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Correspondance AGT pour les opérateurs de surface
AGT correspondence for surface operators

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Résumé. La fonction de partition de théories de jauge supersymétriques avec quatre supercharges sur la sphere à deux dimensions est calculée exactement grâce à la localisation supersymétrique. Pour certaines théories de jauge, les expressions explicites sont égales à des corrélateurs dans la théorie conforme des champs de Toda de dimension deux. Ces égalités trouvent leur place au sein de la correspondance AGT, qui relie des théories de jauge supersymétriques de dimension quatre avec huit supercharges à des corrélateurs de la théorie de Toda. En effet, les théories de jauge à deux dimensions peuvent être insérées le long d'une surface dans une théorie à quatre dimensions, formant ainsi un opérateur de surface à moitié BPS. Une telle insertion correspond à l'ajout d'un opérateur local particulier (un opérateur de vertex dégénéré) dans le corrélateur de Toda.

Cette correspondance enrichie a plusieurs conséquences. D'une part, les symétries des corrélateurs de Toda impliquent des analogues des dualités de Seiberg et de Kutasov–Schwimmer pour les théories de jauge à deux dimensions avec quatre supercharges. D'autre part, les résultats exacts en théorie de jauge fournissent de nouvelles données dans la théorie de Toda. Cela mène à une proposition concrète pour l'échange de deux opérateurs de vertex semi-dégénérés dans la théorie de Toda, qui contient des informations importantes concernant la S-dualité à quatre dimensions.

Mots clef: théorie de jauge, théorie conforme des champs, supersymétrie, opérateur de surface, correspondance AGT, théorie de Toda.

Abstract. The sphere partition function of two-dimensional supersymmetric gauge theories with four supercharges is computed exactly using supersymmetric localization. For some gauge theories, explicit expressions are found to match with correlators in the two-dimensional Toda conformal field theory. This fits into the AGT correspondence, which relates supersymmetric four-dimensional gauge theories with eight supercharges to correlators in the Toda theory. More precisely, the two-dimensional gauge theories can be inserted along a surface in a four-dimensional theory, thus forming half-BPS surface operators: such an insertion corresponds to the addition of a particular local operator (a degenerate vertex operator) in the Toda correlator.

This enriched correspondence has several consequences. On the one hand, symmetries of Toda correlators imply analogues of Seiberg and Kutasov–Schwimmer dualities for two-dimensional gauge theories with four supercharges. On the other hand, exact gauge theory results yield previously unknown data in the Toda theory. This leads to a concrete proposal for the Toda braiding kernel of two semi-degenerate vertex operators, which holds important information about four-dimensional S-duality.

Keywords: gauge theory, conformal field theory, supersymmetry, surface operator, AGT correspondence, Toda theory.

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Chapitre 1F

Présentation des travaux

Le modèle standard de la physique des particules a été confirmé expérimentalement avec une précision remarquable, mais ce n'est pas une description complète de l'Univers. Outre les difficultés bien connues concernant la gravité et certaines observations astrophysiques, un problème plus basique est de déterminer ce que le modèle standard prédit. En effet, tandis qu'à haute énergie tous les couplages de la théorie sont faibles, la constante de couplage de la force nucléaire forte croît à basse énergie. La théorie des perturbations est alors inopérante, puisque les séries en puissances du couplage divergent. Bien qu'en principle la divergence puisse être guérie en prenant en compte des effets non-perturbatifs, ceux-ci sont extrèmement difficiles à étudier dans une théorie quantique des champs générale telle que le modèle standard. Cependant, lorsque la théorie est supersymétrique —une symétrie entre les bosons et les fermions— des résultats non-perturbatifs sont disponibles.

Une théorie supersymétrique de dimension quatre peut être invariante sous $\mathcal{N} = 1$, 2, ou 4 familles de quatre supercharges. Chaque supersymétrie contraint la théorie davantage : par exemple, les théories avec $\mathcal{N} = 4$ supersymétries sont fixées complètement par leur groupe de jauge, tandis que les théories $\mathcal{N} = 2$ décrivent à la fois de la matière et des intéractions de jauge. Malgré leur manque d'applications expérimentales (seul $\mathcal{N} = 1$ est expérimentalement tenable), ces théories permettent d'étudier les phénomènes non-perturbatifs. Elles forment un équilibre entre une diversité des théories et la disponibilité d'expressions exactes.

Des progrès considérables ont eu lieu au cours des vingt dernières années dans la compréhension des théories $\mathcal{N} = 2$. Un développement crucial a été la correspondance AGT (Alday, Gaiotto, Tachikawa) [AGT09], qui relie un grand nombre de théories $\mathcal{N} = 2$ de dimension quatre à des corrélateurs dans la CFT (théorie conforme des champs) de Toda de dimension deux, cousine de la CFT de Liouville. Les boucles de Wilson et d'autres opérateurs non-locaux —dont le support est une variété plutôt qu'un point— dans les théories $\mathcal{N} = 2$ correspondent à divers objets dans la CFT de Toda.

Ce travail étend la correspondance AGT à une classe d'opérateurs de surface dans les théories $\mathcal{N} = 2$, et décrit des prérequis et des conséquences de la correspondance enrichie. Les opérateurs étendus considérés sont construits en couplant la théorie $\mathcal{N} = 2$ de dimension quatre avec une théorie supersymétrique de dimension deux placée sur une surface. Plus précisément, la théorie de dimension deux a $\mathcal{N} = (2, 2)$ supersymétries (4 supercharges), car cela permet aux opérateurs de surface de ne briser que la moitié des 8 supercharges de la supersymétrie $\mathcal{N} = 2$ à quatre dimensions.

Le Chapitre 2 concerne les théories de dimension deux. Il reproduit l'article *Exact Results in $D = 2$ Supersymmetric Gauge Theories* [DGLFL12] (Résultats exacts dans les théories supersymétriques de dimension 2) de Nima Doroud, Jaume Gomis, Sungjay Lee et l'auteur, calculant la fonction de partition sur S^2 de théories de jauge avec $\mathcal{N} = (2, 2)$ supersymétries.

Dans le Chapitre 3, les théories de dimensions deux et quatre sont combinées en les plaçant respectivement sur $S^2 \subset S^4$. Ce chapitre reproduit la première moitié de l'article *M2-brane surface operators and gauge theory dualities in Toda* [GLF14] (Opérateurs de surfaces de M2-branes et dualités de théories de jauge dans Toda) de Jaume Gomis et l'auteur. Nous trouvons de nombreux opérateurs de surfaces $\mathcal{N} = (2, 2)$ dont la fonction de partition sur $S^2 \subset S^4$ est égale à un corrélateur dans la CFT de Toda.

Les symétries manifestes des corrélateurs de la CFT de Toda fournissent ensuite dans le Chapitre 4 (la deuxième moitié de [GLF14]) des dualités non-triviales entre des théories de jauge $\mathcal{N} = (2, 2)$ de dimension deux. Ces analogues de dimension deux de la dualité de Seiberg de dimension quatre indiquent que deux théories de jauge ont la même limite infrarouge. Les deux théories de jauge décrivent donc la même physique à grande distance en termes de différents degrés de liberté microscopiques.

Enfin, le Chapitre 5 explore la CFT de Toda. Il peut être lu indépendamment, bien que certaines expressions ont été initialement obtenues à travers la correspondance AGT. Le résultat principal est une conjecture (soumise à des tests importants) pour le noyau intégral décrivant le tressage[1] de deux opérateurs de vertex semi-dégénérés, définis plus tard. Ce tressage correspond à une dualité importante en quatre dimensions : la S-dualité de la chromodynamique quantique superconforme avec groupe de jauge $SU(N)$.

Ce chapitre, une traduction du Chapitre 1, introduit le contexte du travail et résume chaque chapitre. Il présente les théories $\mathcal{N} = 2$ (Section 1F.1), la CFT de Toda (Section 1F.2) étudiée plus avant dans le Chapitre 5, la localisation supersymétrique et les résultats du Chapitre 2 concernant les fonctions de partition sur S^2 (Section 1F.3), l'inclusion par le Chapitre 3 d'opérateurs de surface dans la correspondance AGT qui relie les théories $\mathcal{N} = 2$ avec la CFT de Toda (Section 1F.4), et les dualités trouvées dans le Chapitre 4 entre théories $\mathcal{N} = (2, 2)$ à deux dimensions (Section 1F.5).

[1] Certains termes techniques issus de l'anglais n'ont pas de traduction canonique.

1F.1 Théories de jauge $\mathcal{N} = 2$ à quatre dimensions

Cette section concerne les théories de jauge de dimension quatre avec $\mathcal{N} = 2$ supersymétries, c'est-à-dire les théories invariantes sous deux familles de 4 supercharges (voir les articles de revue [Tac13 ; Tes14] en anglais). Elle sert de préparatif à la Section 1F.4 sur la correspondance AGT. De nombreuses propriétés des théories $\mathcal{N} = 2$ sont omises, en particulier les courbes de Seiberg–Witten.

Les champs d'une théorie de jauge $\mathcal{N} = 2$ se décomposent en des multiplets vecteurs (multiplets de jauge) et des hypermultiplets (multiplets de matière). On peut décomposer ces deux types de supermultiplets en supermultiplets d'une sous-algèbre $\mathcal{N} = 1$. Un hypermultiplet se compose d'une paire de multiplets chiraux, donc de champs scalaires complexes et de leurs superpartenaires spineurs, qui se transforment tous dans la même représentation du groupe de jauge. Un multiplet vecteur $\mathcal{N} = 2$ se compose d'un multiplet vecteur $\mathcal{N} = 1$ et d'un multiplet chiral, autrement dit d'un boson de jauge et de ses superpartenaires dans la représentation adjointe du groupe de jauge. Les couplages lagrangiens de ces supermultiplets qui préservent $\mathcal{N} = 2$ supersymétries sont plus contraints que dans les théories $\mathcal{N} = 1$, et se réduisent essentiellement à un prépotentiel holomorphe pour le multiplet vecteur.

L'example typique d'une théorie de jauge $\mathcal{N} = 2$ est la SQCD (chromodynamique quantique supersymétrique) $SU(N)$ avec N_f saveurs, constituée d'un multiplet vecteur $\mathcal{N} = 2$ avec groupe de jauge $SU(N)$ couplé à N_f hypermultiplets dans la représentation fondamentale (de dimension N) de $SU(N)$. La fonction beta de la constante de couplage de jauge est proportionnelle à $2N - N_f$ à une boucle, et les théorèmes de renormalisation impliquent que la fonction beta exacte l'est aussi. La théorie est donc asymptotiquement libre pour $N_f < 2N$, exactement conforme pour $N_f = 2N$ (en l'absence de terme de masse), et n'est pas UV-complète pour $N_f > 2N$.

Pour $N > 2$, le groupe de symétries de saveurs de la SQCD $SU(N)$ avec N_f saveurs est $U(N_f)$. Pour $N = 2$, chaque hypermultiplet se scinde en deux demi-hypermultiplets qui sont chacun une représentation de la supersymétrie $\mathcal{N} = 2$, et la symétrie de saveur est augmentée : $SO(2N_f) \supset U(N_f)$. Techniquement, une telle scission et augmentation de la symétrie se produisent pour tout hypermultiplet se transformant dans une représentation pseudo-réelle du groupe de jauge.

Seiberg et Witten [SW94a ; SW94b] ont déterminé en 1994 les vides quantiques de la SQCD $\mathcal{N} = 2$ $SU(2)$ avec $0 \leq N_f \leq 4$ saveurs. Ces auteurs ont déterminé le prépotentiel exact, d'où on peut extraire par exemple les masses des bosons W et des dyons. Ils ont trouvé que la SQCD avec $N_f = 4$ est sujette à la S-dualité : la théorie peut être décrite par des Lagrangiens écrits en termes de différents degrés de liberté fondamentaux.

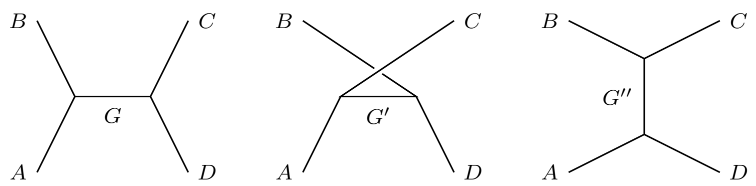


FIGURE 1 : Trois descriptions lagrangiennes de la SQCD $SU(2)$ $N_f = 4$. Les arêtes sont des groupes de symétrie $SU(2)$. Les arêtes extérieures sont des groupes de saveur, tandis que les arêtes intérieures sont des groupes de jauge (donc des multiplets vecteurs). Les sommets où trois arêtes se rencontrent sont des hypermultiplets dans la représentation trifondamentale du groupe $SU(2)^3$ représenté par les trois arêtes.

En termes de la constante de couplage (complexifiée) $\tau = 8\pi\mathrm{i}/g^2 + \vartheta/\pi$, la S-dualité exprime le fait que les Lagrangiens $N_f = 4$ avec un certain couplage τ et avec le couplage dual $\tau^D = -1/\tau$ décrivent la même physique. De cette façon, la S-dualité donne une description faiblement couplée ($\tau^D \to \infty$, $g^D \to 0$) d'une région de l'espace des paramètres où le Lagrangien initial est fortement couplé ($\tau \to 0$, $g \to \infty$). La dualité se généralise en $\tau^D = \frac{a\tau+b}{c\tau+d}$ pour n'importe quelle matrice $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ dans $SL(2,\mathbb{Z})$.

Le groupe $SL(2,\mathbb{Z})$ de S-dualité agit aussi par automorphismes sur le groupe de symétries de saveurs $SO(2N_f) = SO(8)$. Cette action est commodément décrite en séparant les hypermultiplets en deux paires qui ont chacune une symétrie de saveur $SO(4) \overset{\sim}{\twoheadleftarrow} SU(2)^2$. Dans chaque description S-duale de la théorie, la symétrie manifeste $SO(4) \times SO(4)$ est obtenue en groupant les facteurs de $SU(2)^4 \subset SO(8)$ par paire d'une des trois façons possibles. Concrètement, on peut inclure des masses $m_{A,B,C,D}$ pour chacun des facteurs $SU(2)$. Dans une description lagrangienne les $N_f = 4$ hypermultiplets ont pour masses $|m_A \pm m_B|$ et $|m_C \pm m_D|$. Après une S-dualité, les masses sont $|m_A \pm m_C|$ et $|m_B \pm m_D|$, ou $|m_A \pm m_D|$ et $|m_B \pm m_C|$.

Gaiotto [Gai09a] a generalisé la S-dualité à une grande classe de théories $\mathcal{N} = 2$ de dimension quatre $\mathcal{T}_{g,n}$, maintenant nommées théories de classe S.

Les groupes de symétrie de saveur d'une théorie $\mathcal{N} = 2$ peuvent être jaugés par un multiplet vecteur de la même façon que les symétries globales d'une théorie non-supersymétrique sont jaugées par un boson de jauge. Ainsi, n'importe laquelle des quatre symétries de saveur $SU(2)$ de la SQCD $SU(2)$ avec $N_f = 4$ peut être promue à un group de jauge grâce à un multiplet vecteur $SU(2)$. Ce multiplet vecteur additionnel peut être couplé à une paire d'hypermultiplets pour que la théorie reste conforme, et celle-ci possède une symétrie de saveur $SU(2) \times SU(2)$. Répéter la procédure avec les nouvelles symétries $SU(2)$ génère un grand nombre de Lagrangiens $\mathcal{N} = 2$ superconformes : ceux-ci décrivent les théories $SU(2)$ de class S.

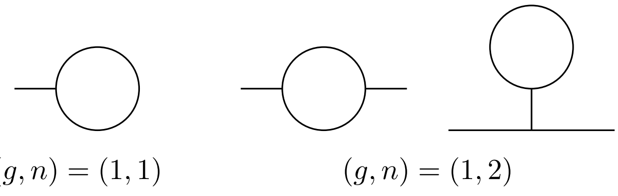


FIGURE 2 : Exemples de graphes trivalents avec g boucles et n arêtes extérieures. Le premier représente un multiplet vecteur $SU(2)$ couplé à un hypermultiplet adjoint, qui a une symétrie de saveur $SU(2)$. Cette théorie est appelée $\mathcal{N} = 2^*$ si l'hypermultiplet est massif et a $\mathcal{N} = 4$ supersymétries en l'absence de massse. Les graphes à droite représentent deux multiplets vecteurs couplés à deux hypermultiplets de façons différentes. Les deux Lagrangiens se trouvent en fait être S-duaux, et décrivent la même théorie.

Chacune des paires d'hypermultiplets de la SQCD $SU(2)$ avec $N_f = 4$ se transforment dans la représentation trifondamentale de groupes de jauge et de saveur $SU(2)^3$. La théorie $N_f = 4$ peut donc être vue comme des hypermultiplets trifondamentaux de $SU(2)_A \times SU(2)_B \times SU(2)_G$ et de $SU(2)_G \times SU(2)_C \times SU(2)_D$ dont la symétrie commune $SU(2)_G$ est jaugée par un multiplet vecteur. Comme décrit ci-dessus, les Lagrangiens S-duaux combinent les $SU(2)_{A,B,C,D}$ en paires de trois façons possibles, représentées par des graphes dans la Figure 1. Chaque hypermultiplet trifondamental est représenté par un sommet relié à trois arêtes représentant des groupes de symétrie $SU(2)$. Les arêtes extérieures sont des symétries de saveur et les arêtes intérieures des groupes de jauge. En utilisant ce dictionaire, chaque graphe trivalent (avec trois arêtes par sommet) correspond à un Lagrangien superconforme composé d'hypermultiplets trifondamentaux et de multiplets vecteurs (voir la Figure 2 pour des exemples).

On considère un graphe trivalent représentant un Lagrangien. Dans la limite où tous les couplages $SU(2)$ sauf un (correspondant à une arête intérieure) sont infiniment petits, la théorie de jauge $SU(2)$ restante, simplement SQCD avec $N_f = 4$, est sujette à la S-dualité. La dualité reconnecte de n'importe quelle façon les quatre arêtes qui touchent l'arête choisie. Cette propriété reste valable même lorsque les autres couplages sont non-nuls. En reconnectant les arêtes par des S-dualités, on peut transformer un graphe trivalent en n'importe quel autre avec le même nombre d'arêtes intérieures et extérieures. Autrement dit, tous les graphes avec g boucles et n arêtes extérieures correspondent à des Lagrangiens qui décrivent la même théorie $\mathcal{T}_{g,n}$ en termes de degrés de liberté différents.

Pour décrire l'action de la S-dualité sur les constantes de couplage de jauge ($\tau \mapsto -1/\tau$ pour la SQCD), il faut plus de structure que les graphes. Les couplages de $\mathcal{T}_{g,n}$ sont en fait décrits par une surface de Riemann $C_{g,n}$ avec genre g et n ponctions. La surface est obtenue de n'importe quel graphe

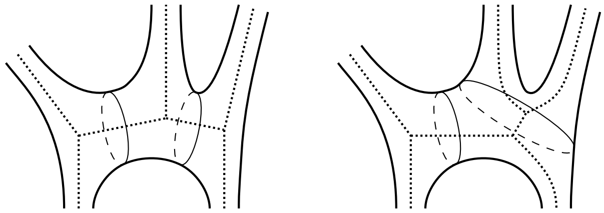


FIGURE 3 : Deux décompositions de la surface de Riemann $C_{0,5}$ en tubes (dessinés comme des ellipses) et trinions (sphères privées de trois points entre les tubes et les ponctions extérieures), et leur graphe trivalent (en pointillés). Les Lagrangiens correspondants décrivent la même théorie $\mathcal{T}_{0,5}$.

avec g boucles et n arêtes extérieures en "épaississant" le graphe, c'est-à-dire en remplaçant chaque arête par un tube et chaque sommet trivalent par un trinion (sphère avec trois ponctions) joignant les trois cylindres. La longueur et l'angle de torsion de chaque tube encode la constante de couplage pour le groupe de jauge associé à cette arête du graphe, de sorte qu'un long cylindre corresponde à un multiplet vecteur faiblement couplé. La S-dualité est alors obtenue en notant que $C_{g,n}$ peut être coupée en tubes et trinions de nombreuses façons, caractérisées par différents graphes trivalents (voir Figure 3). Chaque décomposition de $C_{g,n}$ correspond à une description lagrangienne de $\mathcal{T}_{g,n}$.

Plusieurs théories dans cette classe ont une importance particulière. $\mathcal{T}_{0,3}$ est la théorie de 4 hypermultiplets libres, avec aucun multiplet vecteur puisque son graphe n'a aucune arête intérieure. $\mathcal{T}_{0,4}$ est la SQCD $SU(2)$ avec $N_f = 4$, et depend d'un seul couplage complexifié. La sphère privée de quatre points $C_{0,4}$ est un tube joignant deux trinions. Sa structure complexe ne dépend que du birapport q des quatre points, et changer de décomposition en trinions transforme $q \mapsto 1 - q$ ou $1/q$. Le dernier exemple, $\mathcal{T}_{1,1}$, décrit un multiplet vecteur $SU(2)$ jaugeant deux symétries de saveur $SU(2)$ d'un même hypermultiplet trifondamental : cela résulte en un hypermultiplet dans la représentation adjointe du groupe de jauge et dans la représentation fondamentale du dernier $SU(2)$. Cette théorie, appelée $\mathcal{N} = 2^*$ SYM (super Yang–Mills) lorsque l'hypermultiplet est massif, a une supersymétrie élargie $\mathcal{N} = 4$ lorsque l'hypermultiplet est de masse nulle.

Comme expliqué dans [Gai09a] en utilisant les courbes de Seiberg–Witten, la théorie $\mathcal{T}_{g,n}$ est la réduction à quatre dimensions de la mystérieuse théorie des champs superconforme A_1 $(2, 0)$ à six dimensions placée sur $C_{g,n}$ avec certaines conditions de bords aux n ponctions. Cette théorie n'est pas connue directement, mais ses réductions à diverses dimensions plus basses sont connues. Par exemple, sa réduction sur un cercle est la théorie SYM maximalement supersymétrique avec groupe de jauge $SU(2)$ en dimension cinq, dont la réduction sur un segment est le multiplet vecteur $\mathcal{N} = 2$ associé à chaque cylindre de $C_{g,n}$ dans la description ci-dessus.

En théorie M, la théorie A_1 $(2,0)$ est la théorie de deux M5-branes coïncidentes. Ces branes sont alors enroulées sur la surface de Riemann $C_{g,n}$, dont les ponctions sont réalisées par des M5-branes transverses. Ce type de combinaisons de branes donne des intuitions utiles dans la Section 1F.4 concernant les opérateurs étendus dans les théories $\mathcal{N} = 2$.

Un cousin proche de la théorie A_1 $(2,0)$ est la théorie des champs superconforme A_{N-1} $(2,0)$ à six dimensions, qui décrit N M5-branes coïncidentes. Compactifiée sur une surface de Riemann $C_{g,n}$ avec certaines conditions de bord à chaque ponction, la théorie donne une théorie de jauge $\mathcal{N} = 2$ de dimension quatre, avec des groupes de jauge $SU(N)$. L'ensemble de ces théories $\mathcal{N} = 2$ est appelé classe S. En l'absence de termes de masse ces théories sont superconformes.

L'exemple standard de théorie de class S est la SQCD $\mathcal{N} = 2$ $SU(N)$ avec $2N$ hypermultiplets fondamentaux, obtenue en compactifiant A_{N-1} $(2,0)$ sur la sphère privée de quatre points $C_{0,4}$. Pour $N > 2$, la symétrie de saveur de la SQCD est $U(2N)$, dont un sous-groupe $SU(N) \times U(1) \times U(1) \times SU(N)$ est rendu manifeste par la construction six-dimensionnelle. Une différence importante d'avec le cas $N = 2$ est que les quatre facteurs ne sont pas identiques. En conséquence, les ponctions sur $C_{0,4}$ doivent être munies de conditions de bords différentes. Deux ponctions, dites pleines, supportent une symétrie de saveur $SU(N)$, et les deux autres, dites simples, supportent une symétrie de saveur $U(1)$. De nombreux autres types de ponctions existent.

À nouveau, les descriptions S-duales sont caractérisées par des décompositions de $C_{g,n}$ en trinions. À un trinion avec une ponction simple et deux pleines est associé N^2 hypermultiplets avec pour symétrie de saveur manifeste le groupe $U(1) \times SU(N) \times SU(N)$. Joindre les ponctions pleines de deux trinions correspond à jauger les deux symétries de saveur $SU(N)$ diagonalement. Dans le cas de la SQCD, les deux décompositions de $C_{0,4}$ où chaque trinion a une ponction simple correspondent à des descriptions en terme d'un multiplet vecteur $SU(N)$ couplé à deux fois N hypermultiplets fondamentaux. Lorsque les deux ponctions pleines appartiennent au même trinion, il n'y a aucune description lagrangienne : on couple toujours deux théories en jaugeant une symétrie de saveur commune, mais la théorie correspondant à l'un des trinions n'est pas décrite par un Lagrangien. Plus généralement, tandis que toutes les théories de classe S avec groupes de jauge $SU(2)$ sont décrites par des Lagrangiens, les théories de class S pour $N > 2$ n'ont de description lagrangienne que lorsque chaque trinion d'une décomposition contient une ponction simple.

