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Bogolyubov Institute for Theoretical Physics
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NON-EUCLIDEAN GEOMETRY IN MODERN PHYSICS AND MATHEMATICS

PROCEEDINGS
of the International Conference BGL-4
(Bolyai-Gauss-Lobachevsky)
Nizhny Novgorod, September 7-11, 2004

Edited by L. Jenkovszky and G. Polotovskiy

НЕЕВКЛИДОВА ГЕОМЕТРИЯ В СОВРЕМЕННОЙ ФИЗИКЕ И МАТЕМАТИКЕ

ТРУДЫ
Международной конференции БГЛ-4
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Editorial Foreword

Non-Euclidean Geometry in Modern Physics and Mathematics is a series of biennial conferences, initiated by the Bogolyubov Institute for Theoretical Physics of the National Academy of Science Ukraine. Its shorthand subtitle is BGL, from the abbreviation of the names (in alphabetic order) - Bolyai, Gauss, Lobachevskij - of the founders of the new geometry.

The first conference of this series was held in Uzhgorod (Ukraine) in 1997, the second one - in Nyiregyháza (Hungary), the third was scheduled such as to match the 200-th anniversary of János Bolyai, and was held in Targu-Mures (Marosvásárhely) (Rumania) - hometown of the Bolyai family. The latest one, BGL-4 (see: http://www.unn.ru/bgl4), was held between September 7 and 11, 2004 in Nizhny Novgorod - hometown of Nikolai Ivanovich Lobachevsky.

The present Proceedings contain contributions to BGL-4 that arrived before the dead-line, November 15. To speed up the publication, we minimized the editorial interference to the authors' originals.

The subject of the conference and the contributions, traditionally, can be grouped in three categories: history of the non-Euclidean geometry, its mathematical and physical applications. We thank the participants for their contributions.

We acknowledge the help and support of the institutions and people involved in the organization of BGL-4, namely the Nizhny Novgorod Lobachevsky University, the Bogolyubov Institute for Theoretical Physics, Hungarian Academy of Sciences and, in particular, permanent member of our Organizing Committee Academician István Lovas. We gratefully acknowledge the support by the Russian Foundation of Fundamental Researches (grant 04-01-10107-г) and the firm "TSS" (President V.B. Kosmachev).

The next conference, BGL-5, will be held in Minsk, Byelorussia in 2006. Proposals, applications etc. should be sent to the principle organizer of BGL-5 Professor Yurii Andreevich Kurochkin:

yukuroch@dragon.bas-net.by

and/or to jenk@bitp.kiev.ua , polot@uic.nnov.ru.

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Участники конференции

SCIENTIFIC PROGRAM
of the IV International Conference
"Non-Euclidean Geometry in Modern Physics and Mathematics"

September, 7, Tuesday

10:00 – 10:30 Opening of the Conference (**R.G. Strongin**, Rector of the Lobachevskiy Nizhny Novgorod University; **P. T. Nagy,** Debrecen, Hungary; **L.L. Jenkovszky,** Kiev, Ukraine)

10:30 – 11:10 **H. Terazawa** (Tokyo, Japan). Special Non-Constancy in Pregeometry

11:15 – 11:55 **P. T. Nagy** (Debrecen, Hungary). Riemannian Heisenberg Manifolds.

12:30 – 13:10 **G.M. Polotovskiy** (Nizhny Novgorod, Russia). How did Lobachevsky's biography study.

15:00 – 15:40 **N.I. Zhukova** (Nizhny Novgorod, Russia). Basic automorphisms of Cartan foliations and Cartan orbifolds.

15:45 – 16:15 **Yu.A. Kurochkin** (Minsk, Belarus). Some pecularities of the scattering problem in the Lobachevsky space.

16:15 – 16:45 **V. Magas** (Kiev, Ukraine). Continuation of the dual amplitude with Mandelstam analyticity off mass shell.

17:15 – 17:35 **A.V. Bagaev, N.I. Zhukova** (Nizhny Novgorod, Russia). Influence of curvature onto structure and the isometry group of Riemannian orbifolds.

17:35 – 17:55 **O.N. Pakhareva** (Nizhny Novgorod, Russia). Lax representation of nonlinear sigma-models with reducible metrics.

17:55 – 18:15 **L.L. Jenkovszky** (Kiev, Ukraine). Euclidean parallels in perspective

September, 8, Wednesday

09:00 – 09:40 **D.E. Burlankov** (Nizhny Novgorod, Russia). The curved space dynamics in the theory of gravity.

09:45 – 10:25 **Yu.A. Sitenko** (Kiev, Ukraine). Non-Euclidean geometry in quantum field theory.

10:30 – 11:00 **N.Z. Iorgov** (Kiev, Ukraine). Quantum Toda chain with boundary interaction.

11:30 – 12:00 **V.V. Koryukin** (Yoshkar-Ola, Russia). The differential geometry and the condensed description of Universe.

12:00 – 12:30 **V.V. Kocharovskiy, V. Yu. Martyanov** (Nizhny Novgorod, Russia). Wave-mixing schemes revealing QED vacuum nonlinearity.

September, 9, Thursday

09:00 – 09:40 **Yu.G. Rudoy, A.D. Sukhanov** (Moscow, Russia). Geometrical ideas in statistical thermodynamics.

09:45 – 10:25 **V.Z. Grines** (Nizhny Novgorod, Russia). On interrelation between properties of dynamical systems and foliations on surfaces of negative curvature and geodesic laminations.

10:30 – 11:10 **A.V. Borisov** (Izhevsk, Russia). The 2- and 3-bodies problem in spaces of constant curvature.

11:40 – 12:10 **I.S. Mamaev** (Izhevsk, Russia). Restricted 2-bodies problem in spaces of constant curvature.

12:10 – 12:30 **A. Sabry** (Cairo, Egypt). Some investigations on the quadrupole radiation of a double star.

12:30– 12:50 **O.S. Germanov** (Nizhny Novgorod, Russia). The first integrals of geodesics.

12:50 – 13:20 **P. Akhmetiev** (Moscow, Russia). An integral formula for a higher analog of the linking number of divergent free vector fields.

15:00 – 15:40 **L.M. Lerman** (Nizhny Novgorod, Russia). Symplectic geometry problems inspired by Hamiltonian dynamics.

15:45 – 16:15 **V.D. Gershun** (Kharkov, Ukraine). Nonlocal brackets and integrable models.

16:15 – 16:35 **Z. Kása** (Cluj-Napoca, Romania). The cult of Janos Bolyai in Romania.

17:05 – 17:35 **T. Barbot** (France). Globally hyperbolic spacetimes with constant curvature.

17:35 – 17:55 **R. Lovas (**Debrecen, Hungary). Affine and projective vector fields on spray manifolds.

17:55 – 18:15 **R. Oláh-Gál** (Cluj, Romania). Lobachevsky in Janos Bolyai manuscript, Geodesics on pseudosphere.

September, 10, Friday

09:00 – 09:40 **M. I. Kuznetsov** (Nizhny Novgorod, Russia). Geometrical structures in the theory of simple modular Lie algebras.

09:45 – 10:25 **G.M. Polotovskiy** (Nizhny Novgorod, Russia). What do we know about the topology of plane real algebraic curves?

10:30 – 11:00 **N.G. Fadeev** (Dubna, Russia). Physics beyond Lobachevskiy's parallel lines.

11:00 – 11:20 **D.E.Burlankov** (Nizhny Novgorod, Russia). Inertial systems in the Lobachevskiy Space.

11:45 – 12:25 **E. I. Yakovlev** (Nizhny Novgorod, Russia). Some geometrical and topological methods in dynamics of systems with gyroscopic forces.

12:30 – 13:00 **A.V. Kukushkin** (Nizhny Novgorod, Russia). Group theory approach to the problem of space-time's dimension: post-Maxwellian and post-Einsteinian effects of 5-dimensional group (longitudinal waves of gravitation).

USING THE LOBACHEVSKY PLANE TO STUDY SURFACE FLOWS, FOLIATIONS AND 2-WEBS

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The paper is devoted to exposition of results connected with using Lobachevsky plane to study flows, foliations and 2-webs on closed oriented surface of genus $p \geq 1$. In particular we describe a complete classification of such objects in terms of asymptotic directions of curves on the universal covering of surface (which is Lobachevsky plain) and in terms of special geodesic laminations — frameworks.

1. Introduction

The idea to use the Lobachevsky geometry rises to classical works of G. Hedlund [28] and M. Morse [34] who studied geodesics on surfaces of negative curvature. J. Nilsen [35], [36] applied the Lobachevsky plane and its absolute to give the homotopic classification of homeomorphisms of compact surfaces with negative Euler characteristic. The using of the Lobachevsky plane is based on the fact that this plane is a universal covering space for surfaces of negative Euler characteristic. This surfaces endowed with the metric induced by the metric of the Lobachevsky plane and covering maps is called hyperbolic.

To be precise, a *hyperbolic* surface $M^2 = M$ is a Riemann surface whose universal covering space is the Lobachevsky plane, which we'll consider as the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ endowed with the Poincare metric of the constant curvature -1. The circle $S_\infty = \partial\Delta = (|z| = 1)$ is called a *circle at infinity* or *absolute*. To simplify matters, below we'll consider a closed orientable hyperbolic surface M^2. It is known that given such M^2, there exists a Fuchsian group Γ of orientation-preserving isometries acting freely on Δ such that $\Delta/\Gamma \cong M^2$. The natural projection $\pi : \Delta \to \Delta/\Gamma$ is a universal covering map which induces a Riemann structure on M^2. Geodesics of Δ are the circular arcs orthogonal to S_∞. We suppose that any geodesic is complete and endpoints of geodesics belong to S_∞.

The idea to study two-dimensional dynamical systems and surface foliations with the use of nonlocal asymptotic properties of orbits and leaves is due to A. Weil and D.V. Anosov (see the historical comments in [1] - [6], [5] [14], [37]). In the 1960s, D.V. Anosov put forth the concept that the topological key to the n-onlocal theory of dynamical systems and foliations on M^2 is the study of the arrangement of "infinite" curves without self-intersections (i.e. simple) on M^2 and of the asymptotic behavior of lifts of these curves to the universal covering plane Δ with the use of the absolute S_∞ of this plane. Especially this approaching to study two-dimensional dynamical systems turned up effective for dynamical systems with nontrivially recurrent (in sense, chaotic) motions and nontrivially recurrent invariant manifolds (the most known of such dynamical systems are pseudo-Anosov homeomorphisms, and A-diffeomorphisms with nontrivial attrctors and repellers), and foliations with nontrivially recurrent leaves, see [8] - [12], [23] - [27]. Such approach sometimes is called the Anosov-Weil's theory. The goal of this paper is a review of some aspects of this theory, which generally considers asymptotic properties of simple curves lifted to an universal covering, and their "deviation"from the lines of constant geodesic curvature that have the same asymptotic direction.

It becomes clear that pure geometric methods allow to obtain a significant "topological"information about surface dynamical systems with nontrivially recurrent invariant manifolds, and foliations (in particular, about flows) with nontrivially recurrent leaves (resp., trajectories) on the hyperbolic surfaces. This information is hidden in the special geodesic laminations, so-called geodesic frameworks, built upon such dynamical systems and foliations. Geodesics constituting these laminations define the asymptotic directions which the invariant manifolds or leaves of a given dynamical system or foliation can have. It turns out that geometric properties of such a lamination encode the information on a topological structure of surface dynamical system and foliation.

Let us give a formal definition of asymptotic direction for a curve which, on the side, explains how geodesics appear. Let $l = \{l(t), t \geq 0\}$ be a semi-infinite continuous curve without self-intersections on M, and let $\bar{l}$ be its lifting to Δ. We assume that l endowed with an injective parametrization $[0;\infty) \to l$, $t \to l(t)$. Suppose that $\bar{l}$ tends to precisely one point σ of the absolute S_∞ as $t \to \infty$ in the Euclidean metric on the closed disk $\Delta \cup S_\infty$. In this case, we shall say that the curve $\bar{l}$ has an *asymptotic direction* determined by the point σ (we also shall sometimes say that l has an asymptotic direction), and the point σ is *reached* by the curve $\bar{l}$. Now let $l = \{l(t), t \in \mathbb{R}\}$ be an infinite continuous curve without self-intersections on M, and let $\bar{l}$ be its lifting to Δ. Here we assume that l endowed with an injective parametrization $(-\infty;+\infty) \to l$, $t \to l(t)$. Suppose that $\bar{l}$ has the asymptotic directions determined by the points σ^+ and σ^- as $t \to +\infty$ and $t \to -\infty$ respectively. If $\sigma^+ \neq \sigma^-$, there exists a geodesic $\bar{g}(\bar{l})$ with the ideal endpoints σ^+, σ^- oriented from σ^- to σ^+. This geodesic $\bar{g}(\bar{l})$ is said to be *coasymptotic* for $\bar{l}$. The geodesic $\pi(\bar{g}(\bar{l})) = g(l)$ is said to be *coasymptotic* for l. It can be shown that $g(l)$ has no transversal self-intersections. Hence the topological closure of $g(l)$ is a geodesic lamination [21].

Aranson and Grines [9] and Markley [31] was first who fruitfully applied properties of the Lobachevsky geometry to prove that a nontrivially recurrent trajectory l of any flow on M^2 has a coasymptotic geodesic. As a consequence, for such flow one can construct a special geodesic lamination. This geodesic lamination contains the most part of information about a global topological structure of the quasiminimal set *clos* l. Levitt [29] used special geodesic laminations to get the Whitehead classification of surface foliations.

The main goal of this paper is to represent many old results on surface foliations and flows from a new point of view based on special geodesic laminations, so-called geodesic frameworks. The most results we revisit here belong to the authors, and almost all of them are reformulated in a form different from original one. We suggest that this representation from the common view point of purely geometrical nature opens new investigations in the theory of surface foliations and flows.

Research partially supported by CNRS (France) and RFFI-02-01-00098 (Russia). Most of topics of this survey have been discussed with D. V. Anosov, V. Kaimanovich, F. Laudenbach, V. Medvedev and A. Zorich. It is our great pleasure to thank them for their efficient help and assistance.

2. Main definitions

Rational and irrational points. As we mentioned above, $\Delta/\Gamma \cong M^2$ where Γ is a Fuchsian group of orientation-preserving isometries acting freely on Δ. The group Γ is isomorphic to the fundamental group of M^2. Every isometry of Γ can be extended to a homeomorphism of the closed disk $\Delta \cup S_\infty$. Since M^2 is a closed orientable surface, we have that every isometry $\gamma \in \Gamma$ is a hyperbolic transformation having two fixed points $\gamma^+, \gamma^- \in S_\infty$. A point $\sigma \in S_\infty$ is called *rational* if $\sigma = \gamma^\pm$ for some $\gamma \in \Gamma$, $\gamma \neq id$. Any point of the set

$$S_\infty - \bigcup_{\gamma\in\Gamma} \{\gamma^+, \gamma^-\}$$

is called *irrational.*

Geodesic laminations. Recall that a *geodesic lamination* on a surface M is a foliation of a closed subset of M by geodesics with no self-intersections. Another words, a geodesic lamination is a nonempty collection of mutually disjoint simple geodesics the union of whose is a closed subset of M. Denote by $\mathcal{L}$ the set of geodesic laminations of M. Any union of simple pairwise disjoint closed geodesics forms a *trivial geodesic lamination.* Let us denote the family of trivial geodesic laminations by Λ_{triv}. A lamination is said to be *nontrivial* if it consists of non-closed geodesics. A lamination is *minimal* if it contains no proper sub-laminations. A lamination G on M is said to be *irreducible* if any closed geodesic on M intersects G.

Let G be a geodesic lamination on M. Consider an orientation on the geodesics from G. This orientation is said to be *compatible* if, for any geodesic $l \in G$ and any point $m \in l$, there exists a transversal segment Σ through m endowed with a normal orientation such that the intersection indices of all geodesics from G (intersecting Σ) with Σ are equal. A geodesic lamination is called *orientable* if its geodesics admit a compatible orientation.

We use Λ_{or} (Λ_{non}) to denote the set of nontrivial minimal orientable (respectively, non-orientable) geodesic laminations on M. The families Λ_{or} and Λ_{non} form the set

$$\Lambda = \Lambda_{or} \cup \Lambda_{non}$$

of nontrivial minimal geodesic laminations on M. Every $G \in \Lambda$ consists of nontrivially recurrent geodesics (see definition below) with irrational asymptotic directions and any geodesic from G is dense in G.

We distinguish the subset $\Lambda^{irr} \subset \Lambda$ of irreducible geodesic laminations and call every $G \in \Lambda^{irr}$ *irrational* geodesic lamination.

Let G be a geodesic lamination on M. Clearly, the preimage $\pi^{-1}(G) = \overline{G}$ is a geodesic lamination on Δ. Denote by $G(\infty) \subset S_\infty$ the set of points of the absolute reached by geodesics from the lamination $\overline{G}$. In other words, $G(\infty)$ is the set of ideal endpoints of all geodesics from $\overline{G}$.

Let the quotient

$$GM = Homeo\ (M)/Homeo_0\ (M)$$

be *generalized mapping class group*, where $Homeo\ (M)$ is the group of homeomorphisms of M and $Homeo_0\ (M)$ is the subgroup of homeomorphisms homotopic to the identity. It is known that any $\tau \in GM$ induces a one-to-one map $\tau^* : \mathcal{L} \dashrightarrow \mathcal{L}$ (see, for example, [20], [21]. Given $\lambda \in \mathcal{L}$, the family

$$\lambda_{GM} = \{\tau^*(\lambda) \mid \tau \in GM\}$$

is called an *orbit of the geodesic lamination* λ.

Surface flows. Let f^t be a flow on M meaning that $f^t : M \times R \to M$ is a one-parameter group of homeomorphisms f^t of M. Denote by $l(m) = l$ a trajectory passing through a point $m \in M$ and by $fix(f^t)$ a set of all fixed points of f^t (m is a *fixed point* if $l(m) = m$.

Let $\omega(\alpha)(l)$ be an $\omega(\alpha)$-limit set of l. A trajectory is $\omega(\alpha)$*-recurrent* if it is contained in its $\omega(\alpha)$-limit set. A trajectory l is *recurrent* if it is both ω- and α-recurrent. A recurrent trajectory is *nontrivial* if it is neither a fixed point nor a periodic trajectory. The topological closure of nontrivially recurrent trajectory is called a *quasiminimal set.* Due to the classical Maier's paper [30], any nontrivially recurrent trajectory belonging to a quasiminimal set Q is dense in Q (see the modern proof in [7] and some generalizations in [18]).

According to [22], a flow g^t is called *highly transitive* if every one-dimensional trajectory of g^t is dense in M. A highly transitive flow is *irrational* if it has no fake saddles.

Surface foliations. By a *foliation* F with a set of singularities S on a surface M we mean a decomposition of $M - S$ into pairwise disjoint curves l_α (without self-intersections) locally homeomorphic to a family of parallel straight lines. Any curve l_α is called a *leaf.* Any point of S is called a *singularity.* Let l be a nonclosed leaf of a foliation F. Any point $x \in l$ divides l into two semileaves, say l^+ and l^-. A semileaf $l^{(\cdot)}$ is called *nontrivially recurrent* if its intrinsic topology does not coincide with the induced topology of $l^{(\cdot)}$ from the manifold M. A leaf l is said to be *nontrivially recurrent* if both its semileaves are nontrivially recurrent. The topological closure of a nontrivially recurrent semileaf is called a *quasiminimal set.* The definitions of highly transitive and irrational foliations are similar to the corresponding definitions for flows.

Geodesic frameworks of quasiminimal sets. We give the definition of a geodesic framework for a quasiminimal set of a flow (for a foliation, the construction is similar). Let Q be a quasiminimal set of a flow f^t and let l be a nontrivially recurrent trajectory that is dense in Q. Aranson and Grines [9] proved that the both positive and negative semitrajectories of l have asymptotic directions and this directions are different (i.e. $\alpha(\bar{l}) \neq \omega(\bar{l})$). We give the sketch of proof of this fundamental result to demonstrate the using of the Lobachevsky geometry. Since l is a nontrivially recurrent trajectory, there exists a simple closed transversal C such that $l \cap C \neq \emptyset$ and l intersects C infinitely many times. Then $\bar{l}$ intersects the sequence of curves $\overline{C}_1, \ldots, \overline{C}_n, \ldots \in \pi^{-1}(C)$ as $t \to +\infty$. Since the group Γ is discontinuous, the properties of the Lobachevsky plane Δ imply that the topological limit of the sequence $\overline{C}_n$ is a unique point, say σ, of the absolute S_∞. Hence, $\omega(\bar{l}) = \sigma$. Similarly, $\alpha(\bar{l}) \in S_\infty$. Since C is a transversal, $\alpha(\bar{l}) \neq \omega(\bar{l})$.

Hence there is the coasymptotic geodesic $g(l)$. One can prove that $g(l)$ has no self-intersections. Therefore, the topological closure $clos[g(l)]$ of $g(l)$ is a geodesic lamination [21]. This geodesic lamination is independent on the choice of l. So the following definition is well defined. The geodesic lamination

$$clos[g(l)] \stackrel{\text{def}}{=} G(Q)$$

is called a *geodesic framework* of Q. One can prove that $G(Q)$ is a minimal oriented geodesic lamination consisting of the nontrivially recurrent geodesics each being dense in $G(Q)$.

If f^t is transitive, then $Q = M$. In this case, $G(M) \stackrel{\text{def}}{=} G(f^t)$ is called a *geodesic framework* of f^t. If f^t is highly transitive, then $G(Q)$ is an irrational geodesic lamination.

In general, a geodesic framework of flow is a topological closure of union of all coasymptotic geodesics for trajectories and genealized trajectories (union of saddle fixed points and trajectories that tend to these fixed points). One can prove that a geodesic framework is always a geodesic lamination.

2-web (F_1, F_2) on a surface. *2-web* (F_1, F_2) on surface is a pair of foliations F_1, F_2 such that they have a common singular set and are topologically transversal at all non-singular points. The web theory is a classical area of geometry and is mainly devoted to solving local problems. However, 2-webs also naturally appear in the theory of dynamical systems on surfaces as pairs of stable and unstable foliations of pseudo-Anosov homeomorphisms. The topological equivalence of these webs is clearly a necessary condition for topological conjugacy of these diffeomorphisms and homeomorphisms. 2-web is *irrational* if it consists of a pair of irrational foliations. Two 2-webs (F_1, F_2) and (F_1', F_2') on M are *topologically equivalent* if there is a homeomorphism $f : M \to M$ which maps foliations F_i, $i = 1, 2$, to the corresponding foliations F_i'.

3. Topological classification of irrational flows, foliations and 2-webs

Recall that two flows f_1^t and f_2^t on a surface M are *topologically equivalent* if there exists a homeomorphism h of M which sends the trajectories of f_1^t into the trajectories of f_2^t. It is impossible to classify all surface flows. But if we restrict ourselves to the special classes this problem is manageable. In general, the classification assumes the following (independent) steps.

1. Find a constructive topological invariant which takes the same values on the topologically equivalent flows.

2. Describe all topological invariants which are admissible, i.e. may be realized in the chosen class of flows.

3. Find a standard representative in each equivalence class, i.e. given any admissible invariant, one constructs a flow whose invariant is the admissible one.

An invariant is called *complete* if it takes the same value if and only if two flows are topologically equivalent. The 'if' part only gives a *relative* invariant of flow.

Invariants fall into three major classes: homology (or cohomology), homotopy and combinatorial. Poincaré rotation number is most familiar, which carries an interesting arithmetic information, being at the same time homology and homotopy invariant. Combinatorial invariants (Peixoto and Conley-Lyapunov graphs) are good for description of flows without nontrivially recurrent trajectories. Homology and homotopy invariants (fundamental class of Katok and homotopy rotation class of Aranson–Grines) are convenient for description of flows with nontrivially recurrent trajectories. A homotopy invariant that is most related to the Riemann structure of surface is a geodesic framework. In terms of the geodesic frameworks we can reformulate the Aranson–Grines's [9] classification of irrational flows as follows.

Теорема 3.1. *Let f_1^t, f_2^t be two irrational flows on a closed orientable hyperbolic surface M. Then f_1^t, f_2^t are topologically equivalent via a homeomorphism $M \to M$ homotopic to identity if and only if their geodesic frameworks coincide, $G(f_1^t) = G(f_2^t)$.*

Теорема 3.2. *Let f^t be an irrational flow on a closed orientable hyperbolic surface M. Then its geodesic framework $G(f^t)$ is an irrational orientable geodesic lamination, $G(f^t) \in \Lambda_{\bullet r} \cap \Lambda^{irr}$.*

Теорема 3.3. *Given any irrational orientable geodesic lamination G on a closed orientable hyperbolic surface M, there is an irrational flow f^t on M such that $G(f^t) = G$.*

It follows from these theorems and Nielsen theory [35], [36] that an irrational orientable geodesic framework is a complete invariant for irrational flows (up to the action of the generalized mapping class group GM). Thus an irrational orientable geodesic framework is similar to the Poincare rotation number. Below, we'll see that this similarity keeps for perturbations of a flow.

Remark that the same results is true for closed non-orientable surfaces of genus ≥ 4 [16].

The theorems that are similar to Theorems 3.1-3.3 take place for irrational foliations but one omits the orientability of geodesic framework.

Теорема 3.4. *Let $\mathcal{F}_1$, $\mathcal{F}_2$ be two irrational foliations on a closed orientable hyperbolic surface M. Then $\mathcal{F}_1$, $\mathcal{F}_2$ are topologically equivalent via a homeomorphism $M \to M$ homotopic to identity if and only if their geodesic frameworks coincide, $G(\mathcal{F}_1) = G(\mathcal{F}_2)$.*

Теорема 3.5. *Let $\mathcal{F}$ be an irrational foliation on a closed orientable hyperbolic surface M. Then its geodesic framework $G(\mathcal{F})$ is an irrational geodesic lamination, $G(\mathcal{F}) \in \Lambda^{irr}$.*

Теорема 3.6. *Given any irrational geodesic lamination G on a closed orientable hyperbolic surface M, there is an irrational foliation $\mathcal{F}$ on M such that $G(\mathcal{F}) = G$.*

Thus, an orbit of irrational orientable geodesic framework is a complete invariant for irrational foliations. Let us consider the Aranson-Grines [10] classification of minimal non-trivial sets.

A minimal set of a flow is called *non-trivial (exceptional)* if it is neither a fixed point, nor a closed trajectory, nor the whole surface M. An exceptional minimal set is nowhere dense and consists of continuum nontrivially recurrent trajectories, each being dense in the minimal set. Moreover, an exceptional minimal set is locally homeomorphic to the product of the Cantor set and a segment. Two minimal sets N_1, N_2 of the flows f_1^t, f_2^t respectively are *topologically equivalent* if there exists a homeomorphism $\varphi : M \to M$ such that $\varphi(N_1) = N_2$ and φ maps the trajectories of N_1 onto the trajectories of N_2.

Let N be an exceptional minimal set. A pair of trajectories l_1, $l_2 \subset N$ is called *special* if there exists a simply connected component Ω of $M \setminus N$ such that the accessible boundary of Ω equals $l_1 \cup l_2$.

The most familiar flow with an exceptional minimal set is the Denjoy flow (first constructed by Poincare [38]) on the torus T^2. Such a flow must have special pairs. Conversely, the existence of special pairs on a hyperbolic surface is artificial. Any flow f^t having an exceptional minimal set with special pairs on M can be mapped by a blow-down operation onto the flow with no special pairs. So the first step to classify exceptional minimal sets on M is a classification of this sets with no special pairs.

Теорема 3.7. *Let N_1, N_2 be exceptional minimal sets with no special pairs of flows f_1^t, f_2^t respectively on a closed orientable hyperbolic surface M. Then N_1, N_2 are topologically equivalent via a homeomorphism $M \to M$ homotopic to identity if and only if their geodesic frameworks coincide, $G(N_1) = G(N_2)$. Furthermore, the geodesic framework $G(N)$ of any exceptional minimal set N (possibly,with special pairs) is an orientable nontrivial geodesic lamination, $G(N) \in \Lambda_{or}$, and vise versa, given any geodesic lamination $G \in \Lambda_{or}$, there is a flow f^t with exceptional minimal set N with no special pairs such that $G(N) = G$. Moreover, let N be an exceptional minimal set of flow f^t on M which has no special pairs of trajectories. Then there is a flow f_0^t on M with the following properties:*

1. *The geodesic lamination $G(N)$ is an exceptional minimal set of f_0^t;*
2. *Minimal sets N and $G(N)$ are topologically equivalent via a homeomorphism homotopic to the identity.*

We see that the orbit of nontrivial minimal orientable geodesic lamination (framework) is a complete invariant for an exceptional minimal set with no special pairs of trajectories. One can prove that the orbit of nontrivial minimal orientable geodesic lamination with marked geodesics is a complete invariant for an exceptional minimal set in general case.

Let us show how a "web"of geodesic frameworks helps to classify so-called irrational 2-webs [15].

Теорема 3.8. *Two irrational 2-webs (F_1, F_2) and (F_1', F_2') on a closed orientable hyperbolic surface M are topologically equivalent via a homeomorphism $M \to M$ homotopic to identity if and only if their geodesic frameworks coincide, $G(F_1) = G(F_1')$, $G(F_2) = G(F_2')$.*

Let (F_1, F_2) be an irrational 2-web. Then the pair of geodesic frameworks $(G(F_1), G(F_2))$ has the following properties:

1) The sets $M \setminus G(F_i)$, $i = 1, 2$, have the same number of connected components which equal to the number of (common) singularities of the foliations F_i.

2) For each connected component $D_1 \subset M \setminus G(F_1)$ there is exactly one connected component $D_2 \subset M \setminus G(F_2)$ such that one can lift D_1 and D_2 to geodesic polygons d_1, d_2 with alternating vertices on the absolute.

Two transversal irrational geodesic frameworks $(G(F_1), G(F_2))$ are called *compatible* if conditions 1) and 2) above are satisfied.

Теорема 3.9. *For any irrational 2-web (F_1, F_2) on M the geodesic frameworks $(G(F_1), G(F_2))$ are transversal and form a compatible pair. Conversely, any such pair uniquely (up to a homeomorphism homotopic to identity) determines an irrational 2-web on M.*

4. Deviations

One of the important aspect of the Anosov-Weil theory is a deviation of foliation from its geodesic framework. This aspect is especially nutty for irrational foliations (including flows) and exceptional minimal sets because its geodesic frameworks are complete invariants. Let us give definitions.

Suppose a semi-infinite continuous curve $\bar{l} = \{\bar{l}(t), t \geq 0\}$ has the asymptotic direction $\sigma \in S_\infty$. Take one of the oriented geodesics, say $\overline{g}$, with the same positive direction σ (i.e. σ is one of the ideal endpoints of $\overline{g}$). Such geodesic $\overline{g}$ is called a *representative* of σ. Let $d(t) = \overline{d}(\bar{l}(t), \overline{g})$ be the Poincare distance between $\bar{l}(t)$ and $\overline{g}$. If there is a constant $k > 0$ such that $d(t) \leq k$ for all $t \geq 0$, we'll say that $\bar{l}$ has a *restricted deviation property*. The following theorems was proved in [13], [14].

Теорема 4.1. *Let f^t be a flow with finitely many fixed points on a closed hyperbolic surface M. Let $\bar{l}$ be a semitrajectory of the covering flow $\overline{f}^t$ on Δ. Suppose that $\bar{l}$ has an asymptotic direction. Then $\bar{l}$ has the restricted deviation property.*

Теорема 4.2. *Let F be a foliation on a closed hyperbolic surface M. Suppose that all singularities of F are topological saddles. Let $\overline{L}$ be either a generalized or ordinary leaf of the covering foliation $\overline{F}$. Then $\overline{L}$ has an asymptotic direction and the restricted deviation property.*

After Theorems 4.1, 4.2, it is natural to study the "width"of surface flows and foliations with respect to its geodesic frameworks. Put by definition,

$$\overline{d}_{\overline{L}} = sup_{\overline{m} \in \overline{L}} \overline{d}(\overline{m}, \overline{g}(\overline{L})).$$

Теорема 4.3. *Let F be a foliation on a closed hyperbolic surface M. Suppose that all singularities of F are topological saddles; then*

$$\sup\{\overline{d}_{\overline{L}}\} < \infty,$$

where $\overline{L}$ ranges over the set of all generalized and ordinary leaves of the covering foliation $\overline{F}$.

This theorem means the uniformity of deviations of leaves from a geodesic framework of foliation. The supremum above is called a *deviation of a foliation from its geodesic framework.* As a consequence, we see that the deviation of irrational foliation from its geodesic framework is finite. It is the interesting problem to study the influence of this deviation on dynamical properties of foliation. One can prove that a deviation of exceptional minimal set from its geodesic framework is also finite.

Note that an analytic flow can have a continuum set of fixed points. Nevertheless the strong smoothness allows to prove the following result [19].

Теорема 4.4. *If f^t is an analytic flow on a closed hyperbolic orientable surface M, then any semitrajectory of f^t with an asymptotic direction has the restricted deviation property.*

For plane closed surfaces (the torus and Klein bottle), a similar theorem was proved by Anosov [2], [4].

5. Dynamics and absolute

In this section we show how some properties of points of S_∞ influence on dynamical properties of flows and foliations. In particular, the first theorem says that if a foliation (or flow) with a finite set of singularities has a semi-leaf with an irrational asymptotic direction, then the foliation has a quasiminimal set. More exactly, denote by $\Lambda(\infty) \subset S_\infty$ the set of points reached by the laminations from Λ. Another words, $\Lambda(\infty)$ are points reached by geodesics from minimal nontrivial geodesic laminations. It is known that every point of $\Lambda(\infty)$ is irrational.

Теорема 5.1. *If a foliation $\mathcal{F}$ with a finitely many singularities on M has a semi-leaf with an irrational direction, then $\mathcal{F}$ has a quasiminimal set (in particular, $\mathcal{F}$ has a nontrivially recurrent leaves). Vise versa, if $\mathcal{F}$ has a quasiminimal set, then its geodesic framework reaches a point from $\Lambda(\infty)$.*

Denote by $\Lambda^{irr}(\infty) \subset S_\infty$ the set of points reached by the irrational geodesic laminations.

Теорема 5.2. *Let $\mathcal{F}$ be a foliation with a finitely many singularities on M. If its geodesic framework $G(\mathcal{F})$ reaches a point from $\Lambda(\infty) - \Lambda^{irr}(\infty)$, then $\mathcal{F}$ is not highly transitive and there is a nontrivially homotopic closed curve that is not intersected by any nontrivially recurrent leaf. If $G(\mathcal{F})$ reaches a point from $\Lambda^{irr}(\infty)$, then $\mathcal{F}$ has an irreducible quasiminimal set (i.e. any nontrivially homotopic closed curve on M intersects this quasiminimal set). Moreover, $\mathcal{F}$ is either highly transitive or can be obtained from a highly transitive foliation by a blow-up operation of at least countable set of leaves and by the Whitehead operation. In the last case, when $\mathcal{F}$ is not highly transitive, $\mathcal{F}$ has a unique nowhere dense quasiminimal set.*

Take an irrational geodesic framework G. Then $\pi^{-1}(G) = \overline{G}$ is a geodesic lamination on the Lobachevsky plane Δ. A point $\sigma \in \overline{G}(\infty)$ is a *point of first kind* if there is only one geodesic of $\overline{G}$ with the endpoint σ. Otherwise, σ is called a *point of second kind.* One can prove that this definition does not depend on the choosing of $G \in \Lambda^{irr}$. The following theorem shows that the type of asymptotic direction reflects certain "dynamical"properties of foliation [18].

Теорема 5.3. *Let $\mathcal{F}$ be an irrational foliation on M and let l^+ be a positive semi-leaf of $\mathcal{F}$ such that its lifting $\overline{l}^+$ to Δ has the asymptotical direction $\sigma \in S_\infty$. Then $\sigma \in \Lambda^{irr}(\infty)$. Moreover,*

1. *If σ is a point of first kind then l^+ belongs to a nontrivially recurrent leaf.*
2. *If σ is a point of second kind then l^+ belongs to an α-separatrix of some saddle singularity of $\mathcal{F}$.*

Denote by $\Lambda_{or}(\infty) \subset S_\infty$ the set of points reached by orientable minimal nontrivial laminations. One can reformulate above theorems for flows replacing $\Lambda(\infty)$ by $\Lambda_{or}(\infty)$ and $\Lambda^{irr}(\infty)$ by $\Lambda^{irr}_{or}(\infty)$.

Put by definition, $\Lambda^{irr}(\infty) \cap \Lambda_{non}(\infty) = \Lambda^{irr}_{non}(\infty)$. The set $\Lambda^{irr}_{non}(\infty)$ is dense and has zero Lebesgue measure on S_∞. One holds the following sufficient condition of the existence of continuum fixed points set for flows.

Теорема 5.4. *Suppose a flow f^t on M reaches a point from $\Lambda^{irr}_{non}(\infty)$. Then f^t has a continual set of fixed points. Furthermore, f^t has neither nontrivially recurrent semitrajectories nor closed transversals nonhomotopic to zero.*

6. Absolute and smoothness

In this section we show that some points of S_∞ achieved by C^∞ flows can not be achieved by analytic flows. Recall that $\sigma \in S_\infty$ is called a *point achieved by f^t* if there is a positive (or negative) semitrajectory $l^\pm$ of f^t such that the some covering $\overline{l}^\pm$ for $l^\pm$ has the asymptotic direction defined by σ.

Denote by A_{fl}, A_∞, $A_{an} \subset S_\infty$ the sets of points achieved by all topological, C^∞, and analytic flows respectively. Due to the remarkable result of Anosov [2], $A_{fl} = A_\infty$. Obviously, $A_{an} \subset A_\infty$. It follows from the following theorem that $A_\infty - A_{an} \neq \emptyset$ [19].

Теорема 6.1. *There exists a continual set $U(M) \subset A_\infty$ such that given any C^∞ flow f^t that reaches a point from $U(M)$, is not analytic. The set $U(M)$ is dense and has zero Lebesgue measure on S_∞.*

One can prove that $\Lambda_{triv}(\infty) \subset A_{an} \subset \Lambda_{triv}(\infty) \cup \Lambda_{or}(\infty)$, and $\Lambda_{non}(\infty) \subset A_\infty - A_{an}$.

7. On continuity and collapse of geodesic frameworks

A complete topological invariant of irrational flows on closed hyperbolic orientable surfaces is represented by a homotopy rotation class introduced in 1973 by S. Aranson and V. Grines [9]. On the other hand such an invariant can be represented in terms of a geodesic framework as well. In many cases such a representation is more convenient because the set of geodesic laminations can be endowed with a structure of the topological space. Thus far one can study the parametric families of irrational flows in terms of their geodesic laminations.

Recall that a geodesic framework is *irrational* if it consists of nontrivially recurrent geodesics. A geodesic framework is called *rational* if it does not contain nontrivially recurrent geodesics. Note that a rational geodesic framework contains closed geodesics. A rational geodesic framework is called *strongly rational* if it consists of only closed geodesics. Actually, a strongly rational geodesic framework is a trivial geodesic lamination.

As we saw above, the geodesic framework of highly transitive flow is irrational and consists of nontrivially recurrent geodesics each being dense in the geodesic framework. This irrational geodesic framework is an analog of irrational rotation number of torus flows having nontrivial recurrent trajectories. The results of this section was obtained in collaboration with V. Medvedev [17].

Теорема 7.1. *Let f^t be a highly transitive C^1-flow induced by a vector field $v \in X^1(M)$ on a closed orientable hyperbolic surface M. Suppose that all fixed points of f^t are hyperbolic saddles. Let U be a neighborhood of the geodesic framework $G(f^t)$ of f^t. Then there is a neighborhood $O^1(v)$ of v in the space $X^1(M)$ of all C^1-vector fields such that any flow g^t generated by $w \in O^1(v)$ has a non-empty geodesic framework $G(g^t)$ belonging to U.*

Theorem 7.1 is similar to the assertion that an irrational rotation number of transitive torus flow depends continuously on perturbations of the flow in the space of C^1-flows.

According to Pugh's C^1 Closing lemma, given a torus vector field v with nontrivially recurrent trajectories, there is a vector field w arbitrary close to v in the space $X^1(M)$ such that w has a periodic trajectory nonhomotopic to zero. As a consequence, given a torus vector field with irrational Poincaré rotation number, there is an arbitrary close vector field with rational rotation number. This property is called an instability of rotation number. The following theorem means that an irrational geodesic framework has the similar 'instability'.

Теорема 7.2. *Let f^t be a highly transitive C^1-flow induced by a vector field $v \in X^1(M)$ on a closed orientable hyperbolic surface M. Suppose that all fixed points of f^t are hyperbolic saddles. Then for any neighborhood U of the geodesic framework $G(f^t)$ and any neighborhood $O^1(v)$ of v in the space $X^1(M)$ of C^1-vector fields* *there is a flow* g^t *generated by* $w \in O^1(v)$ *such that the geodesic framework* $G(g^t)$ *is strongly rational and belongs to* U *.*

As far as rational geodesic frameworks is concerned, there are examples both of continuous and discontinuous dependence on parameters of a flow. It is obvious that Morse-Smale flow has rational geodesic framework which does not change under small bifurcations of the flow because any Morse-Smale flow is structurally stable. Two theorems in below describe virtual scenario of the destruction of a rational geodesic framework.

Теорема 7.3. *On a closed hyperbolic orientable surface* M *there is a one-parameter family of* C^∞ *flows* f^t_μ *which depends continuously on the parameter* $\mu \in [0;1]$ *and such that the following conditions are satisfied:*

1. *For all* $\mu \in [0;1)$ *the flow* f^t_μ *has an irrational geodesic framework* $G(f^t_\mu) \neq \emptyset$ *which does not depend on the parameter* μ *.*
2. *The flow* f^t_1 *has a rational geodesic framework* $G(f^t_1)$ *.*
3. *There is a neighborhood* U *of* $G(f^t_1)$ *such that* $G(f^t_\mu) \notin U$ *as* $\mu \in [0;1)$ *.*

Теорема 7.4. *On a closed hyperbolic orientable surface* M *there is a one-parameter family of* C^∞*-flows* f^t_μ *which depends continuously on the parameter* $\mu \in [0;1]$ *such that the following conditions are satisfied:*

1. *For all* $\mu \in [0;1]$ *the flow* f^t_μ *has a rational geodesic framework* $G(f^t_\mu) \neq \emptyset$ *which does not depend on the parameter* μ *as* $\mu \in [0;1)$ *.*
2. *There is a neighborhood* U *of* $G(f^t_1)$ *such that* $G(f^t_\mu) \notin U$ *as* $\mu \in [0;1)$ *.*

Discontinuity of a rational geodesic framework is not surprising, since there are flows on torus (and the Klein bottle) with rational rotation number which varies in a "jump-like"fashion under arbitrarily small perturbations [32], [33].

We formulate now a theorem on the existence of one bifurcation of a geodesic framework which is similar to the 'blue-sky catastrophe' bifurcation of flow and corresponds to a certain family of flows.

Теорема 7.5. *On a closed hyperbolic orientable surface* M *there is a one-parameter family of* C^∞ *flows* f^t_μ *which depends continuously on the parameter* $\mu \in [0;1)$ *such that the following conditions are satisfied:*

1. *For all* $\mu \in [0;1)$ *the flow* f^t_μ *has a rational geodesic framework* $G(f^t_\mu) \neq \emptyset$ *consisting only of closed geodesics.*
2. *The lengths of closed geodesics in* $G(f^t_\mu)$ *tend uniformly to infinity as* $\mu \to 1$ *.*
3. $G(f^t_1) = \emptyset$ *.*

A bifurcation described in Theorem 7.5 we will call a *collapse of geodesic framework*.

The following theorem gives some information on a set of fixed points of a flow under which a collapse of the geodesic framework takes place.

Теорема 7.6. *Let f_μ^t be a one-parameter family of C^∞-flows which depends continuously on the parameter $\mu \in [0;1]$ on a closed hyperbolic orientable surface M. Assume that:*

1. *For all $\mu \in [0;1)$ the flow f_μ^t has a rational geodesic framework $G(f_\mu^t) \neq \emptyset$ consisting only of closed geodesics.*
2. *The lengths of closed geodesics in $G(f_\mu^t)$ tend uniformly to infinity as $\mu \to 1$.*
3. *$G(f_1^t) = \emptyset$.*

Then the flow f_1^t has infinitely many fixed points.

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О ВЫСШЕМ ИНТЕГРАЛЕ СПИРАЛЬНОСТИ

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Приводится отрицательное решение проблемы Арнольда-Новикова о построении высшего аналога интеграла спиральности бездивергентного векторного поля при некоторых естественных дополнительных предположениях. Доказательство следует идее С.С. Подкорытова. Предлагается гипотеза о положительном решении переформулированной проблемы Арнольда-Новикова для случайных бездивергентных векторных полей.

1. Введение

В середине позапрошлого столетия К.Ф. Гаусс предложил интегральную формулу для коэффициента зацепления двух замкнутых кривых в R^3 (см., например, [A-Kh]). В середине прошлого столетия интеграл Гаусса был обобщен и определен как инвариант бездивергентного векторного поля в трехмерном пространстве, быстро убывающего на бесконечности, относительно действия группы диффеоморфизмов с компактным носителем, сохраняющих объем. В этом контексте интеграл Гаусса часто называется интегралом спиральности. Как показал В.И. Арнольд (см. [A-Kh] с последующей ссылкой), интеграл спиральности имеет смысл асимптотического среднего коэффициентов зацепления интегральных траекторий векторного поля. В.И. Арнольд сформулировал в [Arn] проблемы 1984-12, 1990-12, касающиеся обобщений интеграла Гаусса на случай асимптотических высших инвариантов узлов и зацеплений. С.П. Новиков затронул этот круг вопросов в докладе, сделанном в Эдинбурге в сентябре 1998 г. Проблему построения высшего аналога интеграла Гаусса для бездивергентных векторных полей будем называть проблемой Арнольда-Новикова. Построение интегралов, задающих решение проблемы Арнольда-Новикова, означало бы, в частности, построение первых интегралов для вектора завихренности скорости поля в уравнении движения идеальной несжимаемой жидкости, а также первых интегралов для решения системы уравнений в кинематической магнитной гидродинамике.

2. Основная теорема

Обозначим через $\Re$ пространство бездивергентных векторных полей в R^3 с компактным носителем. (Для приложений интереснее рассматривать пространство полей, убывающих на бесконечности как r^{-2}.) Обозначим через $\Re^{link} \subset \Re$ подпространство, состоящее из полей, моделирующих компактные зацепления с замкнутыми компонентами (по поводу полей, моделирующих зацепления, см. [B-F]). Для полей подпространства $\Re^{link}$ проблема Арнольда-Новикова решена положительно в [Akh]. (Строго говоря, в этой работе рассматривается случай полей, моделирующих трехкомпонентные зацепления (см. [B-F] по поводу определения бездивергентного (магнитного) поля, моделирующего зацепление), а общий случай зацепления с произвольным числом компонент, раскрашенных в три цвета, до настоящего времени полностью не изучен.)

Сформулируем естественные (с точки зрения вычислений) условия, при которых можно было бы попытаться искать интегральное выражения для высшего числа спиральности.

Условия конечного порядка. *Скажем, что вещественнозначный функционал* $I : \Re \to R$ *имеет порядок* n, *если этот функционал выражается в виде кратного (возможно, несобственного при вырождении конфигурации точек) интеграла от значения полиномиальной функции фиксированной степени координат векторов поля, приложенных в рассматриваемом наборе точек, по подпространству пространства упорядоченных* n*-точечных конфигураций.*

Замечание. Интеграл спиральности имеет порядок 2 в смысле предыдущего определения, причем интегрирование происходит по пространству двухточечных конфигураций. В магнитной гидродинамике условия конечного порядка являются естественными, т.к. напрямую связаны с корреляционным тензором магнитных полей.

Основной результат выражает запрет на положительное решение проблемы Арнольда-Новикова при некоторых естественных дополнительных предположениях. Этот результат был открыт С.С. Подкорытовым и наше доказательство основано на его идее.

Теорема. *Пусть функционал конечного порядка* I *определен на подпространстве* $\Re^{link} \subset \Re$ *и является инвариантом относительно сохраняющих объем преобразований (т.е. не изменяется при преобразованиях пространства, сохраняющих объем и неподвижных в некоторой окрестности бесконечности). Предположим, кроме того, что этот функционал удовлетворяет следующему дополнительному условию:*

(*) $I(B_0) = 0$, *если* B_0 *моделирует многокомпонентное тривиальное зацепление.*

Тогда $I(B) = 0$, *если векторное поле* B *моделирует двухкомпонентное зацепление Уайтхеда.* (Рисунки векторных полей, моделирующих зацепление Уайтхеда, имеются в [A-Kh], [H-M]).

Доказательство. Рассмотрим $n+1$, $n \geq 2$ векторных полей, обозначаемых через $B_0, B_1, \ldots, B_n$, каждое из которых моделирует некоторое зацепление (см. зацепление на рис.1 слева).

Компоненты зацепления пересекаются между собой по набору отрезков. Векторные поля в окрестности компонент зацепления (эти окрестности принято называть трубками) выбраны так, что поля на общих прямолинейных участках соответствующих трубок противоположны друг другу. В частности, потоки векторов (которые по определению являются положительными характеристиками трубок) во всех трубках совпадают.

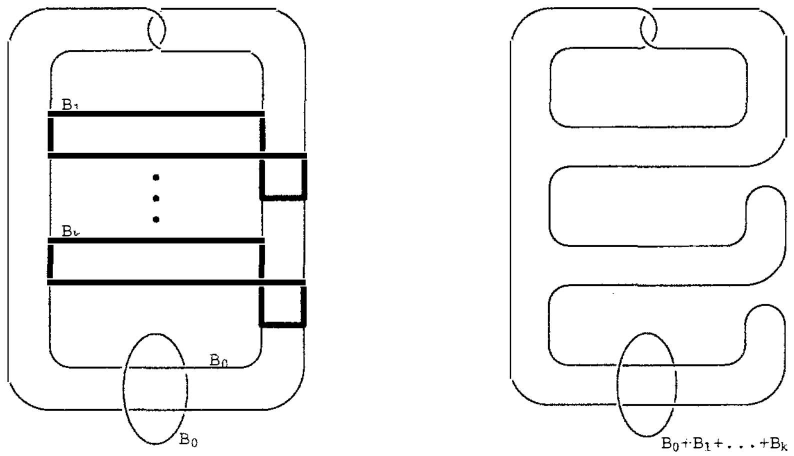

Рис. 1.

Как проиллюстрировано на рис.1 справа, рассматриваемый набор полей удовлетворяет следующему условию: каждое из 2^{n+1} векторных полей $\alpha_0 B_0 + \alpha_1 B_1 + \dots \alpha_n B_n$, где каждое α_i принимает значение 0 или 1, моделирует зацепление, причем зацепление B_0 , которое получается при $\alpha_0 = 1, \alpha_i = 0, 1 \le i \le n$, моделирует зацепление Уайтхеда, а каждое из остальных $2^{n+1} - 2$ полей, получающихся в случае, если хотя бы один из коэффициентов α_i отличен от нуля, моделирует тривиальное многокомпонентное зацепление. Предположим, что рассматриваемый функционал I имеет порядок n . Определим последовательность значений (инвариантов) $I^2(B_0, B_{i_1})$, $i_1 = 1, \dots, n$, по формуле

$$I^2(B_0, B_{i_1}) = I(B_0) + I(B_{i_1}) - I(B_0 + B_{i_1}),$$

$$I^3(B_0, B_{i_1}, B_{i_2}) = I^2(B_0, B_{i_1}) + I^2(B_{i_2}, B_{i_1}) - I^2(B_0 + B_{i_2}, B_{i_1}), i_2 \ne i_1, i_2 = 1, \dots, n$$

и далее, при различных $i_1, \dots, i_s$, принимающих значения $1, \dots, n$, определим

$$I^s(B_0, B_{i_1}, \dots, B_{i_{s-1}}) = I^{s-1}(B_0, B_{i_1}, \dots, B_{i_{s-2}}) +$$

$$+ I^{s-1}(B_{i_{s-1}}, B_{i_2}, \dots, B_{i_{s-1}}) - I^{s-1}(B_0 + B_{i_{s-1}}, B_{i_1}, \dots, B_{i_{n-2}}).$$

Обозначим через $B_{i;a,x}$ первую координату вектора поля B_i в точке $a \in R^3$. Аналогичные обозначения введем для остальных двух координат. Нетрудно проверить, что суммарная степень вхождения координат вектора B_0 в каждое слагаемое подынтегрального ядра понижается на 1 при переходе от I^i к I^{i+1} .

Действительно, если в некоторой точке $(a_1, \ldots, a_r)$ конфигурационного пространства подынтегральное ядро функционала I^s содержит слагаемое $\prod_{i,j,k} B^{\alpha_i}_{0;a_i,x} B^{\beta_j}_{0;a_j,y} B^{\gamma}_{0;a_k,\gamma_k} K$, где K – моном, содержащий компоненты полей $B_1, \ldots, B_n$, то функция I^{s+1} будет содержать слагаемые

$$-\prod_{i,j,k}(B_{0;a_i,x} + B_{i_s;a_i,x})^{\alpha_i}(B_{0;a_j,y} + B_{i_s;a_j,y})^{\beta_j}(B_{0;a_k,z} + B_{i_s;a_k,z})^{\gamma_k}K +$$

$$+\prod_{i,j,k}(B_{0;a_i,x})^{\alpha_i}(B_{0;a_j,y})^{\beta_j}(B_{0;a_k,z})^{\gamma_k}K + \prod_{i,j,k}(B_{i_s;a_i,x})^{\alpha_i}(B_{i_s;a_j,y})^{\beta_i}(B_{i_s;a_k,z})^{\gamma_k}K,$$

причем слагаемые, содержащие старшие степени компонент B_0, после раскрытия скобок взаимно уничтожаются.

По предположению функционал I имеет порядок n. Следовательно, функционал I^{n+1} тождественно равна нулю. С другой стороны, вычисляя этот функционал, получим, что он содержит 3^n слагаемых, одно из которых – слагаемое $I(B_0)$, и другие слагаемые, которые суть значения функционала I на тривиальных многокомпонентных зацеплениях $\alpha_0 B_0 + \cdots + \alpha_n B_n$, $(\alpha_0, \ldots, \alpha_n) \neq (1, 0, \ldots, 0)$, и по условию тождественно обращаются в нуль. Теорема доказана.

Следующая гипотеза естественно связана с обобщением доказанной теоремы.

Гипотеза (А.В.Чернавский). *Произвольный инвариант конечного порядка выражается многочленом от интеграла спиральности.*

Проблему Арнольда-Новикова можно переформулировать на случай упорядоченного семейства бездивергентных векторных полей, что доставляет дополнительные возможности при решении. Интересен случай двух полей (одно из полей – магнитное, второе – поле завихренности скорости) и случай трех полей, которые можно интерпретировать как компоненты поля Черна-Саймона. С топологической точки зрения случай трех полей рассмотрен в [H-M]. В направлении положительного решения проблемы Арнольда-Новикова сформулируем гипотезу. Кратко говоря, утверждается, что проблема Арнольда-Новикова имеет положительное решение в случае ансамбля случайных полей, как принято в теории турбулентности (см. [Fr]).

Обозначим через $\mathfrak{R}^{link} \subset \mathfrak{R}$ подпространство полей, моделирующих зацепления. Обозначим через $F : Diff^0(R^3) \times \mathfrak{R} \to \mathfrak{R}$ действие группы преобразований, сохраняющих элемент объема, на пространстве $\mathfrak{R}$. Определено ограничение $F^{link} : Diff^{\bullet}(R^3) \times \mathfrak{R}^{link} \to \mathfrak{R}^{link}$ действия F на подпространство $\mathfrak{R}^{link}$. Определено пространство $\overline{\mathfrak{R}^n}$ упорядоченных наборов из n полей и его подпространство $(\mathfrak{R}^{link})^n \subset \mathfrak{R}^n$, состоящее из полей, моделирующих многокомпонентные зацепления, компоненты которых окрашены в n цветов (т.е. поле внутри регулярной окрестности любой компоненты зацепления совпадает с ограничением одного из заданных полей).

Гипотеза о положительном решении проблемы Арнольда-Новикова для случайных векторных полей. *Интегральная формула, построенная в ([Akh]), естественным образом определяет инвариант M^{link} на пространстве $(\Re^{link})^3$ троек векторных полей, моделирующих зацепления, компоненты которых раскрашены в 3 цвета относительно преобразований $Diff^0(R^3) \times (\Re^{link})^3 \to (\Re^{link})^3$ пространства, сохраняющих объем.*

При этом формула для интеграла M^{link} имеет вид:

$$M^{link} = M_1 + M_2, \tag{1}$$

где M_1 – функционал порядка 12, определенный на всем пространстве $\Re^3$, удовлетворяющий условию конечного порядка при интегрировании по пространству 12-точечных конфигураций (точки конфигурационного пространства разбиты на 3 группы по 4 точки в каждой группе и значение ядра функционала M_1 преобразуется инвариантно при перестановке точек внутри каждой группы), M_2 – функционал, определенный лишь на пространстве $(\Re^{link})^3$, причем его значения, усредненные по орбите действия $Diff^0(R^3) \times (\Re^{link})^3 \to (\Re^{link})^3$, тождественно обращаются в нуль.

Замечание. Поскольку группа $Diff^o(R^3)$ некомпактна, при усреднении функционала M_2 возникают трудности, которые, вероятно, требуют учета липшицевых констант диффеоморфизмов, определяющих стратификацию пространства группы компактными подпространствами.

3. Заключение

Выражение M_1 в формуле (1) естественно пытаться построить, используя вектор-функции элементарных дипольных источников и их вектор-потенциалы.

С.А. Мелихов в работе [Me] изучил связь милноровских инвариантов с инвариантами Александера крашенного зацепления. Полученные в этой работе результаты позволяют надеяться, что найденный в [Akh] интегральный инвариант в магнитной гидродинамике является одним из инвариантов в бесконечной иерархии интегралов, которая пока не построена. Инвариант для одного векторного поля можно пытаться построить, разлагая это поле в сумму n, $n \geq 3$, случайных полей, для которых затем вычисляются анонсированный в гипотезе и аналогичные инварианты.

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On a higher order analog of the helicity integral

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Podkoritov's Theorem on a negative particular solution of the Arnol'd-Novikov Problem is proved. The conjecture toward a positive solution of a reformulation of the Arnol'd-Novikov Problem for random divergence-free vector fields is presented.

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AFFINELY CONNECTED ORBIFOLDS AND THEM AUTOMORPHISMS

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The automorphism group $\mathcal{A}(N)$ of a n-dimensional affinely connected orbifold N is proved to admit a Lie group structure, and $\dim \mathcal{A}(N) \leq n^2+n$. Estimates are established for dimension of $\mathcal{A}(N)$ depending on the stratification of N.

1. Introduction

Orbifold can be regard as a manifold with singularities; it is a topological space which is locally homeomorphic to a quotient space of manifold Ω by a finite group Γ of diffeomorphisms of Ω. The group Γ is not fixed and can be changed by passing from the one chart of an orbifold to an other chart. Orbifolds appear naturally in many branches of mathematics and mathematical physics. For example, symplectic reduction often gives rise to orbifolds. Orbifolds are used in string theory [1]. Orbifolds arise in foliation theory as "good" spaces of leaves [2].

The problem of a finding of conditions guaranteeing existence of Lie structure for transformation group is one of the central problems of differential geometry [3]. Nomizu [4] proved that the group of all affine transformations of a complete affinely connected manifold is a Lie group. Later Hano and Morimoto [5] have received this result without the assumption of completeness.

We have proved that the automorphism group $\mathcal{A}(N)$ of an arbitrary n-dimensional affinely connected orbifold N admits a Lie group structure. The proof essentially uses the construction of the frame bundle P over an orbifold N. We have shown that the transformation group $\mathcal{A}(N)$ can be realized as an automorphism group of the natural e-structure on P and have applied Kobayashi's theorem [3].

We have investigated an influence of the existence of k-dimensional stratum Δ_k of N on the dimension of the automorphism group $\mathcal{A}(N)$. The presence of orbifold points is shown to sharply decrease the dimension of the transformation group $\mathcal{A}(N)$ of proper orbifold. In general case $\dim \mathcal{A}(N) \leq n^2+n$, and $\dim \mathcal{A}(N) = n^2+n$ if and only if N is the ordinary affine space with the flat affine connection. We have observed that each connected component Δ_k^i of Δ_k formed by points of same orbifold type is invariant relatively the connected component of the unit of the Lie group $\mathcal{A}(N)$. Using this observation we have got some estimates of the dimension of the Lie group $\mathcal{A}(N)$.

2. The category of smooth orbifolds

Let N be a connected Hausdorff topological space with countable base. Let U be an open connected subset of N. Fix natural numbers n and r. A C^r-*chart* of N with coordinate neighborhood U is a triple (Ω, Γ, p) where Ω is a connected open subset of the n-dimensional real vector space $\mathbf{R}^n$, Γ is a finite group of C^r-diffeomorphisms of Ω

and $p\colon \Omega \to U$ is a continuous map so that p is Γ-invariant (i.e. $p \circ \gamma = p, \forall \gamma \in \Gamma$) and p induces a homeomorphism between the orbit space Ω/Γ and U. Let $(\Omega_1, \Gamma_1, p_1)$ and $(\Omega_2, \Gamma_2, p_2)$ are two C^r-charts with coordinate neighborhoods U_1 and U_2 respectively, and $U_1 \subset U_2$. A C^r-embedding $\varphi_{12}\colon \Omega_1 \to \Omega_2$ is called a C^r *-injection* of the C^r-chart $(\Omega_1, \Gamma_1, p_1)$ into the C^r-chart $(\Omega_2, \Gamma_2, p_2)$ if the equality $p_1 = p_2 \circ \varphi_{12}$ is satisfied.

As it is known [6], if φ_{12}, φ'_{12} are two C^r-injections of a C^r-chart $(\Omega_1, \Gamma_1, p_1)$ into a C^r-chart $(\Omega_2, \Gamma_2, p_2)$ then there exists a unique $\gamma_2 \in \Gamma_2$ such that $\varphi'_{12} = \gamma_2 \circ \varphi_{12}$. In particular, since each $\gamma_1 \in \Gamma_1$ can be viewed as a C^r-injection of $(\Omega_1, \Gamma_1, p_1)$ into itself, for the two C^r-injections φ_{12} and $\varphi_{12} \circ \gamma_1$ there exists a unique $\gamma_2 \in \Gamma_2$ so that $\varphi_{12} \circ \gamma_1 = \gamma_2 \circ \varphi_{12}$. Thus the C^r-injection φ_{12} induces the monomorphism group $\psi_{12}\colon \Gamma_1 \to \Gamma_2\colon \gamma_1 \mapsto \gamma_2$.

A C^r *-atlas* on N is defined to be a family of C^r-charts $A = \{(\Omega_i, \Gamma_i, p_i),\ i \in J\}$ such that: (i) the family $\{U_i,\ i \in J\}$ is an open covering of N and (ii) every two charts $(\Omega_i, \Gamma_i, p_i)$ and $(\Omega_j, \Gamma_j, p_j)$ from A are C^r-compatible in the following sense. If $U_i \cap U_j \neq \emptyset$, $U_i = p_i(\Omega_i)$, $U_j = p_j(\Omega_j)$, then for any point $x \in U_i \cap U_j$ there exist a chart (Ω, Γ, p) not necessary belong to A such that $x \in U \subset U_i \cap U_j$ where $U = p(\Omega)$ and C^r-injections $\varphi_i\colon \Omega \to \Omega_i$ and $\varphi_j\colon \Omega \to \Omega_j$ of (Ω, Γ, p) into charts $(\Omega_i, \Gamma_i, p_i)$ and $(\Omega_j, \Gamma_j, p_j)$.

Two C^r-atlases A and B on N are said to be equivalent if the union $A \cup B$ is a C^r-atlas. Applying lemma A.7 from [6] and lemma 4.1.1 from [7] it is not difficult to prove the following.

Proposition 1. *The defined relation is an equivalent relation at the set of indicated C^r-atlases on topological space N.*

An equivalence class of the C^r-atlases on N is called a *n-dimensional orbifold structure* on the topological space N. A C^r-atlas A is said to be maximal if for each C^r-atlas B such that $B \supset A$ it is necessary $A = B$. It is easy to show that the union D of the all C^r-atlases from the same equivalent class is a maximal C^r-atlas on N. If A is a maximal atlas then the chart (Ω, Γ, p) of the definition of C^r-compatible charts belongs to A.

The pair (N, A) where A is a maximal C^r-atlas on N is called a *n-dimensional C^r-orbifold.* The topological space N is called a *underlying space* of an orbifold (N, A).

A C^r *-mapping* or *morphism* from an orbifold (N, A) into an orbifold (N', A') is called a continuous mapping $f\colon N \to N'$ if for every point $x \in N$ there are charts $(\Omega, \Gamma, p) \in A$ and $(\Omega', \Gamma', p') \in A'$ with coordinate neighborhoods U and U' such that $x \in U$ and $f(U) \subset U'$ and there is a C^r-mapping $\bar{f}\colon \Omega \to \Omega'$ satisfying to the equality $f|_U \circ p = p' \circ \bar{f}$. The correctness of this definition, i.e. independence from a choice of charts follows from the C^r-compatible condition of charts from atlas A. We call $\bar{f}$ *the representative* of f in the charts (Ω, Γ, p) and (Ω', Γ', p'). As usually, if $r \geq 1$ then C^r-maps of orbifolds are called smooth ones. We denote the category of orbifolds by **Orb** and the algebra of smooth functions on an orbifold N by $\mathbf{F}(N)$.

It is well known, if x is an arbitrary point of an orbifold (N, A) then there exists a chart $(\Omega, \Gamma, p) \in A$ such that Ω is the n-dimensional real vector space $\mathbf{R}^n$, $p(0) = x$, $0 = (0, \ldots, 0) \in \mathbf{R}^n$, and Γ is a subgroup of the group of orthogonal transformations of $\mathbf{R}^n$. Such chart $(\mathbf{R}^n, \Gamma, p)$ is called a *linearized chart* at x and U is called a *linearized coordinate neighborhood* of x. Further we usually deal with charts satisfying to these conditions.

3. The stratification of smooth orbifolds

Orbifold type of points. A point x of an orbifold (N,A) is called *regular* if there is a chart $(\Omega,\Gamma,p)\in A$ with coordinate neighborhood U, $x\in U$, such that $\Gamma=\{id_\Omega\}$. A nonregular point is called an *orbifold point.* An orbifold having an orbifold point is called a *proper orbifold.* If there are charts at points x and y of N with coordinate neighborhoods isomorphic in the category **Orb**, then x and y are said to have the *same orbifold type.*

Lemma 1. *Let (N,A) be a n-dimensional C^r-orbifold. The subspace N_0 of points of the same orbifold type with the induced topology has a natural C^r-manifold structure, with N_0 is in general disconnected.*

Proof. Let $x\in N_0$ and $(\mathbf{R}^n,\Gamma,p)\in A$ be a linearized chart at x. The fixed-point set $Fix\Gamma:=\{y\in\mathbf{R}^n \mid \gamma(y)=y, \forall\gamma\in\Gamma\}$ of the group Γ is some k-dimensional vector subspace Ω_0 of $\mathbf{R}^n$. Assume that $\Omega_0=\mathbf{R}^k\times\{0\}$. The map $p_0:=p|_{\Omega_0}$ is a homeomorphism of Ω_0 onto the image $U_0:=p_0(\Omega_0)$. As each point $y=p(z)$, $z\in Fix\Gamma$, has the same orbifold type as the point x, so $U_0\subset N_0$. Denote by φ_0 the inverse homeomorphism $p_0^{-1}\colon U_0\to\mathbf{R}^k$. Hence the set N_0 with the induced topology becomes a topological manifold, in general disconnected. The C^r-compatible condition of charts from atlas A implies that the so-defined pairs (U_0,φ_0) determine a differentiable C^r-manifold structure on N_0. Since each two points x and y from N_0 have linearized charts $(\mathbf{R}^n,\Gamma_i,p_i)$ and $(\mathbf{R}^n,\Gamma_j,p_j)$ with coordinate neighborhoods U_i and U_j which are isomorphic in category **Orb**, the fixed-point sets $Fix\Gamma_i$ and $Fix\Gamma_j$ are diffeomorphic and so the points x and y have homeomorphic neighborhoods $p_i(Fix\Gamma_i)$ and $p_j(Fix\Gamma_j)$ respectively. Therefore the dimension of each connected component of N_0 is equal to k. Thus N_0 is a k-dimensional (in general disconnected) C^r-manifold.

Stratification. Observe that the manifolds of orbifold points of different types may have the same dimension. Denote by Δ_k the union of manifolds of orbifold points of dimension k (it is possible that $\Delta_k=\emptyset$) and denote by Δ_n the smooth n-dimensional manifold of regular points. The family

$$\Delta(N)=\{\Delta_k,\ k\in\{0,\dots,n\}\}$$

is called *the stratification* of the orbifold N, and Δ_k themselves are called *strata.*

As it is known the following statement takes place.

Lemma 2. *Let Ψ be a finite subgroup of the orthogonal group $O(n,\mathbf{R})$ and $p\colon\mathbf{R}^n\to\mathbf{R}^n/\Psi$ be a canonical projection onto the orbit space $\mathbf{R}^n/\Psi$. Let $V:=\{x\in\mathbf{R}^n \mid \Psi_x=\{id_{\mathbf{R}^n}\}\}$ be a subset of points with trivial stable subgroups. Then the image $p(V)$ is a connected open and everywhere dense subset of $\mathbf{R}^n/\Psi$.*

Proposition 2. *The stratification $\Delta(N)$ of n-dimensional C^r-orbifold (N,A) satisfies to the following conditions:*

(i) the stratum Δ_k, $k\in\{0,\dots,n-1\}$ is union of k-dimensional C^r-manifolds, and each connected component Δ_k^i of stratum Δ_k is formed by orbifold points of same orbifold type;

(ii) the stratum Δ_n is a connected open and everywhere dense n-dimensional C^r-manifold consisting of the all regular points of N.

Proof. 1. Let $\Delta_k^i=\bigsqcup_{\alpha\in J}N_\alpha$ be a disjoint union where N_α is a set of points of the same orbifold type. According to proof of lemma 1 each N_α is an open subset of Δ_k^i and hence N_α is also closed subset of Δ_k^i. The connection of topological space Δ_k implies that Δ_k^i consists of points of the same orbifold type.

2. Since for each regular point $x \in N$ there is a chart $(\Omega, \Gamma, p) \in A$ such that $\Gamma = id_\Omega$ then the stratum Δ_n is an open set of N.

Demonstrate that the stratum Δ_n is an everywhere dense subset of N. Let x be an arbitrary point in N, let $(\mathbf{R}^n, \Gamma, p) \in A$ be a linearized chart with coordinate neighborhood U at x. The inverse image $p^{-1}(U \cap \Delta_n)$ coincides with subset $V \subset \mathbf{R}^n$ which consists of the set of points with trivial stable subgroups. Therefore according to lemma 2 the image $p(V) = U \cap \Delta_n$ is a connected open and everywhere dense subset of U. So each $x \in N$ is a limiting point of Δ_n, i.e. the closure $\overline{\Delta_n}$ of Δ_n coincides with N. Hence Δ_n is an everywhere dense subset of N.

Show that Δ_n is a connected subset of N. Let $x, y \in \Delta_n$. Since N is a connected topological space, there is a path $h\colon [0,1] \to N$ connecting x and y. As the set $h([0,1])$ is a compact connected subset of N there exists a finite chain $\{U_i, i = 1, \dots, m\}$ covering $h([0,1])$ where U_i are coordinate neighborhoods of charts $(\Omega_i, \Gamma_i, p_i) \in A$, $U_i \cap U_{i+1} \neq \emptyset$. Since Δ_n is an everywhere dense subset of N there are points $z_i \in U_i \cap U_{i+1} \cap \Delta_n$, $i = 1, \dots, m-1$. Put $z_0 := x$, $z_m := y$. The points z_i and z_{i+1} belong to $U_i \cap \Delta_n$, $i = 0, \dots, m$. In accordance with lemma 2 the set $U_i \cap \Delta_n$ is connected, so there is a path $g_i\colon [0,1] \to U_i \cap \Delta_n$ connecting z_i and z_{i+1}. Denote by g the product of paths g_i, $i = 0, \dots, m$. Then g is a path in Δ_n connecting x and y, i.e. the stratum Δ_n is a connected subset of N. The proposition 2 is proved.

The definition of orbifold type points implies that any automorphism $f\colon N \to N$ of an orbifold N in category **Orb** keeps the orbifold type. So $f(\Delta_k) = \Delta_k$ for all $\Delta_k \in \Delta(N)$.

Closures of connected components of strata.

Theorem 1. *Let (N, A) be a C^r-orbifold and Δ_k^i be a connected component of a stratum Δ_k. Then the closure $\overline{\Delta_k^i}$ of Δ_k^i is naturally endowed by C^r-orbifold structure for which Δ_k^i is a set of regular points.*

Proof. Let x be a boundary point of Δ_k^i, i.e. $x \in \partial\Delta_k^i = \overline{\Delta_k} \setminus \Delta_k$. Denote by Δ_m^l the connected component of stratum containing x. Let $(\mathbf{R}^n, \Gamma, p)$ be a linearized chart with coordinate neighborhood U at the point x, $W := p^{-1}(U \cap \Delta_k)$. Take a point $b \in p^{-1}(a)$, $a \in U \cap \Delta_k$. Denote by Γ_1 the stable subgroup Γ_b of group Γ at b. As $\Gamma_1 \subset \Gamma$, $\Gamma_1 \neq \Gamma$, so $m < k$. Consider a decomposition of the group Γ on the contiguous classes by subgroup Γ_1

$$\Gamma = \Gamma_1 \cup \gamma_1\Gamma_1 \cup \ldots \cup \gamma_s\Gamma_1$$

where $\gamma_1, \dots, \gamma_s \in \Gamma \setminus \Gamma_1$. Without loss of generality we may assume that $Fix\Gamma = \mathbf{R}^m \times \{0\}$, $Fix\Gamma_1 = \mathbf{R}^k \times \{0\}$, $m < k$. Show that $\gamma_j|_{\mathbf{R}^k \times \{0\}} \neq id_{\mathbf{R}^k \times \{0\}}$ for $j = 1, \dots, s$. Suppose opposite, $\gamma_j|_{\mathbf{R}^k \times \{0\}} = id_{\mathbf{R}^k \times \{0\}}$, then the group Γ_1 contains γ_j. This contradicts a choice of the element γ_j.

Note that the stable subgroup Γ_y of the group Γ at the point $y \in \mathbf{R}^k \times \{0\}$ coincides with Γ_1 if and only if $\gamma_j(y) \neq y$ for each $j = 1, \dots, s$. Thus the subset $B := \{y \in \mathbf{R}^k \times \{0\} \mid \Gamma_y = \Gamma_1\}$ coincides with set $\mathbf{R}^k \times \{0\} \setminus \cup_{j=1}^s Fix\gamma_j$. As $Fix\gamma_j$ is a k_j-dimensional vector subspace in $\mathbf{R}^k \times \{0\}$, and as shown above $k_j < k$, so B is an open and everywhere dense subset of $\mathbf{R}^k \times \{0\}$. Furthermore $B = W \cap (\mathbf{R}^k \times \{0\})$, so $p(B) \subset p(W) \subset \Delta_k$.

