C_{\text{LR}} \left( 3 - \frac{(12\hat{c} + 2\hat{s}^2)\varphi^2 + 6\hat{c}h^2}{f^2} + \frac{8\hat{c}^2\varphi^2(\varphi^2 + h^2)}{f^4} \right) - g'^2 C_{\text{LR}} \left( 1 - \frac{(4\hat{c} + 2\hat{s}^2)\varphi^2 + 2\hat{c}h^2}{f^2} + \frac{8\hat{c}^2\varphi^2(\varphi^2 + h^2)}{f^4} \right).$$
(F15)

This contribution is the same for all the SU(5)/SO(5) model, including in particular the models we consider in Sec. IV and in Sec. V.

<sup>40</sup>For a lattice calculation of  $C_{LR}$  in a lattice theory that is closely related to the SU(4) model of Sec. V, see Ref. [41].

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