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P. F. Brunet
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1.0 INTRODUCTION

We propose to study the general first order behavior of a non-synchronous rf accelerating structure. Many results are already available as far as the klystron rf wave is concerned[1,2] and as far as the beam induced wave is concerned in constant impedance structures.[1] We will try to find out general expressions and chiefly enhance the behavior of the beam induced wave in constant gradient structures. Theoretical results will be applied to the 2 mile accelerator section leading to a discussion of the rf automatic phasing system. Experimental results will be compared to the theoretical ones.

2.0 HYPOTHESES

2.1 Fields

The fields travel only in the direction, z , of the electron beam and the interaction between the field and the beam is independent of the radial position of the beam. In addition higher space harmonics are neglected.

2.2 Beam

The beam is bunched and travels in the z direction with the velocity of light.

2.3 General

Let, $E(z,t)$ be the field in the structure without beam, and $E_i(z,t)$ be the field induced in the structure by the beam with the klystron off. Since the beam current is not affected by the klystron power, the klystron and beam can be regarded as independent power sources so that with both the klystron and beam on, the total electric field in the structure $E_T(z,t)$, is given by,

$$E_T(z,t) = E_i(z,t) + E(z,t) \tag{1}$$

Similarly the total energy V_T, can be written,

$$V_T = V_i + V \tag{2}$$

where V is the energy gained by the beam with no beam loading, and V_i is the energy lost by beam in a section with no rf input power.

3.0 FIRST ORDER NON-SYNCHRONOUS OPERATION

3.1 Definition

If v_p is the phase velocity of the wave in the structure and v_e is the electron velocity, synchronous operation occurs when,

$$v_p = v_e = c$$

and non-synchronous operation when

$$v_p \neq v_e$$

The phase velocity is a function of the operating frequency and temperature, and the condition $v_p = c$ occurs only at the synchronous frequency f_o for a given temperature. A change in temperature ΔT_p however can be related to a change in frequency, by

$$-\nu \Delta T_p = \frac{\Delta f_o}{f_o} ,$$

where ν is the coefficient of linear expansion of the metal.

In non-synchronous operation, both temperature and frequency variations can be included in the parameter σ, where

$$\sigma = \frac{\Delta f}{f_o} \tag{3}$$

and

$$\Delta f = f - f_o \ .$$

3.2 Slippage Between the Beam and Wave in a Non-Synchronous Section

The phase difference between the rf wave and the beam at a given point z, along the structure is given by,

$$\delta(z) = \delta_o + \omega \left[\frac{z}{c} - \int_o^z \frac{dz}{v_p} \right]$$

where δ_o is the phase difference at $z = o$. If $v_p = c$, then

$$\delta(z) = \delta_o = \text{constant}$$

however, if $v_p \neq c$ then a "slippage" occurs and is given by

$$g(z) = \frac{\omega}{c} z - \int_o^z \frac{dz}{v_p} = \frac{\omega}{c} z - \int_o^z \beta dz$$

For a small frequency variation around the synchronous frequency, $g(z)$ can be expanded in a Taylor series,

$$g(\omega_o + \Delta\omega) = g(\omega_o) + \Delta\omega \frac{dg}{d\omega} + \frac{\Delta\omega^2}{2!} \frac{d^2 g}{d\omega^2}$$

$$= g(\omega_o) + \Delta\omega \left[\frac{z}{c} - \int_o^z \frac{d\beta}{d\omega} dz \right] + \frac{\Delta\omega^2}{2!} \left[- \int_o^z \frac{d^2\beta}{d\omega^2} dz \right] + \ldots$$

Neglecting second order terms, i.e., assuming that

$$\frac{d^2\beta}{d\omega^2} = 0 = \frac{d}{d\omega} \left(\frac{1}{v_g} \right) .$$

[This means that the group velocity is constant around the synchronous frequency and the Brillouin characteristic can be approximated by the tangent at $\omega = \omega_o$. So we neglect all pulse phase and amplitude distortions due to $\frac{d}{d\omega} (v_g) \neq 0$.] * Then,

$$g(z) = \Delta\omega \left[\frac{z}{c} - \int_o^z \frac{dz}{v_g} \right] \quad \text{since} \quad g(\omega_o) = 0$$

or

$$g(z) = \Delta\omega \left[\tau_i(z) - \tau(z) \right]$$

where, $\tau_i(z)$ is the electron transit time to z and τ is the rf filling time to z but since $v_g \approx 0.01\ C$, $\tau_i(z) \ll \tau(z)$ and $\tau_i(z)$ can be

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*A more sophisticated approach by J. M. Leiss[3] takes care of the v_g variations for studying transients in a constant impedance accelerator section, but seems difficult to be used for a constant gradient section and in most of the cases exhibits only small corrections with respect to the first order.

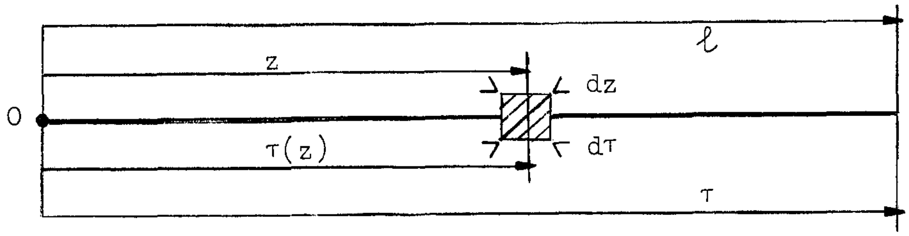
neglected according to the previous hypotheses, then

$$g(z) = - \sigma\omega\tau(z) \tag{4}$$

$g(\ell)$ will be only called g .

4.0 FIELD INDUCED BY THE BEAM IN A NON-SYNCHRONOUS SECTION

Consider an accelerator section of length ℓ and filling time τ .



Let,

r be the shunt impedance/unit length

α be the attenuation coefficient/unit length

Q be the Q of the section

and assume

$$\frac{\partial r}{\partial z} = \frac{\partial Q}{\partial z} = 0$$

and that the field dE_i, induced by the beam in a length dz, is established instantaneously, i.e.,

$$\frac{2Q'}{\omega} \ll \frac{dz}{v_g}$$

where Q' is the Q of an individual cavity which is very low. Then,

$$dE_i = - \alpha rIdz^{(*)} \tag{5}$$

where I is the beam current.

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*See for instance Ref. 4.