Let $\tilde{\Gamma}$ be a subgroup of the group Γ for which $\mathbf{R}^k \times \{0\}$ is invariant subspace, i.e. $\tilde{\Gamma} = \{\gamma \in \Gamma \mid \gamma(\mathbf{R}^k \times \{0\}) = \mathbf{R}^k \times \{0\}\}$. Put $\psi\colon \tilde{\Gamma} \to O(k, \mathbf{R})\colon \gamma \mapsto \gamma|_{\mathbf{R}^k \times \{0\}}$, $\Psi := im\psi$. Then ψ is an epimorphism of the group $\tilde{\Gamma}$ onto the group Ψ. Since $Fix\Gamma_1 = \mathbf{R}^k \times \{0\}$, we have $\Gamma_1 \subset \ker\psi$. Take $\gamma \in \ker\psi$. Then $\gamma \in \Gamma_b = \Gamma_1$ and hence $\ker\psi \subset \Gamma_1$ and $\Gamma_1 = \ker\psi$. This means that the group Ψ is isomorphic to the factor-group $\tilde{\Gamma}/\Gamma_1$ and Ψ is a finite subgroup of the orthogonal group $O(k,\mathbf{R})$ which effectively acts on $\mathbf{R}^k \times \{0\}$. Therefore the set B coincides with set $\{y \in \mathbf{R}^k \times \{0\} \mid \Psi_y = \{id_{\mathbf{R}^k \times \{0\}}\}\}$. Applying lemma 2 to the group Ψ acted on $\mathbf{R}^k \times \{0\}$ and to the factor-map $q\colon \mathbf{R}^k \times \{0\} \to (\mathbf{R}^k \times \{0\})/\Psi$, we see that $p(B)$ is an open and everywhere dense subset of $\mathbf{R}^k \times \{0\}$, with $q(B)$ is connected. Observe that $(\mathbf{R}^k \times \{0\})/\Psi = (\mathbf{R}^k \times \{0\})/\tilde{\Gamma} = p(\mathbf{R}^k \times \{0\})$ and $q(B)$ is homeomorphic to $p(B)$, then $p(B)$ is connected. Inclusion $\mathbf{R}^m \times \{0\} \subset \overline{B}$ implies $p(\mathbf{R}^m \times \{0\}) \subset p(\overline{B})$. By continuously of map p we have $p(\overline{B}) \subset \overline{p(B)} \subset \overline{\Delta_k}$. As $\mathbf{R}^m \times \{0\} \subset \overline{B} = \mathbf{R}^k \times \{0\}$, so $p(\mathbf{R}^m \times \{0\}) \subset p(\overline{B}) \subset \overline{\Delta_k}$. Hence B a connected set then $\overline{B}$ and $p(\overline{B})$ are also connected. Recall that $a \in \Delta_k^i$ and $a \in p(\overline{B})$; therefore $p(\mathbf{R}^m \times \{0\}) \subset p(\overline{B}) \subset \overline{\Delta_k^i}$. Thus the point x belongs to $\overline{\Delta_k^i}$ with the open neighborhood $p(\mathbf{R}^m \times \{0\})$ from $\overline{\Delta_m^l}$. The set $\Delta_m^l \cap \partial\overline{\Delta_k^i}$ where $\partial\overline{\Delta_k^i} := \overline{\Delta_k^i} \setminus \Delta_k^i$ is open in Δ_m^l. It is closed as the trace of closed set $\partial\overline{\Delta_k^i}$ in N. The connection of the topological space Δ_m^l implies the equality $\Delta_m^l \cap \partial\overline{\Delta_k^i} = \Delta_m^l$ and hence $\Delta_m^l \subset \partial\overline{\Delta_k^i} \subset \overline{\Delta_k^i}$.

Identify $\mathbf{R}^k$ with $\mathbf{R}^k \times \{0\}$, then the triple $(\mathbf{R}^k, \Psi, p|_{\mathbf{R}^k \times \{0\}})$ is a linearized chart for $\overline{\Delta_k^i}$ at the point $x \in \partial\overline{\Delta_k^i}$. For any point $y \in \Delta_k^i$ a chart is constructed by the manner specified in lemma 1. A C^r-atlas defined by this a way determines the k-dimensional C^r-orbifold structure on the closure $\overline{\Delta_k^i}$. The theorem 1 is proved.

Examples. 1. Every C^r-manifold is a an orbifold of the same class of smoothness.

2. Note that the domain U of a chart (Ω, Γ, p) which is homeomorhic to Ω/Γ is in itself orbifold. Such orbifolds are called *elementary.*

3. Suppose that a compact Lie group H acts smoothly on a manifold M so that all stationary subgroups of the action are discrete. Then the orbit space M/H is a smooth orbifold.

4. Recall that a group action is locally free if all stationary subgroups are discrete in induced topology. If an isometry group locally free acts on a manifold then orbits form a Riemannian foliation $\mathcal{F}$. If there is an embedded orbit which has a point with a finite stable subgroup then the orbit space is an orbifold. Really, this orbit is the proper leaf $L \in \mathcal{F}$ with a finite holonomy group, it is known [2] in this case the space of leaves which coincides with the orbit space is an orbifold.

4. Bundles over orbifolds

Orbifold bundles. Let (N', A') and (N, A) be two C^r-orbifolds. A C^r-mapping $\pi\colon N' \to N$ is called a ***submersion*** if each representative $\tilde{\pi}\colon \Omega' \to \Omega$ of π in charts (Ω', Γ', p') and (Ω, Γ, p) with coordinate neighborhoods U' and U such that $\pi(U') \subset U$ is a submersion from the manifold Ω' onto the manifold Ω. The correctness of this definition, i.e. independence from a choice of charts follows from C^r-compatible charts of atlas A.

Recall that the map $\lambda\colon G_1 \to G_2$ of a group G_1 into a group G_2 is called an *anti-homomorphism,* if $\lambda(gg') = \lambda(g')\lambda(g)$ for all $g, g' \in G_1$. Let F be a smooth manifold and H be a Lie group. An ***orbifold bundle with standard fiber F and structure group H over an orbifold*** (N, A) is said to be define if:

(1) for each chart $(\Omega_i, \Gamma_i, p_i) \in A$ the following objects are determined: (i) a bundle $\pi_i\colon P_i \to \Omega_i$ with standard fiber F and structure group H; (ii) an anti-homomorphism $h_i\colon \Gamma_i \to AutP_i$ from the group Γ_i into the automorphism group of the above bundle which satisfies to the equality $\gamma^{-1} \circ \pi_i = \pi_i \circ h_i(\gamma)$, $\forall \gamma \in \Gamma_i$;

(2) for injection φ_{ij} from a chart $(\Omega_i, \Gamma_i, p_i)$ into a chart $(\Omega_j, \Gamma_j, p_j)$ with coordinate neighborhoods U_i and $U_j, U_i \subset U_j$, there is an isomorphism $\bar{\varphi}_{ij}: P_j|_{\varphi_{ij}(\Omega_i)} \dashrightarrow P_i$ between bundles $P_j|_{\varphi_{ij}(\Omega_i)}$ and P_i which satisfies to the conditions: (i) $h_i(\gamma) \circ \bar{\varphi}_{ij} = \bar{\varphi}_{ij} \circ h_j(\psi_{ij}(\gamma))$, $\forall \gamma \in \Gamma_i$, where $\psi_{ij}: \Gamma_i \to \Gamma_j$ is the monomorphism group induced by φ_{ij}; (ii) if $U_i \subset U_j \subset U_k$ and φ_{ij} and φ_{jk} are respectively injections then $\overline{\varphi_{jk} \circ \varphi_{ij}} = \bar{\varphi}_{ij} \circ \bar{\varphi}_{jk}$.

Our agreement on charts implies that them coordinate neighborhoods are contractible. Therefore we may presume that the bundles (P_i, π_i, Ω_i) are trivial; i.e. $P_i = \Omega_i \times F$ and π_i is the canonical projection onto the first factor.

For every chart $(\Omega_i, \Gamma_i, p_i) \in A$ the anti-homomorphism h_i gives rise to a smooth action of the group Γ_i on the bundle space P_i. Since Γ_i is a finite group the quotient space $\bar{P}_i := P_i/\Gamma_i$ is a C^r-orbifold of dimension $\dim N + \dim F$, and the following diagram

$$\begin{array}{ccc} P_i & \xrightarrow{\bar{p}_i} & \bar{P}_i \\ \downarrow{\scriptstyle \pi_i} & & \downarrow{\scriptstyle \bar{\pi}_i} \\ \Omega_i & \xrightarrow{p_i} & U_i \end{array}$$

is commutative where $\bar{\pi}_i: \bar{P}_i \dashrightarrow U_i$ is a map translating an orbit $z \cdot \Gamma_i \in \bar{P}_i$, $z \in P_i$, to the point $p_i(\pi_i(z)) \in U_i = p_i(\Omega_i)$. Denote by $\bar{P}$ the disjoint union of $\bar{P}_i$ over all charts $(\Omega_i, \Gamma_i, p_i) \in A$. Endow the set $\bar{P}$ with the following equivalence relation. We will say that two point $\bar{z}_i \in \bar{P}_i$ and $\bar{z}_j \in \bar{P}_j$ are ρ-equivalent if: (i) $\bar{\pi}_i(\bar{z}_i) = \bar{\pi}_j(\bar{z}_j) = x \in U_i \cap U_j$; (ii) there exist two points $z_i \in (\bar{p}_i)^{-1}(\bar{z}_i)$ and $z_j \in (\bar{p}_j)^{-1}(\bar{z}_j)$ and a chart $(\Omega_k, \Gamma_k, p_k) \in A$ with coordinate neighborhood U_k such that $x \in U_k \subset U_i \cap U_j$ and $z_j = (\bar{\varphi}_{kj})^{-1} \circ \bar{\varphi}_{ki}(z_i)$. Demonstrate that the relation ρ is transitive. Let $\bar{z}_i \in \bar{P}_i$, $\bar{z}_j \in \bar{P}_j$, $\bar{z}_l \in \bar{P}_l$, with $\bar{z}_i \overset{\rho}{\sim} \bar{z}_j$ and $\bar{z}_j \overset{\rho}{\sim} \bar{z}_l$. Then $\bar{\pi}_i(\bar{z}_i) = \bar{\pi}_j(\bar{z}_j) = \bar{\pi}_l(\bar{z}_l) = x \in U_i \cap U_j \cap U_l$ and there exist points $z_i \in (\bar{p}_i)^{-1}(\bar{z}_i)$, $z_j \in (\bar{p}_j)^{-1}(\bar{z}_j)$ and $z'_j \in (\bar{p}_j)^{-1}(\bar{z}_j)$, $z_l \in (\bar{p}_l)^{-1}(\bar{z}_l)$ and charts $(\Omega_k, \Gamma_k, p_k)$, $(\Omega_m, \Gamma_m, p_m) \in A$ with coordinate neighborhoods U_k and U_m respectively such that $x \in U_k \subset U_i \cap U_j$, $x \in U_m \subset U_j \cap U_l$ and $z_j = (\bar{\varphi}_{kj})^{-1} \circ \bar{\varphi}_{ki}(z_i)$, $z_l = (\bar{\varphi}_{ml})^{-1} \circ \bar{\varphi}_{mj}(z'_j)$. Put $z_k = \bar{\varphi}_{ki}(z_i)$ and $z_m = \bar{\varphi}_{ml}(z_l)$. Since $z_j, z'_j \in (\bar{p}_j)^{-1}(\bar{z}_j)$, then there exist $\gamma \in \Gamma_j$ such that $z'_j = h_j(\gamma)(z_j)$. So $z_m = \bar{\varphi}_{mj}(z'_j) = \bar{\varphi}_{mj} \circ h_j(\gamma)(z_j) = \bar{\varphi}_{mj} \circ h_j(\gamma) \circ (\bar{\varphi}_{kj})^{-1}(z_k)$. Since the atlas A is maximal for x there exists a chart $(\Omega_r, \Gamma_r, p_r) \in A$ such that $x \in U_r \subset U_k \cap U_m$, and we may assumed that the injection φ_{rk} and φ_{rm} satisfy to the conditions $\pi_k(z_k) \in \varphi_{rk}(\Omega_r)$ and $\pi_m(z_m) \in \varphi_{rm}(\Omega_r)$. The compositions $\varphi_{ki} \circ \varphi_{rk}$ and $\varphi_{ml} \circ \varphi_{rm}$ are injections the chart $(\Omega_r, \Gamma_r, p_r)$ into charts $(\Omega_i, \Gamma_i, p_i)$ and $(\Omega_l, \Gamma_l, p_l)$ respectively. Define a homomorphism $\bar{\varphi}_{rk}$ by the equality $\bar{\varphi}_{rk} := \bar{\varphi}_{rm} \circ \bar{\varphi}_{mj} \circ h_j(\gamma) \circ (\bar{\varphi}_{kj})^{-1}$ where $\bar{\varphi}_{rm}$ is an arbitrary homomorphism satisfying to the conditions of the orbifold bundle definition. Then $z_m = (\bar{\varphi}_{rm})^{-1} \circ \bar{\varphi}_{rk}(z_k)$. By condition (2) of the orbifold bundle definition we have $\overline{\varphi_{ml} \circ \varphi_{rm}} = \bar{\varphi}_{rm} \circ \bar{\varphi}_{ml}$ and $\overline{\varphi_{ki} \circ \varphi_{rk}} = \bar{\varphi}_{rk} \circ \bar{\varphi}_{ki}$. Therefore $z_l = \overline{\varphi_{ml} \circ \varphi_{rm}} \circ (\overline{\varphi_{ki} \circ \varphi_{rk}})^{-1}(z_i)$. Thus the points z_i and z_l are ρ-equivalent and ρ is an equivalent relation. The quotient space $P = \bar{P}/\rho$ is naturally endowed a C^r-orbifold structure. The projection $\pi_i: P_i \dashrightarrow \Omega_i$ define a smooth map $\pi: P \dashrightarrow N$ between the orbifolds which is a submersion. Thus we have the following proposition.

Proposition 3. *Each orbifold bundle with a standard fiber F and a structure group H over a C^r-orbifold N naturally defines a C^r-orbifold P of dimension* $\dim N + \dim F$ *and an orbifold submersion $\pi: P \dashrightarrow N$.*

The C^r-orbifold P is called *the bundle space,* the orbifold submersion $\pi: P \dashrightarrow N$ is called *the projection of the bundle.*

The tangent bundle over orbifold. Let (N, A) be a n-dimensional orbifold, $\Delta(N) =$

$\{\Delta_k\}$ be the stratification of N. Let $(\Omega_i, \Gamma_i, p_i) \in A$ be a chart with coordinate neighborhood U_i. Denote by (P_i, π_i, Ω_i) the tangent bundle over manifold Ω_i. The standard fiber of the tangent bundle P_i is the n-dimensional real vector space $F = \mathbf{R}^n$, the structure group of the tangent bundle P_i is the linear group $H = GL(n, \mathbf{R})$. Define the anti-homomorphism $h_i \colon \Gamma_i \to AutP_i$ from the group Γ_i into the automorphism group of P_i by the equality $h_i(\gamma) := (\gamma^{-1})_*$, $\forall \gamma \in \Gamma_i$, where $(\gamma^{-1})_*$ is differential of the transformation γ^{-1}. If φ_{ij} is an injection from a chart $(\Omega_i, \Gamma_i, p_i)$ into a chart $(\Omega_j, \Gamma_j, p_j)$ with coordinate neighborhoods U_i and U_j, $U_i \subset U_j$, then we determine an isomorphism $\bar{\varphi}_{ij} \colon P_j|_{\varphi_{ij}(\Omega_i)} \to P_i$ between bundles $P_j|_{\varphi_{ij}(\Omega_i)}$ and P_i by the equality $\bar{\varphi}_{ij} := (\varphi_{ij}^{-1})_*$, where $(\varphi_{ij}^{-1})_*$ is differential of the map φ_{ij}^{-1}. We see that so-defined the anti-homomorphisms h_i and the maps $\bar{\varphi}_{ij}$ satisfy to the conditions (1) and (2) of the orbifold bundle definition. Thus we have an orbifold bundle with the standard fiber $F = \mathbf{R}^n$ and the structure group $H = GL(n, \mathbf{R})$ which is called *the tangent bundle over orbifold* (N, A). We denote by TN the orbifold which is the bundle space and the tangent bundle over orbifold N by (TN, π, N).

Let $(\mathbf{R}^n, \Gamma, p)$ be a linearized chart at $x \in N \backslash \Delta_n$. Let $(\bar{P}, \bar{\pi}, \mathbf{R}^n)$ be the tangent bundle over $\mathbf{R}^n$. Then $\gamma(0) = 0$, $\forall \gamma \in \Gamma$, and the fiber $\bar{\pi}^{-1}(0) \subset \bar{P}$ over the point $0 \in \mathbf{R}^n$ is kept by each transformation $h_i(\gamma) = (\gamma^{-1})_*$, $\gamma \in \Gamma$. Therefore $\pi^{-1}(x) \cong \bar{\pi}^{-1}(0)/\Gamma = \mathbf{R}^n/\Gamma$. This implies that the fiber $\pi^{-1}(x)$ over a point $x \in N \setminus \Delta_n$ is not a vector space.

5. The tangent vector space to an orbifold

Let (N, A) be a n-dimensional C^r-orbifold, $r \geq 1$, $\sigma \colon (-\varepsilon, \varepsilon) \to N$ be a C^r-curve in N, $\sigma(0) = q$. Let $(\mathbf{R}^n, \Gamma, p) \in A$ be a linearized chart at the point q. By the definition of the smooth map there exists a C^r-curve $\tilde{\sigma} \colon (-\varepsilon, \varepsilon) \to \mathbf{R}^n$ such that $p \circ \tilde{\sigma} = \sigma$. We call $\tilde{\sigma}$ a representative of the curve σ. Let $f \in \mathbf{F}(N)$. The equality $\tilde{f} := f \circ p$ defines Γ-invariant C^r-function $\tilde{f} \colon \mathbf{R}^n \to \mathbf{R}$. We will designate the algebra of Γ-invariant C^r-function on $\mathbf{R}^n$ by $\mathbf{F}_\Gamma(\mathbf{R}^n)$. Since $f \circ \sigma = f \circ (p \circ \tilde{\sigma}) = (f \circ p) \circ \tilde{\sigma} = \tilde{f} \circ \tilde{\sigma}$, the composition $f \circ \sigma$ is a C^r-function on interval $(-\varepsilon, \varepsilon)$. Denote by Σ_q the set of C^r-curves $\sigma \colon (-\varepsilon, \varepsilon) \to N$ in orbifold N satisfying to the equality $\sigma(0) = q$. Enter on the set Σ_q an equivalence relation. We will say that two curves σ_1 and σ_2 from Σ_q are equivalent if the following equality

$$\left.\frac{d(f \circ \sigma_1)(t)}{dt}\right|_{t=0} = \left.\frac{d(f \circ \sigma_2)(t)}{dt}\right|_{t=0} \tag{5.1}$$

takes place for any $f \in \mathbf{F}(N)$. We may assume that any two curves from Σ_q are defined on the same interval $(-\varepsilon, \varepsilon)$. The class of curves containing a curve σ is designated by $[\sigma]$. Two functions f and g are said to have the same germ at $q \in N$ if there exists an open set U, $q \in U$, such that $f|_U = g|_U$. Note that the value $\frac{d(f \circ \sigma)(t)}{dt}|_{t=0}$, $\sigma \in \Sigma_q$, depends only from the germ of function f at point q. The equality (5.1) is equivalent the following condition. For any linearized chart $(\mathbf{R}^n, \Gamma, p)$ at the point q and any representatives $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ of curves σ_1 and σ_2 respectively in $\mathbf{R}^n$ the equality

$$\left.\frac{d(g \circ \tilde{\sigma}_1)(t)}{dt}\right|_{t=0} = \left.\frac{d(g \circ \tilde{\sigma}_2)(t)}{dt}\right|_{t=0} \tag{5.2}$$

is carried out for all $g \in \mathbf{F}_\Gamma(\mathbf{R}^n)$.

Define an addition of two classes of curves and multiply of class of curves on real numbers by the following way. Let $\tilde{\sigma}_1 : (-\varepsilon, \varepsilon) \to \mathbf{R}^n$ and $\tilde{\sigma}_2 : (-\varepsilon, \varepsilon) \to \mathbf{R}^n$ be representatives of curves σ_1 and σ_2 in a linearized chart $(\mathbf{R}^n, \Gamma, p)$ at q. Then $\tilde{\sigma}_1 + \tilde{\sigma}_2$ is a curve defined by the equality $(\tilde{\sigma}_1 + \tilde{\sigma}_2)(t) := \tilde{\sigma}_1(t) + \tilde{\sigma}_2(t)$, $t \in (-\varepsilon, \varepsilon)$, and $\alpha \cdot \tilde{\sigma}_1$ is a curve defined by the equality $(\alpha \cdot \tilde{\sigma}_1)(t) := \alpha(\tilde{\sigma}_1)(t))$, $t \in (-\varepsilon, \varepsilon)$, where the addition and multiply on real numbers are made in n-dimensional vector space $\mathbf{R}^n$. Put by definition $[\sigma_1] + [\sigma_2] := [p \circ (\tilde{\sigma}_1 + \tilde{\sigma}_2)]$ and $\alpha[\sigma_1] := [p \circ (\alpha \cdot \tilde{\sigma}_1)]$. It is possible to show that $[\sigma_1] + [\sigma_2]$ and $\alpha[\sigma_1]$ do not depend on a choice a linearized chart $(\mathbf{R}^n, \Gamma, p)$ at the point q, on a choice representatives $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ of the curves σ_1 and σ_2 in $\mathbf{R}^n$ and on a choice curves σ_1 and σ_2 in classes $[\sigma_1]$ and $[\sigma_2]$.

Denote by T_qN so-defined vector space of classes of curves. We call T_qN *the tangent vector space* to C^r-orbifold N at the point $q \in N$ where $r \geq 1$. If an orbifold N is a manifold then the given definition of a tangent vector space coincides with well known one [8].

Theorem 2. *The tangent vector space T_qN to C^r-orbifold N, $r \geq 1$, at the point $q \in \Delta_k$ is naturally identified with the tangent vector space $T_q\Delta_k$ to manifold Δ_k at the point q.*

Proof. Let $(\mathbf{R}^n, \Gamma, p) \in A$ be a linearized chart at q, $(x^1, \ldots, x^n)$ be standard coordinates of point $x \in \mathbf{R}^n$. Without loss of generality we assume that the set of fixed points $Fix\Gamma$ coincides with $\mathbf{R}^k \times \{0\}$. Let $[\sigma] \in \Sigma_q$ and $\tilde{\sigma}$ be a representative of σ in $\mathbf{R}^n$. The curve $\tilde{\sigma}$ in the coordinates of $\mathbf{R}^n$ looks like $(\tilde{\sigma}^1, \ldots, \tilde{\sigma}^n)$. Note that the the curve $\sigma^* := p \circ \tilde{\sigma}^*$ where curve $\tilde{\sigma}^*$ has coordinates $((\tilde{\sigma}^*)^1, \ldots, (\tilde{\sigma}^*)^k, 0 \ldots, 0)$ belongs to the class $[\sigma]$. Indeed, for any $g \in \mathbf{F}_\Gamma(\mathbf{R}^n)$ we have $g \circ \gamma = g$, $\forall \gamma \in \Gamma$. Fix an element $\gamma \in \Gamma$. We can regard the diffeomorphism γ of $\mathbf{R}^n$ as a coordinate transformation $y^j = y^j(x^1, \ldots, x^n)$. Differentiating the identity $g \circ \gamma(x) = g(x)$, $x \in \mathbf{R}^n$, we receive

$$\left.\frac{\partial g}{\partial y^j}\right|_{y=0} \left.\frac{\partial y^j}{\partial x^i}\right|_{x=0} = \left.\frac{\partial g}{\partial x^i}\right|_{x=0}, \quad i, j = 1, \ldots, n.$$

As $\gamma(0) = 0$, i.e. $y^j(0, \ldots, 0) = 0$ we have $\frac{\partial g}{\partial y^j}|_{y=0} = \frac{\partial g}{\partial x^j}|_{x=0}$, $j = 1, \ldots, n$. Thus the vector Y with coordinates $\left(\frac{\partial g}{\partial x^1}|_{x=0}, \ldots, \frac{\partial g}{\partial x^n}|_{x=0}\right)$ is kept by each transformation γ_{*0}, $\gamma \in \Gamma$. Therefore Y belongs to the tangent vector space $T_0(Fix\Gamma)$ to manifold $Fix\Gamma = \mathbf{R}^k \times \{0\}$ at $0 \in \mathbf{R}^n$ which is identified with $\mathbf{R}^k \times \{0\}$. Hence $\frac{\partial g}{\partial x^j} = 0$, $j = k+1, \ldots, n$. Since

$$\left.\frac{d(g \circ \tilde{\sigma})(t)}{dt}\right|_{t=0} = \sum_{i=1}^{n} \left.\frac{\partial g}{\partial x^i}\right|_{x=0} \left.\frac{d\tilde{\sigma}^i(t)}{dt}\right|_{t=0} = \sum_{i=1}^{k} \left.\frac{\partial g}{\partial x^i}\right|_{x=0} \left.\frac{d\tilde{\sigma}^i(t)}{dt}\right|_{t=0} =$$

$$\sum_{i=1}^{k} \left.\frac{\partial g}{\partial x^i}\right|_{x=0} \left.\frac{d(\tilde{\sigma}^*)^i(t)}{dt}\right|_{t=0} = \left.\frac{d(g \circ \tilde{\sigma}^*)(t)}{dt}\right|_{t=0}$$

for all $g \in \mathbf{F}_\Gamma(\mathbf{R}^n)$ then $\sigma^* \in [\sigma]$. From here using the the equivalence of the conditions (1) and (2) we have

$$[\sigma_1] = [\sigma_2] \quad \Longleftrightarrow \quad \left.\frac{d\tilde{\sigma}_1^*(t)}{dt}\right|_{t=0} = \left.\frac{d\tilde{\sigma}_2^*(t)}{dt}\right|_{t=0}. \tag{5.3}$$

We will consider $T_q\Delta_k$ as the vector space of equivalence classes $\langle\delta\rangle$ of curves δ in Δ_k. Define the map $\lambda : T_qN \to T_q\Delta_k$ by the formula $\lambda([\sigma]) := \langle\sigma^*\rangle$. Applying the equivalence of conditions (1) and (2) we see that the condition (5.3) implies that $[\sigma_1] = [\sigma_2]$ if and only if $\langle\sigma_1^*\rangle = \langle\sigma_2^*\rangle$ and hence map λ is correctly defined and λ is an injective map, and λ does not depend from a choice chart $(\mathbf{R}^n, \Gamma, p) \in A$ at the point q. It is not difficult to see that λ is a surjective homomorphism of vector spaces T_qN and $T_q\Delta_k$. Then $\lambda\colon T_qN \to T_q\Delta_k$ is canonical isomorphism of vector spaces. Thus the tangent vector spaces T_qN and $T_q\Delta_k$ are identified through λ.

Corollary 1. *Let N be a n-dimensional orbifold.*

1. If $q \in \Delta_k$ then $\dim T_qN = \dim T_q\Delta_k = k$.

2. The point q is a regular point of N if and only if $\dim T_qN = n$.

3. The point q is an isolated orbifold point if and only if $\dim T_qN = 0$.

Let $f\colon N \to N$ be an automorphism of an orbifold N in category **Orb**. By the natural manner we define the differential $f_{*q}\colon T_qN \to T_{f(q)}N$ of map f at a point $q \in N$ setting $f_{*q}([\sigma]) := [f \circ \sigma]$, $[\sigma] \in T_qN$. Since $f(\Delta_k) = \Delta_k$ and $f|_{\Delta_k}\colon \Delta_k \to \Delta_k$ is a diffeomorphism of a manifold Δ_k, using theorem 2 it is easy to show that the map f_{*q} is correctly defined and f_{*q} is an isomorphism of tangent vector spaces T_qN and $T_{f(q)}N$.

Vector fields on orbifolds. Further we will denote the elements of the tangent vector space T_xN of an orbifold N at $x \in N$ by $X_x, Y_x, Z_x, \ldots$ An element X_x, $x \in N$, is an equivalent class $[\sigma]$ where $\sigma \in \Sigma_x$, $\sigma(0) = x$. Put $X_x(f) := \frac{d(f\circ\sigma)(t)}{dt}|_{t=0}$ for any $f \in \mathbf{F}(N)$. A *smooth vector field* on an orbifold N is called a correspondence $X\colon x \mapsto X_x \in T_xN$, $x \in N$, such that for the all function $f \in \mathbf{F}(N)$ on N the function $Xf\colon N \to \mathbf{R}\colon x \mapsto X_x(f)$, $x \in N$, belongs to $\mathbf{F}(N)$. Denote by $\mathbf{X}(N)$ the set of the all smooth vector fields on N. The operations of addition of two vector fields and multiply of a vector field on a real number of $\mathbf{R}$ are defined by the point-wise manner: $(X+Y)_x := X_x + Y_x$, $(\alpha X)_x := \alpha X_x$, $X, Y \in \mathbf{X}(N)$, $\alpha \in \mathbf{R}$, $x \in N$. If N is an orbifold of class C^∞ then the vector space $\mathbf{X}(N)$ of smooth vector fields on N is endowed with Lie algebra structure.

6. Affinely connected orbifolds

The frame bundle over an orbifold. A bundle P with standard fiber F and structure group H over an orbifold N is called *principal* if $F = H$ and the group H acts by right translations on F.

Let $(\Omega_i, \Gamma_i, p_i)$ be a chart of a n-dimensional C^r-orbifold N. Denote by P_i the principal $GL(n, \mathbf{R})$-bundle of frames over Ω_i. Define an anti-homomorphism h_i from the group Γ_i into the automorphism group of bundle P_i as $h_i(\gamma)(z) := (\gamma^{-1})_{*_x} \circ z$, $\gamma \in \Gamma_i$, where $z\colon \mathbf{R}^n \to T_x\Omega_i$ is a frame at $x \in \Omega_i$. If $U_i \subset U_j$ and φ_{ij} is an injection of charts $(\Omega_i, \Gamma_i, p_i)$ and $(\Omega_j, \Gamma_j, p_j)$ with coordinate neighborhoods U_i and U_j respectively then define the homomorphism $\bar{\varphi}_{ij}\colon P_j|_{\varphi_{ij}(\Omega_i)} \to P_i$ between bundles $P_j|_{\varphi_{ij}(\Omega_i)}$ and P_i by the equality $\bar{\varphi}_{ij}(z) := (\varphi_{ij}^{-1})_{*_x} \circ z$ for a frame z at $x \in \varphi_{ij}(\Omega_i)$. The so-constructed h_i and $\bar{\varphi}_{ij}$ define a principal bundle with structure group $GL(n, \mathbf{R})$. It is called the *frame bundle* over the orbifold N and it is designated by (P, π, N).

Proposition 4. *Let (P, π, N) be a frame bundle over n-dimensional orbifold N. Then P is a smooth (n^2+n)-dimensional manifold and the connected components of the fibers of π are leaves of a n-codimensional smooth foliation $\mathcal{F}$ of If (Ω, Γ, p) is a chart at $x \in N$ and the transformations of group Γ keep orientation of Ω then the holonomy group of a leaf $L \subset \pi^{-1}(x)$ is isomorphic to Γ.*

Proof. For every chart $(\Omega_i, \Gamma_i, p_i) \in A$ of orbifold N the group Γ_i is finite. It is easy to see that the Γ_i freely acts through h_i defined above on P_i. Hence the quotient space $\bar{P}_i = P_i/\Gamma_i$ is a C^r-manifold and the quotient map $\bar{p}_i\colon P_i \to \bar{P}_i$ is a covering, with the covering transformation group of $\bar{p}_i$ is isomorphic to the group Γ. Using the proof of proposition 3 we receive that the bundle space P is a C^r-manifold, and the connected components of $\pi^{-1}(x)$, $x \in N$, determine a C^r-foliation $\mathcal{F}$ of codimension n on P. If $(\Omega_i, \Gamma_i, p_i)$ is a linearized chart at $x \in N$ and the transformations of group Γ_i keep an orientation of Ω_i then $\bar{P}_i$ consists of two connected components $\bar{P}_i^1$ and $\bar{P}_i^2$. So the inverse image $\pi_i^{-1}(0)$ consists of two connected components K_j, $j = 1, 2$. The holonomy groups of the leaves $\bar{p}_i(K_j) = L_j \subset \pi^{-1}(x)$, $j = 1, 2$, are isomorphic to Γ.

Affinely connected orbifolds. Let (P, π, N) be the frame bundle over C^r-orbifold N. On P is defined a smooth right action $R\colon P \times GL(n, \mathbf{R}) \to P$ of the group $GL(n, \mathbf{R})$, and $GL(n, \mathbf{R})$ acts on P freely if only if the orbifold N is a manifold ([9]). The global one-parameter group of transformations in $GL(n, \mathbf{R})$ generates the fundamental vector field on P, tangent to the foliation $\mathcal{F}$.

A *connection in* P is a smooth n-dimensional distribution $\mathcal{H}$ on P satisfying to the equalities

$$\mathcal{H}_z \oplus T_z\mathcal{F} = T_zP;$$

$$(R_g)_*(\mathcal{H}_z) = \mathcal{H}_{R_g z},$$

for all $z \in P$, $g \in GL(n, \mathbf{R})$, where $T\mathcal{F}$ is a tangent distribution of $\mathcal{F}$. Each vector $X \in T_zP$ can be uniquely written down as $X = HX + VX$, where $HX \in \mathcal{H}_z$, $VX \in T_z\mathcal{F}$. We call HX *the horizontal component* of X and VX *the vertical component* of X.

Each A from the Lie algebra $\mathbf{gl} := \mathbf{gl}(n, \mathbf{R})$ of $GL(n, \mathbf{R})$ defines the fundamental vector field A^* on P, and the mapping $\mathbf{gl} \to T_z\mathcal{F}\colon A \mapsto A_z^*$ is a vector space isomorphism. Given an arbitrary vector $X \in T_zP$, define $\omega(X)$ to be the only $A \in \mathbf{gl}$ for which A_z^* is equal to VX. The $\mathbf{gl}$-valued 1-form ω is called the *connection form* for $\mathcal{H}$. The connection form ω satisfies to the equalities: (i) $\omega(X^*) = A$ and (ii) $(R_g)^*\omega = Ad(g^{-1})\omega$ where Ad is the adjoint representation of the Lie group $GL(n, \mathbf{R})$ in the Lie algebra $\mathbf{gl}$. Note that if ω is some $\mathbf{gl}$-valued 1-form on P satisfying to these conditions then there is a unique connection $\mathcal{H}$ whose connection form is ω.

The orbifold N with a given connection $\mathcal{H}$ in the frame bundle P is called an *affinely connected orbifold.* The following proposition holds.

Proposition 5. *The connection $\mathcal{H}$ in the frame bundle P over an C^r-orbifold N is given if and only if there exists a mapping $\nabla\colon \mathbf{X}(N) \times \mathbf{X}(N) \to \mathbf{X}(N)$ satisfying to the following conditions:*

$$\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z, \qquad \nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z,$$

$$\nabla_X fY = (Xf)Y + f\nabla_X Y, \qquad \nabla_{fX} Y = f\nabla_X Y,$$

where $X, Y, Z \in \mathbf{X}(N)$, $f \in \mathbf{F}(N)$.

We also call a map ∇ satisfying to the above conditions a *connection or a covariant differential operator* on the orbifold N. Further an affine connected orbifold is denoted by a pair (N, ∇).

Remark 1. A connection ∇ can be viewed as a family $\{\nabla^i, (\Omega_i, \Gamma_i, p_i) \in A\}$ where ∇^i is a connection on a manifold Ω_i satisfying to the following conditions: (i) Γ_i is a transformation group of the affine connected manifold (Ω_i, ∇^i); 2) an injection φ_{ij} a chart $(\Omega_i, \Gamma_i, p_i)$ into chart $(\Omega_j, \Gamma_j, p_j)$ with coordinate neighborhood U_i and U_j, $U_i \subset U_j$, satisfies to the equality $(\varphi_{ij})_*(\nabla^i_X Y) = \nabla^j_{(\varphi_{ij})_* X}(\varphi_{ij})_* Y$ for all vector fields X, Y on Ω_i.

Remark 2. Let $(\Omega_i, \Gamma_i, p_i) \in A$ be a chart, X and Y be Γ_i-invariant vector fields on Ω_i, i.e. $\gamma_* X = X$, $\gamma_* Y = Y$, $\forall \gamma \in \Gamma_i$. The consequence of equalities $\gamma_*(\nabla^i_X Y) = \nabla^i_{\gamma_* X} \gamma_* Y = \nabla^i_X Y$ implies that $\nabla^i_X Y$ is Γ_i-invariant vector field on Ω_i. The equality $'\nabla^i_X Y := \nabla^i_X Y$ where X, Y is vector fields on $Fix\Gamma_i$ defines a connection on $Fix\Gamma_i$. This means that the connection ∇ induces the connection $'\nabla$ on each connected component Δ^l_k of stratum Δ_k. By analogy the connection ∇ induces the connection $''\nabla$ on the closure $\overline{\Delta^l_k}$ which is a k-dimensional C^r-orbifold.

Absolute parallelism on the frame bundle. Let P be the frame bundle over an orbifold N. The canonical form θ on P is the $\mathbf{R}^n$-valued 1-form defined by the following way. For each $X \in T_z P$ and every chart $(\Omega_i, \Gamma_i, p_i)$ at $\pi(z)$ let $\bar{X} \in T_{\bar{z}} P_i$ be such that $(\bar{p}_i)_*(\bar{X}) = X$ where $\bar{p}_i \colon P_i \to \bar{P}_i = P_i/\Gamma_i$ is a quotient mapping and $\bar{z} \colon \mathbf{R}^n \to T_{\bar{\pi}(\bar{z})}\Omega_i$ is a frame at $\bar{\pi}(\bar{z})$ (see [10]). Then we put by definition $\theta(X) := \bar{z}^{-1}(\pi_i)_*(\bar{X})$. A direct check shows that the value of θ is independent of the choice of the chart $(\Omega_i, \Gamma_i, p_i)$ at $\pi(z)$ and of the point $\bar{z} \in P_i$.

Let $\mathcal{H}$ be a connection on P. Note that the canonical form θ defines the linear isomorphism between $\mathcal{H}_z$ and $\mathbf{R}^n$ at every point $z \in P$. For each $\xi \in \mathbf{R}^n$ we define a horizontal vector field $X(\xi)$ on P. Let $X(\xi)_z$ be a uniquely horizontal vector of $\mathcal{H}_z \subset T_z P$ such that $\theta(X(\xi)_z) = \xi$. The vector field $X(\xi)_z$ we call a *standard horizontal vector field on* P.

Let ω be a connection form of connection $\mathcal{H}$ on the frame bundle P over a n-dimensional orbifold N and θ be the canonical form. Denote by $\{B_k,\ k = 1, \ldots, n\}$ the standard horizontal vector fields on P and designate by $\{E^*_{ij},\ i, j = 1, \ldots, n\}$ the fundamental vector fields on P appropriated to standard basis $\{E_{ij},\ i, j = 1, \ldots, n\}$ of the Lie algebra $\mathbf{gl}$. Recall that a family $\{X_1, \ldots, X_n\}$ of vector fields on a n-dimensional manifold M are defined an absolute parallelism on M if $\{(X_1)_x, \ldots, (X_n)_x\}$ is a basis of the tangent vector space $T_x M$ at each point $x \in M$. The absolute parallelism on a manifold M is also called e-structure on M. The following holds:

Proposition 6. *$n^2 + n$ vector fields $\{B_k, E_{ij},\ k, i, j = 1, \ldots, n\}$ define a basis of $T_z P$ at each point $z \in P$. Thus they determine an absolute parallelism on P.*

Proof. Since $\dim P = n^2 + n$, it is enough to prove that these vector fields are linearly independent. Let $(\Omega_l, \Gamma_l, p_l)$ be a chart with coordinate neighborhood U at a point $x \in N$. Put $\tilde{U}_l := \pi^{-1}(U_l)$. Note that the vector fields $\{B_k|_{\tilde{U}_l}, E_{ij}|_{\tilde{U}_l}\}$ are projections of respective vector fields $\{\bar{B}_k, \bar{E}_{ij}\}$ on P_l determining an absolute parallelism on P_l, i.e. $(\bar{p}_l)_*(\bar{B}_k) = B_k|_{\tilde{U}_l}$ and $(\bar{p}_l)_*(\bar{E}_{ij}) = E_{ij}|_{\tilde{U}_l}$. Since the factor-map $\bar{p}_l \colon P_l \to P_l/\Gamma_l \subset P$ is a covering map, the group Γ_l is an automorphism group of P_l and $\{\bar{B}_k, \bar{E}_{ij}\}$ are linearly independent then the vector fields $\{B_k|_{\tilde{U}_l}, E_{ij}|_{\tilde{U}_l}\}$ are also linearly independent. Hence, the the vector fields $\{B_k, E_{ij}\}$ are linearly independent and the family $\{B_k, E_{ij}\}$ determines an absolute parallelism on P.

7. The transformation groups of affinely connected orbifolds

Automorphisms of (N, ∇). Let (N, ∇) be an affinely connected orbifold. An automorphism f of N is said to be an *automorphism of the affinely connected orbifold* (N, ∇) if $f_*(\nabla_X Y) = \nabla_{f_* X} f_* Y$ for all $X, Y \in \mathbf{X}(N)$ where f_* is differential of the automorphism f. Denote by $\mathcal{A}(N)$ the group of the all automorphisms of (N,∇).

Let (P,π,N) be frame bundle over the orbifold N. Let $f_{ij}\colon \Omega_i \to \Omega_j$ be a representative of f in a charts (Ω_i,Γ_i,p_i) and (Ω_j,Γ_j,p_j) with coordinate neighborhoods U_i and U_j respectively, $f(U_i)\subset U_j$. Then it induces homomorphism $\tilde{f}_{ij}\colon P_i\to P_j$ of the frame bundles P_i and P_j. There exists an isomorphism $\psi_{ij}\colon \Gamma_i\to\Gamma_j$ of groups. As the equality

$$h_j(\psi_{ij}(\gamma))\circ\tilde{f}_{ij}=h_i(\gamma)\circ\tilde{f}_{ij} \tag{7.4}$$

is satisfied then a smooth mapping $\hat{f}_{ij}\colon P_i/\Gamma_i\to P_j/\Gamma_j$ is defined. If $f'_{ij}\colon\Omega_i\to\Omega_j$ is other representative of f then there is a transformation $\gamma\in\Gamma_j$ such that $f'_{ij}=\gamma\circ f_{ij}$. Therefore $\hat{f}_{ij}$ and $\hat{f}'_{ij}$ coincide. We may identify P_i/Γ_i and $\pi^{-1}(U_i)$. Thus the morphism $\hat{f}_{ij}$ of the frame bundle $\pi^{-1}(U_i)$ into the frame bundle $\pi^{-1}(U_j)$ is determined. Let $\hat{f}_{kl}$ be a so-defined isomorphism of the frame bundle $\pi^{-1}(U_k)$ into the frame bundle $\pi^{-1}(U_l)$ where U_k and U_l are coordinate neighborhoods of chart (Ω_k,Γ_k,p_k) and (Ω_l,Γ_l,p_l), $f(U_k)\subset U_l$, with $U_i\cap U_k\neq\emptyset$ and $U_j\cap U_l\neq\emptyset$. It is possible to show that the morphisms $\hat{f}_{ij}$ and $\hat{f}_{kl}$ coincide on $\pi^{-1}(U_i\cap U_k)$. This manner the family $\{\hat{f}_{ij}\}$ correctly determines the automorphism $\hat{f}$ of manifold P. Recall that a morphism f of a foliation $(M_1,\mathcal{F}_1)$ into a foliation $(M_2,\mathcal{F}_2)$ is a smooth map $f\colon M_1\to M_2$ which displays leaves of $\mathcal{F}_1$ into leaves $\mathcal{F}_2$. Since $\tilde{f}_{ij}\colon P_i\to P_j$ is a homomorphism of the frame bundles P_i and P_j for which the equality (7.4) takes place then $\hat{f}_{ij}$ is a foliation morphism of $(\pi^{-1}(U_i),\mathcal{F}|_{\pi^{-1}(U_i)})$ and $(\pi^{-1}(U_j),\mathcal{F}|_{\pi^{-1}(U_j)})$. Thus f defines the automorphism $\hat{f}$ of the foliation $(P,\mathcal{F})$, and $\pi\circ\hat{f}=f\circ\pi$.

Lemma 3. *The automorphism $\hat{f}$ of the foliation $(P,\mathcal{F})$ induced by an automorphism f of the affinely connected orbifold (N,∇) keeps invariant the connection form ω and the canonical form θ. Conversely, let h be an automorphism of foliation $(P,\mathcal{F})$, and $h^*\omega=\omega$, $h^*\theta=\theta$. Then h is induced by an automorphism f of (N,∇).*

Proof. Let $\hat{f}$ induce an automorphism f of the affinely connected orbifold (N,∇). Let (Ω_i,Γ_i,p_i) be an arbitrary chart of N with coordinate neighborhood U_i, $\pi_i\colon P_i\to\Omega_i$ be the frame bundle over Ω_i. Denote by $\bar{p}_i\colon P_i\to P_i/\Gamma_i=\bar{P}_i\subset P$ the map onto orbit space $\bar{P}_i$ of the group Γ_i. Note that the connection form ω determines a connection form ω_i on the frame bundle P_i, so that $(\bar{p}_i)^*\omega|_{\pi^{-1}(U_i)}=\omega_i$ where $(\bar{p}_i)^*$ is a codifferential of map $\bar{p}_i$. Since f is an automorphism of the affinely connected orbifold (N,∇), the representative f_{ij} of the automorphism f at the charts (Ω_i,Γ_i,p_i) and (Ω_j,Γ_j,p_j), $f(p_i(\Omega_i))=p_j(\Omega_j)$, is an isomorphism of the affinely connected manifolds (Ω_i,∇^i) and (Ω_j,∇^j). Therefore $(\tilde{f}_{ij})^*\omega_j=\omega_i$ where $\tilde{f}_{ij}\colon P_i\to P_j$ is the homomorphism of the frame bundles P_i and P_j induced by f_{ij}. Then the induced automorphism $\hat{f}$ keeps the connection form ω on P, i.e. $\hat{f}^*\omega=\omega$. By analogy the equality $(\tilde{f}_{ij})^*\theta_j=\theta_i$ where θ_i is a canonical form on P_i, $\bar{p}_i^*\theta|_{\pi^{-1}(U_i)}=\theta_i$ implies that $\hat{f}$ keeps the canonical form θ on P, i.e. $\hat{f}^*\theta=\theta$.

A lie group structure in $\mathcal{A}(N)$.

Lemma 4. *If two automorphisms f and g of an affinely connected n-dimensional orbifold (N,∇) coincide on some open set $U\subset N$ then $f=g$ on the whole orbifold N.*

Proof. Let f and g be two automorphisms of an affinely connected n-dimensional orbifold (N,∇). Let U be a open set of N, and $f|_U=g|_U$. By proposition 6 the forms ω and θ define an absolute parallelism $\{B_k,E_{ij},\ k,i,j=1,\dots,n\}$ or e-structure on P. By lemma 3 the automorphisms $\hat{f}$ and $\hat{g}$ induced f and g keep invariant the connection form ω and the canonical form θ. Hence, $\hat{f}$ and $\hat{g}$ keep invariant the vector fields $\{B_k,E_{ij},\ k,i,j=$

$1, \dots, n\}$. Thus $\hat{f}$ and $\hat{g}$ are the e-structure automorphisms of P. Since $f|_U = g|_U$, the induced automorphisms $\hat{f}$ and $\hat{g}$ coincide on the set $V := \pi^{-1}(U)$. It is well know that if two automorphisms of e-structure on a manifold coincide at one point then they coincide on the whole. Thus $\hat{f} = \hat{g}$. So $f = g$ and lemma 4 is proved.

Theorem 3. *The automorphism group $\mathcal{A}(N)$ of an affinely connected n-dimensional orbifold (N, ∇) admits a Lie group structure, and $\dim \mathcal{A}(N) \leq n^2 + n$.*

Proof. By lemma 1 each automorphism $f \in \mathcal{A}(N)$ induces the e-structure automorphism $\hat{f}$ of P, and the equality $\hat{f} = id_P$ implies $f = id_N$. Thus we have an isomorphism $\eta \colon f \mapsto \hat{f}$ from the group $\mathcal{A}(N)$ onto some subgroup G of the e-structure automorphism group $\mathcal{A}(P)$ of P. Note that $G := \{h \in \mathcal{A}(P) \mid h$ is a foliation automorphism of $(P, \mathcal{F})\}$. By Kobayashi's theorem [3], the group $\mathcal{A}(P)$ of all e-structure automorphism of P is a Lie group with compact-open topology induced from diffeomorphism group $Diff(P)$, and $\dim \mathcal{A}(P) \leq \dim P = n^2 + n$. Demonstrate that G is a closed subgroup of $\mathcal{A}(P)$. Let $\{f_n\} \subset G$ be a consequence converging to f in compact-open topology of $\mathcal{A}(P)$. Then for all $x \in P$ the sequence $\{f_n(x)\}$ converges to $f(x)$ in topology of P. Since $f_n \in G$ there is the sequence $\{g_n\} \subset \mathcal{A}(N)$ such that $\pi \circ f_n(x) = g_n \circ \pi(x)$, $x \in P$. As the map π is continuous and N is a Hausdorff topological space, we have $\pi \circ f(x) = \lim_{n\to\infty}(g_n \circ \pi(x))$. Take any $y \in L(x) \in \mathcal{F}$. The uniquely of limit $\lim_{n\to\infty}(g_n \circ \pi(x))$ implies $\pi \circ f(y) = \lim_{n\to\infty}(g_n \circ \pi(y)) = \lim_{n\to\infty}(g_n \circ \pi(x)) = \pi \circ f(x)$. Furthemore $f(L(x)) = L(f(x))$, i.e. f is a foliation automorphism of $(P, \mathcal{F})$. Hence, $f \in G$ and the group G is a closed subgroup of $\mathcal{A}(P)$. Thus the group G is a Lie subgroup of $\mathcal{A}(P)$. Through the isomorphism η the group $\mathcal{A}(N)$ is endowed by a Lie group structure, with $\dim \mathcal{A}(N) = \dim G \leq \dim \mathcal{A}(P) \leq n^2 + n$. The theorem 3 is proved.

Good orbifolds. Let M be a C^r-manifold and Γ be a group properly disconnected acting on M, then the factor-space M/Γ is a C^r-orbifold. An orbifold N is called a *good orbifold,* if there is an isomorphism of N to an orbifold M/Γ where M is a manifold and Γ is a properly disconnected group of diffeomorphisms of M. If (N, ∇) is a good affinely connected orbifold, $N = M/\Gamma$, then the connection ∇ induces a connection $\tilde{\nabla}$ on manifold M such that the group Γ becomes an automorphism group of the affinely connected manifold $(M, \tilde{\nabla})$.

Proposition 7. *The automorphism group $\mathcal{A}(N)$ of a good affinely connected orbifold (N, ∇), $N = M/\Gamma$, is isomorphic to the factor-group $\mathbf{N}(\Gamma)/\Gamma$ where $\mathbf{N}(\Gamma)$ is normalizer of Γ in the automorphism group $\mathcal{A}(M)$ of the affinely connected manifold $(M, \tilde{\nabla})$.*

8. Estimates of the dimension of $\mathcal{A}(N)$

An influence of the stratification of N on $\dim \mathcal{A}(N)$. Let (N, ∇) be an affinely connected orbifold, let $\mathcal{A}(N)$ be the automorphism group of (N, ∇). According to theorem 3 the group $\mathcal{A}(N)$ is a Lie group. Denote by $\mathcal{A}^e(N)$ the connected component of unit of the Lie group $\mathcal{A}(N)$. Note that $\dim \mathcal{A}(N) = \dim \mathcal{A}^e(N)$.

Proposition 8. *Let Δ_k^i be a connected component of a stratum Δ_k of an affinely connected orbifold N. Then $f(\Delta_k^i) = \Delta_k^i$ for each automorphism $f \in \mathcal{A}^e(N)$.*

Proof. First of all, demonstrate that the action $\Phi \colon \mathcal{A}(N) \times N \to N$ given the equality $\Phi(f, x) := f(x)$, $(f, x) \in \mathcal{A}(N) \times N$, is continuous. Fix $x \in N$, $f \in \mathcal{A}(N)$. Since f is a continuous map, for any open set U' in N, $f(x) \in U'$, there exist an open set U, $x \in U$, such that $f(U) \subset U'$. There are a compact subset K with nonempty interior of $\pi^{-1}(U)$

and a open set U_0 of N such that $U_0 \subset \pi(K) \subset U$. The set $W := \{g \in \mathcal{A}(N) \mid \hat{g}(K) \subset O\}$ where $O := \pi^{-1}(U')$ is an open set of $\mathcal{A}(N)$, and $f \in W$ (see the proof of theorem 3). Then $\Phi(W, U_0) \subset U'$. Thus the group $\mathcal{A}(N)$ continuously acts on N.

Fix $f \in \mathcal{A}^e(N)$. As the group $\mathcal{A}^e(N)$ is connected there exists a continuous path $h\colon [0,1] \to \mathcal{A}^e(N)$ such that $h(0) = id_N$, $h(1) = f$. Let x be a point of a connected component Δ_k^i of a stratum Δ_k. Since the path h and the acting Φ are continuous maps the path $\tilde{h}\colon [0,1] \to N$ defined by the equality $\tilde{h}(t) := \Phi(h(t), x)$ is continuous. As id_N keeps invariant the connected component Δ_k^i and $f(\Delta_k) = \Delta_k$, so $\tilde{h}(t) \in \Delta_k^i$, $\forall t \in [0,1]$. Therefore, the automorphism $h(1) = f$ keeps invariant the connected component Δ_k^i, i.e. $f(\Delta_k^i) = \Delta_k^i$.

Theorem 4. *Let (N, ∇) be a n-dimensional affinely connected orbifold, $\Delta(N) = \{\Delta_k,\ k \in \{0, \ldots, n\}\}$ be the stratification of N.*

1. If $\Delta_k \neq \emptyset$, $k \neq n$, then

$$\dim \mathcal{A}(N) \leq n^2 + n - (n-k)(k+1) < n^2 + n. \tag{8.5}$$

2. The equality $\dim \mathcal{A}(N) = n^2 + n$ is satisfied if and only if (N, ∇) is the ordinary affine space with the flat affine connection.

Proof. 1. According to the equality $\dim \mathcal{A}(N) = \dim \mathcal{A}^e(N)$ it is enough to estimate the dimension of $\mathcal{A}^e(N)$. Suppose that $\Delta_k \neq \emptyset$, $k \neq n$. Fix Δ_k^i. By remark 2 the connection ∇ induced the connection $'\nabla$ on Δ_k^i. Thus $(\Delta_k^i, '\nabla)$ becomes a k-dimensional affinely connected manifold. As it is known $\dim \mathcal{A}(\Delta_k^i) \leq k^2 + k$. According to proposition 8 each automorphism $f \in \mathcal{A}^e(N)$ satisfies to the equality $f(\Delta_k^i) = \Delta_k^i$. So the map $\chi\colon \mathcal{A}^e(N) \to \mathcal{A}(\Delta_k^i)\colon f \mapsto f|_{\Delta_k^i}$ is correctly defined. From to the definitions of Lie group structure on the groups $\mathcal{A}^e(N)$ and $\mathcal{A}(\Delta_k^i)$ it follows that χ is a Lie group homomorphism. From here $\dim \mathcal{A}^e(N) \leq \dim \mathcal{A}(\Delta_k^i) + \dim \ker \chi$. Estimate the dimension of $\ker \chi := \{f \in \mathcal{A}^e(N) \mid f|_{\Delta_k^i} = id_{\Delta_k^i}\}$. Take $f \in \ker \chi$. Suppose that $x \in \Delta_k^i$, $(\mathbf{R}^n, \Gamma, p)$ and $(\mathbf{R}^n, \Gamma', p')$ are linearized charts at x and $\bar{f}\colon \mathbf{R}^n \to \mathbf{R}^n$ is a representative of f in these charts, $\bar{f}(0) = 0$. Without loss of generality, we may assume that $\Gamma = \Gamma'$ and Γ trivially acts on $\mathbf{R}^k \times \{0\}$ and $\bar{f}|_{\mathbf{R}^k \times \{0\}} = id_{\mathbf{R}^k \times \{0\}}$. Then the differential $\bar{f}_{*0}$ of $\bar{f}$ at $0 \in \mathbf{R}^n$ satisfies to the equality $\bar{f}_{*0}|_{\mathbf{R}^k \times \{0\}} = id_{\mathbf{R}^k \times \{0\}}$. Therefore Jacobi matrix of $\bar{f}$ at $0 \in \mathbf{R}^n$ looks like

$$\begin{pmatrix} E & A \\ 0 & B \end{pmatrix} \tag{8.6}$$

where $B \in GL(n-k, \mathbf{R})$, A is a $k \times (n-k)$ matrix, E is the unit of group $GL(k, \mathbf{R})$. Denote by G the Lie subgroup of matrixes of the form (6). Remark that $\dim G = \dim GL(n-k, \mathbf{R}) + k(n-k) = (n-k)^2 + k(n-k) = n(n-k)$. As $\bar{f}$ is an isomorphism of the affine connected manifolds, so if $\bar{f}_{*0} = id_{\mathbf{R}^n}$, there exists an open set $W \ni 0$ such that $\bar{f}|_W = id_W$. As p is open map, then f is equal to id on the open set $p(W)$ of N. By lemma 4 we have $f = id_N$. So the map $\mu\colon \ker \chi \to G\colon f \mapsto \bar{f}_{*0}$ is isomorphism of Lie groups. Furthermore $\dim \ker \chi = \dim G = n(n-k)$. Thus we have $\dim \mathcal{A}(N) = \dim \mathcal{A}^e(N) \leq \dim \mathcal{A}(\Delta_k^i) + \dim \ker \chi \leq k^2 + k + n(n-k) = n^2 + n - (n-k)(k+1)$.

2. The estimate (5) implies that validity of the equality $\dim \mathcal{A}(N) = n^2 + n$ necessitates that only Δ_n in nonempty, i. e. N is a manifold. As it is well known, an affinely connected n-dimensional manifold N has the automorphism group $\mathcal{A}(N)$ of $(n^2 + n)$-dimension if and only if N is the affine space $\mathbf{R}^n$ with the flat affine connection.

Corollary 2. *Let (N, ∇) be a n-dimensional affinely connected orbifold.*

1. If N is a proper orbifold then $\dim \mathcal{A}(N) \leq n^2$.

2. The equality $\dim \mathcal{A}(N) = n^2$ *implies* $\Delta_k = \emptyset$ *for all* $k \in \{1, \dots, n-2\}$.

3. If $\dim \mathcal{A}(N) > n^2$ *then* N *is an affinely connected* n*-dimensional manifold with zero torsion.*

4. If $\dim \mathcal{A}(N) > n^2$ *and* $n \geq 4$ *then* N *is the ordinary affine space with affine connection.*

Proof. Given n and $k \in \{0, \dots, n-1\}$, note that the function $\tau(k) = n^2+n-(n-k)(k+1)$ attains its maximum equal to n^2 at $k = 0$ and $k = n-1$. Therefore, for a proper orbifold N we have the inequality $\dim \mathcal{A}(N) \leq n^2$; moreover, the equality $\dim \mathcal{A}(N) = n^2$ necessitates $\Delta_k = \emptyset$ for all $k \in \{1, \dots, n-2\}$.

According to item 1 the inequality $\dim \mathcal{A}(N) > n^2$ implies $\Delta_k = \emptyset$, $\forall k \in \{0, \dots, n-1\}$. Hence, (N, ∇) is an affinely connected manifold. By theorem 1.3 of chapter 4 [3], we receive that connection ∇ has a zero torsion. Moreover, if $n \geq 4$ then theorem 1.4 of chapter 4 from [3] implies that N is the ordinary affine space $\mathbf{R}^n$ with the flat affine connection.

Some special estimates. Let N be a smooth orbifold, $\Delta(N) = \{\Delta_k, k \in \{0, \dots, n\}\}$, be its stratification. We say that a *connected component* Δ_k^i *of a stratum* Δ_k *does not adjoin to a stratum of a greater dimension,* if the closure $\overline{\Delta_m}$ of the stratum Δ_m, $\forall \Delta_m \in \Delta(N)$, $m < n$, does not contain Δ_k^i. Note that each connected component of Δ_{n-1} satisfies to this condition.

Proposition 9. *Let* (N, ∇) *be a* n*-dimensional affinely connected orbifold.*

1.If there exists a connected component Δ_k^i *of* Δ_k *which does not adjoin to a stratum of a greater dimension, then*

$$\dim \mathcal{A}(N) \leq n^2 + n - (n-k)(2k+1); \tag{8.7}$$

moreover, if $\Delta_k^i \neq \overline{\Delta_k^i}$, *then*

$$\dim \mathcal{A}(N) \leq n^2 + n - (n-k)(2k+1) - k. \tag{8.8}$$

2. The estimates (8.7) and (8.8) are exact.

Proof. 1. In the proof of theorem 4 we have defined the Lie group homomorphism $\chi\colon \mathcal{A}^e(N) \to \mathcal{A}(\Delta_k^i)\colon f \mapsto f|_{\Delta_k^i}$. We have gotten $\dim \mathcal{A}(N) = \dim \mathcal{A}^e(N) \leq \dim \mathcal{A}(\Delta_k^i) + \dim \ker \chi$. Estimate $\dim \ker \chi$.

Let $f \in \ker \chi$, $x \in \Delta_k^i$. Let $(\mathbf{R}^n, \Gamma, p)$ and $(\mathbf{R}^n, \Gamma', p')$ be linearized charts at x and $\bar{f}\colon \mathbf{R}^n \to \mathbf{R}^n$ is a representative of f in these charts, $\bar{f}(0) = 0$. Without loss of generality, we may assume that $\Gamma = \Gamma'$ and $Fix\Gamma = \mathbf{R}^k \times \{0\}$ and $\bar{f}|_{\mathbf{R}^k \times \{0\}} = id_{\mathbf{R}^k \times \{0\}}$. Jacobi matrix of a transformation $\gamma \in \Gamma$ at $0 \in \mathbf{R}^n$ is an orthogonal matrix

$$\begin{pmatrix} E & 0 \\ 0 & C \end{pmatrix}$$

where E is the unit of orthogonal group $O(k, \mathbf{R})$, $C \in O(n-k, \mathbf{R})$. According to the proof of theorem 4, Jacobi matrix of $\bar{f}$ at $0 \in \mathbf{R}^n$ looks like (6). As for an element $\gamma \in \Gamma$ there exists $\gamma' \in \Gamma$ such that $\bar{f} \circ \gamma = \gamma' \circ \bar{f}$, then $\bar{f}_{*0} \circ \gamma_{*0} = \gamma'_{*0} \circ \bar{f}_{*0}$. This implies

$$\begin{pmatrix} E & A \\ 0 & B \end{pmatrix} \begin{pmatrix} E & 0 \\ 0 & C \end{pmatrix} = \begin{pmatrix} E & 0 \\ 0 & C' \end{pmatrix} \begin{pmatrix} E & A \\ 0 & B \end{pmatrix}$$

where $C, C' \in O(n-k, \mathbf{R})$. Hence we receive $AC = A$ or $C^tA^t = A^t$ for all $C \in \tilde{\Gamma} := \{C \in O(n-k, \mathbf{R}) \mid C$ is determined by $\gamma_{*0}, \gamma \in \Gamma\}$, where A^t, C^t are corresponding transposed matrixes. So the lines of the matrix $A = (a_{ij})$ are formed by the vectors $a_i = (a_{i1}, \dots, a_{in-k})$ which are fixed by transformations of the group $\tilde{\Gamma}$. Suppose that there are a vector $X \neq 0$ from the orthogonal complement V to the tangent vector space $T_0(Fix\Gamma)$ for manifold $Fix\Gamma$ at $0 \in \mathbf{R}^n$ and a transformation $\gamma_0 \in \Gamma$, $\gamma_0 \neq id_{\mathbf{R}^n}$, such that $\gamma_{*0}(X) = X$. Let $\Gamma_0 \subset \Gamma$ be the group generated by the element γ_0. As $\Gamma_0 \subset \Gamma$, so $Fix\Gamma \subset Fix\Gamma_0$. Since $X \in V$, we have $\Gamma_0 \neq \Gamma$ and $\dim Fix\Gamma < \dim Fix\Gamma_0$. Then there exists a connected component Δ^l_m of a stratum Δ_m, corresponding to Γ_0, where $m := \dim Fix\Gamma_0$, k, m, n, such that $\overline{\Delta^l_m} \supset \Delta^i_k$. It is opposite to the assumption of proposition 9: Δ^i_k does not adjoin to a stratum of a greater dimension. Hence $X = 0$. Then each element $C \in \tilde{\Gamma}$ keeps only zero vector. Therefore $A = 0$ and $\dim\ker\chi \leq \dim GL(n-k, \mathbf{R}) = (n-k)^2$. So we have $\dim \mathcal{A}(N) = \dim \mathcal{A}^e(N) \leq \dim \mathcal{A}(\Delta^i_k) + \dim\ker\chi \leq k^2+k+(n-k)^2 = n^2+n-(2k+1)(n-k)$.

Let $\Delta^i_k \neq \overline{\Delta^i_k}$. By theorem 1 the subspace $\overline{\Delta^i_k}$ of N is a k-dimensional C^r-orbifold. According to remark 2 the connection ∇ induces the connection $''\nabla$ on the orbifold $\overline{\Delta^i_k}$. Thus $(\overline{\Delta^i_k}, ''\nabla)$ is an affinely connected orbifold. Theorem 4 implies that the automorphism group $\mathcal{A}(\overline{\Delta^i_k})$ of the affinely connected orbifold $(\overline{\Delta^i_k}, ''\nabla)$ is a Lie group, and $\dim \mathcal{A}(\overline{\Delta^i_k}) \leq k^2+k$. Since $\Delta^i_k \neq \overline{\Delta^i_k}$ then $\overline{\Delta^i_k}$ is a proper orbifold and hence by corollary 2 it follows $\dim \mathcal{A}(\overline{\Delta^i_k}) \leq k^2$. According to proposition 8 for each $f \in \mathcal{A}^e(N)$ the equality $f(\Delta^i_k) = \Delta^i_k$ is satisfied. By continuously of f we have $f(\overline{\Delta^i_k}) = \overline{\Delta^i_k}$. So the map $\bar{\chi}\colon \mathcal{A}^e(N) \to \mathcal{A}(\overline{\Delta^i_k})\colon f \mapsto f|_{\overline{\Delta^i_k}}$ is correctly defined. Using the definitions of Lie group structure on the groups $\mathcal{A}^e(N)$ and $\mathcal{A}(\Delta^i_k)$ we receive that $\bar{\chi}$ is a Lie group homomorphism. Let $\chi\colon \mathcal{A}^e(N) \to \mathcal{A}(\Delta^i_k)$ be the above defined homomorphism of the Lie groups. Denote by $\mathcal{A}_N(\Delta^i_k)$ the image $im\chi$ of homomorphism χ and denote by $\mathcal{A}_N(\overline{\Delta^i_k})$ the image $im\bar{\chi}$ of homomorphism $\bar{\chi}$. Since Δ^i_k is the set of regular points of $\overline{\Delta^i_k}$ which is everywhere dense in $\overline{\Delta^i_k}$, then the homomorphism $\psi\colon \mathcal{A}_N(\overline{\Delta^i_k}) \to \mathcal{A}_N(\Delta^i_k)\colon f \mapsto f|_{\Delta^i_k}$ is an monomorphism. Obviously ψ is an epimorphism. Thus the groups $\mathcal{A}_N(\Delta^i_k)$ and $\mathcal{A}_N(\overline{\Delta^i_k})$ are isomorphic. A corresponding check shows that ψ is a Lie group isomorphism. Then $\dim \mathcal{A}_N(\Delta^i_k) = \dim \mathcal{A}_N(\overline{\Delta^i_k}) \leq \dim \mathcal{A}(\overline{\Delta^i_k}) \leq k^2$. Applying the estimate of the dimension of $\ker\chi$ and the inequality $\dim \mathcal{A}^e(N) \leq \dim im\chi + \dim\ker\chi$ we have $\dim \mathcal{A}(N) = \dim \mathcal{A}^e(N) \leq \dim \mathcal{A}_N(\Delta^i_k) + \dim\ker\chi \leq k^2 + (n-k)^2 = n^2 + n - (2k+1)(n-k) - k$.

2. The precision of the estimates (7) and (8) follows from the next examples.

Example 5. Let γ be the reflection of $\mathbf{R}^n$ respective to the subspace $\mathbf{R}^{n-1} \times \{0\}$ given by the matrix

$$C = \begin{pmatrix} E & 0 \\ 0 & -1 \end{pmatrix}$$

where E is the unit of the orthogonal group $O(n-1, \mathbf{R})$, $n > 1$. Let Γ be a group generated by γ. Then the group $\Gamma \cong \mathbf{Z}_2$ acts on $\mathbf{R}^n$ and the quotient space $N_1 := \mathbf{R}^n/\Gamma$ is a n-dimensional orbifold. Since the group Γ fixed the points $(x_1, \dots, x_{n-1}, 0) \in \mathbf{R}^n$ and only them, the stratification of N_1 looks like $\Delta(N_1) = \{\Delta_n, \Delta_{n-1}\}$. The structure of the affine space $\mathbf{R}^n$ induces the flat affine structure on N_1. Thus N_1 is an affine connected orbifold with flat affine connection $\nabla^{(1)}$. Calculate the dimension of transformation group $\mathcal{A}(N_1)$ of $(N_1, \nabla^{(1)})$. According to proposition 7 the group $\mathcal{A}(N_1)$ is isomorphic to the factor-group $\mathbf{N}(\Gamma)/\Gamma$ where $\mathbf{N}(\Gamma)$ is normalizater of group of Γ in the group $Aff(\mathbf{R}^n)$ of all affine transformations of the affine space $\mathbf{R}^n$. The group $Aff(\mathbf{R}^n)$ is semidirect product of the linear group $GL(n, \mathbf{R})$ and the shift group $\mathbf{R}^n$. Therefore a transformation of $Aff(\mathbf{R}^n)$

is demonstrated by a pair $\langle A, a\rangle$ where $A \in GL(n, \mathbf{R})$, $a \in \mathbf{R}^n$, and

$$\langle A, a\rangle \cdot \langle B, b\rangle := \langle AB, Ab + a\rangle, \qquad \langle A, a\rangle, \langle B, b\rangle \in Aff(\mathbf{R}^n).$$

Then the transformation γ can be submitted as $\langle C, 0\rangle$ where $0 = (0, \dots, 0) \in \mathbf{R}^n$. Since $\mathbf{N}(\Gamma) := \{\langle A, a\rangle \in Aff(\mathbf{R}^n) \mid \langle A, a\rangle \cdot \langle C, 0\rangle = \langle C, 0\rangle \cdot \langle A, a\rangle\}$, we receive that $\langle A, a\rangle \in \mathbf{N}(\Gamma)$ if and only if

$$A = \begin{pmatrix} A' & 0 \\ 0 & a_{nn} \end{pmatrix}, \; A' \in GL(n-1, \mathbf{R}), \qquad a = (a_1, \dots, a_{n-1}, 0) \in \mathbf{R}^n.$$

As the group Γ is finite, we have $\dim \mathcal{A}(N_1) = \dim \mathbf{N}(\Gamma) = \dim GL(n-1, \mathbf{R}) + 1 + n - 1 = (n-1)^2 + n = n^2 - n + 1$.

Example 6. Put

$$C_1 := \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; C_2 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The finite group Γ generated by C_1 and C_2 acts on $\mathbf{R}^3$ as a subgroup of $GL(3, \mathbf{R})$. The factor-space $N_2 := \mathbf{R}^3/\Gamma$ is a 3-dimensional orbifold. The subgroup $\Gamma_1 \subset \Gamma$ generated by C_1 fixes points of the axis Oz; the subgroup $\Gamma_2 \subset \Gamma$ generated by C_2 fixes points of the plane Oxy; the group Γ fixes only point $0 = (0,0,0) \in \mathbf{R}^3$. Therefore the stratification of N_2 looks like $\Delta(N_2) = \{\Delta_3, \Delta_2, \Delta_1, \Delta_0\}$. The flat affine connection $\nabla^{(2)}$ on N_2 is induced by the affine connection of the affine space $\mathbf{R}^3$. Calculate the dimension of the transformation group $\mathcal{A}(N_2)$ of the affine connected orbifold $(N_2, \nabla^{(2)})$. By proposition 7 $\mathcal{A}(N_2) \cong \mathbf{N}(\Gamma)/\Gamma$. A direct check shows that $\langle A, a\rangle \in \mathbf{N}(\Gamma)$ if and only if

$$A = \begin{pmatrix} A' & 0 \\ 0 & a_{33} \end{pmatrix}, \; A' \in GL(2, \mathbf{R}), \, a_{33} \in \mathbf{R} \setminus \{0\}, \qquad a = (0,0,0) \in \mathbf{R}^3.$$

So $\dim \mathcal{A}(N_2) = \dim \mathbf{N}(\Gamma) = \dim GL(2, \mathbf{R}) + \dim(\mathbf{R} \setminus \{0\}) = 5$.

Thus examples 5 and 6 imply that the estimates (7) and (8) are exact.

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SOME PECULIARITIES OF QUANTUM-MECHANICAL SCATTERING IN THE LOBACHEVSKY SPACE

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The formulation of the quantum-mechanical scattering problem in the three-dimensional Lobachevsky space is presented. The quantum mechanical problem of the scattering by spherical potential well in the Lobachevsky space is considered. The graphical solution of the problem of bound s-states is given and the influence of curvature on the number of the bound states is investigated. The scattering by the Coulomb center is treated on the basis of exact solution of Schrödinger equation. An expression for the differential cross section is obtained.

1. Introduction

Quantum-mechanical problems in the spaces of a constant positive and negative curvature are the object of interest of researchers since 1940, when Schrödinger [1] was first solved the quantum-mechanical problem about the atom on the three-dimensional sphere (Einstein's Universe). The analogous problem in the three-dimensional Lobachevsky space was first solved by Infeld and Shild [2]. In recent years the quantum-mechanical models based on the geometry of the spaces of constant curvature have attracted considerable attention due to their interesting mathematical features [3, 4, 6] as well as the possibility of application to the physical problems [5]. For example, these models are used for the description of the bound states in nuclear and elementary particle physics [3]. Thus, Kepler problem on the sphere S_3 has been used as a model for description of quarkonium spectra [7]. Kepler – Coulomb problem on the sphere S_3 has been used as a model for description of excitations in quantum dots [8, 9]. Many aspects of this problem in spaces of constant curvature, in particular separation of variables and path integral formulation, have been investigated in the papers [10]–[12]. However, until now,the problem of potential scattering in spaces of a constant curvature was not formulated.

The important problem with the formulation of the scattering problem in the three-dimensional Lobachevsky space was the choice of expression for the incident wave. The use of plane wave of Shapiro related to the representations of the group of motions of Lobachevsky's space, made it possible to formulate and to solve the scattering problem on the Coulomb center [13]. In this paper the formulation of the quantum-mechanical scattering problem in the three-dimensional Lobachevsky space is considered with potential well as the model potential. The graphical solution of the problem of bound s-states is given. The influence of curvature on the number of the bound states is investigated.

2. The formulation of the problem

We use embedding of the Lobachevsky space in 4-dimensional pseudoeuclidean space with coordinates x_μ, $\mu = 1, 2, 3, 4$, given by formulas

$$x_\mu x_\mu = \mathbf{x}^2 + x_4^2 = \mathbf{x}^2 - x_0^2 = -\rho^2, \ \mathbf{x} = \{x_1, x_2, x_3\}, \qquad x_4 = \mathrm{i}x_0. \tag{2.1}$$

Schrödinger equation is $(\hbar = m = 1)$:

$$H\Psi = E\Psi, \ H = \frac{1}{4\rho^2} M_{\mu\nu} M_{\mu\nu} + U, \ M_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu, \tag{2.2}$$

where U is a potential energy.

The scattering solution of the Schrödinger equation behaves at large distances like the superposition of an incident wave and scattered spherical waves. In the flat space the plane wave is considered as the incident wave. In the Lobachevsky space the Schrödinger equation does not have plane wave solutions. The solution of the free equation of Schrödinger of the form closest in its properties to the plane wave as (see [14, 15])

$$\xi(x, \mathrm{n}) = \left(\frac{x_0 - \mathbf{xn}}{\rho}\right)^{-1-\mathrm{i}\eta}, \ \eta = \sqrt{2E\rho^2 - 1}, \tag{2.3}$$

where $\mathbf{n}$ is a unit vector that defines the direction of wave propagation in the Lobachevsky space.