Étant donné qu'une théorie de class S ne dépend que de la structure complexe de $C_{g,n}$ et de données à chaque ponction, n'importe quel observable de la théorie à quatre dimensions peut en principe être obtenu par un calcul sur $C_{g,n}$. En pratique, l'identification est en général obtenue en calculant des observables à quatre dimensions et en trouvant un calcul à deux dimensions donnant le même résultat. La correspondance AGT [AGT09] (Section 1F.4) consiste en un dictionnaire concret entre plusieurs observables obtenus par localisation supersymétrique sur des sphères (Section 1F.3) et des corrélateurs dans la CFT de Toda (Section 1F.2) sur $C_{g,n}$.

1F.2 Théorie conforme des champs de Toda

La théorie de Toda A_{N-1} est une CFT de dimension deux dont l'algèbre de symétrie W_N étend l'algèbre de Virasoro par l'ajout de courants de spin élevé. La théorie de Toda A_1 ($N = 2$) est la CFT bien connue de Liouville, et W_2 est l'algèbre de Virasoro. Cette section rappelle des notions de bases sur les CFT de dimension deux (voir l'article de revue [Rib14]), jusqu'au noyau intégral de tressage d'opérateurs primaires. Elle décrit ensuite l'effet de la symétrie W_N, et la conjecture explicite (5.3.27) pour le noyau intégral de certains opérateurs primaires de W_N. Cette introduction à la CFT de Toda suffit pour lire le reste de la thèse, qui se termine avec une étude plus détaillée de la CFT de Toda dans le Chapitre 5.

La symétrie conforme à deux dimensions implique une action de (deux copies de) l'algèbre de Virasoro sur les états de la théorie. Les deux copies sont dues aux transformations conformes holomorphes et antiholomorphes, et peuvent être traîtées indépendamment. La symétrie conforme implique aussi que la correspondance état-opérateur est une bijection entre les opérateurs ϕ et les états $|\phi\rangle$ obtenus en agissant sur le vide.

L'algèbre de Virasoro est générée par L_n pour $n \in \mathbb{Z}$, sujets à $L_n^\dagger = L_{-n}$ et ayant pour commutateurs $[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m+n}$. Un état de plus haut poids est $|h\rangle$ tel que $L_0\,|h\rangle = h\,|h\rangle$ et $L_n\,|h\rangle = 0$ pour $n > 0$, et l'opérateur correspondant est appelé opérateur primaire de dimension h. L'action des L_{-n} pour $n > 0$ génère un module de Verma : une représentation de l'algèbre de Virasoro dont les états sont des combinaisons linéaires de $L_{-n_1} \cdots L_{-n_p}\,|h\rangle$ pour $n_j > 0$. Un tel état est nommé descendant de $|h\rangle$ au niveau $\sum_j n_j$. Un opérateur primaire et ses descendants forment une famille conforme.

Il est utile de paramétrer la charge centrale par $c = 1+6q^2$ avec $q = b+1/b$, et d'exprimer la dimension $h = \alpha(q-\alpha)$ d'un opérateur primaire V_α en terme d'une impulsion $\alpha \in \mathbb{C}$.

La symétrie conforme exprime les corrélateurs d'opérateurs descendants en termes de corrélateurs d'opérateurs primaires. Elle force le corrélateur de deux opérateurs primaires sur la sphère à s'annuler sauf si les deux opérateurs ont la même dimension. Elle fixe aussi la dépendance de la fonction à trois-points $\langle V_\alpha V_\beta V_\gamma\rangle$ dans les positions des opérateurs, à un facteur global $C(\alpha,\beta,\gamma)$ près. Toutes les fonctions à n-points d'opérateurs primaires sur la sphère sont ensuite fixées comme suit en termes des fonctions à trois points $C(\alpha,\beta,\gamma)$, aussi appelées constantes de structure.

N'importe quelle paire d'opérateurs primaires peut être remplacée par leur OPE (expansion de produit d'opérateurs), une combinaison linéaire d'opérateurs primaires et de descendants dont les coefficients sont fixés par la symétrie conforme en termes des constantes de structure. La fonction à n points est ainsi exprimée comme une intégrale (ou une somme) sur les familles conformes d'une constante de structure multipliée par une fonction à $(n-1)$ points et par un facteur qui prend en compte les contributions de descendants. En répétant le procédé, toute fonction à n points est exprimée comme une intégrale de produits de $n-2$ fonctions à trois points multipliées par un facteur qui est fixé par la symétrie conforme. Ce facteur conforme se factorise en un bloc conforme $\mathcal{F}$ holomorphe dans les positions des opérateurs, multiplié par un bloc conforme antiholomorphe. En omettant certains détails tels que les inverses de fonctions à deux points, on trouve

$$\begin{aligned}\langle V_{\alpha_1}\cdots V_{\alpha_n}\rangle = \int \mathrm{d}\beta_3\cdots\mathrm{d}\beta_{n-1}\, C(\alpha_1,\alpha_2,\beta_3)\cdots C(\beta_{n-1},\alpha_{n-1},\alpha_n)\\ \cdot\left|\mathcal{F}\left[\begin{matrix} & \alpha_2 & & \alpha_3 & & \alpha_{n-2} & & \alpha_{n-1} & \\ \alpha_1 & | & \beta_3 & | & \cdots & | & \beta_{n-1} & | & \alpha_n\end{matrix}\right]\right|^2 .\end{aligned} \tag{1F.2.1}$$

Le graphe trivalent décrit quels OPEs ont été effectués, et garde une trace des impulsion en résultant. Il y a une constante de structure pour chaque sommet de ce graphe trivalent. Les impulsions α_i sont nommées impulsions extérieures, tandis que les impulsions intérieures β_i sont intégrées. Dans un canal différent, c'est-à-dire un choix de quels opérateurs combiner en leur OPE représenté par un autre graphe trivalent, l'expression fait intervenir des constantes de structure et des blocs conformes complètement différents. Pourtant, ces expressions doivent être égales car elles calculent la même fonction à n points. Cette égalité est la symétrie de croisement.

En fait, la symétrie de croisement est impliquée par son cas le plus simple, les fonctions à quatre points. Une transformation conforme globale place les opérateurs en 0, x, 1, et ∞. Prendre l'OPE de l'opérateur en x avec celui en 0, 1, ou ∞ donne respectivement des expressions en terme de blocs conformes dits de canal s, t, ou u :

$$\mathcal{F}^{(\mathrm{s})}_\alpha = \mathcal{F}\left[\begin{matrix}\alpha_3 & & \alpha_2\\ \alpha_4 & \alpha & \alpha_1\end{matrix}\right],\ \mathcal{F}^{(\mathrm{t})}_\alpha = \mathcal{F}\left[\begin{matrix}\alpha_3 & & \alpha_2\\ \alpha_4 & \alpha & \alpha_1\end{matrix}\right],\ \mathcal{F}^{(\mathrm{u})}_\alpha = \mathcal{F}\left[\begin{matrix}\alpha_3 & & \alpha_2\\ \alpha_4 & \alpha & \alpha_1\end{matrix}\right]. \tag{1F.2.2}$$

Il se trouve que la symétrie de croisement et la factorisation holomorphe/anti-holomorphe implique que les blocs conformes holomorphes dans un canal sont des combinaisons linéaires de blocs conformes holomorphes dans un autre canal, après continuation analytique en x. Les combinaisons linéaires prennent la forme d'un noyau de fusion $\mathbf{F}_{\alpha\alpha'}$ et d'un noyau de tressage $\mathbf{B}_{\alpha\alpha'}$:

$$\mathcal{F}^{(\mathrm{s})}_\alpha = \int \mathrm{d}\alpha'\, \mathbf{F}_{\alpha\alpha'}\, \mathcal{F}^{(\mathrm{t})}_{\alpha'} = \int \mathrm{d}\alpha'\, \mathbf{B}_{\alpha\alpha'}\, \mathcal{F}^{(\mathrm{u})}_{\alpha'} \ . \tag{1F.2.3}$$

Ces noyaux intégraux sont égaux à une permutation des α_i près, et prennent la forme [PT99] d'une intégrale d'un rapport de fonctions double sinus de Barnes.

Un dernier mot sur les théories avec symétrie de Virasoro. Un module de Verma dont l'impulsion est $\alpha_{r,s} = (1-r)b/2 + (1-s)/(2b)$ ou $q - \alpha_{r,s}$ pour des entiers $r, s \geq 1$ contient un vecteur nul au niveau rs, c'est-à-dire un état descendant de norme nulle qui est orthogonal à toute la représentation. Le module est donc réductible. De telles impulsions sont dites dégénérées. Les corrélateurs qui incluent des opérateurs dégénérés se simplifient à cause des vecteurs nuls. En utilisant le vecteur nul de niveau rs de $V_{\alpha_{r,s}}$ on peut prouver que les fonctions à trois points $\langle V_{\alpha_{r,s}} V_\beta V_\gamma \rangle$ s'annulent sauf si γ (ou $q-\gamma$) est l'un des rs valeurs $\beta + jb + k/b$ avec $j = \frac{1-r}{2}, \dots, \frac{r-1}{2}$ et $k = \frac{1-s}{2}, \dots, \frac{s-1}{2}$. Ces fonctions à trois points non-nulles contraignent quelles familles conformes peuvent apparaître dans l'OPE de $V_{\alpha_{r,s}}$ et V_β : la règle de fusion est

$$V_{\alpha_{r,s}} \times V_\beta = \sum_{j=(1-r)/2}^{(r-1)/2} \sum_{k=(1-s)/2}^{(s-1)/2} [V_{\beta+jb+k/b}] \tag{1F.2.4}$$

où les crochets dénotent les contributions de descendants, les constantes de structure sont omises, et les sommes ont pour incrément 1.

La description de la CFT de Toda ci-dessous nécessite certaines notations concernant les algèbres de Lie. La sous-algèbre de Cartan $\mathfrak{h}$ de $A_{N-1} = \mathfrak{su}(N)$ est identifiée avec $\mathfrak{h}^*$ par sa forme de Killing. Les poids h_s ($1 \leq s \leq N$) de la représentation fondamentale de A_{N-1} ont pour somme zéro et forment une base surcomplète de $\mathfrak{h}$. Les racines simples sont $e_k = h_k - h_{k+1}$.

Le Lagrangien de Toda A_{N-1} décrit un champ scalaire $\varphi \in \mathfrak{h}$ avec une charge de fond et un potentiel exponentiel. Plus précisément, le terme potentiel est $\sum_{k=1}^{N-1} e^{b\langle e_k, \varphi \rangle}$ en termes d'un paramètre b, et la charge de fond Q est un élément particulier de $\mathfrak{h}$ multiplié par $q = b + 1/b$.

Beaucoup plus important que le Lagrangien de Toda est son invariance sous (deux copies de) l'algèbre W_N, une extension de l'algèbre de Virasoro par des courants de spins $3, \dots, N$. Cette algèbre a $N-1$ familles de générateurs $W_n^{(p)}$ pour $2 \leq p \leq N$, avec $W_n^{(2)} = L_n$. Les opérateurs primaires V_α de l'algèbre W_N sont étiquetés par les valeurs propres de tous les $W_0^{(p)}$ exprimées en termes d'une impulsion $\alpha \in \mathfrak{h}$. Les valeurs propres de $W_0^{(p)}$ ne sont pas affectées par des permutations des composantes $\langle \alpha - Q, h_s \rangle$ de l'impulsion : cette symétrie de Weyl généralise l'invariance $\alpha \mapsto q - \alpha$ des opérateurs primaires de Virasoro. Dans la CFT de Toda, une normalisation convenable $\widehat{V}_\alpha$ (5.4.3) de V_α est invariante par symétrie de Weyl.

Trois types d'impulsions jouent un role dans ce travail. Les impulsions génériques α sont telles que le module de Verma construit en agissant avec

$W_{-n}^{(p)}$, $n > 0$, sur $|\alpha\rangle = V_\alpha |vide\rangle$ n'a aucun vecteur nul. Les impulsions semi-dégénérées sont $\varkappa h_1$ (à symétrie de Weyl près), et leur module de Verma a des vecteurs nuls. Les impulsions dégénérées $-b\omega - \omega'/b$ sont caractérisées par deux poids dominants ω, ω' de A_{N-1}, et leur module de Verma a un nombre maximal de vecteurs nuls. Pour $N = 2$, il n'y a pas de distinction entre les impulsions génériques et semi-dégénérées, et les impulsions dégénérées sont $-\frac{r-1}{2}b - \frac{s-1}{2}b^{-1}$ comme décrit ci-dessus.

Les fonctions à deux points $\langle V_\alpha V_\beta \rangle$ d'opérateurs primaires s'annulent sauf si les deux opérateurs ont les mêmes valeurs propres de tous les $W_0^{(p)}$ à un signe $(-1)^p$ près. En termes d'impulsions, $\beta = 2Q - \alpha$ à une symétrie de Weyl près. La plupart des fonctions à trois points avec un opérateur dégénéré primaire s'annulent : on trouve que l'OPE d'un primaire dégénéré avec un générique est

$$V_{-b\omega-\omega'/b} \times V_\alpha = \sum_{h\in\mathcal{R}(\omega)} \sum_{h'\in\mathcal{R}(\omega')} \left[V_{\alpha-bh-h'/b}\right], \tag{1F.2.5}$$

qui est la généralisation naturelle de l'OPE (1F.2.4) de primaires de Virasoro. Les sommes portent sur les poids h de la représentation $\mathcal{R}(\omega)$ de plus haut poids ω, et de façon semblable pour h'. Une autre règle de fusion utile est

$$V_{-bh_1} \times V_{\varkappa h_1} = [V_{(\varkappa-b)h_1}] + [V_{\varkappa h_1 - bh_2}] \tag{1F.2.6}$$

et sa généralisation (5.5.24) pour la fusion d'un opérateur semi-dégénéré avec n'importe quel dégénéré $V_{-b\omega}$. Toutes ces règles de fusion sont confirmées dans la CFT de Toda grâce au formalisme de gaz de Coulomb, mais l'auteur ne connaît pas de preuve n'utilisant que la symétrie W_N.

Une différence majeure entre l'algèbre de Virasoro et W_N pour $N \geq 3$ est que les corrélateurs de descendants sous W_N ne sont pas fixés en termes de corrélateurs de leurs opérateurs primaires. Le corrélateur de n opérateurs primaires sur la sphère peut encore être décomposée en termes de fonctions à trois points de primaires et de descendants, mais ne se réduit pas plus avant en des fonctions à trois points d'opérateurs primaires multipliées par des blocs conformes factorisés. Pour résoudre complètement une théorie invariante sous W_N, il ne suffit donc pas de trouver toutes les fonction à trois points de primaires de W_N. Bien sûr, connaître toutes les fonctions à trois points d'opérateurs primaires de Virasoro suffit, mais ceux-ci sont beaucoup plus nombreux.

Malgré cette difficulté, les blocs conformes existent si suffisamment d'opérateurs primaires sont semi-dégénérés (ou dégénérés). La fonction à trois points d'un opérateur primaire semi-dégénéré avec deux génériques détermine toutes les fonctions à trois points de leurs descendants, et les blocs conformes existent donc dès que chaque sommet du graphe trivalent définissant le canal fait intervenir une impulsion semi-dégénérée. Par exemple, la fonction à n points (1F.2.1) d'opérateurs primaires garde essentiellement la même forme pour W_N (à condition de remplacer les impulsions scalaires par des vecteurs) si chacune des impulsions $\alpha_2, \ldots, \alpha_{n-1}$ sont prises semi-dégénérées tandis que α_1, α_n et les β_i sont génériques.

On considère maintenant $\langle V_{\alpha_\infty}(\infty) V_{\lambda h_1}(1) V_{-bh_1}(x,\bar{x}) V_{\alpha_0}(0)\rangle$, la fonction à quatre points avec deux impulsions génériques α_0 et α_∞, une semi-dégénérée λh_1, et une dégénérée $-bh_1$ correspondant à la représentation fondamentale $\mathcal{R}(h_1)$ de A_{N-1}. Les opérateurs sont placés en 0, x, 1 et ∞ par une transformation conforme globale. Cette fonction à quatre points a été obtenue initialement dans [FL07] en déterminant grâce à des vecteurs nuls de W_3 que les blocs conformes satisfont à une équation différentielle hypergéométrique (à des facteurs près), puis en écrivant la généralisation correcte pour tout N. La Section 5.2.1 s'attaque directement aux valeurs générales de N via une approche basée sur l'analyticité et les noyaux de fusion et de tressage, puisque les vecteurs nuls de W_N ne sont pas connus explicitement pour tout N.

L'OPE $V_{-bh_1} \times V_{\alpha_0} = \sum_{p=1}^{N} [V_{\alpha_0 - bh_p}]$ implique que le corrélateur se décompose en termes de N blocs conformes du canal s :

$$\langle V_{\alpha_\infty} V_{\lambda h_1} V_{-bh_1} V_{\alpha_0}\rangle = \sum_{p=1}^{N} C_p^{(\mathrm{s})} \left| x^{\Delta(\alpha_0 - bh_p) - \Delta(\alpha_0) - \Delta(-bh_1)} (1+\cdots)\right|^2 \tag{1F.2.7}$$

où les $C_p^{(\mathrm{s})}$ sont des constantes, $\Delta(\alpha) = \langle \alpha, 2Q - \alpha\rangle/2$ est la dimension de V_α, et $(1+\cdots)$ sont N séries en puissances entières positives de x, fixées par la symétrie W_N. De même, la décomposition dans le canal u est

$$\langle V_{\alpha_\infty} V_{\lambda h_1} V_{-bh_1} V_{\alpha_0}\rangle = \sum_{p=1}^{N} C_p^{(\mathrm{u})} \left| x^{\Delta(\alpha_\infty) - \Delta(\alpha_\infty - bh_p) - \Delta(-bh_1)} (1+\cdots)\right|^2 \tag{1F.2.8}$$

en termes de séries en puissances de $1/x$. La décomposition dans le canal t,

$$\langle V_{\alpha_\infty} V_{\lambda h_1} V_{-bh_1} V_{\alpha_0}\rangle = \sum_{p=1}^{2} C_p^{(\mathrm{t})} (|1-x|^2)^{\Delta(\lambda h_1 - bh_p) - \Delta(\lambda h_1) - \Delta(-bh_1)} (1+\cdots) \tag{1F.2.9}$$

est plus délicate : la série $(1+\cdots)$ en puissances de $(1-x)$ et $(1-\bar{x})$ se factorise pour $p=1$ mais pas pour $p=2$. Ceci reflète le fait que $V_{\lambda h_1 - bh_1}$ soit semi-dégénéré mais $V_{\lambda h_1 - bh_2}$ pas, de sorte que les fonctions à trois points de ses descendants avec des opérateurs génériques ne sont pas fixées par les fonctions à trois points d'opérateurs primaires.

Combinées, les expansions (1F.2.7), (1F.2.8), et (1F.2.9) autour de $x = 0$, ∞, et 1 fixent la fonction à quatre points à un facteur indépendant de x près.[2] Cela fixe les rapports de constantes $C_p^{(\mathrm{s,t,u})}$, les blocs conformes dans les

[2] L'auteur remercie Bertrand Eynard pour cette observation dans le cas $N = 2$.

canaux s et u, ainsi que la matrice de tressage reliant ces deux bases de blocs conformes, utile plus tard. Le bloc conforme du canal t avec pour impulsion intérieure $\lambda h_1 - bh_1$ est une combinaison linéaire de blocs conformes du canal s, dont les coefficients (la matrice de fusion) sont eux aussi fixés.

Une fois la fonction à quatre points $\langle V_{\alpha_\infty} V_{\lambda h_1} V_{-bh_1} V_{\alpha_0}\rangle$ connue à une constante indépendante de x près, Fateev et Litvinov [FL07] en ont déduit la fonction à trois points (5.4.28) de deux opérateurs primaires génériques et d'un semi-dégénéré dans la CFT de Toda. La CFT de Toda a au plus un opérateur primaire V_α pour chaque impulsion α, l'opérateur de vertex $e^{\langle\alpha,\varphi\rangle}$. De ce fait, les coefficients $C_p^{(\mathrm{s})}$ dans la décomposition (1F.2.7) du canal s sont

$$C_p^{(\mathrm{s})} = C(\alpha_\infty, \lambda h_1, \alpha_0 - bh_p) C_{-bh_1,\alpha_0}^{\alpha_0 - bh_p} \tag{1F.2.10}$$

où $C(\alpha,\beta,\gamma) = \langle V_\alpha V_\beta V_\gamma\rangle$ dénote une fonction à trois points, et $C_{-bh_1,\alpha_0}^{\alpha_0-bh_p}$ est le coefficient de $V_{\alpha_0-bh_p}$ dans l'OPE de V_{-bh_1} et V_{α_0}, essentiellement une fonction à trois points. Les constantes de structure $C_{-bh_1,\alpha_0}^{\alpha_0-bh_p}$ de la CFT de Toda sont données par le formalisme du gaz de Coulomb. Puisque les ratios des $C_p^{(\mathrm{s})}$ sont fixés par l'analyse ci-dessus, les rapports $C(\alpha_\infty, \lambda h_1, \alpha_0 - bh_p)/C(\alpha_\infty, \lambda h_1, \alpha_0 - bh_s)$ sont eux aussi connus. Puisque l'algèbre W_N ne dépend que de $b + b^{-1}$, il existe une relation de shift analogue avec des impulsions translatées de $b^{-1}(h_p - h_s)$ au lieu de $b(h_p - h_s)$. Pour un b réel et générique, les deux translations sont incommensurables, donc la dépendance de $C(\alpha_\infty, \lambda h_1, \alpha_0)$ en α_0 est déterminée complètement en supposant la continuité. Le même raisonnement utilisant la décomposition du canal u fixe la dépendance en α_∞. Finalement, comparer les constantes dans les canaux s et t fixe la dépendence en l'impulsion semi-dégénérée λh_1. La solution de toutes ces relations de shift est (5.4.28), unique à une normalisation près.

De la matrice de tressage d'un opérateur semi-dégénéré $V_{\lambda h_1}$ autour de V_{-bh_1} (dégénéré) trouvée par Fateev et Litvinov, on peut déduire la matrice $\mathbf{B}_K$ du tressage de $V_{\lambda h_1}$ autour de $V_{-b\omega_K}$, l'opérateur dégénéré correspondant à la K-ième représentation antisymétrique $\mathcal{R}(\omega_K)$ de A_{N-1}. On prouve dans le texte par récurrence sur K que $\mathbf{B}_K$ est égale à une expression explicite (5.2.40) fournie par la correspondance AGT (voir la Section 1F.4). La preuve se base sur la relation de pentagone dessinée dans la Figure 4, qui exprime la matrice de tressage $\mathbf{B}_{K+1}$ de $V_{-b\omega_{K+1}}$ autour de $V_{\lambda h_1}$ en termes des matrices de tressage $\mathbf{B}_K$ et $\mathbf{B}_1$, et de la fusion de $V_{-b\omega_K}$ et V_{-bh_1} donnant $V_{-b\omega_{K+1}}$. Les coefficients de cette dernière fusion sont obtenus en exhibant un bloc conforme de $\langle V_\alpha V_{-bh_1} V_{-b\omega_K} V_\beta\rangle$ avec la monodromie attendue de $V_{-b\omega_K}$ autour de V_{-bh_1}, elle-même le carré d'un cas particulier de $\mathbf{B}_K$.

En principe, la même approche fournit la matrice de tressage d'un opérateur semi-dégénéré avec V_{-Kbh_1}, l'opérateur dégénéré correspondance à la

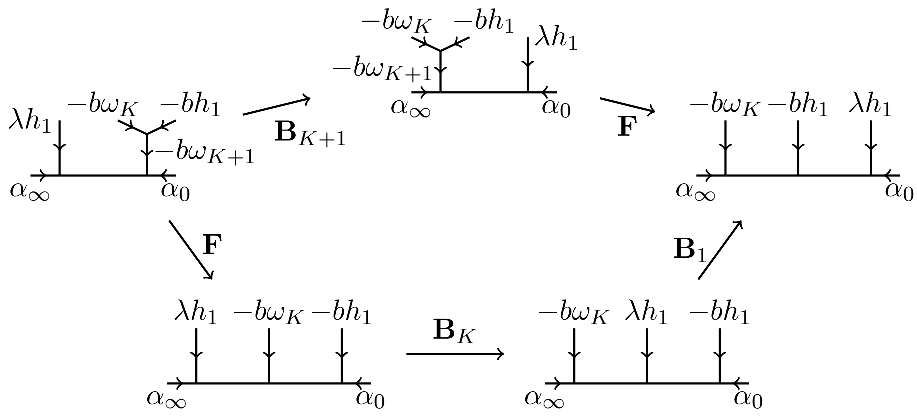


FIGURE 4 : Relation de pentagone utilisée dans l'étape de récurrence pour déduire de la matrice $\mathbf{B}_K$ de tressage de $V_{-b\omega_K}$ autour de $V_{\lambda h_1}$ la matrice de tressage $\mathbf{B}_{K+1}$. La relation fait aussi intervenir les coefficients $\mathbf{F}$ de la fusion de $V_{-b\omega_K}$ et V_{-bh_1} en $V_{-b\omega_{K+1}}$.

K-ième représentation symétrique $\mathcal{R}(Kh_1)$ de A_{N-1}. Cependant, les calculs sont beaucoup plus difficiles que pour $V_{-b\omega_K}$, parce que les poids de $\mathcal{R}(Kh_1)$ ne sont pas simplement des permutations de Weyl d'un unique poids. À la place, une conjecture explicite (5.3.19) est proposée grâce à la correspondance AGT. Cette matrice de tressage est nouvelle.