Equation (5) can be rewritten in terms of the filling time $d\tau$, as follows:

$$dE_i = -\frac{r\omega}{2Q} I \, d\tau = -rId\tau' \qquad (5')$$

where,

$$\tau' = \frac{\omega\tau}{2Q}$$

It should be noted, that with the above assumptions using the filling time as the variable as opposed to the length, that both $dg/d\tau$ and $dE_i/d\tau$ are constant along the section whatever type of structure is considered.

The expression for the total field induced at the end of the section can now be derived as follows:

Let the amplitude of the beam induced wave be governed by $a(z)$, or in terms of the filling time $A(\tau)$ and let the slippage be given by $g(z) = -\sigma\omega\tau(z)$ (Eq. (41)), i.e., when the field travels from $z_1(\tau_1)$ to $z_2(\tau_2)$

the amplitude $\quad E_2 = E_1 \times a(z_2 - z_1) = E_1 \times A(\tau_2 - \tau_1)$

the phase $\quad \varphi_2 = \varphi_1 - \omega(\tau_2 - \tau_1)$ with respect to the beam

$E_i(\ell)$ at the end of the section is the sum of all the $dE_i(z)$ after they travelled from z.

The expression for $E_i(\ell)$ is so given, in the steady state condition by,

$$E_i(\ell) = \int_0^\tau dE_i(z) = \int_0^\tau -\frac{r\omega}{2Q} \; I \; A \,[\tau - \tau(z)]\; e^{-j\sigma\omega[\tau-\tau(z)]} \, d\tau$$

For clarity write $\tau(z) = u$, then,

$$E_i(\ell) = -\frac{\omega r}{2Q} I \int_0^{\tau} A(\tau-u)\, e^{-j\sigma\omega(\tau-u)}\, du \tag{6}$$

Induced power

The travelling power in an accelerator section is given by:

$$P(z) = \frac{E^2(z)}{2\alpha(z) \times r}$$

In the particular case when $E = E_i$, $P(z)$ is the power induced by the beam and if $z = \ell$, end of the section $P(\ell)$ is put out from the section excited by the beam. This holds in the non-synchronous operation as far as we can assume a good output coupler matching and:

$$P_i = \frac{|E_i(\ell)|^2}{2\alpha(\ell) \times r} = \frac{[E_i(\tau)]^2}{2\alpha(\tau) \times r} \tag{7}$$

φ, the phase angle between the beam and the induced wave at the end of the section, is given by;

$$\tan\varphi = \frac{I\,[E_i(\ell)]}{R\,[E_i(\ell)]} = \frac{\int_0^{\tau} A(\tau-u)\,\sin\sigma\omega\,(\tau-u)\,du}{\int_0^{\tau} A(\tau-u)\,\cos\sigma\omega\,(\tau-u)\,du} \tag{8}$$

Remark: At a given point z in the section, E_i, P_i and φ can be determined from Eqs. (6), (7) and (8) by replacing τ with $\tau(z)$ where $\tau(z)$ is the filling time from the input to the point z. It is just like considering each point as the end of a shorter section of filling time $\tau(z)$. This property of course holds during the transient state as well and will be true for all the expressions given in the remaining pages.

The energy lost by beam loading is:

$$V_i = \int_0^{\ell} R\left[E_i(z)\right] dz$$

when $R\left[\;\right]$ means the real part.

5.0 TRANSIENT STATE PHENOMENA

Let the beam pulse of duration D, and amplitude I enter the section (z=o) at t=o and assume that the electron transit time is short compared with the rf filling time.

5.1 Leading Edge $(t < \tau)$

At the end of the section $(z=\ell)$, $0 < t < \tau$, which means that the field is built up only by components coming from z, i.e.

$$t = \tau - \tau(z)\ .$$

E_i and φ, which become $E_i(\ell,t)$ and $\varphi(\ell,t)$ are given by Eqs. (6) and (7) when the lower limit of the integral is replaced by $\tau(z) = \tau - t$.

$$E_i(\ell,t) = -\frac{\omega r}{2Q}\int_{\tau-t}^{\tau} A(\tau-u)\; e^{-j\omega\sigma(\tau-u)}\, du \quad (t < \tau) \qquad (9)$$

It should be noted that the integrals always include the variable u as τ-u so that if τ-u is replaced by a new variable v, the upper limit becomes 0 and the lower limit t. This means that during the transient state, the field is constant at any point in the section provided $\tau(z) > t$.

The energy lost is a function of time and can be calculated in two parts.

from 0 to z $(\tau(z) = t)$ steady state

from z to ℓ, transient state

5.2 Trailing Edge $(D < t < \tau + D)$

The following analysis starts from the steady state condition. At time $t=D$ the beam current stops and the induced field begins to decay. The first part which is lost is that induced in the section at point ℓ. At time t, $(D < t < D + \tau)$ it is the component induced at the point z so that if $\tau(z)$ is the filling time from 0 to z,

$$t - D = \tau - \tau(z)$$

The total field is the steady state field less the field suppressed;

$$E_i(\ell,t) = -\frac{r\omega}{2Q} I \left[\int_0^{\tau} A(\tau-u)\, e^{-j\sigma\omega(\tau-u)}\, du - \int_{\tau+D-t}^{\tau} A(\tau-u)\, e^{-j\sigma\omega(\tau-u)}\, du \right]$$

or,

$$E_i(\ell,t) = -\frac{r\omega}{2Q} I \int_0^{\tau+D-t} A(\tau-u),\ e^{-j\sigma\omega(\tau-u)}\, du \quad (D < t < \tau + D) \qquad (10)$$

We notice that when t goes to $\tau+D$, $E_i(\ell,t)$ goes to zero and, φ to :

$$\varphi(\ell, D+\tau) = -\sigma\omega\tau+\pi$$

Thus the angle of slippage between the beam and the last beam induced wave is given by $g=-\sigma\omega\tau$ since the last component has traveled the full length of the section.

5.3 Particular Case of the Pulse Length Shorter than the Filling Time of the Section ($D < \tau$)

Consider the situation at the end of the section $(z=\ell)$.

$0 < t < D$; This is the leading edge transient state studied in Section 5.1.