The spherical wave is considered as the scattered wave. In flat space this is the outgoing wave, having at large distances r from the center form $f(\theta)\exp(\mathrm{i}kr)/r$. In the Lobachevsky space the Schrödinger equation also has solutions of the form of spherical wave. These solutions can be found by using spherical coordinates

$$\begin{gathered} x_0 = \rho\cosh\beta, x_1 = \rho\sinh\beta\sin\theta\cos\varphi, \\ x_2 = \rho\sinh\beta\sin\theta\sin\varphi, x_3 = \rho\sinh\beta\cos\theta, \\ 0 \le \beta < \infty, \ 0 \le \theta \le \pi, \ 0 \le \varphi < 2\pi. \end{gathered} \tag{2.4}$$

Separating in the solution of the Schrodinger equation dependence on the angles θ and φ by the use of spherical harmonics in the form $\Psi = R_l(\beta) Y_l^m(\theta, \varphi)$, we obtain in the case of $U = 0$ the radial equation

$$\left[\frac{1}{2\rho^2}\left(-\frac{1}{\sinh^2\beta}\frac{\mathrm{d}}{\mathrm{d}\beta}\sinh^2\beta\frac{\mathrm{d}}{\mathrm{d}\beta} + \frac{l(l+1)}{\sinh^2\beta}\right) - E\right] R_l(\beta) = 0. \tag{2.5}$$

The regular solution at $\beta = 0$ of this equation is

$$S_{\eta l}(\beta) = \sqrt{\frac{\pi}{2\sinh\beta}} \frac{\Gamma(\mathrm{i}\eta + l + 1)}{\Gamma(\mathrm{i}\eta + 1)} P^{-\frac{1}{2}-l}_{-\frac{1}{2}+\mathrm{i}\eta}(\cosh\beta). \tag{2.6}$$

The asymptotic form of the solution $S_{\eta l}$ for $\beta \to \infty$ is given by the expression

$$S_{\eta l}(\beta) \approx \frac{1}{2\mathrm{i}\eta\sinh\beta}\left[\mathrm{e}^{\mathrm{i}\eta\beta} - \frac{\Gamma(\mathrm{i}\eta + l + 1)\Gamma(1 - \mathrm{i}\eta)}{\Gamma(l - \mathrm{i}\eta + 1)\Gamma(\mathrm{i}\eta + 1)}\mathrm{e}^{-\mathrm{i}\eta\beta}\right]. \tag{2.7}$$

The solution of equation (2.5) which outgoing spherical wave in the Lobachevsky space is describes on

$$C_{\eta l}(\beta) = \sqrt{\frac{\pi}{2\sinh\beta}}\frac{1}{2}\left[\frac{\Gamma(\mathrm{i}\eta + l + 1)}{\Gamma(\mathrm{i}\eta + 1)}P^{-\frac{1}{2}-l}_{-\frac{1}{2}+\mathrm{i}\eta}(\cosh\beta) + \frac{\Gamma(\mathrm{i}\eta - l)}{\Gamma(\mathrm{i}\eta + 1)}P^{\frac{1}{2}+l}_{-\frac{1}{2}+\mathrm{i}\eta}(\cosh\beta)\right]. \quad (2.8)$$

When $\beta \to \infty$ we have

$$C_{\eta l}(\beta) \approx \frac{1}{2\mathrm{i}\eta\sinh\beta}\mathrm{e}^{\mathrm{i}\eta\beta}. \quad (2.9)$$

We choose vector $\mathbf{n}$ in (2.3) in the form $\mathbf{n} = (0, 0, 1)$. Then the incident wave can be written as

$$\xi(\beta, \theta) = (\cosh\beta - \sinh\beta\cos\theta)^{-1-\mathrm{i}\eta}. \quad (2.10)$$

The incident wave (2.10) can be expressed through the spherical waves (2.6) (see, for example, [14, 15])

$$\xi(\beta, \theta) = \sum_{l=0}^{\infty}(2l + 1)S_{\eta l}(\beta)P_l(\cos\theta). \quad (2.11)$$

The exact wave function which is the solution of the Schrödinger equation with the potential energy $U(\beta)$ takes for $\beta \to \infty$ the form

$$\Psi \approx (\cosh\beta - \sinh\beta\cos\theta)^{-1-\mathrm{i}\eta} + \frac{f(\theta)}{\rho\sinh\beta}e^{\mathrm{i}\eta\beta}. \quad (2.12)$$

Here the function $f(\theta)$ plays the role of the scattering amplitude.

3. Scattering by spherical potential well

As an example let us consider particle scattering on the spherical symmetrical potential well. Let us assign the potential of the well as follows

$$U = \begin{cases} 0 & \text{for} \quad \beta \geq a, \\ -U_0 & \text{for} \quad \beta < a, \end{cases} \quad (3.1)$$

where constant a is the radius of the well.

As the incident wave we will consider the wave of form (2.3). Let $\varphi(\beta, \theta)$ to be wave for $\beta \leq a$ and $\chi(\vec{x}, \vec{n_3})$ is a scattered wave.

We use for the incident wave expansion in series (2.11). The wave inside of the potential well we can expressed as

$$\varphi(\beta, \theta) = \sum_{l=0}^{\infty} A_l S_{\eta' l}(\beta)P_l(\cos\theta), \quad (3.2)$$

where

$$\eta' = \sqrt{2(E + U_0)\rho^2 - 1}. \quad (3.3)$$

The scattered wave is

$$\chi(\beta, \theta) = \sum_{l=0}^{\infty} B_l C_{\eta l}(\beta)P_l(\cos\theta). \quad (3.4)$$

The coefficients of the expansion of A_l and B_l can be determined from the continuity condition of wave functions and their derivatives at the boundary of the well $\beta = \alpha$, which reduces to the following system of linear equations:

$$p_{\eta l}(\cosh a) + q_{\eta l}(\cosh a)B_l = p_{\eta' l}(\cosh a)A_l,$$
$$p'_{\eta l}(\cosh a) + q'_{\eta l}(\cosh a)B_l = p'_{\eta' l}(\cosh a)A_l, \tag{3.5}$$

where we introduce the following notation

$$p_{\eta l}(\cosh\beta) = \sqrt{\frac{2\sinh\beta}{\pi}}S_{\eta l},$$

$$q_{\eta l}(\cosh\beta) = \sqrt{\frac{2\sinh\beta}{\pi}}C_{\eta l} = \frac{1}{2}(s_{\eta l}(\cosh\beta) + s_{\eta,-l-1}(\cosh\beta)),$$

$$p'_{\eta l}(\cosh\beta) = \frac{\mathrm{d}p_{\eta l}(\cosh\beta)}{\mathrm{d}\beta}, \qquad q'_{\eta l}(\cosh\beta) = \frac{\mathrm{d}q_{\eta l}(\cosh\beta)}{\mathrm{d}\beta}.$$

As a result, we obtain for the coefficients B_l and A_l expressions

$$B_l = \frac{p_{\eta' l}(\cosh a)p'_{\eta l}(\cosh a) - p_{\eta l}(\cosh a)p'_{\eta' l}(\cosh a)}{q_{\eta l}(\cosh a)p'_{\eta' l}(\cosh a) - p_{\eta' l}(\cosh a)q'_{\eta l}(\cosh a)}, \tag{3.6}$$

$$A_l = \frac{q_{\eta l}(\cosh a)p'_{\eta l}(\cosh a) - q'_{\eta l}(\cosh a)p_{\eta l}(\cosh a)}{q_{\eta l}(\cosh a)p'_{\eta' l}(\cosh a) - p_{\eta' l}(\cosh a)q'_{\eta l}(\cosh a)}. \tag{3.7}$$

Thus for the scattering amplitude we have an expression

$$f(\theta) = \frac{\rho}{2\mathrm{i}\eta}\sum_{\ell=0}^{\infty} B_l P_l(\cos\theta) \tag{3.8}$$

The poles of B_l in the range of negative energies determine the bound states in the well.

4. The case of the s-states

In particular, when $l = 0$ we have

$$\sqrt{k^2-1}\cot(a\sqrt{k^2-1}) = -\sqrt{\lambda^2+1} = -\sqrt{2\rho^2 U_0 - k^2 + 1}, \tag{4.1}$$

where

$$\lambda = \sqrt{2\rho^2\varepsilon}, k = \sqrt{2\rho^2(U_0-\varepsilon)}, \varepsilon = -E. \tag{4.2}$$

This equation are determines the energy levels of the system.

Let us introduce the variables $\zeta = a\sqrt{k^2-1} \geq 0$ and $\tau = a\sqrt{\lambda^2+1} \geq 0$. Then we obtain

$$\tau = -\zeta\cot\zeta, \qquad \tau^2 + \zeta^2 = 2\rho^2 U_0 a^2. \tag{4.3}$$

The equations (4.3) can be solved is numerically or graphically. Values ζ and γ, which satisfy equations (4.3) are determined by point of intersections of the curve $\gamma = -\zeta\cot\zeta$ with the circle of radius $\rho a\sqrt{2U_0}$.

The curves are represented on the figures 1, 2 (see Section 6). We see that there are such values of curvature for which no stationary states exist. But with increase of the radius curvature ρ appear bound states, number of which rises with the increase of ρ. Also from figures 1, 2 we can observe that with growth of the U_0 the number of bound states increases.

5. Coulomb Scattering in the Lobachevsky space

The Coulomb potential in the Lobachevsky space is given by

$$U = -\frac{\alpha}{\rho} \frac{x_0}{|\mathbf{x}|}, \tag{5.1}$$

where α is a positive constant (we consider of the Coulomb attraction).

In order to find a solution of Schrödinger equation which behaves at large distances like the superposition of an incident plane wave and scattered spherical waves, we use an analog of a parabolic coordinate system [16]

$$\begin{gathered} x_0 = \rho \frac{2 - t_1 - t_2}{2\sqrt{(1-t_1)(1-t_2)}}, x_1 = \rho\sqrt{-t_1 t_2} \cos\varphi, \\ x_2 = \rho\sqrt{-t_1 t_2} \sin\varphi, x_3 = \rho \frac{t_1 + t_2 - 2t_1 t_2}{2\sqrt{(1-t_1)(1-t_2)}}, \\ 0 < t_1 < 1, \ -\infty < t_2 < 0, \ 0 \leq \varphi < 2\pi. \end{gathered} \tag{5.2}$$

Hamiltonian with the Coulomb potential (5.1) takes in these coordinates the form

$$H = \frac{1}{\rho^2}\left[2\frac{1-t_1}{t_2 - t_1}\frac{\partial}{\partial t_1} t_1(1-t_1)\frac{\partial}{\partial t_1} + 2\frac{1-t_2}{t_1-t_2}\frac{\partial}{\partial t_2} t_2(1-t_2)\frac{\partial}{\partial t_2} + \frac{1}{2t_1 t_2}\frac{\partial^2}{\partial \varphi^2}\right] - \\ -\frac{\alpha}{\rho}\frac{2 - t_1 - t_2}{t_1 - t_2}. \tag{5.3}$$

Due to the axial symmetry of the problem, it is sufficient to consider solution of the Schrödinger equation with no dependence on φ, that is solution of the form $\Psi(t_1, t_2) = S_1(t_1) S_2(t_2)$. Substituting this expression into the Schrödinger equation, we find equations for S_1 and S_2

$$(1-t_1)\frac{d}{dt_1}(1-t_1)t_1\frac{dS_1}{dt_1} - \left(\frac{E\rho^2 - \alpha\rho}{2} t_1 + \kappa_1\right) S_1 = 0, \tag{5.4}$$

$$(1-t_2)\frac{d}{dt_2}(1-t_2)t_2\frac{dS_2}{dt_2} - \left(\frac{E\rho^2 + \alpha\rho}{2} t_2 + \kappa_2\right) S_1 = 0, \tag{5.5}$$

where separation constants κ_1 and κ_2 obey the relation $\kappa_1 - \kappa_2 = \alpha\rho$.

Choice of the vector $\mathbf{n} = \{0, 0, 1\}$ means that for $x_3 \to -\infty$ (for $t_2 \to -\infty$) solution of Schrödinger equation with Coulomb potential must tend to solution of equation without potential which describes an incident wave. It is possible if dependence on the coordinate t_1 is taken in the form

$$S_1(t_1) = (1-t_1)^{-i\gamma_- - 1/2}, \tag{5.6}$$

where $\gamma_\pm = \sqrt{(E\rho^2 \pm \alpha\rho)/2 - 1/4}$. This expression corresponds to $\kappa_1 = -i\gamma_- - 1/2$. Then

$$S_2(t_2) = (1-t_2)^{-i\gamma_- - 1/2} \times {}_2F_1(i\gamma_+ - i\gamma_-, -i\gamma_+ - i\gamma_-; 1; t_2) \tag{5.7}$$

is a solution of equation (5.5) with an appropriate behavior.

Solution of the Schrödinger equation can be written in the form $\Psi = AS_1(t_1)S_2(t_2)$, where A is a constant. Using the known asymptotic behavior of the hypergeometric function [17] we find that for $t_2 \to -\infty$

$$\Psi \approx A\frac{\Gamma(-2i\gamma_+)}{\Gamma(1-i\gamma_+ + i\gamma_-)\Gamma(-i\gamma_+ - i\gamma_-)} \times [(1-t_1)(1-t_2)]^{-i\gamma_- -1/2}(-t_2)^{i\gamma_- - i\gamma_+} +$$
$$+A\frac{\Gamma(2i\gamma_+)}{\Gamma(1+i\gamma_+ + i\gamma_-)\Gamma(i\gamma_+ - i\gamma_-)} \times \left(\frac{1-t_2}{1-t_1}\right)^{i\gamma_- +1/2} (-t_2)^{i\gamma_+ - i\gamma_- -1}. \quad (5.8)$$

Since

$$[(1-t_1)(1-t_2)]^{-i\gamma_- -1/2} = [(x_0 - x_3)/\rho]^{-1-2i\gamma_-}$$

we can conclude that the first term in (5.8) describes an incident wave distorted by Coulomb interaction (for $\alpha = 0$ we have $\gamma_+ = \gamma_- = \gamma$ and we arrive at expression (2.3) with $\mathbf{n} = \{0,0,1\}$).

If we set constant A as

$$A = \frac{\Gamma(1-i\gamma_+ + i\gamma_-)\Gamma(i\gamma_+ + i\gamma_-)}{\Gamma(2i\gamma_+)},$$

then the incident wave will have a unity amplitude. The second term in (5.8) that describes a scattered wave for $\beta \to \infty$ can now be written in the form

$$\frac{1}{\rho\sinh\beta}e^{2i\gamma_+\beta}f(\theta),$$

where

$$f(\theta) = \frac{\rho(\gamma_+ - \gamma_-)}{(\gamma_+ + \gamma_-)}\frac{\Gamma(1-i\gamma_+ + i\gamma_-)}{\Gamma(1+i\gamma_+ - i\gamma_-)} \times 2^{-i(\gamma_+ - \gamma_-)}(1-\cos\theta)^{i\gamma_+ - i\gamma_- -1} \quad (5.9)$$

is the scattering amplitude. Finally we arrive at the expression for scattering cross section

$$d\sigma = \frac{\rho^2(\gamma_+ - \gamma_-)^2}{(\gamma_+ + \gamma_-)^2}\frac{d\Omega}{4\sin^4(\theta/2)}.$$

For large ρ we have an approximate expression

$$d\sigma = \left(\frac{\alpha^2}{E^2} + \frac{\alpha^4 + 2\alpha^2 E}{2\rho^2 E^4}\right)\frac{d\Omega}{16\sin^4(\theta/2)}.$$

Partial wave expansion of the scattering amplitude (5.9) is given by

$$f(\theta) = \frac{\rho}{2i(\gamma_+ + \gamma_-)} \times \sum_l (2l+1)\frac{\Gamma(1-i\gamma_+ + i\gamma_- + l)}{\Gamma(1+i\gamma_+ - i\gamma_- + l)}P_l(\cos\theta). \quad (5.10)$$

Scattering amplitude (5.9) has poles at the values of energy defined by relation $-i\gamma_+ + i\gamma_- = -n$, $n = 1, 2, \ldots$. These values correspond to the discrete energy levels of a particle in an attractive Coulomb field in Lobachevsky space.

6. Figures and Tables

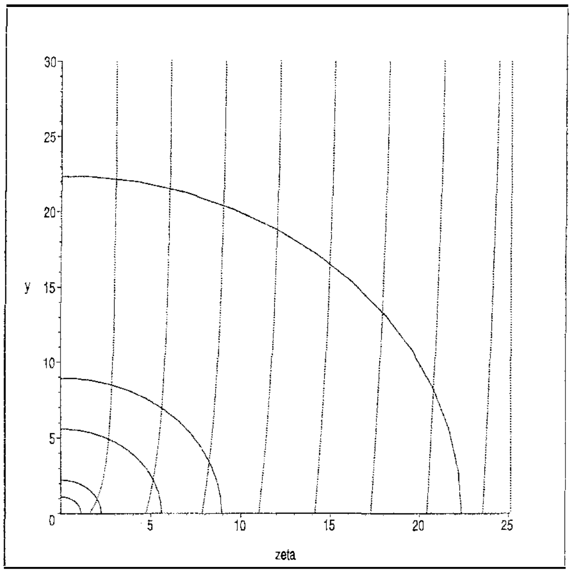

Fig. 1. The graphical solution of equations $\tau = -\zeta \cot \zeta$ and $\tau^2 + \zeta^2 = 2\rho^2 U_0 a^2$, with $a = 0.005, U_0 = 10, \rho = 50, 100, 250, 400, 1000$.

Table 1. The number of bound states depend upon radius of space of curvature. With $a = 0.005$, $U_0 = 10$.

Value ρ	*Value* ε
$\rho = 50$	–
$\rho = 100$	$\varepsilon_1 = 1.8628$
$\rho = 250$	$\varepsilon_1 = 1.5990$, $\varepsilon_2 = 7.7560$
$\rho = 400$	$\varepsilon_1 = 1.4912$, $\varepsilon_2 = 6.0718$, $\varepsilon_3 = 9.0054$
$\rho = 1000$	$\varepsilon_1 = 1.35$, $\varepsilon_2 = 3.5775$, $\varepsilon_3 = 5.5154$, $\varepsilon_4 = 7.12$, $\varepsilon_5 = 8.3763$ $\varepsilon_6 = 9.2773$, $\varepsilon_7 = 9.8192$

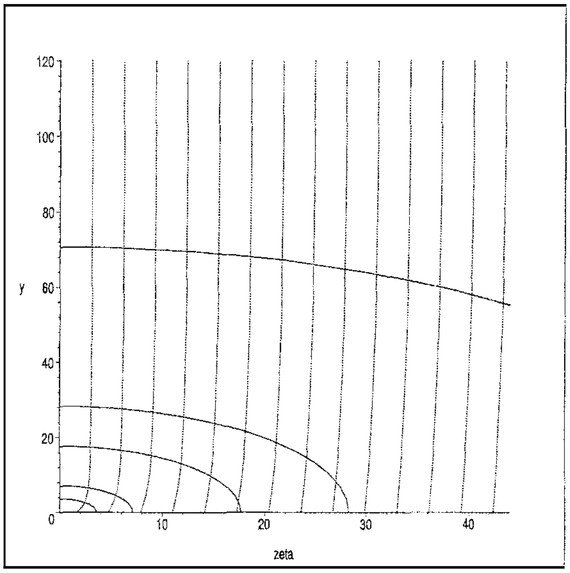

Fig. 2. The graphical solution of equations $\tau = -\zeta \cot \zeta$ and $\tau^2 + \zeta^2 = 2\rho^2 U_0 a^2$, with $a = 0.005, U_0 = 100, \rho = 50, 100, 250, 400, 1000$.

Table 2. The number of bound states depend upon radius of space of curvature. With $a = 0.005,\ U_0 = 100$.

$Value\ \rho$	$Value\ \varepsilon$
$\rho = 50$	$\varepsilon_1 = 54.0462$
$\rho = 100$	$\varepsilon_1 = 41.4281,\ \varepsilon_2 = 84.9497$
$\rho = 250$	$\varepsilon_1 = 2.6466,\ \varepsilon_2 = 30.6285,\ \varepsilon_3 = 55.1936,\ \varepsilon_4 = 74.6678$ $\varepsilon_5 = 97.1223$
$\rho = 400$	$\varepsilon_1 = 8.8385,\ \varepsilon_2 = 27.3263,\ \varepsilon_3 = 44.1024,\ \varepsilon_4 = 71.3362$ $\varepsilon_5 = 81.6279,\ \varepsilon_6 = 89.6548,\ \varepsilon_7 = 95.3989,\ \varepsilon_8 = 98.8492$
$\rho = 1000$	$\varepsilon_1 = 7.98,\ \varepsilon_2 = 15.983,\ \varepsilon_3 = 23.6908,\ \varepsilon_4 = 31.0612$ $\varepsilon_5 = 38.0774,\ \varepsilon_6 = 51.0143\ \varepsilon_7 = 9.8192,\ \varepsilon_8 = 56.9256$ $\varepsilon_9 = 62.4626,\ \varepsilon_{10} = 67.6222,\ \varepsilon_{11} = 72.4034,\ \varepsilon_{12} = 76.8051$ $\varepsilon_{13} = 80.8262,\ \varepsilon_{14} = 84.4661,\ \varepsilon_{15} = 87.7224,\ \varepsilon_{16} = 90.5999$ $\varepsilon_{17} = 93.093,\ \varepsilon_{18} = 95.2029,\ \varepsilon_{19} = 96.9264,\ \varepsilon_{20} = 98.2728$ $\varepsilon_{21} = 99.2323,\ \varepsilon_{22} = 99.808$

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Обобщенная задача двух и четырех ньютоновских центров

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В работе указаны интегрируемые аналоги на сфере потенциала Дарбу, включающего в себя задачу о движении частицы в поле двух и четырех неподвижных ньютоновских центров на плоскости и их обобщения. Полученные результаты могут быть использованы при построении теории движения спутников в поле сплющенного сфероида в пространствах постоянной кривизны.

1. Классическая задача двух центров и ее обобщения

В классической небесной механике хорошо известна задача двух центров, в которой два неподвижных центра с массами m_1, m_2 притягивают некоторую «безмассовую» частицу, движущуюся в их поле, по ньютоновому закону. Интегрируемость этой задачи была показана Эйлером с помощью разделения переменных [14].

Качественный анализ плоской задачи двух центров имеется в книге К. Шарлье [12] (см. также [18]); качественный анализ пространственной задачи двух центров содержится в работе В. М. Алексеева [2]. Отметим также, что еще Лагранж заметил, что задача двух центров остается интегрируемой, если добавить к ней потенциал упругой пружины, которая закреплена в середине отрезка прямой, соединяющей оба центра. Лагранж также рассмотрел предельный случай этой задачи, для которого один из двух центров и его масса устремляются в бесконечность, в пределе получается задача о движении частицы в суперпозиции поля ньютоновского центра (задача Кеплера) и однородного поля. Разделение переменных и качественное исследование этой задачи содержится в книге М. Борна [6] по атомной механике, изучавшего эту задачу в связи с расщеплением спектральных линий атома водорода, помещенного в электрическое поле (эффект Штарка).

Один из более общих случаев интегрируемости потенциальной системы на плоскости, обобщающий задачу двух центров, был найден Г. Дарбу (1901) [17] методом разделения переменных. В этой работе Дарбу также получил условия существования для натуральной системы на плоскости дополнительного квадратичного интеграла, которые впоследствии были также указаны Уиттекером [11].

Рассмотрим частицу единичной массы, движущуюся по плоскости $\mathbb{R}^2 =$

$\{x, y\}$ в потенциальном поле

$$V = \frac{A}{x^2} + \frac{A'}{y^2} + \frac{B}{r} + \frac{B'}{r'} + \frac{B_1}{r_1} + \frac{B_1'}{r_1'} + C\rho^2, \tag{1.1}$$

где A, A', B, B', B_1, B_1', $C = const$, причем r, r' являются действительными расстояниями частицы m от двух одинаковых действительных центров, помещенных в точки $(-c, 0)$, $(c, 0)$ на оси абсцисс, $r = \sqrt{(x-c)^2+y^2}$, $r' = \sqrt{(x+c)^2+y^2}$, ρ — расстояние от m до центра O, r_1, r_1' — «комплексные расстояния» до мнимых центров, помещенных в точки $(0, di)$ и $(0, -di)$, $r_1 = \sqrt{x^2+(y-id)^2}$, $r_1' = \sqrt{x^2+(y+id)^2}$.

Для вещественности потенциала (1.1) необходимо, чтобы B_1' было комплексно сопряжено B_1: $\bar{B_1}' = B_1$. Как показано в [17] если $d = c$, система (1.1) допускает разделение переменных в эллиптических координатах

$$x = c\,\mathrm{ch}\,v\cos u, \qquad y = c\,\mathrm{sh}\,v\sin u$$

и обладает дополнительным первым интегралом, квадратичным по импульсам.

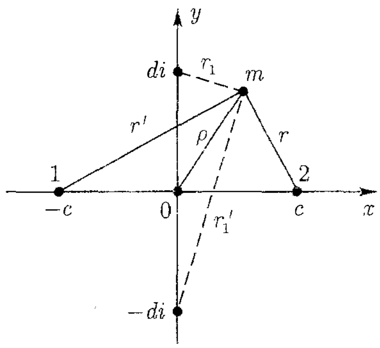

Рис. 1

Остановимся на частных случаях потенциала (1.1). Один случай системы (1.1), для которого $B_1 = B_1' = 0$, был рассмотрен Ж. Лиувиллем (как уже указывалось, еще более частный случай $A = A' = B_1 = B_1' = 0$ был указан Лагранжем).

В работе [1] показано, что задача о движении частицы в поле двух комплексно-сопряженных центров, т.е. для $A = A' = B = B' = C = 0$ в (1.1), интегрируема в трехмерном пространстве и является хорошим приближением к задаче о движении спутника в поле сплющенного сфероида (например, этот потенциал хорошо аппроксимирует потенциал реальной Земли).

В работе И. С. Козлова [9] проинтегрирована в квадратурах и исследована задача о плоском движении частицы в поле четырех неподвижных центров (двух вещественных и двух комплексных). В [9] также предложено несколько интерпретаций этой задачи применительно к реальным вопросам прикладной небесной механики.

2. Задача Кеплера, задача двух центров на сфере и псевдосфере. Исторический комментарий

Систематическое обобщение различных задач классической и небесной механики на пространства постоянной кривизны (включающие как трехмерную сферу S^3, так и псевдосферу — L^3-пространство Лобачевского) содержится в обширной, но, к сожалению, почти забытой работе В. Киллинга [22].

Укажем также, что кроме Киллинга в XIX веке неевклидовой механикой в пространствах постоянной кривизны занимались Р. Липшиц, Ф. Шеринг, Г. Либман. Интересно, что в учебнике Г. Либмана по неевклидовой геометрии [24] целая глава посвящена обобщению ньютонового закона притяжения, исследованию задачи Кеплера и формулировки законов Кеплера на сфере и псевдосфере, тем не менее в XX веке аналогичные результаты вновь и независимо были получены сразу несколькими авторами [23, 15, 20, 8, 26, 21, 25, 16]. Отметим также классическую работу Э. Шредингера [13], в которой он рассматривал квантовый аналог задачи Кеплера в искривленном пространстве, неявно предполагая интегрируемость соответствующей классической задачи. Кстати говоря, аналог закона ньютонового притяжения для L^3 был уже известен П. Серре, Я. Больяи и Н. И. Лобачевскому.

В. Киллинг в [22] также рассматривал вопросы n-мерной динамики в пространствах постоянной кривизны, включая динамику n-мерного твердого тела. Современный анализ можно найти в [19] (см. также [5]).

Обобщение задачи двух центров на пространстве постоянной кривизны также было указано В. Киллингом, проинтегрировавшим эту задачу методом разделения переменных. Независимо эта задача была решена в работе [23], где также рассматривается более общая задача, аналогичная добавлению Лагранжем потенциала упругого взаимодействия в задачу двух центров на плоскости. В работах [7, 27] дан бифуркационный анализ задачи двух центров на сфере и плоскости Лобачевского. В книге [5] авторами разобраны вопросы редукции и интегрируемости пространственной задачи двух центров, а также другие интегрируемые и неинтегрируемые задачи искривленной небесной механики (включая ограниченные задачи двух и трех тел, исследование точек либрации, динамику твердого тела).

Далее мы приведем явное алгебраическое выражение первого интеграла для обобщенной задачи двух центров, рассмотренной в [22, 23], а также укажем новый аналог задачи четырех ньютоновских центров и n гуковских центров. В работе мы ограничимся анализом двумерной сферы S^2, хотя все рассуждения без труда могут быть перенесены на псевдосферу L^2. Некоторые (но не все) результаты обобщаются на случай трехмерной сферы S^3 (псевдосферы L^3).

3. Обобщение задачи двух центров на S^2. Дополнительный квадратичный интеграл

Мы будем предполагать, что единичная сфера S^2 задана в трехмерном пространстве $\mathbb{R}^3 = \{q_1, q_2, q_3\}$ уравнением $|\mathbf{q}|^2 = q_1^2 + q_2^2 + q_3^2 = 1$, векторы $\mathbf{q} = (q_1, q_2, q_3)$, $\mathbf{p} = (p_1, p_2, p_3)$ будут обозначать соответствующие избыточные координаты и импульсы. Если ввести вектор углового момента $\mathbf{M} = \mathbf{p} \times \mathbf{q}$, и положить $\gamma = \mathbf{q}$, то несложно показать [5, 3, 4], что уравнения движения в произвольном потенциале $V = V(\mathbf{q}) = V(\gamma)$ могут быть представлены как гамильтонова система со скобкой Пуассона, определяемой алгеброй $e(3) = so(3) \oplus_s \mathbb{R}^3$:

$$\{M_i, M_j\} = \varepsilon_{ijk} M_k, \quad \{M_i, M_j\} = \varepsilon_{ijk} M_k, \quad \{\gamma_i, \gamma_j\} = 0 \tag{3.1}$$

и гамильтонианом

$$H = \frac{1}{2}(\mathbf{M}, \mathbf{M}) + V(\gamma). \tag{3.2}$$

Уравнения, задаваемые с помощью (3.1), (3.2), имеют вид

$$\dot{\mathbf{M}} = \gamma \times \frac{\partial V}{\partial \gamma}, \qquad \dot{\gamma} = \gamma \times \mathbf{M}$$

и совпадают с уравнениями движения шарового волчка в потенциале $V(\gamma)$ [5, 3]. Скобка (3.1) является вырожденной и обладает двумя функциями Казимира: $F_1 = (\mathbf{M}, \gamma)$, $F_2 = (\gamma, \gamma) = 1$. Для задачи о движении точки на сфере необходимо $F_1 = (\mathbf{M}, \gamma) = (\mathbf{p} \times \gamma, \gamma) = 0$.

Хорошо известно, что аналогами ньютоновского и гуковского потенциала на S^2 соответственно являются $U_1 = \mu \operatorname{ctg} \theta$, $U_2 = c \operatorname{tg}^2 \theta$, $\mu, c = const$, где угол θ отсчитывается от некоторого фиксированного полюса на сфере [22, 23].

Рассмотрим потенциал

$$V = -\mu_1 \operatorname{ctg} \theta_1 - \mu_2 \operatorname{ctg} \theta_2, \tag{3.3}$$

где μ_1, μ_2 — интенсивности ньютоновских центров, θ_i — углы между радиус-вектором частицы и радиус-вектором i-го центра. Поместим ньютоновские центры в точки $\mathbf{r}_1 = (0, \alpha, \beta)$, $\mathbf{r}_2 = (0, -\alpha, \beta)$, $\alpha^2 + \beta^2 = 1$, а также добавим для общности к (3.3) потенциал трех гуковских центров, помещенных на взаимно перпендикулярных осях $\frac{1}{2}\sum c_i/\gamma_i^2$ ($c_i = const$), и дополнительный квадратичный потенциал $C(\alpha^2\gamma_2^2 - \beta^2\gamma_3^2)$, $C \neq 0$, являющийся частным случаем потенциала Неймана. На уровне $(\mathbf{M}, \gamma) = 0$ находим две коммутирующие квадратичные по $\mathbf{M}$ функции $\{H, F\} = 0$ [4, 10]:

$$\begin{aligned} H = \frac{1}{2}\mathbf{M}^2 - \mu_1 \frac{\beta\gamma_3 + \alpha\gamma_2}{\sqrt{\gamma_1^2 + \beta_1\gamma_2^2 + \alpha^2\gamma_3^2 - 2\alpha\beta\gamma_2\gamma_3}} - \\ - \mu_2 \frac{\beta\gamma_3 - \alpha\gamma_2}{\sqrt{\gamma_1^2 + \beta_1\gamma_2^2 + \alpha^2\gamma_3^2 + 2\alpha\beta\gamma_2\gamma_3}} + \frac{1}{2}c_1\frac{\gamma_2^2 + \gamma_3^2}{\gamma_1^2} + \\ + \frac{1}{2}c_2\frac{\gamma_1^2 + \gamma_3^2}{\gamma_2^2} + \frac{1}{2}c_3\frac{\gamma_1^2 + \gamma_2^2}{\gamma_3^2} + C(\alpha^2\gamma_2^2 - \beta^2\gamma_3^2), \end{aligned} \tag{3.4}$$

$$F = \alpha^2 M_2^2 - \beta^2 M_3^2 + 2\alpha\beta(V_1 - V_2) -$$
$$- \frac{c_1}{\gamma_1^2}(\beta^2\gamma_2^2 - \alpha^2\gamma_3^2) - \frac{c_2}{\gamma_2^2}\beta^2\gamma_1^2 + \frac{c_3}{\gamma_3^2}\alpha^2\gamma_1^2 + 2C\alpha^2\beta^2\gamma_1^2, \quad (3.5)$$

где μ_1, μ_2, α, β, c_1, c_2, c_3, $C = const$, а функции V_1, V_2 определяются выражениями

$$V_1 = \frac{\mu_1(\beta\gamma_2 + \alpha\gamma_3)}{\sqrt{\gamma_1^2 + \beta^2\gamma_2^2 + \alpha^2\gamma_3^2 - 2\alpha\beta\gamma_2\gamma_3}}, \quad (3.6)$$

$$V_2 = \frac{\mu_2(\beta\gamma_2 - \alpha\gamma_3)}{\sqrt{\gamma_1^2 + \beta^2\gamma_2^2 + \alpha^2\gamma_3^2 + 2\alpha\beta\gamma_2\gamma_3}}. \quad (3.7)$$

Функция H является гамильтонианом, а F задает дополнительный квадратичный интеграл. Как отмечено в [5], возможность добавления в задачу двух центров (3.3) одного гуковского центра c/γ_3^2, помещенного на дуге между ньютоновскими центрами, тесно связана с интегрируемостью соответствующей трехмерной задачи (т.е. на S^3). Действительно, член c/γ_3^2, $c = const$ возникает в трехмерном случае при редукции по Раусу, использующей циклический интеграл, связанный с инвариантностью уравнений относительно вращений (группа $SO(2)$), в плоскости, перпендикулярной плоскости двух центров.

Система (3.4)-(3.5) принадлежит к лиувиллевскому типу и может быть проинтегрирована в сфероконических координатах u_1, u_2, ($0 < u_1 < \alpha$, $0 < u_2 < \beta$), которые определяются соотношениями

$$\gamma_1 = \sqrt{u_1 u_2}/(\alpha\beta),$$
$$\gamma_2 = \sqrt{(\alpha^2 - u_1)(\alpha^2 + u_2)}/\alpha, \quad (3.8)$$
$$\gamma_3 = \sqrt{(\beta^2 + u_1)(\beta^2 - u_2)}/\beta.$$

Однако отметим нетривиальность задачи получения интегралов (3.4)-(3.5) именно в алгебраической форме, для решения которой необходимо обращать сфероконическое преобразование.

Как нам сообщил А. Албуи, задача двух центров на S^2 (L^2) при помощи центральной (гномонической) проекции и подходящего преобразования времени может быть преобразована в обычную эйлеровскую задачу двух центров. Однако мы не обладаем доказательством этого факта.

4. Задача четырех ньютоновских центров на сфере S^2

Рассмотрим потенциал на сфере вида:

$$
\begin{aligned}
V_{\mathrm{Im}} &= \xi_1 \operatorname{ctg}\theta_1 + \xi_2 \operatorname{ctg}\theta_2 = \\
&= \xi_1 \frac{\mu\gamma_1 + i\nu\gamma_3}{\sqrt{(\mu^2-\nu^2)^2 - (\mu\gamma_1 + i\nu\gamma_3)^2}} + \xi_2 \frac{\mu\gamma_1 - i\nu\gamma_3}{\sqrt{(\mu^2-\nu^2)^2 - (\mu\gamma_1 - i\nu\gamma_3)^2}},
\end{aligned}
\tag{4.1}
$$

где $\mu^2 - \nu^2 = 1$, $\xi_1, \xi_2 = const$.

Этот потенциал соответствует задаче двух центров на сфере, имеющих «комплексные интенсивности» и расположенных на равных удалениях от полюса с комплексно сопряженными расстояниями (рис. 2). Для его действительности необходимо $\bar{\xi}_1 = \xi_2$. Как и в евклидовом случае, потенциал (4.1) может рассматриваться как некоторая аппроксимация задачи о движении частицы в поле сплющенного сфероида в искривленном пространстве.

Система с потенциалом (4.1) также разделяется в сфероконических координатах (3.8) при условии

$$
\mu = \frac{\beta}{1-\alpha^2}, \qquad \nu = \frac{\alpha\beta}{1-\alpha^2}. \tag{4.2}
$$

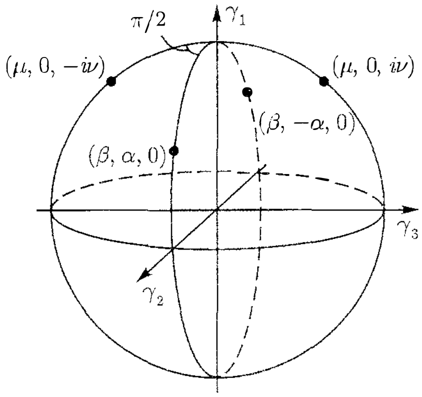

Рис. 2

В координатах (3.8) разделяется также потенциал $V + V_{\mathrm{Im}}$, который (при $c_i = 0$) соответствует задаче четырех неподвижных центров — двух мнимых и двух вещественных, расположенных в двух взаимно перпендикулярных плоскостях, проходящих через полюс (см. рис. 2), при этом, как и в случае плоскости, при фиксированном расстоянии между вещественными центрами расстояние между комплексными центрами также не является произвольным, а определено однозначно с помощью(4.2).

Легко показать, что потенциалы

$$
V_G = \frac{1}{2}\left(\sum c_i/\gamma_i^2\right), \qquad V_N = C(\alpha^2\gamma_2^2 - \beta^2\gamma_3^2), \qquad c_i,\, C = const. \tag{4.3}
$$

могут быть (интегрируемым образом) добавлены в задачу четырех центров и приводят к более общей системе, разделимой в координатах (3.6). Приведем явный вид потенциалов V, V_{Im}, V_G, V_N в переменных (3.6):

$$V=\frac{(\mu_1+\mu_2)\sqrt{(\alpha^2-u_1)(\beta^2+u_1)}+(\mu_1-\mu_2)\sqrt{(\alpha^2+u_2)(\beta^2-u_2)}}{u_1+u_2},$$

$$V_{\mathrm{Im}}=\frac{(\xi_1+\xi_2)\sqrt{u_2(\beta^2-u_2)}+i(\xi_1-\xi_2)\sqrt{u_1(\beta^2+u_1)}}{u_1+u_2},$$

$$\frac{1}{\gamma_1^2}=\beta^2\frac{(\beta^2-u_2)^{-1}-(\beta^2+u_1)^{-1}}{u_1+u_2},$$

$$\frac{1}{\gamma_2^2}=\alpha^2\frac{(\alpha^2-u_1)^{-1}-(\alpha^2+u_2)^{-1}}{u_1+u_2},$$

$$\frac{1}{\gamma_3^2}=\alpha\beta\frac{u_1^{-1}+u_2^{-1}}{u_1+u_2},\qquad V_N=C\frac{u_1^2-u_2^2}{u_1+u_2}.$$

Несложно показать, что при предельном переходе к евклидовой плоскости ($R\to\infty$) суммарный потенциал $V+V_{\Im}+V_G+V_N$ переходит в потенциал Дарбу (1.1). Отметим, что этот потенциал, или даже $V+V_{\mathrm{Im}}$, уже не может быть обобщен до соответствующего интегрируемого потенциала для трехмерной задачи (в S^3) вследствие отсутствия циклического интеграла, хотя каждый из потенциалов V и V_{Im} по отдельности допускает такое обобщение.

5. Задача n гуковских центров на сфере

Укажем еще один интегрируемый вариант задачи о движении материальной точки в поле гуковских потенциалов $c_i/(\gamma,\mathbf{r}_i)^2$, $c_i=const$, при котором гуковские центры притяжения $\mathbf{r}_i$, $i=1,2,\ldots,n$, помещены не по взаимно ортогональным осям, а произвольно располагаются на одном экваторе [10] (рис.3).

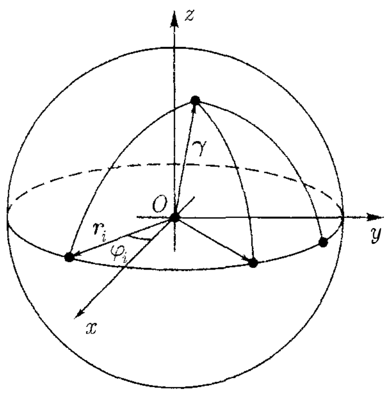

Рис. 3.

Гамильтониан и дополнительный интеграл при $(\mathbf{M}, \gamma) = 0$ имеют вид

$$H = \frac{1}{2}\mathbf{M}^2 + \frac{1}{2}\sum_{i=1}^{n} \frac{c_i}{(bfr_i, \gamma)^2} + U(\gamma_3), \tag{5.1}$$

$$F = M_3^2 + (1-\gamma_3^2)\sum_{i=1}^{n} \frac{c_i}{(bfr_i, \gamma)^2}. \tag{5.2}$$

В выражении (5.1) присутствует произвольная функция $U(\gamma_3)$, которая обозначает добавление произвольного «центрального» поля, центр которого расположен на перпендикуляре к плоскости гуковских потенциалов (рис. 3). В частности, на полюс можно поместить еще один гуковский центр. Из этого следует (см. [4]), что интегрируема также пространственная задача о движении точки на трехмерной сфере S^3 под действием n гуковских центров, расположенных на экваторе.

Отметим, что евклидов аналог рассматриваемой задачи тривиален — разделение возможно уже в декартовых координатах (получается n линейных осцилляторов). При этом расположение гуковских центров на плоскости $\mathbb{R}^2$ произвольно. В криволинейной ситуации, уже на двухмерной сфере, задача о движении в поле трех произвольно расположенных гуковских центров не является интегрируемой. Это показывают численные эксперименты, демонстрирующие хаотическое поведение. Квадратичный интеграл F в (5.2) связан с разделением задачи в сферических координатах (θ, φ). Действительно, гамильтониан H можно записать следующим образом:

$$\begin{aligned} H = \frac{1}{2}\left(p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta}\right) + \frac{1}{2}\sum_{i=1}^{n}\frac{c_i}{\sin^2\theta\cos^2(\varphi-\varphi_i)} + U(\theta) = \\ = \frac{1}{2}p_\theta^2 + \frac{1}{\sin^2\theta}\left[p_\varphi^2 + \sum_{i=1}^{n}\frac{c_i}{\cos^2(\varphi-\varphi_i)}\right] + U(\theta), \end{aligned} \tag{5.3}$$

где θ, φ — координаты движущейся материальной точки, а φ_i задает положение i-го гуковского центра на экваторе (рис. 3). Выражение в квадратных скобках представляет собой дополнительный интеграл движения.

После написания этой работы нам стало известно, что С.Т. Садэтов независимо получил более общие результаты по интегрируемости задачи шести центров на S^2 (к четырем указанным нами центрам добавлены два новых мнимых центра).

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Generalized problem of two and four Newtonian centers

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The paper considers integrable spherical analogs of the Darboux potential, which appears in the problem of planar motion of a particle in a field of two and four Newtonian centers. The obtained results can be useful in constructing a theory of motion of satellites in the field of an oblate spheroid in constant curvature spaces.

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Инерциальные системы в сферическом пространстве

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Трехмерная сфера являеется метрическим многообразием группы SU_2 и векторные поля на сфере, определяющие левые и правые групповые сдвиги, одновременно являются полями Киллинга и геодезическими потоками (*геотоками*). Поток свободных частиц, равномерно движущихся вдоль одного из этих полей, реализует движущуюся систему на трехмерной сфере, сохраняющую метрику, однако в этой системе возникает поле вращения, отличающее движущуюся систему от покоящейся. Построенные на геотоках однородные электрическое или магнитное поля оказываются нестационарными.

1. Инерциальные системы

150 лет назад в лекции 10 июня 1854 г. Риман [1] выдвинул допущение, что наше пространство не обязано быть плоским, а может быть пространством постоянной положительной или отрицательной кривизны. Необходимость постоянства кривизны он обосновывал необходимостью движения и поворота в пространстве материальных тел, обязанных обладать такой же внутренней кривизной. На современном языке – он выдвинул требование однородности и изотропности пространства и показал, что оно может быть не только евклидовым, плоским, но и пространством постоянной положительной или отрицательной кривизны.

Но в таком пространстве исчезает классическое понятие *инерциальных систем* как движущихся друг относительно друга равномерно и прямолинейно. Однако небольшая естественная модификация понятия "*инерциальная система*" позволяет восстановить многообразие инерциальных систем в сферическом мире Римана. В пространстве, являющимся трехмерной сферой, также существует множество пространств, движущихся друг относительно друга инерциально по отношению к законам классической механики. Однако, хотя метрика пространства в движущейся и неподвижной системах одинакова, законы динамики в инерциально движущейся трехмерной сфере отличаются от законов динамики покоящейся, то есть выделить абсолютно покоящееся пространство можно по механическим явлениям.

В классической механике свободное движение в римановом (трехмерном) пространстве есть движение по геодезической.

Пространство можно *материализовать*, поместив в каждую его точку пылинку (в пределе – безмассовую). Покоящиеся свободные пылинки отмечают точки пространства, а расстояния между бесконечно близкими пылинками определяются метрикой пространства или, наоборот, *определяют метрику пространства*. Если в евклидовом пространстве этим частицам придать одинаковые скорости, то расстояния между частицами сохраняются и они в любой момент времени реализуют евклидово пространство. Система, связанная с этими частицами, и является *инерциальной системой* – все ее точки движутся по инерции.

Применим этот же механизм *материализации пространства* к трехмерной сфере. Если построить какое-то поле Киллинга и сдвинуть частички вдоль этого поля, то расстояния между ними – метрика, реализуемая полем этих частичек, – не изменится, останется той же в выбранной системе координат.

2. Поля Киллинга трехмерной сферы

Как известно, трехмерная сфера является метрическим многообразием группы O(3), а однопараметрические подгруппы являются геодезическими полями [2], поэтому свободные частички со скоростями, пропорциональными компонентам какого-то поля Киллинга, будут все время перемещаться вдоль этого поля, реализуя в любой момент трехмерную сферу.

Распишем метрику сферы в углах Эйлера:

$$dl^2 = \frac{r^2}{4}(d\vartheta^2 + d\varphi^2 + d\psi^2 + 2\,\cos\vartheta\, d\varphi\, d\psi).$$

Шесть векторов Киллинга этой метрики собираются в две коммутирующие между собой группы по три вектора:

$$\begin{pmatrix} \xi^\vartheta \\ \xi^\varphi \\ \xi^\psi \end{pmatrix} = \sum_{\alpha=1}^{3} a_\alpha\, \chi_\alpha + \sum_{\alpha=1}^{3} b_\alpha\, \eta_\alpha =$$

$$a_1 \begin{pmatrix} \sin\varphi \\ \cos\varphi\,\mathrm{ctg}\vartheta \\ -\cos\varphi/\sin\vartheta \end{pmatrix} + a_2 \begin{pmatrix} \cos\varphi \\ -\sin\varphi\,\mathrm{ctg}\vartheta \\ \sin\varphi/\sin\vartheta \end{pmatrix} + a_3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \tag{2.1}$$

$$b_1 \begin{pmatrix} \sin\psi \\ -\cos\psi/\sin\vartheta \\ \cos\psi\,\mathrm{ctg}\vartheta \end{pmatrix} + b_2 \begin{pmatrix} \cos\psi \\ \sin\psi/\sin\vartheta \\ -\sin\psi\,\mathrm{ctg}\vartheta \end{pmatrix} + b_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Векторные поля χ^i_α и η^i_α, являющиеся одновременно полями Киллинга и геодезическими потоками, назовем левым и правым *геотоками*.

Векторы χ^i_α назовем векторами *левой группы*, а η^i_α – векторами *правой группы*. Векторы, принадлежащие одной группе, будем называть *односторонними геотоками*, а разным – *разносторонними геотоками*.

2.1. Коммутаторы и роторы

Внутри каждой группы коммутационные соотношения трех полей изоморфны коммутационным соотношениям группы вращений. Полям Киллинга сопоставляются дифференциальные операторы Ли, через которые удобно записывать коммутационные соотношения между полями Киллинга:

$$X_\alpha = \chi^i_\alpha \frac{\partial}{\partial x^i}; \quad Y_\alpha = \eta^i_\alpha \frac{\partial}{\partial x^i}; \quad [X_\alpha, Y_\beta] = 0;$$
$$[X_\alpha, X_\beta] = \varepsilon_{[\alpha\beta\gamma]} X_\gamma; \quad [Y_\alpha, Y_\beta] = \varepsilon_{[\alpha\beta\gamma]} Y_\gamma.$$

$$\mathbf{rot}\, \chi_\beta = -\frac{2}{r}\, \chi_\beta; \quad \mathbf{rot}\, \eta_\beta = \frac{2}{r}\, \eta_\beta \tag{2.2}$$

где r – радиус сферического пространства, а $\varepsilon_{[\alpha\beta\gamma]}$ – абсолютный антисимметричный тензор.

2.2. Теоремы о суперпозиции

Уравнения Киллинга линейны по полям и любая линейная комбинация полей Киллинга есть также поле Киллинга.

Уравнения геодезического потока

$$u^i \nabla_i u^j = 0 \tag{2.3}$$

нелинейны и суперпозиция полей ξ_β уже не является геодезическим потоком. Пусть имеются два геотока u^i и v^i. Их сумма есть вектор Киллинга. Будет ли суммарный поток геодезическим?

$$(u^i + v^i)\nabla_i (u^j + v^j) = u^i \nabla_i u^j + v^i \nabla_i v^j + u^i \nabla_i v^j + u^i \nabla_i u^j = u^i \nabla_i v^j + u^i \nabla_i u^j.$$

Так как геотоки являются полями Киллинга, то $u_{i;k} = -u_{k;i}$ и

$$\nabla_i u^j = \gamma^{jk}\, u_{k;i} = -\gamma^{jk}\, u_{i;k};$$

$$u^i \nabla_i v^j + u^i \nabla_i u^j = -\gamma^{jk}(u^i\, v_{i;k} + v^i\, u_{i;k}) = -\gamma^{jk}\, \nabla_k\, (u^i\, v_i) = -\gamma^{jk}\, \partial_k\, (u^i\, v_i).$$

Будет ли сумма геотоков являться геотоком, определяется их скалярным произведением:

Теорема 2.1.. *Если скалярное произведение двух геотоков постоянно на всем пространстве, то их сумма также является геотоком.*

Скалярные произведения геотоков внутри каждой группы постоянны:

$$(\chi^i_\alpha\,\chi_{i\beta}) = \delta_{\alpha\beta}; \quad (\eta^i_\alpha\,\eta_{i\beta}) = \delta_{\alpha\beta}.$$

Отсюда и из Теоремы 2.1. вытекает

Теорема 2.2.. *Суперпозиция односторонних геотоков является геотоком.*

Прямой проверкой убеждаемся, что скалярное произведение любых левых и правых геотоков не является константой. Это связано с тем, что при суперпозиции геотоков мы можем распоряжаться лишь шестью константами, а множество значений скалярного произведения на сфере бесконечномерно. Исключением является постоянство скалярных произведений односторонних геотоков.

Отсюда следует

Теорема 2.3.. *Сумма любых левого и правого геотоков является полем Киллинга, но не является геодезическим потоком.*

3. Движущиеся системы

Если в каждой точке сферического пространства поместить пылинку и каждой такой пылинке придать скорость, пропорциональную одному из полей геотоков, то вследствие геодезичности такого поля, двигаясь по инерции, частички будут двигаться вдоль этого поля и само поле скоростей с течением времени меняться не будет.

Так как такое поле скоростей является полем Киллинга, то расстояния между пылинками меняться не будет: в каждый момент времени множество пылинок реализует трехмерную сферу.

Поэтому для реализации инерциальной системы – движущейся в себе трехмерной сферы – нужно брать суперпозицию односторонних геотоков.

Однако по своим механическим свойствам такая движущаяся система отличается от покоящейся. В ней метрика, определяемая движущимися пылинками, такая же, как и неподвижная – метрика трехмерной сферы, и поля Киллинга поэтому те же самые. Однако в движущейся системе только три геотока являются односторонними со скоростью. Векторы Киллинга другой группы уже не являются геодезическими, что определимо экспериментально.

4. Электрическое и магнитное поля на S_3

Замечательные дифференциальные и алгебраические свойства геотоков позволяют рассмотреть нетривиальные задачи электродинамики с однородным электрическим и магнитным полями. В евклидовом пространстве очень часто использует для рассмотрения тех или иных электродинамических процессов *постоянные, однородные* электрическое или магнитное поля. Возможны ли такие поля, если пространство является трехмерной сферой S_3 ?

Поле, пропорциональное геотоку, является однородным ($V^i = \chi^i$ или $V^i = \eta^i$):

$$\mathbf{E}(\mathbf{r}, t) = e(t)\,\mathbf{V}; \quad \mathbf{H}(\mathbf{r}, t) = h(t)\,\mathbf{V}.$$

Уравнения Максвелла в вакууме

$$\frac{1}{c}\,\dot{\mathbf{H}} + \mathbf{rot}\,\mathbf{E} = 0, \quad \mathbf{div}\,\mathbf{H} = 0,$$

$$\frac{1}{c}\,\dot{\mathbf{E}} - \mathbf{rot}\,\mathbf{H} = 0, \quad \mathbf{div}\,\mathbf{E} = 0,$$

с учетом соотношения (2.2) для ротора поля Киллинга приводят к зависимости от времени амплитуд электрического и магнитного полей:

$$\dot{h} + \frac{2\,c}{r}\,e = 0; \quad \dot{e} - \frac{2\,c}{r}\,h = 0.$$

$$\ddot{h} + \omega^2\,h = 0; \quad \ddot{e} + \omega^2\,e = 0; \quad \omega = \frac{2\,c}{r}.$$

Если в начальный момент мы задали однородное электрическое поле $\mathbf{E}_0$, то с течением времени оно перекачивается в магнитное по гармоническому закону с частотой $\omega = 2\,c/r$. При этом энергия электромагнитного поля не меняется: $\mathbf{E}^2 + \mathbf{H}^2 = \mathbf{E}_0{}^2$.

При стремлении радиуса сферы к бесконечности период колебаний $T = \pi\,r/c$ стремится к бесконечности – пространство становится евклидовым, а поля становятся статическими.

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Inertial systems in the spherical space

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There are two three-parametric sets of the inertial systems, constructed on the Killing fields, if the space is a three-dimensional sphere. These fields are the geodesic flows at the same time. Their differential and algebraic properties are studied in this article. In the moving system only one set (the one half of all Killing fields) are the geodesic flows. There is the way to distinguish the absolutely quite system from moving ones.

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Global Time and Relativity

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The *Global Time Theory* (GTT) is a next step in the development of the General Relativity (GR). The GTT differs from the GR conceptually, but preserves 90% of the GR mathematical structure and reproduces its main results. The dynamics equations of the GTT are derived from the Lagrangian with nonzero gravitation Hamiltonian. Detailed solutions to the cosmic vortexes are presented. They satisfy a weak principle of superposition and do not have an analog in the GR. The virial theorem of space is formulated and proved. The GTT allows to formulate a quantum theory of gravitation on the basis of the Schroedinger equation, as it is done for other fields. The quantum model of the Big Bang based on the GTT is demonstrated.

1. Introduction

The most recognized theory of space, time and gravity today is the General Theory of Relativity (GR). It treats space and time as a four-dimensional repository which properties can be modified according to the Einstein's equations by the inserted matter. As in the times of Mach, the basis of cosmic dynamics is here a "tangible matter".

Astrophysical observations of XX century showed that dynamics of galaxies and heaps of galaxies cannot be explained by theories based solely on gravitational interactions of visible stars. In order to explain the observed anomalies such notions as "dark matter", "dark energy", massive "dark holes", etc. were introduced but current theories have not been able to accommodate them yet adequately.

GR as the space-time theory suggests only small corrections to the Newtonian dynamics (at its scale). An alternative theory of the space and time, the *the Global Time Theory* (GTT), is introduced here. The GTT differs significantly from the GR in physical postulates it is based on, but preserves 90% of the mathematical structure and main results of the GR. Importantly, the description of the cosmos dynamics, and construction of the quantum gravity are essentially different in the GTT from ones in the GR.

2. The Global Time Theory

In the GTT time is absolute. It flows equally, always and everywhere, and is itself the measure of an equality. The development of *the Entire Universe* occurs in this global time.

The space has three dimension, is Riemanian, and its metric tensor (γ_{ij}) can depend on space coordinates and time. Points of the space are linked with the global time absolutely. The frame of reference in which coordinates of space points do not change is called *the* *global inertial system*. The inertial system allows arbitrary tree-dimensional transformations of coordinates $\tilde{x}^i(x^j)$ (that do not depend on time).

The coordinate transformations that are *time-dependent* lead to the *global non-inertial system* of observation. Meanwhile the time remain global. In the non-inertial system the *vector field of absolute velocities* V^i arises, although it vanish in the inertial system. In the transformation of coordinates, the V^i-field is transformed as a gauge field:

$$\tilde{x}^i = f^i(x,t); \quad \tilde{V}^i = \frac{\partial \tilde{x}^i}{\partial t} + \frac{\partial \tilde{x}^i}{\partial x^k} V^k; \quad \tilde{\gamma}^{ij} = \frac{\partial \tilde{x}^i}{\partial x^k}\frac{\partial \tilde{x}^j}{\partial x^l}\gamma^{kl}. \tag{2.1}$$

2.1. The covariant derivative over the time

We will denote the time derivative in the inertial system as D_t and call it *the covariant derivative over the time*. By the rule of the composite function differentiation

$$D_t\, F = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x^i}\frac{\partial x^i}{\partial t} = \frac{\partial F}{\partial t} + V^i \frac{\partial F}{\partial x^i}, \tag{2.2}$$

what determines the covariant derivative over the time of a scalar field (action, eikonal) in an arbitrary frame with global time.

The structure of covariant derivatives over the time for tensors contains additional terms in the form of the Lie-variation, that are generated by the transformation of coordinates $\delta x^i = -V^i\, dt$ – for returning to the inertial system.

For a tensor of an arbitrary rank

$$D_t\, Q^i_{jk} = \frac{\partial}{\partial t} Q^i_{jk} - V^i_{;s} Q^s_{jk} + V^s_{;j} Q^i_{sk} + V^s_{;k} Q^i_{js} + V^s Q^i_{jk;s}. \tag{2.3}$$

Especially important for the theory is the covariant derivative over time of the metric tensor:

$$D_t\, \gamma_{ij} = \frac{\partial \gamma_{ij}}{\partial t} + V_{i;j} + V_{j;i}. \tag{2.4}$$

2.2. Action and dynamical equations

In the GR, space plays a rather passive role. In contrast, in the GTT, the three-dimensional space is the dynamic field, relative to which there exists an absolute motion, or, on the contrary, there exists a field of space velocities in a given system of coordinates. The equations of motion are derived from the variation principle. The Lagrangian is as usually presented as the difference between the kinetic and potential energy. Introducing the *tensor of the space deformation velocity*

$$\mu_{ij} = \frac{1}{2c} D_t\, \gamma_{ij} = \frac{1}{2\, c}(\dot{\gamma}_{ij} + V_{i;j} + V_{j;i}), \tag{2.5}$$

we can represent this action as preconcerted with the Hilbert action in the GR:

$$S = \frac{c^4}{16\,\pi\, k} \int (\mu^i_j\, \mu^j_i - (\mu^j_j)^2 + R)\sqrt{\gamma}\, d_3\, x\, dt + S_m, \tag{2.6}$$

where S_m is the action of enclosed matter, which adds to dynamic equations the energy-momenta tensor components. The absolute velocities V^i are present only in the kinetic energy term.

By introducing momenta

$$\pi^i_j = \sqrt{\gamma}(\mu^i_j - \delta^i_j\, \mu^s_s),$$

and varying the action over the six components of the spatial metrics, we obtain six equations for the dynamics of the system:

$$\dot{\pi}^i_j = b^i_j + \sqrt{\gamma}\, G^i_j + \sqrt{\gamma}\,(T^i_j - V^i\, T^0_j), \tag{2.7}$$

where b^i_j is what we call *the self-tensor current*

$$b^i_j = -\delta^i_j\, \frac{\sqrt{\gamma}}{2}(\mu^k_l \mu^l_k - \mu^k_k \mu^l_l) - \partial_s(V^s\, \pi^i_j) + V^i{}_{,s}\, \pi^s_j - V^s{}_{,j}\, \pi^i_s, \tag{2.8}$$

G^i_j is the Einstein's tensor of the three-dimensional space, and T^α_β are components of the tensor energy-momenta of the enclosed matter, which determines *the exterior tensor current.*

The variation (only the kinetic part of action) by three components of the field of absolute velocities gives three equations of constraints:

$$\nabla_i\, \pi^i_j = \sqrt{\gamma}\, \frac{8\pi k}{c^4} T^0_j. \tag{2.9}$$

The Hamiltonian differs from the Lagrangian only by the sign in front of the potential component of energy:

$$H = \frac{c^4}{16\pi k} \int \left(\frac{\pi^i_j \pi^j_i - (\pi^i_i)^2/2}{\sqrt{\gamma}} - R\sqrt{\gamma} \right) d_3x. \tag{2.10}$$

Its unique feature is the non-fixed sign, what leads in particular to the possibility of such a phenomena as Friedman cosmological expansion.

2.3. The own time of the moving body

Similar to the GR, the GTT includes the special relativity. At the level of the global time, at which the development of the Entire Universe occur, for the moving observer, there exist its local frame of references and *local time*. All phenomena in the moving system develop in this local time which can be expressed through the square of absolute velocity

$$d\tau = dt\, \sqrt{1 - \gamma_{ij}(\dot{x}^i - V^i)(\dot{x}^j - V^j)}. \tag{2.11}$$

This expression can be represented in four-dimensions by combining the time and space into a unified four-dimensional manifold with the metrics

$$g_{00} = 1 - \gamma_{ij}\, V^i\, V^j; \quad g_{0i} = \gamma_{ij}\, V^j; \quad g_{ij} = -\gamma_{ij}. \tag{2.12}$$

The reverse metric tensor of this four-dimensional manifold is

$$g^{00} = 1; \quad g^{0i} = V^i; \quad g^{ij} = V^i\, V^j - \gamma^{ij}.$$

The first equation here

$$g^{00} = 1. \tag{2.13}$$

is of great significance. This is *the main structural relationship* in the GTT, analogous to the Minkowski metric, which is the main structural relationship in the special relativity.

2.4. General Relativity

If there is a four-dimensional metric $g_{\alpha\beta}$ in an arbitrary four-dimensional space with coordinates x^α, $\alpha = 0..3$, the variable τ must be determined for the reduction to the global time, in order for the main structure relationship (2.13) to hold true. We must transfer the metrics component g^{00} by rule of tensor transformation:

$$\bar{g}^{00} = g^{\alpha\beta} \frac{\partial\tau}{\partial x^\alpha} \frac{\partial\tau}{\partial x^\beta} = 1. \tag{2.14}$$

This differential equation turns out to be the Hamilton-Jacoby differential equation for free-falling bodies (laboratories), the common time for which is τ, which is the global time.

Thus *the equivalence principle* has a place, but, in contrast to the GR, the time of the inertial system exists not only for a local frame, but also for an infinite number of frames in entire space.

For example, the Kerr's metric [4] in the global time has radial and angular components of the absolute velocity field:

$$V^\varphi = -\frac{2\,a\,M\,r}{w}; \quad V^r = \frac{\sqrt{2\,M\,r(r^2 + a^2)}}{\rho^2}, \tag{2.15}$$

where

$$\rho^2 = r^2 + a^2 \cos^2\vartheta; \quad w = (r^2 + a^2)\rho^2 + 2\,M\,r\,a^2 \sin^2\vartheta.$$

The space metrics

$$\gamma_{11} = \frac{(\rho^2)^2}{W}; \quad \gamma_{22} = \rho^2; \quad \gamma_{33} = \frac{w}{\rho^2} \sin^2\vartheta; \quad \sqrt{\gamma} = \rho^2 \sin\vartheta \tag{2.16}$$

has singularity only at $\rho^2 = 0$.

2.5. The energy-momentum tensor of the space

The GTT differs mathematically from the GR only by one equation: since the main structural relationship (2.13) $g^{00} = 1$ prohibits variation of this component, the determined by this component variation tenth Einstein's equation is absent in the GTT . As a result, the difference

$$-2\,\frac{\delta\,S}{\delta\,g^{00}} = \frac{c^4}{8\pi k}\,G_{00} - T_{00} \equiv \rho; \tag{2.17}$$

is nonzero. In tensor terms we will denote this difference as the difference tensor:

$$\theta^\alpha_\beta = \frac{c^4}{8\pi k}\,G^\alpha_\beta - T^\alpha_\beta, \tag{2.18}$$

and as a consequence of (2.17)

$$\theta^0_0 = \rho; \quad \theta^i_0 = \rho V^i. \tag{2.19}$$

Since G^α_β and T^α_β are subject to the Hilbert's identities, the difference tensor θ^α_β is also subject to them:

$$\nabla_\alpha \theta^\alpha_\beta = 0, \tag{2.20}$$

thus (2.19) has the form of the energy-momenta tensor of a dust matter. But if we want to model the GTT in the GR by means of dust, as a result of a non-fixed sign of energy density, the possibility of a negative density of dust in the GR must also be considered.

3. Solutions

The most essential difference between the GTT and the GR is the nonzero Hamiltonian. The dynamic equations conserve the Hamiltonian density, and due to its non-fixed sign, partial solutions with all-around density of zero are possible. These exist also the GR solutions, which comprise *a subset* of the GTT solutions.

Further we describe a small set of the GTT solutions, which illustrate a solving methodology as well as the similarities and differences between the GTT and the GR.

3.1. The Spherical Universe dynamics

The simplest model is a three-dimensional sphere with time-depending radius:

$$dl^2 = r^2(t)\, ds_3^2,$$

where ds_3^2 is the metric of the three-dimensional sphere with a radius of equal to one.

For three-dimensional sphere with radius r, the scalar curvature is

$$R = \frac{3}{r^2}, \quad \sqrt{\gamma} = r^3.$$

The kinetic energy is proportional to

$$T = -3\left(\frac{\dot r}{r}\right)^2 r^3$$

and the Hamiltonian has a negative sign:

$$H = -3\, r\, (\dot r^2 + 1). \tag{3.1}$$

The Hamiltonian conservation leads to a differential equation of the first order:

$$-H = 3\, r\, (\dot r^2 + 1) = 3\, r_{max},$$

which is the Friedman's equation, that has a cycloid solution. In contrast to the classical formulation of the Friedman's problem, this solution is a vacuum one, without matter, and r_{max} is an integration constant independent on the matter density.

3.2. The field of the spherical mass

The inertial system is dynamical, but in global time there exist solutions, that are static from the point of view of some noninertial system.

In a spherically symmetric case, the space metric can be transformed to

$$dl^2 = dr^2 + R^2(r)(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2),$$

and the field of absolute velocities is radial: $V^r = V(r)$. Only the radial constraint-equation is nontrivial:

$$\nabla_i \pi^i_r = \frac{2}{r} R'' = 0,$$

from which $R = r$, and the space turns out to the flat one.

In dynamical equations q^1_1, $q^2_2 = q^3_3$ are non trivial, but if the first equation is satisfied, then the second one is satisfied automatically as a result of the Hilbert identities.

$$q^1_1 = \frac{V\,(2\,r\,V' + V)}{r^2} = \frac{(r\,V^2)'}{r^2}. \tag{3.2}$$

In vacuum $q^1_1 = 0$, from which

$$r\,V^2 = const \equiv 2\,k\,M \geq 0,$$

where k is the gravitational constant, and M is the constant of integration, which can be treated as the mass of a central body. This constant must be positive, whereas in the GR the positive sign of mass is a problem.

The field of radial velocities

$$V^r = V = \sqrt{\frac{2\,k\,M}{r}}$$

leads to the four-dimensional metric

$$ds^2 = \left(1 - \frac{2kM}{r\,c^2}\right) c^2\,dt^2 + 2\sqrt{\frac{2kM}{r}}\,dt\,dr - dr^2 - r^2(d\vartheta^2 + \sin^2\vartheta\,d\varphi^2). \tag{3.3}$$

In 1921 this metric was obtained by Painlevé [3] by transformation of the time variable in Schwarzschild [4] solution of the GR. Painlevé's attention was attracted by the simplicity of the space section $t = const$, which turned out to be a flat Euclidean space.

The reverse metric of this space have $g^{00} = 1$.

3.3. The Vortex field

The problem of space vortexes has no an analog in the GR and is the specific problem of the GTT.

The metric is stationary, axially-symmetric, and thus can be transformed into

$$dl^2 = e^{w(r,\vartheta)}\,(dr^2 + r^2\,d\vartheta^2) + r^2\,\sin^2\vartheta\,d\varphi^2. \tag{3.4}$$

The absolute velocities field is also dependent on r and ϑ, and is the vortex field $V^{\varphi} = \Omega(r, z)$. The kinetic energy is given by

$$T = \frac{c^2}{32\pi k} \int \left(\Omega,_r^2 + \frac{1}{r^2} \Omega,_\vartheta^2 \right) r^4 \sin^3 \vartheta \, dr \, d\vartheta \, d\varphi \tag{3.5}$$

is determined exclusively by the vortex field Ω and is independent of the metric function w.

The unique nontrivial constraint for V^{φ} in the absence of current yields the equation for V^{φ}:

$$\Omega,_{rr} + \frac{4}{r} \Omega,_r + \frac{1}{r^2} \left(\Omega,_{\vartheta\vartheta} + 3 \operatorname{ctg}\vartheta \, \Omega,_\vartheta \right) = 0. \tag{3.6}$$

Note, that this second order *linear* differential equation is independent on the metric function $w(r, \vartheta)$.

The equations for the metric determine the derivatives of the function w:

$$w,_r = \frac{r}{2} \left(\Omega,_\vartheta^2 - r^2 \, \Omega,_r^2 - 2 \operatorname{ctg}\vartheta \, r \, \Omega,_r \, \Omega,_\vartheta \right) \sin^4 \vartheta;$$

$$w,_\vartheta = \frac{r^2}{2} \left(\operatorname{ctg}\vartheta \left(r^2 \, \Omega,_r^2 - \Omega,_\vartheta^2 \right) - 2r \, \Omega,_r \, \Omega,_\vartheta \right) \sin^4 \vartheta. \tag{3.7}$$

The energy density now is expressed solely trough derivatives of Ω:

$$\varepsilon \sqrt{\gamma} = \frac{r^2 c^2}{8\pi k} \left(r^2 \, \Omega,_r^2 + \Omega,_\vartheta^2 \right) \sin^3 \vartheta, \tag{3.8}$$

and the kinetic energy is exactly four times smaller. This is the result of *the space virial theorem.*

The full energy in a given region B without external sources

$$E_B = 2\pi \int_B \varepsilon \sqrt{\gamma} \, dr \, d\vartheta \tag{3.9}$$

is positive and reaches a minima in the equation (3.6) solutions.

3.4. The space virial theorem

Denoting $\pi_i^i \equiv \pi$, the sum of equations (2.7), in absence of external sources (for proper gravitation), gives

$$\dot{\pi} + \partial_s (V^s \pi) = -3T - \frac{1}{2} R \sqrt{\gamma} = -3T + U, \tag{3.10}$$

where T and U are the densities of the kinetic and the potential energy, respectively. The space virial theorem can be applied to the almost stationary fields in space, on the boundaries of which there is no current flow, $V^n = 0$. Averaging (3.10) over time, we obtain the relationship between the average potential, kinetic, and total energies:

$$U = 3T; \quad E = T + U = 4T. \tag{3.11}$$

Under the aforementioned conditions, the kinetic and total energies are positive.

All conditions for application of this theorem to the given task are satisfied.

Determination of the kinetic energy requires knowledge about the field of absolute velocities in a given region. This information may be obtained from visible stars. The virial space theorem allows calculation of the full energy.

3.5. The weak superposition principle

The main part of the vortex problem is to solve the linear differential equation (3.6). Afterwards, equations (3.7) determine the metric function $w(r, \vartheta)$.

Although the overall problem is nonlinear, the first (main) part – determination of the vortex field $\Omega(r, \vartheta)$ – is linear and subject to the superposition principle.

Thus, any field Ω can be represented as the superposition of some basic solutions. However, equations (3.7) for finding the field $w(r, \vartheta)$ contains the square of the field Ω derivatives. The solution as a whole is not a superposition of partial solutions.

3.6. Multipole solutions

The differential equation (3.6) is homogeneous along radius r , thus its common solutions can be found in the form of a power series

$$\Omega(r, \vartheta) = \sum_{l=0}^{\infty} \left(A_l\, r^l + \frac{B_l}{r^{l+3}} \right) P_l(\cos\vartheta). \tag{3.12}$$

The angular part is subject to the differential equation (where $x = \cos\vartheta$):

$$(x^2 - 1)P_l'' + 4\,x\,P_l' - l\,(l+3)P_l = 0. \tag{3.13}$$

The solutions with integer l are the Hegenbauer's polynomials with $\alpha = 3/2$. They are the base of spherical functions in a five-dimensional space. Particularly, at $l = -3$ (as at $l = 0$) the solution of equation (3.13) is a constant – there is a *monopole solution*

$$\Omega_0(r, \vartheta) = \frac{1}{r^3}. \tag{3.14}$$

3.7. The Energy

To get an idea about cosmic energies, we examine the following problem. A globe with radius R is in constant rotation with angular velocity of Ω *coherently.* This means that the globe velocity on the surface coincides with the velocities of space, i.e. the field of angular velocities outside the globe are determined by a monopole solution

$$\omega(r) = \Omega\,\frac{R^3}{r^3}. \tag{3.15}$$

The energy density outside the globe (inside the globe, the field is homogeneous and the energy density is zero):

$$\varepsilon = \frac{9\,\Omega^2\,R^6\,\sin^2\theta}{r^6},$$

and the full energy of space is:

$$E = \frac{c^4}{16\pi k}\,9\,\Omega^2\,R^6\,2\pi \int_0^{\pi} \sin^3\vartheta\,d\vartheta \int_R^{\infty} \frac{r^2\,dr}{r^6} = \frac{R^3\,\Omega^2\,c^2}{2\,k} \equiv M\,c^2, \tag{3.16}$$

where M is the equivalent mass (not the mass of the globe)

$$M = \frac{R^3\,\Omega^2}{2\,k}. \tag{3.17}$$

For example, we examine a globe with diameter 20 cm. ($R = 0.1\,m$), that completes one rotation per second ($\Omega = 2\,\pi\,c^{-1}$). We obtain $M = 300\,000\,000$ kg. To force the space outside the globe to rotate coherently with the globe requires as much energy as is released upon annihilation of 300 000 tons of matter. Hence, laboratory experiments with space vortexes are not realistic.