Le noyau de tressage (5.3.27) de deux opérateurs semi-dégénérés est alors deviné, en généralisant la matrice de tressage précédente à K non-entier. Ce noyau prend une forme semblable aux résultats connus pour l'algèbre de Virasoro ($N = 2$) [PT99], c'est-à-dire une intégrale sur une impulsion à $(N-1)$-composantes d'un rapport de fonctions double sinus de Barnes. La conjecture satisfait aux relations de shifts appropriés, semblables à l'identité pentagonale de la Figure 4. Pour un b réel générique, ces relations de shifts devraient avoir une solution unique, mais l'auteur doit encore finaliser la preuve. Une autre vérification est que la conjecture se réduit au cas d'un opérateur dégénéré symétrique lorsque l'un des opérateurs semi-dégénérés est V_{-Kbh_1}.

Comme expliqué dans la Section 1F.4, la fonction à quatre points de deux opérateurs génériques et deux semi-dégénérés se traduit dans le dictionnaire d'AGT en la SQCD $\mathcal{N} = 2$ $SU(N)$ avec $2N$ saveurs. Les canaux s et u correspondent à différentes descriptions S-duales de la même théorie de dimension quatre, et le noyau de tressage implémente donc la S-dualité.

Le Chapitre 5 décrit aussi diverses règles de fusion dans la Section 5.5 et les ponctions irrégulières obtenues par des collisions d'opérateurs primaires dans la Section 5.6.

1F.3 Localisation supersymétrique sur S^2

La localisation supersymétrique est un outil puissant pour réduire une intégrale de chemin supersymétrique à une intégrale de dimension finie. Depuis son introduction par Witten [Wit88], elle a été utilisée pour évaluer exactement un grand nombre d'observables supersymétriques. Les plus utiles pour ce travail sont la fonction de partition d'instantons de Nekrasov [Nek02 ; NO03] des théories $\mathcal{N} = 2$ à quatre dimensions, et leur fonction de partition sur une sphère ronde [Pes07] ou déformée [HH12], ainsi que la fonction de partition des théories $\mathcal{N} = (2, 2)$ à deux dimensions sur la sphère ronde [BC12 ; DGLFL12] ou déformée [GL12]. De nombreuses autres fonctions de partition et observables dans des théories supersymétriques sur divers espaces ont aussi été calculées, mais une revue du sujet florissant dépasse le cadre de cette thèse.

Cette section commence avec une explication de la localisation supersymétrique. Elle se concentre alors sur la fonction de partition de théories $\mathcal{N} = (2, 2)$ à deux dimensions sur S^2, suivant l'approche de [DGLFL12] reproduit dans le Chapitre 2 : la théorie est introduite, car la construction de théories supersymétriques sur un espace courbe est une tâche non-triviale, et la fonction de partition est localisée de deux façons différentes, ce qui mène à deux expressions explicites distinctes. La fonction de partition sur S^4_b de théories $\mathcal{N} = 2$ à quatre dimensions est présentée ensuite avec une brève explication de comment elle est obtenue par localisation supersymétrique.

L'observation cruciale sous-jacente à la localisation supersymétrique est que les observables supersymétriques ne sont pas affectés par certaines déformations de l'action. Soit $\mathcal{Q}$ une supercharge dans l'algèbre de supersymétrie d'une théorie quantique des champs supersymétrique. Soit V une combinaison des champs de la théorie, qui soit invariante sous la symétrie bosonique $\{\mathcal{Q}, \cdot\}^2$ et telle que la partie bosonique de $\{\mathcal{Q}, V\}$ soit positive semi-définie. Alors pour n'importe quel opérateur invariant de jauge $\mathcal{O}$ qui soit $\mathcal{Q}$-fermé, c'est-à-dire que $\{\mathcal{Q}, \mathcal{O}\} = 0$, la valeur attendue déformée

$$\langle \mathcal{O} \rangle_t = \int [\mathrm{D}\varphi]\, e^{-S[\varphi] - t\{\mathcal{Q}, V[\varphi]\}} \mathcal{O}[\varphi] \tag{1F.3.1}$$

est indépendante du paramètre réel $t \geq 0$ (les signes de $\{\mathcal{Q}, V\}$ et de t assurent que l'intégrale de chemin déformée converge). En effet,

$$\begin{aligned} \partial_t \langle \mathcal{O} \rangle_t &= -\int [\mathrm{D}\varphi]\, e^{-S[\varphi] - t\{\mathcal{Q}, V[\varphi]\}} \mathcal{O}[\varphi] \Big\{ \mathcal{Q}, V[\varphi] \Big\} \\ &= -\int [\mathrm{D}\varphi] \Big\{ \mathcal{Q}, e^{-S[\varphi] - t\{\mathcal{Q}, V[\varphi]\}} \mathcal{O}[\varphi] V[\varphi] \Big\} = 0 \end{aligned} \tag{1F.3.2}$$

où la deuxième égalité utilise $\{\mathcal{Q}, \mathcal{O}\} = \{\mathcal{Q}, \{\mathcal{Q}, V\}\} = \{\mathcal{Q}, S\} = 0$ et la dernière que $\mathcal{Q}$ est une symétrie de la mesure d'intégration.

Grâce à l'invariance en t, la valeur attendue de $\mathcal{O}$, qui est (1F.3.1) pour $t=0$, peut être calculée en prenant $t\to\infty$. Dans cette limite, l'approximation semi-classique autour des points selles φ_0 de $\{\mathcal{Q},V\}$ devient exacte. La valeur attendue prend alors la forme

$$\begin{aligned}\langle\mathcal{O}\rangle_{t=0} = \lim_{t\to\infty}\langle\mathcal{O}\rangle_t = \lim_{t\to\infty}\int_{\text{selles}}[\mathrm{D}\varphi_0]\, e^{-S[\varphi_0]-t\{\mathcal{Q},V[\varphi_0]\}}\mathcal{O}[\varphi_0]Z_{1l}[\varphi_0] \\ = \int_{\{\mathcal{Q},V\}=0}[\mathrm{D}\varphi_0]\, e^{-S[\varphi_0]}\mathcal{O}[\varphi_0]Z_{1l}[\varphi_0]\,, \qquad (1\text{F}.3.3)\end{aligned}$$

où la contribution à une boucle $Z_{1l}[\varphi_0]$ capture la contribution des fluctuations transverses à l'ensemble des points selles. Pour obtenir la dernière ligne, il faut remarquer que les points selles pour lesquels $\{\mathcal{Q},V[\varphi_0]\}>0$ ont des contributions exponentiellement petites lorsque $t\to\infty$, et que les zéros de $\{\mathcal{Q},V\}$ sont automatiquement des points selles par positivité. On dit que l'intégrale de chemin localise aux zéros du terme de déformation, qui peuvent former un espace de dimension finie pour des choix propices de $\mathcal{Q}$ et de V.

Un choix standard est de prendre V égal à la somme de $\psi\overline{\mathcal{Q}\psi}$ pour tous les fermions ψ de la théorie. La partie bosonique de $\{\mathcal{Q},V\}$ est alors une somme de carrés $\mathcal{Q}\psi\overline{\mathcal{Q}\psi}$ dont les zéros sont des points fixes de $\mathcal{Q}$, qui satisfont à $\mathcal{Q}\psi=0$. Ces équations de supersymétrie, ou équations BPS (Bogomol'nyi, Prasad, Sommerfield), sont d'ordre plus bas que les équations définissant les points selles d'autres termes de déformation, et sont donc plus faciles à résoudre. Un autre point de vue sur cette localisation aux points fixes de $\mathcal{Q}$ est que les intégrales le long d'orbites non-triviales de $\mathcal{Q}$ s'annule en tant qu'intégrales fermioniques de constantes.

La contribution à une boucle est déterminée comme suit par une intégrale gaussienne. On décompose $\varphi=\varphi_0+\delta\varphi/\sqrt{t}$ dans (1F.3.1) et on fait une expansion à l'ordre quadratique dans les fluctuations $\delta\varphi$. La normalisation par $\sqrt{t}$ n'affecte pas la mesure d'intégration puisque les Jacobiens des intégrales bosoniques et fermioniques se compensent par supersymétrie. Lorsque $t\to\infty$ seule la partie quadratique de $\{\mathcal{Q},V\}$ demeure ; schematiquement c'est $\delta\varphi\Delta[\varphi_0]\delta\varphi$ pour un certain opérateur Δ. Dans de nombreux cas, sa partie bosonique Δ_b est essentiellement un Laplacien, tandis que sa partie fermionique Δ_f est essentiellement un opérateur de Dirac. L'intégrale gaussienne se réduit aux déterminants de ces opérateurs,

$$Z_{1l}[\varphi_0] = \left(\frac{\det\Delta_\mathrm{b}[\varphi_0]}{\det\Delta_\mathrm{f}[\varphi_0]}\right)^{-1/2}. \qquad (1\text{F}.3.4)$$

Il reste à évaluer ces déterminants.

La première approche, utilisée dans le Chapitre 2, est de décomposer les champs en harmoniques sphériques ou en d'autres modes qui sont pratiques sur l'espace donné. Dans cette décomposition, Δ_b et Δ_f sont typiquement diagonaux par blocs, et chaque bloc fait intervenir un nombre fini de modes.

Les déterminants de tous les blocs sont faciles à calculer, et ils se combinent en des produits infinis dont la régularisation donne la fonction de partition à une boucle (1F.3.4). Au cours de ce procédé, les contributions de nombreux modes bosoniques et fermioniques se compensent.

La seconde approche, utilisée par exemple dans [Pes07], profite de cette compensation. Elle nécessite plus de notions mathématiques, mais est plus systématique. Pour commencer, on trouve une base (X, X') des champs de fluctuation (dénotés $\delta\phi$ ci-dessus) telle que $\mathcal{Q}X = X'$ et $\mathcal{Q}X' = RX$ pour une transformation bosonique $R = \mathcal{Q}^2$. On sépare ensuite les paires (X_0, X'_0) avec X_0 bosonique et X'_0 fermionique des paires (X_1, X'_1) avec les statistiques opposées, et on écrit ensuite la partie de V quadratique dans les fluctuations sous la forme

$$V^{(2)} = X'_0 D_{00} X_0 + X_1 D_{10} X_0 + X'_0 D_{01} X'_1 + X_1 D_{11} X'_1 \,. \tag{1F.3.5}$$

Les opérateurs Δ_{b} et Δ_{f} sont alors obtenus à partir de $\{\mathcal{Q}, V^{(2)}\}$. Après un peu d'algèbre linéaire, la contrainte $\{\mathcal{Q}, \{\mathcal{Q}, V^{(2)}\}\} = 0$ implique que

$$\frac{\det \Delta_{\mathrm{b}}}{\det \Delta_{\mathrm{f}}} = \frac{\det R_0}{\det R_1} = \frac{\det_{\ker D_{10}} R_0}{\det_{\operatorname{coker} D_{10}} R_1} = \prod_i R(i)^{m_i} \,, \tag{1F.3.6}$$

où i décrit les valeurs propres $R(i)$ de R, et m_i est la multiplicité de $R(i)$ dans $\ker D_{10}$ moins celle dans $\operatorname{coker} D_{10}$. Ces valeurs propres et multiplicités sont lues dans l'index R-équivariant

$$\operatorname{ind}_R(D_{10}) = \operatorname{Tr}_{\ker D_{10}} e^{tR} - \operatorname{Tr}_{\operatorname{coker} D_{10}} e^{tR} = \sum_i m_i e^{tR(i)} \,, \tag{1F.3.7}$$

lui-même calculé comme une somme sur les points fixes de R, grâce à la formule de localisation d'Atiyah–Bott–Berline–Vergne [AB84 ; BV82].

Le Chapitre 2 reproduit [DGLFL12], qui applique la localisation supersymétrique à la fonction de partition sur S^2 d'une classe de théories de jauge $\mathcal{N} = (2,2)$. Le calcul a été effectué simultanément dans [BC12] et étendu à la sphère écrasée dans [GL12]. Les théories considérées sont composées de multiplets vecteurs et chiraux, qui dans l'espace plat sont des réductions dimensionelles des multiplets vecteurs et chiraux habituels en supersymétrie $\mathcal{N} = 1$ à quatre dimensions. Il existe aussi des multiplets spécifiques à deux dimensions, tels que les multiplets chiraux twistés, et certaines fonctions de partition sur S^2 ont été étudiées par [DG13].

Il est délicat de préserver la supersymétrie lorsqu'on place une théorie sur une variété courbe. La technique générale [FS11] est d'inclure la théorie dans une théorie de supergravité, puis de geler les valeurs des champs de supergravité à une valeur qui soit invariante sous un ensemble donné de supersymétries. Ceci est l'analogue direct de la façon dont on construit des

Lagrangiens dans un espace courbe, en couplant à des gravitons puis en fixant la métrique à la valeur désirée.

L'approche plus pédestre prise dans la Section 2.2 est de construire le Lagrangien supersymétrique en ajoutant des corrections d'ordre $1/r$ et $1/r^2$ au Lagrangien sur le plan, où r est le rayon de S^2. Écraser cette sphère d'une manière $U(1)$-invariante introduit des corrections supplémentaires au Lagrangien, mais l'intégrale de chemin localisée ne dépend que du rayon r de l'équateur [GL12].

Tout d'abord, le pendant sur S^2 de la supersymétrie de Poincaré $\mathcal{N} = (2,2)$ est obtenu. Comme S^2 est conformément plat, son algèbre superconforme est déduite trivialement de celle du plan, et il s'agit alors de trouver un sous-ensemble des supercharges dont les anti-commutateurs sont les isométries de la sphère, plutôt que toutes les transformations conformes. Le résultat est $SU(2|1)$, dont le sous-groupe bosonique $SU(2) \times U(1)$ agit par rotations de S^2 et une R-symétrie $U(1)$.

On trouve ensuite des analogues sur S^2 de multiplets vecteurs et chiraux, et comment la supersymétrie agit sur leurs composantes. Comme en quatre dimensions, les composantes d'un multiplet vecteur se transforment dans la représentation adjointe du groupe de jauge G tandis que les composantes d'un multiplet chiral se transforment dans une représentation R de G. L'action Lagrangienne renormalisable la plus générale avec $\mathcal{N} = (2,2)$ supersymétries faisant intervenir seulement ces multiplets prend la forme

$$S = S_{\text{v.m.}} + S_{\text{top}} + S_{\text{FI}} + S_{\text{c.m.}} + S_{\text{mass}} + S_W \,. \qquad (1\text{F}.3.8)$$

L'action $S_{\text{v.m.}}$ du multiplet vecteur, l'action $S_{\text{c.m.}}$ du multiplet chiral, et le terme superpotentiel S_W sont des réductions dimensionelles des termes $\mathcal{N} = 1$ à quatre dimensions, avec des corrections d'ordre $1/r$ et $1/r^2$.

Le terme de FI (Fayet–Iliopoulos) S_{FI} associé à chaque facteur $U(1)$ de G est aussi habituel à quatre dimensions. Pour chaque $U(1)$ il y a aussi un terme topologique S_{top} qui mesure le flux B du champs de jauge à travers S^2. Leurs coefficients ξ (paramètre FI) et ϑ (angle theta) se combinent en un paramètre de FI complexifié $z = e^{-2\pi\xi + i\vartheta}$.

Enfin, le terme de masse twistée S_{mass} est obtenu en jaugeant le groupe de symétrie de saveur avec un multiplet vecteur, en donnant aux composantes de celui-ci une valeur constante non-nulle, et en faisant tendre le nouveau couplage vers zéro pour rendre le multiplet vecteur non-dynamique. Les valeurs constantes doivent préserver la supersymétrie, et cela n'autorise qu'un seul paramètre réel m dans l'algèbre de symétrie de saveur. Ce paramètre se combine avec la R-charge q en une masse twistée complexifiée adimensionée[3] $m = r\mathrm{m} + i\frac{q}{2}$ pour chaque multiplet chiral, c'est-à-dire chaque sous-représentation irréductible de G dans R.

[3] Le Chapitre 2 note la masse twistée réelle m et la la masse twistée complexifiée dimensionée $M = m + i\frac{q}{2r}$. Cette introduction utilise à la place la même notation que les chapitres suivants.

La supercharge $\mathcal{Q}$ utilisée pour la localisation est choisie et analysée dans la Section 2.3. Son carré est une combinaison d'une rotation, d'une R-symétrie, et d'une symétrie de jauge. Les deux points fixes de la rotation sont nommés poles nord et sud. Résoudre les équations BPS loin des poles indique que les configurations $\mathcal{Q}$-invariantes sont telles que les scalaires dans le multiplet vecteur sont constants sur la sphère : l'un est proportionnel au flux de jauge B et est discrétisé, tandis que l'autre prend n'importe quelle valeur a dans l'algèbre de jauge réelle, avec $[a, B] = 0$. Une transformation de jauge constante codiagonalise a et B.

L'une des équations BPS est $(a+m)\phi = 0$, où ϕ est la composante scalaire d'un multiplet chiral. De ce fait, pour a générique, les multiplets chiraux s'annulent. L'ensemble des configurations supersymétriques avec $\phi = 0$ est nommée la branche de Coulomb et est paramétrée par a et B. À des points isolés de la branche de Coulomb où des valeurs propres de $-a$ coïncident avec des masses twistées, une branche de Higgs s'ouvre : une analyse des autres équations BPS montre qu'il existe des solutions avec des (anti-)vortex $\phi \neq 0$ de taille nulle aux poles. Il existe aussi parfois des branches mixtes, où certains multiplets chiraux sont non-nuls mais a n'est pas complètement fixé.

Le terme de déformation canonique $\{\mathcal{Q}, V_{\text{can}}\} = |\{\mathcal{Q}, \lambda\}|^2 + |\{\mathcal{Q}, \psi\}|^2$ n'est pas pratique car il brise la symétrie de rotation $SU(2)$ de la sphère en la symétrie de rotation $U(1)$ générée par $\mathcal{Q}^2$. Il se trouve que $S_{\text{v.m.}}$, $S_{\text{c.m.}} + S_{\text{mass}}$ et S_W sont $\mathcal{Q}$-exacts, et peuvent donc être utilisés comme termes de déformation à la place. Une conséquence directe est que la fonction de partition et tout autre observable $\mathcal{Q}$-invariant ne dépend que des paramètres de FI complexifiés, et des masses twistées complexifiées puisqu'elles apparaissent dans l'action de la supersymétrie, mais pas des constantes de couplage de jauge, ni des coefficients du superpotentiel. Le superpotentiel contraint malgré tout les masses twistées complexifiées : S_W n'est supersymétrique que si le superpotentiel W (un polynôme dans les multiplets chiraux) a pour R-charge totale 2 et une masse twistée totale nulle.

La Section 2.4 décrit le résultat de la localisation aux zéros du terme de déformation $\{\mathcal{Q}, V\} = S_{\text{v.m.}} + S_{\text{c.m.}} + S_{\text{mass}}$. Parmi les solutions des équations BPS, les configurations de la branche de Coulomb sont des points selles de $\{\mathcal{Q}, V\}$, tandis que les configurations de la branche de Higgs ne le sont pas. La fonction de partition se localise donc en une intégrale sur la branche de Coulomb,

$$Z_{S^2} = \frac{1}{\mathcal{W}} \sum_B \int_{\mathfrak{t}} \mathrm{d}a \, Z_{\text{cl}}(a, B, z, \bar{z}) Z_{1l}(a, B, m) \,. \tag{1F.3.9}$$

Ici, a est intégré sur l'algèbre de Cartan $\mathfrak{t}$ du groupe de jauge G, et $B \in \mathfrak{t}$ est sommé sur les flux quantifiés GNO, c'est-à-dire $w \cdot B \in \mathbb{Z}$ pour chaque poids w de la représentation R des multiplets chiraux. Cette somme sur tous les B quantifiés divisée par l'ordre $\mathcal{W}$ du groupe de Weyl peut aussi être écrite comme une somme sur des B inéquivalents de jauge avec un terme combinatoire dépendant de B. La contribution classique dans (1F.3.9) est

$$Z_{\mathrm{cl}}(a, B, z, \bar{z}) = z^{\mathrm{Tr}(\mathrm{i}a+\frac{B}{2})}\bar{z}^{\mathrm{Tr}(\mathrm{i}a-\frac{B}{2})}, \tag{1F.3.10}$$

avec un produit implicite faisant intervenir un paramètre de FI complexifié $(z, \bar{z})$ pour chaque facteur $U(1)$ dans G. Le déterminant à une boucle est

$$Z_{1\mathrm{l}}(a, B, m) = \prod_{e>0}\left[\langle e, a\rangle^2 + \frac{\langle e, B\rangle^2}{4}\right]\prod_{w\in R}\left[\frac{\Gamma\left(-\langle w, \mathrm{i}m + \mathrm{i}a + \frac{B}{2}\rangle\right)}{\Gamma\left(1 + \langle w, \mathrm{i}m + \mathrm{i}a - \frac{B}{2}\rangle\right)}\right], \tag{1F.3.11}$$

où le produit sur les racines positives e de G vient des fluctuations du multiplet vecteur, tandis que le produit sur les poids w de R est la contribution du multiplet chiral. Ici, $\langle w, m\rangle$ dénote la masse twistée complexifiée du multiplet chiral auquel correspond le poids w.

La Section 2.5 décrit comment un autre terme de déformation localise la fonction de partition à une intégrale sur la branche de Higgs plutôt que la branche de Coulomb. Les configurations de champs dans la branche de Higgs sont caractérisées par la valeur $a = v$, appelée un vide de Higgs, la configuration des vortex au pole nord, et celle d'anti-vortex au pole sud. Les vortex sont des solutions $\phi \neq 0$ des équations BPS (2.5.1) avec un flux magnétique $k \geq 0$ dans un voisinage infinitésimal du pole nord, tandis que les anti-vortex sont des solutions avec un flux magnétique négatif près du pole sud. Chaque point selle reçoit une contribution classique Z_{cl} et un déterminant à une boucle $Z_{1\mathrm{l}}$. Ces deux facteurs sont chacun le produit d'une contribution de la sphère, égale à (un résidu de) celle de la branche de Coulomb (1F.3.10) et (1F.3.11) en $a = v$ et $B = 0$, et de contributions des vortex et anti-vortex aux poles. En collectant toutes les configurations de vortex pour un vide de Higgs v donné en une fonction de partition de vortex $Z_{\mathrm{v}}(v)$ et de même $Z_{\bar{\mathrm{v}}}(v)$ pour les anti-vortex, on trouve

$$Z_{S^2} = \sum_{v\in\text{vides de Higgs}} Z_{\mathrm{cl}}(v, 0, z, \bar{z}) \operatorname*{res}_{a=v}\left[Z_{1\mathrm{l}}(a, 0, m)\right] Z_{\mathrm{v}}(v, m, z) Z_{\bar{\mathrm{v}}}(v, m, \bar{z}). \tag{1F.3.12}$$

Les contributions Z_{v} et $Z_{\bar{\mathrm{v}}}$ de vortex et d'anti-vortex sont indépendantes car les (anti-)vortex n'influencent pas les champs loin des poles. Les équations BPS impliquent que le terme topologique S_{top} dû au flux magnétique est accompagné d'un terme de FI non-nul S_{FI} tel que $e^{-S_{\mathrm{top}}-S_{\mathrm{FI}}} = z^k$. La fonction de partition de vortex est donc une série en puissances positives de z. Le coefficient de z^k est le volume de l'espace des paramètres de k vortex, seulement connu dans certains cas. De même, la fonction de partition d'anti-vortex est une série en puissances positives de $\bar{z}$. Cette factorisation holomorphe/anti-holomorphe de (1F.3.12) joue un rôle important dans la correspondance avec les corrélateurs de la CFT de Toda dans la Section 1F.4.

L'argument de localisation garantit que l'intégrale de la branche de Coulomb (1F.3.9) et la somme de séries dans la branche de Higgs (1F.3.12) sont égales. L'égalité est prouvée explicitement dans la Section 2.4.2 pour un multiplet vecteur $U(N)$ couplé à des multiplets chiraux fondamentaux et antifondamentaux. L'intégrale de la branche de Coulomb peut aussi être réécrite comme une somme de termes factorisés de la forme (1F.3.12) pour n'importe quels G et R, mais les fonctions de partition de vortex ne sont pas connues. Cette factorisation donne donc des fonctions de partition de vortex autrement inconnues.

Il est fréquent pour une théorie $\mathcal{N} = (2, 2)$ d'admettre plusieurs expansions dans la branche de Higgs, l'une en puissances de $(z, \bar{z})$ et une autre en puissances de $(1/z, 1/\bar{z})$, convergeant respectivement pour $|z| < 1$ et $|z| > 1$. L'intégrale dans la branche de Coulomb est une continuation analytique des deux expressions valide pour tout $(z, \bar{z})$. Les vortex sont essentiels pour l'égalité des expressions de la branche de Coulomb et de celle de Higgs, et donc pour l'égalité d'expansions de Higgs distinctes. La Section 1F.5 décrit plusieurs dualités $\mathcal{N} = (2, 2)$ pour lesquelles les vortex sont tout aussi cruciaux.