$t = D$; Equation (9) gives:

$$E_i(\ell,D) = -\frac{r\omega}{2Q} I \int_{\tau-D}^{\tau} A(\tau-u), \; e^{-j\sigma\omega(\tau-u)} \, du$$

$D < t < \tau$; The induced field is now only a function of the time between z $\left(\tau(z) = t\right)$ and ℓ, and the fields no longer build up. Then,

$$E_i(\ell,t) = E_i(\ell,D) \; A(t-D) \; e^{-j\sigma\omega(t-D)} \quad (D < t < \tau) \quad (11)$$

$t = \tau$;

$$E_i(\ell,\tau) = E_i(\ell,D) \; A(\tau-D) \; e^{-j\sigma\omega(\tau-D)}$$

$\tau < t < \tau+D$; This is the normal case of the trailing edge transient state starting with $E_i(\ell,\tau)$

$$E_i(\ell,t) = E_i(\ell,\tau) + \frac{r\omega}{2Q} I \; A(\tau-D) \; e^{-j\sigma\omega(\tau-D)} \times \int_{2\tau-t}^{\tau} A(\tau-u) \; e^{-j\sigma\omega(\tau-u)} \, du$$

$$E_i(\ell,t) = -\frac{r\omega}{2Q} I \; A(\tau-D) \; e^{-j\sigma\omega(\tau-D)} \times \int_{\tau-D}^{2\tau-t} A(\tau-u) \; e^{-j\sigma\omega(\tau-u)} du \quad (\tau < t < \tau+D) \quad (12)$$

We see that $E_i(\ell,t)$ goes to zero as t goes to $\tau+D$ (i.e., $2\tau-t \to \tau-D$). Let, $t = \tau+D-\epsilon$ where ϵ is small then,

$$E_i(\ell,t)_{\epsilon\to 0} = -\frac{r\omega}{2Q} I\, A(\tau)\, e^{-j\sigma\omega(\tau)}\epsilon$$

So as before and for the same reason the angle of slippage goes to $-\sigma\omega\tau=g$.

6.0 EXPRESSION FOR THE FIELD IN THE CONSTANT IMPEDANCE STRUCTURE

In the constant impedance structure

$$\frac{\partial v_g}{\partial z} = \frac{\partial \alpha}{\partial z} = 0 \quad \text{and} \quad \tau(z) = \frac{z}{v_g} = \frac{2\alpha Q}{\omega} z$$

the amplitude law is given by

$$\begin{cases} a(z) = e^{-\alpha z} \\ A(\tau) = e^{-\alpha v_g \tau} = e^{-\frac{\omega}{2Q}\tau} = e^{-\tau'} \end{cases}$$

From Eq. (6)

$$E_i(\ell) = -\frac{r\omega}{2Q} I \int_0^{\tau} e^{-\frac{\omega}{2Q}(\tau-u)}\, e^{-j\sigma\omega(\tau-u)}\, du$$

$$= r\, I \left[\frac{1-2j\sigma Q}{1+4\sigma^2 Q^2}\right]\left[e^{-\left(\frac{\omega}{2Q}+j\sigma\omega\right)} - 1\right]$$

$$= \frac{rI}{1+4\sigma^2Q^2}\left\{ e^{-\frac{\omega}{2Q}\tau}(\cos g - 2\sigma Q \sin g) - 1 - j\left[e^{-\frac{\omega}{2Q}\tau}(\sin g + 2\sigma Q\cos g) - 2\sigma Q\right]\right\} \qquad (13)$$

The phase of the induced wave with respect to the beam is then given by,

$$\tan \varphi = - \frac{e^{-\frac{\omega}{2Q}\tau}\,[\sin g + 2\sigma Q \cos g] - 2\sigma Q}{e^{-\frac{\omega}{2Q}\tau}\,[\cos g - 2\sigma Q \sin g] - 1} \tag{14}$$

Energy Lost in the Constant Impedance Section by Beam Loading

The energy lost V_i, is given by

$$V_i = \mathrm{Re} \int_0^z E_i(z)\, dz$$

Integrating and expressing V_i as a function of the length ℓ, and of the total angle of slippage, g, gives

$$V_i = \frac{-rI}{1 + 4\sigma^2 Q^2} \left\{ \ell + \frac{1 - 4\sigma^2 Q^2}{\alpha\left(1 + 4\sigma^2 Q^2\right)} \left[e^{-\alpha\ell} (\cos g + 4\sigma Q \sin g) - 1 \right] \right\} \tag{15}$$

It can be shown from the above formula that if $\sigma=0$, then the well-known formulae for the induced field and energy loss obtain and in addition that $\varphi=0$. E_i, φ and V_i can be calculated for the transient state case from Eqs. (9), (10), (11) and (12).

7.0 EXPRESSIONS FOR THE FIELD IN THE CONSTANT GRADIENT SECTION

For the constant gradient section considered let,

$$\frac{\partial E}{\partial z} = 0 \quad \text{for} \quad I = 0$$

then,

$$\alpha = \frac{\alpha_o}{1 - 2\alpha_o z} \qquad \alpha_o = \alpha(o)$$

$$v_g = v_g(o)\left[1 - 2\alpha_o z\right] = \frac{\omega_o}{2\alpha_o z}(1 - 2\alpha_o z)$$

and,

$$\tau(z) = -\frac{Q}{\omega}\log(1 - 2\alpha_o z)$$

in addition,

$$a(z) = A(\tau) = 1$$

7.1 Expression for the Field in the Steady State

From Eq. (6):

$$E_i = -\frac{r\omega}{2Q} I \int_o^{\tau} e^{-j\sigma\omega(\tau-u)}\, du = -\frac{rI}{\sigma Q}\sin\left(\frac{\sigma\omega\tau}{2}\right) \cdot e^{-j\frac{\sigma\omega\tau}{2}}$$

So the E_i amplitude is:

$$\left|E_i\right| = \frac{r\omega}{2Q}\tau I \cdot \frac{\sin\left(\frac{\sigma\omega\tau}{2}\right)}{\frac{\sigma\omega\tau}{2}} \tag{16}$$

and can be written:

$$\left|E_i\right| = \left|E_i\right|_o \times \Gamma \tag{16'}$$

where $\left|E_i\right|_o$ is the field amplitude in the synchronous operation and:

$$\Gamma = \frac{\sin\frac{\sigma\omega\tau}{2}}{\frac{\sigma\omega\tau}{2}} = \frac{\sin\left(\frac{g}{2}\right)}{\left(\frac{g}{2}\right)}$$

The phase of E_i with respect to the electron beam is:

$$\varphi = - \frac{\sigma\omega\tau}{2} + \pi = \frac{g}{2} + \pi \qquad . \tag{17}$$

Remark: Results (16) and (17) can be geometrically found out noticing that the pattern of the vectors $d\vec{E}_i$ added one after another is inscribed in a circle where $\vec{E}_i$ is a cord.