This example also explains why our space is Euclidean with high accuracy: in the energy expression, there is a huge factor $c^4/(16\,\pi\,k)$ in front of the space curvature. This means that the smallest deviation from Euclidean space require tremendous energy.

Our space is (almost) Euclidean not due to the beauty and elegance of Euclidean geometry, but because this space has minimum energy.

4. Big Bang in the GTT

Since the Hamiltonian in the GTT is not equal zero, the effective quantum theory of gravitation can be built on the basis of the Schroedinger equation, as for other fields.

Further we will work in Plankean system of units where light velocity $c = 1$, $8\,\pi\,k = 1$ and $\hbar = 1$. All physical values are dimensionless and the energy E is determined by the dimensionless value e :

$$E = e\,\frac{c^4}{8\,\pi\,k}.$$

4.1. Classical solutions

Now we study the compact cosmological model of Friedman type with space as a three-dimensional sphere with variable radius r , depended on the time t . This Universe is filled by ultra-relativistic matter with the state equation $\varepsilon = 3\,p$.

The Lagrangian of isoenthrophic gas is expressed by integral over space of pressure, determined as function of the chemical potential [5] μ . For ultrarelativistic matter the pressure is proportional to μ^4 , what together with Lagrangian of space yields the full Lagrangian:

$$L = -\frac{r\,(\dot{r}^2 - 1)}{2} + \rho_0 \frac{\dot{\sigma}^4}{4}\,r^3 \tag{4.1}$$

and further the Hamiltonian

$$H = -\dot{r}\,p_r + \dot{\sigma}\,p_\sigma - L = -\frac{p_r^2 + r^2}{2\,r} + \frac{q^2}{2\,r}, \tag{4.2}$$

where q^2 determines the conserved quantity of the ultrarelativistic matter.

The classical equation of motion can be derived from the energy conservation law:

$$\dot{r} = \frac{1}{r}\,\sqrt{q^2 - 2E\,r - r^2}. \tag{4.3}$$

This equation describes the radius oscillations between the maximal and minimal values, which are determined by roots of the subroot expression: $r_{max} = \sqrt{e^2+q^2} - e$, and the second root is negative.

If $q^2 \neq 0$, the energy can be either negative or positive. At $q^2 = 0$ we have a pure gravitational dynamics, the one without matter. In this case the energy can j,tain only negative values,

4.2. Quantum model

The wave function is a function of the radius r. By u' we denote the derivative of the wave function $u(r)$ over r. By symmetryzation the product p^2/r we obtain the cosmological wave equations:

$$-\left(\frac{u'}{r}\right)' + \left(-r + \frac{q^2}{r}\right) u = 2\,E\,u,$$

$$u'' - \frac{u'}{r} + (-r^2 + q^2)\,u = 2\,r\,E\,u. \tag{4.4}$$

This equation has a regular special point $r = 0$ and irregular one $r = \infty$, at that vicinity the wave function behave as the one of the oscillator:

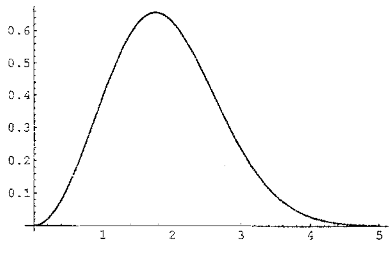
$$u(r) \sim A\,e^{-r^2/2} + B\,e^{r^2/2}.$$

At the some values of E, the coefficient B vanishes – there is normalized solutions of the quantum equation. The function is equal zero at $r = 0$ and infinity, and hence it can have n extremums.

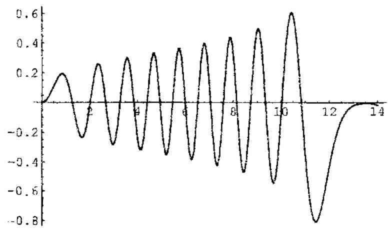
At the vicinity of zero radius all solutions behave as r^2. This means that probability density at $r \to 0$ in any state is zero.

The equation (4.4) have two parameters: q and E.

At $q^2 = 1$ for $n = 1, 18$ the solutions are represented on the graphics:



n=1



n=18

The self energy values for small n at $q^2 = 0, 1, 10$ are represented in following table

n	$q^2=0$	$q^2=1$	$q^2=10$
1	-1.3133	-1.0202	2.6765
2	-1.9243	-1.7122	0.564
3	-2.3863	-2.2107	-0.441
4	-2.773	-2.6193	-1.1208
...	...	...	...
8	-3.9599	-3.8487	-2.8153

The eigenvalues of the energy can be equal to zero (as in the GR), but only at the $q^2 = 4n$. This value of q has only one wave function with n extrema.

This model demonstrates the difference of quantum and classical solutions. The solutions of the classical equation (4.3) describes the radius oscillations between the positive maximal and negative minimal radiuses. The point $r = 0$ in classical solution is usual point. In quantum solutions the point $r = 0$ is special point and the value of the wave function at this point is zero, so the quantum oscillations of radius are between zero and maximal classical radius.

5. ADM-representation

The bridge between the GR and the GTT is the ADM representation of the four-dimensional metric in the GR [6], where the time is explicitly separated from space coordinates.

The ADM representation presents 10 components of the four-metrics through 6 components of the three-metrics γ_{ij}, three-vector V^i (in the GTT notations) and the function of the *time flow* $f(x,t)$:

$$g_{00} = f^2 - \gamma_{ij}V^iV^j; \quad g_{0i} = \gamma_{ij}V^j; \quad g_{ij} = -\gamma_{ij}. \tag{5.1}$$

The inverse metrical tensor components are

$$g^{00} = \frac{1}{f^2}; \quad g^{0i} = \frac{V^i}{f^2}; \quad g^{ij} = \frac{V^iV^j}{f^2} - \gamma^{ij}. \tag{5.2}$$

The variations of the Hilbert action is

$$\delta S = -\int \left(G^{00}\,\delta f + G_{0i}\,\delta V^i + \frac{1}{2}G_{ij}\,\delta\gamma^{ij} \right) \sqrt{\gamma}\, f\, d_3x\, dt. \tag{5.3}$$

The general covariance requires vanishing of all variations, what leads to 10 Einstein equations.

In the GTT, the component $g^{00} = 1$ always and everywhere, and this component cannot be a variable. The action variation by this function is not required to be zero but can be an arbitrary function. This variation is the energy density.

This unique variation differ the GTT from the GR. If we consider the GTT solutions with the energy density equal to zero everywhere, we obtain the GR solutions. In the GR we have $H = 0$ in any region, what is an obstacle for using the Shroedinger's quantum theory in a way similar to that applied for others fields.

6. Conclusion

The GTT is valid competitor to the GR. All known phenomena can be described as by the GR as by the GTT. But the GTT is the dynamical theory of the space with the effective Hamiltonian, what allows to create the quantum gravity in a usual way. The nonzero (and non positive determined) energy density also can explain the different cosmological anomalies.

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Physical Nature of Lobachevsky Parallel Lines and a New Inertial Frame Transformation

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The synchronous process of particle motion and light beams propagation has been found to reveal the physical foundation for violation of the V-th Euclidean postulate in the velocity space. The process revealed also its fruitfulness in solving in a new way the main problem in relativity - the problem of time synchronization for different space points [1]. The first obvious consequences of the new solution - such as simultaneity, proper time, inertial frame coordinate transformation and relativistic velocity summation law - are also presented in this paper.

1. Introduction

The physical nature of Lobachevsky parallel lines (LPL) remains unknown despite of the fact that the Lobachevsky velocity space is widely used to study particle scattering processes in modern high energy physics. As the existence of LPL is based on the denial of the Euclidean V-th postulate, then a physical foundation for its violation is also not known. At present, LPL have only a geometrical interpretation - either as infinite lines on a pseudospherical surface or as hordes on the Euclidean circle [2].

Further developments of the approach published earlier in [3] have been described in this paper. We consider light propagation according to the Huygens principle and the independency of the light beams. So, the phenomena of light diffraction and interference are not considered. It is assumed that the time counting for a space point starts when a light front comes to that point. This is also the moment of a secondary light hemisphere emission, according to the Huygens principle. We accept the constant light velocity principle and we use the same plane light fronts as widely used to explain the light reflection and refraction phenomena. The basic knowledge of Lobachevsky geometry [2, 4] is assuming.

2. Physical nature of Lobachevsky parallel lines

Let us consider two inertial frames K and K_s. Each of the frames may be associated with a particle. The space axises of both frames are parallel and K_s is moving with constant velocity V along the X-axis of frame K. It is assumed that their origins, O and O_s, coincide when the plane light front directed at the parallel angle θ_L reaches the point O (a lateral beam is moving from bottom to top in XY-plane as shown in Fig.1a). At this initial moment a light sphere (hemisphere to the falling front) starts to spread out from O. The parallel angle θ_L is defined as

$$\cos\theta_L \equiv \cos\Pi(\rho/k) = th(\rho/k) = V/c \equiv \beta, \qquad (k = c) \tag{2.1}$$

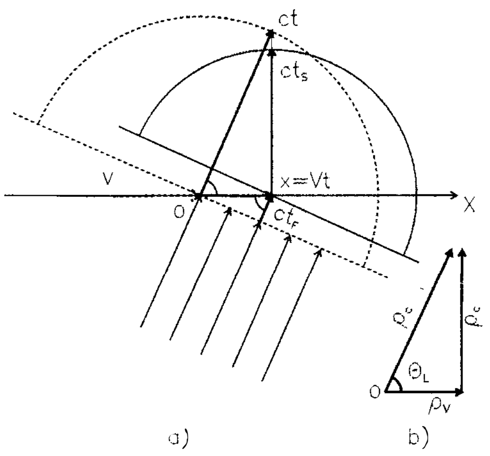

Fig. 1. a) Synchronization of the K_s-motion (Vt) and the light rays (ct and ct_s) propagation by the side light beam. b) Lobachevsky parallel lines in the velocity space plane corresponding to synchronous motions of ct, ct_s and Vt in Euclidean plane ($c = 1$ is used for rapidities).

here β is the velocity V in units of c, ρ/k is a value of rapidity ρ in units of $k = c$, $\Pi(\rho/k) \equiv \theta_L$ is a parallel angle, k is the Lobachevsky constant, c is the velocity of light. The second equality $\beta = th(\rho/c)$ in (1.1) is known from the Beltarami model [2] and used to define a particle rapidity:

$$\rho/c = 1/2 \ln((1+\beta)/(1-\beta)). \tag{2.2}$$

The first equality in (1.1) can be rewritten as

$$\theta_L \equiv \Pi(\rho/k) = 2arctg\, e^{-\rho/c}, \tag{2.3}$$

known as the Lobachevsky function. It is seen from (1.1) that for any rapidity (and its velocity) there is a definite angle θ_L . For the negative argument of the Lobachevsky function the parallel angle θ_L changes to $\pi - \theta_L$ [2], which corresponds to the same velocity but for the opposite direction.

Let us consider a space-time point $(x = Vt, t)$ in frame K . The light ray from the origin O will get to this point in time x/c (Einstein's signal) but the lateral beam's ray will come there first with some delay (relatively to O) in the moment of time t_F as

$$ct_F = x\cos\theta_L = Vt\cos\theta_L = ct\cos^2\theta_L, \tag{2.4}$$

and then a new light sphere starts to spread out from the x-point. By the given moment of time t a new sphere will spread out to the radius

$$ct_s = ct - ct_F = ct - x\cos\theta_L = ct - xV/c, \qquad t_s = t - xV/c^2, \tag{2.5}$$

and for $x = Vt$:

$$ct_s = ct - ct\cos^2\theta_L = ct\sin^2\theta_L = ct\,(1 - V^2/c^2), \tag{2.6}$$

where ct is the light sphere radius from origin O, so that $ct_s < ct$.

Let us choose two light rays from these two spheres: one, ct, emitted from O under the angle θ_L to the X-axis in some plane, and the other, ct_s, emitted from O_s (located at x) perpendicular to the X-axis in the same plane (see Fig.1a). Three segments ct, Vt and ct_s form a rectangular triangle. But two sides of triangle, ct and ct_s, have no common (intersection) point at no moment of time t, so they are parallel in any chosen Euclidean plane. As rapidity (1.1) for the light velocity is the infinity, then the obtained triangle transforms into the LPL or, more precisely, into the parallel lines in one side on the Lobachevsky plane in the velocity space as it is illustrated in Fig.1b.

Thus, the LPL in a velocity space corresponds to the light rays ct and ct_s emitted (according to the Huygens principle) from different points and different times and synchronized with particle motion Vt by the side light beam. The physical reason for the lack of intersection point is the time delay t_F (see (3.)). This time delay is an obvious physical foundation for the violation of the V-th postulate in the velocity space. As the value of t_F for given x defines by c (with changing V the θ_L changes but not the c) then one can conclude that the basic reason for the V-th postulate violation is the constant light velocity principle.

To find out light rays corresponding to LPL in another side, one can consider a lateral beam to another direction (from top to bottom) in the same plane (as shown in Fig.2a and Fig.2b).

For light rays corresponding to the LPL (in both sides) for negative argument of Lobachevsky function (for $V < 0$), one should use a pair of lateral beams directed opposite to X-axis, i.e. from right to left (for $V > 0$ the beams were directed from left to right), as shown in Fig.2c and Fig.2d. A complete set of light rays synchronized with the particle motion Vt (both for $V > 0$ and $V < 0$) which corresponds to LPL in the velocity space is presented in Fig.3.

Thus, the moving reference frame (for $V > 0$ and/or $V < 0$) can be associated with the definite lateral light beams. The rest frame ($V = 0$) is associated with the direct beams at $\theta_L = \pi/2$ (as shown in Fig.2). Lobachevsky function has the same form for the rest frame and for the moving ones, i.e. it follows the principle of relativity. So, Lobachevsky function expresses the constant light velocity principle at $k = c$.

The synchronization method used to reveal the physical nature of Lobachevsky parallel lines is also fruitful in solving the main difficulty of relativity - the problem of time synchronization for different space points.

3. x and t- coordinate transformation and light ether concept

Let us continue with the inertial frames K and K_s for $V > 0$. One can assume that a pair of direct beams (from top and bottom) reaches X-axis at the same moment of time

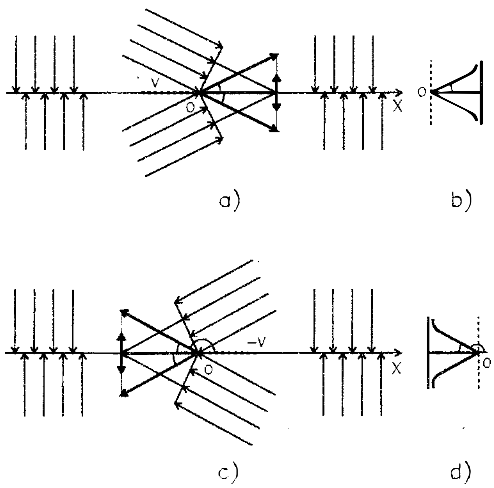

Fig. 2. a) Two lateral light beams (for $V > 0$) give two pairs of light rays ct and ct_s for both sides of the plane (top and bottom), synchronous with K_s-motion Vt. b) Parallel lines in both sides on Lobachevsky plane, corresponding to synchronous motions in a). The plots for $V < 0$ are shown in c) and d).

as a pair of lateral beams (from left to right) reaches the point where both origins coincide. All x-points (including O) are "exited"simultaneously, and this moment of time is usually chosen as the initial one for K frame (the same for all coordinates). The initial moment of time for any x-point is delayed by t_F relative to the lateral beams (see (3.)) so that time t_s at a given moment of time t (in K) is defined by (3.1). Thus, due to the synchronization of K and K_s frames (by the corresponding pairs of direct and lateral fronts) two moments of time, t and t_s, can be defined at any x point. For the chosen event (x,t) time t_s depends only on the velocity of the moving frame K_s.

Let us define the time t in the fixed frame via the distance ct passed by the light ray emitted from the point O at the parallel angle θ_L to X-axis in some plane. It is seen from Fig.1-Fig.4 that for any event (x,t) the delay time ct_F is just a projection of the given x-point on the chosen light ray ct.

Obviously, the displacement of K_s origin $Vt = ct\ cos\theta_L$ is just a projection of the light ray ct on the X-axis. So, for any given coordinate x at a given time t a value x_s relative to the origin O_s is

$$x_s = x - Vt = x - ct\cos\theta_L. \tag{3.1}$$

For any event $(x = Vt,\ t)$ a relative coordinate is $x_s = 0$. It means that time t_s (see (3.1) and (3.2)) is the proper time of K_s, i.e. the time "measured"by means of a "moving clock", when one spectator observes the light sphere with the radius ct in K and in the same time t a moving spectator observes another light sphere with the radius ct_s (both spheres are

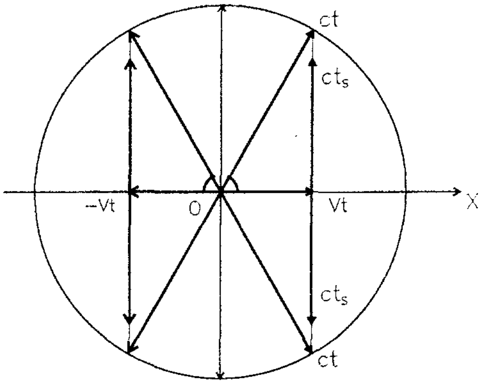

Fig. 3. An illustration for synchronous propagation of the corresponding light rays and particle motion (for $V > 0$ or $V < 0$) in K -frame.

triggered off by the lateral light beams). For the event (x, t) the corresponding moment of time t_s is the time "measured"by means of the "moving clock"located at the point x_s of K_s . Unlike of t in K , the time t_s defined for O_s is not all the same for the points on X_s -axis.

Indeed, from (3.) one can see that the initial moment of time (provoked by the lateral light front) propagates along X -axis with the velocity v_F :

$$v_F \equiv x/t_F = c/\cos\theta_L = c^2/V = c/\beta > c. \tag{3.2}$$

So, for $0 < V < c$ any two events (x_1, t) and (x_2, t) have different time t_s in K_s . For $V \to 0$ ($\theta_L \to \pi/2$ for side beams) the velocity $v_F \to \infty$ and one comes to the Newton time $t_s \to t$, and for $V = c$ ($\theta_L = 0$) the proper time $t_s = 0$.

Thus, for any event (x, t) in K the corresponding coordinates in K_s are simple shifts (see (3.1) and (3.3)). To obtain the values of shifts, one should make symmetrical projections as described above. We have used this symmetry to find out the Lorentz coordinates x' and t' for a moving frame. To get them, one has to find the crossing point O' of two perpendiculars producing the projections for any (x, t) event (see Fig.4). Then the length of the interval from O' to x corresponds to x' :

$$x' = (x - ct\cos\theta_L)/\sin\theta_L = (x - Vt)/\sqrt{1 - V^2/c^2}, \qquad x_s = x'\sin\theta_L, \tag{3.3}$$

and the distance from O' to the ct corresponds to ct' :

$$ct' = (ct - x\cos\theta_L)/\sin\theta_L = (ct - xV/c)/\sqrt{1 - V^2/c^2}, \qquad ct_s = ct'\sin\theta_L. \tag{3.4}$$

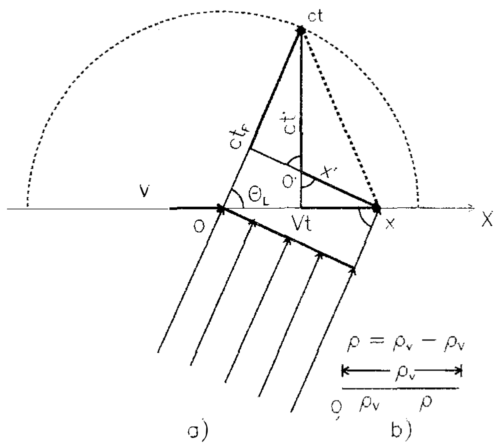

Fig. 4. a) An illustration of the inertial frame x and t coordinate transformation (including Lorentz transformation). b) A velocity space diagram corresponding to x and t shifts. The x-coordinate is the x-position of a particle, moving with a velocity of $v = x/t$ in K frame by the moment of time t.

It is seen from (3.5) and (3.6) that primed and shifted coordinates are related as the corresponding projections. But the point O', which is always considered as the origin of the moving frame, does not coincide in space with O_s. It is also seen that the line $O'x'$ is not parallel to the X-axis. So, it seems obvious that the primed values can not be regarded as the coordinates in a moving frame.

The distance between the given points x and ct (dashed line in Fig.4) can be defined via the primed and unprimed values:

$$l^2 = c^2t^2 + x^2 - 2ctx\cos\theta_L = c^2t'^2 + x'^2 + 2ct'x'\cos\theta_L \equiv l'^2\,, \tag{3.5}$$

or as a sum of two terms, either as $l^2 = s_1^2 + s_2^2$ (to get it one should add $\pm x^2$ to the left part of (3.7) and $\pm x'^2$ to its right part), or as $l^2 = -s_1^2 + s_3^2$ (add $\pm c^2t^2$ to the left part of (3.7) and $\pm c^2t'^2$ to the right part), where:

$$s_1^2 = c^2t^2 - x^2 = c^2t'^2 - x'^2 = \gamma^2(c^2t_s^2 - x_s^2)\,, \qquad \gamma = 1/sin\theta_L = 1/\sqrt{1 - V^2/c^2}\,, \tag{3.6}$$

$$s_2^2 = 2x(x - ct\cos\theta_L) = 2x'(x' \pm ct'\cos\theta_L)\,, \qquad s_3^2 = 2ct(ct - x\cos\theta_L) = 2ct'(ct' \pm x'\cos\theta_L)\,. \tag{3.7}$$

Term s_1^2 is known as an invariant interval. Obviously, it is only a part of the full distance l^2 and is a result of cancelling of two equal values, either s_2^2, or s_3^2 in the expressions for $l^2 = l'^2$. Terms s_2^2 and s_3^2 may differ by sign: (+)/(-) corresponds to the point O' located inside/outside the cone defined by the angle θ_L. For an event $(x = Vt, t)$ term s_2^2 is equal to zero (as $x_s = x' = 0$) and $s_3^2 = 2s_1^2$, so $l^2 \equiv s_1^2 \equiv l'^2$. The Lorentz coordinate transformations for this particular case have being usually presented in the manuals (e.g. [5]).

From (3.9) one can find (using the second formulae in (3.5, 3.6))

$$x = (x_s + ct_s \cos\theta_L)/\sin^2\theta_L = (x_s + Vt_s)/(1 - V^2/c^2), \tag{3.8}$$

and

$$ct = (ct_s + x_s \cos\theta_L)/\sin^2\theta_L = (ct_s + Vx_s/c)/(1 - V^2/c^2), \tag{3.9}$$

which are the reverse transformation from the moving frame to the rest frame. To check that, one can solve (3.1) and (3.3) for x and ct (once the factor $1/sin\theta_L$ is inserted into the brackets then the terms in brackets became the lengths of perpendiculars corresponding to the mentioned projection symmetry).

It is seen from (3.1),(3.3) and (3.10-3.) that the direct and reverse transformations are different: the latter could not be obtained by changing V to $-V$. This means that one already knows that the frame either moves, or not. When changing V on $-V$ one should also choose an appropriate lateral light beam direction for a moving frame. So, if K_s moves backward to X ($V < 0$) one should change the sign in (3.1), (3.3) and in nominators of the reverse formulae (3.10-3.). Thus, for any two frames one frame can be regarded as a moving frame and other one as the rest frame and vise versa by choosing the corresponding direct and lateral light beams (according to the known parallel angles).

A possible way to realize these opportunities is to make an assumption about the presence of many light streams of any directions. One may assume an ether, not a restful one, but the moving light ether. The absence of the absolute frame testifies upon the absence restful ether and does not contradict the presence of the moving light ether.

Thus, the relation between space and time coordinates expresses through the parallel angle or through the corresponding velocities. So, this relation is generated by the presence of the corresponding light streams and particles.

4. y, z- coordinate transformation and invariants

Let us consider event $(x, y, z = 0, t)$ in K frame. The lateral light beam is reaching X-axis in XY-plane as shown in Fig.5, i.e. it spreads from bottom to top, first enters the plane point (x, y) and then the point $(x, y = 0)$ at the X-axis (if y-coordinate has an opposite sign, then one can choose another lateral beam heading from top to bottom).

The secondary light sphere spreads out from the first point to the point $(x, y = 0)$ at the X-axis in a time of y/c. The lateral beam ray reaches this point in a moment of time $y\, sin\theta_L/c$ (since the secondary sphere starts to spread out from the first point). So, the light way difference is

$$c\Delta t \equiv \Delta y = y - y\sin\theta_L. \tag{4.1}$$

To compensate for this difference and make the initial moment of time counting caused by the lateral beam to be the same for x_s and y_s, the origin of K_s frame should be shifted along the Y-axis by the value of Δy (4.1). Then the y-coordinate in K_s frame is

$$y_s = y - \Delta y = y\sin\theta_L = y\sqrt{1 - V^2/c^2} \tag{4.2}$$

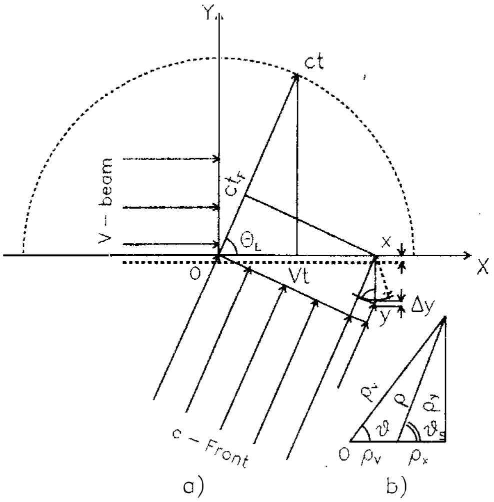

Fig. 5. a) An illustration of the Δy-shift origin due to the light way difference, and b) a corresponding velocity space diagram (see note in Fig.4b).

and the transverse coordinate

$$z_s = z - \Delta z = z \sin\theta_L = z\sqrt{1 - V^2/c^2}\,. \tag{4.3}$$

The reverse transformation is also obvious:

$$y = y_s/\sin\theta_L = y_s/\sqrt{1 - V^2/c^2}\,, \qquad z = z_s/\sin\theta_L = z_s/\sqrt{1 - V^2/c^2}\,. \tag{4.4}$$

Then for the non-invariant interval (see(3.8)) one can get

$$c^2t^2 - x^2 - y^2 - z^2 = \gamma^2(c^2t_s^2 - x_s^2 - y_s^2 - z_s^2)\,. \tag{4.5}$$

The obtained coordinate transformation leads to the contracted interval but this does not contradict to the relativistic velocity summation law.

So, for any event (x, y, z, t) in K there is the "parallel"event (x_s, y_s, z_s, t_s) corresponding to the moving K_s frame shifted in space and time in an appropriate way. These two sets of coordinates are related by the equation (4.5).

As it is seen from Fig.4 the point O' looks as a center of projectivity and the X-axis with the chosen light ray ct may be considered as a projective lines [4]. Let us consider x and ct values as corresponding projective coordinates. The projectivity, or the projective transformation, establishes some definite correspondence or projective equivalence of the point-like systems of 1, 2 and 3 - dimensions, namely, between the points of two projective lines. The main invariant for projectivity is the complex fraction of any four elements of two multitude, i.e. of any four corresponding points for two projective lines [4]:

$$(x_1, x_2, x_3, x_4) = \frac{x_3 - x_1}{x_2 - x_3} : \frac{x_4 - x_1}{x_2 - x_4} = \frac{ct_3 - ct_1}{ct_2 - ct_3} : \frac{ct_4 - ct_1}{ct_2 - ct_4} = (t_1, t_2, t_3, t_4) \tag{4.6}$$

According to the main theorem of projective geometry the projectivity is determined when any three pairs of corresponding elements are defined. In our case it means that, in this case for any given x one can find the corresponding t from (4.6) and vise versa.

Let us consider two simple cases:

- for three known events $x_1 = 0$, $t_1 = 0$, $x_2 = Vt$, $t_2 = t$ and $x_4 = \infty$, $t_4 = \infty$ (4.6) becomes [4]

$$(x_1, x_2, x_3, \infty) = \frac{x_3 - x_1}{x_2 - x_3} = \frac{t_3 - t_1}{t_2 - t_3} = (t_1, t_2, t_3, \infty) \tag{4.7}$$

and for any x_3, t_3 one can find corresponding

$$t_3 = x_3 t_2 / x_2 = x_3 / V, \qquad x_3 = V t_3 . \tag{4.8}$$

Obviously, this result corresponds to the projection of the X-axis onto the ct ray and vise versa, made by the beam of lines with the center at infinity (just like the direct light beam used for the rest frame K).

- for $x_1 = 0$, $t_1 = 0$, $x_2 = x$, $t_2 = x cos\theta_L / c$ and $x_4 = \infty$, $t_4 = \infty$ one can get

$$t_3 = x_3 cos\theta_L / c, \qquad x_3 = c t_3 / cos\theta_L . \tag{4.9}$$

This result corresponds to the lateral projections with the center at infinity and with the beam direction turned by $\pi/2$ to that as it was used for the K_s frame.

The projectivity allows one to find the values of shifts. It follows from the comparison of (3.1),(3.3) and (2.9-4.9), but needs more study.

Let us rewrite (4.6) as

$$\frac{(x_1, x_2, x_3, x_4)}{(t_1, t_2, t_3, t_4)} = \frac{\beta_{x31}}{\beta_{x23}} : \frac{\beta_{x41}}{\beta_{x24}} = 1, \qquad \beta_{xik} = \frac{x_i - x_k}{c(t_i - t_k)}, \tag{4.10}$$

and use reverse transformation (3.10-3.) for each event (x_i, t_i). Then one can find:

$$\beta_{xik} = \frac{\beta'_{xik} + V/c}{1 + \beta'_{xik} V/c}, \qquad \beta'_{xik} = \frac{x_{si} - x_{sk}}{c(t_{si} - t_{sk})}. \tag{4.11}$$

One can carry through the same calculations for y and z coordinates:

$$\beta_{yik} = \frac{y_i - y_k}{c(t_i - t_k)} = \frac{\beta'_{yik}\sqrt{1 - V^2/c^2}}{1 + \beta'_{xik} V/c}, \qquad \beta'_{yik} = \frac{y_{si} - t_{sk}}{c(t_{si} - t_{sk})}, \tag{4.12}$$

$$\beta_{zik} = \frac{z_i - z_k}{c(t_i - t_k)} = \frac{\beta'_{zik}\sqrt{1 - V^2/c^2}}{1 + \beta'_{xik} V/c}, \qquad \beta'_{zik} = \frac{z_{si} - t_{sk}}{c(t_{si} - t_{sk})}. \tag{4.13}$$

Relations (4.11-4.13) are known as the relativisic velocity summation law. From these relations one can find the expressions for primed components and get for each of them:

$$\frac{\beta_{31}}{\beta_{23}} : \frac{\beta_{41}}{\beta_{24}} = \frac{\beta'_{31}}{\beta'_{23}} : \frac{\beta'_{41}}{\beta'_{24}} = 1 . \tag{4.14}$$

Thus, instead of a noninvariant interval (4.5) one may consider a well known invariant (4.6), the main invariant in projective geometry which allows to establish the correspondence between space and time coordinates if any three events are defined. The interval (4.5), which is defined by two events, does not satisfy this requirement.

5. Relativistic effects and wave character of the initial moment of time propagation

Let us consider two events (x_1, t_1) and (x_2, t_2) in K frame and two events (x_{s1}, t_{s1}) and (x_{s2}, t_{s2}) corresponding to them in K_s frame. Then, according to the new transformation ($V > 0$), one can get for shifted coordinates:

$$\Delta x_s = \Delta x - c\Delta t cos\theta_L \,, \qquad c\Delta t_s = c\Delta t - \Delta x cos\theta_L \,, \tag{5.1}$$

and for unshifted coordinates:

$$\Delta x = \gamma^2(\Delta x_s + c\Delta t_s cos\theta_L)\,, \qquad c\Delta t = \gamma^2(c\Delta t_s + \Delta x_s cos\theta_L)\,, \tag{5.2}$$

where $\Delta x_s = x_{s2} - x_{s1}$, $\Delta t_s = t_{s2} - t_{s1}$, and $\Delta x = x_2 - x_1$, $\Delta t = t_2 - t_1$. Let us also remind the relation between the primed and shifted values (see (3.5-3.6)):

$$\Delta x' = \gamma \Delta x_s \,, \qquad c\Delta t' = \gamma c \Delta t_s \,. \tag{5.3}$$

If Δx is a length of some rod in the rest frame, then its length in the moving frame will be the difference of its coordinates Δx_s at the same moment of time $\Delta t_s = 0$ (by definition). Then one can find from (5.1) (using (5.3) for the primed values) that

$$\Delta x = \gamma^2 \Delta x_s = \gamma \Delta x' \,. \tag{5.4}$$

So, for the new transformation, the length of a rod becomes shorter even in comparison with the primed value.

But, it is seen from the second formula of (5.2) that the requirement $\Delta t_s = 0$ gives

$$c\Delta t = \Delta x \cos\theta_L \equiv \Delta t_F \,, \tag{5.5}$$

i.e. $\Delta t \neq 0$ (where Δt_F is the time delay difference for the points x_1 and x_2). It means that the moving frame has two identical moments of times $t_{s1} = t_{s2}$ in the two different space points Vt_1 and Vt_2 corresponding to the t_1 and t_2 in K frame (see Fig.6a). So, due to the definition of the rod length the measurements of two coordinates are performed from two (shifted) locations of frame K_s .

The requirement of simultaneity $\Delta t_s = 0$ for the moving frame can also reveal a wave character of the initial moment of time propagation along X -axis. Indeed, the wave propagation is characterized by the fact that the value of excitation function (depending on x and t) can be the same in different (x, t) points in two moments of time [6]. If time t_s (see (3.3)) is an argument of the function, then it may happen when

$$t - xV/c^2 = (t + \Delta t) - (x + \Delta x)V/c^2 \quad \Rightarrow \quad c\Delta t = \Delta x \cos\theta_L = c\Delta t_F \,, \tag{5.6}$$

and if $\Delta t_s = 0$, then $\Delta t = \Delta t_F$ (see Fig.6a). One can get from here:

$$\Delta x/\Delta t = \Delta x/\Delta t_F = c/\cos\theta_L \equiv v_F > c \,, \tag{5.7}$$

the same velocity v_F as in (3.4). So, for $V \neq 0$ the initial moment of time counting propagates as a wave with the finite velocity v_F along X -axis. It is known [6], that a differential wave equation is defined by the structure of an argument of the excitation

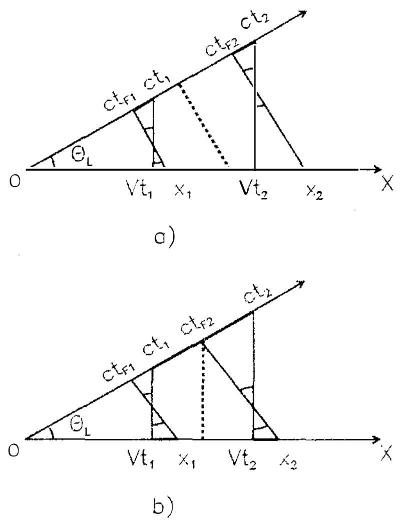

Fig. 6. Illustrations of the Lorentz contraction: a) for a length - two simultaneous events (x_{s1}, t_s) and (x_{s2}, t_s) in a moving frame; b) for a time interval - two events (x_s, t_{s1}) and (x_s, t_{s2}) in the same point in a moving frame.

function $\psi = \psi(x,t)$. As an argument of the light excitation function ψ has a form of $\psi(x,t) = \psi(t \pm xV/c^2)$, then this wave propagation (along X-axis) should follow an equation

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = \frac{1}{\beta^2}\frac{\partial^2\psi}{\partial x^2}. \tag{5.8}$$

For $\beta = 1$ (or $v_F = c$) it is the same as the known wave equation for the light, $\psi = \psi(t \pm x/c)$. When $\beta = 0$ (or $v_F = \infty$, i.e. when the lateral front becomes the direct one) the excitation function ψ does not depend on x, and the initial moment of time counting is the same for any x-point (Newton time): $\psi = \psi(t)$.

It is clear from the first equation of (5.1) that the rod's length is the same for the both frames: $\Delta x_s = \Delta x$, if one takes two events (x_1, t) and (x_2, t) in the fixed frame at $\Delta t = 0$. In this case the corresponding coordinates x_{s1} and x_{s2} in a moving frame are measured at different moments of time t_{s1} and t_{s2}, but from the same position Vt of K_s corresponding to the choosen t. It is possible to choose $t_{s1} = -t_{s2}$ and then $ct = (x_1 + \Delta x/2)\cos\theta_L$. This moment of time t corresponds to the projection of the rod's center onto the ct ray (see dashed line on Fig.6a).

Now, let us consider two events in a moving frame located in the same place, $\Delta x_s = 0$, but separated by the time interval $\Delta t_s = t_{s2} - t_{s1}$. The corresponding time interval in K frame one can get from the second formula of (5.2):

$$\Delta t = \gamma^2 \Delta t_s = \gamma \Delta t', \tag{5.9}$$

i.e. the new transformation again makes a time interval shorter for a moving frame. But one can see from the first formula of (5.1) that $\Delta x = c\Delta t \cos\theta_L = V\Delta t$ for $\Delta x_s = 0$, i.e. $\Delta x \neq 0$ in K frame. It means, that the moving frame has two identical coordinates $x_{s1} = x_{s2}$ in two different space points x_1 and x_2 (in K, see Fig.6b). The time delay difference for them, $c\Delta t_F = c\Delta t \cos^2\theta_L$ (see (3.)), is non-zero, so similar to (5.9) one has $c\Delta t_s = c\Delta t - c\Delta_F$. Again, due to the $\Delta x_s = 0$ requirement the measurements of two moments of time are made from two K_s-frames shifted in space with $c\Delta t_F \neq 0$ in rest frame.

It is clear from the second equation of (5.1) that an interval of time for the both frames is the same ($\Delta t_s = \Delta t$) if one considers two events (x, t_1) and (x, t_2) in the rest frame at $\Delta x = 0$, i.e. in the same space point. In this case the two corresponding moments of time t_{s1} and t_{s2} in a moving frame are measured also from two different points x_{s1} and x_{s2}, but $c\Delta t_F = 0$. One can choose $x_{s1} = -x_{s2}$ and find $x = V(t_1 + \Delta t/2) = c(t_1 + \Delta t/2)\cos\theta_L$. This x-coordinate corresponds to the projection of a middle point of the time interval $c\Delta t$ onto the X-axis (see dashed line in Fig.6b).

Thus, the nature of relativistic effects is not in changing the scales of space or time for a moving frame, but in changing of the reference points for the space and time coordinates. One can find the same values for the space or time intervals in the rest and moving frames by changing the way of measurement.

6. Lorentz energy-momentum transformation

The Lorentz transformation for particle energy-momentum in this consideration is a direct consequence of the relativistic velocity summation law or of the aditivity law for particle rapidity:

$$\rho' = \rho - \rho_o, \qquad \rho = \rho' + \rho_o, \tag{6.1}$$

where ρ' is a particle rapidity in the moving frame and ρ - in the rest, ρ_o is a rapidity corresponding to the velocity $\beta_o = V/c$ of a moving frame (we use $c = 1$ units for all rapidities). Then $\beta' = th\rho'$ and $\beta = th\rho$ are particle velocities in the moving and in the rest frames. Hyperbolic tangent of (6.1) leads to the relativistic velocity summation law:

$$th\rho' = \frac{th\rho - th\rho_o}{1 - th\rho th\rho'}, \qquad th\rho = \frac{th\rho' + th\rho_o}{1 + th\rho' th\rho_o}. \tag{6.2}$$

The particle velocity has to be transformed according to (6.2). This requirement can be satisfied by defining energy and momentum of the particle through its velocity $\beta = th\rho = (msh\rho)/(mch\rho) = P/E$, where m is a particle mass, $P = msh\rho = m\beta/\sqrt{1-\beta^2}$ is the momentum and $E = mch\rho = m/\sqrt{1-\beta^2}$ is the particle energy. It is easy to find values for sines and cosines from (6.1):

$$sh\rho' = sh\rho\, ch\rho_o - ch\rho\, sh\rho_o, \qquad ch\rho' = ch\rho\, ch\rho_o - sh\rho\, sh\rho_o, \tag{6.3}$$

Multiplying (3.) by the particle mass m and taking into account the energy and momentum definitions one can find the Lorentz transformation:

$$P' = (P - \beta_o E)/\sqrt{1-\beta_o^2}, \qquad E' = (E - \beta_o P)/\sqrt{1-\beta_o^2}. \tag{6.4}$$

A reverse transformation can be found in the same way from the second equation of (6.1). In general case, when direction of the particle velocity β does not coinside with the direction of the moving frame, one should assume longitudinal rapidity in (6.1) and longitudinal momentum in (6.4).

Thus, the requirement for the particle velocity expressed through its energy-momentum to be transformed according to the (6.2), may consider as some condition for their definition. As their relativistic definition is in agreement with (6.2), then the Lorentz energy-momentum transformation is a straightforward consequence of the relativistic velocity (rapidity) summation law.

7. Conclusions

- A complete correspondence has been established between Lobachevsky parallel lines in the velocity space and the synchronous process of particle and light beams propagation in the Euclidean space.
- The constant light velocity principle has been found as the physical reason for the violation of the V-th postulate in the Lobachevsky velocity space.
- Lobachevsky function has been shown as a tool to express the constant light velocity principle.
- A new method of time synchronization for different space points have been found and a new contents of the simultaneity conception, common time and proper time, have been formulated.
- A new inertial frame coordinate transformation, as the simple shifts, has been found. It leads to the known relativistic velocity summation law and requires the existence of the light (moving) "ether".
- It has been shown, that the initial moment of time counting for the moving frame propagates in space in the same direction with a finite velocity greater than the velocity of light.
- The relativistic effects have been shown to take place due to the coordinate and time shifts of the origin point. One can find the values of space or time intervals to be the same in the moving and the rest frames by changing the measurement way.
- It has been shown, that Lorentz energy-momentum transformation is a straightforward consequence of the relativistic velocity summation law.
- The four elements complex fraction invariant and a possible wave equation have been presented.

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Первые интегралы геодезических

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Строятся римановы пространства и пространства Вейля, уравнения геодезических линий которых допускают первый интеграл 2-го порядка (квадратичный и дробно-квадратичный соответственно).

Исследуются римановы пространства $V_n\Big(g_{ij}(xk)\Big)$ (g_{ij} — метрический тензор пространства, x^k — координаты пространства, $i, j, k = \overline{1, n}$, $n = \dim V_n$), допускающие существование так называемого [1] вейлево-геодезического поля конусов направлений. Эти поля направлений X^i определяются с помощью невырожденного симметрического тензора a_{ij} уравнением

$$a_{ij}\, X^i X^j = 0, \tag{0.1}$$

причем в данной связности тензор a_{ij} удовлетворяет вместе с некоторыми полями M_k и R_k условию

$$\nabla_{(k}\, a_{ij)} = M_{(k}\, a_{ij)} + R_{(k}\, g_{ij)} \tag{0.2}$$

(∇_k — символ ковариантного дифференцирования в данной связности, скобки, как обычно, означают симметрирование по индексам, содержащимся в них).

При $R_k = 0$ вейлево-геодезическое поле (0.1)–(0.2) является геодезическим полем конусов направлений [2], характеризующихся тем, что все геодезические линии, направления которых совпадают в некоторой точке с направлением, определяемым (0.1)–(0.2) при $R_k = 0$, сохраняют это свойство на всем своем протяжении.

К тому же наличие вейлево-геодезического поля конусов (0.1)–(0.2) хотя бы в одном римановом пространстве гарантирует существование такого же поля во всех пространствах Вейля $W_n(g_{ij}, \omega_k)$ (g_{ij} — основной тензор, ω_k — дополнительный вектор пространства), конформных данному V_n [3], и среди этих пространств Вейля существует однозначно определяемое пространство, в котором поле конусов (0.1)–(0.2) является геодезическим [3]. Это обстоятельство, кстати, и объясняет название поля (0.1)–(0.2).

Еще один любопытный факт связан с исследуемыми полями конусов в пространстве Вейля: имея решения уравнений (0.2) в римановом пространстве $V_n(g_{ij})$, при неградиентном поле M_k ($M_k \neq \partial_k M$) мы можем построить связность Вейля, основной тензор которой совпадает с метрическим g_{ij}, а дополнительный вектор ω_k строится [4] по вектору M_k. Полученная связность Вейля $W_n(g_{ij}, \omega_k)$ интересна тем, что её аффинная подвижность (сохранение связности некоторой группой Ли преобразований) является следствием её конформной подвижности (сохранения той же группой преобразований основного тензора связности) [4].

Как известно [5], если в (0.2) $M_k = R_k = 0$, то соотношение

$$a_{ij}\,\frac{dx^i}{ds}\,\frac{dx^j}{ds} = \mathrm{const}$$

(s — аффинный параметр (дуга) геодезической) является первым квадратичным (или 2-го порядка) интегралом дифференциальных уравнений геодезических линий данного риманова пространства $V_n(g_{ij})$. Если же поле (0.1)–(0.2) задано в пространстве Вейля $W_n(g_{ij}, \omega_k)$ и $R_k = 0$, а $M_k = 2\omega_k$, то соотношение

$$\frac{a_{ij}\,dx^i\,dx^j}{g_{ij}\,dx^i\,dx^j} = \text{const}$$

является первым дробно-квадратичным (и также 2-го порядка) интегралом уравнений геодезических линий W_n [6].

Таким образом, знание римановых пространств, допускающих вейлево-геодезическое поле конусов, позволяет в конечном счете построить аффинно-связные пространства, уравнения геодезических линий которых допускают первый интеграл 2-го порядка (квадратичный для римановых и дробно-квадратичный для пространств Вейля). Построение таких пространств и является целью исследования.

В основу построения положено следующее обстоятельство.

Первой известной римановой связностью, уравнения геодезических линий которых допускают первый квадратичный интеграл, является, по-видимому, связность поверхности Лиувилля [7].

В подходящей системе координат (x, y) метрику этой поверхности можно записать так:

$$ds^2 = \Big(Y(y) - X(x)\Big)(dx^2 + dy^2),$$

тогда тензор a_{ij}, определяющий первый интеграл геодезических, приводится к виду

$$a_{11} = (Y - X)Y, \quad a_{22} = (Y - X)X, \quad a_{12} = 0.$$

Здесь x, y — координаты на V_2; $X(x)$, $Y(y)$ — произвольные функции указанных переменных. Ясно, что эти функции являются (действительными) корнями так называемого [8] характеристического уравнения поля конусов

$$|a_{ij} - \lambda g_{ij}| = 0 \tag{0.3}$$

и служат характеристикой поверхности. Например, если одна из этих функций (один из корней (0.3)) постоянна, то такая поверхность Лиувилля является, как легко видеть, поверхностью вращения.

Поэтому при построении римановых пространств, допускающих вейлево-геодезическое поле конусов, учитываются свойства корней уравнения (0.3). Эти корни могут быть как действительными, так и комплексно-сопряженными функциями.

Предполагается, что эти корни обладают следующими свойствами: (i) размерности векторных подпространств, соответствующих действительным корням, равны их кратности; размерность векторных подпространств, натянутых на собственные векторы, соответствующие паре комплексно-сопряженных корней, равна удвоенной кратности корней; (ii) подпространства эти неизотропны; (iii) поля подпространств — голономны.

Поля конусов (0.1)–(0.2), характеристическое уравнение (0.3) которых обладает вышеперечисленными свойствами, называются специальными.

Аналоги поверхностей Лиувилля — римановы поверхности, допускающие специальное вейлево-геодезическое поле конусов (сеть), уравнение (0.3) которых имеет как пару комплексно-сопряженных корней, так и один действительный кратности 2, построены в [1]. Там же построены и псевдолиувиллевы поверхности — двумерные пространства

Вейля, уравнения геодезических которых допускают первый дробно-квадратичный интеграл, а уравнение (0.3) обладает теми же свойствами.

Отметим, что псевдолиувиллевы поверхности, уравнения (0.3) которых имеют пару действительных различных корней, построены в [6].

В [3] классифицированы трехмерные римановы пространства, допускающие существование специального вейлево-геодезического поля конусов. На основании этих результатов построены и трехмерные пространства (как Вейля, так и римановы), уравнения геодезических которых допускают первый интеграл 2-го порядка.

Римановы пространства V_n, допускающие специальное вейлево-геодезическое поле конусов направлений (0.1)–(0.2), характеристическое уравнений (0.3) которого имеет $k \leq n = \dim V_n$ различных действительных корней соответствующей кратности, полностью исследованы в [9]. Здесь же построены и первые интегралы 2-го порядка уравнений геодезических линий соответствующих пространств. Заметим, что полученные результаты не позволяют построить n-мерные римановы пространства, допускающие специальное поле конусов (0.1)–(0.2), уравнение (0.3) которого имеет единственный действительный корень, кратность которого равна размерности пространства.

Такие пространства можно получить, рассматривая римановы пространства с полем (0.1)–(0.2), уравнения (0.3) которых имеют хотя бы одну пару комплексно-сопряженных корней.

Оказывается, в том случае, когда такая пара корней единственна, строение пространства, допускающее подобное поле конусов, уравнение (0.3) которого имеет n-кратный действительный корень, практически ничем не отличается от строения трехмерных пространств, допускающих подобное поле конусов [3]. Остальные возможности подлежат дальнейшему изучению.

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The first integrals of the geodesics

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In this paper we consider Riemannian and Weyl spaces which admit 2th order first integrals of the geodesic (quadratic and quadratic-fractional respectly).

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INTEGRABLE STRING AND HYDRODYNAMICAL TYPE MODELS AND NONLOCAL BRACKETS

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The closed string model in the background gravity field is considered as a bi-Hamiltonian system in assumption that string model is the integrable model for particular kind of the background fields. The dual nonlocal Poisson brackets (PB), depending of the background fields and of their derivatives, are obtained. The integrability condition is formulated as the compatibility of the bi-Hamiltonity condition and the Jacobi identity of the dual Poisson bracket. It is shown that the dual brackets and dual Hamiltonians can be obtained from the canonical PB and from the initial Hamiltonian by imposing the second kind constraints on the initial dynamical system, on the closed string model in the constant background fields, as example. The hydrodynamical type equation was obtained. Two types of the nonlocal brackets are introduced. Constant curvature and time-dependent metrics are considered, as examples. It is shown, that the Jacobi identities for the nonlocal brackets have particular solution for the space-time coordinates, as matrix representation of the simple Lie group.

1. Introduction

The bi-Hamiltonian approach to the integrable systems was initiated by Magri [1] for the investigation of the integrability of the KdV equation. This approach was generalized by Das, Okubo [2].

Определение 1.1.. *A finite dimensional dynamical system with* $2N$ *degrees of freedom* $x^a, a = 1, \ldots, 2N$ *is integrable, if it is described by the set of the* n *integrals of motion* $F_1, \ldots, F_n$ *in involution under some Poisson bracket (PB)*

$$\{F_i, F_k\}_{PB} = 0.$$

The dynamical system is completely solvable, if $n = N$ *. Any of the integral of motion (or any linear combination of them) can be considered as the Hamiltonian* $H_k = F_k$ *.*

Определение 1.2.. *The bi-Hamiltonity condition* [2] *has following form:*

$$\dot{x}^a = \frac{dx^a}{dt} = \{x^a, H_1\}_1 = ... = \{x^a, H_N\}_N. \tag{1.1}$$

The hierarchy of new PB is arose in this connection:

$$\{,\}_1, \{,\}_2, \ldots, \{,\}_N.$$

The hierarchy of new dynamical systems arises under the new time coordinates t_k :

$$\frac{dx^a}{dt_{n+k}} = \{x^a, H_n\}_{k+1} = \{x^a, H_k\}_{n+1}. \tag{1.2}$$

The new equations of motion describe the new dynamical systems, which are dual to the original system, with the dual set of the integrals of motion.

There is another approach to the bi-Hamiltonian systems [1]. Two PB $\{,\}_1$ and $\{,\}_2$ are called compatible if any linear combination of these PB $c_1\{,\}_1 + c_2\{,\}_2$ is PB. It is possible to find two corresponding Hamiltonians H_1 and H_2 which are satisfy to bi-Hamiltonity condition.

We used first approach to the closed string models as the bi-Hamiltonity systems. Second approach was used to description of the hydrodynamical type models.

We consider the dynamical systems with constraints. In this case, first kind constraints are generators of the gauge transformations and they are integrals of motion. First kind constrains $F_k(x^a) \approx 0$, $k = 1, 2...$ form the algebra of constraints under some PB.

$$\{F_i, F_k\}_{PB} = C^l_{ik} F_l \approx 0.$$

The structure functions C^l_{ik} may be functions of the phase space coordinates in general case. The second kind constraints $f_k(x^a) \approx 0$ are the representations of the first kind constraints algebra. The second kind constraints is defined by the condition

$$\{f_i, f_k\} = C_{ik} \neq 0.$$

The reversible matrix C_{ik} is not constraint and also it is a function of phase space coordinates. The second kind constraints take part in deformation of the $\{,\}_{PB}$ to the Dirac bracket $\{,\}_D$. As rule, such deformation leads to nonlinear and to nonlocal brackets. The bi-Hamiltonity condition leads to the dual PB that are nonlinear and nonlocal brackets as a rule. We suppose, that the dual brackets can be obtained from the initial canonical bracket under the imposition of the second kind constraints. We have applied [3, 4, 5, 6], [7, 8, 9, 10] bi-Hamiltonian approach to the investigation of the integrability of the closed string model in the arbitrary background gravity field and antisymmetric B-field. The bi-Hamiltonity condition and the Jacobi identities for the dual brackets were considered as the integrability condition for a closed string model. They led to some restrictions on the background fields.

The plan of the paper is the following. In the second section we briefly considered papers about hydrodynamical type nonlocal brackets. In the third section we considered closed string model in the arbitrary background gravity field. We suppose that this model is an integrable model for some configurations of the background fields. The bi-Hamiltonity condition and the Jacobi identities for the dual PB resulted in to the integrability condition, which restrict the possible configurations of the background fields. As examples we considered constant curvature space and time-dependent metric space. In the fourth section we considered closed string model in the constant background gravity field. We obtained hydrodynamical type equation for the string model on the second kind constraints as configuration subspace embedded in a phase space.

2. Hydrodynamical type models

Mokhov and Ferapontov introduced the nonlocal PB [11]. The Ferapontov nonlocal PB (or hydrodynamical type nonlocal PB) [12] is:

$$\{u^i(x), u^k(y)\} = g^{ik}(u)\frac{\partial}{\partial x}\delta(x-y) - g^{ij}\Gamma^k_{jl}u^l_x\delta(x-y) + \sum_{s=1}^{L}\omega^{(s)i}_j(u(x))u^j_x\nu(x-y)\omega^{(s)k}_l(u(y))u^l_y, \tag{2.1}$$

here $\nu(x-y) = sgn(x-y) = (\frac{d}{dx})^{-1}\delta(x-y)$, $u^i(x)$ are local coordinates, $u^i_x(x) = \partial_x u^i(x), i = 1...N$. The coefficients $g^{ik}(x), \Gamma^k_{jl}(x), \omega^{(s)i}_k(x)$ are smooth functions of local coordinates. This nonlocal PB is satisfy the Jacoby identity if and only if $g^{ik}(u)$ is the pseudo-Riemannian metric without torsion and also the coefficients satisfy the following relations:

1. $\Gamma^k_{jl}(u)$ is the Levi-Civita connection;
2. $g^{ik}(u)\omega^{(s)j}_k(u) = g^{jk}(u)\omega^{(s)i}_k(u)$;
3. $\nabla_k\omega^{(s)i}_l(u) = \nabla_l\omega^{(s)i}_k$, where ∇_k is the covariant differential;
4. $R^{ij}_{kl}(u) = \sum_{s=1}^{L}[\omega^{(s)i}_l\omega^{(s)j}_k - \omega^{(s)j}_l\omega^{(s)i}_k]$, where R^{ij}_{kl} is Riemannian curvature tensor of the metric g^{ik};
5. $\omega^{(s)i}_k\omega^{(t)k}_l = \omega^{(t)i}_k\omega^{(s)i}_k$.

This nonlocal PB corresponds to an N-dimensional surface with flat normal bundle embedded in a pseudo-Euclidean space E^{N+L} [13]. There metric g^{ik} is the first fundamental form, $\omega^{(s)i}_k$ is Weingarten operator of this embedded surface, which is define the second fundamental form. The relations 2-4 are the Gauss-Peterson-Codazzi equations. The relations 5 are correspond to the Ricci equations for this embedded surface.

Dubrovin and Novikov have considered the local dual PB of the similar type [14] in the application to the Hamiltonian hydrodynamical models. Dubrovin-Novikov PB (or the hydrodynamical type local PB) can be obtained from the nonlocal PB (2.3) under condition $\omega^{(s)i}_k = 0$.

The Jacobi identity for this PB is satisfied if g_{ik} is the Riemann metric without torsion, the curvature tensor is equal to zero. The metric tensor is constant, locally.

It need to consider the linear combination of the local and the nonlocal Poisson brackets to obtain the hydrodynamical type equations [15]. There we consider Mokhov, Ferapontov, nonlocal PB [11] for the metric space of constant Riemannian curvature K, as example:

$$\{u^i(x), u^k(y)\} = c_1\eta^{ik}\frac{d}{dx}\delta(x-y) + c_2(\frac{\partial h^k}{\partial u^i} + \frac{\partial h^i}{\partial u^k} - Ku^iu^k)\frac{d}{dx}\delta(x-y)$$
$$+(\frac{\partial^2 h^k}{\partial u^i\partial u^l} - K\delta^i_l u^k)u^l_x\delta(x-y) + Ku^i_x\nu(x-y)u^k_y. \tag{2.2}$$

The Jacobi identity is satisfied on the following relations:

$$\frac{\partial^2 h^i}{\partial u^k\partial u^n}\frac{\partial^2 h^j}{\partial u^n\partial u^l} = \frac{\partial^2 h^j}{\partial u^k\partial u^n}\frac{\partial^2 h^i}{\partial u^n\partial u^l},$$
$$(\frac{\partial h^n}{\partial u^i} + \frac{\partial h^i}{\partial u^n} - Ku^iu^n)\frac{\partial^2 h^k}{\partial u^j\partial u^n} = \{i \longleftrightarrow j\}.$$

First of this equations is the WDVV [16, 17] consistence local condition. The system of hydrodynamical type is a bi-Hamiltonian system with the PB $\{,\}_{FM}$ and $\{,\}_{ND}$ if:

$$\dot{u}^i(x) = \{u^i(x), H_1\}_{FM} = \{u^i(x), H_2\}_{ND}.$$

Here Hamiltonians H_1 and H_2 are following:

$$H_1 = \frac{1}{2}\int u^i(x)u^i(x)dx,\ H_2 = \int[h^i(u(x))u^i(x) - \frac{K}{8}u^iu^iu^ku^k]dx.$$

3. Closed string in the background fields.

The string model in the background gravity field is described by the system of the equations:

$$\ddot{x}^a - x''^a + \Gamma^a_{bc}(x)(\dot{x}^b\dot{x}^c - x'^b x'^c) = 0,\ g_{ab}(x)(\dot{x}^a\dot{x}^b + x'^a x'^b) = 0,\ g_{ab}(x)\dot{x}^a x'^b = 0,$$

where $\dot{x}^a = \frac{dx^a}{d\tau}$, $x'^a = \frac{dx^a}{d\sigma}$. We will consider the Hamiltonian formalism. The closed string in the background gravity field is described by first kind constraints in the Hamiltonian formalism:

$$h_1 = \frac{1}{2}g^{ab}(x)p_a p_b + \frac{1}{2}g_{ab}(x)x'^a x'^b \approx 0,\ h_2 = p_a x'^a \approx 0, \tag{3.1}$$

where $a, b = 0, 1, ...D-1$, $x^a(\tau,\sigma), p_a(\tau,\sigma)$ are the periodical functions on σ with the period on π. The original PB are the symplectic PB:

$$\{x^a(\sigma), p_b(\sigma')\}_1 = \delta^a_b\delta(\sigma - \sigma'), \{x^a(\sigma), x^b(\sigma')\}_1 = \{p_a(\sigma), p_b(\sigma')\}_1 = 0.$$

The Hamiltonian equations of motion of the closed string, in the arbitrary background gravity field under the Hamiltonian $H_1 = \int_0^\pi h_1 d\sigma$ and PB $\{,\}_1$, are

$$\dot{x}^a = g^{ab}p_b,\ \dot{p}_a = g_{ab}x''^b - \frac{1}{2}\frac{\partial g^{bc}}{\partial x^a}p_b p_c - \frac{1}{2}\frac{\partial g_{bc}}{\partial x^a} + \frac{\partial g_{ac}}{\partial x^b}.$$

The dual PB are obtained from the bi-Hamiltonity condition

$$\dot{x}^a = \{x^a, \int_0^\pi h_1(\sigma')d\sigma'\}_1 = \{x^a, \int_0^\pi h_2(\sigma')d\sigma'\}_2,$$

$$\dot{p}_a = \{p_a, \int_0^\pi h_1(\sigma')d\sigma'\}_1 = \{p_a, \int_0^\pi h_2(\sigma')d\sigma'\}_2. \tag{3.2}$$

They have the following form:

Proposition 1.

$$\{A(\sigma), B(\sigma')\}_2 = \frac{\partial A}{\partial x^a}\frac{\partial B}{\partial x^b}[[\omega^{ab}(\sigma) + \omega^{ab}(\sigma')]\nu(\sigma' - \sigma) + [\Phi^{ab}(\sigma) + \Phi^{ab}(\sigma')]\frac{\partial}{\partial\sigma'}\delta(\sigma' - \sigma)$$
$$+[\Omega^{ab}(\sigma) + \Omega^{ab}(\sigma')]\delta(\sigma' - \sigma)] + \frac{\partial A}{\partial p_a}\frac{\partial B}{\partial p_b}[[\omega_{ab}(\sigma) + \omega_{ab}(\sigma')]\nu(\sigma' - \sigma) +$$
$$+[\Phi_{ab}(\sigma) + \Phi_{ab}(\sigma')]\frac{\partial}{\partial\sigma'}\delta(\sigma' - \sigma) + [\Omega_{ab}(\sigma) + \Omega_{ab}(\sigma')]\delta(\sigma' - \sigma)] +$$
$$+[\frac{\partial A}{\partial x^a}\frac{\partial B}{\partial p_b} + \frac{\partial A}{\partial p_b}\frac{\partial B}{\partial x^a}][[\omega^a_b(\sigma) + \omega^a_b(\sigma')]\nu(\sigma' - \sigma) + [\Phi^a_b(\sigma) + \Phi^a_b(\sigma')]\frac{\partial}{\partial\sigma'}\delta(\sigma' - \sigma)]$$
$$+[\frac{\partial A}{\partial x^a}\frac{\partial B}{\partial p_b} - \frac{\partial A}{\partial p_b}\frac{\partial B}{\partial x^a}][\Omega^a_b(\sigma) + \Omega^a_b(\sigma')]\delta(\sigma' - \sigma).$$

The arbitrary functions $A, B, \omega, \Phi, \Omega$ are the functions of the $x^a(\sigma), p_a(\sigma)$. The functions ω^{ab}, ω_{ab}, Φ^{ab}, Φ_{ab} are the symmetric functions on a, b and Ω^{ab}, Ω_{ab} are the antisymmetric functions to satisfy the condition $\{A, B\}_2 = -\{B, A\}_2$.

The equations of motion under the Hamiltonian $H_2 = \int\limits_0^\pi h_2(\sigma')d\sigma'$ and PB $\{,\}_2$ are

$$\dot{x}^a = -\omega^a_b x^b + 2\omega^{ab}p_b + 2\Phi^{ab}p_b^{''} - 2\Phi^a_b x^{''b} + 2\Omega^a_b x^{'b} - 2\Omega^{ab}p_b^{'} +$$

$$+\int\limits_0^\pi d\sigma'[\omega^a_b x^{'a} + \frac{d\omega^{ab}}{d\sigma'}p_b]\nu(\sigma'-\sigma) + \frac{d\Phi^{ab}}{d\sigma}p_b^{'} - \frac{d\Phi^a_b}{d\sigma}x^{'b},$$

$$\dot{p}_a = -\omega_{ab}x^b - 2\Phi_{ab}x^{''b} + 2\Omega_{ab}x^{'b} + 2\omega^b_a p_b + 2\Phi^b_a p_b^{''} + 2\Omega^b_a p_b^{'} +$$

$$+\int\limits_0^\pi d\sigma'[\omega_{ab}x^{'b} + \frac{d\omega^b_a}{d\sigma'}p_b]\nu(\sigma'-\sigma) - \frac{d\Phi_{ab}}{d\sigma}x^{'b} + \frac{d\Phi^b_a}{d\sigma}p_b^{'}.$$

The bi-Hamiltonity condition (3.2) is led to the two constraints

$$-\omega^a_b x^b + 2\omega^{ab}p_b + 2\Phi^{ab}p_b^{''} - 2\Phi^a_b x^{''b} + 2\Omega^a_b x^{'b} - 2\Omega^{ab}p_b^{'} +$$

$$\int\limits_0^\pi d\sigma'[\omega^a_b x^{'a} + \frac{d\omega^{ab}}{d\sigma'}p_b]\nu(\sigma'-\sigma) + \frac{d\Phi^{ab}}{d\sigma}p_b^{'} - \frac{d\Phi^a_b}{d\sigma}x^{'b} = g^{ab}p_b,$$

$$-\omega_{ab}x^b - 2\Phi_{ab}x^{''b} + 2\Omega_{ab}x^{'b} + 2\omega^b_a p_b + 2\Phi^b_a p_b^{''} + 2\Omega^b_a p_b^{'} +$$

$$+\int\limits_0^\pi d\sigma'[\omega_{ab}x^{'b} + \frac{d\omega^b_a}{d\sigma'}p_b]\nu(\sigma'-\sigma) - \frac{d\Phi_{ab}}{d\sigma}x^{'b} + \frac{d\Phi^b_a}{d\sigma}p_b^{'} =$$

$$+g_{ab}x^{''b} - \frac{1}{2}\frac{\partial g^{bc}}{\partial x^a}p_b p_c - \frac{1}{2}\frac{\partial g_{bc}}{\partial x^a}x^{'b}x^{'c} + \frac{\partial g_{ac}}{\partial x^b}x^{'b}x^{'c}.$$

In really, there is the list of the constraints depending on the possible choice of the unknown functions ω, Ω, Φ. In the general case, there are both the first kind constraints and the second kind constraints. Also it is possible to solve the constraints equations as the equations for the definition of the functions ω, Φ, Ω. We considered the latter possibility and we obtained the following consistent solution of the bi-Hamiltonity condition:

$$\Phi^{ab} = 0,\ \Omega^{ab} = 0,\ \Phi^a_b = 0,\ \Omega^a_b = 0,\ \frac{\partial\omega^{ab}}{\partial x^c}x^c + 2\omega^{ab} = g^{ab},$$

$$\omega_{ab} = \frac{1}{2}\frac{\partial^2\omega^{cd}}{\partial x^a\partial x^b}p_c p_d,\ \omega^a_b = -\frac{\partial\omega^{ac}}{\partial x^b}p_c,$$

$$\Phi_{ab} = -\frac{1}{2}g_{ab}, \Omega_{ab} = \frac{1}{2}(\frac{\partial\Phi_{bc}}{\partial x^a} - \frac{\partial\Phi_{ac}}{\partial x^b})x^{'c},\ \frac{\partial\omega^{ab}}{\partial p_c} = 0.$$

Remark 1. In distinct from the PB of the hydrodynamical type, we need to introduce the separate PB for the coordinates of the Minkowski space and for the momenta because, the gravity field is not depend of the momenta. Although, this difference is vanished under the such constraint as $f(x^a, p_a) \approx 0$.

Consequently, the dual PB for the phase space coordinates are

$$\{x^a(\sigma), x^b(\sigma')\}_2 = [\omega^{ab}(\sigma) + \omega^{ab}(\sigma')]\nu(\sigma'-\sigma),$$

$$\{p_a(\sigma), p_b(\sigma')\}_2 = [\frac{\partial^2\omega_{cd}(\sigma)}{\partial x^a\partial x^b}p_c p_d + \frac{\partial^2\omega_{cd}(\sigma')}{\partial x^a\partial x^b}p_c p_d]\nu(\sigma'-\sigma)-$$

$$-\frac{1}{2}[g_{ab}(\sigma)+g_{ab}(\sigma')]\frac{\partial}{\partial\sigma'}\delta(\sigma'-\sigma)+[\frac{\partial g_{ac}}{\partial x^b}-\frac{\partial g_{bc}}{\partial x^a}]x'^c(\sigma)\delta(\sigma'-\sigma)$$

$$\{x^a(\sigma), p_b(\sigma')\}_2 = -[\frac{\partial\omega^{ac}(\sigma)}{\partial x^b}p_c + \frac{\partial\omega^{ac}(\sigma')}{\partial x^b}p_c]\nu(\sigma'-\sigma),$$

$$\{p_a(\sigma), x^b(\sigma')\}_2 = -[\frac{\partial\omega^{bc}(\sigma)}{\partial x^a}p_c + \frac{\partial\omega^{bc}(\sigma')}{\partial x^c}p_c]\nu(\sigma'-\sigma). \tag{3.3}$$

The function $\omega^{ab}(x)$ is satisfied on the equation:

$$\frac{\partial\omega^{ab}}{\partial x^c}x^c + 2\omega^{ab} = g^{ab}. \tag{3.4}$$

The Jacobi identities for the PB $\{,\}_2$ are led to the nonlocal consistence conditions on the unknown function $\omega^{ab}(\sigma)$. We can calculate unknown metric tensor $g^{ab}(\sigma)$ by substitution of the solution of the consistence condition for ω^{ab} to the equation (3.4).

The Jacobi identity

$$\{x^a(\sigma), x^b(\sigma')\}x^c(\sigma'')\}_J \equiv \tag{3.5}$$

$$\{x^a(\sigma), x^b(\sigma')\}x^c(\sigma'')\} + \{x^c(\sigma''), x^a(\sigma)\}x^b(\sigma')\} + \{x^b(\sigma'), x^c(\sigma'')\}x^a(\sigma\}) = 0$$

is led to the following nonlocal analogy of the WDVV [16, 17] consistence condition:

$$[\frac{\partial\omega^{ab}(\sigma)}{\partial x^d}[\omega^{dc}(\sigma)+\omega^{dc}(\sigma'')] - \frac{\partial\omega^{ac}(\sigma)}{\partial x^d}[\omega^{db}(\sigma)+\omega^{db}(\sigma')]]\nu(\sigma'-\sigma)\nu(\sigma''-\sigma)+$$

$$[\frac{\partial\omega^{cb}(\sigma')}{\partial x^d}[\omega^{da}(\sigma')+\omega^{da}(\sigma)] - \frac{\partial\omega^{ab}(\sigma')}{\partial x^d}[\omega^{dc}(\sigma')+\omega^{dc}(\sigma'')]]\nu(\sigma-\sigma')\nu(\sigma''-\sigma')+$$

$$[\frac{\partial\omega^{ac}(\sigma'')}{\partial x^d}[\omega^{db}(\sigma'')+\omega^{db}(\sigma')] - \frac{\partial\omega^{cb}(\sigma'')}{\partial x^d}[\omega^{da}(\sigma'')+\omega^{da}(\sigma)]]\nu(\sigma-\sigma'')\nu(\sigma'-\sigma'') = 0. \tag{3.6}$$

This equation has the particular solution of the following form:

$$\frac{\partial\omega^{ab}(\sigma)}{\partial x^d}[\omega^{dc}(\sigma)+\omega^{dc}(\sigma'')] - \frac{\partial\omega^{ac}(\sigma)}{\partial x^d}[\omega^{db}(\sigma)+\omega^{db}(\sigma')] =$$

$$[T^b, T^c]T^a]f(\sigma,\sigma',\sigma'')\nu(\sigma''-\sigma)\nu(\sigma'-\sigma),$$

where $T^a, a = 0,1,...D-1$ is the matrix representation of the simple Lie algebra and $f(\sigma,\sigma',\sigma'')$ is arbitrary function. The Jacobi identity is satisfied on the Jacobi identity of the simple Lie algebra in this case:

$$([T^a,T^b]T^c] + [T^c,T^a]T^b] + [T^b,T^c]T^a])f(\sigma,\sigma',\sigma'') = 0$$

and we used the relation $\nu^2(\sigma'-\sigma) = 1$. The local solution of the Jacobi identities leads to the constant metric tensor. The rest Jacobi identities are cumbrous and we do not reduce this expressions here. The symmetric factor of σ, σ' of the antisymmetric functions $\nu(\sigma'-\sigma)$; $\frac{\partial}{\partial\sigma}\delta(\sigma-\sigma')$ in the right side of the PB can be both sum of the functions of σ and σ', and production of them. Last possibility can be used in the vielbein formalism.

Proposition 2. *The bi-Hamiltonity condition can be solved in the terms PB* $\{,\}_2$ *, which have the following form:*

$$\{x^a(\sigma), x^b(\sigma')\}_2 = e^a_\mu(\sigma)e^b_\mu(\sigma')\nu(\sigma' - \sigma),$$

$$\{x^a(\sigma), p_b(\sigma'\}_2 = -e^a_\mu(\sigma)\frac{\partial e^c_\mu(\sigma'}{\partial x^b}p_c(\sigma')\nu(\sigma' - \sigma),$$

$$\{p_a(\sigma), p_b(\sigma')\}_2 = \frac{\partial e^c_\mu(\sigma)}{\partial x^a}p_c(\sigma)\frac{\partial e^d_\mu(\sigma')}{\partial x^b}p_d(\sigma')\nu(\sigma' - \sigma) - e^\mu_a(\sigma)e^\mu_b(\sigma')\frac{\partial}{\partial \sigma'}\delta(\sigma' - \sigma) +$$

$$+[\frac{\partial e^\mu_a}{\partial x^c}e^\mu_b - \frac{\partial e^\mu_b}{\partial x^c}e^\mu_a - \frac{\partial e^\mu_c}{\partial x^a}e^\mu_b + \frac{\partial e^\mu_c}{\partial x^b}e^\mu_a]x'^c(\sigma)\delta(\sigma' - \sigma), \qquad (3.7)$$

where veilbein e^a_μ *is satisfied on the additional conditions:*

$$g^{ab} = \eta^{\mu\nu}e^a_\mu e^b_\nu, \; g_{ab} = \eta_{\mu\nu}e^\mu_a e^\nu_b$$

and $\eta^{\mu\nu}$ *is the metric tensor of the flat space.*

The particular solution of the Jacobi identity is

$$\frac{\partial e^a_\mu(\sigma)}{\partial x^d}e^b_\mu(\sigma')e^d_\nu(\sigma)e^c_\nu(\sigma'') - \frac{\partial e^a_\mu(\sigma)}{\partial x^d}e^c_\mu(\sigma'')e^d_\nu(\sigma)e^b_\nu(\sigma') =$$

$$[T^b, T^c]T^a]f(\sigma, \sigma', \sigma'')\nu(\sigma'' - \sigma)\nu(\sigma' - \sigma).$$

As example let me consider the the constant curvature space.