La localisation supersymétrique a aussi été appliquée [Pes07 ; HH12] aux fonctions de partition de théories lagrangiennes $\mathcal{N} = 2$ à quatre dimensions sur la sphère (écrasée) S^4_b, c'est-à-dire la variété $U(1) \times U(1)$-invariante

$$\frac{x_0^2}{r^2} + \frac{x_1^2 + x_2^2}{\ell^2} + \frac{x_3^2 + x_4^2}{\tilde{\ell}^2} = 1 \qquad (1\text{F}.3.13)$$

où $\ell/\tilde{\ell} = b^2$. Une telle théorie décrit un multiplet vecteur avec groupe de jauge G, et des hypermultiplets dans une représentation R. La théorie est placée supersymétriquement sur S^4_b en suivant la procédure générale de la coupler à un multiplet de supergravité [FS11].

La supercharge de localisation $\mathcal{Q}$ a pour carré une rotation $U(1) \times U(1)$, une R-symétrie, et une transformation de jauge. La rotation a pour points fixes les poles nord et sud en $x_0 = \pm r$. Loin des poles, les équations BPS imposent à tous les champs de s'annuler, sauf l'un des scalaires dans le multiplet vecteur, qui est une constante $a \in \mathfrak{g}$ dans l'algèbre de Lie réelle de G. Aux poles, les équations BPS se réduisent à des équations d'(anti-)instantons étudiées de longue date : le tenseur de jauge $F_{\mu\nu}$ est (anti-)self-dual. Elles admettent des solutions instantoniques ponctuelles pour a générique, contrairement au cas de dimension deux où les vortex n'existent que pour des a discrets. La fonction de partition prend donc la forme

$$Z = \int_{\mathfrak{g}} \mathrm{d}a\, Z_{\text{cl}}(a, z, \bar{z}) Z_{\text{1l}}(m, a) Z_{\text{inst}}(m, a, z) Z_{\text{anti-inst}}(m, a, \bar{z})\,. \qquad (1\text{F}.3.14)$$

où l'intégrale sur $\mathfrak{g}$ peut être réduite à une intégrale sur sa sous-algèbre de Cartan grâce à la symétrie de jauge. Le déterminant à une boucle est un produit de fonctions spéciales (fonctions Upsilon), une par racine de G et une par poids de R. La contribution classique est essentiellement $(z\bar{z})^{\langle a,a\rangle}$ en termes du couplage complexifié $z = e^{2\pi i\tau}$ qui combine le couplage de jauge et l'angle theta topologique. Comme en deux dimensions, les contributions d'instantons et d'anti-instantons se factorisent en des fonctions (anti-)holomorphes de z. Ces fonctions sont des séries en puissances positives de z et $\bar{z}$. Elles sont connues explicitement lorsque G est un produit de groupes unitaires et dans quelques autres cas.

La S-dualité prédit que différents Lagrangiens décrivent la même théorie $\mathcal{N} = 2$, donc leurs fonctions de partition sur S^4_b doivent être égales. Étant donné que le résultat (1F.3.14) de la localisation supersymétrique est aussi valable pour les Lagrangiens fortement couplés, il peut servir pour tester la S-dualité. Hélas, les expressions localisées de Lagrangiens S-duaux restent difficiles à comparer parce que les fonctions de partition d'instantons ont des expansions en puissances de différents couplages z et z^D. Contrairement au cas de la dimension deux, aucune expression de la fonction de partition n'interpole de manière commode entre deux telles expressions.

Les formes (1F.3.14) et (1F.3.12) des fonctions de partition sur S^4_b et S^2 rappellent la factorisation de corrélateurs dans les CFT à deux dimensions en des blocs conformes holomorphes/anti-holomorphes. La correspondance AGT expliquée ci-après confirme que la fonction de partition sur S^4_b est en effet égale à un corrélateur de la CFT de Toda.

1F.4 Correspondance AGT et opérateurs étendus

Les théories de classe S sont des réductions à quatre dimensions avec $\mathcal{N} = 2$ supersymétries [Gai09a] de la théorie superconforme A_{N-1} $(2,0)$ sur une surface de Riemann $C_{g,n}$ de genre g avec n ponctions (voir la Section 1F.1). Tous les observables d'une théorie de classe S sont en principe complètement fixés par $C_{g,n}$ et des données à chaque ponction. La correspondance AGT [AGT09] (voir aussi [Wyl09]) est un dictionnaire concret pour les observables d'une théorie de classe S placée sur une sphère écrasée S^4_b.

Le premier résultat dans ce dictionnaire est que la fonction de partition sur S^4_b est égale à un corrélateur de la CFT de Toda A_{N-1} sur $C_{g,n}$ avec un opérateur de vertex à chacune des n ponctions :

$$Z_{S^4_b} = \left\langle \widehat{V}_{\alpha_1} \cdots \widehat{V}_{\alpha_n} \right\rangle^{\text{Toda } A_{N-1}}_{C_{g,n}} . \qquad (1\text{F}.4.1)$$

Les multiplets de matière sont rendus massifs dans une théorie de classe S en jaugeant faiblement des symétries de saveur. Le paramètre m de masse pour une ponction pleine, correspondant à une symétrie de saveur $SU(N)$, appartient à l'algèbre de Cartan de $SU(N)$, et est encodé par une impulsion générique $\alpha = Q + im$. Une ponction simple avec une symétrie de saveur $U(1)$ n'a qu'un seul paramètre de masse, encodé par une impulsion semi-dégénérée.

Le premier exemple de (1F.4.1) concerne la sphère avec deux ponctions pleines et une simple : la fonction à trois points (5.4.28) de deux opérateurs de vertex génériques et d'un semi-dégénéré dans la CFT de Toda [FL07] est égale à la fonction de partition (3.2.4) sur S_b^4 de N^2 hypermultiplets libres [HH12].

Pour chaque décomposition de $C_{g,n}$ en trinions, le corrélateur dans la CFT de Toda prend la forme d'une intégrale de produits de blocs holomorphes et anti-holomorphes. Si chaque trinion fait intervenir une ponction simple (cela nécessite $g = 0$ ou 1), alors l'intégrale peut être écrite explicitement (1F.2.1) :

$$\left\langle \widehat{V}_{\alpha_1} \cdots \widehat{V}_{\alpha_n} \right\rangle_{C_{g,n}}^{\text{Toda } A_{N-1}} = \int \mathrm{d}^{3g-3+n}\beta \Big[C(\alpha,\beta)\, \mathcal{F}(\alpha,\beta,z)\, \mathcal{F}(\alpha,\beta,\bar{z}) \Big] \tag{1F.4.2}$$

où il y a une impulsion générique β pour chaque tube et $C(\alpha,\beta)$ est le produit d'une fonction à trois points pour chaque trinion et de l'inverse d'une fonction à deux points pour chaque tube. Les blocs conformes $\mathcal{F}(z)$, qui sont fixés par l'algèbre de symétrie W_N de la CFT de Toda, contiennent toute la dépendence en la structure complexe de $C_{g,n}$, parametrée par z. Ils ont une expansion dans la région $z \to 0$ de l'espace des paramètres où les tubes de la décomposition sont fins.

De même, à chaque décomposition de $C_{g,n}$ correspond une description de la théorie de classe S en termes de multiplets vecteurs (les tubes) jaugeant des symétries de saveur de théories de matière (les trinions). Pour $N = 2$ ces descriptions sont toujours lagrangiennes, tandis que pour $N > 2$ elles ne le sont que si chaque trinion fait intervenir une ponction simple. Pour une description lagrangienne donnée, la localisation supersymétrique exprime la fonction de partition sur S_b^4 comme l'intégrale (1F.3.14)

$$Z_{S_b^4} = \int \mathrm{d}^{3g-3+n}a \Big[Z_{11}(m,a) Z_{\text{cl,inst}}(m,a,z) Z_{\text{cl,anti-inst}}(m,a,\bar{z}) \Big] . \tag{1F.4.3}$$

L'intégrale sur la branche de Coulomb est paramétrée par un scalaire a dans l'algèbre de Cartan de $SU(N)$ pour chaque multiplet vecteur, et m dénote les masses. La contribution classique dans (1F.3.14) est ici combinée avec les fonctions de partition d'(anti-)instantons, qui ont des expansions en séries autour du point faiblement couplé $z \to 0$ de cette décomposition de $C_{g,n}$ en trinions.

Le parallèle entre (1F.4.2) et (1F.4.3) est clair. Outre l'égalité (1F.4.1) de la fonction de partition sur S_b^4 et du corrélateur dans la CFT de Toda, la correspondance AGT indique que les intégrands coïncident. La structure complexe de $C_{g,n}$ encode les constantes de couplage de jauge complexifiées, les impulsions extérieures α correspondent aux masses m, et les impulsions intérieures β sont les paramètres a de la branche de Coulomb.

Une vérification simple est que $C(\alpha, \beta)$ est égal à $Z_{11}(m, a)$. Le premier est un produit d'une fonction à trois points pour chaque trinion, et de l'inverse d'une fonction à deux points pour chaque tube. Le second est le produit des déterminants à une boucle de chaque multiplet vecteur et hypermultiplet dans la théorie $\mathcal{N} = 2$. Les déterminants à une boucle des multiplets vecteurs reproduisent les inverses de fonctions à deux points et les déterminants à une boucle des hypermultiplets reproduisent les fonctions à trois points. Cette dernière égalité est simplement la correspondance AGT pour N^2 hypermultiplets libres. Comparer les blocs conformes $\mathcal{F}(z)$ et les fonctions de partition d'instantons est plus délicat. Les blocs conformes peuvent être péniblement évalués ordre par ordre, et les quelques premiers coefficients s'identifient à ceux des fonctions de partition d'instantons convenables.

La S-dualité relie des descriptions d'une théorie $\mathcal{N} = 2$ associées à différentes décompositions de $C_{g,n}$ en trinions. Les fonctions de partition d'instantons pertinentes sont des expansions en série complètement distinctes autour de différents coins de l'espace des constantes de couplage. Les fonctions de partition d'instantons holomorphes et anti-holomorphes doivent malgré tout se combiner en un objet invariant par S-dualité, la fonction de partition sur S_b^4. Son invariance par S-dualité se traduit élégamment comme l'invariance modulaire de la CFT de Toda : un corrélateur ne dépend pas du canal dans lequel il est écrit en termes de blocs conformes. L'invariance modulaire de la CFT de Liouville, prouvée dans [PT99 ; Tes03 ; HJS09], confirme donc la S-dualité pour les théories de classe S avec groupes de jauge $SU(2)$.

Dans le calcul de localisation, on peut inclure n'importe quel observable invariant de jauge qui préserve la supercharge $\mathcal{Q}$ utilisée pour localiser. De nombreuses constructions d'opérateurs non-locaux supportés sur des courbes, des surfaces, ou des murs tridimensionels possèdent une traduction AGT.

L'opérateur le plus simple de ce type est la boucle de Wilson, plus précisément la version supersymétrique de $W_R = \operatorname{Tr}_R \operatorname{Pexp} \oint_\gamma A$, avec pour support un cercle γ invariant sous $\mathcal{Q}^2$. C'est l'holonomie, autour du cercle γ, de la 1-forme de jauge A d'un multiplet vecteur $SU(N)$ dans une représentation R, autrement dit la trace de l'exponentielle ordonnée de son intégrale. Cette définition dépend d'un choix de description S-duale, c'est-à-dire un choix de décomposition de $C_{g,n}$ en trinions. Sur S_b^4, la valeur attendue $\langle W_R \rangle$ prend la même forme que la fonction de partition (1F.3.14) avec un facteur supplémentaire $\operatorname{Tr}_R \exp(-2\pi b^{\pm 1} a)$. Comme remarqué dans [Ald+09 ; DGOT09], ce facteur peut être réalisé dans la CFT de Toda comme l'ajout d'une boucle de Verlinde. Schematiquement un tel opérateur de boucle est construit en insérant un opérateur de vertex dégénéré étiqueté par R et en le déplaçant le long d'une courbe qui entoure le tube de $C_{g,n}$ correspondant au multiplet vecteur $SU(N)$.

N'importe quelle courbe sans auto-intersection sur $C_{g,n}$ entoure un tube dans une certaine décomposition en trinions, et correspond donc à une boucle de Wilson dans une des descriptions S-duales de la théorie $\mathcal{N} = 2$. La S-dualité envoie les boucles de Wilson, qui mesurent l'effet de l'insertion d'une sonde infiniment massive chargée electriquement, sur des boucles de 't Hooft (ou dyoniques), qui mesurent l'effet d'une sonde chargée (électriquement et) magnétiquement. L'insertion de boucles de Verlinde sur des courbes arbitraires dans le corrélateur de la CFT de Toda (1F.4.1) devrait donc donner la valeur attendue de boucles de 't Hooft et de boucles dyoniques. Cette prédiction a été confirmée dans [GOP11] en définissant et en localisant la fonction de partition sur S^4 en présence d'une boucle de 't Hooft, et en comparant dans certains cas le résultat à un calcul en CFT effectué dans [GLF10] par Jaume Gomis et l'auteur. Pour obtenir n'importe quelle boucle dyonique, il faut généraliser les boucles de Verlinde à des réseaux topologiques.

Les opérateurs de surface $\mathcal{Q}$-invariants peuvent avoir pour support deux $S^2 \subset S^4_b$ écrasées, définies par $x_1 = x_2 = 0$ ou $x_3 = x_4 = 0$ dans (1F.3.13). Deux constructions d'opérateurs de surface à moitié BPS, c'est-à-dire préservant 4 des 8 supercharges d'une théorie de classe S, sont connues.

La première construction [GW06], qui donne les opérateurs de surface dits de M5-branes, est moralement semblable aux boucles de 't Hooft. Elle remplace l'intégrale de chemin par une intégrale sur des configurations des champs avec une condition au bord non-triviale autour de la surface. L'intégrale de chemin modifiée correspond elle aussi à un corrélateur sur $C_{g,n}$, mais dans une théorie autre que la CFT de Toda, comme conjecturé dans [BFFR10]. Des résultats préliminaires [GKLFN15] avec Jaume Gomis, Hee-Cheol Kim, et Satoshi Nawata indiquent que tous ces opérateurs de M5-branes peuvent être obtenus comme des cas particuliers de la construction suivante.

La seconde construction est de coupler la théorie $\mathcal{N} = 2$ de dimension quatre à une théorie $\mathcal{N} = (2,2)$ de dimension deux restreinte à la surface. Comme décrit plus loin dans cette section, la première partie de [GLF14] (le Chapitre 3) exhibe de tels opérateurs de surface, dits de M2-branes, qui correspondent à l'insertion dans le corrélateur de la CFT de Toda d'un opérateur de vertex dégénéré étiqueté par n'importe quelle représentation de $SU(N)$. Ceci est en accord avec une conjecture faite dans [Ald+09] pour l'opérateur de vertex le plus simple ; davantage de références sont données dans le Chapitre 3.

Enfin, les murs de domaine $\mathcal{Q}$-invariants peuvent être construits en faisant varier continûment les couplages de jauge près de l'équateur pour relier deux hémisphères avec des constantes de couplage différentes. En présence d'un mur de domaine, la contribution de chaque hémisphère à la fonction de partition est une fonction de partition d'instantons, égale à un bloc conforme de la CFT de Toda. Le mur de domaine change la manière dont les contributions holomorphes et anti-holomorphes sont combinées. Ceci est reproduit dans la

CFT de Toda par l'insertion d'un défaut topologique [DGG10]. Lorsque les couplages sur les deux hémisphères sont réglés de sorte que les deux théories sont S-duales, on peut appliquer la S-dualité à un côté du mur pour obtenir la même théorie des deux côtés : cela mène au mur de domaine de S-dualité. Puisque la S-dualité agit sur les fonctions de partition d'instantons comme les transformations modulaires sur les blocs conformes, la fonction de partition en présence d'un mur de domaine est

$$Z = \int \mathrm{d}\alpha\, \mathcal{F}^{\sigma}_{\alpha}(z)\, \mathcal{F}^{\sigma}_{\alpha}(\bar{z}^{D}) = \int \mathrm{d}\alpha\, \mathrm{d}\alpha'\, \mathcal{F}^{\sigma}_{\alpha}(z)\, \mathbf{B}^{\sigma\sigma'}_{\alpha\alpha'}\, \mathcal{F}^{\sigma'}_{\alpha'}(\bar{z}) \qquad (1\text{F}.4.4)$$

où σ et σ' denotent des décompositions de $C_{g,n}$ en trinions reliées par la S-dualité, $\mathcal{F}$ sont des blocs conformes, et $\mathbf{B}$ est la transformation modulaire. Les déterminants à une boucle sont ici absorbés dans la mesure d'intégration $dd\alpha$.

Le même mur de S-dualité peut aussi être réalisé en couplant à la théorie de dimension quatre une théorie $\mathcal{N} = 2$ à trois dimensions sur l'équateur. La théorie tridimensionnelle adaptée a été déterminée pour $\mathcal{N} = 4$ SYM dans [GW08a], pour $\mathcal{N} = 2^*$ SYM dans [HLP10], et pour la SQCD $SU(2)$ avec $N_f = 4$ dans [TV12]. Comme expliqué dans [DGG10], sa fonction de partition sur la sphère de dimension trois doit être égale au noyau modulaire $\mathbf{B}_{\sigma\sigma'}$ de sorte que la coupler à la théorie de dimension quatre sur chaque hémisphère donne (1F.4.4). Il devrait être possible de déduire du nouveau noyau de tressage (5.3.27), obtenu dans la Section 5.3.2, la théorie tridimensionelle sur le mur de S-dualité pour la SQCD $SU(N)$ avec $N_f = 2N$.

Tous les opérateurs étendus décrits ci-dessus peuvent aussi être construits à partir de branes dans la théorie M, et cela aide à comprendre leur correspondance avec des observables de la CFT de Toda [DGG10]. On rappelle que les théories de classe S sont obtenues comme la théorie de N M5-branes sur une surface de Riemann Σ. En ne brisant que la moitié de la supersymétrie, on peut insérer des M5-branes transverses avec quatre directions en commun avec les N M5-branes, ou des M2-branes qui se terminent sur une surface de dimension deux. Une intersection de M5-branes est étiquetée par une impulsions continue (générique ou partiellement dégénérée), tandis qu'une intersection de M2 et M5-branes est étiquetée par une impulsion dégénérée, ou de manière équivalente par une représentation de $SU(N)$.

Du point de vue de la théorie superconforme A_{N-1} $(2,0)$ de dimension six, les M5-branes transverses insèrent des défauts de codimension 2. Les quatre directions communes peuvent être distribuées de diverses manières parmi Σ et l'espace.

- Un point sur Σ et l'espace tout entier. Ceci insère un opérateur de vertex dans la CFT et altère la théorie de dimension quatre : avec de telles branes transverses on construit des théories de classe S correspondant à des surfaces de Riemann épointées $\Sigma = C_{g,n}$.

- Une courbe sur Σ et un mur de domaine à quatre dimensions. Ceci insère un défaut topologique au sein du corrélateur dans la CFT, correspondant à un mur de domaine de la théorie de classe S. En particulier le mur de S-dualité est réalisé en tressant une ponction le long d'une courbe.

- La totalité de Σ et une surface dans l'espace. La CFT de dimension deux sur Σ est altérée. L'opérateur de surface de M5-branes introduit des conditions de bord non-triviales sur les champs à quatre dimension, qui brise le groupe de jauge $SU(N)$ au commutant de l'impulsion.

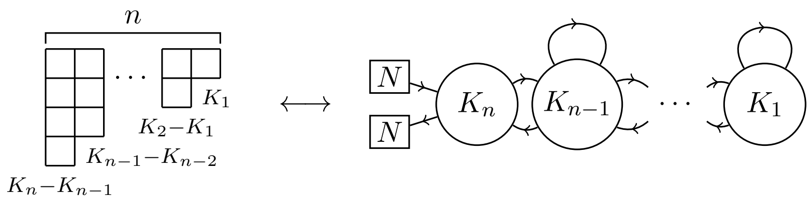
Des défauts de codimension 4 sont construits à partir de collections de M2-branes finissant sur les N M5-branes.

- La totalité de Σ et un point de l'espace. Ceci devrait altérer la CFT de dimension deux et insérer un opérateur local en quatre dimensions, mais l'auteur ne connait pas de résultats quantitatifs dans cette direction.

- Une courbe sur Σ et une boucle à quatre dimensions. Cette configuration mène à une correspondance entre boucles de Wilson/'t Hooft loop en théorie de jauge et boucles de Verlinde dans la CFT de Toda, toutes les deux étiquetées par une représentation de $SU(N)$.

- Un point sur Σ et une surface à quatre dimensions. Les M2-branes insèrent dans le corrélateur un opérateur vertex dégénéré étiqueté par une représentation de $SU(N)$, et insèrent un opérateur de surface dans la théorie de dimension quatre en la couplant à une théorie de dimension deux décrite ci-après.

Le Chapitre 3 [GLF14] associe une théorie de jauge avec $\mathcal{N} = (2,2)$ supersymétries en dimension deux à chaque représentation R de $SU(N)$:



(1F.4.5)

avec les notations suivantes.[4] Le diagramme de Young de R a n colonnes de longueurs respectives $K_n - K_{n-1} \geq \cdots \geq K_2 - K_1 \geq K_1 \geq 0$. Le carquois à droite de l'égalité représente une théorie de jauge $\mathcal{N} = (2,2)$ en dimension deux : les carrés sont des symétries de saveur $U(N)$, les cercles sont des multiplets vecteurs $U(K_j)$, et les flèches sont des multiplets chiraux dans la représentation bifondamentale des groupes à leurs extrémités. Un

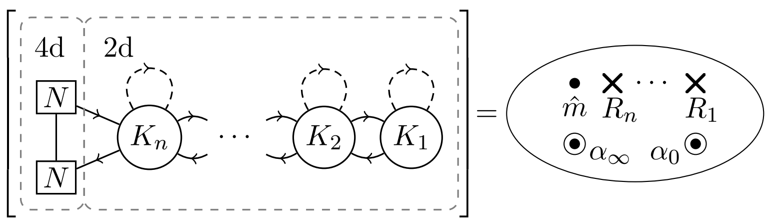
[4]Puisque le Chapitre 3 et le Chapitre 4 se concentrent sur les théories de dimension deux, les rangs N et K_j sont notés là-bas N_f et N_j.

superpotentiel cubique couple chaque multiplet chiral adjoint (les boucles du carquois) aux multiplets chiraux bifondamentaux voisins, et tous les paramètres de FI sauf celui de $U(K_n)$ sont pris nuls. Notons que la symétrie de saveur est réduite de $U(N) \times U(N)$ à $S[U(N) \times U(N)] = SU(N) \times U(1) \times SU(N)$ puisque le $U(1)$ diagonal est une transformation de jauge.

En plaçant le système de branes sur une surface de Riemann Σ, on obtient un opérateur de surface dans la théorie de classe S définie par Σ. La théorie à deux dimensions est couplée en identifiant son groupe de symétrie de saveur $S[U(N) \times U(N)]$ avec des symétries de saveur (ou de jauge) d'un hypermultiplet dans la théorie de classe S, puis en ajoutant un superpotentiel cubique pour cet hypermultiplet et les multiplets chiraux (anti-)fondamentaux (les flèches les plus à gauche du carquois). Lorsque la théorie de classe S est placée sur S^4_b, l'insertion d'un opérateur de surface sur une sphère $S^2 \subset S^4_b$ invariante par $U(1) \times U(1)$ correspond à l'insertion d'un opérateur vertex dégénéré étiqueté par R dans le corrélateur de la CFT de Toda.

Le paramètre de FI (complexifié) de $U(K_n)$ contrôle la position de l'opérateur vertex dégénéré sur Σ. Si on choisit de donner des valeurs non-nulles aux paramètres de FI de $U(K_j)$ pour $j < n$ dans l'identification ci-dessus, la ponction dégénérée étiquetée par R est remplacée par n ponctions dégénérées dont les positions sont contrôlées par les paramètres de FI : si le j-ième paramètre de FI s'annule alors la j-ième et la $(j+1)$-ième ponctions sont placées au même point. La j-ième ponction est étiquetée par la $(K_j - K_{j-1})$-ième représentation antisymétrique : le diagramme de Young (1F.4.5) s'est séparé en ses colonnes. Plus généralement, l'insertion de n ponctions étiquetées par des représentations symétriques ou antisymétriques (et pas seulement antisymétriques) peut être réalisée en n'incluant de multiplet chiral adjoint que pour certains sommets du carquois, comme indiqué dans (1F.4.6).

La correspondance est vérifiée en considérant des opérateurs de surface dans la théorie de classe S la plus simple : N^2 hypermultiplets libres. Étant donné que dans ce cas les théories de dimensions deux et quatre ne sont couplées que par des symétries de saveur plutôt qu'à travers des champs dynamiques, la fonction de partition du système 4d/2d se décompose en $Z_{S^2\subset S^4_b} = Z_{S^4_b} Z_{S^2}$. Le Chapitre 3 confirme que la fonction de partition sur $S^2 \subset S^4_b$ du système 4d/2d est égale à une fonction à $(n+3)$ points de la CFT de Toda, avec n opérateurs vertex dégénérés ✕, un semi-dégénéré •, et deux génériques ⦿ :

$Z_{S^2\subset S^4_b}$ 

(1F.4.6)

En utilisant les expressions explicites des fonctions de partition sur S^2 et S^4_b, l'égalité est testée dans toutes les limites où deux opérateurs de la CFT de Toda se recontrent, et est prouvée pour le cas d'un unique opérateur vertex dégénéré antisymétrique. Les expressions dans le texte incluent des facteurs qui sont ici omis car ils peuvent être absorbés par une normalisation des opérateurs vertex, et dans des ambiguïtés dans la définition de Z_{S^2}. Le Tableau 3.1 résume la correspondance, et des cas particuliers pour $n = 1$ analysés dans les premières sections du Chapitre 3.