7.2 Induced power

From Eqs. (7) and (16')

$$P_i = \left|E_i\right|_o^2 \Gamma^2 \frac{1}{2\alpha(\ell) \cdot r}$$

or

$$P_i = P_{io} \Gamma^2 \tag{18}$$

Where P_{io} is the induced power in the synchronous operation and can be written as a function of τ (Ref. 5)

$$P_{io} = \frac{r}{8\alpha_o} \left(\frac{\omega}{Q} \tau\right)^2 \cdot e^{-\frac{\omega}{Q}\tau}$$

The Γ factor in both E_i and P_i expression shows that E_i and P_i can go to zero several times along the section $\left(\frac{\sigma\omega\tau}{2} = k\pi\right)$ if σ is large enough. That means the power is balanced between the field and the beam; in the first part the field is induced by the beam and reaches a maximum, then it accelerates the beam again and goes to zero and so on.

7.3 Energy Lost by Beam Loading

$$dz = \frac{1}{2\alpha_o} \cdot \frac{\omega}{2Q} e^{-\frac{\omega}{Q}\tau} d\tau$$

The energy lost V_i is given by,

$$V_i = R \int_0^z E_i \, dz$$

$$R \left\{ \frac{rI\omega}{4\sigma Q^2 \alpha_o} \int_0^\tau \left[\sin \sigma\omega u - j(1 - \cos \sigma\omega u) \right] e^{-\frac{\omega}{Q} u} \, du \right\}$$

$$= - \frac{rI\omega}{4\alpha_o \sigma Q^2} \int_0^\tau e^{-\frac{\omega}{Q} u} \sin \sigma\omega u \, du$$

$$= \frac{rI}{4\alpha_o} \cdot \frac{e^{-\frac{\omega}{Q} \tau} (\sin g + \sigma Q \cos g) - \sigma Q}{\sigma Q (1 + \sigma^2 Q^2)} \tag{19}$$

We can check that V_i goes to the synchronous expression[5] when σ goes to zero since,

$$\alpha_o = \frac{1 - e^{-\frac{\omega}{Q} \tau}}{2\ell}$$

7.4 Leading and Trailing Edge Transients

At the leading edge at a time $t < \tau$, the induced field is given from Eq. (5) by

$$E_i(t, \ell) = - \frac{r\omega}{2Q} I \int_{\tau - t}^{\tau} e^{-j\sigma\omega(\tau - u)} \, du$$

$$E_i(t) = - \frac{rI}{2\sigma Q} \left[\sin \sigma\omega t - j (1 - \cos \sigma\omega t) \right]$$

$$= - \frac{rI}{\sigma Q} \sin \frac{\sigma\omega t}{2} \, e^{-\frac{j\sigma\omega t}{2}} \tag{20}$$

and,

$$\varphi = -\frac{\sigma\omega t}{2} + \pi$$

It can be seen from these expressions that both the phase and amplitude are functions only of time and not of distance.

In the case of the trailing edge, with $D < t < D + \tau$, the induced field is given from Eq. (6) by

$$E_i(\ell,t) = -\frac{r\omega}{2Q} I \int_0^{\tau+D-t} e^{-j\sigma\omega(\tau-u)} du$$

$$E_i(\ell,t) = -\frac{rI}{\sigma Q} \sin \sigma\omega \left(\frac{D + \tau - t}{2}\right) e^{-j\sigma\omega\left(\frac{\tau-D+t}{2}\right)} \qquad (21)$$

and

$$\varphi = -\sigma\omega \left(\frac{\tau-D+t}{2}\right)$$

It should be noted that as t goes to $\tau + D$, the amplitude goes to zero and the phase to $-\sigma\omega\tau = g$.

7.5 Particular Case of Beam Pulse Shorter than the Section Filling Time

In this case $D < \tau$

For $t \leq D$

The expressions for the field amplitude and phase are given by Eqs. (20) and (21) of Section 7.4.

For $D < t < \tau$

$$E_i(t) = -\frac{rI}{\sigma Q} \sin \frac{\sigma\omega D}{2} e^{-j\sigma\omega\left(t - \frac{D}{2}\right)}$$

hence,

$$\left|E_i(t)\right| = \text{constant}$$

$$\varphi(t) = -\sigma\omega\left(t - \frac{D}{2}\right) + \pi$$

<u>For $t = \tau$</u>

$$E_i(\ell,\tau) = -\frac{rI}{\sigma Q}\sin\frac{\sigma\omega D}{2}\, e^{-j\sigma\omega\left(\tau - \frac{D}{2}\right)}$$

and

$$\varphi(\ell,\tau) = -\sigma\omega\left(\tau - \frac{D}{2}\right) + \pi$$

<u>For $\tau < t < \tau + D$</u>

From Equation (12), Section 5.3

$$E_i(\ell,t) = -\frac{rI\omega}{2Q}\, e^{-j\sigma\omega(\tau-D)}\int_{\tau-D}^{2\tau-t} e^{-j\sigma\omega(\tau-u)}\, du$$

$$= -\frac{rI}{\sigma Q}\sin\sigma\omega\left(\frac{\tau+D-t}{2}\right) e^{-\frac{j\sigma\omega}{2}(\tau-D+t)} \qquad (22)$$

and

$$\varphi(\ell,t) = -\frac{\sigma\omega}{2}\left[\tau-D+t\right] + \pi$$

It can be seen from Eq. (22) that as t goes to $\tau+D$ the amplitude goes to zero and the phase $\varphi(t)$ to $-\sigma\omega\tau = g$.

7.6 <u>Energy Lost by Beam Loading in Transient State</u>

From 0 to $z(t)$ so that $\tau(z) = t$, it is steady state and V_i is given by Eq. (19) where τ is t.

From $z(t)$ to ℓ, E_i is $E_i(t)$ given by Eq. (20) and V_i is $R\left\{E_i(t) \times \left[\ell - z(t)\right]\right\}$, so:

$$V_i(t) = \frac{rI\ell}{2\sigma Q} \cdot \frac{1}{\left(1-e^{-\frac{\omega}{Q}\tau}\right)} \left[\frac{e^{-\frac{\omega}{Q}\tau}(\sin\sigma\omega t + \sigma Q\cos\sigma\omega t) - \sigma Q}{1+\sigma^2 Q^2}\right.$$

$$\left. - \sin\sigma\omega t\left(e^{-\frac{\omega}{Q}t} - e^{-\frac{\omega}{Q}\tau}\right)\right] \qquad (23)$$

When $t \geq \tau$ Eq. (23) becomes Eq. (19) again. We can have at each time the total energy gained by the electrons travelling through the sections, remembering that:

$$V_T = V + V_i(t) -$$

7.7 Graphical Representation

φ and $|E_i|$ are plotted versus time in Figure 1. $|E_i|$ varies as $\sin\frac{\sigma\omega t}{2}$ from 0 to τ (or D') and the curve is symmetrical with respect to the vertical axis $t = \frac{\tau + D}{2}$ (or $\frac{\tau+D'}{2}$). From a good "square" beam pulse both represented φ and $|E_i|$ curves can be observed on a scope.