Example 1. The constant curvature space is described by the metric tensor $g_{ab}(x(\sigma))$ and by it inverse tensor g^{-1}_{ab} :

$$g_{ab} = \eta_{ab} + \frac{kx_a x_b}{1 - kx^2}, \; g^{ab} \equiv g^{-1}_{ab} = \eta_{ab} - kx_a x_b.$$

Proposition 3. *Dual (PB)* $\{,\}_2$ *are:*

$$\{x_a(\sigma), x_b(\sigma')\} = [\eta_{ab} - kx_a(\sigma)x_b(\sigma')]\nu(\sigma' - \sigma),$$

$$\{x_a(\sigma), p_b(\sigma')\} = kx_a(\sigma)p_b(\sigma')\nu(\sigma' - \sigma),$$

$$\{p_a(\sigma), p_b(\sigma')\} = -kp_a(\sigma)p_b(\sigma')\nu(\sigma' - \sigma)$$

$$-\frac{1}{2}[2\eta_{ab} + \frac{kx_a x_b}{1 - kx^2}(\sigma) + \frac{kx_a x_b}{1 - kx^2}(\sigma')]\frac{\partial}{\partial \sigma'}\delta(\sigma' - \sigma) + \frac{x_a x'_b - x_b x'_a}{2(1 - kx^2)}\delta(\sigma' - \sigma). \qquad (3.8)$$

The Jacobi identity (3.5) is led to the equation

$$[\eta_{ab}x_c(\sigma'') - \eta_{ac}x_b(\sigma')]\nu(\sigma' - \sigma)\nu(\sigma - \sigma'') + [\eta_{bc}x_a(\sigma) - \eta_{ba}x_c(\sigma'')]\nu(\sigma - \sigma')\nu(\sigma' - \sigma'') +$$

$$[\eta_{ca}x_b(\sigma') - \eta_{cb}x_a(\sigma)]\nu(\sigma' - \sigma'')\nu(\sigma'' - \sigma) = 0.$$

The particular solution of this equation is:

$$\eta_{ab}x_c(\sigma'') - \eta_{ac}x_b(\sigma') = [T_b, T_c]T_a]f(\sigma, \sigma', \sigma'')\nu(\sigma'' - \sigma)\nu(\sigma' - \sigma). \qquad (3.9)$$

Consequently, the space-time coordinate $x_a(\sigma)$ is the matrix representation of the simple Lie algebra.

The Jacobi identity $\{x_a(\sigma), x_b(\sigma')\} p_c(\sigma'')\}_J$ is led to the equation

$$k\eta_{ab}p_c(\sigma'')\nu(\sigma'-\sigma)[\nu(\sigma''-\sigma)+\nu(\sigma''-\sigma')]=0. \tag{3.10}$$

These results can be obtained from the veilbein formalism under the following ansatz for the veilbein of the constant curvature space:

$$e_\mu^{a(s)}=n_\mu(m_1^{(s)}n^a+\sqrt{-k}m_2^{(s)}x^a), e_a^{\mu(s)}=n^\mu g_{ab}(m_1^{(s)}n^b+\sqrt{-k}m_2^{(s)}x^b),$$

where $n_\mu^2=1$, $m_1^{(s)}m_1^{(s)}=1$, $m_2^{(s)}m_2^{(s)}=1$, $m_1^{(s)}m_2^{(s)}=0$, $n^a n^b=\delta^{ab}$ and (s) is number of the solution of the equations

$$e_\mu^a e_\mu^b = g^{ab},\ e_a^\mu e_b^\mu = g_{ab},\ e_\mu^a e_b^\mu = \delta_b^a.$$

The following example is time-dependent metric space.

Example 2. The time-dependent metric in the light-cone variables has form:

$$ds^2=g_{ik}(x^+)dx^i dx^k + g_{++}(x^+)dx^+dx^+ + 2g_{+-}dx^+dx^-.$$

We are used Poisson brackets (3.3) for the space coordinates $x^a=\{x^i,x^+,x^-\}$, $i=1,2,\ldots,D-2$. We introduced the light-cone gauge as two first kind constraints:

$$F_1(\sigma)=x'^+\approx 0,\ F_2(\sigma)=p'_-\approx 0,$$

and we imposed them on the equations of motion and on the Jacobi identities. The Jacobi identities are reduced to the simple equation

$$\frac{\partial\omega^{ab}}{\partial x^+}\omega^{+c}-\frac{\partial\omega^{ac}}{\partial x^+}\omega^{+b}=0.$$

We obtained following result from this equation and additional condition (3.4): there is constant background gravity field only for the non-degenerate metric.

4. Constant background fields ($g_{ab}=const$)

In this section we are supplemented the bi-Hamiltonity condition (3.2) by the mirror transformations of the integrals of motion:

$$\dot{x}^a=\{x^a,\int_0^\pi h_1 d\sigma'\}_1=\{x^a,\int_0^\pi \pm h_2 d\sigma'\}_{\pm 2}.$$

The dual PB are

$$\{x^a(\sigma),x^b(\sigma')\}_{\pm 2}=\pm g^{ab}\nu(\sigma'-\sigma),\ \{x^a(\sigma),p_b(\sigma')\}_{\pm 2}=0,$$

$$\{p_a(\sigma),p_b(\sigma')\}_{\pm 2}=\mp g_{ab}\frac{\partial}{\partial\sigma'}\delta(\sigma'-\sigma).$$

The dual dynamical system

$$\dot{x}^a = \{x^a, \pm H_2\}_1 = \{x^a, H_1\}_{\pm 2}.$$

is the left(right) chiral string

$$\dot{x}^a = \pm x^{'a}, \ \dot{p}_a = \pm p_a^{'}.$$

Another way to obtain the dual brackets is the imposition of the second kind constraints on the initial dynamical system, by such manner, that $F_i = F_k$ for $i \neq k, i, k = 1, 2, \dots$ on the constraints surface $f(x^a, p_a) = 0$.

Example 3. The constraints $f_a^{(-)}(x,p) = p_a - g_{ab}x'^b \approx 0$ or $f_a^{(+)} = p_a + g_{ab}x'^b \approx 0$ (do not simultaneously) are the second kind constraints.

$$\{f_a^{(\pm)}(\sigma), f_b^{(\pm)}(\sigma')\}_1 = C_{ab}^{(\pm)}(\sigma - \sigma') = \pm 2 g_{ab} \frac{\partial}{\partial \sigma'} \delta(\sigma' - \sigma).$$

The inverse matrix $(C^{(\pm)})^{-1}$ has following form $C^{(\pm)ab}(\sigma - \sigma') = \pm\frac{1}{2} g^{ab} \nu(\sigma' - \sigma)$. There is only one set of the constraints, because consistency condition

$$\{f^{(\pm)}(\sigma), H_1\}_1 = f^{'(\pm)}(\sigma) \approx 0, \ \dots, \{f^{(\pm)(n)}(\sigma), H_1\}_1 = f^{(\pm)(n+1)}(\sigma) \approx 0.$$

is not produce the new sets of constraints. By using the standard definition of the Dirac bracket, we are obtained following Dirac brackets for the phase space coordinates.

$$\{x^a(\sigma), x^b(\sigma')\}_D = \pm\frac{1}{2} g^{ab} \nu(\sigma' - \sigma), \{p_a(\sigma), p_b(\sigma')\}_D = \mp\frac{1}{2} g_{ab} \frac{\partial}{\sigma'} \delta(\sigma' - \sigma),$$

$$\{x^a(\sigma), p_b(\sigma')\}_D = \frac{1}{2} \delta_b^a \delta(\sigma' - \sigma).$$

equation The equations of motion under the Hamiltonians $H_1 = h_1, H_2 = h_2$ and Dirac bracket

$$\dot{x}^a = \{x^a, H_1\}_D = \{x^a, H_2\}_D = g^{ab} p_b = \pm x'^a,$$

$$\dot{p}_a = \{p_a, H_1\}_D = \{p_a, H_2\}_D = g_{ab} x'^b = \pm p'_a.$$

are coincide on the constraints surface. The dual brackets $\{,\}_{\pm 2}$ are coincide with the Dirac brackets also. The contraction of the algebra of the first kind constraints means that the integrals of motion $H_1 = H_2$ are coincide on the constraints surface too.

Example 4. Constraints $f_a(\sigma) = p_a - h_{ac} x'^c(\sigma)$,where metric tensor of second fundamental form $h_{ac} = const., h_{ab} = h_{ba}$, $h_{ab} h^{bc} = \delta_a^c$ are the second kind constraints:

$$\{f_a(\sigma), f_b(\sigma')\} = C_{ab}(\sigma - \sigma') = 2 h_{ab} \frac{\partial}{\partial \sigma'} \delta(\sigma' - \sigma).$$

Inverse matrix $C^{ab}(\sigma - \sigma')$ has form:

$$C^{ab}(\sigma - \sigma') = \frac{1}{2} h^{ab} \nu(\sigma' - \sigma).$$

Dirac bracket of arbitrary function $A(\sigma), B(\sigma)$ is

$$\{A(\sigma), B(\sigma')\}_{D1} = \{A(\sigma), B(\sigma')\}_{PB} - \int \{A(\sigma), f_a(\sigma'') C^{ab}(\sigma'' - \sigma''')\{f_b(\sigma'''), B(\sigma')\} d\sigma'' d\sigma'''.$$

Therefore, we obtained the only Dirac bracket

$$\{x^a(\sigma), x^b(\sigma'\}_{D1} = -\frac{1}{2}h^{ab}\nu(\sigma - \sigma')$$

on the surface $p_a - h_{ab}x^{'b} = 0$. The equations of motion under the Dirac bracket are

$$\dot{x}^a(\sigma) = \int \{x^a(\sigma), h_1(\sigma'\}d\sigma'\}_{D1} = h^a_b x'^b, \ \dot{x}^a(\sigma) = \int \{x^a(\sigma), h_2(\sigma'\}_{D1}d\sigma' = x'^a(\sigma),$$

where $h^a_b = g^{ac}h_{cb}$ is Weingarten operator. The equation of motion

$$\dot{x}^a = h^a_b x'^b \tag{4.1}$$

under the Hamiltonian h_1 is the hydrodynamical type equation [18]. The equation (4.1) for the diagonal operator $h^a_b = \delta^a_b h_b$ was considered as the Hamiltonian equation under the local bracket for the sphere embedded in a pseudo-Euclidean space E^N . It has the following form in the sphere-conic coordinates R^a [13, 19]:

$$\dot{R}^a = (2R^a + \sum_{k=1}^{N-1} R^k - \sum_{K=1}^{N} h^a)R'^a.$$

The bi-Hamiltonity condition

$$\dot{x}^a = \int \{x^a(\sigma), h^1(\sigma')\}_{D1}d\sigma' = \int \{x^a(\sigma), h_2(\sigma')\}_{D2}d\sigma'$$

led to the following dual Dirac brackets:

$$\{x^a(\sigma), x^b(\sigma')\}_{D2} = \frac{1}{2}g^{ab}\nu(\sigma' - \sigma), \ \{p_a(\sigma), p_b(\sigma')\}_{D2} = -\frac{1}{2}g_{ab}\frac{\partial}{\partial\sigma'}\delta(\sigma' - \sigma), \tag{4.2}$$
$$\{x^a(\sigma), p_b(\sigma')\}_{D2} = \frac{1}{2}h^a_b\delta(\sigma - \sigma').$$

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Quantum Toda Chain With Boundary Interaction

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In this contribution, we give an integral representation of the wave functions of the quantum N-particle Toda chain with boundary interaction. In the case of the Toda chain with one-boundary interaction, we obtain the wave function by an integral transformation from the wave functions of the open Toda chain. The kernel of this transformation is given explicitly in terms of Γ-functions. The wave function of the Toda chain with two-boundary interaction is obtained from the previous wave functions by an integral transformation. In this case, the difference equation for the kernel of the integral transformation admits separation of variables. The separated difference equations coincide with the Baxter equation.

1. Introduction

Recently, some progress in the derivation of the eigenfunctions of the Hamiltonians of some integrable quantum chains with finite number of particles has been achieved [1]–[7]. It is connected with the development of the method of separation of variables [1] for quantum integrable models. The first steps in the elaboration of this method were taken by Gutzwiller [2], who has found a solution of the eigenvalue problem for $N = 2, 3, 4$-particle periodic Toda chain.

Using the R-matrix formalism, Sklyanin [3] proposed an algebraic formulation of the method of separation of variables applicable to a broader class of integrable quantum chains. The next important step was taken by Kharchev and Lebedev [4], who combined the analytic method of Gutzwiller and algebraic approach of Sklyanin. They obtained the eigenfunctions of the N-particle periodic Toda chain by some integral transformation of the eigenfunctions of an auxiliary problem, the open ($N-1$)-particle Toda chain. It turned out that the kernel of this transformation admits separation of variables. The separated equations coincide with the Baxter equation. A solution of this equation has been found in [8] (see also [4]).

Later Kharchev and Lebedev [5] have found a remarkable recurrence relation between the eigenfunctions of the N-particle and ($N-1$)-particle open Toda chains. Understanding these formulas from the viewpoint of the representation theory [6] made it possible to extend their approach to other integrable systems [6, 7].

In this paper, we apply this method to the derivation of the eigenfunctions of the commuting Hamiltonians of the N-particle quantum Toda chain with boundary interaction. We use the Sklyanin approach [9] to the boundary problems for the quantum integrable models. The N-particle eigenfunctions of the quantum Toda chain in which the first and last particles exponentially interact with the walls (the two-boundary interaction) is constructed by means of an integral transformation of the eigenfunctions for the Toda chain with one-boundary interaction (the auxiliary problem). These eigenfunctions, in turn, are constructed using the eigenfunctions of the N-particle open Toda chain. Such a complicated hierarchy allows one to separate the variables in the difference equation for the kernel of the mentioned integral transformation reducing it to a version of the Baxter equation. We note that, for the classical Toda chain with general boundary interaction, the separation of variables was performed by Kuznetsov [10].

2. Integrals of motion of the open Toda chain

To describe the integrals of motion of the quantum N-particle open Toda chain, we use the L-operators (one for each particle)

$$L_k(u) = \begin{pmatrix} u - p_k & e^{-q_k} \\ -e^{q_k} & 0 \end{pmatrix}, \qquad k = 1, 2, \ldots, N,$$

where N is the number of particles in the chain, p_k and q_k are the operators of momentum and position of the k-th particle, respectively. The monodromy matrix is defined as

$$T(u) := L_N(u) L_{N-1}(u) \cdots L_2(u) L_1(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \tag{2.1}$$

The commutation relations for the matrix elements of $T(u)$ follow from the canonical commutation relations

$$[p_k, q_l] = -\mathrm{i}\frac{1}{2}\delta_{kl}$$

and can be written as

$$R(u-v)\,(T(u) \otimes \mathbf{1})\,(\mathbf{1} \otimes T(v)) = (\mathbf{1} \otimes T(v))\,(T(u) \otimes \mathbf{1})\,R(u-v), \tag{2.2}$$

where $R(u)$ is the rational R-matrix:

$$R(u) = \begin{pmatrix} 1 + \frac{\mathrm{i}\frac{1}{2}}{u} & 0 & 0 & 0 \\ 0 & 1 & \frac{\mathrm{i}\frac{1}{2}}{u} & 0 \\ 0 & \frac{\mathrm{i}\frac{1}{2}}{u} & 1 & 0 \\ 0 & 0 & 0 & 1 + \frac{\mathrm{i}\frac{1}{2}}{u} \end{pmatrix}. \tag{2.3}$$

From (2.1) it follows that $A(u)$ is a polynomial of degree N in u:

$$A(u) = \sum_{m=0}^{N} (-1)^m u^{N-m} H_m(p_1, q_1; p_2, q_2; \ldots; p_N, q_N) =$$
$$= u^N - H_1 u^{N-1} + H_2 u^{N-2} - \cdots + (-1)^N H_N.$$

In particular, relations (2.2) give

$$[A(u), A(v)] = 0,$$

and, therefore, $[H_m, H_k] = 0$, that is, $A(u)$ is a generating function for the commuting operators H_m. Since

$$H_1 = \sum_{k=1}^{N} p_k, \qquad H_2 = \sum_{\substack{k,l \\ k<l}} p_k p_l - \sum_{k=1}^{N-1} e^{q_k - q_{k+1}},$$

we get the Hamiltonian for the open Toda chain in the form

$$H = H_1^2/2 - H_2 = \sum_{k=1}^{N} \frac{p_k^2}{2} + \sum_{k=1}^{N-1} e^{q_k - q_{k+1}}.$$

Therefore, the operators H_m are Hamiltonians for the open Toda chain.

3. Wave functions for the open Toda chain

Let a wave function $\psi(q_1, \ldots, q_N)$ for the open Toda chain be a common eigenfunction of the commuting Hamiltonians H_m:

$$H_m \psi(q_1, \ldots, q_N) = E_m \psi(q_1, \ldots, q_N).$$

Then

$$A(u)\psi_{\gamma_N}(q_1, \ldots, q_N) = \prod_{l=1}^{N} (u - \gamma_{Nl}) \psi_{\gamma_N}(q_1, \ldots, q_N),$$

where $\gamma_N = (\gamma_{N1}, \gamma_{N2}, \ldots, \gamma_{NN})$ are the quantum numbers of the N-particle system, $E_m = e_m(\gamma_{N1}, \gamma_{N2}, \ldots, \gamma_{NN})$, and e_m is the m-th elementary symmetric polynomial. For every set γ_N, the space of eigenfunctions is $N!$ dimensional. The physical eigenfunction ψ_{γ_N} is fixed by the requirement that ψ_{γ_N} rapidly decreases in the classically forbidden region, that is, for $q_k >> q_{k+1}$ for some k. For $q_1 << q_2 << \cdots << q_N$, ψ_{γ_N} is a superposition of plane waves.

Recently, Kharchev and Lebedev [5] have found a recursive procedure of constructing the N-particle wave function $\psi_{\gamma_N}(q_1, q_2, \ldots, q_N)$ through the $(N-1)$-particle wave functions $\psi_{\gamma_{N-1}}(q_1, q_2, \ldots, q_{N-1})$. The recurrence relation is

$$\psi_{\gamma_N}(q_1, q_2, \ldots, q_N) = \int d\gamma_{N-1} \mu(\gamma_{N-1}) Q(\gamma_{N-1}|\gamma_N) \psi_{\gamma_{N-1}}(q_1, q_2, \ldots, q_{N-1}) \times e^{\frac{i}{\frac{1}{2}}(\sum_{j=1}^{N} \gamma_{N,j} - \sum_{k=1}^{N-1} \gamma_{N-1,k}) q_N}, \tag{3.1}$$

where integration is carried out with respect to $\gamma_{N-1,k}$, $k = 1, 2, \ldots, N-1$, along any set of the lines parallel to the real axis and such that

$$\min_k \operatorname{Im} \gamma_{N-1,k} > \max_j \operatorname{Im} \gamma_{N,j}, \qquad j = 1, \ldots, N, \tag{3.2}$$

$$Q(\gamma_{N-1}|\gamma_N) = \prod_{k=1}^{N-1} \prod_{j=1}^{N} \frac{1}{2}^{\frac{\gamma_{N-1,k} - \gamma_{N,j}}{i\frac{1}{2}}} \Gamma\left(\frac{\gamma_{N-1,k} - \gamma_{N,j}}{i\frac{1}{2}}\right),$$

$$\mu^{-1}(\gamma_{N-1}) = \prod_{\substack{k,l \\ k \neq l}} \Gamma\left(\frac{\gamma_{N-1,k} - \gamma_{N-1,l}}{i\frac{1}{2}}\right). \tag{3.3}$$

In a similar way, the $(N-1)$-particle wave functions can be expressed through the $(N-2)$-particle wave functions, and so on. The wave function for the 1-particle open Toda chain is just a plane wave:

$$\psi_{\gamma_{11}}(q_1) = e^{\frac{i}{\frac{1}{2}} \gamma_{11} q_1}.$$

In what follows, we use the notation $\gamma := \gamma_N$, $\gamma_k := \gamma_{N,k}$. As shown in [5], the wave function ψ_γ satisfies the relations

$$A(u)\psi_\gamma = \prod_{l=1}^{N}(u-\gamma_l)\,\psi_\gamma, \tag{3.4}$$

$$B(u)\psi_\gamma = \mathrm{i}^{N-1}\sum_{p=1}^{N}\left(\prod_{l\neq p}\frac{u-\gamma_l}{\gamma_p-\gamma_l}\right)\psi_{\gamma+p}, \tag{3.5}$$

$$C(u)\psi_\gamma = \mathrm{i}^{-N-1}\sum_{p=1}^{N}\left(\prod_{l\neq p}\frac{u-\gamma_l}{\gamma_p-\gamma_l}\right)\psi_{\gamma-p}, \tag{3.6}$$

where $\psi_{\gamma\pm p} := \psi_{\gamma_1,\gamma_2,\ldots,\gamma_p\pm\mathrm{i}\hbar,\ldots,\gamma_N}$.

In order to find the action of $D(u)$ on ψ_γ , we use the following property of the quantum determinant of $T(u)$ for the Toda chain:

$$D(u)A(u-\mathrm{i}\hbar)-C(u)B(u-\mathrm{i}\hbar)=1. \tag{3.7}$$

The result is

$$D(u)\psi_\gamma = \sum_{p=1}^{N}\left(\prod_{l\neq p}\frac{u-\gamma_l}{\gamma_p-\gamma_l}\right)\frac{1}{\mathrm{i}\hbar}\left(\frac{1}{\prod_{l\neq p}(\gamma_p-\gamma_l+\mathrm{i}\hbar)}-\frac{1}{\prod_{l\neq p}(\gamma_p-\gamma_l-\mathrm{i}\hbar)}\right)\psi_\gamma -$$

$$-\sum_{\substack{p,q\\ p\neq q}}\frac{1}{\gamma_q-\gamma_p-\mathrm{i}\hbar}\frac{1}{\prod_{l\neq p}(\gamma_p-\gamma_l)}\prod_{l\neq p,q}\frac{u-\gamma_l}{\gamma_q-\gamma_l}\,\psi_{\gamma+p,-q}\,. \tag{3.8}$$

4. Integrals of motion of the Toda chain with boundary interaction

In this section, we give a sketch of the R-matrix formalism for the quantum Toda chain with boundary interaction proposed by Sklyanin [9]. This formalism is important for the construction of wave functions. The key object in this approach is the matrix

$$U(u) := T(u)K^{(-)}(u-\frac{\mathrm{i}\frac{1}{2}}{2})\tilde{T}(-u) = \begin{pmatrix} \mathcal{A}(u) & \mathcal{B}(u) \\ \mathcal{C}(u) & \mathcal{D}(u)\end{pmatrix}, \tag{4.1}$$

where $T(u)$ is the monodromy matrix (2.1) of the N-particle open Toda chain, and

$$\tilde{T}(-u)=\sigma_2T^t(-u)\sigma_2=(\sigma_2L_1^t(-u)\sigma_2)(\sigma_2L_2^t(-u)\sigma_2)\cdots(\sigma_2L_N^t(-u)\sigma_2).$$

Here, σ_2 is the Pauli matrix. The matrix $K^{(-)}(u-\mathrm{i}\frac{1}{2}/2)$ is

$$K^{(-)}(u-\frac{\mathrm{i}\frac{1}{2}}{2}) = \begin{pmatrix} \alpha_1 & u-\frac{\mathrm{i}\frac{1}{2}}{2} \\ -\beta_1(u-\frac{\mathrm{i}\frac{1}{2}}{2}) & \alpha_1\end{pmatrix}. \tag{4.2}$$

As shown in [9], the matrix $U(u)$ satisfies the reflection equation

$$R(u-v)\,(U(u)\otimes\mathbf{1})\,R(u+v-\mathrm{i}\frac{1}{2})\,(\mathbf{1}\otimes U(v)) =$$
$$(\mathbf{1}\otimes U(v))\,R(u+v-\mathrm{i}\frac{1}{2})\,(U(u)\otimes\mathbf{1})\,R(u-v), \quad (4.3)$$

where $R(u)$ is given by (2.3).

This equation implies $\mathcal{B}(u)\mathcal{B}(v) = \mathcal{B}(v)\mathcal{B}(u)$. Therefore, the expansion of $\mathcal{B}(u)$ in powers of u gives commuting operators which, in fact, are the Hamiltonians of the one-boundary Toda chain

$$\mathcal{B}(u) = (-1)^N(u-\mathrm{i}\frac{1}{2}/2)\left(u^{2N} - u^{2N-2}H_1^{\mathrm{B}} + u^{2N-4}H_2^{\mathrm{B}} - \cdots + (-1)^N H_N^{\mathrm{B}}\right), \quad (4.4)$$

where

$$H_1^{\mathrm{B}} = \sum_{k=1}^{N} p_k^2 + 2\sum_{k=1}^{N-1} e^{q_k-q_{k+1}} - 2\alpha_1 e^{-q_1} + \beta_1 e^{-2q_1}.$$

Here the last two terms describe interaction of the first particle with the wall.

The Sklyanin's transfer-matrix

$$t(u) := \mathrm{Tr}\, K^{(+)}(u+\mathrm{i}\frac{1}{2}/2)U(u), \quad (4.5)$$

where

$$K^{(+)}(u+\frac{\mathrm{i}\frac{1}{2}}{2}) = \begin{pmatrix} \alpha_N & \beta_N(u+\frac{\mathrm{i}\frac{1}{2}}{2}) \\ -(u+\frac{\mathrm{i}\frac{1}{2}}{2}) & \alpha_N \end{pmatrix},$$

satisfies the commutation relation [9]

$$t(u)t(v) = t(v)t(u). \quad (4.6)$$

Hence, $t(u)$ is a generating function for commuting operators which, in fact, are the Hamiltonians of the two-boundary Toda chain.

For simplicity, in what follows, we fix $\beta_1 = \beta_N = 0$ and use the notation $e^{\kappa_1} := -2\alpha_1$, $e^{-\kappa_N} := -2\alpha_N$. In this case, we have

$$t(u) = (-1)^{N-1}(u^2 + \frac{1}{2}^2/4)\times$$
$$\left(u^{2N} - u^{2N-2}H_1^{\mathrm{BB}} + u^{2N-4}H_2^{\mathrm{BB}} - \cdots + (-1)^N H_N^{\mathrm{BB}}\right) + 2\alpha_1\alpha_N, \quad (4.7)$$

where

$$H_1^{\mathrm{BB}} = \sum_{k=1}^{N} p_k^2 + 2\sum_{k=1}^{N-1} e^{q_k-q_{k+1}} - 2\alpha_1 e^{-q_1} - 2\alpha_N e^{q_N}.$$

In the case of the Toda chain, the matrix $U(u)$ has some additional symmetry (unitarity) [9]:

$$\begin{pmatrix} \mathcal{A}(-u) & \mathcal{B}(-u) \\ \mathcal{C}(-u) & \mathcal{D}(-u) \end{pmatrix} = \frac{1}{2u-\mathrm{i}\frac{1}{2}} \begin{pmatrix} -\mathrm{i}\frac{1}{2}\mathcal{A}(u) + 2u\mathcal{D}(u) & -(2u+\mathrm{i}\frac{1}{2})\mathcal{B}(u) \\ -(2u+\mathrm{i}\frac{1}{2})\mathcal{C}(u) & 2u\mathcal{A}(u) - \mathrm{i}\frac{1}{2}\mathcal{D}(u) \end{pmatrix}. \quad (4.8)$$

In particular, this leads to

$$\mathcal{A}(u) = \frac{1}{u}\left((u - \frac{\mathrm{i}\frac{1}{2}}{2})\mathcal{D}(-u) + \frac{\mathrm{i}\frac{1}{2}}{2}\mathcal{D}(u)\right). \tag{4.9}$$

Therefore, using this equality and (4.5), we obtain

$$t(u) = \alpha_N \frac{(u + \frac{\mathrm{i}\frac{1}{2}}{2})}{u}\, \mathcal{D}(u) + \alpha_N \frac{(u - \frac{\mathrm{i}\frac{1}{2}}{2})}{u}\, \mathcal{D}(-u) - (u + \frac{\mathrm{i}\frac{1}{2}}{2})\mathcal{B}(u). \tag{4.10}$$

Using (4.1), we obtain the following expressions for the matrix elements of $U(u)$ in terms of the matrix elements of the monodromy matrix $T(u)$ for the N-particle open Toda chain:

$$\mathcal{A}(u) = \alpha_1\left(A(u)D(-u) - B(u)C(-u)\right) - \left(u - \mathrm{i}\frac{\frac{1}{2}}{2}\right) A(u)C(-u), \tag{4.11}$$

$$\mathcal{B}(u) = -\alpha_1\left(A(u)B(-u) - B(u)A(-u)\right) + \left(u - \mathrm{i}\frac{\frac{1}{2}}{2}\right) A(u)A(-u), \tag{4.12}$$

$$\mathcal{C}(u) = \alpha_1\left(C(u)D(-u) - D(u)C(-u)\right) - \left(u - \mathrm{i}\frac{\frac{1}{2}}{2}\right) C(u)C(-u), \tag{4.13}$$

$$\mathcal{D}(u) = \alpha_1\left(D(u)A(-u) - C(u)B(-u)\right) + \left(u - \mathrm{i}\frac{\frac{1}{2}}{2}\right) C(u)A(-u). \tag{4.14}$$

We give some examples:

$\underline{N=1:}$

$$\mathcal{B}(u) = -(u - \mathrm{i}\frac{1}{2}/2)(u^2 - (p_1^2 + e^{\kappa - q_1})),$$

$$t(u) = (u^2 + \frac{1}{2}^2/4)\left(u^2 - (p^2 + e^{\kappa - q_1} + e^{q_1 - \kappa'})\right) + 2\alpha\alpha';$$

$\underline{N=2:}$

$$\mathcal{B}(u) = (u - \mathrm{i}\frac{1}{2}/2)\left(u^4 - u^2(p_1^2 + p_2^2 + 2e^{q_1-q_2} + e^{\kappa_1 - q_1}) + (p_1p_2 - e^{q_1-q_2})^2 - \alpha_1 p_2^2 e^{-q_1} - 2\alpha_1 e^{-q_2}\right.$$

$$t(u) = -(u^2 + \frac{1}{2}^2/4)\left(u^4 - u^2(p_1^2 + p_2^2 + 2e^{q_1-q_2} + e^{\kappa_1-q_1} + e^{q_2-\kappa_2}) + \cdots\right) + 2\alpha_1\alpha_2.$$

5. Wave functions for the one-boundary Toda chain

We define the function $\Psi_{\boldsymbol\lambda} \equiv \Psi_{\lambda_1,\ldots,\lambda_N}$ as

$$\Psi_{\boldsymbol\lambda}(q_1,\ldots,q_N) = \int d\gamma_1\cdots d\gamma_N \mu(\gamma) Q(\gamma|\boldsymbol\lambda) e^{-\frac{\mathrm{i}\kappa_1(\gamma_1+\cdots+\gamma_N)}{\frac{1}{2}}} \psi_{\boldsymbol\gamma}(q_1,\ldots,q_N), \tag{5.1}$$

where $e^{\kappa_1} = -2\alpha_1$ and

$$Q(\gamma|\boldsymbol{\lambda}) = \frac{\prod_{k,l}\Gamma\left(\frac{\lambda_l-\gamma_k}{\mathrm{i}\frac{1}{2}}\right)\Gamma\left(\frac{-\lambda_l-\gamma_k}{\mathrm{i}\frac{1}{2}}\right)}{\prod_{\substack{k,l\\k<l}}\Gamma\left(-\frac{\gamma_k+\gamma_l}{\mathrm{i}\frac{1}{2}}\right)}\prod_k \frac{1}{2}^{-\frac{(N+1)\gamma_k}{\mathrm{i}\frac{1}{2}}},$$

$$\mu^{-1}(\gamma) = \prod_{\substack{k,l\\k\neq l}}\Gamma\left(\frac{\gamma_k-\gamma_l}{\mathrm{i}\frac{1}{2}}\right). \tag{5.2}$$

We show that this is a wave function for the quantum one-boundary Toda chain, and

$$\mathcal{B}(u)\Psi_{\boldsymbol{\lambda}}(q_1,\ldots,q_N) = (-1)^N(u-\frac{\mathrm{i}\frac{1}{2}}{2})\prod_{l=1}^{N}(u^2-\lambda_l^2)\Psi_{\boldsymbol{\lambda}}(q_1,\ldots,q_N), \tag{5.3}$$

where the structure of the right-hand side corresponds to (4.4). The integration in (5.1) is carried out along any set of lines parallel to the real axis and such that

$$\max_k \operatorname{Im}\gamma_k < -\min_j \operatorname{Im}\lambda_j, \qquad k=1,2,\ldots,N, \quad j=1,\ldots,N. \tag{5.4}$$

First, we prove the absolute convergence in (5.1). For this, we use the inequalities

$$|\Gamma(x+\mathrm{i}y)| \le \Gamma(x)p_x(|y|)e^{-\frac{\pi|y|}{2}}, \qquad x>0,$$

where $p_x(|y|)$ is some polynomial in $|y|$ with degree linearly depending on x,

$$\frac{1}{|\Gamma(x+\mathrm{i}y)|} \le \frac{\left(1+\frac{|y|}{x}\right)e^{\frac{\pi|y|}{2}}}{\Gamma(x)}, \qquad x>0,$$

and also inequality

$$\begin{aligned}\sum_{k,l=1}^{N}\left(|\tilde\lambda_k-\tilde\gamma_{N,l}|+|\tilde\lambda_k+\tilde\gamma_{N,l}|\right)+\sum_{r=1}^{N-1}\sum_{k,l}|\tilde\gamma_{r+1,k}-\tilde\gamma_{r,l}|-\\ -2\sum_{r=2}^{N}\sum_{k<l}|\tilde\gamma_{r,k}-\tilde\gamma_{r,l}|-\sum_{k<l}|\tilde\gamma_{N,k}+\tilde\gamma_{N,l}|\ge\\ \ge -2N\sum_{k=1}^{N}|\tilde\lambda_k|+2\sum_{k<l}\left(|\tilde\lambda_k-\tilde\lambda_l|+|\tilde\lambda_k+\tilde\lambda_l|\right)+\frac{2}{N}\sum_{r=1}^{N}\sum_{k=1}^{r}|\tilde\gamma_{r,k}|,\end{aligned} \tag{5.5}$$

which is valid for any set of real variables $\tilde\lambda_k$, $k=1,2,\ldots,N$; $\tilde\gamma_{r,l}$, $l=1,2,\ldots,r$, $r=1,2,\ldots,N$. A proof of the last inequality is given in Appendix A of [11]. For our purposes, we fix $\tilde\lambda_k$ (respectively, $\tilde\gamma_{r,l}$) to be equal to $\operatorname{Re}\lambda_k$ (respectively, $\operatorname{Re}\gamma_{r,l}$).

Presenting (5.1) as

$$\Psi_{\boldsymbol{\lambda}}(q_1,\ldots,q_N) = \int\prod_{r=1}^{N}\prod_{k=1}^{r}d\tilde\gamma_{r,k}F(\gamma_1,\gamma_2,\ldots,\gamma_N,\boldsymbol{\lambda};q_1,\ldots,q_N),$$

we obtain the following inequality for the dependence of the integrand on $\gamma_{r,k}$:

$$|F(\gamma_1,\gamma_2,\ldots,\gamma_N,\boldsymbol{\lambda};q_1,\ldots,q_N)| \leq P(\{\tilde{\gamma}_{r,k}\}) \exp\left(-\frac{\pi}{\frac{1}{2}N}\sum_{r=1}^{N}\sum_{k=1}^{r}|\tilde{\gamma}_{r,k}|\right), \tag{5.6}$$

where $P(\{\tilde{\gamma}_{r,k}\})$ has polynomial dependence on the variables $\tilde{\gamma}_{r,k}$ and certain dependence on the other variables. Estimate (5.6) leads to absolute convergence of the integral on the right-hand side of (5.1). We would like to mention that integral (5.1) does not depend on the values of the imaginary parts of $\gamma_{r,k}$ (that is, lines of integration) provided the mentioned inequalities (3.2) and (5.4) for them are satisfied. This follows from two facts. First, we do not encounter poles as we shift the integration contour. Second, due to estimate (5.6), the integrand is vanishing at the infinities of the integration contours. This justifies the correctness of shifting of the integration contours which we use in what follows.

From the physical viewpoint, the function $\Psi_{\boldsymbol{\lambda}}(q_1,\ldots,q_N)$ given by (5.1) has correct asymptotic behaviour rapidly decreasing in the classically forbidden region, that is, where $q_k >> q_{k+1}$ for some k or where $q_1 << 0$. In the region $0 << q_1 << q_2 << \cdots << q_N$, the function $\Psi_{\boldsymbol{\lambda}}(q_1,\ldots,q_N)$ is a superposition of plane waves.

The formulas for the action of the matrix elements of $U(u)$ on $\Psi_{\boldsymbol{\lambda}}$, in particular (5.3), are derived in Appendix B of [11]. Other action formulas proved there are

$$\mathcal{D}(u)\Psi_{\boldsymbol{\lambda}} = \alpha_1\sum_{p=1}^{N}\left(\prod_{l\neq p}\frac{u^2-\lambda_l^2}{\lambda_p^2-\lambda_l^2}\right)\times$$
$$\left[\frac{(u+\lambda_p)}{2\lambda_p}\frac{(u-\frac{\mathrm{i}\frac{1}{2}}{2})}{(\lambda_p-\frac{\mathrm{i}\frac{1}{2}}{2})}\Psi_{\boldsymbol{\lambda}^{-p}}+\frac{(u-\lambda_p)}{2\lambda_p}\frac{(u-\frac{\mathrm{i}\frac{1}{2}}{2})}{(\lambda_p+\frac{\mathrm{i}\frac{1}{2}}{2})}\Psi_{\boldsymbol{\lambda}^{+p}}\right]+\alpha_1\left(\prod_{l=1}^{N}\frac{\lambda_l^2-u^2}{\lambda_l^2+(\frac{1}{2})^2}\right)\Psi_{\boldsymbol{\lambda}}, \tag{5.7}$$

$$\tilde{t}(u)\Psi_{\boldsymbol{\lambda}} := (-1)^{N-1}\frac{t(u)-2\alpha_1\alpha_N}{u^2+\left(\frac{\frac{1}{2}}{2}\right)^2}\Psi_{\boldsymbol{\lambda}} = \prod_{l=1}^{N}(u^2-\lambda_l^2)\Psi_{\boldsymbol{\lambda}}+(-1)^{N-1}\alpha_1\alpha_N\times$$
$$\sum_{p=1}^{N}\left(\prod_{l\neq p}\frac{u^2-\lambda_l^2}{\lambda_p^2-\lambda_l^2}\right)\left[\frac{1}{\lambda_p(\lambda_p-\frac{\mathrm{i}\frac{1}{2}}{2})}\Psi_{\boldsymbol{\lambda}^{-p}}+\frac{1}{\lambda_p(\lambda_p+\frac{\mathrm{i}\frac{1}{2}}{2})}\Psi_{\boldsymbol{\lambda}^{+p}}-\frac{2}{\lambda_p^2+\left(\frac{\frac{1}{2}}{2}\right)^2}\Psi_{\boldsymbol{\lambda}}\right]. \tag{5.8}$$

In particular, formula (5.7) gives

$$\mathcal{D}(\lambda_r)\Psi_{\boldsymbol{\lambda}} = \alpha_1\Psi_{\boldsymbol{\lambda}^{-r}}, \qquad \mathcal{D}(-\lambda_r)\Psi_{\boldsymbol{\lambda}} = \alpha_1\Psi_{\boldsymbol{\lambda}^{+r}}, \qquad \mathcal{D}(\mathrm{i}\frac{1}{2}/2)\Psi_{\boldsymbol{\lambda}} = \alpha_1\Psi_{\boldsymbol{\lambda}}. \tag{5.9}$$

The action of $\mathcal{A}(u)$ and $\mathcal{C}(u)$ on $\Psi_{\boldsymbol{\lambda}}$ can be derived using (4.9) and Sklyanin determinant [9] for $U(u)$, respectively.

Here we give some heuristic explanation of formulas (5.9). Let $\Psi_{\boldsymbol{\lambda}}(q_1,\ldots,q_N)$ be an eigenfunction of $\mathcal{B}(u)$ satisfying (5.3). Then the commutation relation

$$\left(u^2-(v-\mathrm{i}\frac{1}{2})^2\right)\mathcal{D}(v)\mathcal{B}(u)-(u^2-v^2)\mathcal{B}(u)\mathcal{D}(v) =$$
$$= \mathrm{i}\frac{1}{2}(u+v-\mathrm{i}\frac{1}{2})\mathcal{D}(u)\mathcal{B}(v)+\mathrm{i}\frac{1}{2}(u-v)\mathcal{A}(u)\mathcal{B}(v), \tag{5.10}$$

which follows from (4.3), gives

$$\mathcal{B}(u)\mathcal{D}(\lambda_r)\Psi_{\boldsymbol{\lambda}} = (-1)^N(u - \frac{\mathrm{i}\frac{1}{2}}{2})(u^2 - (\lambda_r - \mathrm{i}\frac{1}{2})^2)\prod_{\substack{k=1\\k\neq r}}^{N}(u^2 - \lambda_k^2)\cdot\mathcal{D}(\lambda_r)\Psi_{\boldsymbol{\lambda}}$$

at $v = \lambda_r$, and, therefore, $\mathcal{D}(\lambda_r)\Psi_{\boldsymbol{\lambda}}$ is an eigenfunction of $\mathcal{B}(u)$ with λ_r replaced by $(\lambda_r - \mathrm{i}\frac{1}{2})$. Clearly, this argumentation is not sufficient to prove the relation $\mathcal{D}(\lambda_r)\Psi_{\boldsymbol{\lambda}} = \alpha_1\Psi_{\boldsymbol{\lambda}^{-r}}$. As mentioned before, a proof of this relation is given in Appendix B of [11].

6. Wave functions for the two-boundary Toda chain

Taking into account (4.7), it is useful to introduce

$$\tilde{t}(u) := (-1)^{N-1}\frac{t(u) - 2\alpha_1\alpha_N}{u^2 + \left(\frac{1}{2}\right)^2} = u^{2N} - u^{2N-2}H_1^{\mathrm{BB}} + u^{2N-4}H_2^{\mathrm{BB}} - \cdots + (-1)^N H_N^{\mathrm{BB}}.$$

Let $\Phi_{\boldsymbol{\rho}}(\boldsymbol{q})$ be a wave function for the two-boundary Toda chain:

$$\tilde{t}(u)\Phi_{\boldsymbol{\rho}}(\boldsymbol{q}) = \prod_{k=1}^{N}(u^2 - \rho_k^2)\Phi_{\boldsymbol{\rho}}(\boldsymbol{q}) =: \tilde{t}(u|\boldsymbol{\rho})\Phi_{\boldsymbol{\rho}}(\boldsymbol{q}),$$

where $\boldsymbol{\rho} = \{\rho_1, \rho_2, \ldots, \rho_N\}$ are the quantum numbers of the corresponding state.
We look for $\Phi_{\boldsymbol{\rho}}(\boldsymbol{q})$ in the form

$$\Phi_{\boldsymbol{\rho}}(\boldsymbol{q}) = \int d\lambda_1\cdots d\lambda_N\ \tilde{\mu}(\boldsymbol{\lambda})C(\boldsymbol{\lambda}|\boldsymbol{\rho})\Psi_{\boldsymbol{\lambda}}(\boldsymbol{q}), \tag{6.1}$$

where

$$\tilde{\mu}^{-1}(\boldsymbol{\lambda}) = \prod_{\substack{i,j\\i<j}}\left(\Gamma\left(\frac{\lambda_i - \lambda_j}{\mathrm{i}\frac{1}{2}}\right)\Gamma\left(-\frac{\lambda_i - \lambda_j}{\mathrm{i}\frac{1}{2}}\right)\right)\prod_{\substack{i,j\\i\leq j}}\left(\Gamma\left(\frac{\lambda_i + \lambda_j}{\mathrm{i}\frac{1}{2}}\right)\Gamma\left(-\frac{\lambda_i + \lambda_j}{\mathrm{i}\frac{1}{2}}\right)\right),$$

and the integration with respect to $\{\lambda_k\}$ is carried out along arbitrary lines parallel to the real axis. Using (5.8) and $\frac{\tilde{\mu}(\boldsymbol{\lambda}^{+p})}{\tilde{\mu}(\boldsymbol{\lambda})} = \frac{(\lambda_p+\mathrm{i}\frac{1}{2})}{\lambda_p}\prod_{l\neq p}\frac{(\lambda_p+\mathrm{i}\frac{1}{2})^2-\lambda_l^2}{\lambda_p^2-\lambda_l^2}$, we obtain

$$
\begin{aligned}
(-1)^{N-1}\tilde{t}(u|\boldsymbol{\rho})\Phi_{\boldsymbol{\rho}}(\boldsymbol{q}) &= \int d\lambda_1\cdots d\lambda_N\ \Psi_{\boldsymbol{\lambda}}(\boldsymbol{q})\times \\
&\times\left[\alpha_1\alpha_N\sum_{p=1}^{N}\left[\frac{\tilde{\mu}(\boldsymbol{\lambda}^{+p})C(\boldsymbol{\lambda}^{+p}|\boldsymbol{\rho})}{(\lambda_p+\mathrm{i}\frac{1}{2})(\lambda_p+\frac{\mathrm{i}\frac{1}{2}}{2})}\left(\prod_{l\neq p}\frac{u^2-\lambda_l^2}{(\lambda_p+\mathrm{i}\frac{1}{2})^2-\lambda_l^2}\right)+\right.\right. \\
&+\frac{\tilde{\mu}(\boldsymbol{\lambda}^{-p})C(\boldsymbol{\lambda}^{-p}|\boldsymbol{\rho})}{(\lambda_p-\mathrm{i}\frac{1}{2})(\lambda_p-\frac{\mathrm{i}\frac{1}{2}}{2})}\left(\prod_{l\neq p}\frac{u^2-\lambda_l^2}{(\lambda_p-\mathrm{i}\frac{1}{2})^2-\lambda_l^2}\right)- \\
&\left.\left.-\frac{2\tilde{\mu}(\boldsymbol{\lambda})C(\boldsymbol{\lambda}|\boldsymbol{\rho})}{(\lambda_p^2+\frac{\frac{1}{2}^2}{4})}\left(\prod_{l\neq p}\frac{u^2-\lambda_l^2}{\lambda_p^2-\lambda_l^2}\right)\right]-\prod_{l=1}^{N}(\lambda_l^2-u^2)\right]= \\
&= \int d\lambda_1 d\lambda_2\cdots d\lambda_N\ \mu(\boldsymbol{\lambda})C(\boldsymbol{\lambda}|\boldsymbol{\rho})\Psi_{\boldsymbol{\lambda}}(\boldsymbol{q})\left[\alpha_1\alpha_N\sum_{p=1}^{N}\left(\prod_{l\neq p}\frac{u^2-\lambda_l^2}{(\lambda_p+\mathrm{i}\frac{1}{2})^2-\lambda_l^2}\right)\times\right. \\
&\left.\times\left[\frac{1}{\lambda_p(\lambda_p+\frac{\mathrm{i}\frac{1}{2}}{2})}\frac{C(\boldsymbol{\lambda}^{+p}|\boldsymbol{\rho})}{C(\boldsymbol{\lambda}|\boldsymbol{\rho})}+\frac{1}{\lambda_p(\lambda_p-\frac{\mathrm{i}\frac{1}{2}}{2})}\frac{C(\boldsymbol{\lambda}^{-p}|\boldsymbol{\rho})}{C(\boldsymbol{\lambda}|\boldsymbol{\rho})}-\frac{2}{(\lambda_p^2+\frac{\frac{1}{2}^2}{4})}\right]-\prod_{l=1}^{N}(\lambda_l^2-u^2)\right].
\end{aligned}
\tag{6.2}
$$

We set $u=\lambda_p$. Then the previous relation is satisfied if

$$
\begin{aligned}
(-1)^{N-1}\tilde{t}(\lambda_p|\boldsymbol{\rho}) &= \frac{t(\lambda_p|\boldsymbol{\rho})-2\alpha_1\alpha_N}{\lambda_p^2+\frac{\frac{1}{2}^2}{4}} = \\
&= \alpha_1\alpha_N\left[\frac{1}{\lambda_p(\lambda_p+\frac{\mathrm{i}\frac{1}{2}}{2})}\frac{C(\boldsymbol{\lambda}^{+p}|\boldsymbol{\rho})}{C(\boldsymbol{\lambda}|\boldsymbol{\rho})}+\frac{1}{\lambda_p(\lambda_p-\frac{\mathrm{i}\frac{1}{2}}{2})}\frac{C(\boldsymbol{\lambda}^{+p}|\boldsymbol{\rho})}{C(\boldsymbol{\lambda}|\boldsymbol{\rho})}-\frac{2}{(\lambda_p^2+\frac{\frac{1}{2}^2}{4})}\right],
\end{aligned}
$$

where $t(u|\boldsymbol{\rho}) = (-1)^{N-1}(u^2+\frac{1}{2}^2/4)\prod_{k=1}^{N}(u^2-\rho_k^2)+2\alpha_1\alpha_N$. This multidimensional difference equation admits separation of variables. Namely, we suppose the factorization property

$$
C(\boldsymbol{\lambda}|\boldsymbol{\rho}) = \prod_{p=1}^{N} c(\lambda_p|\boldsymbol{\rho}).
$$

Then $c(\lambda|\boldsymbol{\rho})$ satisfies the Baxter equation

$$
\frac{1}{\lambda(\lambda+\frac{\mathrm{i}\frac{1}{2}}{2})}c(\lambda+\mathrm{i}\frac{1}{2}|\boldsymbol{\rho})+\frac{1}{\lambda(\lambda-\frac{\mathrm{i}\frac{1}{2}}{2})}c(\lambda-\mathrm{i}\frac{1}{2}|\boldsymbol{\rho}) = \frac{t(\lambda|\boldsymbol{\rho})c(\lambda|\boldsymbol{\rho})}{\alpha_1\alpha_N(\lambda^2+\frac{\frac{1}{2}^2}{4})},
$$

or, equivalently,

$$
(\lambda-\frac{\mathrm{i}\frac{1}{2}}{2})c(\lambda+\mathrm{i}\frac{1}{2}|\boldsymbol{\rho})+(\lambda+\frac{\mathrm{i}\frac{1}{2}}{2})c(\lambda-\mathrm{i}\frac{1}{2}|\boldsymbol{\rho}) = \frac{\lambda\ t(\lambda|\boldsymbol{\rho})\ c(\lambda|\boldsymbol{\rho})}{\alpha_1\alpha_N}.
$$

Solutions of this equation can be constructed in terms of ratios of infinite-dimensional determinants as it was done in the case of the periodic Toda chain [8, 4]. We expect that, similarly to the case of the periodic Toda chain [8, 4], the requirement of the analytical properties of $c(\lambda|\rho)$ (which is important, in particularly, for the convergence of integral (6.1)) restricts possible values of ρ to the discrete spectrum of the quantum two-boundary Toda chain.

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The Cult of János Bolyai in Transylvania

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János Bolyai (1802-1860) is the greatest Hungarian mathematician who after recognizing the impossibility to prove Euclid's fifth (the so called parallel) postulate from Euclid's others, developed the absolute geometry (maybe the first non-Euclidian geometry) that is independent of the fifth postulate. He was only 21 years old when in 1823 he reported his finding to his father, Farkas Bolyai: "*I have created a new, different world out of nothing.*" His discovery was published in 1832 as an appendix to his father's book *Tentamen*, so generally reffered to as *Appendix*. For more than hundred years his mathematical activity was identified with the *Appendix*, but he was not only a geometer. He also developed in the unpublished *Responsio* a rigorous geometric concept of complex numbers as ordered pairs of real numbers. Although he never published more than the 26-page *Appendix*, mainly because he was unable to gain recognition for his work, he left more than 14000 pages of manuscript of mathematical work when he died. Recently these have been thoroughly researched by Elemér Kiss with surprising success: mathematical gems have been found, mainly results in number theory and algebra which were new in Bolyai's time [1].

1. János Bolyai's life and activity

János Bolyai was born on December 15, 1802 in Kolozsvár (now Cluj-Napoca) in the middle of Transylvania (see Fig.1). Since 1804 the Bolyai family lives in Marosvásárhely (now

Fig. 1. The map of Romania.

Târgu Mures) where Farkas Bolyai was invited as a teacher in the Reformed College. Here spent János Bolyai his childhood. In 1818 he began his study at the Academy of Military Engineers in Vienna. In 1823, after finishing his study, was nominated sub-lieutenant and assigned to Temesvár (now Timisoara). He wrote in November 3, 1823 in his famous letter to his father: "*I have so much to write you about my new findings... now I cannot say anything else: I have created a new, different world out of nothing.*" In 1832 his father published in Latin his book *Tentamen* with the *Appendix: Scientia spatii absolute veram exhibens* of János Bolyai. This *Appendix* was published in 1831 as a preprint. This 26-page paper contains the exposition of the absolute geometry, which is independent of the Euclid's fifth postulate. Farkas Bolyai sent a copy to Gauss, his youthful friend. The Gauss' reply on the *Appendix* was a crushing one: "*If I praised it, I should praise myself since the whole content... coincide almost entirely with my reflections over 30-35 years*". But in letter to his friend Gerling he wrote: "*I consider the young geometer Bolyai as a genius of first order.*" In this letter he recognizes that his ideas in 1798 were far from the maturity found in the work of János Bolyai.

After retiring in 1833 János Bolyai lived in Domáld (now Viisoara) and in Marosvásárhely, where he still worked mathematics as the remaining manuscripts attest. Between 1835–1840 he clearly expressed the idea which later came to be known as the "geometrization of physics".

In 1848 he got Lobachevsky's work on parallels published in 1840 in German. First, he was suspicious that he had been stolen, but after reading the entire book he made enthusiastic comments on it.

János Bolyai died in 1860. In the *Kolozsvári Közlöny* (Bulletin of Kolozsvár) has been written: "*It is our enormous loss, that the life of such a brilliant man and owner of deep knowledge has passed almost with no use among us, and being by nature odd and avoiding people, he lived exclusively being engaged in his vast ideas. May he rest in peace!*" [4]

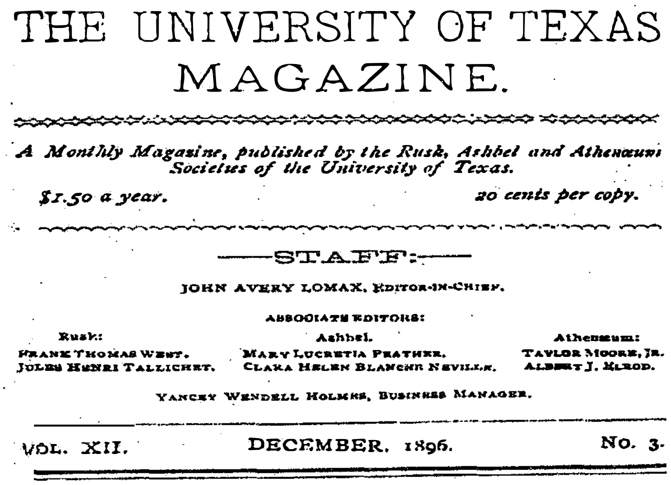
2. The Cult of János Bolyai

His scientific activity was unrecognized in his life. Only after his death the scientific world finds out his outstanding results in geometry. The *Appendix* was translated in several languages, as Italian and French (1867), English (1891), Hungarian (1897), Russian (1950), Romanian (1954) etc.

The first who took note of the importance of the *Appendix* was Richard Baltzer from Drezda. Under his influence G.J. Hoüel from Bordeaux started to study the geometry of Bolyai. Gyula Vályi (1855–1913) was the first professor at the new created university in Kolozsvár (in 1872) who held lectures about the *Appendix* in the second semester of the academic year 1891–92, which lectures were repeated several times every four years. In order to compare the absolute geometry with the hyperbolic geometry he sometimes borrowed some of Lobachevsky's results [2].

In the summer of the year 1896 George Bruce Halsted (1853–1922), professor at Texas University in Austin, who have been published the Lobachesky's treatis and the Bolyai's Appendix, made a trip to Marosvásárhely and after this to Kazan. In *The University Of Texas Magazine* was published a paper by J. A. Lomax, the editor in chief, about this very interesting summer trip [3] (see Fig. 2).

From this interesting paper let us point here only two facts on Bolyai and Lobachevsky: "*For many years Dr. Halsted, our professor in mathematics, has been interested in the*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VOL. XII. DECEMBER, 1896. NO. 3.

Fig. 2. Lomax article's magazine.

study of non-Euclidean geometry. In the pursuance of these study he has discovered to the English-speaking world two great characters... Nicolai Ivanovich Lobachevsky, a Russian, and Bolyai Janos, a Hungarian... " [3] (see Fig. 3).

"*Among the other interesting facts about his family life, unknown before, it was learned that Lobachevsky had a son and daughter now living. The son is a political exile in Siberia, and the daughter, poor in purse, lives in St. Petersburg.*" Halsted became a benefactor, because the Czar promptly awarded to the Lobachevsky's daughter an annual pension, and Halsted also ask the Czar to pardon the son [3] (see Fig. 4).

The house where János Bolyai was born (Fig.5) had been tracked down and marked by a memorial plaque only in 1902, when 100 years since the birth was celebrated. This period is the real beginning of the Bolyai-cult.

The Bolyai Prize was established by the Hungarian Academy of Sciences to be awarded to an outstanding mathematician every five years by an international committee. In 1905 Henri Poincaré, the great French mathematician was honored, and in 1910 the famous German David Hilbert. For the third prize was recommended Albert Einstein, but because of the I. World War the prize was not awarded. The Bolyai Prize was renewed in 1991 and awarded again only in 2000 to S. Shelah for his monography *Cardinal Arithmetic* [5].

The first book on the two Bolyais' life was published by P. Stäckel in German in 1913 and translated in Hungarian one year later.

After the II. World War, when Transylvania became again a part of Romania, in 1945 a Hungarian university was established in Cluj (actually this was the continuation of the former Hungarian university), which later was named after the two Bolyais (Farkas, the father and János, the son) *Bolyai University*. This university was merged by force in 1959 with the *Victor Babes University* in Romanian language. The new university got the name *Babes-Bolyai University* (Fig.6), which still exists and now is one of the largest university in Romania.

DR. HALSTED AND HIS SUMMER TRIP.

For many years Dr. Halsted, our professor of mathematics, has been interested in the study of non-Euclidean geometry. In the pursuance of these studies he has discovered to the English-speaking world two great characters. These two men, of different nationalities, unknown to each other, living thousands of miles apart, made a strange discovery that, as one of them says, "From nothing creates another wholly new world." Stranger still, the same conclusions were hit upon almost at the same moment. Nicolai Ivanovich Lobachevsky, a Russian, and Bolyai Janos, a Hungarian, have gone back of Euclid, and evolved from pure reason a conception of the universe, so new and so strange that time alone can enable us to grasp its full import. The ages have been blindly trusting an assumption. Dr. Halsted is bringing to light the works of the two men of all the world who were brave and independent enough to demand the proofs. This not forthcoming, they boldly created a new world equally valid with the old.

Fig. 3.

Again was Dr. Halsted successful in finding out much that is new to the world about this now famous character. Among other interesting facts about his family life, unknown before, it was learned that Lobachevsky had a son and daughter now living. The son is a political exile in Siberia, and the daughter, poor in purse, lives in St. Petersburg. To her Dr. Halsted had already unconsciously been a benefactor. Through some of his work on Lobatchevsky, the attention of the minister of education was called to the great Russian. He in turn reported the matter to the Czar, who became interested, traced the daughter and promptly awarded her an annual pension for life of six hundred roubles. Dr. Halsted should now petition for the pardon of the son. It may be that he has already done so. An unpublished history of non-Euclidean Geometry up to his time, written by Lobachevsky himself, was found, and is now being translated in Austin, to be shortly given to the scientific world.

Fig. 4.

The life and activity of János Bolyai has been studied in the last decades by several researchers in Transylvania as Elemér Kiss, Tibor Weszely, Tibor Toró, Róbert Oláh-Gál, Samu Benkő (the non-mathematical papers) and are studied further on.

In 2002 we celebrated 200 years since the birth of János Bolyai by a lot of commemorative conferences held in Romania. The first one was the BGL-3 conference in Târgu Mures on July 3-6 (the first two editions had been held in Uzhgorod, Ucraine and in Nyíregyháza, Hungary). The second conference of the year dedicated to Bolyai was on October 1-5 in Cluj-Napoca (*International Conference on Geometry and Topology*) organized by *Babes-*

Fig. 5. J. Bolyai's birthhouse.

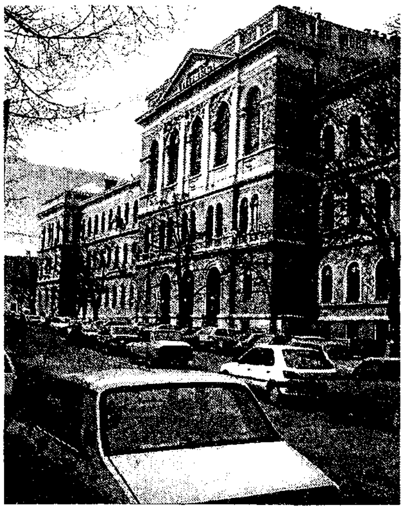

Fig. 6. Babes-Bolyai University.

Bolyai University with the cooperation a *Farkas Bolyai Highschool* in Târgu Mures. In october an Bolyai-exposition was achieved by the university of Cluj-Napoca. The Romanian Academy had organized a commemorative session on December 13 in Bucharest. The last events in the Bolyai Year were a conference in Hungarian at Cluj-Napoca on December 14, a remembrance spectacle at the Hungarian Opera in Cluj-Napoca on the same evening, and on December 15 in Târgu Mures the inauguration of the so called *Pseudosphere* monument (Fig.7) in front of the Bolyai Museum, projected by the mathematician Sándor Horváth. Every year on November 3 at noon, if the sun is shining, a mirror system lightens the inscription *I have created a new, different world out of nothing*, written on the base of the monument.

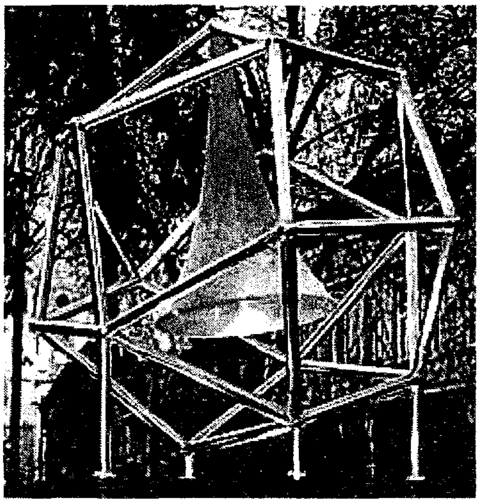

Fig. 7. The *Pseudosphere* monument.

In 2002 a great number of papers on Bolyai's life and acticity in Hungarian, English and Romanian have been appeared in different journals and nice books have been published. It is worth to mention a book edited by Polis Publishing House with 27 poems dedicated to János Bolyai by 8 Hungarian poets from Transylvania.

After a long slight, nowadays János Bolyai took his worthy place in the history of sciences.

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Дифференциальная геометрия и сжатое описание Вселенной

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Будем предполагать, что эволюция Вселенной определяется частицами, большая часть которых в настоящее время находится в связанном состоянии и которые проявляют себя лишь посредством слабых взаимодействий. Для описания бозонных состояний мы будем использовать гладкие поля $\mathcal{B}^c_a$. Вероятно, ранг матрицы плотности ρ полей $\mathcal{B}^c_a$ равен n, но нельзя исключить, что данное равенство есть следсвие пренебрежения некоторыми компонентами матрицы плотности. Учитывая неразличимость большей части бозонных состояний, мы будем использовать редуцированный набор полей $\{\Phi^{(j)}_i, A^{\underline{b}}_i\}$ вместо полного набора $\{\mathcal{B}^a_b\}$. Конечно, необходимо учитывать, что в лагранжиане появятся постоянные, играющие роль весовых множителей — такие, как $1/G_N$ (G_N — гравитационная постоянная). В результате уравнения полей $\Phi^{(j)}_i$ могут быть записаны как уравнения Эйнштейна. Это позволяет определить пространство–время M^o_n как риманово многообразие, основной тензор g_{ij} которого мы будем вводить посредством редуцированной матрицы плотности ρ'.

1. Введение

В ноябре 2003 года была опубликована статья Б.Б. Кадомцева [1] по материалам его лекций 1997 года, которая показала, что дискуссия 30-х годов XX столетия между Бором и Эйнштейном, касающаяся основополагающих принципов квантовой механики, не потеряла своего значения и в настоящее время. Как известно, Эйнштейн предполагал, что вероятностные законы квантовой механики являются следствием неполноты описания физических систем. При этом неполнота может быть устранена посредством введения дополнительных скрытых параметров. Тем самым предполагалось наличие классических закономерностей на более глубоком субквантовом уровне материи. Напротив, Бор предполагал принципиальную невозможность достичь этого, так как многие характеристики микромира проявляются исключительно благодаря наличию макроскопических приборов и не могут быть приписаны элементарным частицам в отсутствии измерения.

Заметим, что в XX веке были сделаны громадные попытки разрушить иллюзию детерминизма, которая утвердилась в науке в конце XIX века. Конечно, главную лепту в это внесла квантовая механика, создание которой было инициировано результатами экспериментов в атомной и ядерной физики. Но и в основе основ, на которой базировался детерминизм — классической механике — были отмечены “недостатки”, приводящие к утрате иллюзий [2]. Несмотря на то, что с иллюзиями было покончено, от идеи детерминизма трудно отказаться, так как планирование физических экспериментов основано на расчетах, опирающихся на методы, наработанные в науке при его господстве. К этим методам, в первую очередь, необходимо отнести исчисление бесконечно малых. Успехи в этой области трудно переоценить. Можно указать на одну лишь область в математике — теорию групп Ли, которая оказала огромное влияние на всю теоретическую физику. Конечно же, полезные результаты могли быть здесь получены благодаря "хорошим" свойствам пространства-времени. В первую очередь это свойство хаусдорфовости (отделимости), которое постулируется, несмотря на квантовый характер законов, действующих в микромире. Во–вторых, в теории групп Ли важную роль играет наличие гладких конгруэнций, получаемых как решения дифференциальных уравнений. В то же время нельзя не отметить, что в квантовой механике отрицается само существование траекторий элементарных частиц. Именно поэтому вместо производных Ли становится необходимым использовать более общие операторы, которые индуцировали бы и более общие по сравнению с группами Ли алгебраические структуры, в частности, локальные лупы Ли [3, 4], и которые позволили бы учесть отсутствие детерминизма в реальных физических процессах. Заметим, что неадекватность описания физических систем при помощи гладких полей в дифференцируемых многообразиях ведет к необходимости дать вероятностную интерпретацию геометрическим объектам. Вследствие этого мы будем рассматривать решения дифференциальных уравнений лишь как наиболее правдоподобные функции, применяемые для описания этих систем. Конечно, при этом мы учитываем законы, действующие в микромире, и считаем их более фундаментальными, чем те, которые применяются для описания движения макроскопических тел.

Мы будем опираться на подход, предложенный Шредингером [5], который ввел набор неортогональных друг другу волновых функций Ψ, описывающих нерасплывающийся волновой пакет для квантового осциллятора. Позднее Глаубер [6] показал возможность описания когерентных явлений в оптике при помощи введенных Шредингером состояний, которые назвал когерентными. Данный подход далее был развит в работах Переломова [7], который ввел определение обобщенных когерентных состояний как состояний, возникающих при действии оператора представления некоторой группы преобразований на какой-либо фиксированный вектор в пространстве этого представления. По нашему мнению, именно это и позволяет дать физическую интерпретацию калибровочным преобразованиям как преобразованиям, позволяющим получать обобщенные когерентные состояния, характеризующиеся непрерывными и, возможно, скрытыми параметрами [8].

Итак, рассмотрим волновые пакеты $\{\Upsilon(\omega)\}$ "эмпирических" функций $\Upsilon(\omega)$, являющихся амплитудами вероятности физической системы, находящейся в состоянии, которое характеризуется параметрами ω. Переходы между состояниями будем задавать при помощи инфинитезимальных подстановок

$$\Upsilon \to \Upsilon + \delta\Upsilon = \Upsilon + \delta T(\Upsilon) \tag{1.1}$$

локальной лупы Ли, где δT является инфинитезимальным оператором перехода. Введение макроскопического наблюдателя заставляет нас искать представление операторов перехода дифференциальными операторами. В результате становится желательным использование дифференцируемого многообразия M_r, в области Ω_r которого будем искать гладкие "теоретические" поля $\Upsilon(\omega)$ ($\omega \in \Omega_r \subset M_r$) как решения дифференциальных уравнений, что в общем случае является нереальной задачей (как известно, даже в классической динамике наиболее интересные проблемы не сводятся к интегрируемым системам [9]). Именно поэтому представляет интерес более простая задача поиска сужений $\Upsilon(x)$ "теоретических" полей на многообразии M_n ($x \in M_n \subset M_r, n \leq r$).

Для этого через некоторую точку $\omega \in M_r$ проведем гладкие кривые, с помощью которых определим соответствующее множество векторных полей $\{\delta\xi(\omega)\}$, а с их помощью определим отклонение полей $\Upsilon(\omega)$ в точке $\omega \in M_r$ в виде

$$\delta_0\Upsilon = \delta X(\Upsilon) = \delta T(\Upsilon) - \delta\xi(\Upsilon). \tag{1.2}$$

Если $\delta_0\Upsilon = 0$, то мы получаем аналог кинетического уравнения Больцмана, где член $\delta T(\Upsilon)$ играет роль интеграла столкновения. Так как мы не надеемся в общем случае получить интегрируемую систему, то будем требовать, чтобы эти отклонения (1.2) хотя бы в "среднем" были минимальны [10].

Итак, определим квадрат полунормы $\mid X(\Upsilon) \mid$ в векторном пространстве с полускалярным произведением как интеграл

$$\mathcal{A} = \int\limits_{\Omega_n} \mathcal{L}\, d_nV = \int\limits_{\Omega_n} \kappa\overline{X(\Upsilon)}\varrho X(\Upsilon) d_nV \tag{1.3}$$

($\mathcal{A}$ — действие; $\mathcal{L}(\Upsilon)$ — лагранжиан; κ — постоянная, ϱ — матрица плотности, черта сверху означает дираковское сопряжение являющееся суперпозицией эрмитового сопряжения и пространственной инверсии), являющийся аналогом дисперсии сужений "теоретических" полей $\Upsilon(x)$ в области $\Omega_n \subset M_n$. Решения $\Upsilon(x)$ (возможно, даже одно решение) уравнений, которые получаются из требования минимальности интеграла (1.3), могут быть использованы для построения полного набора функций $\{\Upsilon(x)\}$ (генерируемых операторами перехода), описывающих волновой пакет.

Конечно, для этой цели можно использовать аналог метода наибольшего правдоподобия, применяемый в математической статистике. Как известно, согласно гипотезе Фейнмана, амплитуда вероятности перехода системы из состояния $\Upsilon(x)$ в состояние $\Upsilon'(x')$ равна следующему интегралу

$$K(\Upsilon, \Upsilon') = \int\limits_{\Omega(\Upsilon,\Upsilon')} \exp{(i\mathcal{A})}\, \mathcal{D}\Upsilon =$$

$$= \lim_{N\to\infty} I_N \int d\Upsilon_1 ... \int d\Upsilon_k ... \int d\Upsilon_{N-1} \exp\left(i \sum_{k=1}^{N-1} \mathcal{L}(\Upsilon(x_k))\, \triangle V_k \right) \tag{1.4}$$

(используется система единиц $h/(2\pi) = c = 1$, где h — постоянная Планка, c — скорость света; $i^2 = -1$; величина I_N выбирается так, чтобы предел существовал). Вследствие этого функции $\Upsilon(x)$, получаемые из требования минимальности действия $\mathcal{A}$ и используемые для описания квантовых систем, также являются лишь наиболее правдоподобными. В этом подходе лагранжиан играет более фундаментальную роль при описании физических систем, чем дифференциальные уравнения, которые из него получаются.

2. Локальные лупы Ли

Далее мы будем рассматривать пространство M_r как многообразие, параметры ω^a ($a, b, c, d, e = 1, 2, \ldots, r$) — как координаты произвольной точки $\omega \in M_r$, а поля $\Upsilon(\omega)$ будем задавать в некоторой области Ω_r данного многообразия ($\omega \in \Omega_r \subset M_r$). Пусть область Ω_r содержит подобласть Ω_n с точкой ω, при этом область Ω_n принадлежит определенному дифференцируемому многообразию M_n (хотя, возможно, удобно определять многообразие M_n отдельно от многообразия M_r). Более того, пусть множество гладких кривых, принадлежащих многообразию M_n , имеет общую точку ω . Определим также набор векторных полей $\xi(x)$, являющихся касательными к этим кривым, и будем считать, что $x \in \Omega_n$, а на области Ω_n определена собственная координатная система.

Пусть $\delta\Omega_r$ является достаточно малой окрестностью точки ω , в связи с этим задается и достаточно малая окрестность $\delta\Omega_n$ точки x ($x \equiv \omega \in \delta\Omega_n \subset \delta\Omega_r$). Координаты точки x запишем как x^i ($i, j, k, l, p, q = 1, 2, \dots, n$). Используя векторные поля $\delta\xi(x)$, координаты соседней точки $x' = x + \delta x \in \delta\Omega_n$ перепишем в виде

$$x'^i = x^i + \delta x^i \cong x^i + \delta\omega^a(x)\xi^i_a(x). \tag{2.1}$$

Сравнивая значения полей $\Upsilon'(x')$ и $\Upsilon(x')$, где

$$\Upsilon'(x') = \Upsilon + \delta\Upsilon = \Upsilon + \delta T(\Upsilon) \cong \Upsilon + \delta\omega^a T_a(\Upsilon), \tag{2.2}$$

$$\Upsilon(x') = \Upsilon(x + \delta x) \cong \Upsilon + \delta\omega^a\xi^i_a\partial_i\Upsilon \tag{2.3}$$

(∂_i — частные производные), мы видим, что они отличаются переменными

$$\delta_o\Upsilon(x) = \delta\omega^a X_a(\Upsilon) = \delta\omega^a[T_a(\Upsilon) - \xi^i_a\partial_i\Upsilon], \tag{2.4}$$

которые можно интерпретировать как отклонения полей $\Upsilon(x)$, полученных с помощью подстановок (2.2).

Далее мы будем рассматривать область $\delta\Omega_r \subset M_r$ как область локальной лупы Ли G_r (которая, в частности, может иметь и структуру локальной группы Ли, если мы потребуем для нее свойство ассоциативности), индуцированной множеством $\{T\}$, при этом будем рассматривать выражение (2.2) как инфинитезимальный закон подстановок локальной лупы Ли полей $\Upsilon(x)$. Отметим, что структура локальной лупы Ли будет характеризовать степень когерентности рассматриваемых квантовых систем. При этом максимальная степень достигается для простой группы Ли, а минимальная — для абелевой. В последнем случае мы будем иметь некогерентную смесь.