La théorie de dimension deux décrit un multiplet vecteur $U(K_1) \times \cdots \times U(K_n)$ couplé à des multiplets chiraux fondamentaux, antifondamentaux, adjoints, et bifondamentaux tels que décrits par le carquois. Pour chaque facteur du groupe de jauge, soit il y a un multiplet chiral adjoint et deux termes cubiques de superpotentiel le couplant aux multiplets chiraux bifondamentaux voisins, soit il y a un terme quartique de superpotentiel pour les quatre multiplets chiraux bifondamentaux voisins, et pas de multiplet chiral adjoint. Les N multiplets chiraux fondamentaux et N antifondamentaux de $U(K_n)$ sont exempts de tels superpotentiels.

Le couplage 4d/2d fixe les masses des hypermultiplets, donc $Z_{S^4_b}$, en termes des masses twistées des multiplets chiraux fondamentaux et antifondamentaux de $U(K_n)$. Ces $2N$ masses twistées, notées m_s et $\widetilde{m}_s$ pour $1 \leq s \leq N$, sont redondantes : une transformation de jauge globale $U(1)$ les translate toutes. Les impulsions generiques α_0 et α_∞ encodent $2(N-1)$ masses twistées :

$$\alpha_0 = Q - \frac{1}{b}\sum_{s=1}^{N} \mathrm{i} m_s h_s \,, \qquad \alpha_\infty = Q - \frac{1}{b}\sum_{s=1}^{N} \mathrm{i}\widetilde{m}_s h_s \,, \tag{1F.4.7}$$

où les poids h_s de la représentation fondamentale de A_{N-1} ont pour somme zéro. La masse twistée restante apparaît dans l'impulsion semi-dégénérée

$$\hat{m} = (\varkappa + K_n b) h_1 \,, \qquad \varkappa = \frac{1}{b}\sum_{s=1}^{N} (1 + \mathrm{i} m_s + \mathrm{i}\widetilde{m}_s) \,. \tag{1F.4.8}$$

Une transformation conforme place les opérateurs vertex avec des impulsions α_0, α_∞ et $\hat{m}$ en 0, ∞ et 1 respectivement. La théorie de jauge a une symétrie de saveur $U(1)$ additionnelle sous laquelle les adjoints ont pour charge ± 2 et les bifondamentaux ± 1. La correspondance nécessite que la masse twistée associée soit $-\mathrm{i}b^2$.

Finalement, les opérateurs vertex dégénérés décrivent l'information restante de la théorie de jauge. Leurs n impulsions $-b\Omega_j$ donnent les rangs des groupes de jauge et le contenu en matière. Pour $1 \leq j \leq n$,

$$\Omega_j = \omega_{K_j - K_{j-1}} \quad \text{ou} \quad (K_j - K_{j-1}) h_1 \tag{1F.4.9}$$

est le plus haut poids de la $(K_j - K_{j-1})$-ième représentation antisymétrique ou symétrique de A_{N-1}. Le contenu en matière est lu de la façon suivante :

le facteur $U(K_j)$ a un multiplet chiral adjoint si Ω_j et Ω_{j+1} sont tous deux symétriques ou tous deux antisymétriques, et sinon il n'en a pas. Les paramètres de FI complexifiés sont (à un signe près)

$$\hat{z}_j = x_j/x_{j+1} \tag{1F.4.10}$$

en termes des positions x_j des opérateurs vertex dégénérés. Pour simplifier les notations ci-dessus, $K_0 = 0$, Ω_{n+1} est considéré comme symétrique, et $x_{n+1} = 1$.

Ceci conclut la description de (1F.4.6).

1F.5 Dualités $\mathcal{N} = (2,2)$ à deux dimensions

La section précédente identifie des opérateurs de surface dans des théories de classe S à des corrélateurs de la CFT de Toda enrichis par l'inclusion d'opérateurs vertex dégénérés. En prenant pour inspiration le fait que l'invariance modulaire dans la CFT de Toda correspond à la S-dualité, les symétries de corrélateurs enrichis sont traduites dans le Chapitre 4 en des dualités entre des paires de théories $\mathcal{N} = (2,2)$ à deux dimensions. Les dualités, semblables à la dualité de Seiberg $\mathcal{N} = 1$ à quatre dimensions, déclarent que différents Lagrangiens ont la même limite infrarouge.

Certaines symétries sont évidentes de part et d'autre de la correspondance (1F.4.6). Les opérateurs vertex generiques sont invariants sous les symétries de Weyl, qui permutent les composantes de $\alpha - Q$: étant donné (1F.4.7), les masses twistées correspondantes sont simplement permutées. Une autre symétrie simple est l'invariance conforme sous $x \mapsto x^{-1}$. Elle échange $\alpha_0 \leftrightarrow \alpha_\infty$, donc $m \leftrightarrow \widetilde{m}$, et $\hat{z}_j \mapsto \hat{z}_j^{-1}$ donc correspond à la conjugaison de toutes les charges dans la théorie de jauge.

Deux symétries de la CFT de Toda se traduisent en des dualités intéressantes de la théorie de jauge : la conjugaison de toutes les impulsions, définie ci-dessous, et les permutations d'opérateurs vertex dégénérés. Pour chaque dualité, les fonctions de partition sur S^2 de théories duales sont égales parce que les corrélateurs correspondants (1F.4.6) dans la CFT de Toda sont égaux.

La Section 4.2 se concentre sur le cas le plus simple de (1F.4.6), c'est-à-dire l'insertion d'un unique opérateur vertex dégénéré $\widehat{V}_{-b\omega_K}$ étiqueté par la K-ième représentation antisymétrique de A_{N-1}. La théorie de jauge correspondante est la SQCD $U(K)$ avec N saveurs, décrite par un multiplet vecteur $U(K)$ couplé à N multiplets chiraux fondamentaux et N antifondamentaux. Son paramètre de FI complexifié est $\hat{z} = (-1)^N z = x$, et les masses twistées sont fixées par (1F.4.7) et (1F.4.8).

La fonction à quatre points de la CFT de Toda est invariante par la conjugaison de toutes les impulsions, qui agit par $h_s \mapsto h_s^C = -h_{N+1-s}$. Sous cette transformation, l'impulsion dégénérée devient $-b\omega_K^C = -b\omega_{K^D}$

avec $K^D = N - K$; les impulsions génériques deviennent $\alpha \mapsto 2Q - \alpha$, à une symétrie de Weyl sans importance près ; et l'impulsion semi-dégénérée devient (à une symétrie de Weyl près) $(\varkappa^D + K^D b)h_1$ avec $\varkappa^D = Nb^{-1} - \varkappa$. Le corrélateur dans la CFT de Toda devient donc une autre fonction à quatre points de la même forme, avec un opérateurs vertex dégénéré antisymétrique, un semi-dégénéré, et deux generiques.

En termes de théorie de jauge, la conjugaison des impulsions donne $K^D = N - K$ et

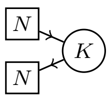
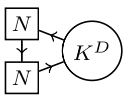
$$\begin{aligned} m_s^D &= \mathrm{i}/2 - m_s \\ \widetilde{m}_s^D &= \mathrm{i}/2 - \widetilde{m}_s \\ \hat{z}^D &= \hat{z} \end{aligned} \qquad \text{ou} \qquad \begin{aligned} m_s^D &= \mathrm{i}/2 - \widetilde{m}_s \\ \widetilde{m}_s^D &= \mathrm{i}/2 - m_s \\ \hat{z}^D &= \hat{z}^{-1} \end{aligned} \tag{1F.5.1}$$

où le second ensemble de paramètres est obtenu en appliquant une transformation conforme supplémentaire $x \mapsto x^{-1}$, en d'autre termes la conjugaison de charges dans la théorie de jauge. La Section 4.2 utilise le premier choix de paramètres, tandis que cette introduction et d'autres articles utilisent le second choix, plus pratique pour les carquois. L'égalité des corrélateurs se traduit par

$$Z_{S_b^4} Z_{S^2}^{U(K),N}(\hat{z}, \overline{\hat{z}}, m, \widetilde{m}) = Z_{S_b^4}^D Z_{S^2}^{U(K^D),N}(\hat{z}^D, \overline{\hat{z}}^D, m^D, \widetilde{m}^D)\,. \tag{1F.5.2}$$

Les deux fonctions de partition sur S_b^4 sont différentes parce que les hypermultiplets sont couplés à des multiplets chiraux en dimension deux avec des masses twistées différentes. Un calcul explicite montre que leur rapport est un déterminant à une boucle de N^2 multiplets chiraux libres. Ces multiplets ont des masses twistées $m_t + \widetilde{m}_s = \mathrm{i} - m_s^D - \widetilde{m}_t^D$ pour $1 \leq s, t \leq N$, et se transforment dans la représentation bifondamentale du groupe de saveur $S[U(N) \times U(N)]$. La relation entre les masses twistées de multiplets fondamentaux q_s^D, antifondamentaux $\widetilde{q}_t^D$, et libres M_{st}^D dans la théorie $U(K^D)$ peut être imposée par un superpotentiel cubique $W = \sum_{s,t} \operatorname{Tr} q_s^D M_{st}^D \widetilde{q}_t^D$.

La conclusion est que la SQCD $U(K)$ avec N saveurs (appelée théorie électrique) et la SQCD $U(K^D)$ avec N saveurs et N^2 multiplets chiraux libres sujette à un superpotentiel cubique (appelée théorie magnétique) ont des fonctions de partition sur S^2 égales. En termes diagrammes,

$Z_{S^2}\Big[$  $\Big|\,\hat{z}, \overline{\hat{z}}, m, \widetilde{m}\Big] = Z_{S^2}\Big[$  $\Big|\,\hat{z}^D, \overline{\hat{z}}^D, m^D, \widetilde{m}^D\Big]\,.$ (1F.5.3)

L'égalité des fonctions de partition sur S^2 est une forte indication que les deux théories sont duales. Cette dualité, la dualité de Seiberg pour les théories $\mathcal{N} = (2,2)$ à deux dimensions, est un analogue direct de la dualité de Seiberg $\mathcal{N} = 1$ à quatre dimensions bien connue. Le rang dual $K^D = N - K$ et le superpotentiel cubique sont identiques, et dans les deux dualités tous les nombres quantiques des multiplets chiraux libres M^D_{st} dans la théorie magnétique coïncident avec les nombres quantiques des mésons $\widetilde{q}_s q_t$ dans la théorie électrique.

Il est intéressant de considérer la décomposition (1F.3.12) de la branche de Higgs de la fonction de partition sur S^2. Elle a un terme pour chaque vide de la branche de Higgs, c'est-à-dire chaque solution des équations BPS dues à la localisation supersymétrique avec un paramètre de FI non-nul ($|\hat{z}| \neq 1$). En supposant que $|\hat{z}| < 1$ sans perte de généralité, on trouve $\binom{N}{K}$ vides, étiquetés par des ensembles de K saveurs parmi $[\![1, N]\!]$. Ce nombre est égal à la dimension de la représentation de A_{N-1} avec plus haut poids ω_K. Par conséquent la décomposition de la branche de Higgs de Z_{S^2} et la décomposition de la fonction à quatre points de la CFT de Toda dans le canal s ont un nombre égal de termes. En fait, les deux sommes sont égales terme à terme : les vides de la branche de Higgs correspondent aux impulsions internes permises par la fusion de $\widehat{V}_{-b\omega_K}$ avec $\widehat{V}_{\alpha_0}$, les déterminants à une boucle correspondent aux produits de fonctions à trois points, et les fonctions de partition classiques et de vortex correspondent à des blocs conformes. Ces identifications fonctionnent de la même manière pour tous les cas de la correspondance (1F.4.6), et donnent de nouvelles expressions pour certains blocs conformes.

À quatre dimensions, la dualité de Seiberg $\mathcal{N} = 1$ a de nombreuses généralisations, dont plusieurs ont des analogues pour les théories $\mathcal{N} = (2, 2)$ à deux dimensions. Les dualités de Kutasov–Schwimmer décrites ci-dessous s'appliquent à la SQCD enrichie par un multiplet chiral adjoint sujet à un superpotentiel. Pour un autre choix de superpotentiel, on trouve une dualité dite plus tard de type $\mathcal{N} = (2, 2)^*$. Comme expliqué ci-après, toutes ces dualités s'appliquent aussi à des groupes de jauge individuels dans les théories de jauge de carquois.

La Section 4.3 considère le cas suivant le plus simple de (1F.4.6), avec un unique opérateur vertex dégénéré $\widehat{V}_{-Kbh_1}$ étiqueté par la K-ième représentation symétrique de A_{N-1}. La théorie de jauge est la SQCD $U(K)$ avec N saveurs et un adjoint, c'est-à-dire un multiplet vecteur $U(K)$ couplé à N multiplets chiraux fondamentaux et N antifondamentaux ainsi qu'un multiplet chiral adjoint. Son paramètre de FI complexifié est $\hat{z} = x$, la masse twistée m_X de l'adjoint est donnée par $\mathrm{i}m_X = b^2$, et les autres masses twistées sont fixées par (1F.4.7) et (1F.4.8).

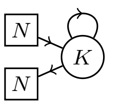
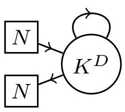
Résoudre les équations BPS montre que la théorie a $\binom{N-1+K}{K}$ vides dans la branche de Higgs. Contrairement à la SQCD sans matière adjointe, ce nombre n'est pas invariant sous $K \to N - K$: il croît indéfiniment avec K. Ainsi, la SQCD $U(K)$ avec un adjoint ne peut pas exhiber de dualité $K \mapsto K^D$ en géneral. Des dualités sont obtenues ci-dessous en incluant un terme de superpotentiel qui réduit le nombre de vides.

Un autre point de vue sur l'absence de dualité en général est de noter que contrairement à l'impulsion dégénérée antisymétrique $-b\omega_K$, dont le conjugué est $-b\omega_{K^D}$, le conjugué de l'impulsion dégénérée symétrique $-Kbh_1$, c'est-à-dire Kbh_N, n'est ni symétrique ni antisymétrique. Ainsi, la conjugaison des impulsions ne donne pas une fonction à quatre points dans la CFT de Toda de la même forme que le corrélateur original. Malgré cela, il y a deux cas où la fonction à quatre points a des symétries.

Dans le premier cas (voir Section 4.3.1), l'impulsion semi-dégénérée $\hat{m}$ de (1F.4.8) est rendue dégénérée en prenant $\varkappa = -Lb$ d'où $\hat{m} = -K^D bh_1$ avec $K^D = L-K$. La présence de deux impulsions dégénérées dans la fonction à quatre points lie les deux impulsions géneriques : $\alpha_0 + \alpha_\infty$ ne peut prendre qu'un nombre fini de valeurs. La théorie de jauge correspondante est la SQCD $U(K)$ avec N multiplets chiraux fondamentaux q_t, N antifondamentaux $\widetilde{q}_t$, et un adjoint X sujets à un superpotentiel

$$W = \sum_{t=1}^{N} \widetilde{q}_t X^{l_t} q_t \qquad (1F.5.4)$$

pour des entiers $l_t \geq 0$ avec $L = \sum_t l_t$. La symétrie de croisement des deux opérateurs dégénérés échange $K \leftrightarrow K^D = L - K$, donc les SQCD $U(K)$ et $U(K^D)$ avec matière adjointe et le superpotentiel (1F.5.4) sont duales. Contrairement à la dualité de Seiberg ci-dessus, les impulsions externes ne sont pas altérées donc la contribution sur S_b^4 ne change pas, et la dualité ne nécessite pas de multiplet chiral libre supplémentaire. On conclut

Z_{S^2} [ | $\hat{z}, \bar{\hat{z}}, m, \widetilde{m}$; $W = \widetilde{q} X^l q$] $= Z_{S^2}$ [ | $\hat{z}^D, \bar{\hat{z}}^D, m^D, \widetilde{m}^D$; $W = \widetilde{q}^D (X^D)^l q^D$] (1F.5.5)

avec $\hat{z}^D = \hat{z}^{-1}$, $m_t^D = m_t$ et $\widetilde{m}_t^D = \widetilde{m}_t$. Lorsque $l_t = 1$ pour tout t, cette théorie est la SQCD $\mathcal{N} = (2,2)^*$, une déformation massive de la SQCD $\mathcal{N} = (4,4)$, d'où le nom dualité de type $\mathcal{N} = (2,2)^*$ pour cette dualité.

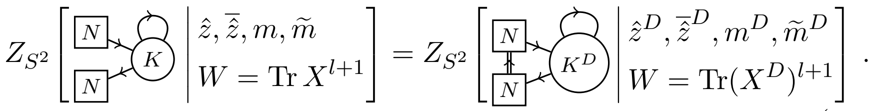
Dans le second cas (voir Section 4.3.2), on prend $\mathrm{i}m_X = b^2 = -1/(l+1)$ pour un entier $l \geq 1$. En théorie de jauge, ceci rend le superpotentiel

$$W = \operatorname{Tr} X^{l+1} \qquad (1F.5.6)$$

supersymétrique. Il se trouve qu'à une symétrie de Weyl près, le conjugué de l'impulsion dégénérée symétrique $-Kbh_1$ est alors aussi une impulsion dégénérée symétrique : $-K^D bh_1$ avec $K^D = lN - K$. En suivant les mêmes étapes que pour une impulsion dégénérée antisymétrique, la conjugaison des impulsions se traduit en la dualité de Kutasov–Schwimmer. Les paramètres de la théorie duale sont $m_X^D = m_X = -\mathrm{i}b^2 = \mathrm{i}/(l+1)$, et

$$\begin{aligned} m_s^D &= m_X - m_s \\ \widetilde{m}_s^D &= m_X - \widetilde{m}_s \\ \hat{z}^D &= \hat{z} \end{aligned} \qquad \text{ou} \qquad \begin{aligned} m_s^D &= m_X - \widetilde{m}_s \\ \widetilde{m}_s^D &= m_X - m_s \\ \hat{z}^D &= \hat{z}^{-1} \end{aligned} \qquad (1F.5.7)$$

où le second ensemble de paramètres est obtenu en conjuguant les charges dans la théorie de jauge. La théorie magnétique inclut aussi lN^2 multiplets chiraux libres M^D_{jst} dont les masses twistées sont égales à celles des mésons dans la théorie électrique, c'est-à-dire $m_t + \widetilde{m}_s + jm_X$ pour $0 \leq j < l$. Ces masses twistées sont fixées dans la théorie magnétique par un superpotentiel. En terme de diagrammes,

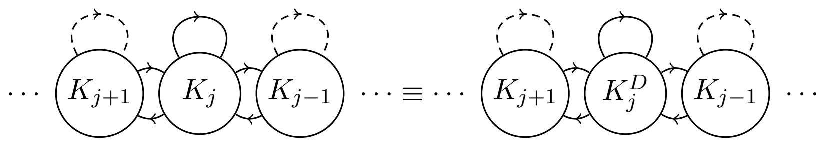


(1F.5.8)

La Section 4.4 décrit des dualités de théories de jauge de carquois qui apparaissent dans la correspondance (1F.4.6). Elle est basée sur l'observation que les dualités de Seiberg, de Kutasov–Schwimmer, et de type $\mathcal{N} = (2,2)^*$ ci-dessus sont encore valides si les symétries de saveur de leurs multiplets chiraux fondamentaux et antifondamentaux sont jaugées. En d'autres termes, si les multiplets chargés sous $U(K_j)$ forment une théorie pour laquelle une dualité est disponible, alors cette dualité s'applique effectivement.

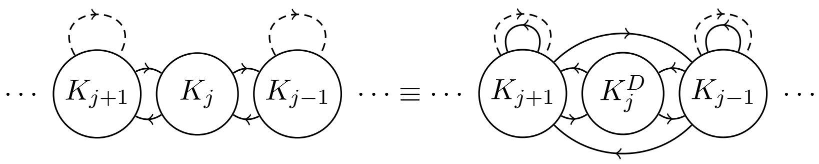
Considérons un sommet $U(K_j)$ du carquois (1F.4.6) avec $j < n$. Soit il y a un multiplet chiral adjoint et des superpotentiels cubiques, soit un superpotentiel quartique et pas d'adjoint. Les deux cas présentent une dualité.

Lorsqu'il y a un multiplet chiral adjoint, les multiplets chargés sous le groupe $U(K_j)$ forment la SQCD $U(K_j)$ $\mathcal{N} = (2,2)^*$ avec $K_{j-1} + K_{j+1}$ saveurs. La théorie duale $\mathcal{N} = (2,2)^*$ a donc $K^D_j = K_{j-1} + K_{j+1} - K_j$ et $\hat{z}^D_j = \hat{z}_j^{-1}$, et une analyse plus précise montre que les paramètres de FI des sommets voisins sont aussi affectés : $\hat{z}^D_{j\pm1} = \hat{z}_j \hat{z}_{j\pm1}$. Les masses twistées sont données dans le texte. Schématiquement,



(1F.5.9)

Lorsqu'il n'y a pas de multiplet chiral adjoint, les multiplets chargés sous $U(K_j)$ forment la SQCD $\mathcal{N} = (2,2)$ avec $K_{j-1} + K_{j+1}$ saveurs, et la dualité de Seiberg s'applique. Les rangs et paramètres de FI se transforment comme dans le cas précédent. La seule différence est que la théorie duale a de la matière supplémentaire avec des charges égales à celle des mésons de $U(K_j)$ dans la théorie initiale :



(1F.5.10)

Les étapes suivantes simplifient la théorie duale. Le superpotentiel quartique de la théorie initiale devient un superpotentiel quadratique pour les multiplets bifondamentaux $M^D_{j-1,j+1}$ et $M^D_{j+1,j-1}$ de $U(K_{j-1}) \times U(K_{j+1})$ dans la théorie duale, et ces multiplets chiraux disparaissent donc dans l'infrarouge. Les intégrer combine les superpotentiels cubiques qui les couple aux bifondamentaux de $U(K^D_j) \times U(K_{j\pm1})$, c'est-à-dire $\mathrm{Tr}(M^D_{j-1,j+1} q^D_{j+1,j} q^D_{j,j-1}) + \mathrm{Tr}(M^D_{j+1,j-1} q^D_{j-1,j} q^D_{j,j+1})$, en un terme quartique pour ces bifondamentaux. Par ailleurs, les nouveaux multiplets chiraux adjoints de $U(K_{j\pm1})$ passent de l'absence à la présence d'adjoints pour ces sommets, ou vice-versa : en effet, si la théorie initiale n'a pas d'adjoint pour $U(K_{j\pm1})$ alors la théorie duale a un adjoint, tandis que si la théorie initiale a un adjoint alors la théorie duale en a deux, sujets à un superpotentiel quadratique qui les fait disparaître dans l'infrarouge.

Au total, la théorie duale a $K^D_j = K_{j-1} + K_{j+1} - K_j$ et $\hat{z}^D_j = \hat{z}^{-1}_j$ ainsi que $\hat{z}^D_{j\pm1} = \hat{z}_j \hat{z}_{j\pm1}$, et si le sommet $U(K_j)$ n'a pas d'adjoint alors pour chacun des sommets $U(K_{j\pm1})$, la présence ou absence d'adjoint est échangée.

Cette prescription assez élaborée se traduit magnifiquement en l'échange de deux opérateurs vertex dégénérés $\widehat{V}_{-b\Omega_j}(x_j, \bar{x}_j)$ et $\widehat{V}_{-b\Omega_{j+1}}(x_{j+1}, \bar{x}_{j+1})$ dans la CFT de Toda. Ceci est obtenu comme suit. La dualité échange $K_j - K_{j-1}$ avec $K_{j+1} - K_j$, et l'échange $x_j \leftrightarrow x_{j+1}$ reproduit la règle de transformation pour les paramètres de FI. Si Ω_j et Ω_{j+1} sont tous deux symétriques ou tous deux antisymétriques, alors leur échange garde la même distribution de représentations symétriques et antisymétriques de A_{N-1} ; en accord avec cela, la dualité ne modifie pas le contenu en matière car la théorie de jauge a un adjoint de $U(K_j)$. Sinon, l'échange affecte le fait que Ω_j et Ω_{j+1} soit du même type (symétrique/antisymétrique) ou non que leurs voisins Ω_{j-1} et Ω_{j+2} ; en accord avec cela, la théorie de jauge n'a pas d'adjoint, et la dualité de Seiberg affecte le contenu en matière des sommets voisins.

La traduction dans la CFT de Toda donne immédiatement l'ensemble des descriptions duales obtenues par des suites de dualités sur les groupes $U(K_j)$ for $j < n$. Celles-ci correspondent aux $n!$ permutations des n opérateurs vertex dégénérés. Les dualités du facteur $U(K_n)$, explorées dans la Section 4.4.2, sont plus délicates car les multiplets chiraux fondamentaux et antifondamentaux de $U(K_n)$ ne sont sujets à aucun couplage superpotentiel.

Pour chaque dualité décrite dans le Chapitre 4, les fonctions de partition sur S^2 de théories duales sont aussi prouvées égales sans aucune référence à la CFT de Toda, en comparant les décompositions dans la branche de Higgs. L'étape difficile de comparer les fonctions de partition de vortex est effectuée dans l'Appendice 4.A et l'Appendice 4.B. Cette preuve directe permet d'attaquer certaines dualités qui n'ont aucune traduction en termes de CFT de Toda.