The observed $|E_i(t)|$ is about a trapezoid; in the synchronous case the sinusoid only becomes a linear function of the time.[6,4] If σ is small and considering the non-perfect detector characteristic it is difficult to see the difference.

The observed $\varphi(t)$ is about the theoretical one = an accurate reading would require an amplitude free phase detector and a very stable pulse phase. For the same reason it is difficult to get a horizontal straight line when $\sigma=0$ which could be an absolute indication of synchronism (as already suggested by K. Mallory).

8.0 APPLICATION TO THE BEAM INDUCTION TECHNIQUE OF PHASING LINEAR ACCELERATORS

The beam induction technique, which has been previously described, consists basically of comparing the phase of the klystron wave and the

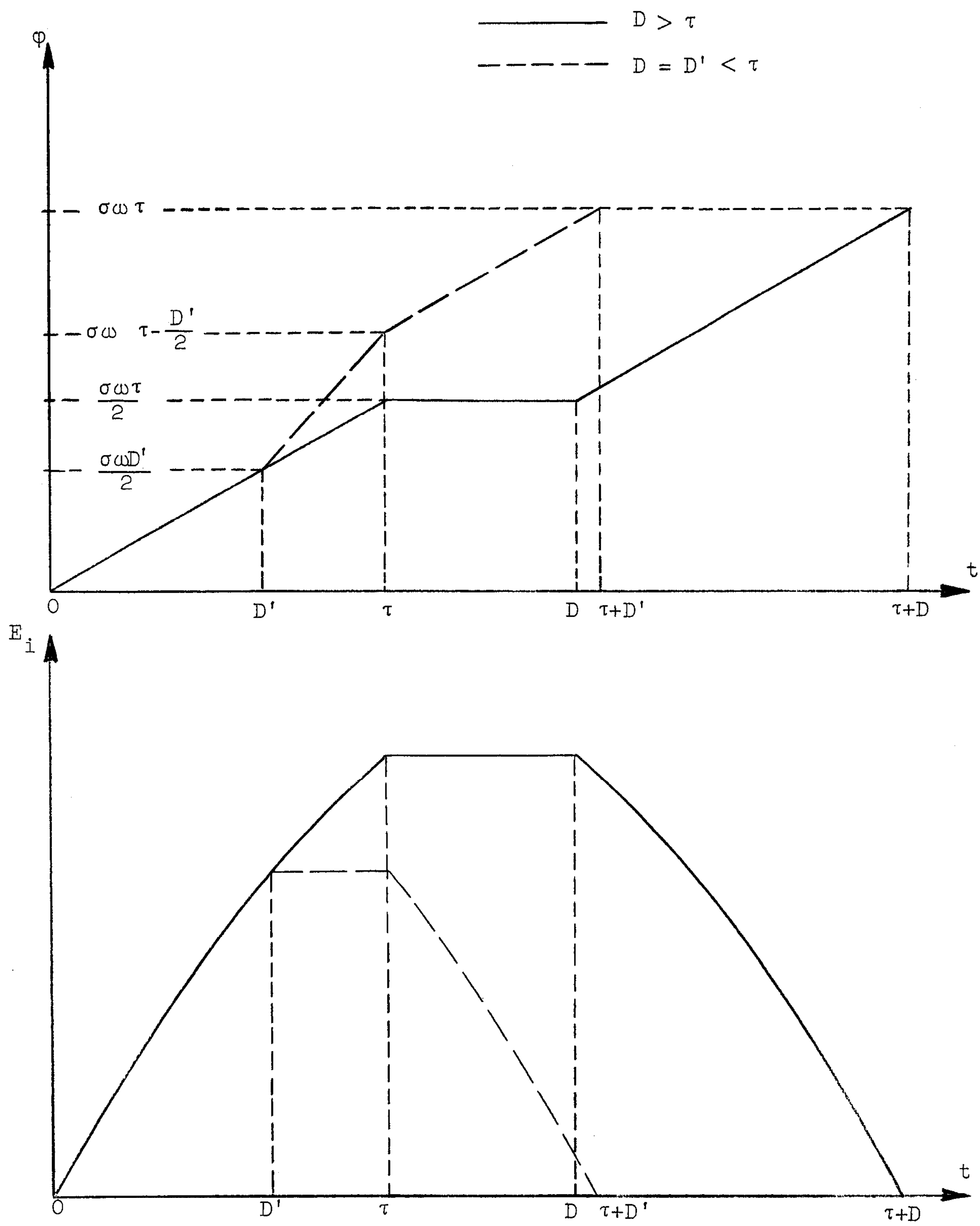
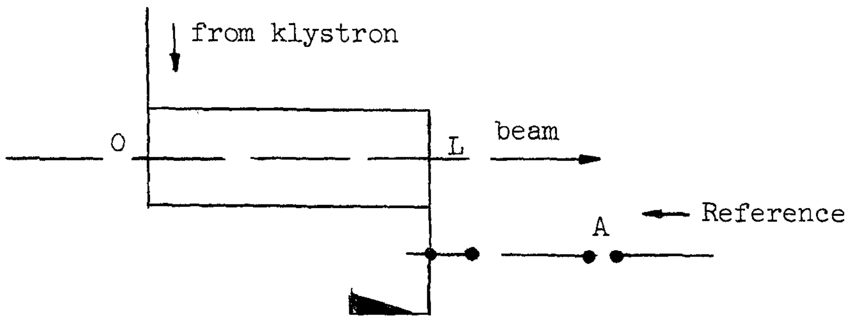


Figure 1

beam induced wave at the output of the accelerator section. It has been shown in this note that the phase at the output of the section is a function of the synchronism of the beam. Synchronism is determined by the operating frequency and the temperature of the section. It is important therefore to consider what is the effect of non-synchronism when the beam induction technique of phasing is employed.

8.1 The Effect of Non-Synchronism on the Beam Induction Technique of Phasing



Let,

φ_w be the phase angle between O and L for the klystron wave

φ_b be the phase angle between O and L for the beam induced wave

φ_o be the phase angle between L and A

$$\varphi_w = \frac{\omega \ell}{c} - \sigma\omega\tau \quad \text{(from Eq. 4)}$$

and,

$$\varphi_b = \frac{\omega \ell}{c} + \varphi$$ φ is calculated in Section 7.1, Eq. (17)

Now if,

ψ_w is the phase angle at A of the klystron wave

and,

ψ_b is the phase angle at A of the beam induced wave

then,

$$\psi_w = \delta + \varphi_w + \varphi_o$$

and

$$\psi_b = \varphi_b + \varphi_o$$

where δ is the phase angle between the wave and the beam at the input of the section which can be adjusted by the klystron phase shifter.