Так как невозможно пренебречь взаимодействием между частицами, мы должны уметь отбирать те взаимодействия, которые нас интересуют. Именно поэтому имеет смысл выбрать множество операторов, которые в дальнейшем будут играть роль связности. Конечно, мы принимаем во внимание зависимость систем отсчета от физических свойств инструментов (включая эталоны) и, более того, что часть переходов являются не наблюдаемыми. Пусть

$$L_a(\Upsilon) = T_a(\Upsilon) + \xi^i_a\Gamma_i\Upsilon. \tag{2.5}$$

Тогда формула (2.4) примет вид

$$\delta_o\Upsilon = \delta\omega^a X_a(\Upsilon) = \delta\omega^a(L_a(\Upsilon) - \xi^i_a\nabla_i\Upsilon), \tag{2.6}$$

где ∇_i — ковариантные производные в отношении связности $\Gamma_i(x)$. Заметим, если $L_a(\Psi) = L_a\Psi$, то должны иметь место следующие соотношения [11]

$$\xi^i_a\ \nabla_i\xi^k_b - \xi^i_b\ \nabla_i\xi^k_a - 2\ S^k_{ij}\ \xi^i_a\ \xi^j_b = -C^c_{ab}\ \xi^k_c, \tag{2.7}$$

$$L_aL_b - L_bL_a - \xi^i_a\ \nabla_iL_b + \xi^i_b\ \nabla_iL_a + R_{ij}\ \xi^i_a\ \xi^j_b = C^c_{ab}\ L_c, \tag{2.8}$$

где $S^k_{ij}(x)$ — компоненты кручения многообразия M_n, а $R_{ij}(x)$ — компоненты кривизны связности $\Gamma_i(x)$, определяемые как

$$S^k_{ij} = (\Gamma^k_{ij} - \Gamma^k_{ji})/2, \tag{2.9}$$

$$R_{ij} = \partial_i\Gamma_j - \partial_j\Gamma_i + \Gamma_i\Gamma_j - \Gamma_j\Gamma_i. \tag{2.10}$$

Здесь и далее ${\Gamma_{ij}}^k(x)$ — компоненты внутренней связности многообразия M_n. При этом компоненты $C^c_{ab}(x)$ структурного тензора локальной лупы Ли G_r должны удовлетворять тождествам

$$C^c_{ab} + C^c_{ba} = 0, \tag{2.11}$$

$$C^d_{[ab}\, C^e_{c]d} - \xi^i_{[a}\, \nabla_{|i|} C^e_{bc]} + {R_{ij[a}}^e\, \xi^i_b\, \xi^j_{c]} = 0, \tag{2.12}$$

где ${R_{ija}}^e(x)$ — компоненты кривизны связности $\Gamma^{\,b}_{i a}(x)$, определяемые в виде

$${R_{ijb}}^a = \partial_i\Gamma^{\,a}_{j b} - \partial_j\Gamma^{\,a}_{i b} + \Gamma^{\,a}_{i c}\Gamma^{\,c}_{j b} - \Gamma^{\,a}_{j c}\Gamma^{\,c}_{i b}. \tag{2.13}$$

Мы рассматриваем дифференцируемое многообразие M_n, не интерпретируя его физически. Конечно, мы предполагаем рассматривать многообразие M_n как пространство–время M_4. В то же время нельзя не учитывать возможность фазовых переходов системы, в результате которых можно ожидать появления когерентных состояний. Вследствие этого удобно не фиксировать размерность многообразия M_n. Можно считать, что при коллапсе макроскопическая система достигает именно такого состояния. В результате мы имеем классический аналог когерентного состояния квантовой системы. Кроме того, необходимо заметить, что в квантовой теории поля имеется достаточно развитый аппарат — размерная регуляризация, использующая пространства с изменяющейся размерностью.

3. Калибровочные поля

Рассмотрим гипотезу Больцмана рождения Вселенной вследствие гигантской флуктуации, но не в пустом пространстве, а в среде, состоящей из слабо взаимодействующих частиц, характеризующихся нулевой температурой и образующих бозе–конденсат. Конечно, если частицы являются фермионами, они должны находиться в связанном состоянии. Для описания такого состояния материи Вселенной, которое будем считать чистым, необходимо ввести амплитуду вероятности $\mathcal{B}$ с компонентами $\mathcal{B}^b_a$ и матрицу плотности $\rho(\mathcal{B})$ (для чистого состояния $\operatorname{rank}\rho(\mathcal{B}) = 1$), которая определяется стандартным образом:

$$\mathcal{B}\mathcal{B}^+ = \rho\,\mathrm{tr}(\mathcal{B}\mathcal{B}^+) \tag{3.1}$$

($\mathrm{tr}\,\rho = 1$, $\rho^+ = \rho$, верхний индекс "+" есть символ эрмитового сопряжения).

Пусть в результате каких-либо причин произойдет распад бозе–конденсата с образованием "свободных" фермионов (для их описания введем амплитуду вероятности Ψ) и с увеличением давления в некоторой локальной области Вселенной (при этом некоторое время температура фоновых частиц должна оставаться равной (или близкой) нулю — так называемый период инфляции). В результате ранг матрицы плотности ρ начнет расти, что характеризует появление смешанных состояний. Обратный процесс релаксации, характеризуемый образованием бозе–конденсата и уменьшением давления, должен идти с выделением энергии, которая пойдет на разогрев ферми–жидкости с образованием возбужденных состояний — известных заряженных фермионов (кварков и лептонов). С этого момента можно вводить метрику и использовать результаты, полученные для горячей модели Вселенной (с возможными инфляционными модификациями), интерпретируя эволюцию Вселенной как процесс, характеризуемый ростом энтропии $S = -\operatorname{tr}(\rho \ln \rho)$. В настоящее время материя наблюдаемой области Вселенной находится на той стадии эволюции, когда преобладающее число частиц, которые будем описывать сжатым образом, вернулось в бозе–конденсатное состояние, проявляясь лишь при слабом взаимодействии с частицами видимой материи.

Возможно, ранг матрицы плотности ρ равен n, но нельзя исключить, что данное равенство выполняется лишь приближенно, когда некоторыми компонентами матрицы плотности можно пренебречь. В любом случае будем считать, что среди полей $\mathcal{B}_a^b$ выделились смеси Π_a^i с ненулевыми вакуумными средними h_a^i, которые определяют дифференцируемые векторные поля $\xi_a^i(x)$ для рассматриваемой области Ω_n в виде

$$\Pi_a^i = \mathcal{B}_a^b\, \xi_b^i \tag{3.2}$$

(поля $\xi_\bullet^i(x)$ определяют дифференциал $d\pi$ проекции из $\Omega_r \subset M_r$ в Ω_n). Это позволяет определить риманово пространство–время M_n^o, основной тензор $g_{ij}(x)$ которого введем посредством редуцированной матрицы плотности $\rho'(x)$. В результате можно будет впоследствии "спрятать" часть полей при помощи нетривиальной геометрической структуры.

Итак, пусть компоненты ρ_i^j редуцированной матрицы плотности $\rho'(x)$ определяются следующим образом:

$$\rho_i^j = \xi^{+a}_{\ i}\ \rho_a^b\ \xi_b^j\ /\ (\xi^{+c}_{\ k}\rho_c^d\xi_d^k) = \Pi^{+a}_{\ i}\Pi_a^j\ /\ (\Pi^{+b}_{\ k}\Pi_b^k), \tag{3.3}$$

и пусть поля

$$g^{ij} = \eta^{k(i}\rho_k^{j)}\ (g^{lm}\eta_{lm}) \tag{3.4}$$

являются компонентами тензора обратного основному тензору пространства–времени M_n^o. При этом компоненты $g_{ij}(x)$ основного тензора должны являться решениями следующих уравнений:

$$g^{ij}g_{ik} = \delta_k^j \tag{3.5}$$

(здесь и далее η_{ij} — компоненты метрического тензора касательного пространства к M_n^o, а η^{ik} определяются как решения уравнений $\eta^{ij}\eta_{ik} = \delta_k^j$; δ_j^i — символы Кронекера).

Запишем интеграл (1.3) следующим образом

$$\mathcal{A}_t = \int\limits_{\Omega_n} \mathcal{L}_t d_n V = \int\limits_{\Omega_n} [\mathcal{L}_o(\mathcal{B}) + \mathcal{L}_1(\Psi)] d_n V, \tag{3.6}$$

где

$$\mathcal{L}_1 = \kappa\overline{X^b}(\Psi)\rho_b^a X_a(\Psi) = \kappa\overline{D^a}\Psi D_a\Psi\ /\ (\mathcal{B}^{+c}_{\ b}\mathcal{B}_c^b). \tag{3.7}$$

Пусть поля

$$D_a\Psi = -\mathcal{B}_a^c X_c(\Psi) = \mathcal{B}_a^c(\xi_c^i\nabla_i\Psi - L_c\Psi). \tag{3.8}$$

изменяются аналогично полям $\Psi(x)$ в точке $x \in M_n$, то есть

$$\delta_o D_a\Psi = \delta\omega^b\ (L_b D_a\Psi - L_{ba}^{\ c}\ D_c\Psi - \xi_b^i\ \nabla_i D_a\Psi) \tag{3.9}$$

(поля $L_{b}{}_{a}^{c}(x)$ удовлетворяют соотношениям, аналогичным (2.8)). В результате изменения $\delta_o \mathcal{B}_c^{\bullet}$ запишутся в виде:

$$\delta_o \mathcal{B}_a^d = \delta\omega^b \, (C_{cb}^d \, \mathcal{B}_a^c - L_{b}{}_{a}^{c} \, B_c^d - \xi_b^i \, \nabla_i \mathcal{B}_a^d) + \Pi_a^i \, \nabla_i \delta\omega^d, \tag{3.10}$$

что вследствие появления последнего слагаемого в правой части формулы (3.10) позволяет называть поля $\mathcal{B}(x)$ калибровочными.

Так как действие (3.6) должно быть инвариантно при инфинитезимальных подстановках локальной лупы Ли G_r, то лагранжиан $\mathcal{L}_o(\mathcal{B})$ должен зависеть от калибровочных (бозонных) полей $\mathcal{B}(x)$ ([10, 12]) посредством напряженностей $\mathcal{F}_{ab}^c(B)$, имеющих вид

$$\mathcal{F}_{ab}^c = \Theta_d^c \, (\Pi_a^i \, \partial_i \mathcal{B}_b^d - \Pi_b^i \, \partial_i \mathcal{B}_a^d + \Xi_{ab}^d), \tag{3.11}$$

где

$$\Theta_b^c = \delta_b^c - \xi_b^i \, \Pi_i^d \, (\mathcal{B}_d^c - \beta_d^c), \quad \Xi_{ad}^b = (\mathcal{B}_a^c \, L_{c}{}_{d}^{e} - \mathcal{B}_d^c \, L_{c}{}_{a}^{e}) \, \mathcal{B}_e^b - \mathcal{B}_a^c \, \mathcal{B}_d^e \, C_{ce}^b. \tag{3.12}$$

Здесь и далее выбор полей $\Pi_i^{\bullet}$ и β_c^a ограничен соотношениями

$$\Pi_j^a \, \Pi_a^i = \delta_j^i, \quad \beta_c^a \, \xi_a^i = h_c^i. \tag{3.13}$$

Далее удобно воспользоваться лагранжианом

$$\mathcal{L}_o(\mathcal{B}) = \frac{\kappa_o'}{4} \mathcal{F}_{ab}^c \mathcal{F}_{de}^f [t^{ad}(s_c^e s_f^b - \upsilon s_c^b s_f^e) + t^{be}(s_f^a s_c^d - \upsilon s_c^a s_f^d) + u_{cf}(t^{ad} t^{be} - \upsilon t^{ab} t^{de})] \tag{3.14}$$

(κ_o', υ — постоянные) [12]. Если $s_a^b = \delta_a^b$, $t^{ab} = \eta^{ab}$, $u_{ab} = \eta_{ab}$ (η_{ab} — компоненты метрического тензора плоского пространства, а η^{ab} — компоненты тензора обратного к основному), то данный лагранжиан наиболее применим для описания горячей стадии эволюции материи наблюдаемой области Вселенной, так как является наиболее симметричным относительно напряженностей калибровочных полей $\mathcal{F}_{ab}^c$. Более того, мы будем требовать выполнения соотношений

$$L_{a}{}_{c}^{b} \, \eta^{cd} + L_{a}{}_{c}^{d} \, \eta^{cb} = 0, \tag{3.15}$$

чтобы операторы перехода $L_{a}{}_{c}^{b}$ генерировали симметрию, которая следует из сделанных предположений. В отсутствии полей $\Pi_a^i(x)$ и $\Psi(x)$ на ранней стадии эволюции материи наблюдаемой области Вселенной полный лагранжиан $\mathcal{L}_t$ становится даже более симметричным ($\mathcal{L}_t \propto \mathcal{B}^4$), так что образование фермионов (появление полей $\Psi(x)$ в полном лагранжиане $\mathcal{L}_t$) из первичных бозонов является необходимым (хотя и не достаточным) условием перехода материи наблюдаемой области Вселенной к современной стадии ее развития со спонтанным нарушением симметрии. Только образование бозе–конденсата из пар некоторого класса фермионов (возможно, из нейтрино различных ароматов) привело к заметному росту масс покоя тех векторных бозонов (W^+, W^-, Z^o), которые взаимодействовали с данным классом фермионов. Параллельно мог идти рост масс покоя и других фундаментальных частиц, хотя и не всех (фотон, непосредственно с нейтрино не взаимодействующий, не имеет массы покоя).

Свяжем ненулевые вакуумные средние β_a^b калибровочных полей $\mathcal{B}_a^b$ со спонтанным нарушением симметрии, которое произошло в ранней Вселенной и которое необходимо рассматривать как фазовый переход с образованием бозе–конденсата из фермионных пар. Переход к современной стадии эволюции материи наблюдаемой области Вселенной, для которой предполагается наличие кластерных состояний слабо взаимодействующих частиц, будет выражаться в следующей формуле для тензоров s_a^b, t^{ab}, u_{ab} и h_i^a :

$$s_a^b = s\ \xi_a^i\ h_i^b + \xi_a^{\underline{c}}\ \varepsilon_{\underline{c}}^b, \qquad t^{ab} = t\ \varepsilon_{(i)}^a\ \varepsilon_{(j)}^b\ \eta^{(i)(j)} + \varepsilon_{\underline{c}}^a\ \varepsilon_{\underline{d}}^b\ \eta^{\underline{cd}},$$

$$u_{ab} = u\ \xi_a^i\ \xi_b^j\ h_i^c\ h_j^d\ \eta_{cd} + \xi_a^{\underline{c}}\ \xi_b^{\underline{d}}\ \eta_{\underline{cd}}, \qquad h_i^a = h_i^{(j)}\ \varepsilon_{(j)}^a \tag{3.16}$$

($(i),(j),(k),(l),... = 1,2,\ldots,n;$ $\underline{a},\underline{b},\underline{c},\underline{d},\underline{e} = n+1, n+2,\ldots,n+\underline{r};$ $\underline{r}/r \ll 1$), где поля $h_i^{(j)}(x)$, принимая во внимание соотношения (3.16), однозначно определяются из уравнений $h_k^a\ h_a^i = \delta_k^i$. Подобным образом тензоры $\eta^{(i)(j)}$, $\eta^{\underline{ab}}$ определяются из уравнений $\eta^{(i)(k)}\eta_{(j)(k)} = \delta_{(j)}^{(i)}$, $\eta^{\underline{ab}}\eta_{\underline{cb}} = \delta_{\underline{c}}^{\underline{a}}$, в то время как тензоры $\eta_{(i)(j)}$, $\eta_{\underline{ab}}$ определяются следующим образом: $\eta_{(i)(k)} = \eta_{ab}\ \varepsilon_{(i)}^a\ \varepsilon_{(k)}^b$, $\eta_{\underline{ab}} = \eta_{cd}\ \varepsilon_{\underline{a}}^c\ \varepsilon_{\underline{b}}^d$. Мы свяжем постоянные $\varepsilon_{(i)}^a$, $\varepsilon_{\underline{b}}^a$ с выбором калибровочных полей $\Pi_i^a(x)$, переписывая их в виде

$$\Pi_i^a = \Phi_i^{(j)}\ \varepsilon_{(j)}^a + P_i^{\underline{b}}\ \varepsilon_{\underline{b}}^a, \tag{3.17}$$

и пусть $\varepsilon_{\underline{b}}^a = 0$. Кроме того, мы будем применять разложение полей $B_b^a(x)$ в виде

$$\mathcal{B}_c^a = \zeta_i^a\ \Pi_c^i + \zeta_{\underline{b}}^a\ A_c^{\underline{b}}, \tag{3.18}$$

где $A_{\underline{b}}^{\underline{a}} = B_b^c\ \xi_c^{\underline{a}}$. Отметим, что мы разбиваем физическую систему, описываемую полями $\mathcal{B}_c^a(x)$, на две подсистемы. Одна из них, описываемая полями $\Pi_c^i(x)$, будет играть роль медленной подсистемы. При этом компоненты промежуточных тензорных полей $\xi_a^i(x)$, $\xi_a^{\underline{b}}(x)$, $\zeta_i^a(x)$, $\zeta_{\underline{b}}^a(x)$ должны быть связаны соотношениями $\zeta_i^a\xi_a^j = \delta_i^j$, $\zeta_i^a\xi_a^{\underline{b}} = 0$, $\zeta_{\underline{b}}^a\xi_a^j = 0$, $\xi_a^{\underline{c}}\zeta_{\underline{b}}^a = \delta_{\underline{b}}^{\underline{c}}$. Именно это и будет первым шагом при построении сжатого описания [13] для современной стадии эволюции материи наблюдаемой области Вселенной. Итак, учитывая неразличимость физических состояний слабо взаимодействующих частиц, мы будем использовать уменьшенный набор полей $\{\Pi_c^i, A_c^{\underline{b}}\}$ вместо полного набора $\{\mathcal{B}_c^a(x)\}$. Естественно, необходимо учитывать, что в лагранжиане появятся постоянные, исполняющие роль весовых множителей, такие как $1/G_N$, где G_N — гравитационная постоянная Ньютона.

4. Поляризационные поля и пропагатор векторного бозона

Пусть $n = 4$, $v = 2$, $tu = s^2$, $L_{c(k)}^a = L_{cb}^a\varepsilon_{(k)}^b = L_{c(k)}^{(i)}\varepsilon_{(i)}^a$, $L_{i(j)}^{(k)} = \zeta_i^a L_{a(j)}^{(k)}$, $L_{\underline{b}(j)}^{(k)} = \zeta_{\underline{b}}^a L_{a(j)}^{(k)}$, так что полный лагранжиан (3.6) перепишется следующим образом:

$$\mathcal{L}_t = \mathcal{L}(\Psi, D\Psi) + \eta^{(j)(m)}\ [\kappa_o\ E_{(i)(j)}^{\underline{a}}\ E_{(k)(m)}^{\underline{b}}\ \eta^{(i)(k)}\ \eta_{\underline{ab}} +$$

$$+\kappa_1\ (F_{(i)(j)}^{(k)}\ F_{(l)(m)}^{(n)}\ \eta^{(i)(l)}\ \eta_{(k)(n)} + 2\ F_{(i)(j)}^{(k)}\ F_{(k)(m)}^{(i)} - 4\ F_{(i)(j)}^{(i)}\ F_{(k)(m)}^{(k)})]/4, \tag{4.1}$$

где

$$\kappa_o = \kappa_o'\ t^2, \qquad \kappa_1 = \kappa_o'\ t\ s^2. \tag{4.2}$$

$$F_{(i)(j)}^{(k)} = F_{ab}^c\ \varepsilon_{(i)}^a\ \varepsilon_{(j)}^b\ h_c^l\ h_l^{(k)} = \Phi_l^{(k)}\ F_{mn}^l\ \Phi_{(i)}^m\ \Phi_{(j)}^n =$$

$$= (\Phi(i)^l \nabla_l \Phi^m_{(j)} - \Phi^l_{(j)} \nabla_l \Phi^m_{(i)})\ \Phi^{(k)}_m + \Phi^l_{(i)} L l^{(k)}_{(j)} - \Phi^l_{(j)} L l^{(k)}_{(i)} + A^{\underline{a}}_{(i)}\ L_{\underline{a}(j)}^{(k)} - A^{\underline{a}}_{(j)}\ L_{\underline{a}(i)}^{(k)}, \quad (4.3)$$

$$E^{\underline{a}}_{ij} = E^{\underline{a}}_{(k)(l)}\ \Phi^{(k)}_i\ \Phi^{(l)}_j = F^d_{bc}\ \xi^{\underline{a}}_d\ \varepsilon^b_{(k)}\ \varepsilon^c_{(l)}\ \Phi^{(k)}_i\ \Phi^{(l)}_j =$$

$$= \nabla_i A^{\underline{a}}_j - \nabla_j A^{\underline{a}}_i + A^{\underline{b}}_i\ A^{\underline{c}}_j\ C^{\underline{a}}_{\underline{bc}} + C^{\underline{a}}_{i\underline{b}}\ A^{\underline{b}}_j - C^{\underline{a}}_{j\underline{b}}\ A^{\underline{b}}_i + C^{\underline{a}}_{ij}, \quad (4.4)$$

$$\Phi^i_{(k)} = \Pi^i_a\ \varepsilon^a_{(k)}, \quad A^{\underline{b}}_i = A^{\underline{b}}_{(j)}\ \Phi^{(j)}_i, \quad (4.5)$$

$$C^{\underline{c}}_{\underline{ab}} = \zeta^e_{\underline{a}} \zeta^d_{\underline{b}} C^g_{ed}\ \xi^{\underline{c}}_e,\ C^{\underline{c}}_{i\underline{a}} = (\zeta^b_i \zeta^d_{\underline{a}} C^e_{bd} + \partial_i \zeta^e_{\underline{a}}) \xi^{\underline{c}}_e,\ C^{\underline{c}}_{ij} = (\zeta^a_i \zeta^b_j C^d_{ab} + \nabla_i \zeta^d_j - \nabla_j \zeta^d_i) \xi^{\underline{c}}_d. \quad (4.6)$$

В результате уравнения полей $\Phi^i_{(j)}(x)$ можно получить стандартным образом [14] в виде гравитационных уравнений Эйнштейна

$$D_i \Psi\ \frac{\partial \mathcal{L}}{\partial D_j \Psi}\ g_{jk} - g_{ik}\ \mathcal{L}(\Psi, D_m \Psi) + \kappa_o\ \eta_{\underline{ab}}\ g^{jl}\ (E^{\underline{a}}_{ij}\ E^{\underline{b}}_{kl} - \frac{1}{4}\ g_{ik}\ g^{mn}\ E^{\underline{a}}_{jm}\ E^{\underline{b}}_{ln}) =$$

$$= \kappa_1\ (2\ R_{jik}{}^j - g_{ik}\ g^{lm}\ R_{jlm}{}^j) \quad (4.7)$$

($R_{ijk}{}^l$ — тензор кривизны связности Γ^k_{ij} риманова пространства-времени M^o_n; $\kappa_o = 1/(4\pi)$, $\kappa_1 = 1/(16\pi G_N)$). Естественно, что уравнения Эйнштейна отражают современное физическое состояние материи Вселенной. Все это подтверждает возможность интерпретации полей $\Phi^i_{(j)}(x)$ или полей $\Phi^{(j)}_i(x)$ как гравитационных потенциалов, но учитывая зависимость их от свойств среды (вакуума), а также исторически сложившееся мнение считать компоненты $g_{ij}(x)$ метрического тензора пространства–времени потенциалами гравитационного поля, имеет смысл называть $\Phi^i_{(j)}(x)$ и $\Phi^{(j)}_i(x)$ поляризационными полями. Именно данные поля, описывающие медленную подсистему, можно спрятать, вводя риманову структуру пространства–времени, тем самым получая возможность применять методы дифференциальной геометрии при сжатом описании физических систем.

Рассмотрим приближение, в котором пространство–время можно считать пространством Минковского, поля $\Phi^{(k)}_i$, $\Phi^i_{(k)}$ являются постоянными и пусть $\underline{r} = 1$, что предполагает $C^{\underline{c}}_{\underline{ab}} = 0$. Для получения уравнений поля $A^{\underline{b}}_i(x)$ в фейнмановской теории возмущений калибровка должна быть фиксирована, для чего добавим к лагранжиану (4.1) следующее слагаемое:

$$\mathcal{L}_q = \kappa_o\ q_{\underline{bb}}\ g^{ij}\ g^{kl}\ (\partial_i A^{\underline{b}}_j - q_o\ C_i\ A^{\underline{b}}_j)\ (\partial_k A^{\underline{b}}_l - q_o\ C_k\ A^{\underline{b}}_l)/2, \quad (4.8)$$

где $q_o = \eta_{\underline{bb}}/q_{\underline{bb}}$, $C_i = C^{\underline{b}}_{i\underline{b}}$. Кроме того, пусть

$$T_{a}{}^{(i)}_{(k)}\ \eta^{(j)(k)} + T_{a}{}^{(j)}_{(k)}\ \eta^{(i)(k)} = \varepsilon^{\underline{b}}_a\ t_{\underline{b}}\ \eta^{(i)(j)}. \quad (4.9)$$

В результате уравнения векторного поля $A^{\underline{b}}_i(x)$ запишутся в виде:

$$g^{jk}[\partial_j \partial_k A^{\underline{a}}_i - (1 - 1/q_o) \partial_i \partial_j A^{\underline{a}}_k + (1 - q_o) C_i C_j A^{\underline{a}}_k] + m^2 A^{\underline{a}}_i = I^{\underline{a}}_i/\kappa_o, \quad (4.10)$$

где $I^{\underline{a}}_i = (g_{ij}/\eta_{\underline{aa}})\ (\partial \mathcal{L}(\Psi))/(\partial A^{\underline{a}}_j)$ и

$$m^2 = (n-1)(n-2) \kappa_1 t^2_{\underline{a}}/(2 \kappa_o \eta_{\underline{aa}}) - g^{jk} C_j C_k. \quad (4.11)$$

Отметим, что вследствие поляризации вакуума ($C_i \neq 0$) пропагатор векторного бозона имеет довольно громоздкий вид [15]

$$D_{ij}(p) = [(1-q_o)\frac{(p_ip_j - C_iC_j)(p^kp_k - q_om^2) + (1-q_o)p^kC_k(p_iC_j + C_ip_j)}{(p^lp_l - q_om^2)^2 + (1-q_o)^2(p^lC_l)^2} -$$

$$-g_{ij}]/(p^mp_m - m^2), \tag{4.12}$$

который упрощается и принимает знакомую форму ($-g_{ij}/(p^kp_k - m^2)$, p^k — 4-импульс, а m — масса векторного бозона) лишь в фейнмановской калибровке ($q_o = 1$).

Итак, переход к горячему состоянию материи Вселенной был связан с разрушением бозе–конденсата (при этом некоторое время температура фоновых частиц Вселенной могла оставаться равной или близкой к нулю — так называемый период инфляции) и увеличением, соответственно, давления ферми–газа. В результате массы покоя W^+, W^-, Z^o бозонов уменьшились так, что слабое взаимодействие перестало быть слабым и все (или почти все) частицы из основного (вакуумного) состояния стали участвовать в установлении термодинамического равновесия. Данное явление и стало причиной кажущегося увеличения плотности частиц. Предполагая, что средняя плотность n_o частиц во Вселенной при этом не менялась, а сценарий горячей модели ее эволюции в общем верен, мы приходим к следующей ее оценке: $n_o \sim m_\pi^3 \sim 10^{-3}$ ГэВ ^{3} (m_π — масса π -мезона). Этот результат позволяет дать объяснение известному соотношению [16] $H_o/G_N \approx m_\pi^3$, если считать, что постоянная Хаббла H_o дает оценку $1/H_o$ длины $l \sim 1/(n_o\sigma_\nu)$ свободного пробега частицы в "вакууме" на современной стадии эволюции Вселенной (σ_ν — сечение рассеяния нейтрино на заряженной частице), и учесть оценку, данную ранее [17] гравитационной постоянной G_N ($G_N \sim \sigma_\nu \propto G_F^2T_\nu^2$, G_F — постоянная Ферми, T_ν — температура фоновых нейтрино Вселенной).

На большую плотность частиц во Вселенной, взаимодействующих лишь слабым образом, указывает и значительная величина масс покоя m_W и m_Z , соответственно, $W^\pm$ и Z^o бозонов, генерирующих слабое взаимодействие. Здесь мы имеем аналог сверхпроводника первого рода с большой длиной когерентности (ее роль может играть $1/H_o$) и малой лондоновской глубиной проникновения слабого поля (ее роль может играть $1/m_Z$). Применяя аналог известной формулы для лондоновской глубины проникновения магнитного поля ($\lambda_L^2 = m_qc^2/(4\pi n_q q^2)$, где λ_L — лондоновская глубина проникновения, m_q — масса куперовской пары, n_q — плотность куперовских пар, q — заряд куперовской пары), можно сделать грубую оценку массы покоя частиц основного состояния, экранирующих слабое поле: $m \sim 10^{-8}$ ГэВ, что по порядку близко к предполагаемой массе покоя электронного нейтрино.

5. Заключение

Таким образом, материя в бозе–конденсатном состоянии должна намного превосходить все остальные формы материи и мы должны переосмыслить не только природу гравитационных явлений и, в частности, гравитационных волн, но и обсудить возможность детектирования звуковых и температурных волн в космических масштабах. Вследствие этого, возможно, следует переосмыслить и стандартную интерпретацию результатов наблюдений, выполненных космическим аппаратом WMAP. С нашей точки зрения, поводов для этого предостаточно и, в частности, по наблюдениям когерентных явлений в причинно несвязанных областях Вселенной. Отметим, что неспособность стандартной космологической модели объяснить поведение космических объектов заставляет вводить в теорию такие экзотические объекты, как темная материя или темная энергия, а также предполагать сверхсветовые скорости и тому подобное. Кроме того, индуцированный характер гравитационных взаимодействий и их зависимость от слабых делают актуальным мониторинг процессов, генерированных слабыми взаимодействиями, и предполагают поиск их корреляций с гравитационными явлениями, наличие которых делает зависимость гравитационной "постоянной" G_N от времени не вызывающей сомнений.

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The differential geometry and the condensed description of Universe

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We shall assume that the Universe evolution is defined by particles the most of which is in the bound state at present and which's manifest oneself by weak interactions only. For the description of boson states we shall use smooth fields $\mathcal{B}_a^c$. Probably the rank of the density matrix ρ of fields $\mathcal{B}_a^c$ equals n , but it is impossible to eliminate that the generally given equality is satisfied only approximately when some components of a density matrix can be neglected. Considering the indistinguishability of the most of boson states we shall use the reduced set of fields $\{\Phi_i^{(j)}, A_i^b\}$ instead of the full set $\{\mathcal{B}_b^a\}$. Naturally, it is necessary to take into account that the constants performing the role of weighting coefficients such as $1/G_N$ (G_N is the gravitational constant) appear in the Lagrangian. As a result of an equations of fields $\Phi_i^{(j)}$ can be written down as the Einstein equations. It allows to define a space–time M_n^o as the Riemannian manifold, the basic tensor g_{ij} of which we shall introduce by a reduced density matrix ρ' .

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A GROUP THEORETICAL APPROACH TO THE PROBLEM OF SPACE-TIME DIMENSIONALITY. POST-MAXWELLIAN AND POST-EINSTEINIAN EFFECTS IN THE L_5 GROUP

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Physical consequences of "splitting"of the single time co-ordinate of Minkovsky world into two independent time-like co-ordinates are investigated. Within this approach, first, the generalized Maxwell equations, unifying electric, magnetic, gravitational and scalar fields, are derived. Second, their post–Maxwellian solutions are found, which correspond to the longitudinal electrograviscalar waves, propagating in vacuum at the speed of light. Third, an investigation of the newly discovered "dark matter"effects in cosmology is suggested on this basis.

1. Introduction

We consider a non-compact extension of Minkovsky space-time by way of transition from the single time co-ordinate to the τ – plane of two Cartesian time-like co-ordinates, t_4 and t_5. It is essential that the physical time, τ, in the theory suggested is not a co-ordinate of the "flat"five dimensional space-time, but the parameter connected with the two time-like co-ordinates invariantly (in a linear way (1.3)). By this we partly return to the principles of the pre-relativistic physics where the physical time acts as a parameter not a co-ordinate. However, as it is in the classical special relativistic theory (SRT), the general part is played by the Cartesian co-ordinates, which are five in number here, and the metric signature is the following $(+++--)$.

There are examples of conformally invariant unification theories of Kaluza-Klein's type where two and even more time co-ordinates were introduced; among those there are works by Pavsic [1] and Ingraham [2]. Also A. D. Sakharov [3] introduced compact time co-ordinates and explored cosmological inferences of such introduction. The main result of those attempts is the renormalization of various masses at the Planck scale. But, as Yu. S. Vladimirov says [4], to use his own words "one would rightly expect it to be not a formal calculation trick, but a step discovering new aspects of the reality. This step should be seriously grounded."We share this point of view and, moreover, show how the concept of two time axes may be used for making more valuable phisical predictions. There has recently appeared a series of works by Bars [5], where he does seriously ground the additional time dimension. He confronts " $1-T$ "and " $2-T$ physics"using the so-called M – theory, the non-commutative quantum field theory, and the supergravitation concept. We are rather against " $2-T$ physics". We make it a point here that any theory in its final expression must allow to reduce itself to the $1-T$ physics with the single time variable as in the classical SRT.

The τ – plane used in the present paper is shown in Fig. 1. The physical time parameter is actually an algebraic length of the time-like vector $\mathbf{T}$. This is shown in Fig. 1 in two positions: $\mathbf{T}_1$ and $\mathbf{T}_2$. The first takes place if $t_{4,5} > 0$, the second if $t_{4,5} < 0$. In the first

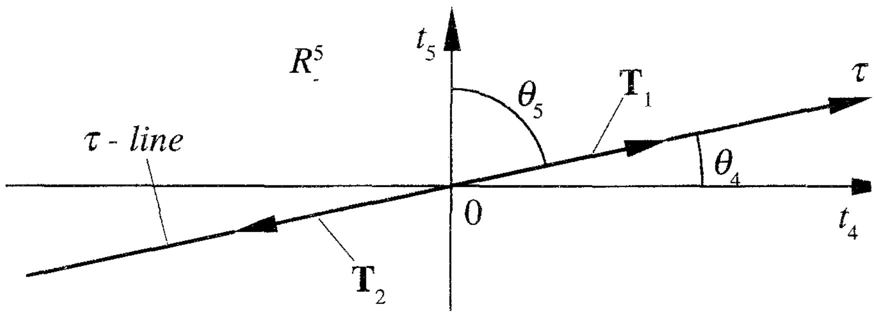

Fig. 1. τ – plane of the R^5_- space

case the τ_1 time is the $\mathbf{T}_1$ vector length, in the second it is the $\mathbf{T}_2$ vector length with minus. Mathematically it is expressed by usual scalar proportions:

$$t_4 = \tau \cdot \cos\theta_4, \tag{1.1}$$

$$t_5 = \tau \cdot \sin\theta_4, \tag{1.2}$$

$$\tau = t_4 \cos\theta_4 + t_5 \sin\theta_4, \tag{1.3}$$

which will be used everywhere below, particularly when transitioning from the Lorentz-Poincarè co-ordinate transformations of the L_4 group to the generalized co-ordinate transformations of the 5-dimensional L_5 group. Conventionally it could be imagined that Minkovsky space were embedded into the R^5 space. But in fact, and that is crucial, this is not so. The fictitious co-ordinate τ has no unit vector. The basis of tensor algebra in R^5 are the five unit vectors of the corresponding five co-ordinates. In any other inertial 5-dimensional reference frame, that is, moving relative to the K frame at velocity, $\mathbf{V}$, shown as K' in Fig.2, the τ'-plane, will look absolutely the same as in Fig.1, for we postulate that the θ_4 parameter is invariant. This postulate is one of the main in the theory. Inversion of the θ_4 parameter sign creates an additional R^5_- space (τ-plane in Fig. 1 belongs to it) which will act in the theory as well as R^5 . The following symbolic scheme of extension of Minkovsky space-time satisfies all the above said:

$$(3+1) \Rightarrow (3 + 1 \cdot \cos\theta_4 + 1 \cdot \cos\theta_5)\,. \tag{1.4}$$

where $\cos\theta_5 = \sin\theta_4$. The realization of scheme (1.4) is a serious step: in the case of two time co-ordinates in the frame, the electric and gravitational charge can mix in the relativistical sense. As a result, the law of conservation of charge may not hold. Besides we distinguish between the notions of the "gravitation mass"and the "inertial mass", and the weak principle of equivalence (WPE) is reconsidered for the case of the moving reference frames. As is well known, the WPE has been experimentally tested under laboratory conditions only.

But the theoretical inferences of scheme (1.4) realization must not contradict the experimental data on the conservation of an elementary charge, e, in the moving reference

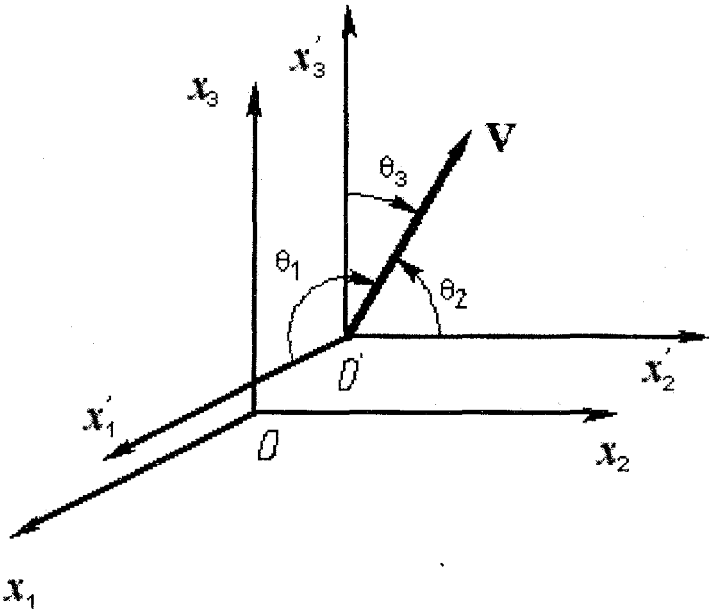

Fig. 2. Relative position of the K and K' frames at the moment of time fixed by the watch in the rest K frame. Along (in the general case) curved trajectory following the origin of the K' frame, the axes of the two frames remain parallel (only the angles θ_i and the length of $\mathbf{V}$ vector change their values).

frames. If we suppose for the sake of clearness that the parameter θ_4 equals the ratio of the two constants of the gravitational and electromagnetic interactions:

$$\theta_4 = \sqrt{\kappa} m_{p,e}/|e| \sim 10^{-17}, 10^{-20} \tag{1.5}$$

(where $m_{p,e}$ is the proton (electron) gravitational mass, κ the Newton gravitational constant), the value of novel expected effects is suppressed by the extremely small value of the ratio. Thus, the effects remains and will remain unobserved in near future, since the energy necessary for this observation is unachievable at present.

The paper is devoted to the theoretical consequences of such generalization of Minkovsky space-time. In particular, the latter give new interpretation of the substance, which is conventionally called "dark matter"in astrophysics and cosmology. The point is that the realization of scheme (1.4) makes one to put forward the hypothesis, that the stable elementary particles of usual matter and anti-matter are manifestation of the interior motions of some substance deprived of electric charge but not of gravitational charge. We may (though not necessarily) identify this hypothetical substance as the known "dark matter"and consider transformation of "dark matter elements"(DME) into "usual matter elements"(UME). The group theoretical analysis of the DME physics shows that the WPE does not hold for those. This matter possesses very low inertia, that is why its transformation into UME (understood as the interior motion of DME) can take place at low energies.

One more important remark is the following. The group theoretical approach distinguishes between the inertial mass and the gravitational mass. The inertial mass, unlike gravitational, is not a charge, that is why it has nothing to mix with, whatever dimensionality of the world we would introduce. Thus, all the kinematic theorems of the Lorentz-Poincarè group should retain their standard 4-dimensional expression in the new R^5 space-time.

2. Co-ordinate transformations equations for the new L_5 group and its 4-dimensional kinematics

First, let us consider the general 4-dimensional co-ordinate transformation from the primed frame into the unprimed one (Fig. 2) in Poincarè's form [6]:

$$x_i = x_i' + \cos\theta_i \cdot \mu_s' + \sigma_i' \qquad (i = 1, 2, 3), \tag{2.1}$$

$$x_0 = x_0' + \mu_\tau' + \sigma_0' \qquad (x_0 = c\tau), \tag{2.2}$$

where $\sigma_{\bullet . i}'$ are constants fixed by initial conditions, $\cos\theta_i$ the guiding cosines (θ_i is shown in Fig. 2) of the velocity vector, $\mathbf{V}$, and

$$\mu_s' = (\gamma - 1)\, r_s' + \beta\gamma x_0', \tag{2.3}$$

$$\mu_\tau' = (\gamma - 1)\, x_0' + \beta\gamma r_s', \tag{2.4}$$

$$r_s' = \sum_1^3 x_i' \cos\theta_i, \quad \left(\sum_1^3 \cos^2\theta_i = 1\right) \tag{2.5}$$

$$\gamma = \left(1 - \beta^2\right)^{-1/2}, \quad \beta = |\mathbf{V}|/c. \tag{2.6}$$

The first three equations for the new L_5 group are deduced from Eqns. (2.1), (2.3) by way of simple substitution the right–hand side of primed Eqn.(1.3) instead of $\tau'(x_0')$. To get two new equations from single Eqn. (2.2), we multiply Eqn. (2.2) first by $\cos\theta_4$, then by $\cos\theta_5 (= \sin\theta_4)$. Using then primed and unprimed Eqns. (1.1) and (1.2), we get two time co-ordinate transformation equations for $i = 4, 5$. Thus we get five new equations for the L_5 group co-ordinate transformations:

$$x_i = x_i' + \cos\theta_i \cdot \left\{ \begin{array}{c} \mu_s' \\ \mu_\tau' \end{array} \right\} + \sigma_i'. \tag{2.7}$$

It is supposed that Eqns. (2.7) includes μ_s' , when $i = 1, 2, 3$, and μ_τ' , when $i = 4, 5$. The fundamental part in the following argument belongs henceforth to Eqns. (2.7), which is in fact but a widened form of the Lorentz-Poincarè classical transformations.

The θ_4 parameter's invariance (due to which the τ-line does not rotate) conserves all the group theory kinematics theorems for the L_4 group (time and space dilatation, composition of velocities theorem) in the L_5 group. The general and consistent proof of the fact that the 4-dimensional kinematics remains valid in L_5 is the following. Let us consider the motion of a substance of arbitrary propagated inertial mass density or, in other words, partially differentiable inertial mass density in the R^5 space. The relativistically invariant in the R^4 space tensor method to get the left-hand side of the vortical motion equation for such substance was offered by the author in [7]. To solve this problem in R^5 space we should invariantly define the three 5-vectors: the velocity 5-vector, the momentum density 5-vector and the 5-dimensional Hamilton operator. It is evident that the invariantly defined velocity 5-vector is

$$\mathbf{v} = \gamma \left\{\mathbf{V}, -c\cos\theta_4, -c\sin\theta_4\right\}. \tag{2.8}$$

The momentum density is easily deduced from this by multiplying (13) by the inertial mass density scalar function μ . The co-ordinate expression for the operator ∇ in the covariant form is obvious:

$$\nabla = \{grad, \partial/c\partial t_4, \partial/c\partial t_5\}.$$

Using henceforth this co-ordinate expression of ∇, we get the left-hand side of the motion equation wherein there are partial derivatives by the time-like t_4 and t_5. Neither has any physical sense, unlike the physical time τ, which is connected with the $t_{4,5}$ by invariant Eqn. (1.3). Using Eqn. (1.3) which leads to

$$\partial \Big/ \partial t_{4,5} = \frac{\partial}{\partial \tau} \cdot \partial \tau \Big/ \partial t_{4,5} = \cos\theta_i \frac{\partial}{\partial \tau}, \tag{2.9}$$

where $i = 4, 5$, we can represent the 5-vector ∇ in the *parametric* form:

$$\nabla = \{grad, \cos\theta_4 \cdot \partial/c\partial\tau, \sin\theta_4 \cdot \partial/c\partial\tau\}, \tag{2.10}$$

which includes the τ physical time as a parameter and does not the physically senseless variables.

It is worth noting that this trick will be used on in the argument and in the field theory as well. It enables one to represent the 5-dimensional equations in the 4-dimensional form, not co-ordinate but parametric, as it took place in the pre-relativistic physics. This trick helps to overcome that above spoken of fundamental difficulty, which all theories with two time co-ordinates experienced.

Making operations with the three 5-vectors in the same way as was done in the R^4 space [7], as a result we get the left-hand side of the substance motion equation in the R^5 space:

$$\gamma \left\{ \frac{d\mathbf{P}}{d\tau} \left(\cos^2\theta_4 + \sin^2\theta_4\right), -\mathbf{Ve}\cos\theta_4, -\mathbf{Ve}\sin\theta_4 \right\} = \gamma \left\{ \frac{d\mathbf{P}}{d\tau}, -\mathbf{Ve}\cos\theta_4, -\mathbf{Ve}\sin\theta_4 \right\}, \tag{2.11}$$

where

$$\frac{d\mathbf{P}}{d\tau} = c\mathbf{e} - [\mathbf{V} \times rot\mathbf{p}],$$

$$\mathbf{e} = c^{-1}\left(gradK + \partial\mathbf{p}/\partial\tau\right), \left(\mathbf{p} = \mu\gamma\mathbf{V}, \quad K = \mu\gamma c^2\right).$$

There is no θ_4 parameter in the space-like part of 5-vector (2.11) (in the right-hand member of Eqn. (2.11)), so this part is identical to the respective part of the 4-vector in R^4 (see ref. [7]). In other words, the θ_4 parameter (an invariant additional to the speed of light in the L_5 group) does not influence the classical Lorentz-Poincarè group kinematics, Q. E. D.

3. Post–Maxwellian effects of the L_5 group: gravi–electrical and electrogravitational inductions

Henceforth we suppose that electric and gravitational *charges* (along with corresponding convection currents), or "electromagnetic units", are meant by the field sources.

The use of the Lorentz form for co-ordinate transformations (2.7) will be sufficient to describe local effects of the infinitesimal transformations:

$$x_i = x_i' + \cos\theta_i \cdot \left\{ \begin{array}{c} \mu_s' \\ \mu_\tau' \end{array} \right\}. \tag{3.1}$$

Now we suppose that matter in the primed frame is characterized by either type of charge density, the frame being co–moving with the substance element. In other words, the contravariant hypercurrent density 5-vector possesses but two nonzero components, that is, the two time-like components:

$$\mathbf{J}' = \left\{0, 0, 0, -4\pi\rho', -4\pi\sqrt{\kappa}\mu_g'\right\}. \tag{3.2}$$

As the result of convection, in the K rest frame there appears the hypercurrent density 3-vector $\mathbf{j}$. The unprimed hypercurrent density 5-vector will have all the five components:

$$\mathbf{J} = \left\{4\pi/c \cdot \mathbf{j}, -4\pi\rho, -4\pi\sqrt{\kappa}\mu_g\right\}, \tag{3.3}$$

where in accordance with Eqn. (3.1)

$$\mathbf{j} = \left(\sqrt{\kappa}\mu_g' \sin\theta_4 + \rho' \cos\theta_4\right)\gamma V = \sigma' \cos\left(\theta_4 - \bar{\theta}_4\right)\gamma V, \tag{3.4}$$

$$\begin{aligned} \rho = \sqrt{\kappa}\mu_g' (\gamma - 1)\sin\theta_4\cos\theta_4 + \rho'\left[1 + (\gamma - 1)\cos^2\theta_4\right] = \\ = \sigma'\left[\cos\bar{\theta}_4 + (\gamma - 1)\cos\theta_4\cos\left(\theta_4 - \bar{\theta}_4\right)\right], \end{aligned} \tag{3.5}$$

$$\begin{aligned} \sqrt{\kappa}\mu_g = \rho'(\gamma - 1)\sin\theta_4\cos\theta_4 + \sqrt{\kappa}\mu_g'\left[1 + (\gamma - 1)\sin^2\theta_4\right] = \\ = \sigma'\left[\sin\bar{\theta}_4 + (\gamma - 1)\sin\theta_4\cos\left(\theta_4 - \bar{\theta}_4\right)\right]. \end{aligned} \tag{3.6}$$

Here

$$\sigma' = \sqrt{(\kappa\mu_g')^2 + (\rho')^2}, \qquad tg\bar{\theta}_4 = \sqrt{\kappa}m_g'/q',$$

m_g', q' are the gravitational mass and the electrical charge in the K' frame. It is clear from these formulae that in the general case neither the gravitational mass nor the electrical charge are conserved. But if we grant $\bar{\theta}_4 = \theta_4$, then from Eqns (3.5), (3.6) it follows that

$$\rho = \gamma\rho', \qquad \mu_g = \gamma\mu_g'.$$

This proves that in this particular case the electrical charge and the *gravitational mass* are conserved, for the post-Maxwellian effects of gravi–electrical and elctrogravitational induction are mutually compensated. This theorem seems to testify that the θ_4 parameter could be measured experimentally in case it were possible to define what charge of what elementary particle is conserved at however high energies.

We shall start however with the following approximation. Let us attribute such a volume to the θ_4 parameter in the R^5 space that will provide for the electron charge and mass conservation:

$$tg\theta_4 \cong \theta_4 = \sqrt{\kappa}m_e/e \sim -10^{-20}. \tag{3.7}$$

The effect of gravi–electrical induction shows itself in Eqn. (3.2) when $\rho' = 0$. Then the following formulae take place:

$$\mathbf{j} = \sqrt{\kappa}\mu_g' \sin\theta_4 \gamma \mathbf{V},) \tag{3.8}$$

$$\rho = \sqrt{\kappa}\mu_g' (\gamma - 1)\sin\theta_4 \cos\theta_4, \tag{3.9}$$

$$\mu_g = \mu_g' \left[1 + (\gamma - 1)\sin^2\theta_4\right]. \tag{3.10}$$

Among the consequences of the L_5 group the effect of the gravi–electrical induction deserves the closest concern. Theoretically *electricity may be a relativistical manifestation of gravitation.* For the "dark matter", where there are neither negative nor positive electricity, Eqns. (3.8)–(3.10) are exact. As a first approximation we will consider that these formulae are applied for neutral particles of usual matter.

However, here we are interested mainly in low energy effects expressed by Eqns. (3.9) and (3.10). First we consider them for the usual matter, when the Lorentz factor γ due to the comparatively large initial inertia does not reach high volumes. Chargeless UMEs have large inertia, that is why the Lorentz factor cannot be of a high value at low energies. Thus, integration by volume in Eqns. (3.9) and (3.10) or (3.6) for $\beta \ll 1$ results in

$$q \approx \sqrt{\kappa} m_g' \sin\theta_4 \cdot \beta^2/2, \tag{3.11}$$

$$m_g \approx m_g' \left(1 - \beta^2/2\right). \tag{3.12}$$

As follows from Eqn. (3.12) gravi–electrical induction is substantially suppressed by the $\sin\theta_4$ coefficient. This reduces to zero all the chances to observe the effect in the experiments with elementary particles.

Eqn. (3.12) proves, that the group theoretical approach discloses it that the WPE does not hold for the moving neutral usual matter objects, for

$$m_i/m_g \approx \left(1 + \beta^2/2\right)/\left(1 - \beta^2/2\right) \approx 1 + \beta^2.$$

In experiments with elementary particles this cannot be observed as gravitation is greatly suppressed by electrodynamics.

Let us suppose now that at the basis of *low energy* transformation of DME into UME there lies the relativistic effect of gravi–electrical induction. Let us put the question, what qualities a DME should possess in order to become, say, an electron, that is, an element of the elementary charge and its rest inertial mass equal to its gravitational mass, as a result of such low energy transformation. Let us integrate Eqns. (3.9) and (3.10) by volume and apply the relativistic formula on the inertial mass to the final expressions:

$$q \cong \sqrt{\kappa} m_g' \theta_4 \left(1 - \gamma^{-1}\right), \tag{3.13}$$

$$m_g \cong m_g' \left[\gamma^{-1} + \theta_4^2 \left(1 - \gamma^{-1}\right)\right], \tag{3.14}$$

$$m_i = \gamma m_i', \tag{3.15}$$

where m_g', m_i' are respectively gravitational and inertial masses of DME at rest. Since in Eqn. (3.14) $\theta_4^2 \cong 10^{-40}$, then, supposing that in Eqn. (3.14) $\gamma^{-1} \leq 10^{-41}$, and equating m_i to the electron gravitational mass (in accordance with WPE for usual matter at rest), it follows from (3.13) (or (3.14)) and (3.7) that

$$m_g' \approx 10^{13} g, \qquad m_i' \leq 10^{-68} g. \tag{3.16}$$

The result is striking. It testifies to a gigantic discrepancy between the inertial mass and gravitational mass values of the primary matter element. It is worth mentioning that whatever value of the θ_4 parameter we choose, whether to preserve the electron gravitational mass and charge or that of the proton or of the quark or anything else, the discrepancy will remain gigantic.

Deducing (3.16) we have used WPE for usual matter at rest. It means that the ultra-relativistic motion of the DME with parameters (3.16) must progress by laws of irregular finite motion and by the present days evaluations within a volume of the radius less than 10^{-16} cm.

The very evaluation (3.16) is related to the parameters of the substance at rest, an element of which possesses a gigantic gravitational charge and negligible inertial mass ("starter"rest energy). By this it is possible to conclude that within the substance there is gigantic primary tension (interior potential energy), which must dynamically manifest itself as a practically momentary collapse since initial inertial mass is negligible. The supposition, that this mighty factor exist, can make us change the habitual evaluations of the balance between gravitational and electrical forces, and of the role which the balance plays in the stability of charged UME (electron, proton, quark, ets).

To an observer it will seem that "dark matter"sub-elements different by mass reach one and the same reversal point (there will be but two in a flat trajectory, where $\gamma = 1$) at different moments of time. It is significant that the observer will register the sub-elements of the substance stay still "for a moment"at those points, and therefore get the qualities of the primary substance with negligible inertial mass, large gravitational charge and zero electrical charge. Without those reversal points there is induced electric charge in the substance sub-elements and thus the centrifugal acceleration appears that bars the further collapse. Without going into detail we may suppose that dynamic balance between centripetal and centrifugal forces may be reached in such a model, while in the old static model gravitation (or surface tension as in Poincarè's model) was insufficient for holding the charged matter within a finite volume.

Of course, the above expressed arguments are purely speculative and need grounding by calculations before coming to so strong assertions. Nevertheless, this hypothesis may be useful for the interpretation of the generalized Maxwell equations in the next section.

Formulae (3.15) and (3.16) testify to the purely mechanical genesis of the UME rest energy. It should also be mentioned here that the Lorentz factor γ may reach gigantic values ($\gamma \geq 10^{41}$ at *low energies* of ~ 0.5 MeV) due to the primary environment tension and to the negligible value of the DME inertial mass.

It is very important, that due to the irregular mass distribution along the trajectory (which must appear due to the self-action of the substance along the current lines) and due to the great range of the factor γ ($\gamma \in [1, 10^{41}]$) the integrate effect of the dm_g' and dm_i' transformation will be different at different moments of time for all the system taken as a whole. Therefore the full mass and charge of UME will be **time dependent**, periodicaliy or quasi-periodically. This consequence of the hypothesis cannot be observed experimentally nowadays and above all it should be considered when building up a field theory in the R^5 space. Experimentally only average mass and charge values of a UME can be observed since their radius is very small ($r \leq 10^{-16}$ cm) and the velocity of the DME periodical motion is virtually the speed of light. Therefore the T period can be evaluated as $T \cong 2\pi r/c \leq 10^{-26}$ s, that is, much smaller than the integrating constant of any macroscopic instrument. Pulse time dependences of the electron charge and its two masses are shown in Fig. 3. The plots, built with the help of the Gaussian function, show what one can expect the qualitative aspects of the process to be.

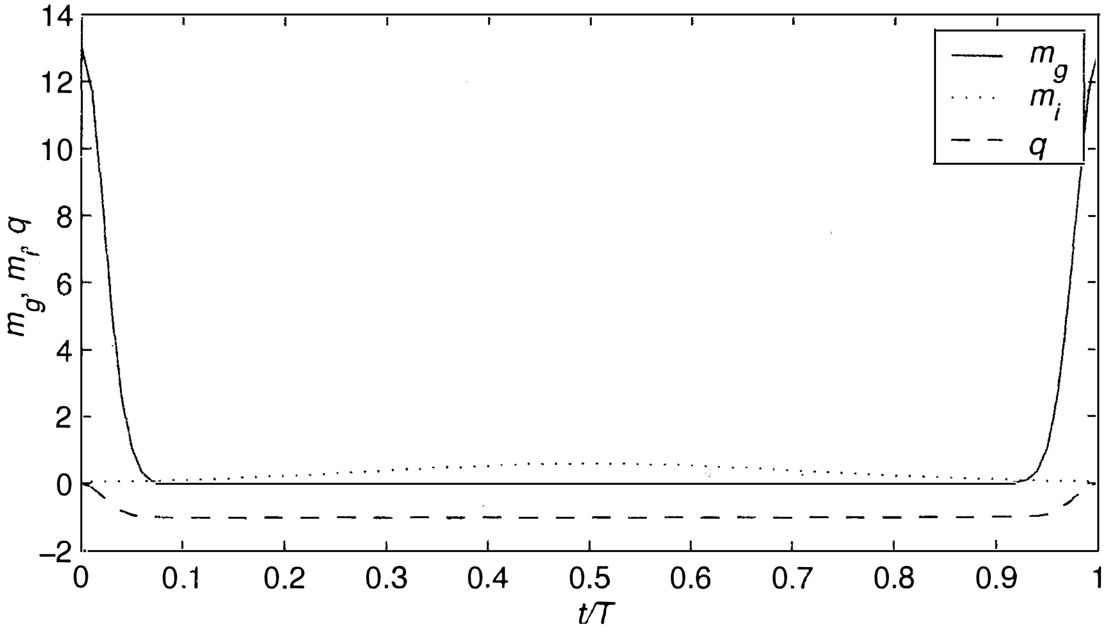

Fig. 3. Pulse time dependences of the electron charge and its two masses (qualitatively); logarithmic scale on the vertical axis for m_g, m_i and q are not the same.

The solution to the problem on anti–matter particles' origin is contained in the symmetry relative to the m_g' sign inversion in (3.9) and (3.10). That is, there is supposed to be "dark matter"of negative gravitational charge. It follows from (3.9) and (3.10) that the negatively charged gravitating particle and its respective anti–particle belong to the $R^5(\theta_4 < 0)$. Vice versa, the positively charged particle and its respective anti–gravitating anti–particle belong to the space $R^5_{-}(\theta_4 > 0)$. Therefore, all charged anti–particles must have negative gravitational charge. Thus the L_5 group's qualities include the operation of charge conjugation relative to the gravitational charge.

4. The generalized Maxwell equations

The Maxwell equations general form for any dimensionality space of arbitrary metrics is well known [8]:

$$RotF_{ik} = 0 \qquad (\nabla \times F = 0)\,, \tag{4.1}$$

$$DivF_{ik} = J_k \qquad (\nabla \cdot F = \mathbf{J})\,, \tag{4.2}$$

where J_k are the components of hypercurrent density N-vector $\mathbf{J}$, rotor and divergency are supposed to be definite in N dimensions, F_{ik} bivector components of hyperfield F .

In the R^5 flat space-time with metric signature $(+ + + + -)$ the problem of the Maxwell equations generalization in vacuum was solved by Corben [9]. But it was not stated in a physically clear and self-sufficient way what the premises for introduction of the five-dimensional space-time were. They were rather copied more or less formally from those that were in Kaluza-Klein non-Euclidian 5-dimenssional space-time. Let us see now what this new metric signature $(+++--)$ will grant us. Following in every detail the algebraic scheme for field equation deduction, suggested by Minkovsky, we suppose that the field is completely defined by the 5-potential

$$\alpha = \{\mathbf{A}, -A_4, A_5\}, \tag{4.3}$$

where $A_{4,5}$ are respectively electric and gravitational potentials, $\mathbf{A}$ is the 3-vector hyperfield potential. The F -field tensor is connected with the 5-potential (4.3) in the usual way:

$$F = \nabla \times \alpha, \tag{4.4}$$

where the parametrical form for the operator ∇ in the R^5 space is suggested above (in Eqn.(2.10)). Therefore, the F – tensor components are

$$\mathbf{E} = -gradA_4 - \cos\theta_4 \partial \mathbf{A}/c\partial\tau \qquad (F_{i4}) \tag{4.5}$$

$$\mathbf{G} = -gradA_5 + \sin\theta_4 \partial \mathbf{A}/c\partial\tau \qquad (F_{i5}) \tag{4.6}$$

$$\mathbf{H} = rot\mathbf{A} \qquad (F_{23}, F_{31}, F_{12}) \tag{4.7}$$

$$\Psi = \sin\theta_4 \partial A_4/c\partial\tau + \cos\theta_4 \partial A_5/c\partial\tau \quad (F_{45}), \tag{4.8}$$

where $\mathbf{E}, \mathbf{G}, \mathbf{H}, \Psi$ are the electric, gravitational, magnetic and scalar fields respectively (in Corben theory the additional field is pseudoscalar). Putting the right-hand side of (4.2) in (4.3) and using the potential Lorentz gauge

$$div\mathbf{A} + \partial\xi/c\partial\tau = 0 \qquad (Div\alpha = 0), \tag{4.9}$$

where $\xi = -A_5 \sin\theta_4 + A_4 \cos\theta_4$, we come to the standard wave equations:

$$\Box\mathbf{A} = -4\pi/c \cdot \mathbf{j} \tag{4.10}$$

$$\Box A_4 = -4\pi\rho, \tag{4.11}$$

$$\Box A_5 = 4\pi\sqrt{\kappa}\mu_g. \tag{4.12}$$

Let us mention it again (see Eqn. (2.9)) that the 5-dimensional form of equations is reduced to 4-dimensional, as it was in kinematics, due to

$$\partial^2/\partial t_4^2 + \partial^2/\partial t_5^2 \equiv \partial^2/\partial\tau^2.$$

Now we put components of tensor (4.4) into (4.1) and (4.2) and come to the generalized form of the Maxwell equations in vacuum:

$$
\begin{aligned}
&rot\mathbf{E} = -\frac{\cos\theta_4}{c}\frac{\partial\mathbf{H}}{\partial\tau},\\
&rot\mathbf{G} = \frac{\sin\theta_4}{c}\frac{\partial\mathbf{H}}{\partial\tau},\\
&div\mathbf{H} = 0,\\
&rot\mathbf{H} = \frac{\cos\theta_4}{c}\frac{\partial\mathbf{E}}{\partial\tau} - \frac{\sin\theta_4}{c}\frac{\partial\mathbf{G}}{\partial\tau} + \frac{4\pi}{c}j,\\
&grad\Psi = -\frac{\cos\theta_4}{c}\frac{\partial\mathbf{G}}{\partial\tau} - \frac{\sin\theta_4}{c}\frac{\partial\mathbf{E}}{\partial\tau},\\
&div\mathbf{E} = -\frac{\sin\theta_4}{c}\frac{\partial\Psi}{\partial\tau} + 4\pi\rho,\\
&div\mathbf{G} = -\frac{\cos\theta_4}{c}\frac{\partial\Psi}{\partial\tau} - 4\pi\sqrt{\kappa}\mu_g,
\end{aligned}
\tag{4.13}
$$

where $\mathbf{j}$ is the electrogravitational current density.

Relatively to the τ physical time, Eqns. (4.13) are written down in parametrical not co-ordinate form, as it is in the classical Maxwell equations of relativistic physics (in the same equations of pre-relativistic physics time was considered a parameter). However Eqns. (4.13) set is essentially 5-dimensional and invariant relatively to the new L_5 group. Since the gravimagnetism in Eqns. (4.13) is significantly suppressed by the $\sin\theta_4$ coefficient, in case of $\mu_g' = 0$ and $\partial\rho/\partial\tau = 0$ the system of Eqns. (4.13) is transformed under condition $\theta_4 \to 0$ into the Maxwell system. The first four equations of (4.13) are practically in Maxwellian form, since $\cos\theta_4 \cong 1, \sin\theta_4 \cong 0$.

Using accessory expressions

$$
\begin{aligned}
\mathbf{T}_E &= \mathbf{E}\cos\theta_4 - \mathbf{G}\sin\theta_4,\\
\mathbf{L}_G &= \mathbf{E}\sin\theta_4 + \mathbf{G}\cos\theta_4,
\end{aligned}
\tag{4.14}
$$

the system of Eqns. (4.13) is transformed into a more laconic, but a more particular form of two independent sub-sets:

$$
\begin{aligned}
&rot\mathbf{T}_E = -\partial\mathbf{H}/c\partial\tau,\\
&div\mathbf{H} = 0,\\
&rot\mathbf{H} = \partial\mathbf{T}_E/c\partial\tau + 4\pi/c\cdot\mathbf{j},
\end{aligned}
\tag{4.15}
$$

$$
\begin{aligned}
&grad\Psi = -\partial\mathbf{L}_G/c\partial\tau,\\
&div\mathbf{L}_G = -\partial\Psi/c\partial\tau + 4\pi\sigma_{eg},
\end{aligned}
\tag{4.16}
$$

where $\sigma_{eg} = \sqrt{\kappa}\mu_g\cos\theta_4 - \rho\sin\theta_4$. The major qualities of system (4.13) show themselves most clearly in those two artificially deduced systems. Subsystem (4.15) is the Maxwell system as it is, but wanting one equation which has transferred to (4.16). But the three equations are sufficient to describe usual transverse waves (electrogravimagnetic as it is here (see Eqns. (4.14)). Subset (4.16) on the contrary describes longitudinal field of gravitation and electricity (see Eqns. (4.14)) and their retarding propagation in vacuum. It is clearly seen that the system is post-Maxwellian in structure. The sources of those time-dependent waves are varying in time but rest gravitational and electric charges.

Applying the energy conservation law for arbitrary wave processes in vacuum out of sources together with the following equality (taken from vector analysis)

$$\mathbf{L}_G grad\Psi + \Psi div\mathbf{L}_G = div\left(\Psi \cdot \mathbf{L}_G\right),$$

to Eqns. (4.16) the Pointing vector and longitudinal wave energy density are easily found:

$$\mathbf{\Pi} = c/4\pi \cdot \left(\Psi \cdot \mathbf{L}_G\right), \qquad W = 1/8\pi \cdot \left[\mathbf{L}_G^2 + \Psi^2\right]. \tag{4.17}$$

It follows from Eqns. (4.17) that the $L_g\Psi$ field is a field of a longitudinal gravi–electroscalar wave propagating in vacuum at the speed of light carrying away some interior energy of the rest UME. Actually it is a post–Maxwellian and altogether post–Einsteinian effect of the L_5 group, which is not to be found in Corben's theory. If we just integrate now usual wave Eqns. (4.11) and (4.12) by volume (for instance, in case of electron) where in their right–hand sides the charge densities are time–dependant (pulse functions) we will get longitudinal electrogravitoscalar waves. If we calculate then the time average volumes of the matter interacting **E** and **G** fields, we will get what is observed experimentally: longitudinal electrostatic and gravistatic fields (of the electron). Now that we have studied all the qualities of the L_5 group we know there are retarding fast–oscillating processes (electrograviscalar waves) underlying the static electric and gravitational processes; that satisfies all the conditions of the classical SRT.

It is absolutely obvious that those waves carry away some of the interior energy of an elementary particle at rest. If we take the hypothesis that the rest UME is the primary substance interior motion manifestation, where there is a gigantic interior tension, evidently the latter, and it alone, can play the part of the compensating mechanism, which replenishes the energy loss (radiation friction).

In conclusion it should be said that this undoubtedly heuristic method by which set (4.16) was deduced needs experimental tests of its consequences (gravimagnetism, for instance). We believe, however, that a good argument for those equations is that they enable one to view electro– and gravistatics as macroscopic manifestation of longitudinal electrograviscalar dynamics, which satisfies all the conditions of the classical SRT.

5. Conclusions

From the methodological point of view it is denial to view the τ co-ordinate as the fourth co-ordinate of Minkovsky world. Introduction of two independent time-like co-ordinates firmly bound with the τ variable by the postulate of the θ_4 parameter invariability enabled us to get a vector equation system for the unified field of the only one τ variable (which is present in the equations as a parameter) and to stay within the scope of 1-T physics.

The study of time-space (not gauge ones!) symmetries in the L_5 group enables us to put forward the hypothesis that electricity is a special manifestation of some primary substance gravitation (maybe "dark matter") which differs from the usual matter, in particular, the WPE does not hold in the rest frames.

The analysis of structure of the unified field equations showes that there are solutions, which correspond to the longitudinal waves of gravitation and electricity propagating in vacuum at the speed of light.

6. Acknowledgments

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CONTINUATION OF THE DUAL AMPLITUDE WITH MANDELSTAM ANALYTICITY OFF MASS SHELL

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The off mass shell continuation of dual amplitude with Mandelstam analyticity (DAMA) is proposed. The modified DAMA (M-DAMA) preserves all the attractive properties of DAMA, such as its pole structure and Regge asymptotics, and leads to a generalized dual amplitude $A(s,t,Q^2)$. In such a way we complete a unified "two-dimensionally dual"picture of strong interaction [1, 2, 3, 4]. This generalized amplitude can be checked in the known kinematical limits, i.e. it should reduce to the ordinary dual amplitude on mass shell, and to the nuclear structure function when $t = 0$. We fix the Q^2 -dependence in M-DAMA by comparing the structure function F_2, resulting from it, with phenomenological parameterizations. The results of M-DAMA are in qualitative agreement with the experiment in all studied regions, i.e. in the large and low x limits as well as in the resonance region.

1. Introduction

This work is devoted to modeling of the scattering amplitude for inelastic electron-proton scattering. The kinematics of inclusive ep scattering, applicable to both high energies, typical of HERA, and low energies as at JLab, is shown in Fig. 1. We introduce virtuality

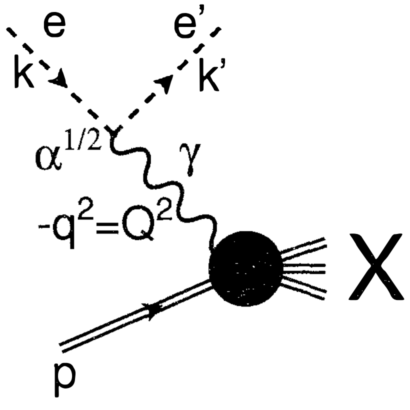

Fig. 1. Kinematics of inelastic electron-proton scattering.

Q^2, $Q^2 = -q^2 = -(k-k')^2 \geq 0$, and Bjorken variable $x = Q^2/2p \cdot q$. These variables x, Q^2 and Mandelstam variable s (of the $\gamma^* p$ system), $s = (p+q)^2$, obey the relation:

$$s = Q^2(1-x)/x + m^2 , \tag{1.1}$$

where m is the proton mass. And Fig. 2 shows how inelastic $\gamma^* p$ scattering is related to the forward elastic (t=0) $\gamma^* p$ scattering, and then the latter is decomposed into a sum of the s-channel resonance exchanges.

About thirty years ago Bloom and Gilman [5] observed that the prominent resonances in inelastic $e^- p$ scattering (see Fig. 1) do not disappear with increasing photon virtuality Q^2, but fall at roughly the same rate as background. Furthermore, the smooth scaling limit proved to be an accurate average over resonance bumps seen at lower Q^2 and s, this is so called Bloom-Gilman or hadron-parton duality. Since the discovery, the hadron-parton duality was studied in a number of papers [6] and the new supporting data has come from the recent experiments [7, 8]. These studies were aimed mainly to answer the questions: in which way a limited number of resonances can reproduce the smooth scaling behaviour? The main theoretical tools in these studies were finite energy sum rules and perturbative QCD calculations, whenever applicable. Our aim instead is the construction of an explicit dual model combining direct channel resonances, Regge behaviour typical for hadrons and scaling behaviour typical for the partonic picture. Some attempts in this direction have already been done in Refs. [1, 2, 3, 4], which we will discuss in more details below.

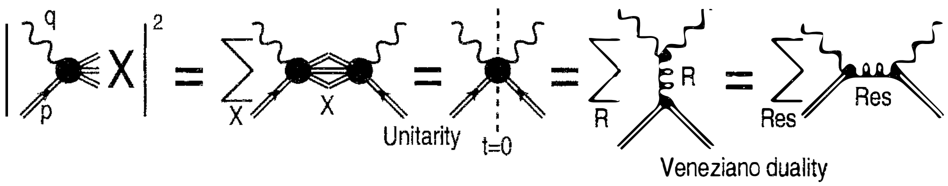

Fig. 2. (From [2].) According to the Veneziano (or resonance-reggeon) duality a proper sum of either t-channel or s-channel resonance exchanges accounts for the whole amplitude.

The possibility that a limited (small) number of resonances can build up the smooth Regge behaviour was demonstrated by means of finite energy sum rules [9]. Later it was confused by the presence of an infinite number of narrow resonances in the Veneziano model [10], which made its phenomenological application difficult, if not impossible. Similar to the case of the resonance-reggeon duality [9], the hadron-parton duality was established [5] by means of the finite energy sum rules, but it was not realized explicitly like the Veneziano model (or its further modifications).

First attempts to combine resonance (Regge) behaviour with Bjorken scaling were made [11, 12, 13] at low energies (large x), with the emphasis on the right choice of the Q^2-dependence, such as to satisfy the required behaviour of form factors, vector meson dominance (the validity (or failure) of the (generalized) vector meson dominance is still disputable) with the requirement of Bjorken scaling. Similar attempts in the high-energy (low x) region became popular recently stimulated by the HERA data. These are discussed in section 3.

Recently in a series of papers [1, 2, 3, 4] authors made attempts to build a generalized Q^2-dependent dual amplitude $A(s, t, Q^2)$. This amplitude, a function of three variables, should have correct known limits, i.e. it should reduce to the on shell hadronic scattering amplitude on mass shell, and to the nuclear structure function (SF) when $t = 0$. In such a way we could complete a unified "two-dimensionally dual"picture of strong interaction [1, 2, 3, 4] – see Fig. 3.

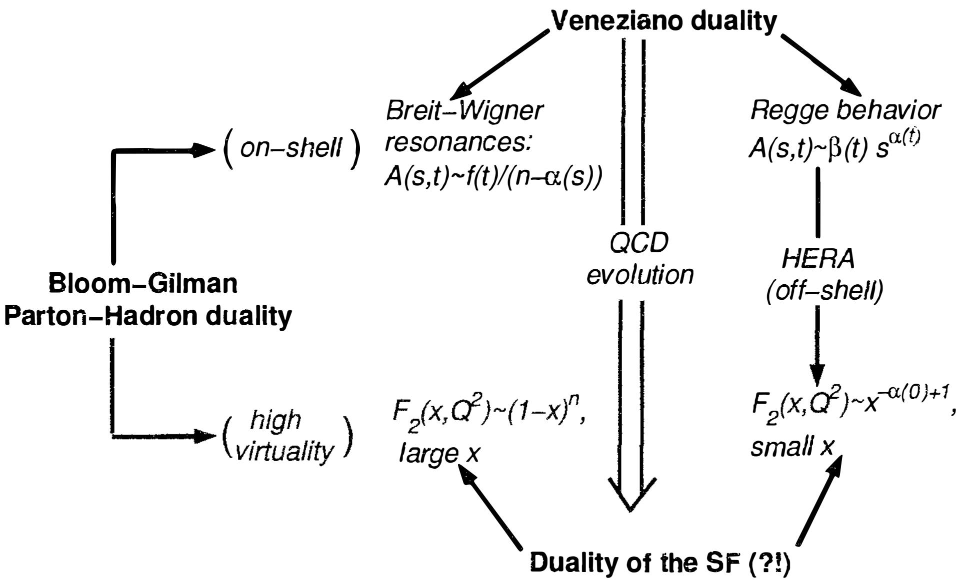

Fig. 3. Veneziano, or resonance-reggeon duality [10] and Bloom-Gilman, or hadron-parton duality [5] in strong interactions. From [2].

In Ref. [1, 2] the authors tried to introduce Q^2-dependence in Veneziano amplitude [10] or more advanced Dual Amplitude with Mandelstam Analyticity (DAMA) [14]. The Q^2-dependence can be introduced either through a Q^2-dependent Regge trajectory [1], leading to a problem of physical interpretation of such an object, or through the g parameter of DAMA [1, 2]. This last way seems to be more realistic [2], but it is allowed only in the limited range of Q^2 due to the DAMA model requirement $g > 1$ [14] (see [2] for details).

In the papers [3, 4] the authors went in an opposite direction – they built a Regge-dual model with Q^2-dependent form factors, inspired by the pole series expansion of DAMA, which fits the SF data in the resonance region[1]. The hope was to reconstruct later the Q^2-dependent dual amplitude, which would lead to such an expansion.

A consistent treatment of the problem requires the account for the spin dependence, which we ignore in this paper for the sake of simplicity. Our goal is rather to check qualitatively the proposed new way of constructing the "two-dimensionally dual"amplitude.

[1]It is important that DAMA not only allows, but rather requires nonlinear complex Regge trajectories [14]. Then the trajectory with restricted real part lead to a limited number of resonances.