Des résultats préliminaires (qui ne sont pas inclus dans cette thèse) indiquent que la SQCD avec deux adjoints X et Y sujets au superpotentiel

$W = \mathrm{Tr}(X^{l+1} + XY^2)$ exhibe une dualité avec $K^D = 3lN - K$. Ceci est un analogue direct de la dualité de Kutasov–Schwimmer $\mathcal{N} = 1$ de type D à quatre dimensions.

Des dualités de théories avec un groupe de jauge orthogonal ou symplectique sont aussi connues, mais l'auteur n'a pas tenté de comparer leurs fonctions de partition sur S^2.

La forme du carquois peut aussi être généralisée à des graphes arbitraires. Considérons pour simplifier un carquois $\mathcal{N} = (2,2)^*$, où chaque sommet a un multiplet chiral adjoint et des termes cubiques de superpotentiel avec les bifondamentaux voisins. Le groupe de dualités d'un tel carquois semble être le groupe de Coxeter dont le diagrame de Coxeter a les même sommets que le carquois et une arête pour chaque paire de multiplets chiraux bifondamentaux. Ainsi, un carquois $\mathcal{N} = (2,2)^*$ a un nombre infini de descriptions duales sauf si sa forme est celle d'un diagrame ADE. Les carquois de forme A (carquois linéaires) sont étudié ci-dessus grâce à la correspondance avec la CFT de Toda, et il serait intéressant de découvrir si les carquois de type D ou E ont des réalisations semblables. Incidemment, on peut réaliser tous les groupes de Coxeter finis ABCDEFG comme groupes de dualité en considérant des théories de type $\mathcal{N} = (2,2)^*$ où les superpotentiels cubiques $\widetilde{q}Xq$ sont replacés par $\widetilde{q}X^lq$ (1F.5.4) avec différent exposants l pour chaque arête du carquois. Cependant, il faut noter qu'avoir un nombre fini ou infini de descriptions duales est probablement une propriété anecdotique de la théorie de jauge.

Une autre direction de recherche est de considérer plus avant les opérateurs de surface dans une théorie de classe S intéragissante qui correspond à une surface de Riemann Σ plus complexe que la sphère privée de trois points. La fonction de partition sur $S^2 \subset S^4_b$ n'a pas encore été calculée, car on doit pour cela déterminer les volumes d'espaces de paramètres de combinaisons de vortex et d'instantons. La correspondance prédit que cette fonction de partition sera égale à un corrélateur dans la CFT de Toda avec l'insertion d'opérateurs vertex dégénérés. Les permutations d'opérateurs vertex dégénérés se traduisent en des dualités à deux dimensions décrites ci-dessus. Changer la décomposition en trinions de la surface de Riemann Σ correspond à la S-dualité pour la théorie à quatre dimensions : cela pourrait aider à déterminer comment la S-dualité agit sur les opérateurs de surface. Enfin, certaines transformations modulaires déplacent les ponctions dégénérées d'un trinion de Σ à un autre. Celles-ci se traduisent en un changement de quel hypermultiplet est couplé à la théorie de dimension deux. La correspondance prédit donc une dualité 4d/2d de “saut de nœud” : coupler la théorie de dimension deux à différents hypermultiplets décrit le même opérateur de surface.

L'introduction touche à sa fin. J'espère qu'elle aura éveillé un intérêt pour les intéractions entre les théories de jauge supersymétriques et la CFT de Toda découvertes au sein de la correspondance AGT. Dans le présent travail, cette relation traduit des symétries explicites de la CFT de Toda en de nouvelles dualités de théories de jauge, et elle fournit de nouvelles données dans la CFT de Toda qui mènent à un noyau de tressage auparavant inconnu. Développer complètement les conséquences de la correspondance AGT prendra encore de nombreuses années.

Pour décider de sa prochaine destination (en Anglais), le lecteur est envoyé à la table des matières, ou au bref survol de la thèse en page ii. Pour récapituler, le Chapitre 2 calcule les fonctions de partition sur S^2, le Chapitre 3 les identifie avec des corrélateurs dans la CFT de Toda, le Chapitre 4 déduit des dualités de théories de jauge à partir de symétries de Toda, et le Chapitre 5 explore de nouveaux résultats dans la CFT de Toda.

Chapter 1

Introduction and summary

The Standard Model of particle physics has been confirmed to remarkable accuracy in collider experiments over the last forty years, but it is not a complete description of the Universe. Besides the commonly mentionned lack of gravitation and incompatibilities with some astrophysical observations, a more basic issue is to even determine what the Standard Model predicts. Indeed, while at high energy all coupling constants in the theory are small, the coupling constant for the strong force becomes large at low energy. This renders perturbation theory useless, as series in powers of the coupling constant diverge. While the divergence can in principle be cured by taking into account non-perturbative effects, these are exceedingly difficult to study in a general quantum field theory such as the Standard Model. However, when the quantum field theory exhibits supersymmetry —a symmetry between bosons and fermions of the theory— non-perturbative results are known.

A supersymmetric four-dimensional (interacting) quantum field theory can be invariant under $\mathcal{N} = 1$, 2, or 4 sets of four supercharges. Each additional supersymmetry constrains the theory further: $\mathcal{N} = 4$ supersymmetric theories are in fact uniquely fixed by their gauge group. Theories with $\mathcal{N} = 2$ supersymmetry are more varied, as they involve both gauge interactions and matter. Despite their lack of experimental application (only $\mathcal{N} = 1$ supersymmetry is experimentally viable), $\mathcal{N} = 2$ theories form a good testing ground to understand non-perturbative phenomena in interacting quantum field theories. They strike a balance between the diversity of theories and the availability of exact expressions.

Tremendous progress has occurred over the last two decades in the understanding of $\mathcal{N} = 2$ theories. A crucial development is the AGT (Alday, Gaiotto, Tachikawa) correspondence [AGT09], which relates a large class of four-dimensional $\mathcal{N} = 2$ theories to correlators in the two-dimensional Toda CFT (conformal field theory), a close cousin of the Liouville CFT. Wilson loops and other non-local operators in $\mathcal{N} = 2$ theories —supported on a manifold rather than at a point— correspond to various Toda CFT objects.

The present work describes the AGT translation of a class of surface operators in four-dimensional $\mathcal{N} = 2$ theories, as well as prequisites and consequences of the enriched correspondence. The non-local operators of interest are constructed by coupling to the four-dimensional $\mathcal{N} = 2$ theory a supersymmetric two-dimensional gauge theory supported on a surface. More precisely, the two-dimensional gauge theory has $\mathcal{N} = (2, 2)$ supersymmetry (4 supercharges), as this allows the resulting surface operator to only break half of the 8 supercharges of $\mathcal{N} = 2$ supersymmetry.

The first step is to consider the two-dimensional theories in isolation. Chapter 2 reproduces the article *Exact Results in $D = 2$ Supersymmetric Gauge Theories* [DGLFL12] by Nima Doroud, Jaume Gomis, Sungjay Lee and the author, where we computed the partition function on S^2 of $\mathcal{N} = (2, 2)$ supersymmetric gauge theories.

In Chapter 3, the two-dimensional and four-dimensional theories are combined by placing them on $S^2 \subset S^4$ respectively. This chapter reproduces the first half of the article *M2-brane surface operators and gauge theory dualities in Toda* [GLF14] by Jaume Gomis and the author. We find many $\mathcal{N} = (2, 2)$ surface operators whose $S^2 \subset S^4$ partition functions are equal to Toda CFT correlators.

Next, manifest symmetries of Toda CFT correlators are leveraged to deduce non-trivial dualities between two-dimensional $\mathcal{N} = (2, 2)$ gauge theories in Chapter 4, the second half of [GLF14]. These two-dimensional analogues of the four-dimensional Seiberg duality state that two gauge theories flow to the same infrared fixed point. The two gauge theories thus describe the same large-distance physics but have different microscopic degrees of freedom.

Finally, Chapter 5 explores the Toda CFT. It can be read independently, although some explicit expressions were originally obtained using the AGT correspondence. The chapter culminates with a (well tested) proposal for the braiding kernel of two so-called semi-degenerate vertex operators and two generic ones. This braiding corresponds through the AGT dictionary to an important four-dimensional duality, namely S-duality of $SU(N)$ superconformal QCD (quantum chromodynamics), defined below.

Chapter 1F is a French translation of the present chapter.

This introduction intertwines background material with summaries of each chapter. It presents $\mathcal{N} = 2$ gauge theories (Section 1.1), the Toda CFT (Section 1.2) studied further in Chapter 5, supersymmetric localization and the results of Chapter 2 on S^2 partition functions (Section 1.3), how Chapter 3 includes surface operators in the AGT correspondence relating $\mathcal{N} = 2$ theories to the Toda CFT (Section 1.4), and finally dualities found in Chapter 4 between pairs of two-dimensional $\mathcal{N} = (2, 2)$ theories (Section 1.5).

1.1 Four-dimensional $\mathcal{N} = 2$ gauge theories

This section is about four-dimensional gauge theories with $\mathcal{N} = 2$ supersymmetries, that is, theories invariant two sets of 4 supercharges (see the reviews [Tac13; Tes14]). The aim is to prepare for the AGT correspondence (in Section 1.4), thus many properties of $\mathcal{N} = 2$ theories are skipped, most importantly Seiberg–Witten curves.

The field content of an $\mathcal{N} = 2$ gauge theory decomposes into vector multiplets (gauge multiplets) and hypermultiplets (matter multiplets). Both types of $\mathcal{N} = 2$ supermultiplets can be split into supermultiplets of an $\mathcal{N} = 1$ subalgebra. A hypermultiplet is composed of a pair of chiral multiplets, thus of complex scalars and their spinor superpartners, all in the same representation of a gauge group. An $\mathcal{N} = 2$ vector multiplet is composed of an $\mathcal{N} = 1$ vector multiplet and a chiral multiplet, in other words a gauge boson and its superpartners in the adjoint representation of the gauge group. Lagrangian couplings of these supermultiplets which preserve $\mathcal{N} = 2$ supersymmetry are more restricted than in $\mathcal{N} = 1$ theories, and boil down to the so-called holomorphic prepotential for the vector multiplet.

The prime example of an $\mathcal{N} = 2$ gauge theory is $SU(N)$ SQCD (super quantum chromodynamics) with N_f flavours, which consists of an $\mathcal{N} = 2$ vector multiplet with gauge group $SU(N)$ coupled to N_f hypermultiplets in the fundamental (dimension N) representation of $SU(N)$. The one-loop beta function of the gauge coupling constant is proportional to $2N - N_f$, and non-renormalization theorems imply that the exact beta function also is. The theory is thus asymptotically free for $N_f < 2N$, exactly conformal for $N_f = 2N$ (in the absence of mass terms), and it is not UV complete for $N_f > 2N$.

For $N > 2$, the flavour symmetry of $\mathcal{N} = 2$ $SU(N)$ SQCD with N_f flavours is $U(N_f)$. For $N = 2$, each hypermultiplet splits into two half-hypermultiplets which are both representations of the $\mathcal{N} = 2$ superalgebra, and the flavour symmetry enhances to $SO(2N_f) \supset U(N_f)$. Technically, such a splitting and symmetry enhancement occurs whenever a hypermultiplet transforms in a pseudo-real representation of the gauge group.

Seiberg and Witten [SW94a; SW94b] worked out in 1994 the quantum vacua of $\mathcal{N} = 2$ $SU(2)$ SQCD with $0 \leq N_f \leq 4$ fundamental flavors. These authors determined the exact prepotential of $\mathcal{N} = 2$ $SU(2)$ SQCD, from which one can extract for instance masses of W-bosons and dyons. They found that the $N_f = 4$ SQCD Lagrangian exhibits S-duality: the theory can be described by Lagrangians written in terms of different sets of fundamental degrees of freedom.

In terms of the (complexified) gauge coupling $\tau = 8\pi\mathrm{i}/g^2 + \vartheta/\pi$, S-duality states that the $N_f = 4$ Lagrangians with a given coupling τ and with the

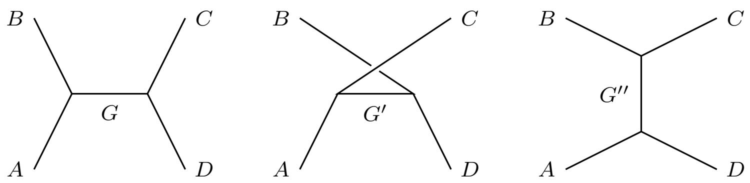


Figure 1.1: Three Lagrangian descriptions of $N_f = 4$ $SU(2)$ SQCD. Lines are $SU(2)$ symmetry groups. External lines are flavour groups while internal lines are gauge groups (hence vector multiplets). Vertices at which three lines connect are hypermultiplets in the trifundamental representation of the $SU(2)^3$ group depicted by the three lines.

dual coupling $\tau^D = -1/\tau$ describe the same physics. In this way, S-duality provides a weakly coupled description ($\tau^D \to \infty$, $g^D \to 0$) of a region of the parameter space where the initial theory is strongly coupled ($\tau \to 0$, $g \to \infty$). The duality generalizes to give $\tau^D = \frac{a\tau+b}{c\tau+d}$ for any matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $SL(2,\mathbb{Z})$.

The S-duality group $SL(2,\mathbb{Z})$ also acts by automorphisms of the flavour symmetry group $SO(2N_f) = SO(8)$. This action is conveniently described by splitting the hypermultiplets into two pairs, each with an $SO(4) \xleftarrow{\sim} SU(2)^2$ flavour symmetry. In every S-dual description of the theory, the manifest $SO(4) \times SO(4)$ symmetry results from pairing the factors of $SU(2)^4 \subset SO(8)$ in one of three possible ways. Concretely, one can include masses $m_{A,B,C,D}$ for the four $SU(2)$ factors. In some Lagrangian description the $N_f = 4$ hypermultiplets have masses $|m_A \pm m_B|$ and $|m_C \pm m_D|$. After S-duality, masses are $|m_A \pm m_C|$ and $|m_B \pm m_D|$, or $|m_A \pm m_D|$ and $|m_B \pm m_C|$.

Gaiotto [Gai09a] generalized S-duality to a wide class of four-dimensional $\mathcal{N} = 2$ theories $\mathcal{T}_{g,n}$, now called class S theories.

Flavour symmetry groups of an $\mathcal{N} = 2$ theory can be gauged by a vector multiplet in the same way as global symmetries of non-supersymmetric theories are gauged by a gauge boson. Thus, any of the four $SU(2)$ flavour symmetry groups of $SU(2)$ SQCD with $N_f = 4$ can be promoted to a gauge group with an $SU(2)$ vector multiplet. The additional vector multiplet can be coupled to a pair of hypermultiplets to keep the theory conformal, and these come with $SU(2) \times SU(2)$ flavour symmetry. Repeating the procedure with the new $SU(2)$ symmetries generates a large number of $\mathcal{N} = 2$ superconformal Lagrangians: their mass deformations describe $SU(2)$ class S theories.

Both pairs of hypermultiplets in $N_f = 4$ $SU(2)$ SQCD transform in the trifundamental representation of $SU(2)^3$ flavour and gauge groups. The $N_f = 4$ theory can thus be regarded as trifundamental hypermultiplets of $SU(2)_A \times SU(2)_B \times SU(2)_G$ and $SU(2)_G \times SU(2)_C \times SU(2)_D$ whose common $SU(2)_G$ symmetry is gauged by a vector multiplet. As described above, S-dual Lagrangians group the $SU(2)_{A,B,C,D}$ in three possible pairings,

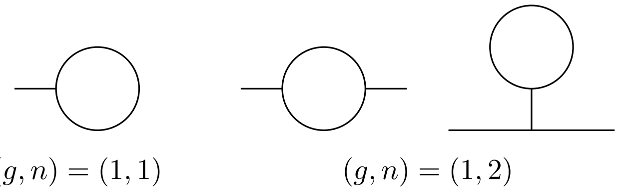


Figure 1.2: Examples of trivalent graphs with g loops and n external lines. The first represents an $SU(2)$ vector multiplet coupled to an adjoint hypermultiplet, which has an $SU(2)$ flavour symmetry. This theory is called $\mathcal{N} = 2^*$ if the hypermultiplet is massive, and it has $\mathcal{N} = 4$ supersymmetry in the massless case. The graphs on the right represent two vector multiplets coupled to two hypermultiplets in different ways. In fact, the two Lagrangians turn out to be S-dual, and describe the same theory.

depicted as graphs in Figure 1.1. Each trifundamental hypermultiplet is represented as a vertex connected to three line representing $SU(2)$ symmetry groups. External lines are flavour symmetries, while internal lines are gauge groups. Using this dictionary, any trivalent graph (three edges per vertex) corresponds to a superconformal Lagrangian composed of trifundamental hypermultiplets and vector multiplets (see Figure 1.2 for examples).

Consider a trivalent graph representing a Lagrangian. In the limit where all $SU(2)$ gauge couplings except one (corresponding to an internal edge) are vanishingly small, the remaining $SU(2)$ gauge theory is simply $N_f = 4$ SQCD and it obeys S-duality. The duality reconnects in any pairing the four edges touching the chosen internal edge. This is expected to hold even when other gauge couplings are non-vanishing. Reconnecting edges through S-dualities leads from a trivalent graph to any other with the same numbers of internal and external lines. In other words, all graphs with g loops and n external lines correspond to Lagrangians which describe the same theory $\mathcal{T}_{g,n}$ in terms of different degrees of freedom.

Properly keeping track of how S-duality acts on the gauge coupling constants ($\tau \mapsto -1/\tau$ for SQCD) requires more structure than the graphs. The correct data to describe couplings of $\mathcal{T}_{g,n}$ is in fact a Riemann surface $C_{g,n}$ with genus g and n punctures. The surface is obtained from any graph with g loops and n external edges by "fattening" the graph, in other words by replacing each edge by a tube and each trivalent vertex by a smooth trinion (three-punctured sphere) joining the three cylinders. The length and twisting angle of each tube encode the coupling constant for the gauge group attached to this edge of the graph, so that a long cylinder corresponds to a weakly coupled vector multiplet. S-duality is then retrieved by noting that $C_{g,n}$ can be cut into tubes and trinions in many ways, labelled by the various trivalent graphs (see Figure 1.3). Each decomposition of $C_{g,n}$ corresponds to a Lagrangian description of $\mathcal{T}_{g,n}$.

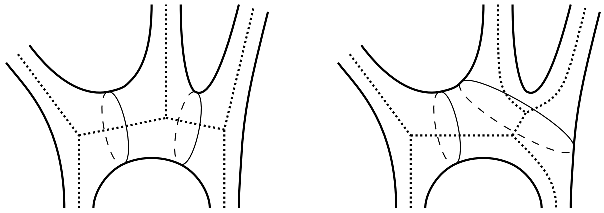


Figure 1.3: Two decompositions of the Riemann surface $C_{0,5}$ into tubes (drawn as ellipses) and trinions (three-punctured spheres between the tubes and external punctures), and their trivalent graphs (in dotted lines). The two corresponding Lagrangians describe the same theory $\mathcal{T}_{0,5}$.

Several theories in this class are worth noting. $\mathcal{T}_{0,3}$ is the theory of 4 free hypermultiplets, with no vector multiplet since its graph has no internal edge. $\mathcal{T}_{0,4}$ is $SU(2)$ SQCD with $N_f = 4$, and depends on a single complexified coupling. The four-punctured sphere $C_{0,4}$ is a tube joining two trinions. Its complex structure only depends on the cross-ratio q of the four punctures, and changing trinion decomposition maps $q \mapsto 1-q$ or $1/q$. As a last example, $\mathcal{T}_{1,1}$ describes an $SU(2)$ vector multiplet gauging two $SU(2)$ flavour symmetries of a single trifundamental hypermultiplet: this results in a hypermultiplet in the adjoint representation of the gauge group and in the fundamental representation of the last $SU(2)$. The theory, called $\mathcal{N} = 2^*$ SYM (super Yang–Mills) when the hypermultiplet is massive, has an enhanced $\mathcal{N} = 4$ supersymmetry when the hypermultiplet is massless.

As argued in [Gai09a] using Seiberg–Witten curves, the theory $\mathcal{T}_{g,n}$ is the four-dimensional reduction on $C_{g,n}$ of the mysterious A_1 $(2,0)$ six-dimensional superconformal field theory with some boundary conditions at the punctures. This theory is not known directly, but its reductions to various lower dimensions are known. For instance, its reduction on a circle is the maximally supersymmetric $SU(2)$ Yang–Mills theory in five dimensions, which reduces further to the four-dimensional $\mathcal{N} = 2$ vector multiplet associated to each cylinder of $C_{g,n}$ in the description above.

In M-theory, the A_1 $(2,0)$ theory is the world-volume theory of two coincident M5-branes. These branes are then wrapped around the Riemann surface $C_{g,n}$, whose punctures are realized by transverse M5-branes. Brane constructions give useful intuitions in Section 1.4 on extended operators of $\mathcal{N} = 2$ theories.

A close cousin of A_1 $(2,0)$ is the six-dimensional A_{N-1} $(2,0)$ superconformal field theory, the world-volume theory of N coincident M5-branes. Compactifying it on a Riemann surface $C_{g,n}$ with some boundary conditions at punctures yields a four-dimensional $\mathcal{N} = 2$ gauge theory with $SU(N)$ gauge groups. The set of all such $\mathcal{N} = 2$ theories is dubbed class S. In the absence of mass terms these theories are superconformal.

The standard example of a class S theory is $\mathcal{N} = 2$ $SU(N)$ SQCD with $2N$ fundamental hypermultiplets, obtained when $C_{g,n} = C_{0,4}$ is the four-punctured sphere. For $N > 2$ the flavour symmetry of SQCD is $U(2N)$, with an $SU(N) \times U(1) \times U(1) \times SU(N)$ subgroup made manifest by the six-dimensional construction. An important difference with the $N = 2$ case is that the four factors are not identical: correspondingly the punctures on $C_{0,4}$ come with different boundary conditions. Two punctures carry an $SU(N)$ flavour symmetry and are called full, and the other two carry a $U(1)$ flavour symmetry and are called simple. Many other types of punctures exist.

As before, S-dual descriptions are labelled by trinion decompositions of $C_{g,n}$. A trinion with one simple and two full punctures is associated to N^2 hypermultiplets and makes a $U(1) \times SU(N) \times SU(N)$ flavour symmetry explicit. Joining full punctures of two trinions corresponds to gauging the two $SU(N)$ flavour symmetries diagonally. In the case of SQCD, the two decompositions of $C_{0,4}$ where both trinions have a simple puncture correspond to descriptions as an $SU(N)$ vector multiplet coupled to two sets of N fundamental hypermultiplets. When the two full punctures belong to the same trinion, there is no Lagrangian description: one is still coupling two theories by gauging a common flavour symmetry, but the building block corresponding to one of the trinions is non-Lagrangian. More generally, while all $SU(2)$ class S theories are described by Lagrangians, class S theories for $N > 2$ only have a Lagrangian description in duality frames where every trinion involves a simple puncture.

Since a class S theory only depends on the complex structure of $C_{g,n}$ and on data at each puncture, any observable of the four-dimensional theory can in principle be derived from a computation on $C_{g,n}$. In practice, the identification is typically worked out by computing four-dimensional observables and finding a matching two-dimensional calculation. The AGT correspondence [AGT09] (Section 1.4) consists in a concrete dictionary between several observables obtained through supersymmetric localization on spheres (Section 1.3) and correlators in the Toda CFT (Section 1.2) on $C_{g,n}$.

1.2 Toda conformal field theory

The A_{N-1} Toda theory is a two-dimensional CFT whose symmetry algebra W_N is an extension of the Virasoro algebra by higher spin currents. The A_1 Toda theory ($N = 2$) is the well-known Liouville CFT, and W_2 is the Virasoro algebra. This section recalls basic notions of two-dimensional CFT (reviewed in [Rib14]) up to the braiding kernel of primary operators. It then describes effects of the W_N symmetry, and the explicit proposal (5.3.27) for the braiding kernel of some W_N primary operators. This introduction to the Toda CFT is enough to read the thesis, which concludes with a detailed study of the theory in Chapter 5.

Two-dimensional conformal symmetry implies an action of (two copies of) the Virasoro algebra on states of the theory. The two copies are due to holomorphic and antiholomorphic conformal transformations, and can be treated independently. Conformal symmetry also implies that the state-operator correspondence is a bijection between operators ϕ and states $|\phi\rangle$ obtained by acting on the vacuum.

The Virasoro algebra has generators L_n for $n \in \mathbb{Z}$ subject to $L_n^\dagger = L_{-n}$ and the commutation relations $[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n}$. A highest-weight state is $|h\rangle$ such that $L_n |h\rangle = 0$ for $n > 0$ and $L_0 |h\rangle = h |h\rangle$, and the corresponding operator is called a primary operator of dimension h. Acting with L_{-n} for $n > 0$ yields a Verma module: a representation of the Virasoro algebra whose states are linear combinations of $L_{-n_1} \cdots L_{-n_p} |h\rangle$ for $n_j > 0$. Such a state is called a descendant of $|h\rangle$ at level $\sum_j n_j$. A primary operator and its descendants form a conformal family.

For convenience, the central charge is parametrized as $c = 1 + 6q^2$ with $q = b + 1/b$, and the dimension $h = \alpha(q - \alpha)$ of a primary operator V_α is expressed in terms of a momentum $\alpha \in \mathbb{C}$.