The automatic phasing system adjusts the angle between the klystron wave and the beam induced wave so that,

$$\psi_w - \psi_b = - \pi$$

whence,

$$\delta = \psi - \pi + \sigma\omega\tau$$

Assume now, that the beam pulse is larger than the rf filling time of the section and the induced wave is sampled during the steady state condition, then,

$$\delta = \sigma\omega\tau - \frac{\sigma\omega\tau}{2} = \frac{\sigma\omega\tau}{2} = - \frac{g}{2} \tag{23}$$

thus if σ is zero than δ is zero as would be expected.

A more interesting feature appears however, when the phase of the beam with respect to the wave δ, is compared with the optimum phase angle δ_{opt}, obtained theoretically.[2]

$$\tan \delta_{opt} = \frac{\sin \sigma\omega\tau + \sigma Q \left(\cos \sigma\omega\tau - e^{(\omega/Q)\tau}\right)}{\cos \sigma\omega\tau - \sigma Q \sin \sigma\omega\tau - e^{(\omega/Q)\tau}} \tag{24}$$

Substitute,

$$\tan \frac{\sigma\omega\tau}{2} = x \ , \quad \tan \delta_{opt} = \frac{N}{D}$$

Let us calculate $\delta - \delta_{opt} = \Delta$ then,

$$\tan(\Delta) = \frac{xD-N}{D+xN}$$

where,

$$D = \left[\frac{1-x^2}{1+x^2} - \frac{2\sigma Q x}{1+x^2} - e^{\frac{\omega}{Q}\tau}\right]$$

and

$$N = \frac{2x}{1+x^2} + \sigma Q\left[\frac{1-x^2}{1+x^2} - e^{\frac{\omega}{Q}\tau}\right]$$

therefore,

$$\tan(\Delta) = \frac{xP-\sigma QM}{M+\sigma QxP} \tag{25}$$

when $M = e^{\frac{\omega}{Q}\tau} + 1$ and $P = e^{\frac{\omega}{Q}\tau} - 1$ are not functions of σ. It can be immediately checked that,

$$\tan(\Delta) \to 0 \quad \text{or} \quad \delta-\delta_{opt} \to 0$$

when,

$$\sigma \to 0 \; (x \to 0)$$

In Section 8.2 it is shown that for practical values of σ, $\delta-\delta_{opt}$ is small.

8.2 Numerical Example for the Two Mile Machine

The following values are assumed

$$\omega = 2\pi \times 2.856 \times 10^9$$

$$\tau = 0.83 \; \mu\text{sec}$$

$$Q = 13{,}000$$

Then,

$$P = e^{\frac{\omega T}{Q}} + 1 = 4.14$$

and

$$M = e^{\frac{\omega T}{Q}} - 1 = 2.14$$

In the following table Δ is calculated for several values of σ or Δf.

$\Delta\omega/2\pi$ (kc/s)	σ (10^{-5})	δ (rad)	δ d^o	x	δ_{opt} (d^o)	Δ (d^o)
0	0	0	0	0	0	0
50	1.75	.1304	7.465 $\sim$ 7.5	.13	6.17	1.29
100	3.50	.2608	14.925 $\sim$ 15	.267	12.11	2.81
200	7.0	.5216	29.9 $\sim$ 30	.575	24.2	5.7
400	14.0	1.0432	59.8 $\sim$ 60	1.718	47.8	12.0

Within the tuning range of the accelerator, which is $\pm$ 100 kc/sec, $\Delta \leq 3$ degrees, which means that for practical purposes the beam induction technique keeps δ close to the optimum value.

8.3 Loss of Beam Energy

$$V_t = V + V_i$$

and since

$$\frac{\partial V_i}{\partial \delta} = 0 \qquad (26)$$

then,

$$\frac{\partial V_t}{\partial \delta} = \frac{\partial V}{\partial \delta} \tag{27}$$

It has been shown[2]

$$V(\partial,\sigma) = I_c \cos\delta - I_s \sin\delta \tag{28}$$

where I_c and I_s are not functions of δ. When $\delta \approx \delta_{opt}$, V can be expanded in a Taylor's series as follows,

$$V(\delta_{opt}+\epsilon) = V(\delta_{opt}) + \epsilon \frac{\partial V}{\partial \delta}(\delta_{opt}) + \frac{\epsilon^2}{2} \frac{\partial^2 V}{\partial \delta^2}(\delta_{opt})$$

but from the definition of δ_{opt},

$$\frac{\partial V}{\partial \delta}(\delta_{opt}) = 0$$

and,

$$\frac{\partial^2 V}{\partial \delta^2} = - V$$

$$\therefore \quad V(\delta_{opt}+\epsilon) = V(\delta_{opt}) \left[1 - \frac{\epsilon^2}{2}\right]$$

so,

$$\frac{\Delta V}{V} = 1 - \frac{\epsilon^2}{2} \tag{29}$$

whence for $\epsilon = \Delta < 3^\circ$

$$\frac{\Delta V}{V} < 0.15\%$$

Which means that for practical purposes the beam induction technique of phasing has the property that frequency and temperature drifts are automatically compensated for if the machine is rephased.

9.0 EXPERIMENTAL RESULTS

An experiment has been performed on the Mark IV accelerator for checking the behavior of a two-mile accelerator section in the non-synchronous case.

We used the second section of Mark IV. The corresponding klystron is turned off. We then turned the beam on and analyzed the section output signal in phase with respect to the beam and in amplitude.

The non-synchronous state is obtained by changing the section temperature rather than the frequency in order to keep all the other parameters constant.

So, $\sigma = -1.6 \cdot 10^{-5} \times \frac{5}{9} \Delta T_p$. When $1.6 \cdot 10^{-5}$ is the linear expansion coefficient of the copper and ΔT_p is the temperature change in degrees Farenheit. The beam reference is picked up out of a cavity excited by the beam.

9.1 Phase

The theoretical change in phase is a linear function of the frequency change (Eq. (17)) and the slope calculated in the previous Section 8.2 was found to be

$$\frac{\Delta\varphi}{\Delta f} = 149.25 \ d^{o}/Mc/s$$

On the figure N'_2 we plotted the measured $\Delta\varphi$ against the equivalent $\Delta f = \sigma f$. We come up with a good straight line, the average slope of which is = 152 d^{o}/Mc/s. This result can be considered as a good confirmation of the theory within our hypotheses and the limited accuracy of the temperature setting up and measurement.