2. Modified DAMA model

The DAMA integral is a generalization of the integral representation of the B-function used in the Veneziano model [14] [2]:

$$D(s,t) = \int_0^1 dz \left(\frac{z}{g}\right)^{-\alpha_s(s')-1} \left(\frac{1-z}{g}\right)^{-\alpha_t(t'')-1}, \tag{2.1}$$

where $a' = a(1-z)$, $a'' = az$, and g is a free parameter, $g > 1$, and $\alpha_s(s)$ and $\alpha_t(t)$ stand for the Regge trajectories in the s- and t-channels.

In this paper we propose a modified definition of DAMA (M-DAMA) with Q^2-dependence [15]. It also can be considered as a next step in generalization of the Veneziano model. M-DAMA preserves all the attractive features of DAMA, such as pole decompositions in s and t, Regge asymptotics etc., yet it gains the Q^2-dependent form factors, correct large and low x behaviour for $t = 0$ etc.

The proposed M-DAMA integral reads [15]:

$$D(s,t,Q^2) = \int_0^1 dz \left(\frac{z}{g}\right)^{-\alpha_s(s')-\beta(Q^{2''})-1} \left(\frac{1-z}{g}\right)^{-\alpha_t(t'')-\beta(Q^{2'})-1}, \tag{2.2}$$

where $\beta(Q^2)$ is a smooth dimensionless function of Q^2, which will be specified later on from studying different regimes of the above integral.

The on mass shell limit, $Q^2 = 0$, leads to the shift of the s- and t-channel trajectories by a constant factor $\beta(0)$ (to be determined later), which can be simply absorbed by the trajectories and, thus, M-DAMA reduces to DAMA. In the general case of the virtual particle with mass M we have to replace Q^2 by $(Q^2 + M^2)$ in the M-DAMA integral.

Now all the machinery developed for the DAMA model (see for example [14]) can be applied to the M-DAMA integral. Below we shall report briefly only some of its properties, relevant for the further discussion.

2.1. Singularities in M-DAMA

The dual amplitude $D(s,t,Q^2)$ is defined by the integral (2.2) in the domain $\mathcal{R}e\ (\alpha_s(s') + \beta(Q^{2''})) < 0$ and $\mathcal{R}e\ (\alpha_t(t'') + \beta(Q^{2'})) < 0$. For monotonically decreasing function $\mathcal{R}e\ \beta(Q^2)$ (or non-monotonic function with maximum at $Q^2 = 0$) and for increasing or constant real parts of the trajectories these equations, applied for $0 \le z \le 1$, mean $\mathcal{R}e\ (\alpha_s(s)+\beta(0)) < 0$ and $\mathcal{R}e\ (\alpha_t(t)+\beta(0)) < 0$. To enable us to study the properties of M-DAMA in the domains $\mathcal{R}e\ (\alpha_s(s') + \beta(Q^{2''})) \ge 0$ and $\mathcal{R}e\ (\alpha_t(t'') + \beta(Q^{2'})) \ge 0$, which are of the main interest, we have to make an analytical continuation of M-DAMA. This leads to the appearance of two moving poles ${}_s(s(1-z_n)) + \beta(Q^2 z_n) = n$ and $\alpha_t(tz_m)+\beta(Q^2(1-z_m)) = m, \quad n,m = 0,1,2...polesThesingularitiesofthedualamplitudear$

The collision of a moving pole $z = z_n$ with the branch point $z = 0$ results in a pole at $s = s_n$, where s_n is defined by

$$\alpha_s(s_n) + \beta(0) = n\,. \tag{2.3}$$

[2] There are several integral representations of DAMA [14], here we shall use the most common one.

Please, notice the presence of an extra (in comparison to DAMA) term $\beta(0)$. It can be considered as a shift of the trajectory. If $\beta(0)$ is an integer number, then the modification is trivial.

The collision of a moving pole $z = z_n$ with the branch point $z = 1$ results in a pole at $Q^2 = Q_n^2$, defined by

$$\alpha_s(0) + \beta(Q_n^2) = n\,. \tag{2.4}$$

In this sense we can think about $\beta(Q^2)$ as of a kind of trajectory, but we do not mean that it describes real physical particles. Also we will see later that with a proper choice of $\beta(Q^2)$ we can avoid these unphysical poles, and $\beta(Q^2)$ required by the low x behaviour of the nucleon SF is exactly of this type.

Similarly, the collision of a moving pole $z = z_m$ with the branch point $z = 1$ results in a pole at $t = t_m$, defined by

$$\alpha_t(t_m) + \beta(0) = m\,. \tag{2.5}$$

The collision of a moving pole $z = z_m$ with the branch point $z = 0$ results in a pole at $Q^2 = Q_m^2$, defined by

$$\alpha_t(0) + \beta(Q_m^2) = m\,. \tag{2.6}$$

Note that if $\alpha_s(0) = \alpha_t(0)$ the poles in Q^2 will be degenerate. For further discussion we shall consider a non-degenerated case.

2.2. Pole decompositions

Similarly as for DAMA [14], case 1 from the above results into pole decomposition of M-DAMA amplitude with the following expression for the pole term [15]:

$$D_{s_n}(s,t,Q^2) = g^{n+1} \sum_{l=0}^{n} \frac{[\beta'(0)Q^2 - s\alpha_s'(s)]^l C_{n-l}(t,Q^2)}{[n - \alpha_s(s) - \beta(0)]^{l+1}}\,, \tag{2.7}$$

where

$$C_l(t,Q^2) = \frac{1}{l!}\frac{d^l}{dz^l}\left[\left(\frac{1-z}{g}\right)^{-\alpha_t(tz)-\beta(Q^2(1-z))-1}\right]_{z=0}. \tag{2.8}$$

Formula (2.7) shows that our $D(s,t,Q^2)$ does not contain ancestors and that an $(n+1)$-fold pole emerge on the n-th level. The crossing-symmetric term can be obtained in a similar way by considering the case 3 from the list above.

The modifications with respect to DAMA are A) the shift of the trajectory $\alpha_s(s)$ by the constant factor of $\beta(0)$ (we can easily remove this shift including $\beta(0)$ into trajectory); B) the coefficients C_l are now Q^2-dependent and can be directly associated with the form factors. The presence of the multipoles, eq. (2.7), does not contradict the theoretical postulates. On the other hand, they can be removed without any harm to the dual model by means the so-called Van der Corput neutralizer[3]. This procedure [14] seems to work for M-DAMA equally well as for DAMA and will result in a "Veneziano-like"pole structure:

$$D_{s_n}(s,t,Q^2) = g^{n+1}\frac{C_n(t,Q^2)}{n-\alpha_s(s)-\beta(0)}\,. \tag{2.9}$$

The Q^2 -pole terms can be obtained by considering cases 2 and 4 from section 2.1., but, as we shall see later in section 4., with our choice of $\beta(Q^2)$ we avoid Q^2 poles.

2.3. Asymptotic properties of M-DAMA

Let us now discuss the asymptotic properties of M-DAMA. Using exactly the same method as in [14] it is possible to show that if the trajectory satisfies some restriction on its increase, then we obtain the Regge asymptotic behaviour [15]:

$D(s,t,Q^2) \sim s^{\alpha_t(t)+\beta(0)} g^{\beta(Q^2)}\,, \quad s\to\infty .W0So, intheReggelimitM{-}DAMAhasthe$

$\beta(0)$).

It is more interesting to study the new regime, which does not exist in DAMA – the limit $Q^2 \to \infty$, with constant s , t . We assume that $\beta(Q^2) \to -\infty$ for $Q^2 \to \infty$. Then [15],

$$D(s,t,Q^2)|_{Q^2\to\infty} \approx (2g)^{2\beta(Q^2/2)+\alpha_s(s/2)+\alpha_t(t/2)+2}\sqrt{\frac{2\pi}{W}}\,, \tag{2.10}$$

where $W \approx 8\gamma\ln(Q^2/Q_0^2)$. For DIS, as we shall see below, if s and t are fixed and $Q^2 \to \infty$ then $u = -2Q^2 \to -\infty$, as it follows from the kinematic relation $s+t+u = 2m^2-2Q^2$. So, we need also to study the $D(u,t,Q^2)$ term in this limit. If $|\alpha_u(-2Q^2)|$ is growing slower than $|\beta(Q^2)|$ or terminates when $Q^2 \to \infty$, then the previous result (eq. (2.10), s to be changed to $u = -2Q^2$) is still valid.

3. Nucleon structure function

The total cross section of $\gamma^* p$ scattering is related to the SF by

$$F_2(x,Q^2) = \frac{Q^2(1-x)}{4\pi\alpha(1+4m^2x^2/Q^2)}\sigma_t^{\gamma^* p}\,, \tag{3.10}$$

[3]In brief, the procedure [14] is to multiply the integrand of (2.2) by a function $\varphi(z)$, which has the following properties:

$$\varphi(0)=0,\quad \varphi(1)=1,\quad \varphi^n(1)=0,\quad n=1,2,3,...$$

The function $\varphi(z) = 1-exp\left(-\frac{z}{1-z}\right)$, for example, satisfies the above conditions.

where α is the fine structure constant. In eq. (3.) we neglected $R(x, Q^2) = \sigma_L(x, Q^2)/\sigma_T(x, Q^2)$, which is a reasonable approximation.

The total cross section is related to the imaginary part of the scattering amplitude

$$\sigma_t^{\gamma^* p}(x, Q^2) = \frac{8\pi}{P_{CM}\sqrt{s}} \mathcal{I}m\ A(s(x, Q^2), t = 0, Q^2) \ . \tag{3.10}$$

where P_{CM} is the center of mass momentum of the reaction, $P_{CM} = \frac{s-m^2}{2(1-x)}\sqrt{\frac{1+4m^2x^2/Q^2}{s}}$ for DIS. Thus, we have

$$F_2(x, Q^2) = \frac{4Q^2(1-x)^2}{\alpha\,(s-m^2)\,(1+4m^2x^2/Q^2)^{3/2}} \mathcal{I}m\ A(s(x, Q^2), t = 0, Q^2) \,. \tag{3.10}$$

The minimal model for the scattering amplitude is a sum [17]

$$A(s, 0, Q^2) = c(s-u)(D(s, 0, Q^2) - D(u, 0, Q^2)), \tag{3.10}$$

providing the correct signature at high-energy limit, where c is a normalization coefficient. As it was said at the beginning, we disregard the symmetry properties of the problem (spin and isospin), concentrating on its dynamics.

In the low x limit: $x \to 0$, $t = 0$, $Q^2 = const$, $s = Q^2/x \to \infty$, $u = -s$ we obtain from eqs. (??,3.):

$$\mathcal{I}m\ A(s, 0, Q^2)|_{s\to\infty} \sim s^{\alpha_t(t)+\beta(0)+1} g^{\beta(Q^2)} \,. \tag{3.10}$$

Our philosophy in this section is the following: we specify a particular choice of $\beta(Q^2)$ in the low x limit and then we use M-DAMA integral (2.2) to calculate the dual amplitude, and correspondingly SF, in all kinematical domains. We will see that the resulting SF has qualitatively correct behaviour in all regions. Even more – our choice of $\beta(Q^2)$ will automatically remove Q^2 poles.

According to the two-component duality picture [18], both the scattering amplitude A and the structure function F_2 are the sums of the diffractive and non-diffractive terms. At high energies both terms are of the Regge type. For $\gamma^* p$ scattering only the positive-signature exchanges are allowed. The dominant ones are the Pomeron and f Reggeon, respectively. The relevant scattering amplitude is as follows:

$$B(s, Q^2) = iR_k(Q^2)\Big(\frac{s}{m^2}\Big)^{\alpha_k(0)}, \tag{3.10}$$

where α_k and R_k are Regge trajectories and residues and k stands either for the Pomeron or for the Reggeon. The residue is chosen to satisfy approximate Bjorken scaling for the SF [19, 20]. From eqs. (3.,3.) SF is given as:

$$F_2(x, Q^2) \sim Q^2 R_k(Q^2)\Big(\frac{s}{m^2}\Big)^{\alpha_k(0)-1} \,. \tag{3.10}$$

Bjorken variable $x = Q^2/s$ for $s \to \infty$ and thus, Regge asymptotics and scaling behaviour require that

$$R_k(Q^2) \sim (Q^2)^{-\alpha_k(0)} \,. \tag{3.10}$$

Actually, it could be more involved if we require the correct $Q^2 \to 0$ limit to be respected and the observed scaling violation (the "HERA effect") to be included. Various models to cope with the above requirements have been suggested [16, 19, 20]. At HERA, especially at large Q^2, scaling is so badly violated that it may not be explicit anymore.

In the phenomenological models which are used nowadays to fit F_2 data [19, 20, 7, 8, 24] (also [3, 4] were discussed in introduction) the Q^2-dependence is introduced "by hands", via residue in the form (3.), parameters of which are then fitted to the data. Now we have a model which contains Q^2-dependence from the very beginning and automatically gives a correct behaviour of the residues.

Data show that the Pomeron exchange leads to a rising structure function at large s (low x). To provide for this we have two options: either to assume supercritical Pomeron with $\alpha_P(0) > 1$ or to assume a critical ($\alpha_P(0) = 1$) dipole (or higher multipole) Pomeron [16, 21, 22]. The latter leads to the logarithmic behaviour of the SF:

$$F_{2,P}(x, Q^2) \sim Q^2 R_P(Q^2) \ln\left(\frac{s}{m^2}\right), \tag{3.10}$$

which proves to be equally efficient [16, 22].

Let us now come back to M-DAMA results. Using eqs. (3.,3.) we obtain:

$$F_2 \sim s^{\alpha_t(0)+\beta(0)} Q^2 g^{\beta(Q^2)}. \tag{3.10}$$

Choosing

$$\beta(0) = -1 \tag{3.10}$$

we restore the asymptotics (3.) and this allows us to use trajectories in their commonly used form. Now we have to find such a $\beta(Q^2)$, which can provide for Bjorken scaling. If we choose $\beta(Q^2)$ in the form

$$\beta(Q^2) = d - \gamma \ln(Q^2/Q_0^2), \tag{3.10}$$

with

$$\gamma = (\alpha_t(0) + \beta(0) + 1)/\ln g = \alpha_t(0)/\ln g, \tag{3.10}$$

where d, Q_0^2 are some parameters, we get the exact Bjorken scaling.

Actually, the expression (3.) might cause problems in the $Q^2 \to 0$ limit. To avoid this, it is better to use a modified expressions

$$\beta(Q^2) = \beta(0) - \gamma \ln\left(\frac{Q^2 + Q_0^2}{Q_0^2}\right) = -1 - \frac{\alpha_t(0)}{\ln g} \ln\left(\frac{Q^2 + Q_0^2}{Q_0^2}\right). \tag{3.10}$$

This choice leads to

$$F_2(x, Q^2) \sim x^{1-\alpha_t(0)} \left(\frac{Q^2}{Q^2 + Q_0^2}\right)^{\alpha_t(0)}, \tag{3.10}$$

where slowly varying factor $\left(\frac{Q^2}{Q^2+Q_0^2}\right)^{\alpha_t(0)}$ is typical for the Bjorken scaling violation (for example [20]).

Now let us turn to the large x limit. In this regime $x \to 1$, s is fixed, $Q^2 = \frac{s-m^2}{1-x} \to \infty$ and correspondingly $u = -2Q^2$. Using eqs. (2.10,3.,3.) we obtain:

$$F_2 \sim (1-x)^2 Q^4 g^{2\beta(Q^2/2)} \sqrt{\frac{2\pi}{W}} \left(g^{\alpha_s(s/2)} - g^{\alpha_u(-Q^2)} \right) . \tag{3.10}$$

For $Q^2 \to \infty$ factors $\left(g^{\alpha_s(s/2)} - g^{\alpha_u(-Q^2)} \right)$ and W are slowly varying functions of Q^2 under our assumption about $\alpha_u(-Q^2)$. Thus, we end up with qualitatively correct behaviour

$$F_2 \sim \left(\frac{2Q_0^2}{Q^2} \right)^{2\gamma \ln 2g} \sim (1-x)^{2\alpha_t(0) \ln 2g / \ln g} . \tag{3.10}$$

Let us now study F_2 given by M-DAMA in the resonance region. The existence of resonances in SF at large x is not surprising by itself: as it follows from (3.) and (3.) they are the same as in $\gamma^* p$ total cross section, but in a different coordinate system.

For M-DAMA the resonances in s-channel are defined by the condition (2.3). For simplicity let us assume that we performed the Van der Corput neutralization and, thus, the pole terms appear in the form (2.9). In the vicinity of the resonance $s = s_{Res}$ only the resonance term $D_{Res}(s, 0, Q^2)$ is important in the scattering amplitude and correspondingly in the SF.

Using $\beta(Q^2)$ in the form (3.), which gives Bjorken scaling at large s, we obtain from eq. (2.8):

$$C_1(Q^2) = \left(\frac{g Q_0^2}{Q^2 + Q_0^2} \right)^{\alpha_t(0)} \left[\alpha_t(0) + \ln g \frac{Q^2}{Q^2 + Q_0^2} - \frac{\alpha_t(0)}{\ln g} \ln \left(\frac{Q^2 + Q_0^2}{Q_0^2} \right) \right] . \tag{3.10}$$

The term $\left(\frac{Q_0^2}{Q^2+Q_0^2} \right)^{\alpha_t(0)}$ gives the typical Q^2-dependence for the form factor (the rest is a slowly varying function of Q^2).

If we calculate higher orders of C_n for subleading resonances, we will see that the Q^2-dependence is still defined by the same factor $\left(\frac{Q_0^2}{Q^2+Q_0^2} \right)^{\alpha_t(0)}$. Here comes the important difference from the Regge-dual model [3, 4] motivated by introducing Q^2-dependence through the parameter g. As we see from eq. (2.9), g enters with different powers for different resonances on one trajectory – the powers are increasing with the step 2. Thus, if $g \sim \left(\frac{Q_0^2}{Q^2+Q_0^2} \right)^{\triangle}$, then the form factor for the first resonance is ($n = 0$) $\sim \left(\frac{Q_0^2}{Q^2+Q_0^2} \right)^{\triangle}$, and for the second one ($n = 2$) it is $\sim \left(\frac{Q_0^2}{Q^2+Q_0^2} \right)^{3\triangle}$ etc. As discussed in [4] the present accuracy of the data does not allow to discriminate between the constant powers of form factor (for example Refs. [23, 7, 8, 24], and this work) and increasing ones.

4. How to avoid Q^2 poles?

General study of the M-DAMA integral allows the existence of Q^2 poles (see cases 2, 4 in section 2.1.) which would be unphysical. The appearance and properties of these singularities depend on the particular choice of the function $\beta(Q^2)$, and for our choice, given by eq. (3.), the Q^2 poles can be avoided.

We have chosen $\beta(Q^2)$ to be a decreasing function, then, according to conditions (2.4,2.6), there are no Q^2 poles in M-DAMA in the physical domain $Q^2 \geq 0$, if

$$\mathcal{R}e\ \beta(0) < -\alpha_s(0)\,, \quad \mathcal{R}e\ \beta(0) < -\alpha_t(0)\,. \tag{4.10}$$

We have already fixed $\beta(0) = -1$, eq. (3.), and, thus, we see that indeed we do not have Q^2 poles, except for the case of supercritical Pomeron with the intercept $\alpha_P(0) > 1$. Such a supercritical Pomeron would generate one unphysical pole at $Q^2 = Q^2_{pole}$ defined by equation

$$-1 - \frac{\alpha_P(0)}{\ln g} \ln\left(\frac{Q^2+Q_0^2}{Q_0^2}\right) + \alpha_P(0) = 0 \quad \Rightarrow \quad Q^2_{pole} = Q_0^2(g^{\frac{\alpha_P(0)-1}{\alpha_P(0)}} - 1)\,. \tag{4.10}$$

Therefore we can conclude that M-DAMA does not allow a supercritical trajectory – what is good property from the theoretical point of view, since such a trajectory violates the Froissart-Martin limit [25].

As it was discussed above there are other phenomenological models which use dipole Pomeron with the intercept $\alpha_P(0) = 1$ and also fit the data (see for example [16]). This is a very interesting case – ($\alpha_t(0) = 1$) – for the proposed model. At the first glance it seems that we should anyway have a pole at $Q^2 = 0$. It should result from the collision of the moving pole $z = z_0$ with the branch point $z = 0$, where $\alpha_t(0) + \beta(Q^2(1 - z_0)) = 0$ in our case. Then, checking the conditions for such a collision:

$$\alpha_t(0) - t\,\alpha_t'(0) z_0 + \beta(Q^2) - \beta'(Q^2) Q^2 z_0 = 0 \ \Rightarrow \ z_0 = \frac{-\alpha_t(0) - \beta(Q^2)}{t\,\alpha_t'(0) - Q^2\beta'(Q^2)}\,,$$

we see that for $t = 0$ and for $\beta(Q^2)$ given by eq. (3.) the collision is simply impossible, because $z_0(Q^2)$ does not tend to 0 for $Q^2 \to 0$. Thus, for the Pomeron with $\alpha_P(0) = 1$ M-DAMA does not contain any unphysical singularity.

On the other hand, a Pomeron trajectory with $\alpha_P(0) = 1$ does not produce rising SF (3.), as required by the experiment. So, we need a harder singularity and the simplest one is a dipole Pomeron. A dipole Pomeron produces poles of the second power – $D_{dipole}(s, t_m) \propto \frac{C(s)}{(m-\alpha_P(t)+1)^2}$, see for example ref. [21] and references therein. Formally such a dipole Pomeron can be written as $\frac{\partial}{\partial \alpha_P} \frac{C(s)}{(m-\alpha_P(t)+1)}$, and generalizing this – $D_{dipole}(s,t) = \frac{\partial}{\partial \alpha_P} D(s,t)$, where $D(s,t)$ can be given for example by DAMA or M-DAMA. Applying this expression to the asymptotic formula of M-DAMA, eq. (??), we obtain a term $g^{\beta(Q^2)} s^{\alpha_t(t)+\beta(0)} \ln s$, which then leads to a logarithmically rising SF (for $\alpha_P(0) + \beta(0) = 0$) – the one given by eq. (3.).

For $\beta(Q^2)$ in the form (3.) M-DAMA will generate an infinite number of the Q^2 poles concentrated near the "ionization point" $Q^2 = -Q_0^2$. Although these are in the unphysical region of negative Q^2 , such a feature of the model

A) makes us think about $\beta(Q^2)$ as about a kind of trajectory, what is not the case, as it was stressed above, and

B) might create a problem for a general theoretical treatment, for example for making analytical continuation in Q^2. To avoid this we can redefine $\beta(Q^2)$ in the nonphysical Q^2 region, for example in the following way:

$$\beta(Q^2) = \begin{cases} -1 - \frac{\alpha_t(0)}{\ln g} \ln\left(\frac{Q^2+Q_0^2}{Q_0^2}\right), & \text{for } Q^2 \geq 0, \\ -1 - \frac{\alpha_t(0)}{\ln g} \ln\left(\frac{Q_0^2-Q^2}{Q_0^2}\right), & \text{for } Q^2 < 0. \end{cases} \tag{4.10}$$

This function has a maximum at $Q^2 = 0$, $\beta(0) = -1$. M-DAMA with $\beta(Q^2)$ given by eq. (4.) preserves all its good properties, discussed above, and does not contain any singularity in Q^2 (except for the supercritical Pomeron case, which we do not allow).

5. Conclusions

A new model for the Q^2-dependent dual amplitude with Mandelstam analyticity is proposed. The M-DAMA preserves all the attractive properties of DAMA, such as its pole structure and Regge asymptotics, but it also leads to generalized dual amplitude $A(s,t,Q^2)$ and in this way realizes a unified "two-dimensionally dual"picture of strong interaction [1, 2, 3, 4] (see Fig. 3). This amplitude, when $t = 0$, can be related to the nuclear SF, and in this way we fix the function $\beta(Q^2)$, which introduces the Q^2-dependence in M-DAMA, eq. (2.2). Our analyzes shows that for both large and low x limits as well as for the resonance region the results of M-DAMA are in qualitative agreement with the experiment.

In the proposed formulation a Q^2-dependence is introduced into DAMA through the additional function $\beta(Q^2)$. Although in the integrand this function stands next to Regge trajectories, this, as it was stressed already, does not mean that it also corresponds to some physical particles. There is no qualitative difference between two ways of introducing Q^2-dependence into DAMA: through the Q^2-dependent parameter g, i.e. function $g(Q^2)$ [1, 2] or through the function $\beta(Q^2)$. On the other hand the second way, i.e. M-DAMA, is applicable for all range of Q^2 and it results into physically correct behaviour in all tested limits.

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WAVE-MIXING SCHEMES REVEALING QED VACUUM NONLINEARITY

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Mode coupling and combinative frequency generation due to quantum-electrodynamical nonlinearity of vacuum are considered for specially designed microwave-optical cavities.

1. Introduction

According to quantum electrodynamics (QED), there is photon-photon scattering in vacuum; see, e.g., [1, 2, 3, 4, 5]. It is owing to virtual electron-positron pairs and makes vacuum a nonlinear medium. Nonlinear terms in Maxwell equations vanish in the limit of parallel propagation of plane waves. However, observation of nonlinear vacuum effects is possible when eigenmodes of a cavity or crossed beams are used (some recent proposals are given in [6, 7, 8, 9]).

In the present paper, several new wave-mixing schemes revealing this nonlinearity are considered, including third harmonic generation in high-Q microwave (MW) cavities and quasi-optical resonators, and combinative frequency generation due to coupling of MW fields with laser beams.

It is easy to understand why the electrodynamics in vacuum is nonlinear. First, there is a well-known process of annihilation of a charged particle and its anti-particle, for example, electron and positron, that produces a pair of photons. Consequently, there must exist an inverse process in which two photons collide and produce an electron-positron pair. This pair can immediately disappear, or be virtual, and in turn produce two photons. The process as a whole can be thought of as scattering of one photon on another (Fig.1), and since photons interact, the evolution of electromagnetic field cannot be described by linear equations. If such an interaction is

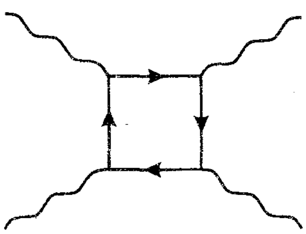

Рис. 1. Photon-photon scattering due to virtual pairs.

taken into account, it gives an additional term in the Lagrange function, which was calculated by Heisenberg and Euler in 1936 [10]:

$$\mathcal{L}_{HE} = \frac{\mathbf{E}^2-\mathbf{B}^2}{8\pi} +$$
$$+\frac{1}{8\pi^2}\int\limits_0^\infty \left\{1+\frac{e^2\eta^2}{3}\left(\mathbf{B}^2-\mathbf{E}^2\right) - e^2\eta^2\mathbf{BE}\times\frac{\mathrm{Re}\ \mathrm{ch}\left(e\eta\sqrt{\mathbf{B}^2-\mathbf{E}^2+2i\mathbf{BE}}\right)}{\mathrm{Im}\ \mathrm{ch}\left(e\eta\sqrt{\mathbf{B}^2-\mathbf{E}^2+2i\mathbf{BE}}\right)}\right\} e^{-m^2\eta}\frac{d\eta}{\eta^3}.$$

Here the Planck's constant $\hbar$ and the velocity of light c are equal to unity, e is the electron charge, m the electron mass, $\alpha \equiv e^2/\hbar c = e^2$ the fine structure constant. We are taking only electrons and positrons into account because they have the smallest mass and thus are most readily produced in vacuum by electromagnetic fields $\mathbf{E}$, $\mathbf{B}$. Over the years, various authors have studied the feasibility of using high-intensity radiation in vacuum to observe some nonlinear effects, such as four-wave mixing [7], self-action [11], vacuum birefringence [12], etc.

2. Nonlinearity in low-frequency weak fields

Electron-positron pair creation is exponentially small if the work of electric field over Compton's wavelength is much smaller than the electron rest energy or, equivalently, the field is smaller than the so-called critical field, $E_c = mc^2/e\lambdabar$, where $\lambdabar = \hbar/mc$.

Under laboratory conditions pair creation is hardly possible, with the exception of some extreme cases such as collision of two nuclei with relativistic velocities or x-rays focusing [3, 9, 13]. Hereafter we assume the fields to be much weaker than the critical ones, and the frequencies to be not too high so that dispersive effects owing to quantum nature of light can be neglected [4, 5, 11]:

$$E, B \ll E_c = B_c = m^2c^3/e\hbar = 4.4\cdot 10^{13}\mathrm{G} \approx 1.3\cdot 10^{16}\ \mathrm{V/cm},$$

$$\lambda \sim c/\omega \gg mc^2/e(E+B), \quad \text{i.e.,} \quad \hbar\omega \ll \lambdabar e(E+B).$$

Both assumptions are valid for modern powerful microwave and optical sources. Then one can use perturbation theory and obtain the following well-known form of the Lagrange function [1]

$$8\pi\mathcal{L} = \mathbf{E}^2 - \mathbf{B}^2 + \frac{\alpha}{45\pi B_c^2}\left[\left(\mathbf{E}^2-\mathbf{B}^2\right)^2 + 7\left(\mathbf{EB}\right)^2\right],$$

and come to the standard Maxwell equations, where polarization and magnetization depend cubically on the electric and magnetic fields:

$$\mathrm{div}\mathbf{B} = 0, \qquad \mathrm{div}(\mathbf{E}+4\pi\mathbf{P}) = 0,$$

$$\mathrm{rot}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \qquad \mathrm{rot}(\mathbf{B} - 4\pi\mathbf{M}) = \frac{1}{c}\frac{\partial(\mathbf{E} + 4\pi\mathbf{P})}{\partial t};$$

$$\mathbf{P} = \frac{\xi}{4\pi}\left[2(E^2 - B^2)\mathbf{E} + 7(\mathbf{E}\cdot\mathbf{B})\mathbf{B}\right],$$

$$\mathbf{M} = -\frac{\xi}{4\pi}\left[2(E^2 - B^2)\mathbf{B} - 7(\mathbf{E}\cdot\mathbf{B})\mathbf{E}\right].$$

Nonlinear constant is defined by the electron charge and mass:

$$\xi = \frac{\hbar e^4}{45\pi m^4 c^7} \approx 0.26 \cdot 10^{-31} \text{ e.s.u.},$$

$$E_{NL} \sim \xi^{-1/2} \approx 0.6 \cdot 10^{16} \text{ e.s.u.} \approx 2 \cdot 10^{18} \text{ V/cm}.$$

In all the processes we are going to discuss, no real electron-positron pairs are produced, the pairs are virtual and manifest themselves only through the nonlinearity of Maxwell equations. In other words, there are no actual charge and current sources in the equations. It is important to note again that the nonlinear terms are exactly equal to zero for a plane electromagnetic wave, or for a group of plane waves propagating in one direction.

In sections 3-6 we briefly discuss some of the predicted nonlinear vacuum effects and proposed schemes for their experimental detection. In sections 7, 8 we discuss the possibility of observing the vacuum-produced third harmonic of microwave radiation, and in sections 9, 10 we present new combined MW-laser schemes, which are a natural generalization of previously proposed schemes based on either MW or optical radiative processes; cf., e.g., [7, 8, 11, 13].

3. Magnetized vacuum as an anisotropic medium

There are products of three electromagnetic fields in the expressions for vacuum polarization and magnetization, and by fixing some of them to be constant external fields we can obtain linear, quadratic, and cubic effects.

One of these linear effects is birefringence in magnetized vacuum. In the presence of strong constant magnetic field, the two eigenmodes of Maxwell equations have different indices of refraction. Dependence of ordinary and extraordinary indices of refraction on external magnetic field may be easily calculated and is shown in Fig.2, where θ is the angle between a wave vector and the magnetic field, $\mathbf{B}_0$. When the field is much weaker then the critical value, $B_0 \ll B_c$, and the wave frequency is not too high, $\hbar\omega \ll mc^2 |\sin\theta| B_c/B_0$, the correction to the phase velocity of electromagnetic waves is quadratic on the external magnetic field (see, e.g., [14, 15]):

$$\begin{cases} n_e \approx 1 + (7/2)a\sin^2\theta \\ n_0 \approx 1 + 2a\sin^2\theta \end{cases}, \qquad a = \frac{\alpha}{45\pi}\frac{B_0^2}{B_c^2}.$$

Thus, vacuum in a strong magnetic field can influence the polarization of electromagnetic waves propagating through an inhomogeneous magnetoactive plasma. So, observing polarization peculiarities of radiation coming from astrophysical sources (e.g., neutron stars) with high magnetic fields may provide an indirect evidence of nonlinearity of vacuum. However, there are no firm observations of this kind nowadays, though there were some observational attempts and many discussions on this account; see, e.g., [14, 15, 16, 17, 18].

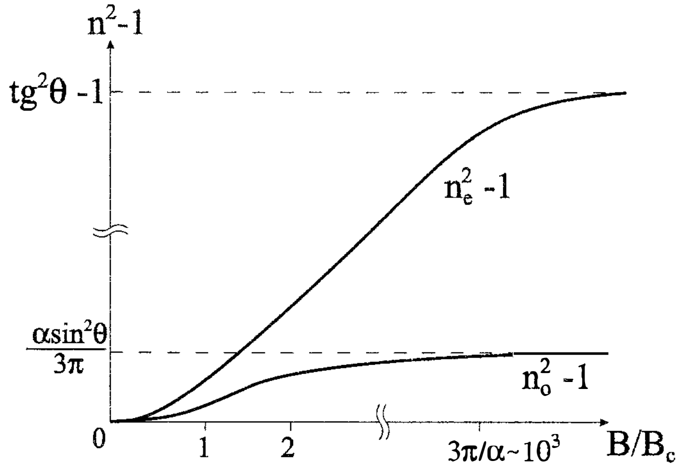

Рис. 2. Birefringence in magnetized vacuum; n_e and n_0 are refractive indices of x- and o- modes, respectively.

4. Nonlinear phase shift

Phase shift leading to change of the wave polarization can also occur when a radiation beam propagates through oscillating electromagnetic field. Consider the setup depicted in Fig.3 (see [12]): a focused high power beam with electric field amplitude E and another probe beam are propagating in the opposite directions. Due to nonlinear interaction the probe beam will experience a phase shift proportional to the length of interaction region, l, and this shift will depend on the initial polarization of the probe beam: $\delta\Phi \sim \alpha(|E^2|/E_c^2)(l/\lambda)$. According to estimates [12], for a wavelength $\lambda \sim 1\mu$m, a powerful beam of cross-section $S \sim (300\mu\mathrm{m})^2$ and length $l \sim (S/\lambda) \sim 10$cm, with energy $\mathcal{E} \sim 10$kJ, produces a phase shift of about $\delta\Phi \sim 10^{-10}$rad, which is very hard to detect. Many other well-known nonlinear opti-

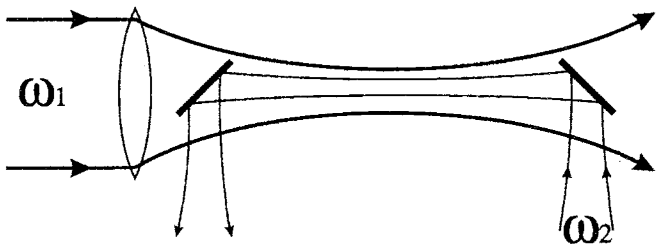

Рис. 3. Scheme for observation of nonlinear phase shift.

cal effects are possible in vacuum, including second harmonic generation, parametric instabilities, self-focusing and channelling in counter-propagating beams, see, e.g., [6, 7]. In particlar, the critical power for self-focusing instability is of the order of [7] $P_{cr} \sim 10^3 E_c^2 \lambda^2 \sim 2.5 \cdot 10^{24}$ W.

Another interesting nonlinear effect is photon splitting in magnetized vacuum [19]. An ordinary photon can decay into two extraordinary photons (Fig.4). This 3-wave interaction process has also been widely studied; in particular, self-similar solutions of photon transfer equations are found [19].

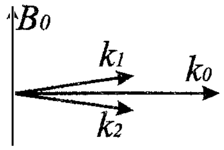

Рис. 4. Photon splitting $\gamma_0 \to \gamma_e + \gamma_e$ in magnetized vacuum.

5. 4-wave mixing in vacuum

Virtual electron-positron pair creation and annihilation leads to photon-photon scattering $\gamma_1 + \gamma_2 \to \gamma' + \gamma''$ (Fig.1). Its effectiveness is characterized by well-known cross-section, σ, which is calculated by quantum electrodynamical methods and has the following asymptotics [1, 2]

$$\sigma \approx 0.03\alpha^2 \left(\frac{e^2}{mc^2}\right)^2 \left(\frac{\hbar\omega}{mc^2}\right)^6, \qquad \hbar\omega \ll mc^2;$$

$$\sigma \approx 4.7\alpha^4(c/\omega)^2, \qquad \hbar\omega \gg mc^2.$$

A maximum, $\sigma_{max} \sim 10^{-30}\mathrm{cm}^2$, is found at $\hbar\omega \approx 1.5mc^2$. This extremely small value of σ_{max} explains why the effects of vacuum nonlinearity are so hard to observe.

Generation of sum of three frequencies, $\omega_s = \omega_1 + \omega_2 + \omega_3$, is forbidden in vacuum. In this case, to satisfy the phase matching condition together with the vacuum dispersion relation, all three initial wavevectors and scattered wavevector must be parallel, $\mathbf{k}_s \upuparrows \mathbf{k}_1 \upuparrows \mathbf{k}_2 \upuparrows \mathbf{k}_3$, and for waves propagating in the same direction vacuum nonlinearity vanishes.

However, for combinative frequency generation, $\omega_s = \omega_1 + \omega_2 - \omega_3$, the phase matching condition, $\mathbf{k}_s = \mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3$, can be met. It is illustrated in Fig.5 [11]. If

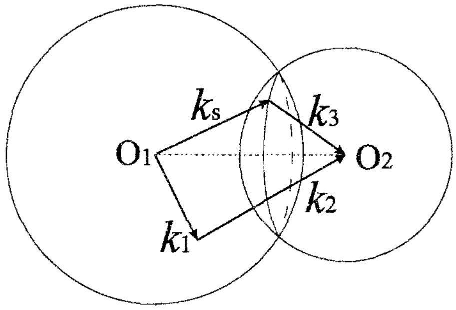

Рис. 5. Schematics of the phase matching for combinative frequency generation (after [11]).

we fix the three initial frequencies, and therefore the lengths of all four wavevectors, and if we fix vectors $\mathbf{k}_1$ and $\mathbf{k}_2$, the two spheres with radii k_s and k_3 intersect each other. Then, taking any point in the intersection we would satisfy the phase matching condition. The electric field of the scattered wave will be proportional to nonlinear polarization, scattered differential frequency, ω_s, and grows linearly with coordinate along the scattered wavevector, $\mathbf{k}_s$: $E_{sx}(z) \propto i\omega_s z(P_{sx}^{NL} + M_{sy}^{NL})$.

6. Wave conjugation scheme

One special case of the four-wave mixing scheme is so-called wave conjugation. It is realized when

$$\mathbf{k}_s = -\mathbf{k}_3, \quad \mathbf{k}_1 = -\mathbf{k}_2, \quad \omega_s = \omega_1 = \omega_2 = \omega_3 = \omega.$$

In this case the scattered wave goes exactly in the opposite direction to the third wave, and that is where the name “conjugation” comes from (Fig.6). This scheme is widely used in nonlinear optics, and one finds for the scattered wave amplitude [11] $E_s(l) \sim i\xi(l\omega/c)(3+\cos^2\psi)E_1E_2E_3^*$, where $\xi \approx (\alpha/B_c)^2$, and l is the path of the scattered wave along four wave overlapping region.

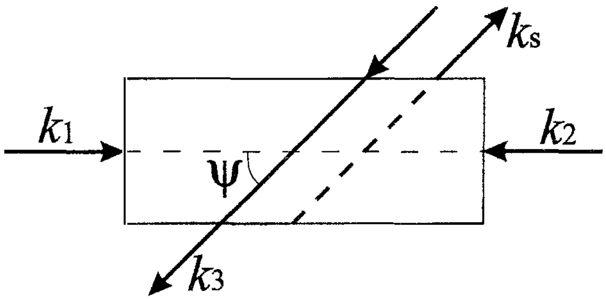

Рис. 6. Geometry of wave conjugation in vacuum.

For vacuum nonlinearity, it has been shown that if the intensities of the three laser pulses are equal $I_1 = I_2 = I_3 = \mathcal{E}/S\tau$, then the scattered energy, $\mathcal{E}_s$, is independent of pulse duration, τ, and an estimate of this energy has been obtained [11]. In the case of small angle of conjugation, $\psi \ll 1$, the scattered energy $\mathcal{E}_s \sim 10^{-4}c^2\mathcal{E}^3/(E_cS\lambda)^2$. For $\lambda \sim 1\mu$m and $S \sim (100\mu\text{m})^2$ one would need laser pulse energy $\mathcal{E} > 1$kJ to get a few scattered photons.

All of the optical manifestations of vacuum nonlinearity described above have been analyzed theoretically, but none of them is anywhere near experimental realization.

In the radio frequency and microwave radiation band, which we would denote MW range, there have been proposals to detect vacuum nonlinearity by use of waveguides [8]. As an eigenmode propagates along a waveguide, the nonlinear polarization induced by it gradually excites the third harmonic, and theoretically the latter can be measured, but estimates show that this would require vacuum tubes more than a million kilometers in length. Therefore, it is more practical to use a cavity instead of a waveguide.

7. Third harmonic generation in a high-Q cavity

If we take a cavity with high quality factor Q, in which the fundamental eigenmode at frequency Ω is excited to sufficiently large amplitude, the nonlinear vacuum polarization and magnetization will be oscillating at frequency 3Ω. If this is also an eigenfrequency of the cavity, then there is a resonance and this third mode will also grow, until it reaches some saturated level, $E_{3\Omega}$. To detect the third harmonic more easily, it is convenient to design the cavity in such a way that the fundamental mode has negligible amplitude in some part of the cavity, and hence does not interfere with measurements of the third harmonic. This can be accomplished by separating the cavity into two parts and connecting the parts with a filter waveguide for which the fundamental frequency of out cavity is below the cutoff (Fig.7 and also [8]).

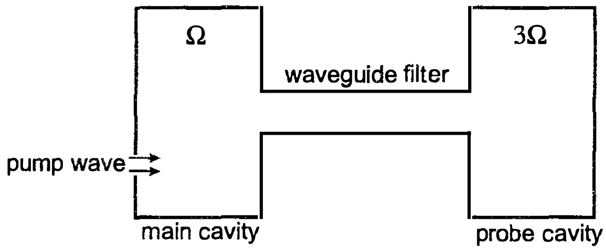

Рис. 7. MW cavity for third harmonic detection.

For the sake of simplicity, let us consider a cavity with very simple geometry, so all the computations can be carried out analytically. Namely, we choose a basic mode $m_1 n_1 0$ of a rectangular cavity (Fig.8).

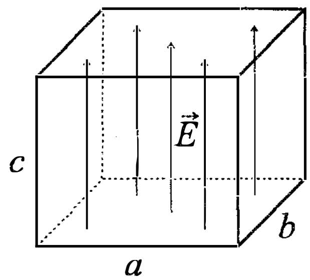

Рис. 8. Model rectangular cavity.

In this case it is also very simple to derive the condition that both fundamental and triple frequencies are eigenfrequencies of the cavity:

$$\frac{m_3^2}{a^2} + \frac{n_3^2}{b^2} = 9\left(\frac{m_1^2}{a^2} + \frac{n_1^2}{b^2}\right).$$

Stationary amplitude of the excited third harmonic, owing to the nonlinear sources P_3 and M_3, can be found by standard perturbation theory methods:

$$E_{3\Omega} = \frac{-4\pi \cdot 3k_1}{(9k_1^2 - k_3^2)} \left[\frac{3k_1 \int \mathbf{P}_3 \mathbf{E}^{*(m)} dV}{\int \mathbf{E}^{(m)} \mathbf{E}^{*(m)} dV} + \frac{k_3 \int \mathbf{M}_3 \mathbf{H}^{*(m)} dV}{\int \mathbf{H}^{(m)} \mathbf{H}^{*(m)} dV}\right],$$

where $E^{(m)}(\mathbf{r})$ and $H^{(m)}(\mathbf{r})$ are normalized fields of the excited mode. Taking into account the finite quality factor, Q, of the superconductive cavity at the frequency 3Ω, one finds $E_{3\Omega} \sim \xi E_{\Omega}^{3} Q$. Here and in the following estimates we take into account only the term with nonlinear polarization of vacuum, $\mathbf{P}_3$. The term with nonlinear magnetization is of the same order, and this two effects can in principle partially cancel each other, especially in paraxial schemes such as discussed below in sections 9 and 10.

This estimate obviously means that the more power we inject into the cavity the easier it is to detect the third harmonic, but in practice, metallic walls cannot withstand high electric fields, and the emitted electrons will spoil the vacuum. To prevent this negative effects, the electromagnetic fields should be kept under about 0.4MV/cm, or, equivalently, 1300 e.s.u.. So, we arrive at an estimate for the third harmonic amplitude which depends only on the quality factor:

$$E_{3\Omega} < 0.6 \cdot 10^{-22} \text{ e.s.u.} \cdot Q.$$

What quality factor do we need to be able to detect the nonlinear field? At least, this field should be above the thermal fluctuation level, which for typical cavity volume $V \sim 10^2$ cm^3 at liquid helium temperatures is

$$E_{3\Omega\text{therm}} \sim \sqrt{\frac{8\pi\kappa T}{2V}} \sim 0.8 \cdot 10^{-8} \text{ e.s.u..}$$

And to exceed that, we need quality factor $Q > 10^{14}$, which can hardly be achieved even in superconductive cavities, and demands unattainable stability of an MW oscillator.

8. Third harmonic generation in quasi-optical microwave resonator

To overcome the principal limitation of the closed cavity scheme, we can use a quasi-optical resonator, so the focusing of the beams gives much higher field amplitudes in the interaction region than the walls can tolerate. The third harmonic we are trying to detect will be excited in a smaller probe resonator aligned at an angle to the high-power resonator (Fig.9).

This alignment is convenient for detection of nonlinear field components, since the high-power radiation has no contact with cooled surfaces of the high-Q probe resonator, and the harmonic field detector is away from the strong electric field. To counter the positive effect of this geometry, note, however, that the interaction

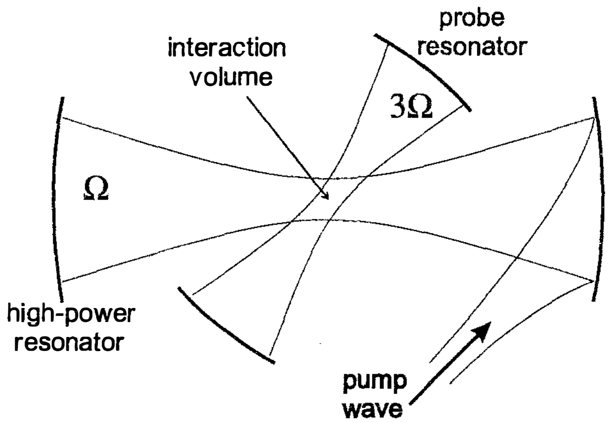

Рис. 9. Generation and detection of the third harmonic using quasi-optical resonators.

volume becomes much smaller than the resonator volume, while in the case of a cavity they were comparable.

Assuming the beams to be Gaussian and neglecting their divergence, we can obtain an analytical estimate of the saturated amplitude of the third harmonic. For perpendicularly aligned resonators the third harmonic amplitude is

$$E_{3\Omega} \sim \frac{\sqrt{\pi}}{8}\frac{Qa_3}{L_3}\frac{a_1^2}{a_3^2}\xi E_\Omega^3 \exp\left(\frac{-4a_1^2k_1^2}{3}\right),$$

where Q and L_3 are the quality factor and the length of the probe resonator, a_1 and a_3 radii of waists of quasi-optical beams at frequencies Ω and 3Ω, respectively. Decreasing the angle between resonator axes can improve the above estimate only by a few times, mostly due to $\exp(...) \to 1$. This result is of similar nature to the cavity estimate: the same third power of pump wave field, nonlinear constant and quality factor, which in this case comes in a combination that is independent of the length of resonator and characterizes the property of the material that the walls are made of. The rest are geometrical factors of order unity, except for the exponent. The latter reflects phase matching or mismatching between the modes of the two resonators, and also can be made close to unity.

With sensible mirror sizes (about a meter) of quasi-optical resonator and wavelength ~ 1 cm, the electric field in the interaction volume is bound by $E_\Omega \sim 4 \cdot 10^4$ e.s.u.. For $a_1^2/a_3^2 \sim 10$, $L_3/a_3 \sim 10^2$ we obtain

$$E_{3\Omega} \sim Q \cdot 1.6 \cdot 10^{-19} \exp(...) \text{ e.s.u..}$$

Thermal fluctuations in the probe resonator are

$$E_{3\Omega\text{therm}} \sim \sqrt{\frac{8\pi\kappa T}{2L_3 a_3^2}} \sim 3 \cdot 10^{-8} \text{ e.s.u.}.$$

To exceed thermal threshold we need the quality factor $Q > 10^{11}$, which can in principle be realized in superconductive resonators, but the requirements for the stability of an MW oscillator remain prohibitive.

9. Mixing of two MW waves with an optical wave

Since our chances for detection of vacuum nonlinearity in the radio or microwave frequency band do not look optimistic, let us consider combining it with laser radiation in the optical band. Suppose we have the same high-power quasi-optical MW resonator and we shine a laser beam through it. Then, nonlinear polarization and magnetization are induced in the interaction volume, and as they coherently oscillate, they produce radiation. The effect can be thought of as coherent scattering of the laser pulse on MW radiation, with a corresponding shift in frequency ($\pm 2\Omega$) and in direction of propagation (φ, Fig.10).

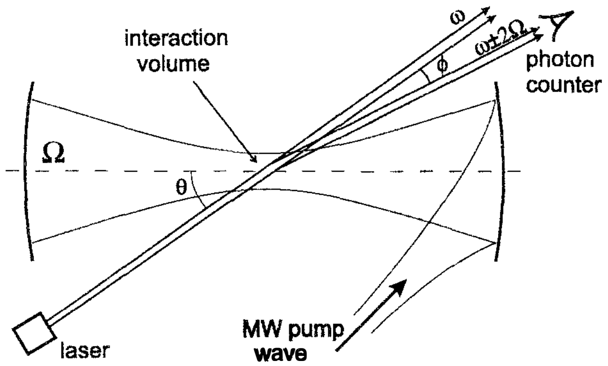

Рис. 10. Coherent scattering of a laser beam on an MW field in a quasi-optical resonator.

For this process to be efficient, the phase matching condition must be satisfied. If we think of it as four-wave mixing, the sum of wavevectors of three partial pump waves must be equal to the scattered wavevector. Exact phase matching condition, $\mathbf{k}+2\,\mathbf{k}_{\text{mw}} = |\mathbf{k}|+2\,|\mathbf{k}_{\text{mw}}|$, can only be fulfilled if the laser radiation propagates along the axis of the resonator, when vacuum polarization drops to zero. Competition of these two effects determines the optimal angle between the laser pulse and the resonator axis, $\theta \approx \sqrt{2/(k_{\rm mw}a_{\rm mw})}$, where $a_{\rm mw}$ is the radius of the waist of MW beam (see Figs. 10 and 11).

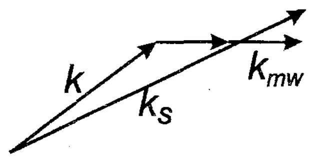

Рис. 11. Wavevector mismatch maximizing vacuum nonlinear effect for the geometry of Fig.10.

Maximum scattered power turns out to be proportional to the power of optical pump, P:

$$P_s \approx \left[\frac{\sqrt{\pi}}{16}\frac{k}{k_{\rm mw}}\xi E_\Omega^2 k_{\rm mw}a_{\rm mw}\frac{\exp\left(-k_{\rm mw}a_{\rm mw}/2\right)}{\sqrt{2k_{\rm mw}a_{\rm mw}}}\right]^2 P.$$

For $k/k_{\rm mw} \sim 10^4$, $k_{\rm mw}a_{\rm mw} \approx \pi$, $E_\Omega \sim 4 \cdot 10^4$ e.s.u. we obtain $P_s \sim 1.5 \cdot 10^{-38}$ P. The scattered photon beam is of similar optical quality as the pump beam and is deflected by a small angle, $\varphi \sim k_{\rm mw}/k$.

To be experimentally measurable, the scattered radiation must contain at least one photon (the energy of an optical photon is $\hbar\omega \sim 2 \cdot 10^{-19}$ J). The latter requires $\sim 10^{19}$ J of energy of the pump wave. This is hardly possible in laboratory conditions.

10. Mixing of two optical waves with an MW wave

As an improvement of the previous scheme, we can use two laser pulses that interact in the presence of an MW field. Again, this should produce scattered beams.

For the beam at frequency $2\omega - \Omega$ (Fig.12) we can fulfill the phase matching condition $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_s + \mathbf{k}_{\rm mw}$ by aligning the wavevectors as shown in the diagram of Fig.13, where the following condition is assumed:

$$c\left|\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_{\rm mw}\right| = 2\omega - \Omega \equiv c\left(|\mathbf{k}_1| + |\mathbf{k}_2| - |\mathbf{k}_{\rm mw}|\right).$$

In this scheme we have far greater freedom of choosing the angles, and for the sake of computational simplicity we will assume $\mathbf{k}_s \upharpoonleft\downharpoonright \mathbf{k}_{\rm mw}$ and $\theta_1 = \theta_2 = \theta$ (see Fig.12). Then the phase matching condition,

$$2k\cos\theta + k_{\rm mw} = 2k - k_{\rm mw},$$

determines the angle θ uniquely, $\theta \approx \sqrt{2k_{\rm mw}/k}$.

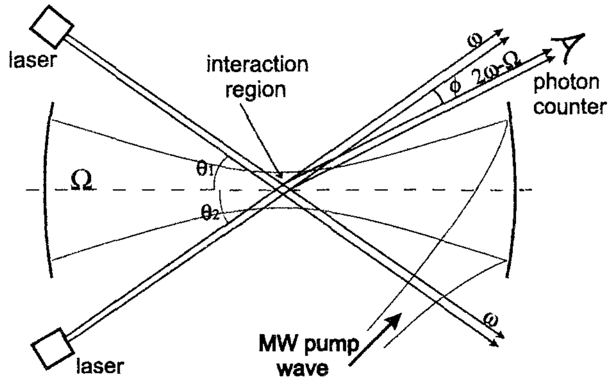

Рис. 12. Interaction of two beams with a standing electromagnetic wave.

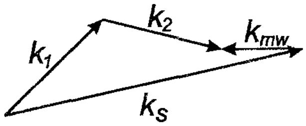

Рис. 13. Wavevector alignment for Fig.12.

Solving the corresponding electrodynamical problem we arrive at the following scattered power at frequency $2\omega - \Omega$:

$$P_s \approx \pi\,(\xi E_\Omega E_\omega)^2\,(k_{\rm mw}a_{\rm opt})^2\left(\frac{k}{k_{\rm mw}}\right)^3 P \approx 4\pi(\xi E_\Omega)^2\frac{k^3}{ck_{\rm mw}}P^2,$$

where both laser beams are assumed to have radius $a_{\rm opt}$ in the interaction region.

Now the scattered power is proportional to the second power of pump laser power, so, for a given pump energy we should make the laser pulse as short as possible. But it cannot be shorter than the period of MW oscillations or the bandwidth of the laser pulse will be greater than the MW frequency, the coherent effect of this oscillations will be lost, and the shift in the frequency and angle would be impossible to measure.

Finally, the constraint $P_s > P_{s\min} \sim 2\cdot 10^{-10}$ W should be fulfilled, which means that during the action of the laser pulse (~ 1 ns) at least one photon is scattered. Hence, with the same beam parameters as in the previous scenario (($k/k_{\rm mw}) \sim 10^4$, $(2\pi/k) \sim 1\mu$m , $E_\Omega \sim 4\cdot 10^4$ e.s.u.), we obtain the following requirement for the power of the optical pump:

$$P > \left[\frac{P_{s\min}}{4\pi(\xi E_\Omega)^2}\frac{ck_{\rm mw}}{k^3}\right]^{1/2} \sim 3\cdot 10^{16}\ {\rm W}.$$

The improvement over the previous non-resonant scheme is due to the numerical factor $\sim 10^4$, and two large factors $(k/k_{\rm mw}) \sim 10^4$ and $(E_\omega/E_\Omega)^2$. Nevertheless, the required energy in one nanosecond laser pulse $\sim 3 \cdot 10^7$ J, corresponding to electric field $E_\omega \gtrsim 3 \cdot 10^7$ e.s.u., is still beyond the capabilities of modern laser systems. Nanosecond lasers available now have energies in the kilojoule range, so an increase in energy of about three orders of magnitude is required.

However, the progress in laser optics and microwave electronics over the past two decades has been tremendous, and if its rate remains the same in the future, vacuum nonlinearity will probably be detected over the course of a few decades.

11. Acknowledgments

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GEODESICS ON THE PSEUDOSPHERE

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In the present contribution we gave an elementary technology for drawing the geodesics, paracycles and hypercycleson the pseudosphere.

1. Introduction

We have mentioned several times, that the Bolyai-Lobachevsky plane geometry doesn't materialize in the third dimensional space (theorem of D. Hilbert). A lot of mathematicians submit that the Bolyai-Lobachevsky plane geometry can be easily presented by the logical models (Pioncare-Klein, Cayley). Unfortunately this is not so, and it seems that, physicists accept this information more easily than mathematicians do.

It's a well known fact that the pseudosphere is one of the local models where the Bolyai-Lobachevsky plane geometry is materialized in little. But as we will see, right here on the pseudosphere we can present, that the model is indeed local. One surface is complete, if the geodesics are extendable in any orientation and in any length. Well, this doesn't materialize on the pseudosphere. Furthermore, we haven't seen any drawing that would present the hypercycle or the paracycle on the pseudosphere. We know that the paracycles must be congruent, namely every paracycle is congruent as well as the lines are. Consequently on the Bolyai-Lobachevsky plane geometry there exist a line and a paracycle ruler. So we can see that the paracycles are not congruent on the pseudosphere, so there isn't a paracycle ruler on them, because of this the way reproducing the Bolyai-Lobachevsky plane geometry is very distorted.

When we named as our target to draw the famed lines of the Bolyai-Lobachevsky plane geometry, on the pseudosphere we first looked for some special literature. We have found one remarkable book, whereof sure many don't know. We have worked from this book. We have calculated and illustrated these geodesics.

We can say that these geodesics haven't been illustrated correctly, probably because nobody troubled himself to draw the paracycles and hypercycles with hands and with the help of representative geometry design until now.

The differential equations of geodetics are:

$$\dot{u}\ddot{v} - \ddot{u}\dot{v} + C_{11}^{2}\dot{u}^{3} + \left(C_{12}^{2} - C_{11}^{1}\right)\dot{u}^{2}\dot{v} - \left(2C_{12}^{1} - C_{22}^{2}\right)\dot{v}^{2}\dot{u} - C_{22}^{1}\dot{v}^{3} = 0.$$

In case of pseudosphere:

$$ds^2 = \left(\frac{\cos u}{\sin u}\right)^2 du^2 + \sin^2 u dv^2,$$

$$C_{11}^{1} = \frac{-1}{\sin u \cos u}, C_{22}^{1} = \frac{-\sin^3 u}{\cos u}, C_{12}^{2} = \frac{\cos u}{\sin u},$$

$$\dot{u}\ddot{v} - \ddot{u}\dot{v} + 2\frac{\cos u}{\sin u}\dot{u}^{2}\dot{v} + \frac{1}{\sin u \cos u}\dot{u}^{2}\dot{v} + \frac{\sin^3 u}{\cos u}\dot{v}^{3} = 0.$$

This differential equation can be solved analytically, and the solutions can be represented in the pseudosphere. However, the figures of the geodesics also depend on the choise of two constants, and we don't obtain nice looking figures. One of the authors has already done the mathematical study of this in paper [3]. But the analytical approach of the paracycles and hipercycles is very difficult.

The differentiable geometry's equation of paracycle and hypercycle is very complicated, and may not be given in a closed form.

An elementary technology for drawing the geodesics, paracycles, and hypercycles is: In every case, first we represent the famed lines in the Poincare half-plane model, then we transform these lines inside of a circle, and from here we project them on the pseudosphere.

The steps of drawing the geodesics: First we draw the geodesic in the Poincare half-plane model, what is defined in the following way. Let e be a line in the Euclidean plane, and be named boundary line. For the points of the half-plane model we consider one of the half-plane's, which is defined by this boundary line. The geodesics are the half-circles which are range (located) in this half-plane and intersect in right angle the e boundary line, as well as the half-lines (figure 4.3) which are perpendicular on the e boundary line. Without loss of generality we can presume that the chosen e line is the ordinate and the center of the represented geodesic (half-circle) is in origin. We take the band between the lines $x = \pi$ and $x = -\pi$ (see figure 1).

If the radius of the half-circle's is bigger then π, then we project the half-circle symmetrically back regarding the lines $x = \pi$ and $x = -\pi$, until every part of the half-circle will be in the $[-\pi, \pi]$ band (see figure 2).

This is the following:

$$\text{if } x < -\pi \text{ then } x = -2 * \pi - x,$$
$$\text{if } x > \pi \text{ then } x = 2 * \pi - x.$$

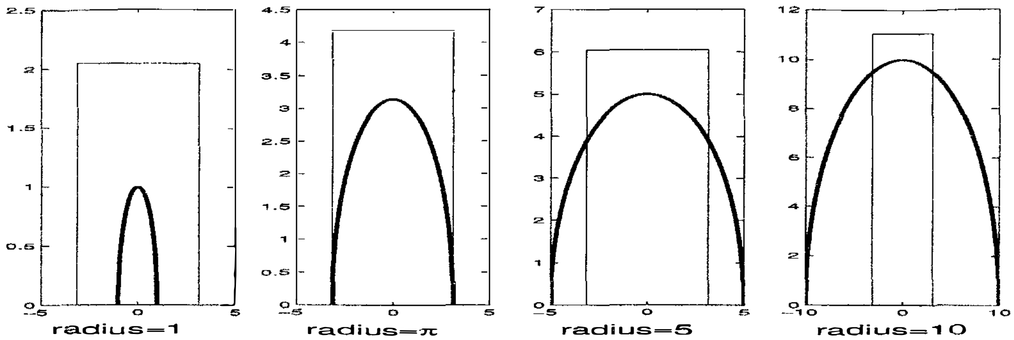

Рис. 1. Geodesics in the Poincare half-plane.

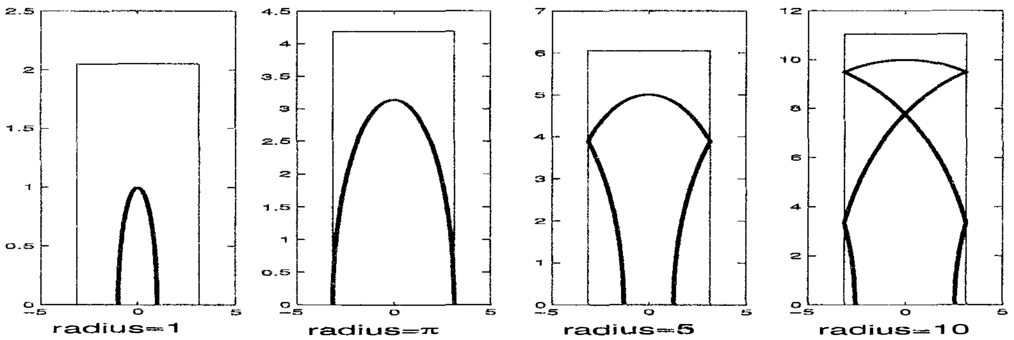

Рис. 2. Geodesics in the $[-\pi,\pi]$ band.

The next step is the transformation of the obtained curve inside of one circle. We transform the strip between the lines $y=0$ and $y=r+1$ of the band given by the Euclidean $x=\pi$ and $x=-\pi$ lines (the rectangle obtained in this way contains the whole curve) into one circle, where r is the radius of the half-circle. We shrink the line segment between the points $(-\pi, r+1)$ and $(\pi, r+1)$ of the obtained rectangle into one point, which will be the center of the circle. We elongate the line segment between $(-\pi, 0)$ and $(\pi, 0)$ to the length of $2\pi(r+1)$ and we join the ends of them, to obtain a circle (see figure 3).

Mathematical we obtain this with the

$$\begin{cases} x_2 = (r+1-y)\cdot\cos x, \\ y_2 = (r+1-y)\cdot\sin x \end{cases}$$

replacement, where the x, y are the drawing's curve.

We only have to project the points of the curve inside the circle into the pseudo-

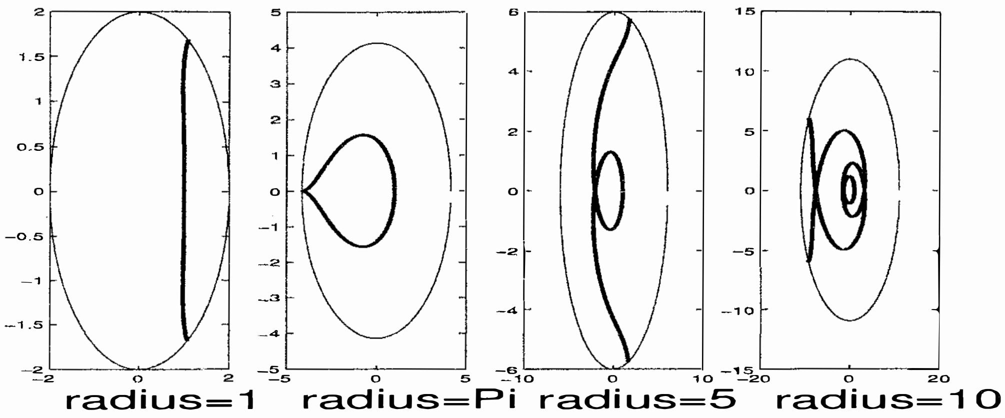

Рис. 3. Geodesics in the circle.

sphere (see figure 4), knowing the formula of the pseudosphere

$$\begin{cases} x = ru\cos(v), \\ y = ru\sin(v), \\ z = r\log\frac{r+\sqrt{r^2-(y^2+x^2)}}{\sqrt{y^2+x^2}} - \sqrt{r^2-(y^2+x^2)}. \end{cases}$$

where u is the height (altitude) of the parallel circles, and v is the angel inside of these parallel circles which value is in $[-\pi, \pi]$ interval.

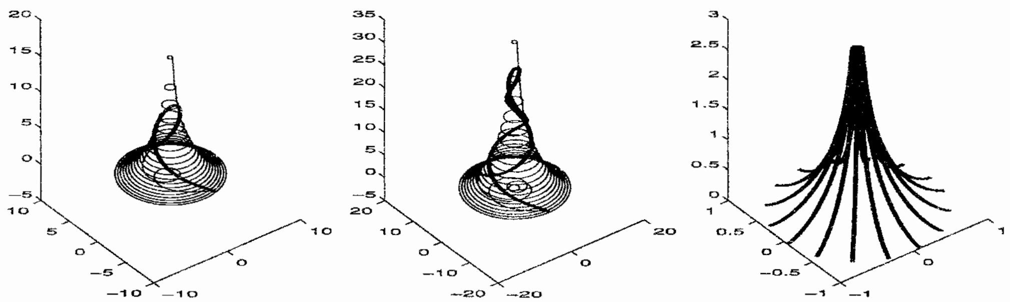

Рис. 4. Geodesic on the pseudosphere.

The course of drawing the paracycles is similar to that of the geodesics. The paracycles in the Poincare half-plane model are the circles which contact the boundary line, respectively the parallel lines with the e boundary line. The drawing of the paracycles which are parallel to the boundary line is not a problem because these are the parallel circles on the pseudosphere (figure 6.3). Thus, the drawing of others paracycles: let consider those specific circles which have the center on the line that is perpendicular on the boundary line in the origin. We wedge the paracycles into the $[-\pi, \pi]$ band as same as in case of the drawing of geodesics. Henceforth we proceed the same way as we have described above (see figures 5, 6).

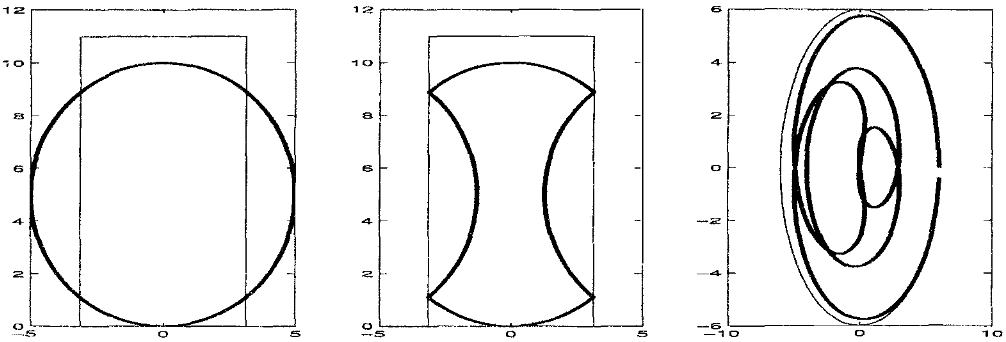

Рис. 5. Paracycle in the Poincare half-plane for R=5.

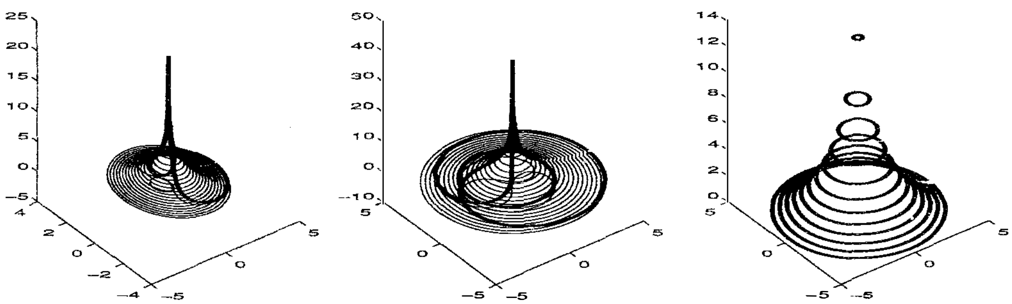

Рис. 6. Paracycle on the pseudosphere.

The drawing of the hypercycle. Let be one line in the Poincare half-plane and we draw the curve which is at distance 1. In the course of this draft we take the lines (half-circles) in sense of Poincare which are perpendicular to our line (half-circle). The points in these lines which are at distance 1 at the given line constitute the wanted curve. In this logical model the distance, metrics is defined by

$$D(A,B) = k \log(ABVU)$$

formula. Here k is one arbitrary positive constant, A, B are two points on the half-circle (line) and V, U are the intersection points of the half-circles and the boundary line, with $(ABVU)$ we sign the cross ratio

$$(ABVU) = \frac{AV}{BV} \cdot \frac{BU}{AU}.$$

The algorithm for finding the points which are at distance 1 to the given line:

$$rad = r\tan(o - 3\tfrac{\pi}{2});$$
$$kp = -\sqrt{r^2 + rad^2};$$

rad is the radius and $(0, kp)$ is the center of the half-circle which is perpendicular on the given line (half-circle) at point which location is in angel o.

$x = rx; y = ry;$
$dist = 0;$
while $(dist - 1 <= 0)$ do
begin
$dist = \frac{1}{2}\log\frac{((kp-rad-x)^2+y^2)((kp+rad-rx)^2+ry^2)}{((kp+rad-x)^2+y^2)((kp-rad-rx)^2+ry^2)};$
$fi = fi - \frac{\pi}{1000};$
$x = kp + rad * sin(fi);$
$y = rad * cos(fi);$
end
$sx = x;$
$sx = y.$

Here (rx, ry) is the point of the given half-circle at the angle o, the (x, y) is the point of the half-circle which is perpendicular on the given half-circle at point (rx, ry), and the (sx, sy) is the point at distance 1 at (rx, ry).

Henceforward the wedge in $[-\pi, \pi]$ band, the transformation into the circle and the projection on the pseudosphere are the same as in the previous cases. The obtained curve is on figure 8.

2. Conclusions

We can assure about the correctness of the geodesic's drawings by the evocation of theorem of Clairaut. (A. Clairaut (1713-1765) wrote the first study about space curve's: Traité des courbes ' a la double courbure, 1731). According Clairaut's theorem the product of the radius of parallel circle and the cosine of angle between

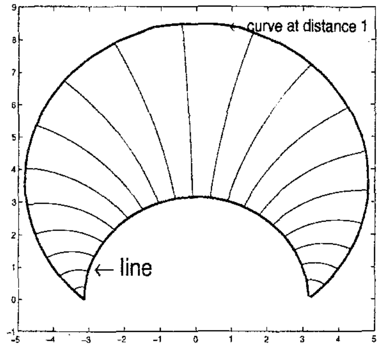

Рис. 7. Hypercycle in the Poincare half-plane.

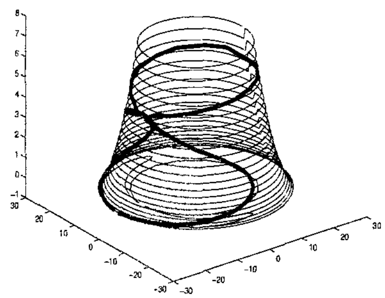

Рис. 8. Hypercycle on the pseudosphere.

geodesics and parallel circles is constant in case of surface of revolution. Based on this theorem every meridian curve is geodesic, since the meridian curves are perpendicular to the parallel circles, so this way the cosine of angle is zero and the constant in the theorem is zero too and independent of the radius. It's easy to educe that in case of the cylinder the parallel circles are geodesics. So in the case of the pseudosphere the meridian curves are geodesics, moreover even those curves that satisfy Clairaut's theorem are geodesics too. Namely they wrap gradually around the pseudosphere until they reach oscillatory one parallel circle, because then $\cos 0 = 1$ so the prescribed constant is the radius of the reached parallel circle.

With the exemplification of the famed lines on the pseudosphere we wanted to prove, that the Bolyai-Lobachevsky plane geometry in the 3-dimensional Euclidean space doesn't materialize physically.

3. Acknowledgments

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Рукописи Яноша Бояи о Лобачевском

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Рассказывается о малоизвестных в России рукописях Яноша Бояи.

Многие русские учёные уверены (см., например, [1]), что Янош Бояи ревновал к Лобачевскому. В русской научной литературе даже распространено мнение, что Бояи обвинял Гаусса в переработке и издании его труда под псевдонимом "Лобачевский". Верно, Янош Бояи пережил и такое волнение, но это было до того, как он прочитал работы Лобачевского на немецком языке. К сожалению, это ему удалось очень поздно, только в 1848 году. После знакомства с работами Лобачевского Янош Бояи полностью изменил своё мнение и стал думать и писать о Лобачевском только с величайшим признанием и уважением. Приведем доказательства этого.

Бояи написал прекрасную работу "Замечания о работе Лобачевского"(40 рукописных страниц в [2]). Кроме этого, есть много маленьких записок, в которых Бояи пишет о Лобачевском с величайшим почтением и ставит себе цель приобрести все работы Лобачевского. На рис. 1 приведена копия подлинного документа, текст которого следующий: "Приобрести все работы Лобачевского из Казани".

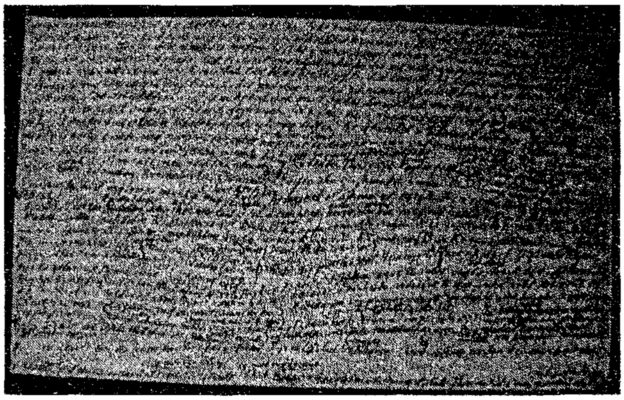

Fig. 1. "Приобрести все работы Лобачевского из Казани"

Янош Бояи писал свои заметки и труды в области математики на венгерском, латинском или немецком языках. Интересно, что наиболее трудно разбираются его рукописи

[4]Венгерское название - Коложвар.
[5]Венгерское название - Чиксереда.