Conformal symmetry expresses correlators of descendant operators in terms of correlators of primary operators. It forces sphere two-point functions of primary operators to vanish unless the two operators have the same dimension. It also fixes the coordinate dependence of sphere three-point functions $\langle V_\alpha V_\beta V_\gamma \rangle$, but not an overall factor $C(\alpha, \beta, \gamma)$. All n-point functions of primary operators on the sphere are then fixed as follows in terms of the three-point functions $C(\alpha, \beta, \gamma)$, also called structure constants.

Any pair of primary operators can be replaced by their OPE (operator product expansion), a linear combination of primary operators and descendants whose coefficients are fixed in terms of the structure constants by conformal symmetry. The n-point function gets recast as an integral (or sum) over conformal families of a structure constant multiplied by an $(n-1)$-point function and by a factor keeping track of descendant contributions. Repeating the procedure expresses any n-point function as an integral of products of $n-2$ three-point functions multiplied by a factor that is fixed by conformal symmetry. This conformal factor factorizes as a conformal block $\mathcal{F}$ holomorphic in the positions of operators times an antiholomorphic conformal block. Glossing over details such as inverse two-point functions,

$$\langle V_{\alpha_1} \cdots V_{\alpha_n} \rangle = \int \mathrm{d}\beta_3 \cdots \mathrm{d}\beta_{n-1}\, C(\alpha_1, \alpha_2, \beta_3) \cdots C(\beta_{n-1}, \alpha_{n-1}, \alpha_n) \cdot \left| \mathcal{F}\left[\begin{array}{c} \alpha_2 \quad \alpha_3 \quad \cdots \quad \alpha_{n-2} \quad \alpha_{n-1} \\ \alpha_1 \quad \beta_3 \quad \cdots \quad \beta_{n-1} \quad \alpha_n \end{array} \right] \right|^2 . \tag{1.2.1}$$

The trivalent graph describes which OPEs were performed, and keeps track of the resulting momenta. There is one structure constant for each vertex of this trivalent graph. The momenta α_i are called external momenta, while β_i are internal momenta and are integrated over. In a different channel, that is, a choice of which operators to pair into OPEs represented by another trivalent graph, the expression involves completely different structure constants and conformal blocks $\mathcal{F}$. Crossing symmetry states that the expressions must be equal, as they both compute the same n-point function.

In fact, crossing symmetry is implied by its simplest case, four-point functions. A global conformal transformation places the operators at 0, x, 1, and ∞. Taking the OPE of the operator at x with that at 0, 1, or ∞ yields expressions in terms of s-, t-, and u-channel conformal blocks, respectively:

$$\mathcal{F}^{(\mathrm{s})}_{\alpha} = \mathcal{F}\begin{bmatrix} \alpha_3 & & \alpha_2 \\ \alpha_4 & \alpha & \alpha_1 \end{bmatrix}, \ \mathcal{F}^{(\mathrm{t})}_{\alpha} = \mathcal{F}\begin{bmatrix} \alpha_3 & & \alpha_2 \\ \alpha_4 & \alpha & \alpha_1 \end{bmatrix}, \ \mathcal{F}^{(\mathrm{u})}_{\alpha} = \mathcal{F}\begin{bmatrix} \alpha_3 & & \alpha_2 \\ \alpha_4 & \alpha & \alpha_1 \end{bmatrix}. \tag{1.2.2}$$

It turns out that crossing symmetry and the holomorphic/antiholomorphic factorization imply that holomorphic conformal blocks in one channel are linear combinations of holomorphic conformal blocks in another channel, after analytic continuation in x. The linear combinations are expressed as a fusion kernel $\mathbf{F}_{\alpha\alpha'}$ and a braiding kernel $\mathbf{B}_{\alpha\alpha'}$:

$$\mathcal{F}^{(\mathrm{s})}_{\alpha} = \int \mathrm{d}\alpha' \, \mathbf{F}_{\alpha\alpha'} \, \mathcal{F}^{(\mathrm{t})}_{\alpha'} = \int \mathrm{d}\alpha' \, \mathbf{B}_{\alpha\alpha'} \, \mathcal{F}^{(\mathrm{u})}_{\alpha'} \,. \tag{1.2.3}$$

These kernels are related by a permutation of the α_i, and were determined in [PT99] as an integral of ratios of Barnes double sine functions.

One last word on theories with Virasoro symmetry. A Verma module whose momentum is one of $\alpha_{r,s} = (1-r)b/2 + (1-s)/(2b)$ or $q - \alpha_{r,s}$ for integers $r, s \geq 1$ contains a null-vector at level rs, namely a zero-norm descendant state that is orthogonal to the whole representation, hence the module is reducible. These momenta are called degenerate. Correlation functions which include degenerate primary operators simplify because of null-vectors. Using the level rs null-vector of $V_{\alpha_{r,s}}$, the three-point functions $\langle V_{\alpha_{r,s}} V_\beta V_\gamma \rangle$ are found to vanish unless γ (or $q - \gamma$) is one of the rs values $\beta + jb + k/b$ with $j = \frac{1-r}{2}, \ldots, \frac{r-1}{2}$ and $k = \frac{1-s}{2}, \ldots, \frac{s-1}{2}$. These non-zero three-point functions constrain what conformal families can appear in the OPE of $V_{\alpha_{r,s}}$ with V_β: the fusion rule is

$$V_{\alpha_{r,s}} \times V_\beta = \sum_{j=(1-r)/2}^{(r-1)/2} \sum_{k=(1-s)/2}^{(s-1)/2} [V_{\beta+jb+k/b}] \tag{1.2.4}$$

where brackets denote contributions from descendants, structure constants are omitted, and sums run in steps of 1.

Describing the Toda CFT requires some Lie algebra notations. The Cartan subalgebra $\mathfrak{h}$ of $A_{N-1} = \mathfrak{su}(N)$ is identified to $\mathfrak{h}^*$ using its Killing form. The weights h_s $(1 \leq s \leq N)$ of the fundamental representation of A_{N-1} sum to zero and form an overcomplete basis of $\mathfrak{h}$. Simple roots are $e_k = h_k - h_{k+1}$.

The A_{N-1} Toda Lagrangian describes a scalar field $\varphi \in \mathfrak{h}$ with a background charge and an exponential potential term. More precisely, the potential term is $\sum_{k=1}^{N-1} e^{b\langle e_k, \varphi\rangle}$ in terms of a parameter b, and the background charge Q is a fixed element of $\mathfrak{h}$ multiplied by $q = b + 1/b$.

Much more important than the Toda Lagrangian is its invariance under (two copies of) the W_N algebra, a higher-spin generalization of the Virasoro algebra. This algebra has $N-1$ sets of generators $W_n^{(p)}$ for $2 \leq p \leq N$, with $W_n^{(2)} = L_n$. Primary operators V_α of the W_N algebra are labelled by the eigenvalues of all $W_0^{(p)}$ expressed in terms of a momentum $\alpha \in \mathfrak{h}$. Permuting the components $\langle \alpha - Q, h_s\rangle$ of the momentum does not change the eigenvalues of $W_0^{(p)}$: this Weyl symmetry generalizes the $\alpha \mapsto q - \alpha$ invariance of Virasoro primary operators. In the Toda CFT, an appropriate normalization $\widehat{V}_\alpha$ (5.4.3) of V_α is invariant under Weyl symmetries.

Three types of momenta play a role in the present work. Generic momenta α are such that the Verma module constructed by acting with $W_{-n}^{(p)}$, $n > 0$, on $|\alpha\rangle = V_\alpha\, |vacuum\rangle$ has no null-vector. Semi-degenerate momenta take the form $\varkappa h_1$ (up to Weyl symmetries), and their Verma modules have some null-vectors. Degenerate momenta $-b\omega - \omega'/b$ are characterized by two dominant weights ω, ω' of A_{N-1}, and Verma modules have a maximal number of null-vectors. For $N = 2$, there is no distinction between generic and semi-degenerate momenta, and degenerate momenta reproduce the degenerate momenta $-\frac{r-1}{2}b - \frac{s-1}{2}b^{-1}$ of Virasoro.

Two-point functions $\langle V_\alpha V_\beta\rangle$ of primaries vanish unless the two operators have equal eigenvalues of all $W_0^{(p)}$ up to a sign $(-1)^p$. In terms of momenta, $\beta = 2Q - \alpha$ or a Weyl permutation thereof. Three-point functions with one degenerate primary operator vanish in most cases: accordingly the OPE of a degenerate and a generic primaries is

$$V_{-b\omega-\omega'/b} \times V_\alpha = \sum_{h\in\mathcal{R}(\omega)} \sum_{h'\in\mathcal{R}(\omega')} \left[V_{\alpha-bh-h'/b}\right], \tag{1.2.5}$$

the natural extension of the OPE (1.2.4) of Virasoro primaries. Here, sums run over weights h of the representation $\mathcal{R}(\omega)$ with highest weight ω, and similarly for h'. Another useful fusion rule is

$$V_{-bh_1} \times V_{\varkappa h_1} = \left[V_{(\varkappa-b)h_1}\right] + \left[V_{\varkappa h_1 - bh_2}\right] \tag{1.2.6}$$

and its generalization (5.5.24) to the fusion of a semi-degenerate operator with any degenerate $V_{-b\omega}$. All of these fusion rules are confirmed in the Toda CFT through the Coulomb gas formalism, but the author does not know of a proof using only W_N symmetry.

A major difference between the Virasoro algebra and W_N for $N \geq 3$ is that correlators of W_N descendants are not fixed in terms of correlators of their W_N primary operators. Sphere n-point functions of primary operators can still be decomposed in terms of three-point functions of primary and descendant operators, but do not decompose further to three-point functions of primaries multiplied by factorized conformal blocks. To solve a W_N-invariant theory, it is thus not enough to find all three-point functions of W_N primaries. Of course, knowing the three-point functions of all Virasoro primaries suffices, but these are much more numerous.

Despite this difficulty, conformal blocks exist if enough primary operators are semi-degenerate (or degenerate). The three-point function of a semi-degenerate and two generic operators fixes all three-point functions of their descendants, hence conformal blocks exist whenever each vertex of the trivalent graph defining the channel has a semi-degenerate momentum. For instance, the n-point function (1.2.1) of Virasoro primaries keeps essentially the same form for W_N primaries (replacing momenta by vectors) if all $\alpha_2, \ldots, \alpha_{n-1}$ are taken to be semi-degenerate and α_1, α_n and the β_i to be generic.

Consider the four-point function $\langle V_{\alpha_\infty}(\infty) V_{\lambda h_1}(1) V_{-bh_1}(x,\bar{x}) V_{\alpha_0}(0)\rangle$ with two generic momenta α_0 and α_∞, a semi-degenerate λh_1, and a degenerate $-bh_1$ labelled by the fundamental representation $\mathcal{R}(h_1)$ of A_{N-1}. Operators are placed at 0, x, 1 and ∞ through a global conformal transformation. This four-point function was originally determined in [FL07] by working out using null-vectors of W_3 that conformal blocks obey a hypergeometric differential equation (up to some factors), then writing the correct generalization for all N. Section 5.2.1 directly attacks the general N case through a bootstrap approach since null-vectors are not known explicitly for W_N.

Due to the OPE $V_{-bh_1} \times V_{\alpha_0} = \sum_{p=1}^{N} [V_{\alpha_0 - bh_p}]$, the correlator decomposes in terms of N s-channel conformal blocks:

$$\langle V_{\alpha_\infty} V_{\lambda h_1} V_{-bh_1} V_{\alpha_0}\rangle = \sum_{p=1}^{N} C_p^{(\mathrm{s})} \Big| x^{\Delta(\alpha_0 - bh_p) - \Delta(\alpha_0) - \Delta(-bh_1)} (1 + \cdots) \Big|^2 \tag{1.2.7}$$

where $C_p^{(\mathrm{s})}$ are constants, $\Delta(\alpha) = \langle \alpha, 2Q - \alpha\rangle/2$ is the dimension of V_α and $(1 + \cdots)$ are N series in non-negative integer powers of x, fixed by W_N symmetry. Similarly, the u-channel decomposition is

$$\langle V_{\alpha_\infty} V_{\lambda h_1} V_{-bh_1} V_{\alpha_0}\rangle = \sum_{p=1}^{N} C_p^{(\mathrm{u})} \Big| x^{\Delta(\alpha_\infty) - \Delta(\alpha_\infty - bh_p) - \Delta(-bh_1)} (1 + \cdots) \Big|^2 \tag{1.2.8}$$

in terms of series $(1 + \cdots)$ in powers of $1/x$. The t-channel decomposition

$$\langle V_{\alpha_\infty} V_{\lambda h_1} V_{-bh_1} V_{\alpha_0}\rangle = \sum_{p=1}^{2} C_p^{(\mathrm{t})} \big(|1 - x|^2\big)^{\Delta(\lambda h_1 - bh_p) - \Delta(\lambda h_1) - \Delta(-bh_1)} (1 + \cdots) \tag{1.2.9}$$

is more intricate: the series $(1+\cdots)$ in powers of $(1-x)$ and $(1-\bar{x})$ factorizes for $p=1$ but does not for $p=2$. This is because $V_{\lambda h_1 - bh_1}$ is semi-degenerate but $V_{\lambda h_1 - bh_2}$ is not, and three-point functions of its descendants with generic operators are not fixed by three-point functions of the primary operators.

Together, the expansions (1.2.7), (1.2.8), and (1.2.9) around $x = 0$, ∞, and 1 fix the four-point function[1] up to an overall factor independent of x. This fixes s-channel and u-channel conformal blocks and ratios of the constants $C_p^{(\mathrm{s,t,u})}$, as well as the braiding matrix relating the two bases of conformal blocks, which is useful later. The t-channel conformal block with an internal momentum $\lambda h_1 - bh_1$ is a linear combination of s-channel conformal blocks, whose coefficients (the fusion matrix) are also fixed.

Knowing the four-point function $\langle V_{\alpha_\infty} V_{\lambda h_1} V_{-bh_1} V_{\alpha_0}\rangle$ up to an overall x-independent factor, Fateev and Litvinov [FL07] deduced the three-point function (5.4.28) of two generic and one semi-degenerate operators in the Toda CFT. The Toda CFT has at most one primary operator V_α for each momentum α, the vertex operator $e^{\langle\alpha,\varphi\rangle}$. Therefore, the coefficients $C_p^{(\mathrm{s})}$ in the s-channel decomposition (1.2.7) are

$$C_p^{(\mathrm{s})} = C(\alpha_\infty, \lambda h_1, \alpha_0 - bh_p) C_{-bh_1,\alpha_0}^{\alpha_0 - bh_p} \tag{1.2.10}$$

where $C(\alpha,\beta,\gamma) = \langle V_\alpha V_\beta V_\gamma\rangle$ denotes a three-point function, and $C_{-bh_1,\alpha_0}^{\alpha_0-bh_p}$ is the coefficient of $V_{\alpha_0 - bh_p}$ in the OPE of V_{-bh_1} and V_{α_0}, essentially a three-point function. The Toda CFT structure constants $C_{-bh_1,\alpha_0}^{\alpha_0-bh_p}$ are given by the Coulomb gas formalism. Since ratios of $C_p^{(\mathrm{s})}$ are known from the analysis above, ratios $C(\alpha_\infty, \lambda h_1, \alpha_0 - bh_p)/C(\alpha_\infty, \lambda h_1, \alpha_0 - bh_s)$ are known. Since the W_N algebra only depends on $b + b^{-1}$, an analoguous shift relation with shifts by $b^{-1}(h_p - h_s)$ holds. For generic real b the two shifts are not commensurable, hence the α_0-dependence of $C(\alpha_\infty, \lambda h_1, \alpha_0)$ is completely determined assuming continuity. Similarly, the u-channel decomposition fixes the α_∞-dependence. Finally, comparing constants in the s- and t-channel fixes the dependence on the semi-degenerate momentum. The solution of all these shift relations is (5.4.28), unique up to a normalization.

From the braiding matrix of a semi-degenerate operator $V_{\lambda h_1}$ with the degenerate V_{-bh_1} found by Fateev and Litvinov one can deduce the braiding matrix $\mathbf{B}_K$ of $V_{\lambda h_1}$ with $V_{-b\omega_K}$, the degenerate operator labelled by the K-th antisymmetric representation $\mathcal{R}(\omega_K)$ of A_{N-1}. The approach is to prove by induction that $\mathbf{B}_K$ is equal to an explicit expression (5.2.40) provided by the AGT correspondence (see Section 1.4). The proof is based on the pentagon relation depicted in Figure 1.4, which expresses the braiding matrix $\mathbf{B}_{K+1}$ of $V_{-b\omega_{K+1}}$ with $V_{\lambda h_1}$ in terms of the braiding matrices $\mathbf{B}_K$ and $\mathbf{B}_1$, and of the fusion of $V_{-b\omega_K}$ and V_{-bh_1} into $V_{-b\omega_{K+1}}$. The required fusion coefficients

[1] The author thanks Bertrand Eynard for pointing this out in the $N = 2$ case.

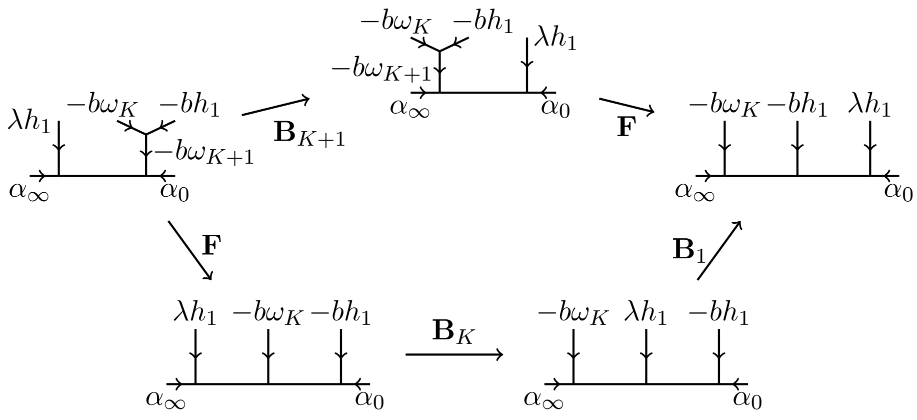


Figure 1.4: Pentagon relation used in the induction step to go from the braiding matrix $\mathbf{B}_K$ of $V_{-b\omega_K}$ with $V_{\lambda h_1}$ to the braiding matrix $\mathbf{B}_{K+1}$. The relation also involves fusion coefficients $\mathbf{F}$ of $V_{-b\omega_K}$ and V_{-bh_1} into $V_{-b\omega_{K+1}}$.

are found by exhibiting a conformal block of $\langle V_\alpha V_{-bh_1} V_{-b\omega_K} V_\beta\rangle$ which has the expected monodromy when braiding $V_{-b\omega_K}$ around V_{-bh_1} (twice): this braiding is itself a particular case of $\mathbf{B}_K$.

In principle the same approach yields the braiding matrix of a semi-degenerate operator with V_{-Kbh_1}, the degenerate operator labelled by the K-th symmetric representation $\mathcal{R}(Kh_1)$ of A_{N-1}. However, computations are much more tedious than for $V_{-b\omega_K}$, because weights of $\mathcal{R}(Kh_1)$ are not just Weyl permutations of a single weight. Instead, an explicit proposal (5.3.19) is derived from the AGT correspondence. This braiding matrix is new.

The braiding kernel (5.3.27) of two semi-degenerate operators is then guessed by generalizing the previous braiding matrix to non-integer K. It takes a form similar to the known Virasoro ($N = 2$) case [PT99], namely an integral over an $(N-1)$-component momentum of a ratio of Barnes double sine functions. The proposal obeys relevant shift relations, akin to the pentagon identity of Figure 1.4. For generic real b these shift relations should have a unique solution, but the author has yet to prove it. Another check is that the proposal reduces to the previous braiding when one of the semi-degenerate operators is tuned to be the degenerate V_{-Kbh_1}.

As explained in Section 1.4, the four-point function of two generic and two semi-degenerate operators translates through the AGT dictionary to four-dimensional $\mathcal{N} = 2$ $SU(N)$ SQCD with $2N$ flavours. The s- and u-channels correspond to different S-dual descriptions of the same four-dimensional theory, hence the braiding kernel implements S-duality.

Chapter 5 also discusses various fusion rules in Section 5.5 and irregular punctures obtained from collisions of primary operators in Section 5.6.

1.3 Supersymmetric localization on S^2

Supersymmetric localization is a powerful tool to reduce a supersymmetric path integral to a finite-dimensional integral. Since its introduction by Witten [Wit88], it has been used to evaluate exactly a large number of supersymmetric observables. The most relevant to the present work are Nekrasov's instanton partition function [Nek02; NO03] of four-dimensional $\mathcal{N} = 2$ theories, and their partition functions on round [Pes07] and deformed [HH12] four-spheres, as well as the partition function of two-dimensional $\mathcal{N} = (2, 2)$ theories on the round [BC12; DGLFL12] and deformed [GL12] two-sphere. Numerous other partition functions and observables of supersymmetric theories on various manifolds were also computed, but a review of the blossoming subject is beyond the scope of this thesis.

This section begins with an explanation of supersymmetric localization. It then concentrates on the partition function of two-dimensional $\mathcal{N} = (2, 2)$ theories on S^2, following [DGLFL12] reproduced in Chapter 2: the theory is introduced, as constructing supersymmetric theories on a curved space is non-trivial, and the partition function is localized in two different ways, yielding different explicit expressions. The S^4_b partition function of four-dimensional $\mathcal{N} = 2$ theories is presented afterwards with a brief explanation of how it arises from supersymmetric localization.

The crucial observation underlying supersymmetric localization is that supersymmetric observables are not affected by some deformations of the action. Consider a supersymmetric quantum field theory, and select a supercharge $\mathcal{Q}$ in its supersymmetry algebra. Find a combination V of the fields which is invariant under the bosonic symmetry $\{\mathcal{Q}, \cdot\}^2$ and which is such that the bosonic part of $\{\mathcal{Q}, V\}$ is positive semi-definite. Then for any $\mathcal{Q}$-closed observable $\mathcal{O}$, that is, $\{\mathcal{Q}, \mathcal{O}\} = 0$, the deformed expectation value

$$\langle \mathcal{O} \rangle_t = \int [\mathrm{D}\varphi]\, e^{-S[\varphi] - t\{\mathcal{Q}, V[\varphi]\}} \mathcal{O}[\varphi] \tag{1.3.1}$$

is independent of the real parameter $t \geq 0$ (signs of $\{\mathcal{Q}, V\}$ and t ensure that the deformed path integral converges). Indeed,

$$\begin{aligned} \partial_t \langle \mathcal{O} \rangle_t &= - \int [\mathrm{D}\varphi]\, e^{-S[\varphi] - t\{\mathcal{Q}, V[\varphi]\}} \mathcal{O}[\varphi] \Big\{ \mathcal{Q}, V[\varphi] \Big\} \\ &= - \int [\mathrm{D}\varphi] \Big\{ \mathcal{Q}, e^{-S[\varphi] - t\{\mathcal{Q}, V[\varphi]\}} \mathcal{O}[\varphi] V[\varphi] \Big\} = 0 \end{aligned} \tag{1.3.2}$$

where the second equality uses that $\{\mathcal{Q}, \mathcal{O}\} = \{\mathcal{Q}, \{\mathcal{Q}, V\}\} = \{\mathcal{Q}, S\} = 0$ and the last that $\mathcal{Q}$ is a symmetry of the path integral measure.

Thanks to t-invariance, the expectation value of $\mathcal{O}$, which is (1.3.1) for $t = 0$, can be computed by taking $t \to \infty$. In this limit, the semi-classical approximation around saddle points φ_0 of $\{\mathcal{Q}, V\}$ becomes exact. The expectation value then takes the form

$$\begin{aligned}\langle\mathcal{O}\rangle_{t=0} = \lim_{t\to\infty}\langle\mathcal{O}\rangle_t = \lim_{t\to\infty}\int_{\text{saddles}}[\mathrm{D}\varphi_0]\, e^{-S[\varphi_0]-t\{\mathcal{Q},V[\varphi_0]\}}\mathcal{O}[\varphi_0]Z_{1l}[\varphi_0] \\ = \int_{\{\mathcal{Q},V\}=0}[\mathrm{D}\varphi_0]\, e^{-S[\varphi_0]}\mathcal{O}[\varphi_0]Z_{1l}[\varphi_0]\,, \qquad (1.3.3)\end{aligned}$$

where the one-loop contribution $Z_{1l}[\varphi_0]$ encapsulates the contribution from fluctuations transverse to the saddle point locus. To get the last line, note that saddle points with $\{\mathcal{Q}, V[\varphi_0]\} > 0$ are exponentially suppressed as $t \to \infty$, and that zeros of $\{\mathcal{Q}, V\}$ are automatically saddle points by positivity. One says that the path integral localizes to zeros of the deformation term, which may form a finite-dimensional set for well-chosen $\mathcal{Q}$ and V.

A standard option is to take V as the sum of $\psi\overline{\mathcal{Q}\psi}$ over fermions ψ of the theory. The bosonic part of $\{\mathcal{Q}, V\}$ is then a sum of squares $\mathcal{Q}\psi\overline{\mathcal{Q}\psi}$ whose zeros are fixed points of $\mathcal{Q}$, obeying $\mathcal{Q}\psi = 0$. These supersymmetry equations, or BPS (Bogomol'nyi, Prasad, Sommerfield) equations, have a lower order than equations defining saddle points of a generic deformation term, thus are easier to solve. Another point of view on this localization to fixed points of $\mathcal{Q}$ is that integrals along non-trivial orbits of $\mathcal{Q}$ vanish by virtue of being fermionic integrals of constants.