9.2 Amplitude

The relative output power is

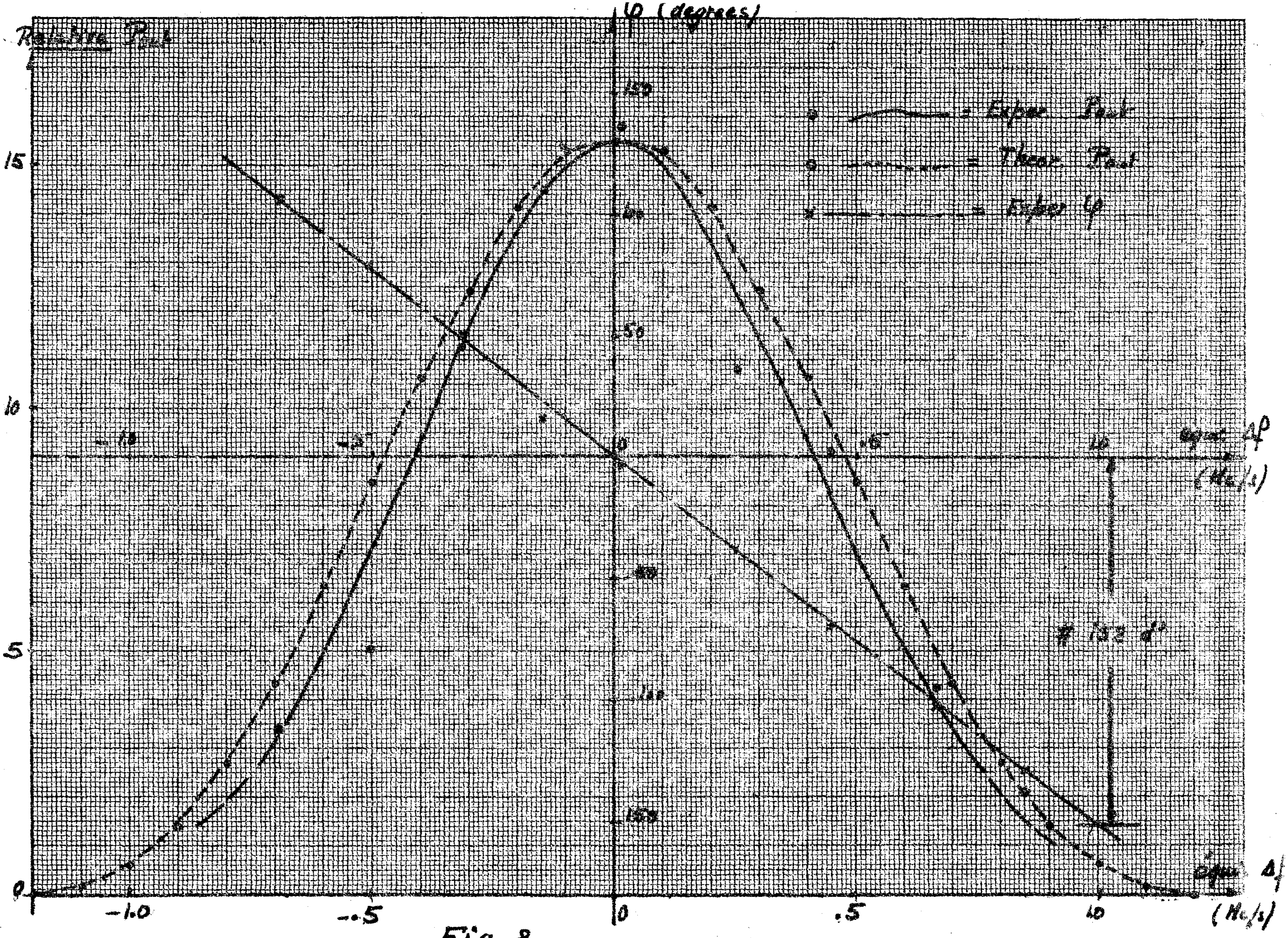
$$\Gamma^2 = \left[\frac{\sin\left(\frac{\sigma\omega\tau}{2}\right)}{\frac{\sigma\omega\tau}{2}} \right]^2$$

which is tabulated below.

Δf (Mc/s)	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0	1.1	1.2
Γ^2	1	.97	.912	.801	.687	.548	.41	.281	.174	.09	.04	.009	0

We cannot cover all the ranges since $\Delta f = 1.2$ Mc/s is equivalent to $\Delta T_p = 47.5$ d^{o}F and cannot be reached by the section heating system. However we plotted both theoretical and experimental variation on the same paper (Fig. 2) and we found them not far from each other. The experimental value seems to be systematically smaller than the theoretical one reaching a maximum difference of about 1.2 db at half amplitude points due maybe to the output coupler mismatch.

The synchronous point from the experimental amplitude curve is found to be $f_o = 2856.3$ Mc/s at $T_p = 102$ d^{o} F.



φ (degrees)
Relative Pout
= Exper. Pout
= Theor. Pout
= Exper. φ
equi. Δf (Mc/s)
132 d°
KEUFFEL & ESSER CO. MADE IN U.S.A.