на венгерском языке, поскольку он создал и использовал специальные знаки и буквы. Его немецкие рукописи написаны готическими и латинскими буквами, а наиболее легко читаются рукописи на латыни, поскольку они являются чистовыми экземплярами готовых математических работ. У Бояи есть много похвальных замечаний о Лобачевском, самым выразительным из них представляется следующее: "Пока в России живут такие красивые, добрые и стремящие к благородству души, как Лобачевский, есть повод и основание надеяться на высшее просвещении России."(BJ-93/1; см. Рис.2).

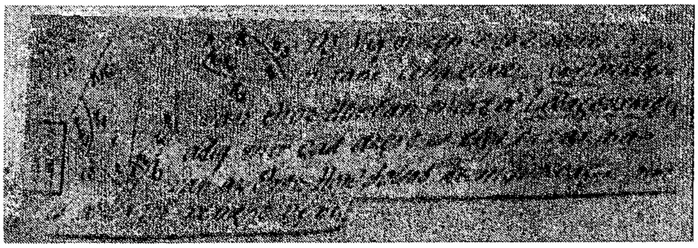

Fig. 2. Лестное мнение Яноша Бояи о Лобачевском

Нужно знать, что в Марошвашархе е[6] сохранены 14000 рукописных страниц Яноша Бояи, которые содержают его самые личные исповеди: у Яноша не было сочувственной жены или друга, поэтому обычно ему оставалась только бумага, чтобы говорить о своих самых личных мыслях и чувствах. В Интернете [3] доступна полная переписка Фаркаша Бояи[7] с Гауссом, которая свидетельствует о том, как поздно узнали оба Бояи о Лобачевском. Здесь обратим внимание только на два письма. Первое из них - письмо Гаусса Фаркашу Бояи, второе - от Фаркаша Бояи к Гауссу. Письмо Фаркаша Бояи интересно тем, что он пишет: "В арифметике я опирался на Ньютона, геометрию начал с Лобачевским. Не было у меня счастья, чтобы стать открывателем дорог. . . "Так что и Фаркаш Бояи понял и принял труд Лобачевского в полной его глубине. Это является и ответом на мнение, согласно которому Фаркаш Бояи не понял якобы сущность неевклидовой геометрии и её эпохальную роль. Если он "геометрию начал с Лобачевским", то у него должны были быть довольно современные взгляды. И он провёл параллель с Ньютоном! В 1853 очень мало математиков строили свою геометрическую учебную систему на Лобачевском. Возможно, только один Фаркаш Бояи? Мы хотели бы, чтобы наши русские коллеги и друзья поняли, что Янош Бояи был единомышленник, духовный друг и поклонник Лобачевского. В Нижнем Новгороде мы почувствовали большой интерес и сердечность со стороны русских коллег при наших докладах о Бояи, поэтому мы поставили перед собой цель разобрать и другие рукописи Яноша Бояи, касающиеся Лобачевского, и издать их в переводе на русский язык.

[6]Официальное (румынское) название - Тыргу Муреш.

[7]Фаркаш Бояи (отец Яноша Бояи) был тоже математик, преподователь в Реформатском Лицее г.Марошвашархей. Эпохальная работа Яноша Бояи появилась как приложение к книге Тентамен (Tentamen) Фаркаша Бояи.

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János Bolyai's manuscripts about N.I. Lobachevsky

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About little-known in Russia János Bolyai's manuscripts.

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Представление Лакса нелинейных σ-моделей с приводимыми метриками

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Предложен способ, позволяющий ассоциировать с каждым симметрическим пространством G/H представление Лакса некоторой системы дифференциальных уравнений. Получены достаточные условия, при которых данная конструкция приводит к лагранжевым системам. Особо выделен случай симметрических пространств вида $G/(H_1 \times H_2)$. Рассмотрены примеры допускающих представление Лакса систем с приводимыми метриками.

1. Введение

Рассматриваются системы дифференциальных уравнений в частных производных

$$U^a_{xy} + \Gamma^a_{bc} U^b_x U^c_y + Q^a = 0, \tag{1.1}$$

где x,y — независимые переменные; Γ^a_{bc}, Q^a — некоторые гладкие функции от $U^1, U^2, ..., U^n$; индексы принимают значения от 1 до n. Системы такого вида называются общими нелинейными σ-моделями или системами кирального типа, если они принадлежат классу лагранжевых систем, т.е. являются системами уравнений Эйлера-Лагранжа для лагранжианов

$$L = p_{ab} U^a_x U^b_y + Q, \tag{1.2}$$

где p_{ab}, Q — гладкие функции от $U^1, U^2, ..., U^n$ и симметрическая часть матрицы $\|p_{ab}\|$ невырожденная. Если $p_{[ab]} = 0$, то p_{ab} можно рассматривать как метрический тензор некоторого риманова многообразия V^n, заданный в локальной системе координат $U^1, ..., U^n$. При этом коэффициенты Γ^a_{bc} в системе (2.1) совпадают с символами Кристоффеля связности на V^n, согласованной с метрикой p_{ab}.

Будем называть систему (2.1) системой с приводимой метрикой, если в некоторой локальной системе координт координаты можно разделить на две группы $(U^\alpha, U^{\alpha'})$ так, что лагранжиан примет вид

$$L = p_{\alpha\beta}(U^\gamma) dU^\alpha dU^\beta + p_{\alpha'\beta'}(U^{\gamma'}) dU^{\alpha'} dU^{\beta'} + Q.$$

Здесь и далее не предполагаются выполненными условия $p_{[\alpha\beta]} = 0$, $p_{[\alpha'\beta']} = 0$.

Пусть Θ^i — левоинвариантные дифференциальные формы некоторой группы Ли G, удовлетворяющие структурным уравнениям Маурера-Картана

$$d\Theta^i = \frac{1}{2} C^i_{jk} \Theta^k \wedge \Theta^j, \tag{1.3}$$

где C^i_{jk} — структурные константы алгебры Ли $\mathbf{g}$ группы G, индексы i, j, k принимают значения от 1 до r, $(r \geq n)$. Говорят, что система (2.1) допускает представление нулевой кривизны или представление Лакса с группой Ли G, если существуют функции A^i, B^i, зависящие от U^a, их производных $U^a_x, U^a_y, U^a_{xx}, \ldots$ и некоторого параметра λ, такие, что система уравнений

$$\Theta^i = A^i dx + B^i dy \tag{1.4}$$

является вполне интегрируемой (в смысле Фробениуса) на решениях системы (2.1), т.е., если подстановка соотношений (1.4) в уравнения (1.3) приводит к тождествам в силу системы (2.1).

В [1] указан класс систем, допускающих представление Лакса с группами Ли, структурные уравнения которых могут быть записаны в виде

$$d\omega^A = D^A_{Ba}\theta^a \wedge \omega^B, \tag{1.5}$$

$$d\theta^a = C^a_{bc}\theta^c \wedge \theta^b + R^a_{BC}\omega^C \wedge \omega^B, \tag{1.6}$$

где $a, b, c = \overline{1, n}$, $A, B = \overline{n+1, n+r}$; θ^a, ω^A — левоинвариантные дифференциальные формы группы G. Из тождеств Якоби для группы G следует, что структурные константы удовлетворяют условиям

$$D^A_{Ca}C^a_{bc} + D^A_{B[b}D^B_{|C|c]} = 0, \tag{1.7}$$

$$C^a_{bc}R^c_{BD} + R^a_{[B|C|}D^C_{D]b} = 0, \tag{1.8}$$

$$D^A_{[B|a|}R^a_{DC]} = 0. \tag{1.9}$$

Уравнения (2.4), (2.5) можно также интерпретировать как структурные уравнения некоторого локально симметрического пространства G/H, где C^a_{bc} — структурные константы группы изотропии H. Справедлива теорема (см. [1]):

Теорема 1.1.. *Пусть C^a_{bc} — структурные константы группы изотропии H некоторого локально симметрического пространства G/H, структурные уравнения которого имеют вид* (2.4), (2.5). *Предположим, что существуют матрицы $\|T^{\ a}_{1b}\|$, $\|T^{\ a}_{2b}\|$ и функции M^A, N^A, удовлетворяющие условиям*

$$T^{\ a}_{i[b,c]} = C^a_{de}T^{\ d}_{ib}T^{\ e}_{ic} \quad (i = 1, 2), \tag{1.10}$$

$$det\|T^{\ a}_{1b} - T^{\ a}_{2b}\| \neq 0, \tag{1.11}$$

$$M^A_{,b} = D^A_{Ba}T^{\ a}_{2b}M^B, \tag{1.12}$$

$$N^A_{,b} = D^A_{Ba}T^{\ a}_{1b}N^B. \tag{1.13}$$

Тогда система (2.1) *допускает представление Лакса, если функции Γ^a_{bc}, Q^a имеют вид*

$$\Gamma^a_{bc} = \tilde{P}^a_d[P^d_{(b,c)} + 2C^d_{fg}S^f_bS^g_c - 2C^d_{fg}P^g_{(c}S^f_{b)}],$$

$$Q^a = -\tilde{P}^a_f R^f_{AB} N^B M^A,$$

где $S^a_b = \frac{1}{2}(T_{1b}^{\ a} + T_{2b}^{\ a})$, $P^a_b = \frac{1}{2}(T_{1b}^{\ a} - T_{2b}^{\ a})$, $\tilde{P}$ *— обозначение матрицы, обратной к* P .

Замечание. С каждым локально симметрическим пространством G/H можно ассоциировать систему вида (2.1), допускающую представление Лакса. Действительно, пусть C^a_{bc} — структурные константы группы H относительно базиса левоинвариантных дифференциальных форм $\Psi^a = T^a_b dU^b$. Выберем $T_{1b}^{\ a} = T^a_b$, $T_{2b}^{\ a} = 0$ (или $T_{1b}^{\ a} = 0$, $T_{2b}^{\ a} = T^a_b$). Тогда условия (2.8), (2.9) выполнены и системы (1.12), (2.10) совместны в силу (2.8), (1.7). Полагая

$$\omega^A = \lambda M^A dx + \frac{1}{\lambda} N^A dy, \tag{1.14}$$

$$\theta^a = T_{1b}^{\ a} U^b_x dx + T_{2b}^{\ a} U^b_y dy, \tag{1.15}$$

получим представление Лакса системы (2.1), коэффициенты которой имеют вид, указанный в теореме. Однако, не всегда полученная таким образом система принадлежит классу лагранжевых систем.

2. Представление Лакса лагранжевых систем

Как известно (см., например, [2]), каждое симметрическое пространство G/H порождает симметрическую алгебру Ли $(\mathbf{g},\ \mathbf{h},\ \xi)$, где $\mathbf{g}, \mathbf{h}$ — алгебры Ли групп G и H соответственно, ξ — инволютивный автоморфизм для $\mathbf{g}$. Всегда существует разложение

$$\mathbf{g} = \mathbf{h} \oplus \mathbf{m}$$

на собственные подпространства инволютивного автоморфизма ξ, которое называется каноническим разложением симметрической алгебры $(\mathbf{g},\ \mathbf{h},\ \xi)$, причем выполняются следующие включения

$$[\mathbf{h}, \mathbf{h}] \subset \mathbf{h}, \quad [\mathbf{h}, \mathbf{m}] \subset \mathbf{m}, \quad [\mathbf{m}, \mathbf{m}] \subset \mathbf{h}.$$

В сформулированной ниже теореме указывется условие на $\mathbf{h}$ и $\mathbf{m}$, выполнение которого гарантирует, что с данным симметрическим пространством можно ассоциировать представление Лакса лагранжевой системы.

Далее используются следующие обозначения:

1) (U^a) — локальная координатная система на группе H ;

2) C^a_{bc} — структурные константы группы Ли H относительно базиса левоинвариантных дифференциальных форм $\Psi^a = T^a_b dU^b$;

3) h^0_{ab} — форма Киллинга алгебры $\mathbf{h}$, заданная в двойственном базисе;

4) g_{ab} — метрика Киллинга группы H , т.е.

$$g_{ab} = h^0_{cd} T^c_a T^d_b;$$

5) $\sigma = a_{ab} dU^a \wedge dU^b$ — 2-форма, удовлетворяющая условию

$$d\sigma = \frac{2}{3} h^0_{ab} C^b_{cd} \Psi^a \wedge \Psi^d \wedge \Psi^c.$$

Теорема 2.1.. *Пусть G/H — локально симметрическое пространство с полупростой группой Ли H и структурными уравнениями* (2.4), (2.5). *Пусть $\mathbf{g} = \mathbf{h} \oplus \mathbf{m}$ — каноническое разложение и для любых элементов $h_i \in \mathbf{h}$, $m_i \in \mathbf{m}$ $(i = 1, 2)$ справедливо равенство*

$$([[m_1, h_1], m_2], h_2) = ([[m_2, h_2], m_1], h_1),$$

где круглые скобки обозначают скалярное произведение в $\mathbf{h}$ относительно метрики Киллинга h^0 .

Тогда существует функция Q такая, что система уравнений Эйлера-Лагранжа для лагранжиана

$$L = g_{ab} U^a_x U^b_y + a_{ab} U^a_x U^b_y + Q(U^a)$$

допускает представление Лакса.

Замечание. Непосредственная проверка показывает, что симметрические пространства AI: $SU(n)/SO(n)$, DIII: $SO(2n)/U(n)$, CI: $Sp(n)/U(n)$ удовлетворяют условиям теоремы.

3. Представление Лакса систем с приводимыми метриками

Чтобы получить представления Лакса нелинейных σ-моделей с приводимыми метриками, рассмотрим симметрические пространства вида $G/(H_1 \times H_2)$. Разделим координаты (U^a) на две группы $(U^\alpha, U^{\alpha'})$ в соответствии с разложением группы $H = H_1 \times H_2$. Пусть
1) $C^\alpha_{\beta\gamma}, C^{\alpha'}_{\beta'\gamma'}$ — структурные константы групп H_1, H_2 относительно базисов левоинвариантных дифференциальных форм $\Phi^\alpha = T^\alpha_\beta dU^\beta, \Phi^{\alpha'} = T^{\alpha'}_{\beta'} dU^{\beta'}$;
2) $h^0_{\alpha\beta}, h^0_{\alpha'\beta'}$ — формы Киллинга алгебр Ли $\mathbf{h}_1, \mathbf{h}_2$;
3) $g_{\alpha\beta}, g_{\alpha'\beta'}$ — метрики Киллинга групп H_1, H_2 ;
4) $\delta = a_{\alpha\beta} dU^\alpha \wedge dU^\beta$, $\delta' = a_{\alpha'\beta'} dU^{\alpha'} \wedge dU^{\beta'}$ — 2-формы, удовлетворяющие условиям

$$d\delta = \frac{2}{3} h^0_{\alpha\beta} C^\alpha_{\gamma\delta} \Phi^\alpha \wedge \Phi^\delta \wedge \Phi^\gamma, \; d\delta' = \frac{2}{3} h^0_{\alpha'\beta'} C^{\alpha'}_{\gamma'\delta'} \Phi^{\alpha'} \wedge \Phi^{\delta'} \wedge \Phi^{\gamma'}.$$

Теорема 3.1.. *Пусть G/H — локально симметрическое пространство, $H = H_1 \times H_2$, H_1, H_2 — полупростые группы Ли. И пусть для любых элементов $h_i^{'}, h_i^{''} \in \mathbf{h}_i$, $m_i \in \mathbf{m}$ $(i = 1, 2)$ справедливы равенства*

$$1)\ \ ([[m_1, h_i^{'}], m_2], h_i^{''}) = ([[m_2, h_i^{''}], m_1], h_i^{'}),$$

$$2)\ \ R([[m_1, h_1^{'}], m_2], h_2^{'}) = S([[m_2, h_2^{'}], m_1], h_1^{'}),$$

где R, S — фиксированные ненулевые константы, а круглые скобки обозначают скалярное произведение в $\mathbf{h}$ относительно метрики Киллинга h^0.

Тогда существует функция $Q(U^{\alpha}, U^{\alpha'})$ такая, что системы уравнений Эйлера-Лагранжа для лагранжианов

$$L = S[g_{\alpha\beta}(U^{\gamma}) + a_{\alpha\beta}(U^{\gamma})]U_x^{\alpha}U_y^{\beta} + R[g_{\alpha'\beta'}(U^{\gamma'}) + \varepsilon a_{\alpha'\beta'}(U^{\gamma'})]U_x^{\alpha'}U_y^{\beta'} + Q,$$

$$\varepsilon = \pm 1,$$

допускают представления Лакса.

Замечание. Непосредственная проверка показывает, что симметрические пространства

BDI: $SO(p+q)/(SO(p) \times SO(q))$, $p, q \geq 3$, CII: $Sp(p+q)/(Sp(p) \times Sp(q))$, G: $G_2/(SO(3) \times SO(3))$ удовлетворяют условиям теоремы.

4. Примеры

Рассмотрим примеры построений представлений Лакса систем, ассоциированных с симметрическими пространствами вида $SO(p+q)/(SO(p) \times SO(q))$ $(p, q \geq 3)$.

Пример 1.($SO(6)/(SO(3) \times SO(3))$)

Выберем в качестве локальных координат U^1, U^2, U^3 на группе $H_1 = SO(3)$ углы Эйлера. Тогда

$$\theta^1 = -\cos U^2 dU^3 - \sin U^2 \sin U^3 dU^1,$$

$$\theta^2 = \sin U^3 \cos U^2 dU^1 - \sin U^2 dU^3,$$

$$\theta^3 = -\cos U^3 dU^1 - dU^2$$

— базис левоинвариантных дифференциальных форм и

$$d\theta^1 = \theta^3 \wedge \theta^2, \quad \theta^2 = \theta^1 \wedge \theta^3, \quad \theta^3 = \theta^2 \wedge \theta^1.$$

— структурные уравнения группы $SO(3)$. Локальные координаты U^4, U^5, U^6 и левоинвариантные формы $\theta^4, \theta^5, \theta^6$ на $H_2 = SO(3)$ выберем аналогичным образом.

Запишем структурные уравнения симметрического пространства $SO(6)/(SO(3)\times SO(3))$ в виде

$$d\Omega=\Omega\wedge\Omega,\ \Omega=\left\|\begin{array}{cc}\theta^{\alpha}_{\beta} & \omega^{\alpha}_{\beta'}\\ \omega^{\alpha'}_{\beta} & \theta^{\alpha'}_{\beta'}\end{array}\right\|, \tag{4.1}$$

где

$$\theta^{\alpha}_{\beta}=-\theta^{\beta}_{\alpha},\theta^{\alpha'}_{\beta'}=-\theta^{\beta'}_{\alpha'},\quad \omega^{\alpha'}_{\beta}=-\omega^{\beta}_{\alpha'}\ (\alpha,\beta=\overline{1,3},\ \alpha',\beta'=\overline{4,6}).$$

Положим

$$\omega^{\alpha}_{\alpha'}=\lambda M^{\alpha}_{\alpha'}dx+\frac{1}{\lambda}N^{\alpha}_{\alpha'}dy,$$

$$\theta^1_2=(-\cos U^3U^1_y-U^2_y)dy,\ \theta^1_3=(-\sin U^3\cos U^2U^1_y+\sin U^2U^3_y)dy,$$

$$\theta^2_3=(-\cos U^2U^3_y-\sin U^2\sin U^3U^1_y)dy,\ \theta^4_5=(-\cos U^6U^4_x-U^5_x)dx, \tag{4.2}$$

$$\theta^4_6=(-\sin U^6\cos U^5U^4_x+\sin U^5U^6_x)dx,\ \theta^5_6=(-\cos U^5U^6_x-\sin U^5\sin U^6U^4_x)dx.$$

Подставляя выражения (4.2) в (4.1), получим систему уравнений для определения функций $M^{\alpha}_{\alpha'},N^{\alpha}_{\alpha'}$. В простейшем случае эти функции можно выбрать в следующем виде: $M^1_6=\sin U^2\sin U^3$, $M^2_6=-\cos U^2\sin U^3$, $M^3_6=\cos U^3$, $N^3_4=n\sin U^2\sin U^3$, $N^3_5=-n\cos U^5\sin U^6$, $N^3_6=n\cos U^6$, $n=const$, а остальные положить равными нулю. Тогда выражения (4.2) определяют представление Лакса системы уравнений Эйлера для лагранжиана

$$L_1=\sum_{a=1}^{3}U^a_xU^a_y+2\cos U^3U^1_yU^2_x+\sum_{a'=4}^{6}U^{a'}_xU^{a'}_y+$$

$$+\cos U^6(U^4_yU^5_x+U^4_xU^5_y)+\varepsilon\cos U^6(U^4_yU^5_x-U^4_xU^5_y)+2n\cos U^3\cos U^6,$$

где $\varepsilon=-1$.

Пример 2.(Двойной комплексный sin-Gordon)

Система уравнений Эйлера для лагранжиана

$$L_2=V^1_xV^1_ytg^2\frac{V^2}{2}+V^2_xV^2_y+V^3_xV^3_ytg^2\frac{V^4}{2}+V^4_xV^4_y+2n\cos V^2\cos V^4\quad(n=const)$$

получена при помощи редукции и преобразований Бэклунда из системы с лагранжианом L_1. Ее можно рассматривать как "двойной комплексный sin-Gordon", поскольку подстановка $V^1=V^3,V^2=V^4$ приводит к обычному комплексному sin-Gordon. Данная система также допускает представление Лакса вида (4.1), где

$$\theta^1_2=\frac{1}{2\cos\frac{V^2}{2}}(V^1_xdx+V^1_ydy),\ \theta^1_3=-\frac{\sin\frac{V^2}{2}}{2\cos^2\frac{V^2}{2}}(V^1_xdx-V^1_ydy),$$

$$\theta_3^2 = \frac{1}{2}(V_x^2 dx - V_y^2 dy), \ \theta_5^4 = \frac{1}{2\cos\frac{V^4}{2}}(V_x^3 dx + V_y^3 dy),$$

$$\theta_6^4 = -\frac{\sin\frac{V^4}{2}}{2\cos^2\frac{V^4}{2}}(V_x^3 dx - V_y^3 dy), \ \theta_6^5 = \frac{1}{2}(V_x^4 dx - V_y^4 dy),$$

$$\omega_{\alpha'}^{\alpha} = \lambda M_{\alpha'}^{\alpha} dx + \frac{1}{\lambda} N_{\alpha'}^{\alpha} dy,$$

$$M_4^{\alpha} = M_{\alpha'}^1 = 0, \quad N_4^{\alpha} = N_{\alpha'}^1 = 0,$$

$$M_5^2 = \sin\frac{V^2}{2}\sin\frac{V^4}{2}, \ M_6^2 = -\sin\frac{V^2}{2}\cos\frac{V^4}{2},$$

$$M_5^3 = -\cos\frac{V^2}{2}\sin\frac{V^4}{2}, \ M_6^3 = \cos\frac{V^2}{2}\cos\frac{V^4}{2},$$

$$N_5^2 = n\sin\frac{V^2}{2}\sin\frac{V^4}{2}, \ N_6^2 = n\sin\frac{V^2}{2}\cos\frac{V^4}{2},$$

$$N_5^3 = n\cos\frac{V^2}{2}\sin\frac{V^4}{2}, \ N_6^3 = n\cos\frac{V^2}{2}\cos\frac{V^4}{2}.$$

Пример 3.($SO(p+3)/(SO(p)\times SO(3))$)

Доказано (см. [3]), что с каждым симметрическим пространством $SO(p+3)/(SO(p)\times SO(3))$ можно ассоциировать представление Лакса системы уравнений Эйлера для лагранжиана

$$L_3 = g_{ab}U_x^a U_y^b + a_{ab}U_x^a U_y^b - 2(p-2)(V_x^1 V_y^1 tg^2\frac{V^2}{2} + V_x^2 V_y^2) + Q, \quad a,b = \overline{1, \frac{p(p-1)}{2}},$$

где Q — некоторая гладкая функция от U^a, V^1, V^2 .

Пример 4.($SO(p+2)/(SO(p)\times SO(2))$)

Для каждого из симметрических пространств $SO(p+2)/(SO(p)\times SO(2))$ существует функция $Q(U^a, V)$ такая, что система уравнений Эйлера-Лагранжа для лагранжиана

$$L_4 = g_{ab}U_x^a U_y^b + a_{ab}U_x^a U_y^b - 2(p-2)k^2 V_x V_y + Q \quad (k = const)$$

допускает представление Лакса (см. [3]).

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Lax representation of nonlinear σ-models with reducible metrics

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The way to associate to some symmetric spaces the Lax representations of nonlinear σ-models is proposed. Some examples of σ-models with reducible metrics are considered.

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Some Investigations on the Quadrupole Radiation of a Double Star

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The trajectory of a double star is obtained by applying the special theory of relativity equations of motion, and that in the center of mass frame. Similar to the original Bohr-Sommerfeld quantum mechanical application of the one electron atom, the excited energy levels of the double star can be obtained. Gravitation waves are then emitted during the passage to the ground state, when the two stars collide.

1. Introduction

The application of the general theory of relativity has already been applied to the motion of a double star, to explain the slowing of the orbital motion period. The emission of gravitational waves is causing the Earth to slowly spiral towards the Sun, but it would take , according to the application of general relativity 10^{27} years for them to collide. In 1975 Russel Hulse and Joseph Taylor [2] discovered the binary pulsar PSR 1963+16 in a system of two neutron stars, orbiting each other with a maximum separation of only one solar radius. The change predicted by general relativity is in excellent agreement with careful observations by Hulse and Taylor of the orbital parameters, indicates that since 1975, the period has shortened by only 10 seconds in 1993. They were awarded the Noble Prize for this confirmation of the general relativity.

In the case when the pair of double stars approach each other, leading to very high orbiting velocity, the trajectories can no longer be approximated by Kepler ellipses. Before the collapse, the application of quantum mechanics to the system becomes quite essential. This can be approximated by applying the Bohr-Sommerfeld quantum conditions to the trajectories, similar to what happened in the beginning of the discovery of quantum mechanics. As it was known that the Sommerfeld quantum conditions already explained roughly the energy levels of the one electron atom, as later verified by the exact solution of Dirac relativistic equation, the present application of the relativistic Sommerfeld quantum conditions can lead roughly to the excited energy levels of the double star.

2. Relativistic Two Body Trajectories in the Center of Mass Frame

The momenta and energies $\vec{p}_i$, $m_i c^2$ $(i = 1,2)$ of two bodies of rest masses m_{0i} $(i = 1,2)$, measured in a system O , where

$$\vec{p}_1 + \vec{p}_2 = \vec{P}, \quad (m_1 + m_2)c^2 = Mc^2, \tag{2.1}$$

$\vec{P}$, Mc^2 are their total momentum and energy in the system.

These can simply be expressed in their center of mass frame O', moving at velocity $\vec{V}$ with respect to O

$$\vec{V} = \frac{\vec{P}}{M} \tag{2.2}$$

by the momenta and energies respectively

$$\vec{p'}, m_1'c^2; -\vec{p'}, \quad m_2'c^2, \tag{2.3}$$

where

$$m_1' = \frac{M'^2 + m_{01}^2 - m_{02}^2}{2M'}, \quad m_2' = \frac{M'^2 - m_{01}^2 + m_{02}^2}{2M'}, \tag{2.4}$$

$$p'^2 = \frac{c^2}{4M'^2}[M'^2 - (m_{01} + m_{02})^2][M'^2 - (m_{01} - m_{02})^2]. \tag{2.5}$$

The total momentum in the center of mass system vanishes and $M'c^2$ is the total energy in the center of mass system, given by

$$M'c^2 = c\sqrt{M^2c^2 - P^2}. \tag{2.6}$$

If $\vec{x_1'}$, $\vec{x_2'}$ denote the particle positions in the center of mass frame, then we can express the momenta $\vec{p'}$, $-\vec{p'}$ in this frame by

$$\vec{p'} = m_{01}\frac{d\vec{x_1'}}{d\tau}, \quad -\vec{p'} = m_{02}\frac{d\vec{x_2'}}{d\tau}, \tag{2.7}$$

where $d\tau(= \sqrt{dt^2 - \frac{(d\vec{x'})^2}{c^2}})$ is the element of proper time. In terms of their relative position $\vec{x'} = \vec{x_1'} - \vec{x_2'}$ in their center of mass frame, we can then express

$$\vec{x_1'} = \frac{m_{02}}{m_{01} + m_{02}}\vec{x'}, \quad \vec{x_2'} = -\frac{m_{01}}{m_{01} + m_{02}}\vec{x'} \tag{2.8}$$

and hence

$$\vec{p'} = \frac{m_{01}m_{02}}{m_{01} + m_{02}}\frac{d\vec{x'}}{d\tau}. \tag{2.9}$$

The equation of constant total energy E' in the center of mass system is given

$$(M' - m_{01} - m_{02})c^2 - \frac{\gamma m_{01}m_{02}}{r'} = E', \tag{2.10}$$

where $r' = |\vec{x'}|$ is the magnitude of $\vec{x'}$. Using Eqs. (2.9) and (2.5), we get

$$(\frac{d\vec{x'}}{d\tau})^2 = (\frac{m_{01} + m_{02}}{m_{01}m_{02}})^2 p'^2, \tag{2.11}$$

where p'^2 as function of $\vec{M'}$ is given by Eq. (2.5).

Let the orbital angular momentum vector in the center of mass frame be $\frac{m_{01}m_{02}}{m_{01}+m_{02}}\vec{L}$, then we can express

$$\frac{m_{01}m_{02}}{m_{01} + m_{02}}\vec{L} = \vec{x'} \times \vec{p'} \tag{2.12}$$

satisfies, according to Eq. (2.9),

$$\frac{m_{01}m_{02}}{m_{01}+m_{02}}\frac{d}{d\tau}\vec{L}=\vec{x'}\times\frac{d}{d\tau}\vec{p'}. \tag{2.13}$$

For a central force acting along $\vec{x'}$, it follows

$$\frac{d}{d\tau}\vec{L}=0 \tag{2.14}$$

showing that the orbital angular momentum remains constant. Using polar coordinates r',θ in the plane of motion, we then find

$$r'^2\frac{d\theta}{d\tau}=L, \tag{2.15}$$

where L is the constant angular momentum per unit rest mass in the center of mass frame. Also the left hand side of Eq. (2.11) becomes

$$(\frac{d\vec{x'}}{d\tau})^2=(\frac{dr'}{d\tau})^2+r'^2(\frac{d\theta}{d\tau})^2. \tag{2.16}$$

Thus substituting for p'^2 from Eq. (2.5) into Eq. (2.11), we get for $(\frac{dr'}{d\tau})^2$:

$$(\frac{dr'}{d\tau})^2=\frac{c^2(m_{01}+m_{02})^2}{4m_{01}^2m_{02}^2M'^2}[M'^4-2(m_{01}^2+m_{02}^2)M'^2+(m_{01}^2-m_{02}^2)^2]-\frac{L^2}{r'^2}. \tag{2.17}$$

Using further Eq. (2.10) to substitute M' for $\frac{1}{r'}$

$$\frac{L}{r'}=\frac{Lc^2}{\gamma m_{01}m_{02}}[M'-m_{01}-m_{02}-\frac{E'}{c^2}], \tag{2.18}$$

we get

$$(\frac{dr'}{d\tau})^2=\frac{c^2}{4m_{01}^2m_{02}^2M'^2}[AM'^4+BM'^3-CM'^2+D], \tag{2.19}$$

where

$$A=(m_{01}+m_{02})^2-m_\gamma^2,\quad B=2m_\gamma^2(m_{01}+m_{02}+\frac{E'}{c^2}), \tag{2.20}$$

$$C=2(m_{01}+m_{02})^2(m_{01}^2+m_{02}^2)+m_\gamma^2(m_{01}+m_{02}+\frac{E'}{c^2})^2, D=(m_{01}+m_{02})^2(m_{01}^2-m_{02}^2)^2, \tag{2.21}$$

$$m_\gamma=\frac{2Lc}{\gamma}. \tag{2.22}$$

Using Eq. (2.15) and noticing that from Eq. (2.18) that

$$\frac{-L}{r'^2}dr'=\frac{Lc^2}{\gamma m_{01}m_{02}}dM', \tag{2.23}$$

we finally obtain

$$\frac{d\theta}{dM'}=-\frac{m_\gamma M'}{\sqrt{AM'^4+BM'^3-CM'^2+D}}. \tag{2.24}$$

The polar equation of the trajectory between r', θ is then obtained, by integrating Eq. (2.24) to give the relation between θ and M'

$$\theta = -\int \frac{m_\gamma M' dM'}{\sqrt{AM'^4 + BM'^3 - CM'^2 + D}}, \tag{2.25}$$

and then using the relation between M' and r' given by Eq. (2.10). The above integral represents an elliptic integral [1], whose value depends on the roots of the fourth order polynomial under the square root. When the four roots of the biquadratic equation are all real, the value of the integral is given as a simple sum of elliptic integrals $F(\varphi, k)$ of the first kind and $\Pi(\varphi, \alpha^2, k)$ of the third kind, where the modulus k and α^2 are given as functions of the four real roots, also the angle φ is expressed as function of M' and the four roots [1]. When there are two, or four complex roots of the biquadratic equation, the value of the integral is exprcssed apart from elliptic integrals of the first and third kinds, an additional contribution depending on a logarithmic or arctangent function of an expression depending on the values of the roots. The evaluation of the roots of the biquadratic equation as is well known, depends on obtaining the three roots of the cubic equation

$$A^3y^3 + A^2Cy^2 - 4A^2Dy - 4D(AC + B^2) = 0. \tag{2.26}$$

3. The Simple Case of Two Equal Masses

Instead of going through the complicated expression of the value of the integral in the general case, the behaviour of the trajectory can simply be found by considering the special case when the masses of the pair of stars are equal:

$$m_{01} = m_{02} = m_0. \tag{3.1}$$

In this simple case, as the value of the coefficient D vanishes, the value of the integral simplifies to

$$\theta = -\int \frac{m_\gamma dM'}{\sqrt{A_s M'^2 + B_s M' - C_s}}, \tag{3.2}$$

where now

$$A_s = 4m_0^2 - m_\gamma^2, \quad B_s = 2m_\gamma^2(2m_0 + \frac{E'}{c^2}), \quad C_s = 16m_0^4 + m_\gamma^2(2m_0 + \frac{E'}{c^2})^2. \tag{3.3}$$

The integral can then be readily evaluated, giving (for $A_s < 0$)

$$-\theta = \frac{m_\gamma}{\sqrt{-A_s}} \arccos \frac{(M' + \frac{B_s}{2A_s})}{\sqrt{(\frac{B_s^2}{4A_s^2} + \frac{C_s}{A_s})}}. \tag{3.4}$$

On using Eq. (3.3), we find

$$\frac{B_s^2}{4A_s^2} + \frac{C_s}{A_s} = \frac{16m_0^2}{4A_s^2}[16m_0^4 + m_\gamma^2(\frac{E'^2}{c^4} + 4m_0\frac{E'}{c^2})]. \tag{3.5}$$

Further using Eq. (2.10) and Eq. (2.22), we find

$$M' + \frac{B_s}{2A_s} = \frac{4m_0^2}{4m_0^2 - m_\gamma^2}(2m_0 + \frac{E'}{c^2}) + \frac{\gamma m_0^2}{c^2 r'}. \tag{3.6}$$

Thus from Eq. (3.4), we find the equation of the trajectory for $A_s < 0$

$$\frac{\gamma m_0}{2c^2 r'} = \frac{1}{m_\gamma^2 \lambda^2}[2m_0(2m_0 + \frac{E'}{c^2}) + m_\gamma \sqrt{(2m_0 + \frac{E'}{c^2})^2 - 4m_0^2\lambda^2 cos(\lambda\theta)}], \tag{3.7}$$

where for $0 < \lambda < 1$

$$\lambda^2 = 1 - \frac{4m_0^2}{m_\gamma^2} \tag{3.8}$$

and for $A_s > 0$ we find in a similar way, for $0 < \lambda' < \infty$,

$$\frac{\gamma m_0}{2c^2 r'} = \frac{1}{m_\gamma^2 \lambda'^2}[-2m_0(2m_0 + \frac{E'}{c^2}) + m_\gamma \sqrt{(2m_0 + \frac{E'}{c^2})^2 + 4m_0^2\lambda'^2 cosh(\lambda'\theta)}], \tag{3.9}$$

where, for $0 < \lambda' < \infty$,

$$\lambda'^2 = -\lambda^2. \tag{3.10}$$

The equation of the trajectory in polar coordinates r', θ, as given by Eq. (3.7) and Eq. (3.9), for $A_s < 0$, and $A_s > 0$, respectively, shows that the double star spiral, lowering their distance r' apart, until they collide. One notices that the expression $\frac{\gamma m_0}{2c^2 r'}$ on the left side of Eq. (3.7), or Eq. (3.9) is a measure of the ratio of the potential energy $\frac{\gamma m_0^2}{r'}$ to the total rest energy $2m_0c^2$ of the double star. In the applications of the general theory of relativity, this ratio is known as the gravitational parameter.

4. Application of Quantum Mechanics in the Form of Bohr-Sommerfeld Quantum Conditions

In the case of neglecting the small deviation from a closed path during one complete rotation of the double star, we apply the original form of the Bohr-Sommerfeld quantum conditions in the form

$$\frac{1}{2\pi}\oint p'_{r'}dr' = n_1\hbar, \quad \frac{1}{2\pi}\oint p'_\theta d\theta = n_2\hbar, \tag{4.1}$$

where the momenta $p'_{r'}$, p'_θ are given according to Eq. (2.9) and Eq. (2.15)

$$p'_{r'} = \frac{m_{01}m_{02}}{m_{01}+m_{02}}(\frac{dr'}{d\tau}), \tag{4.2}$$

$$p'_\theta = \frac{m_{01}m_{02}}{m_{01}+m_{02}}r'^2(\frac{d\theta}{d\tau}) = \frac{m_{01}m_{02}}{m_{01}+m02}L. \tag{4.3}$$

Restricting now to the simple case, when the two masses are equal, we get

$$\frac{1}{\pi}\int_{r_2}^{r_1}\frac{1}{2}m_0(\frac{dr'}{d\tau})dr' = n_1\hbar, \quad \frac{1}{2}m_0 L = n_2\hbar, \tag{4.4}$$

where from Eq. (2.19) and Eq. (2.21)

$$(\frac{dr'}{d\tau}) = \frac{c}{2m_0^2}\sqrt{A_s M'^2 + B_s M' - C_s}. \tag{4.5}$$

Substituting for M' in terms of r' from Eq. (3.6)

$$M' = 2m_0 + \frac{E'}{c^2} + \frac{\gamma m_0}{c^2 r'} \tag{4.6}$$

the expression in Eq. (4.4) for $n_1 \hbar$ becomes

$$n_1 \hbar = \frac{cK}{2\pi} \int_{r_2}^{r_1} \frac{dr'}{r'} \sqrt{-r'^2 + \frac{2\gamma m_0^2}{c^2 K^2}(2m_0 + \frac{E'}{c^2})r' - \frac{\gamma^2 m_0^2}{4c^4 K^2}(m_\gamma^2 - 4m_0^2)}, \tag{4.7}$$

where

$$K^2 = 4m_0^2 - (2m_0 + \frac{E'}{c^2})^2. \tag{4.8}$$

On using the standard integral

$$\frac{1}{\pi} \int_{r_2}^{r_1} \frac{dr}{r} \sqrt{(r - r_2)(r_1 - r)} = \frac{1}{2}(r_1 + r_2) - \sqrt{r_1 r_2} \tag{4.9}$$

expression in Eq. (4.7), then becomes

$$n_1 \hbar = \frac{\gamma m_0^2}{2cK}(2m_0 + \frac{E'}{c^2}) - \frac{\gamma m_0}{4c} \sqrt{m_\gamma^2 - 4m_0^2}. \tag{4.10}$$

Using Eq. (2.22) and Eq. (4.4), one can express m_γ as

$$m_\gamma = \frac{4c}{m_0 \gamma} \hbar n_2. \tag{4.11}$$

Hence the eigen values of energy E', following Eq. (4.9) and Eq. (4.11), can finally be expressed

$$\frac{E'}{c^2} + 2m_0 = \frac{2m_0(n_1 + \sqrt{n_2^2 - N^2})}{\sqrt{N^2 + (n_1 - \sqrt{n_2^2 - N^2})^2}}, \tag{4.12}$$

where the number N is given by

$$N = \frac{\gamma m_0^2}{2c\hbar}. \tag{4.13}$$

For a fixed value of $n_2 > N$, the value of $\frac{E'}{c^2} + 2m_0$, as function of n_1, starts with the value $2m_0 \frac{\sqrt{n_2^2 - N^2}}{n_2}$ for $n_1 = 0$, then increases to a maximum, followed by a continuous decrease to zero as $n_1 \to \infty$. The maximum value E'_m of E', as function of n_1 (for given n_2), is given

$$E'_m = 2m_0(-1 + \frac{\sqrt{4n_2^2 - 3N^2}}{N}) \tag{4.14}$$

at n_1, given by

$$n_1 = \frac{2n_2^2 - N^2}{2\sqrt{n_2^2 - N^2}}. \tag{4.15}$$

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ON EFFECTS OF NON-EUCLIDEAN GEOMETRY IN QUANTUM THEORY

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Theory of scattering of a quantum-mechanical particle on a cosmic string is developed. S-matrix and scattering amplitude are determined as functions of the flux and the tension of the string. We reveal that, in the case of the nonvanishing tension, the high-frequency limit of the differential scattering cross section does not coincide with the differential cross section for scattering of a classical pointlike particle on a string.

1. Introduction

Usually, the effects of non-Euclidean geometry are identified with the effects which are due to the curvature of space. It can be immediately shown that this is not the case and there are spaces which are flat but non-Euclidean.

A simplest example is given by a twodimensional space (surface) which is obtained from a plane by cutting a segment of a certain angular size and then sewing together the edges. The resulting surface is the conical surface which is flat but has a singular point corresponding to the apex of the cone. To be more precise, the intrinsic (Gauss) curvature of the conical surface is proportional to the twodimensional delta-function placed at the apex; the coefficient of proportionality is the deficit angle. Usual cones correspond to positive values of the deficit angle, i.e. to the situation when a segment is deleted from the plane. But one can imagine a situation when a segment is added to the plane; then the deficit angle is negative, and the resulting flat surface can be denoted as a saddle-like cone. The deleted segment is bound by the value of 2π, whereas the added segment is unbounded. Thus, deficit angles for possible conical surfaces range from $-\infty$ to 2π.

It is evident that an apex of the conical surface with the positive deficit angle can play a role of the convex lens, whereas an apex of the conical surface with the negative deficit angle can play a role of the concave lens. Really, two parallel trajectories coming from infinity towards the apex from different sides of it, after bypassing it, converge (and intersect) in the case of the positive deficit angle, and diverge in the case of the negative deficit angle. This demonstrates the non-Euclidean nature of conical surfaces. It is interesting that this item provides a basis for understanding such physical objects as cosmic strings. In the present paper we shall discuss peculiarities of quantum theory and its quasiclassical limit, which are due to non-Euclidean geometry of locally flat space-times.

2. Space-time in the presence of a cosmic string

Cosmic strings are topological defects which are formed as a result of phase transitions with spontaneous breakdown of symmetries at early stages of evolution of the universe, see, e.g., reviews in Refs.[1, 2]. In general, a cosmic string is characterized by two quantities: flux

$$\Phi = \int\limits_{\text{core}} d^2x\sqrt{g}B^3, \tag{2.1}$$

and tension

$$\mu = \frac{1}{16\pi G}\int\limits_{\text{core}} d^2x\sqrt{g}R; \tag{2.2}$$

here the integration is over the transverse section of the core of the string, B^3 is the field strength which is directed along the string axis, R is the scalar curvature, G is the gravitational constant, and units $\hbar = c = 1$ are used. The space-time metric outside the string core is

$$ds^2 = dt^2 - (1-4G\mu)^{-1}d\tilde{r}^2 - (1-4G\mu)\,\tilde{r}^2\,d\varphi^2 - dz^2 = dt^2 - dr^2 - r^2d\,\tilde{\varphi}^2 - dz^2, \tag{2.3}$$

where

$$\tilde{r} = r\sqrt{1-4G\mu}, \quad 0 \le \varphi < 2\pi, \quad 0 \le \tilde{\varphi} < 2\pi(1-4G\mu). \tag{2.4}$$

A surface which is transverse to the axis of the string is isometric to the surface of a cone with a deficit angle equal to $8\pi G\mu$. Such space-times were known a long time ago (M. Fierz, unpublished, see footnote in Ref.[3]) and were studied in detail by Marder [4]. In the present context, as cosmological objects and under the name of cosmic strings, they were introduced in seminal works of Kibble [5] and Vilenkin [6]. A cosmic string resulting from a phase transition at the scale of the grand unification of all interactions is characterized by the values of tension

$$\mu \sim (10^{-7} \div 10^{-6})G^{-1}. \tag{2.5}$$

The nonvanishing of the string tension leads to various cosmological consequences and, among them, to a very distinctive gravitational lensing effect. A possible observation of such an effect has been reported recently [7], and this has revived an interest towards cosmic strings.

The flux parameter (2.1) is nonvanishing for the so-called gauge cosmic strings, i.e. strings corresponding to spontaneous breakdown of local symmetries. If tension vanishes ($\mu = 0$), then a gauge cosmic string becomes a magnetic string, i.e. a tube of the magnetic flux lines in Euclidean space. If the tube is impenetrable for quantum-mechanical charged particles, then scattering of the latter on the magnetic string depends on flux Φ periodically with period $2\pi e^{-1}$ (e is the coupling constant – charge of the particle). This is known as the Bohm-Aharonov effect [8], which has no analogue in classical physics, since the classical motion of charged particles cannot be affected by the magnetic flux from the impenetrable for the particles region. The natural question is, how the nonvanishing string tension ($\mu \neq 0$) influences scattering of quantum-mechanical particles on the string. Thus, the subject of cosmic strings, in addition to tantalizing phenomenological applications, acquires a certain conceptual importance.

3. Quantum scattering on a cosmic string

Due to nonvanishing flux Φ and tension μ, the quantum scattering of a test particle by a cosmic string is a highly nontrivial problem. It is impossible to choose a plane wave as the incident wave, because of the long-range nature of the interaction inherent in this problem. A general approach to quantum scattering in the case of long-range interactions was elaborated by Hormander [9]. This approach covers the cases of scattering on a Coulomb center and on a magnetic string ($\mu = 0$), but is not applicable to the case of scattering on a cosmic string ($\mu \neq 0$). Therefore the last case needs a special consideration and a thorough substantiation.

When the effects of the core structure of a cosmic string are neglected and the transverse size of the core is negligible, the field strength and the scalar curvature are presented by twodimensional delta-functions. Scattering of a quantum-mechanical particle on an idealized (without structure) cosmic string was considered in Refs.[10, 11, 12, 13]. A general theory of quantum-mechanical scattering on a cosmic string, permitting to take into account the effects of the core structure, was elaborated in Ref.[14]. According to this theory, the S-matrix in the momentum representation is

$$S(k,\varphi;\ k',\varphi') = \frac{1}{2}\frac{\delta(k-k')}{\sqrt{kk'}}\left\{\Delta(\varphi-\varphi'+\frac{4G\mu\pi}{1-4G\mu})\exp\left[-\frac{ie\Phi}{2(1-4G\mu)}\right]+\right.$$
$$\left.+\,\Delta\left(\varphi-\varphi'-\frac{4G\mu\pi}{1-4G\mu}\right)\exp\left[\frac{ie\Phi}{2(1-4G\mu)}\right]\right\}+\delta(k-k')\sqrt{\frac{i}{2\pi k}}f(k,\ \varphi-\varphi'), \tag{3.1}$$

where the initial $(\mathbf{k})$ and final $(\mathbf{k}')$ twodimensional momenta of the particle are written in polar variables, $f(k,\ \varphi-\varphi')$ is the scattering amplitude, and $\Delta(\varphi) = \frac{1}{2\pi}\sum\limits_{n=-\infty}^{\infty} e^{in\varphi}$ is the angular part of the twodimensional delta-function. Note that in the case of short-range interaction one has $2\Delta(\varphi-\varphi')$ instead of the figure brackets in Eq.(3.1). Thus, one can see that, due to the long-range nature of interaction, even the conventional relation between S-matrix and scattering amplitude is changed, involving now a distorted unity matrix (first term in Eq.(3.1)) instead of the usual one, $\delta(k-k')\Delta(\varphi-\varphi')(kk')^{-1/2}$.

In view of the comparison with the Bohm-Aharonov effect [8], we shall be interested in the situation when the string core is impenetrable for the particle. The scattering amplitude in this case takes form:

$$f(k,\ \varphi) = f_0(k,\ \varphi) - \sqrt{\frac{2}{\pi i k}}\sum_{n=-\infty}^{\infty}\exp[in\varphi - i(\alpha_n - |n|)\pi]\frac{J_{\alpha_n}(kr_c)}{H^{(1)}_{\alpha_n}(kr_c)}, \tag{3.2}$$

where r_c is the radius of the string core, $J_\nu(u)$ and $H^{(1)}_\nu(u)$ are the Bessel and the first-kind Hankel functions of order ν,

$$\alpha_n = \left|n - \frac{e\Phi}{2\pi}\right|(1-4G\mu)^{-1}, \tag{3.3}$$

and

$$f_0(k,\varphi) = \frac{1}{\sqrt{2\pi i k}}\left\{\frac{\exp\left[i[[\frac{e\Phi}{2\pi}]](\varphi+\frac{4G\mu\pi}{1-4G_\mu}) - \frac{ie\Phi}{2(1-4G\mu)}\right]}{1-\exp\left[-i\left(\varphi+\frac{4G\mu\pi}{1-4G\mu}\right)\right]} - \right.$$
$$\left. - \frac{\exp\left[i[[\frac{e\Phi}{2\pi}]]\left(\varphi-\frac{4G\mu\pi}{1-4G\mu}\right)+\frac{ie\Phi}{2(1-4G\mu)}\right]}{1-\exp\left[-i\left(\varphi-\frac{4G\mu\pi}{1-4G\mu}\right)\right]}\right\} \tag{3.4}$$

is the amplitude of scattering on an idealized (without structure) cosmic string, $[[u]]$ is the integer part of u. Sum over n in Eq.(3.2) describes the core structure effects. In the low-frequency limit $(k \to 0)$ these effects die out, and the differential cross section (i. e. the square of the absolute value of the amplitude) takes form

$$\frac{d\sigma}{d\varphi} = \frac{1}{4\pi k}\left\{\frac{1}{2\sin^2\left[\frac{1}{2}\left(\varphi+\frac{4G\mu\pi}{1-4G\mu}\right)\right]}+\frac{1}{2\sin^2\left[\frac{1}{2}\left(\varphi-\frac{4G\mu\pi}{1-4G\mu}\right)\right]}\right.$$
$$\left.-\frac{\cos\left[\frac{e\Phi}{1-4G\mu}-\left(2[[\frac{e\Phi}{2\pi}]]+1\right)\frac{4G\mu\pi}{1-4G\mu}\right]}{\sin\left[\frac{1}{2}\left(\varphi+\frac{4G\mu\pi}{1-4G\mu}\right)\right]\sin\left[\frac{1}{2}\left(\varphi-\frac{4G\mu\pi}{1-4G\mu}\right)\right]}\right\}. \quad (3.5)$$

4. Differential cross section in the limit of high frequency of scattered particle

In the high-frequency limit $(k \to \infty)$ the first term in Eq.(3.2) dies out, and the differential cross section takes form

$$\frac{d\sigma}{d\varphi} = \frac{1}{2}r_c(1-4G\mu)^2\left|\sum_l\sqrt{\cos[\frac{1}{2}(1-4G\mu)(\varphi-\pi+2l\pi)]}\times\right.$$
$$\left.\times\ \exp\{ie\Phi l-2ikr_c\cos[\frac{1}{2}(1-4G\mu)(\varphi-\pi+2l\pi)]\}\right|^2, \quad (4.1)$$

where the summation is over integer l satisfying condition

$$-\frac{\varphi}{2\pi}-\frac{2G\mu}{1-4G\mu}<l<-\frac{\varphi}{2\pi}+1+\frac{2G\mu}{1-4G\mu}. \quad (4.2)$$

Note that results (3.5) and (4.1) are periodic in the value of flux Φ with period equal to $2\pi e^{-1}$. This feature is common with the scattering on a purely magnetic string ($\mu = 0$). The difference is that the Bohm-Aharonov differential cross section in the low frequency limit ($k \to 0$) diverges in the forward direction, $\varphi = 0$, while Eq.(3.5) diverges in two symmetric directions, $\varphi = \pm 4G\mu(1-4G\mu)^{-1}$. The difference becomes much more crucial in the high-frequency limit $(k \to \infty)$. In the $\mu = 0$ case one gets

$$\frac{d\sigma}{d\varphi} = \frac{1}{2}r_c\sin\frac{\varphi}{2}, \quad (4.3)$$

which is the cross section for scattering of a classical pointlike particle by an impenetrable cylindrical shell of radius r_c; evidently, the dependence on fractional part of $e\Phi(2\pi)^{-1}$ disappears in this limit. In the $\mu \neq 0$ case the dependence survives, see Eq.(4.1). In particular, if $0 < \mu < (8G)^{-1}$, which is most interesting from the phenomenological point of view, then the cross section at $k \to \infty$ takes the following form in the region of the cosmic string shadow, $-\frac{4G\mu\pi}{1-4G\mu} < \varphi < \frac{4G\mu\pi}{1-4G\mu}$:

$$\frac{d\sigma}{d\varphi} = r_c(1-4G\mu)^2 \Bigg(\cos[\frac{1}{2}(1-4G\mu)\varphi]\sin(2G\mu\pi) + \\ + \sqrt{\sin^2(2G\mu\pi) - \sin^2[\frac{1}{2}(1-4G\mu)\varphi]} \times \\ \cos\left\{ e\Phi + 4kr_c \sin[\frac{1}{2}(1-4G\mu)\varphi]\cos(2G\mu\pi) \right\}. \quad (4.4)$$

Integrating Eq.(4.4) over the region of the shadow and the appropriate expression (which is independent of Φ) over the region out of the shadow, we obtain the total cross section in the $k \to \infty$ limit:

$$\sigma_{\rm tot} = 2r_c(1-4G\mu). \quad (4.5)$$

The high-frequency limit is usually identified with the quasiclassical limit. Although this identification is valid for the total cross section, it is found to be invalid for the differential cross section, see Eqs.(4.1) and (4.4) revealing the periodic dependence on the flux, which is a purely quantum effect.

These results are generalized to the case of scattering of a particle with spin.

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SPECIAL INCONSTANCY IN PREGEOMETRY

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A theory of special inconstancy, in which some fundamental physical constants such as the fine-structure and gravitational constants may vary, is proposed in pregeometry. In the theory, the alpha-G relation of $\alpha = 3\pi/[16\ln(4\pi/5GM_W^2)]$ between the varying fine-structure and gravitational constants (where M_W is the charged weak boson mass) is derived from the hypothesis that both of these constants are related to the same fundamental length scale in nature. Furthermore, it leads to the prediction of $\dot{G}/G = (0.4 \pm 0.4) \times 10^{-12} yr^{-1}$ from the recent observation of $\dot{\alpha}/\alpha = (5 \pm 5) \times 10^{-15} yr^{-1}$ by Webb *et al.*, which is not only consistent with the most precise limit of $\dot{G}/G = (-0.6 \pm 2.0) \times 10^{-12} yr^{-1}$ by Thorsett but also feasible for future experimental tests. In special inconstancy, the past and present of the Univese are explained and the future of it is predicted, which is quite different from that in the Einsein theory of gravitation. The contents of this talk include the following:
1. Introduction
2. Pregeometry
3. Special Inconstancy
4. Further Discussions and Future Prospects.

1. Introduction

Is a physical constant really constant? In 1937, Dirac [1] discussed possible time variation in the fundamental constants of nature. He made not only the large number hypothesis (LNH) but also, as a consequences of the LNH, the astonishing prediction that the gravitational constant G varies as a function of time. Since then, Jordan [2] and many others [3,4] have tried to construct new theories of gravitation or general relativity in order to accomodate such a time-varying G. Although the LNH has been inspiring many theoretical developments and has recently led myself [5] to many new large number relations, the prediction of the varying G has not yet received any experimental evidence. Recently, Thorsett [6] has shown that measurements of the masses of young and old neutron stars in pulsar binaries lead to the most precise limit of

$$\dot{G}/G = (-0.6 \pm 2.0) \times 10^{-12} yr^{-1}$$

at the 68% confidence level.

More recently, on the other hand, Webb *et al.* [7] have investigated possible time variation in the fine structure constant α by using quasar spectra over a wide range of epochs, spanning redshifts $0.2 < z < 3.7$, in the history of our Universe, and derived the remarkable result of

$$\dot{\alpha}/\alpha = (6.40 \pm 1.35) \times 10^{-16} yr^{-1}$$

for $0.2 < z < 3.7$, which is consistent with a time-varying α. Note, however, that in 1976 Shylakhter [8] obtained the very restrictive limit of $|\Delta\alpha/\alpha| < 10^{-7}$ or, more precisely,

$$\Delta\alpha/\alpha\Delta t = (-0.2 \pm 0.8) \times 10^{-17} yr^{-1}$$

for $z \sim 0.16$ (but over a narrower and latest range of epochs between now and about 1.8 billion years ago) from the "Oklo natural reactor". Very lately, Srianand *et al.* [9] have made a detailed many-multiplet analysis performed on a new sample of Mg II systems observed in high quality quasar spectra obtained using the Very Large Telescope and found a null result of $\Delta\alpha/\alpha = (-0.06 \pm 0.06) \times 10^{-5}$ for the fractional change in α or a 3σ constraint of

$$-2.5 \times 10^{-16} yr^{-1} \leq (\Delta\alpha/\alpha\Delta t) \leq +1.2 \times 10^{-16} yr^{-1}$$

for $0.4 \leq z \leq 2.3$, which seems to be inconsistent with the result of Webb *et al.* [7]. However, a careful comparison of these different results [7-9] indicates that they are all consistent with a time-varying α as

$$\dot{\alpha}/\alpha = (5 \pm 5) \times 10^{-15} yr^{-1}$$

for $2.2 < z < 3.7$.

In this talk, I am going to propose a theory of special inconstancy, in which some fundamental physical constants such as the fine-structure and gravitational constants may vary. In the theory, the alpha-G relation of

$$\alpha = 3\pi/[16 \ln(4\pi/5GM_W^2)]$$

(where M_W is the charged weak boson mass) is derived from the hypothesis that both of α and G are related to the same fundamental length scale in nature. Furthermore, from the above result on $\dot{\alpha}$, it leads to the prediction of

$$\dot{G}/G = (0.4 \pm 0.4) \times 10^{-12} yr^{-1}$$

which is not only consistent with the limit on $\dot{G}$ by Thorsett [6] but also feasible for future experimental tests. I will organize this talk as follows: in Section II, I will briefly review pregeometry in which a theory of special inconstancy is constructed. In Section III, I will present the theory and its predictions. Finally in Section IV, I will present further discussions and future prospects. In addition, in special inconstancy I will explain the history of our Universe and predict the future of it, which is quite different from that in the conventional Einstein theory of gravitation.

2. Pregeometry

Pregeometry is a theory in which Einstein's geometrical theory of gravity in general relativity can be derived from a more fundamental principle as an effective and approximate theory at low energies (or at long distances). In 1967, Sakharov [10] suggested possible approximate derivation of the Einstein-Hilbert action from quantum fructuations of matter. A decade later, we [11] demonstrated that not only Einstein's theory of gravity in general relativity but also the standard model of strong and electroweak interactions in quantum chromodynamics and in the unified gauge theory can be derived as an effective and approximate theory at low energies from the more fundamental unified composite model of all fundamental particles and forces [12].

Let us explain what pregeometry means more explicitly in a simple model of

$$S_0 = \int d^4x\sqrt{-g}L_0(g_{\mu\nu}(x), A_\mu(x), \varphi_i(x))$$

where $g_{\mu\nu}$ is the space-time metric, $g = det(g_{\mu\nu})$, A_μ is an Abelian gauge field, and φ_i ($i = 1 \sim n$) are n complex scalar fields of matter with the charge e. The fundamental Lagrangian L_0 consists of the gauge-invariant kinetic terms of the matter fields only as

$$L_0 = g^{\mu\nu}[(\partial_\mu + iA_\mu)\varphi_i^\dagger](\partial_\nu - iA_\nu)\varphi_i - F^{-1}$$

(where F is an arbitrary constant) but does not contain either the kinetic term of the space-time metric or that of the gauge field so that both of $g_{\mu\nu}$ and A_μ are auxiliary fields. The effective action for the space-time metric and gauge field can be defined by the path-integral over the matter fields as

$$exp(iS_{eff}) = \int \prod_i [d\varphi_i^\dagger][d\varphi_i]exp(iS_0)$$

and it can be expressed formally as

$$S_{eff} = -iTrln[(\partial_\nu - iA_\nu)\sqrt{-g}g^{\mu\nu}(\partial_\mu + iA_\mu)] - \int d^4x\sqrt{-g}F^{-1}$$

after the path-integration over φ_i. For small scalar curvature R and Ricci curvature tensor $R_{\mu\nu}$, the effective action can be calculated to be

$$S_{eff} = \int d^4x\sqrt{-g}[2\lambda + (1/16\pi G)R + c(R^2 + dR^{\mu\nu}R_{\mu\nu}) + (1/4e^2)F^{\mu\nu}F_{\mu\nu} + ...]$$

with

$$\begin{aligned} 2\lambda &= [n\Lambda^4/8(4\pi)^2] - F^{-1}, \\ (1/16\pi G) &= n\Lambda^2/24(4\pi)^2, \\ c &= nln\Lambda^2/240(4\pi)^2, \\ d &= 2, \end{aligned}$$

and

$$(1/4e^2) = nln\Lambda^2/3(4\pi)^2,$$

where λ and Λ are the cosmological constant and the momentum cut-off of the Pauli-Villars type, respectively. Note that the arbitrary constant F^{-1} plays a role of counter term so that the cosmological constant may become as small as it is observed. Note also that the momentum cut-off Λ must be of order of the Planck mass $G^{-1/2}$ ($\sim 10^{19}GeV$). Furthermore, not only the R^2 and $R^{\mu\nu}R_{\mu\nu}$ terms but also the remaining terms in the expansion of S_{eff} are practically negligible. This completes a simple demonstration that not only the Einstein-Hilbert action of gravity but also the Maxwell action of electromagnetism in general relativity can be derived as an effective and approximate theory at low energies from the simple model in pregeometry, provided that there exists a natural momentum cut-off at around the Planck mass in nature [13].

One of the most remarkable consequences of pregeometry is the alpha-G relation, a simple relation between the fine-structure and gravitaional constant, which can be easily derived from the results for α and for G by eliminating the momentum cut-off Λ. In our unified quark-lepton model of all fundamental forces [14,15], the alpha-G relation is given by [16]

$$\alpha = 3\pi / \sum_i Q_i^2 ln(12\pi/nGm_i^2),$$

where Q_i and m_i are the charge and mass of quarks and leptons, respectively. For three generations of quarks and leptons and their mirror- or super-partners, the alpha-G relation simply becomes

$$\alpha \cong 3\pi/16ln(4\pi/5GM_W^2)$$

where M_W is the charged weak boson mass. Note that this alpha-G relation is very well satisfied by the experimental data of $\alpha \cong 1/137$, $G^{-1/2} \cong 1.22 \times 10^{19} GeV$, and $M_W \cong 80.4 GeV$.

3. Special Inconstancy

Special inconstancy is a principle in which some fundamental physical constants such as the fine-structure and gravitational constants may vary. Let us first make it clear that in this talk we use the natural unit system of $h/2\pi = c = 1$ (where h is the Planck constant and c is the speed of light in vacuum). Note, however, that it does not mean that, in discussing the relevant possibility of the varying fine-structure and gravitational constants [17], we exclude another intriguing possibility of the varying light velocity recently discussed by some authors [18] since varying either h or c is inevitably related to varying the fine-structure constant $\alpha\,(\equiv e^2/2hc)$(if the unit charge e stays constant). It simply means that we must set up a certain reference frame on which we can discuss whether physical quantities such as the fine-structure, gravitational, and cosmological [19] constants be really constant. Our basic hypothesis is that both of the fine-structure and gravitational constants are related to the more fundamental length scale of nature as in the unified (pregauge [20] and) pregeometric [10-12] theory (or "pregaugeometry" in short) of all fundamental forces[14,15] reviewed in the last Section.

To be more explicit, in the simple model of pregaugeometry discussed in the last Section, assert that

$$< (\partial_\mu + iA_\mu)\sqrt{-g}g^{\mu\nu}(\partial_\nu - iA_\nu)\varphi_i >_\Lambda = 0,$$

$$g_{\mu\nu} = F < [(\partial_\mu + iA_\mu)\varphi_i^\dagger](\partial_\nu - iA_\nu)\varphi_i >_\Lambda,$$

and

$$A_\mu = (i/2) < [\varphi_i^\dagger \partial_\mu \varphi_i - (\partial_\mu \varphi_i^\dagger)\varphi_i]/(\varphi_j^\dagger \varphi_j) >_\Lambda,$$

where $<>_\Lambda$ denotes the expectation value in the space-time with the fundamental length scale parameter of Λ^{-1}. The first equation is the usual field equation for φ_i while the last two can be taken either as the "equations of motion" for $g_{\mu\nu}$ and A_μ, which can be derived from the fundamental action S_0, or as the "fundamental field equations", which can reproduce the effective Einstein-Hilbert-Maxwell action S_{eff} at low energies($\ll \Lambda$) or at long distances($\gg \Lambda^{-1}$).

The most important consequence of special inconstancy in pregaugeometry is the $\dot{\alpha}-\dot{G}$ relation for the varying fine-structure and gravitational constant of

$$(\dot{G}/G)+2(\dot{M}_W/M_W)=(3\pi/16)(\dot{\alpha}/\alpha^2),$$

which can b e derived from differentiating both hand sides of the alpha-G relation with respect to any parameter for varying fundamental physical constants. This immediately leads to the remarkable predictions of

$$\dot{G}/G=(0.4\pm0.4)\times10^{-12}yr^{-1}$$

for constant M_W and

$$\dot{M}_W/M_W=(0.2\pm0.2)\times10^{-12}yr^{-1}$$

for constant G from the experimental data of $\dot{\alpha}/\alpha=(5\pm5)\times10^{-15}yr^{-1}$ by Webb *et al.* [7]. The first prediction is not only consistent with the limit of $\dot{G}/G=(-0.6\pm2.0)\times10^{-12}yr^{-1}$ by Thorsett [6] but also feasible for future experimental tests. The second prediction, however, seems too small to be feasible for experimental tests in the near future although such prediction for the possible varying particle masses seems extremely interesting at least theoretically. Note that the varying M_W is perfectly possible through the varying electroweak gauge coupling constant g (which is related to the fine-structure constant in the standard unified electroweak gauge theory of Glashow-Salam-Weinberg [21]) and/or the varying vacuum expectation value of the Higgs scalar v (which is related to the momentum cut-off Λ in the unified composite model of the Nambu-Jona-Lasinio type for all fundamental forces [22]) since $M_W=gv/2$.

4. Further Discussions and Future Prospects

In this talk, I have proposed a theory of special inconstancy in which some fundamental physical constants such as the fine-structure and gravitational constants may vary, based on the hypothesis that these constants are related to the fundamental length scale in nature. In the pregaugeometric theory, I have derived the simple relation between the varying α and G, predicted the value of $\dot{G}/G$ from the $\dot{\alpha}$ - $\dot{G}$ relation and the experimental data on $\dot{\alpha}/\alpha$, and found that the prediction is not only consistent with the present experimental limit on $\dot{G}/G$ but also feasible for future experimental tests.

Let us first add that in some pregaugeometric model [23] the alpha-G relation is not of the type of $\alpha\sim1/ln(1/G)$ but of the type of $\alpha\sim GM^2$ (where M is a parameter of mass dimension) so that the $\dot{\alpha}$ - $\dot{G}$ relation becomes

$$(\dot{G}/G)+2(\dot{M}/M)=\dot{\alpha}/\alpha.$$

This type of relation predicts

$$\dot{G}/G=(5\pm5)\times10^{-15}yr^{-1}$$

for constant M and

$$\dot{M}/M=(2.5\pm2.5)\times10^{-15}yr^{-1}$$

for constant G from the experimental data by Webb *et al.* [7]. We suspect that either one of these predicted values for $\dot{G}/G$ and $\dot{M}/M$ seems too small to be feasible for experimental tests in the near future although the first prediction is consistent with the limit by Thorsett [6].

Next, remember that in the principle of special inconstancy we do not assert that physical constants may vary as a function of time but do that they may vary in general, depending on any parameters including the cosmological time, temperature, *etc.*. What is the origin of varying the physical constants? The answer to this question may be related to the answer to another fundamental question: What is the origin of the fundamental length scale Λ^{-1} in nature? It can be spontaneous breakdown of scale-invariance in the Universe, which has been proposed by myself [5] for the last quarter century. It can be the natural, dynamical, automatic, *a priori*, but somewhat "wishful-thinking " cut-off at around the Planck length $G^{1/2}$ where gravity would become as strong as electromagnetism, which was suggested by Landau [13] in 1955. It can also be due to the Kaluza-Klein extra dimension [24], which is supposed to be compactified at an extremely small length scale of the order of $G^{1/2}$ or at a relatively large length scale of the order of $1/TeV$ recently emphasized by Arkani-Hamed *et al.* [25]. It seems, however, the most natural and likely that the origin of the fundamental length comes from the substructure of fundamental particles including quarks, leptons, gauge bosons, Higgs scalars, *etc.* [26,27]. In the unified composite model of all fundamental particles and forces [27], the fundamental energy scale Λ in pregaugeometry can be related to some even more fundamental parameters such as the masses of subquarks, the more fundamental constituents of quarks and leptons, and the energy scale in quantum subchromodynamics, the more fundamental dynamics confining subquarks into a quark or a lepton. In either way, the fundamental length scale Λ^{-1} can be idetified with the size of quarks and leptons, the fundamental particles.

In pregeometric special inconstancy, let us briefly explain the past and present of the Universe and predict the future of it, which may differ from that in the conventional Einstein theory of gravitation [28]. The history of our Universe goes as follows: Long, long time ago, there was no physical space-time, in which the space-time metric was finite and non-vanishing so that the distance was well defined, but the only matter "existed" in the mathematical space-time. Suddenly, there appeared the big bang of our Universe as a phase transition of the space-time from the pregeometric phase to the geometric one due to quantum fluctuations of matter, as suggested by us [29] in the early nineteen eighties, and our Universe had happened to be either flat or open. Then, not only all fundamental particles but also all fundamental forces between them were created and they started obeying the effective theory of all fundamental particles and forces including the Einstein theory of gravity with the non-vanishing and varying cosmological constant. In the eariest era during which the matter density had been extremely small, our Universe had been expanding almost exponentially. It had been the "almost inflationary Universe". In the next era of the radiation dominated Universe, our Universe was expanding less fast. Furthermore, in the last era of the matter dominated Universe, our Universe has still been expanding even faster. This history of our Universe is well simulated by a simple model of $(\Omega_m, \Omega_\lambda, -q) = (0,1,1)$, $(1/3, 2/3, 1/3)$, or $(1/3, 2/3, 1/2)$ for the early inflationary era, for the radiation dominated era, or for the matter dominated era, respectively, where Ω_m, Ω_λ, and q are the "pressureless-matter-density", "scaled cosmological constant", and deceleration parameter of the Universe, respectively. Note that there must be another "phase transition" in which Ω_λ changed from 1 to $2/3$ in between the early inflationary era and the radiation dominated era. Concerning the cosmological constant, I have been most impressed by the recent observation of the "farthest supernova ever seen" by Hubble Space Telescope [30]. "This supernova shows us the universe is behaving like a driver who slows down approaching a red stoplight and then hits the accelerator when the light turns green." Note that this behavior of the Universe is what our model simulates. Note also that our model of the Universe is consistent with the recent measurement of the cosmological mass density from clustering in the Two-Degree-Field Galaxy Redshift Survey [31] which strongly favors a low density Universe with $\Omega_m \cong 0.3$. Very lately, The CBI Collaboration [32] has found $\Omega_m = 0.64 + 0.11/-0.14$ and $\Omega_m + \Omega_\lambda = 0.99 \pm 0.12$. More lately, the Wilkinson Microwave Anisotropy Probe (WMAP) team [33] has found 1) the first generation of stars to shine in the Universe first ignited only 200 million years after the big bang, 2) the age of the Universe is 13.7 ± 0.2 years old, and 3) $\Omega_m = 0.27 \pm 0.04$ and $\Omega_\lambda = 0.73 \pm 0.04$.

The future of the Universe in our special inconstant picture can be quite different from that expected in the Einstein-Friedmann picture: 1) Since the cosmological constant may vary in special inconstancy, the space-time of our Universe which is almost flat and expanding faster and faster may not continue to be flat and accelerating forever. Our Universe may even encounter a "topological phase transition", which was first discussed by Wheeler [34] in 1959, from the open Universe to the closed one. 2) If the gravitational constant increases, the expansion of the space-time may not contitue forever. The Universe may well stop expanding, start contracting, and even be bouncing forever. If G decreases, it will be more accelerated ever. 3) If the fine-structure constant (and/or other fundamental coupling constants such as the strong and weak coupling constants) varies, our Universe may encounter an "obsolete phase transition" from the matter-dominated Universe to the radiation-dominated one. In short, we can expect anything about the future of our Universe or, in other words, we can predict nothing definite on the destiny of our Universe.

In conclusion, let us point out that not only continuous physical constants such as α and G but also discrete physical numbers such as the number of the space-time dimensions n, the number of quark colors N_c, the number of quark-lepton generations N_g, *etc.* may vary. In fact, an astonishing "dimensional phase transition", which was discussed by myself [35] about two decades ago, may be possible in the history of our Universe. If n is related to N_c as in the "space-color corespondence", which was proposed by myself about three decades ago [36], both of these fundamental physical natural numbers must vary simultaneously. Before concluding this talk, let me ask the following question: Are no constants of nature constant? After all, it may be that nothing is constant or permanent in the Unverse as emphasized by the Greek and Indian philosophers about two and a half millennia ago!

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IV-th International Conference
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Решение
IV Международной конференции «Неевклидова геометрия в современной физике и математике» (BGL-4)

Нижний Новгород, 7 – 11 сентября 2004 г.

1. 12 февраля 2006 года исполнится 150 лет со дня смерти выдающегося русского ученого, одного из первооткрывателей неевклидовой геометрии Николая Ивановича Лобачевского. Участники IV Международной конференции «Неевклидова геометрия в современной физике и математике» считают необходимым увековечение памяти Н.И. Лобачевского на его родине в г. Нижнем Новгороде.

В связи с вышеизложенным участники IV Международной конференции «Неевклидова геометрия в современной физике и математике» просят ректорат Нижегородского государственного университета им. Н.И. Лобачевского возобновить усилия, направленные на увековечение памяти Н.И. Лобачевского в Нижнем Новгороде, в частности, вновь обратиться в местные органы власти с инициативой об установке в Нижнем Новгороде в 2006 году памятника Н.И. Лобачевскому на месте дома, где он родился 1 декабря 1792 года.

Справка: с инициативой об увековечении памяти Н.И. Лобачевского в Нижнем Новгороде более полувека назад выступил академик А.А. Андронов. В результате его деятельности в этом направлении в 1956 г. (к столетию со дня смерти Н.И. Лобачевского) Нижегородскому государственному университету было присвоено имя Н.И. Лобачевского, но вопрос с установкой памятника так и остается нерешенным. В настоящее время на месте дома, где родился Н.И. Лобачевский (пересечение улиц Алексеевской и Октябрьской, в историческом центре города, недалеко от Кремля) функционирует вещевой рынок, территория архитектурно неблагоустроенна.

2. Провести следующую, V Международную конференцию «Неевклидова геометрия в современной физике и математике» (BGL-5), в 2006 году в Минске, Республика Беларусь.

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