Fig. 2

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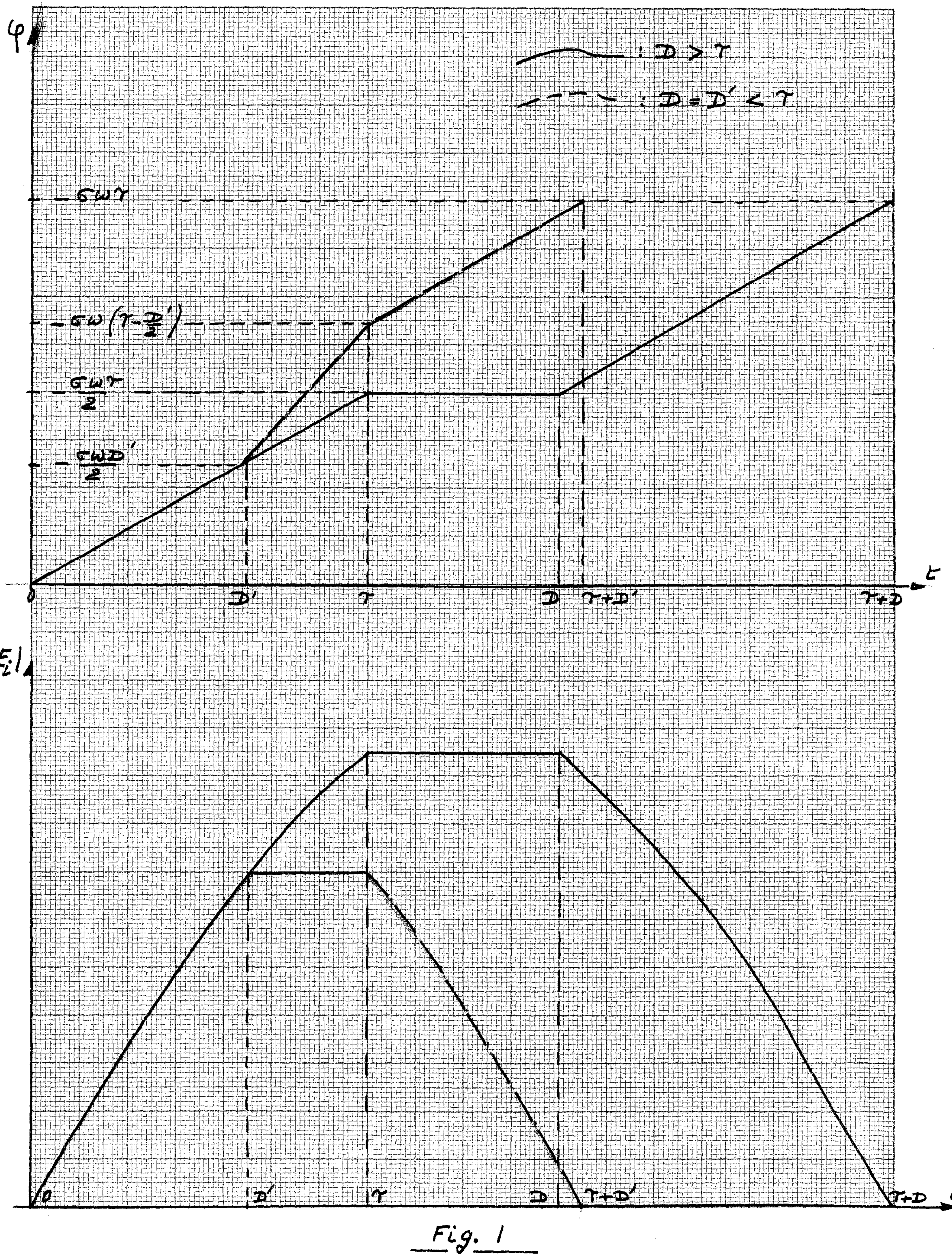

Fig. 1

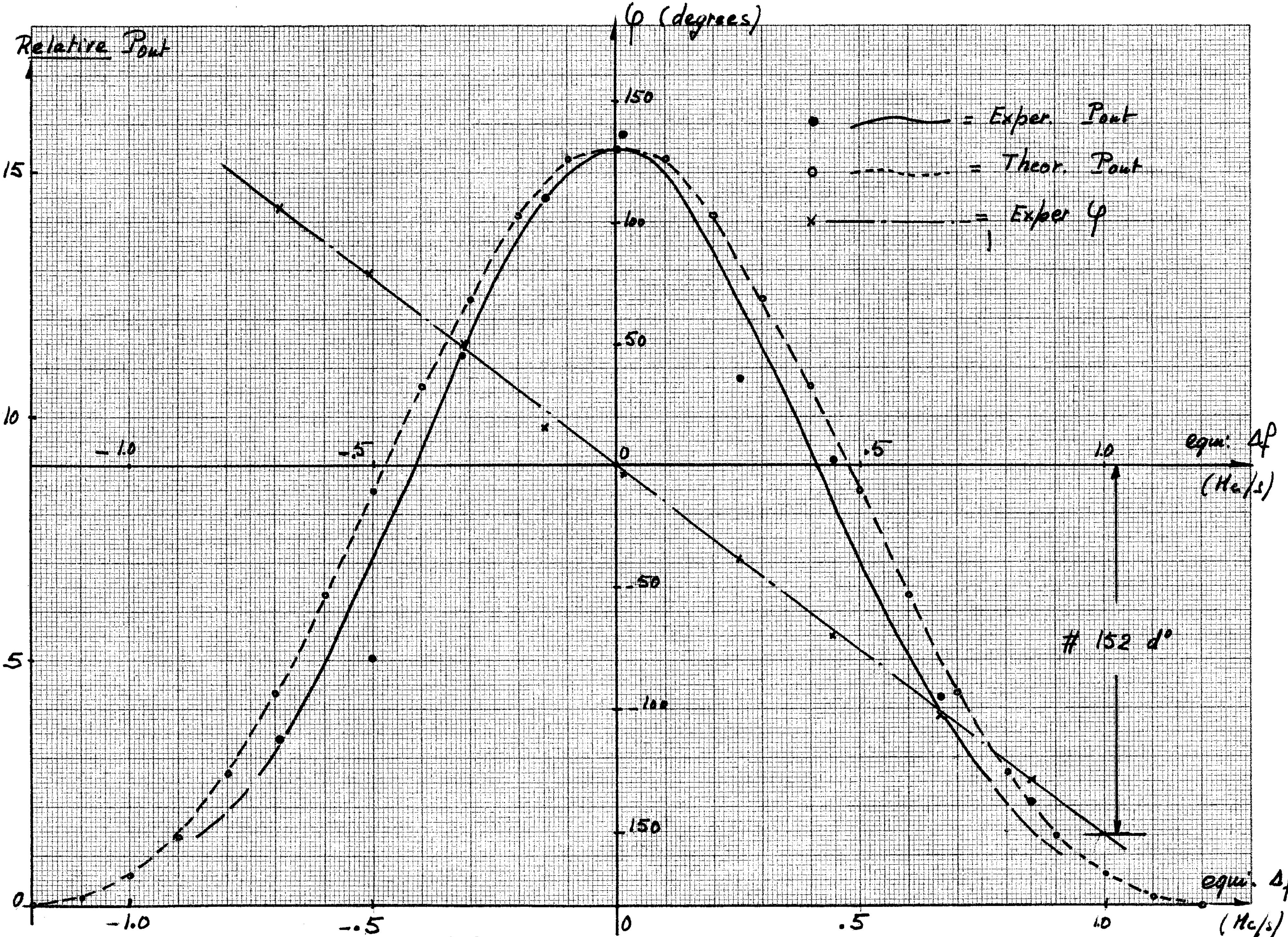
Relative Pout
φ (degrees)
150
100
50
-50
-100
150
15
10
5
0
= Exper. Pout
= Theor. Pout
= Exper φ
equ. Δf
(Mc/s)
-1.0
-.5
0
.5
1.0
152 d°

Fig. 2