The one-loop contribution is determined as a Gaussian integral as follows. Set $\varphi = \varphi_0 + \delta\varphi/\sqrt{t}$ in (1.3.1) and expand to quadratic order in the fluctuations $\delta\varphi$. The normalization by $\sqrt{t}$ does not affect the integration measure, as Jacobians in bosonic and fermionic integrals cancel because of supersymmetry. As $t \to \infty$ only the quadratic part of $\{\mathcal{Q}, V\}$ remains; schematically it is $\delta\varphi\Delta[\varphi_0]\delta\varphi$ for some operator Δ. In many cases, its bosonic part Δ_b is essentially a Laplacian, while its fermionic part Δ_f is essentially a Dirac operator. The Gaussian integral reduces to determinants of these operators,

$$Z_{1l}[\varphi_0] = \left(\frac{\det \Delta_\mathrm{b}[\varphi_0]}{\det \Delta_\mathrm{f}[\varphi_0]}\right)^{-1/2}. \qquad (1.3.4)$$

The remaining task is to evaluate these determinants.

The first approach, used in Chapter 2, is to decompose fields into spherical harmonics, or other modes that are convenient on the given manifold. In this decomposition Δ_b and Δ_f are typically block diagonal, with blocks involving a finite number of modes. The determinants of all blocks are straightforward to evaluate, and they combine into an infinite product whose regularization gives the one-loop partition function (1.3.4). In this process, the contributions of many bosonic and fermionic modes cancel.

The second approach, used for instance in [Pes07], takes advantage of this cancellation. It requires some mathematical machinery but is more systematic. To begin with, find a basis (X, X') of the fluctuation fields such that $\mathcal{Q}X = X'$

and $\mathcal{Q}X' = RX$ where $R = \mathcal{Q}^2$ is a bosonic transformation. Separate pairs (X_0, X_0') with X_0 bosonic and X_0' fermionic from pairs (X_1, X_1') with opposite statistics, and write down the part of V quadratic in fluctuations as

$$V^{(2)} = X_0' D_{00} X_0 + X_1 D_{10} X_0 + X_0' D_{01} X_1' + X_1 D_{11} X_1' \,. \tag{1.3.5}$$

The operators Δ_{b} and Δ_{f} are read from $\{\mathcal{Q}, V^{(2)}\}$. After some linear algebra, the constraint $\{\mathcal{Q}, \{\mathcal{Q}, V^{(2)}\}\} = 0$ implies that

$$\frac{\det \Delta_{\mathrm{b}}}{\det \Delta_{\mathrm{f}}} = \frac{\det R_0}{\det R_1} = \frac{\det_{\ker D_{10}} R_0}{\det_{\operatorname{coker} D_{10}} R_1} = \prod_i R(i)^{m_i} \,, \tag{1.3.6}$$

where i indexes eigenvalues $R(i)$ of R, and m_i is the multiplicity of $R(i)$ in $\ker D_{10}$ minus that in $\operatorname{coker} D_{10}$. These eigenvalues and multiplicities are read from the R-equivariant index

$$\operatorname{ind}_R(D_{10}) = \operatorname{Tr}_{\ker D_{10}} e^{tR} - \operatorname{Tr}_{\operatorname{coker} D_{10}} e^{tR} = \sum_i m_i e^{tR(i)} \,, \tag{1.3.7}$$

itself computed as a sum over fixed points of R, thanks to the Atiyah–Bott–Berline–Vergne equivariant localization formula [AB84; BV82].

Chapter 2 reproduces [DGLFL12], which applies supersymmetric localization to the S^2 partition function of a class of $\mathcal{N} = (2,2)$ gauge theories. The calculation was done simultaneously in [BC12] and extended to a squashed sphere in [GL12]. The theories of interest are composed of vector and chiral multiplets, which in flat space are dimensional reductions of the usual four-dimensional $\mathcal{N} = 1$ vector and chiral multiplets. Multiplets specific to two dimensions, such as twisted chiral multiplets, were considered in [DG13].

It is difficult to preserve supersymmetry when placing a theory on a curved manifold. The general technique [FS11] is to embed the theory in a supergravity theory then freeze the values of supergravity fields to a value that is invariant under a chosen set of supersymmetries. This is the direct analogue of how curved space Lagrangians are obtained by coupling to a metric then fixing it to the desired background.

The more pedestrian route taken in Section 2.2 is to construct the supersymmetric Lagrangian by adding $1/r$ and $1/r^2$ corrections to the Lagrangian on the plane, where r is the radius of the S^2. Squashing in a $U(1)$-invariant way induces further corrections to the Lagrangian, but the localized integral only depends on the radius r of the equator [GL12].

First, the S^2 counterpart of the $\mathcal{N} = (2,2)$ Poincaré supersymmetry is determined. Since S^2 is conformally flat, its superconformal algebra is trivially deduced from that of the plane, and there remains to find a subset of supercharges which close onto isometries of the sphere, rather than all conformal transformations. The result is $SU(2|1)$, whose bosonic subgroup $SU(2) \times U(1)$ acts as rotations of S^2 and a $U(1)$ R-symmetry.

One then finds S^2 analogues of vector and chiral multiplets, and how supersymmetry acts on their components. Just as in four-dimensions, vector multiplet components transform in the adjoint representation of a gauge group G, while chiral multiplet components transform in some representation R of G. The most general renormalizable Lagrangian action with $\mathcal{N} = (2,2)$ supersymmetry involving only these multiplets takes the form

$$S = S_{\text{v.m.}} + S_{\text{top}} + S_{\text{FI}} + S_{\text{c.m.}} + S_{\text{mass}} + S_W \,. \tag{1.3.8}$$

The vector multiplet action $S_{\text{v.m.}}$, the chiral multiplet action $S_{\text{c.m.}}$, and the superpotential term S_W are dimensional reductions of four-dimensional $\mathcal{N} = 1$ terms, with some $1/r$ and $1/r^2$ corrections on S^2.

The FI (Fayet–Iliopoulos) term S_{FI} associated to each $U(1)$ factor of G is also familiar from four dimensions. Additionally, for each $U(1)$ there is a topological term S_{top} measuring the gauge field flux B through S^2. Their coefficients ξ (FI parameter) and ϑ (theta angle) combine into complexified FI parameter $z = e^{-2\pi\xi + i\vartheta}$.

Finally, the twisted mass term S_{mass} is obtained by gauging the flavour symmetry group with a vector multiplet, giving it non-zero background values, and sending the gauge coupling to zero to make it non-dynamical. The background values must preserve supersymmetry, and this only allows one real parameter m in the flavour symmetry algebra. This parameter combines with the $U(1)$ R-charge q into a dimensionless complexified twisted mass[2] $m = r\mathrm{m} + i\frac{q}{2}$ for each chiral multiplet, that is, each irreducible representation of G in R.

The localization supercharge $\mathcal{Q}$ is selected and analyzed in Section 2.3. Its square combines a rotation, an R-symmetry, and a gauge symmetry. The two fixed points of the rotation are dubbed north and south poles. Solving the BPS equations away from the poles yields that $\mathcal{Q}$-invariant field configurations are such that vector multiplet scalars are constant on the sphere: one of the scalars is proportional to the gauge flux B and is discrete, while the other takes any value a in the real gauge algebra with $[a, B] = 0$. A constant gauge transformation diagonalizes a and B.

One of the BPS equations reads $(a+m)\phi = 0$ where ϕ are chiral multiplet scalars. Thus, for generic a, chiral multiplets vanish. The set of supersymmetric configurations with $\phi = 0$ is referred to as the Coulomb branch, and is parametrized by a and B. At isolated points of the Coulomb branch where eigenvalues of $-a$ coincide with some twisted masses, a Higgs branch opens up: an analysis of other BPS equations shows that there exist solutions with point-like (anti-)vortices $\phi \neq 0$ at the poles. There can also exist mixed branches, where some chiral multiplets are non-zero while a is not completely constrained.

[2] Chapter 2 denotes the real twisted mass as m and the dimensionful complexified twisted mass as $M = m + i\frac{q}{2r}$. This introduction uses the same notation as other chapters instead.

The canonical deformation term $\{\mathcal{Q}, V_{\text{can}}\} = |\{\mathcal{Q}, \lambda\}|^2 + |\{\mathcal{Q}, \psi\}|^2$ is not convenient, as it breaks the $SU(2)$ rotation symmetry of the sphere down to the $U(1)$ rotation generated by $\mathcal{Q}^2$. It turns out that $S_{\text{v.m.}}$, $S_{\text{c.m.}} + S_{\text{mass}}$ and S_W are $\mathcal{Q}$-exact, hence can be used as deformation terms instead. A direct consequence is that the partition function and other $\mathcal{Q}$-invariant observables only depend on the complexified FI parameters, and on complexified twisted masses since they appear in supersymmetry transformations, but not on the gauge coupling constants or on coefficients in the superpotential. Nevertheless, the superpotential constrains complexified twisted masses: S_W is only supersymmetric if the superpotential W (a polynomial in chiral multiplets) has total R-charge 2 and vanishing twisted mass.

Section 2.4 describes the result of localizing with respect to the deformation term $\{\mathcal{Q}, V\} = S_{\text{v.m.}} + S_{\text{c.m.}} + S_{\text{mass}}$. Among solutions of the BPS equations, Coulomb branch field configurations are saddle points of $\{\mathcal{Q}, V\}$, while Higgs branch configurations are not. The partition function thus localizes to an integral over the Coulomb branch,

$$Z_{S^2} = \frac{1}{\mathcal{W}} \sum_B \int_{\mathfrak{t}} \mathrm{d}a \, Z_{\text{cl}}(a, B, z, \bar{z}) Z_{1l}(a, B, m) \,. \tag{1.3.9}$$

Here, a is integrated over the Cartan algebra $\mathfrak{t}$ of the gauge group G, and $B \in \mathfrak{t}$ summed over GNO quantized fluxes, namely $w \cdot B \in \mathbb{Z}$ for all weights w of the chiral multiplet representation R of G. This sum over all quantized B divided by the order $\mathcal{W}$ of the Weyl group is occasionally written as a sum over gauge inequivalent B with a B-dependent combinatorical factor. The classical contribution in (1.3.9) is

$$Z_{\text{cl}}(a, B, z, \bar{z}) = z^{\operatorname{Tr}(\mathrm{i}a + \frac{B}{2})} \bar{z}^{\operatorname{Tr}(\mathrm{i}a - \frac{B}{2})} \,, \tag{1.3.10}$$

with an implicit product involving one complexified FI parameter $(z, \bar{z})$ for each $U(1)$ factor in G. The one-loop determinant is

$$Z_{1l}(a, B, m) = \prod_{e>0} \left[\langle e, a \rangle^2 + \frac{\langle e, B \rangle^2}{4} \right] \prod_{w \in R} \left[\frac{\Gamma\bigl(-\langle w, \mathrm{i}m + \mathrm{i}a + \frac{B}{2} \rangle\bigr)}{\Gamma\bigl(1 + \langle w, \mathrm{i}m + \mathrm{i}a - \frac{B}{2} \rangle\bigr)} \right] , \tag{1.3.11}$$

where the product over positive roots e of G comes from fluctuations of the vector multiplet, while the product over weights w of R is the chiral multiplet contribution. Here, $\langle w, m \rangle$ stands for the twisted mass of the chiral multiplet which the weight w corresponds to.

Section 2.5 describes how a different deformation term localizes the partition function to an integral over the Higgs branch instead of the Coulomb branch. Field configurations in the Higgs branch are characterized by the value $a = v$, called a Higgs branch vacuum, the vortex configuration at the north pole, and the anti-vortex configuration at the south pole. Vortices are solutions $\phi \neq 0$ to the BPS equations (2.5.1) with a magnetic flux $k \geq 0$

in an infinitesimal neighborhood of the north pole, while anti-vortices are solutions with non-positive magnetic flux near the south pole. The integrand consists in a classical contribution Z_{cl} and a one-loop determinant Z_{11}. Both Z_{cl} and Z_{11} are products of a contribution from the bulk, equal to (a residue of) the Coulomb branch ones (1.3.10) and (1.3.11) at $a = v$ and $B = 0$, and contributions from vortices and anti-vortices. Collecting vortex contributions in each vacuum v as $Z_{\text{v}}(v)$ and anti-vortex contributions as $Z_{\bar{\text{v}}}(v)$,

$$Z_{S^2} = \sum_{v \in \text{Higgs vacua}} Z_{\text{cl}}(v, 0, z, \bar{z}) \operatorname*{res}_{a=v} [Z_{11}(a, 0, m)]\, Z_{\text{v}}(v, m, z) Z_{\bar{\text{v}}}(v, m, \bar{z})\,. \quad (1.3.12)$$

The contributions Z_{v} and $Z_{\bar{\text{v}}}$ from vortices and anti-vortices are independent because (anti-)vortices do not affect fields away from the poles. BPS equations imply that the topological term S_{top} due to the magnetic flux is accompanied by a non-zero FI term S_{FI} such that $e^{-S_{\text{top}} - S_{\text{FI}}} = z^k$. The vortex partition function is thus a series in non-negative powers of z. The coefficient of z^k is the volume of the moduli space of k vortices, only known in some cases. Similarly, the anti-vortex partition function is a series in non-negative powers of $\bar{z}$. This holomorphic/anti-holomorphic factorization of (1.3.12) plays an important role in the correspondence with Toda CFT correlators in Section 1.4.

The localization argument guarantees that the Coulomb branch integral (1.3.9) and the Higgs branch sum of series (1.3.12) are equal. This is shown explicitly in Section 2.4.2 for a $U(N)$ vector multiplet coupled to fundamental and antifundamental chiral multiplets. The Coulomb branch integral can also be recast as a sum of factorized terms of the form (1.3.12) for any G and R, but vortex partition functions are not known. This factorization thus provides otherwise unknown vortex partition functions.

It is quite common for an $\mathcal{N} = (2, 2)$ theory to admit several Higgs branch expansions, one in powers of $(z, \bar{z})$ and another in powers of $(1/z, 1/\bar{z})$, converging for $|z| < 1$ and $|z| > 1$ respectively. The Coulomb branch integral continues the two expressions to all complex z. Vortices are essential for the equality of Coulomb branch and Higgs branch expressions, hence for the equality of distinct Higgs branch expansions. Section 1.5 discusses several $\mathcal{N} = (2, 2)$ dualities for which vortices are again crucial.

Supersymmetric localization has also been applied [Pes07; HH12] to the partition function of a four-dimensional $\mathcal{N} = 2$ Lagrangian gauge theory on a (squashed) four-sphere S_b^4, namely the $U(1) \times U(1)$ symmetric manifold

$$\frac{x_0^2}{r^2} + \frac{x_1^2 + x_2^2}{\ell^2} + \frac{x_3^2 + x_4^2}{\tilde{\ell}^2} = 1 \quad (1.3.13)$$

where $\ell/\tilde{\ell} = b^2$. Such a theory describes a vector multiplet with a gauge group G, and hypermultiplets in a representation R. The theory is placed supersymmetrically on S_b^4 following the general procedure of coupling it to a background supergravity multiplet [FS11].

The localization supercharge $\mathcal{Q}$ squares to a $U(1) \times U(1)$ rotation, an R-symmetry and a gauge transformation. The rotation has fixed points at the north and south poles $x_0 = \pm r$. Away from the poles, the BPS equations impose that all fields vanish, except one of the vector multiplet scalars which is a constant $a \in \mathfrak{g}$ in the Lie algebra of G. At the poles, the BPS equations reduce to well-studied (anti-)instanton equations: the gauge field strength is (anti-)self-dual. They admit point-like instanton solutions for generic a, contrarily to the two-dimensional case where vortices only exist for discrete values of a. The partition function thus takes the form

$$Z = \int_{\mathfrak{g}} \mathrm{d}a\, Z_{\mathrm{cl}}(a, z, \bar{z}) Z_{1l}(m, a) Z_{\mathrm{inst}}(m, a, z) Z_{\text{anti-inst}}(m, a, \bar{z})\,. \qquad (1.3.14)$$

where the integral over $\mathfrak{g}$ can be reduced to an integral over its Cartan subalgebra thanks to gauge symmetry. The one-loop determinant is a product of special functions (Upsilon functions), one per root of G and one per weight of R. The classical contribution is essentially $(z\bar{z})^{\langle a,a\rangle}$ in terms of the complexified coupling $z = e^{2\pi i\tau}$ which combines the gauge coupling and topological theta angle. As in two dimensions, contributions from instantons and anti-instantons factorize into (anti-)holomorphic functions of z. These functions are series in non-negative powers of z and $\bar{z}$. They are known explicitly when G is a product of unitary groups and in some other cases.

S-duality predicts that different Lagrangians describe the same $\mathcal{N} = 2$ theory, thus their S_b^4 partition functions must be equal. Since the supersymmetric localization result (1.3.14) holds even for strongly coupled Lagrangians, it can serve as a test of S-duality. Unfortunately the localized results of S-dual Lagrangians remain difficult to compare because instanton partition functions are expanded in terms of different couplings z. In contrast to the two-dimensional setting, no known expression of the partition function interpolates conveniently between two such expansions.

The forms (1.3.14) and (1.3.12) of S_b^4 and S^2 partition functions are reminiscent of the factorization of two-dimensional CFT correlators into holomorphic/anti-holomorphic conformal blocks. The AGT correspondence explained next confirms that the S_b^4 partition function is indeed equal to a Toda CFT correlator.

1.4 AGT correspondence and extended operators

Class S theories are four-dimensional $\mathcal{N} = 2$ reductions [Gai09a] of the six-dimensional A_{N-1} $(2,0)$ superconformal theory on a Riemann surface $C_{g,n}$ with genus g and n punctures (see Section 1.1). All observables of a class S theory are in principle fully determined by $C_{g,n}$ and data at each puncture. The AGT correspondence [AGT09] (see also [Wyl09]) is a concrete dictionary for observables of the class S theory placed on a squashed four-sphere S_b^4.

The first entry in the dictionary is that the S_b^4 partition function is equal to a correlator in the A_{N-1} Toda CFT on $C_{g,n}$ with a vertex operator at each puncture:

$$Z_{S_b^4} = \Big\langle \widehat{V}_{\alpha_1} \cdots \widehat{V}_{\alpha_n} \Big\rangle_{C_{g,n}}^{A_{N-1}\text{ Toda}} . \tag{1.4.1}$$

Matter multiplets are given masses in a class S theory by weakly gauging flavour symmetries. The mass parameters m for a full $SU(N)$ flavour symmetry belong to its Cartan subalgebra, and are encoded as a generic momentum $\alpha = Q + im$. Simple punctures with $U(1)$ flavour symmetry only have one mass parameter, encoded in a semi-degenerate momentum.

The simplest instance of (1.4.1) concerns the sphere with two full and one simple punctures: the Toda CFT three-point function (5.4.28) of two generic and one semi-degenerate vertex operators [FL07] is equal to the S_b^4 partition function (3.2.4) of N^2 free hypermultiplets [HH12].

For each trinion decomposition of $C_{g,n}$, the Toda CFT correlator factorizes as an integral of the product of a holomorphic and an anti-holomorphic conformal blocks. If each trinion involves a simple puncture (this requires $g = 0$ or 1) then the expansion can be written explicitly (1.2.1) as

$$\Big\langle \widehat{V}_{\alpha_1} \cdots \widehat{V}_{\alpha_n} \Big\rangle_{C_{g,n}}^{A_{N-1}\text{ Toda}} = \int d^{3g-3+n}\beta \Big[C(\alpha,\beta)\, \mathcal{F}(\alpha,\beta,z)\, \mathcal{F}(\alpha,\beta,\bar{z}) \Big] \tag{1.4.2}$$

where there is one generic momentum β for each tube and $C(\alpha, \beta)$ is the product of one three-point function per trinion and one inverse two-point function per tube. Conformal blocks $\mathcal{F}(z)$, fixed by the W_N symmetry algebra of the Toda CFT, capture the dependence on the complex structure of $C_{g,n}$ parametrized by z. They have series expansions around the corner $z \to 0$ of the moduli space where the decomposition has thin tubes.

Similarly, to each decomposition of $C_{g,n}$ corresponds a description of the class S theory in terms of vector multiplets (tubes) gauging flavour symmetries of matter theories (trinions). For $N = 2$ these descriptions are always Lagrangian, while for $N > 2$ they only are if each trinion involves a simple puncture. Given a Lagrangian description, supersymmetric localization expresses the S_b^4 partition function as an integral (1.3.14)

$$Z_{S_b^4} = \int d^{3g-3+n} a \Big[Z_{1l}(m,a) Z_{\text{cl,inst}}(m,a,z) Z_{\text{cl,anti-inst}}(m,a,\bar{z}) \Big] . \tag{1.4.3}$$

The integral ranges over the Coulomb branch, parametrized by a in the Cartan algebra of $SU(N)$ for each vector multiplet scalar, and m stands for masses. The classical contribution in (1.3.14) is combined here with the (anti-)instanton partition functions, which have series expansions around the weakly coupled point $z \to 0$ of this trinion decomposition of $C_{g,n}$.

The parallel between (1.4.2) and (1.4.3) is clear. Beyond stating the equality (1.4.1) of the S_b^4 partition function and the Toda CFT correlator, the AGT correspondence states that the integrands coincide. The complex structure of $C_{g,n}$ encodes complexified gauge coupling constants, external momenta α correspond to masses m, and internal momenta β to the Coulomb branch parameters a.

A straightforward check is that $C(\alpha, \beta)$ is equal to $Z_{11}(m, a)$. The former is a product of a three-point function for each trinion, and an inverse two-point function for each tube. The latter is the product of one-loop determinants of all vector and hypermultiplets in the $\mathcal{N} = 2$ theory. Vector multiplet one-loop determinants reproduce inverse two-point functions and hypermultiplet one-loop determinants reproduce three-point functions. This second point is not surprising: it is the AGT correspondence for N^2 free hypermultiplets. Conformal blocks $\mathcal{F}(z)$ and instanton partition functions are more difficult to compare: conformal blocks can be evaluated tediously order by order, and the first few coefficients match with those of the appropriate instanton partition functions.

S-duality relates descriptions of the $\mathcal{N} = 2$ theory associated to different trinion decompositions of $C_{g,n}$. The relevant instanton partition functions are completely different power series expansions around different corners of the moduli space of coupling constants. Holomorphic and antiholomorphic instanton partition functions must nevertheless assemble into an S-duality invariant object, the S_b^4 partition function. Its invariance under S-duality translates elegantly as modular invariance in the Toda CFT: correlators do not depend on the channel in which they are expanded into conformal blocks. Modular invariance in the Liouville CFT, shown in [PT99; Tes03; HJS09], thus confirms S-duality for class S theories with $SU(2)$ gauge groups.

In the localization computation, one can include any gauge-invariant observable that preserves the supersymmetry $\mathcal{Q}$ used to localize. Many constructions of non-local operators supported on curves, surfaces, or three-dimensional walls have found AGT translations over the years.

The simplest such operator is the Wilson loop operator, namely the supersymmetric version of $W_R = \operatorname{Tr}_R \operatorname{Pexp} \oint_\gamma A$, supported on a $\mathcal{Q}^2$-invariant circle γ. It is defined by integrating the gauge one-form A of one of the $SU(N)$ vector multiplets, and tracing the path-ordered exponential in a representation R of $SU(N)$. This definition relies on a choice of S-duality frame, in other words a trinion decomposition of $C_{g,n}$. The expectation value $\langle W_R \rangle$ takes the same form as the S_b^4 partition function (1.3.14) with an additional factor $\operatorname{Tr}_R \exp(-2\pi b^{\pm 1} a)$. As observed in [Ald+09; DGOT09], this factor can be realized in the Toda CFT as the addition of a Verlinde loop operator. Schematically, such a loop operator is constructed by inserting a degenerate vertex operator labelled by R, and moving it along a curve which wraps the tube of $C_{g,n}$ corresponding to the $SU(N)$ vector multiplet.

Any non-self-intersecting curve on $C_{g,n}$ wraps a tube in some trinion decomposition hence corresponds to a Wilson loop in some S-duality frame of the $\mathcal{N} = 2$ theory. S-duality maps Wilson loops, which measure the effect of inserting massive electrically charged probes, to 't Hooft (or dyonic) loops, measuring the effect of (electrically and) magnetically charged probes. The insertion of Verlinde loop operators on arbitrary curves in the Toda CFT correlator (1.4.1) should thus give the expectation value of 't Hooft loops. This prediction was confirmed in [GOP11] by defining and localizing the partition function on S^4 in the presence of a 't Hooft loop operator, then comparing in some cases to a CFT calculation done in [GLF10] by Jaume Gomis and the author. Topological webs instead of Verlinde loop operators are needed to capture arbitrary dyonic loops.

Surface operators invariant under $\mathcal{Q}$ may be supported on two (squashed) $S^2 \subset S^4_b$, defined by $x_1 = x_2 = 0$ or $x_3 = x_4 = 0$ in (1.3.13). Half-BPS surface operators, namely operators preserving 4 of the 8 supercharges of a class S theory, are constructed in two ways.

The first construction [GW06], which yields so-called M5-brane surface operators, is similar in spirit to 't Hooft loop operators. It replaces the path integral by an integral over field configurations with a non-trivial boundary condition around the surface. The altered path integral is captured by a correlator on $C_{g,n}$ of a theory other than the Toda CFT, as conjectured in [BFFR10]. Preliminary results [GKLFN15] with Jaume Gomis, Hee-Cheol Kim, and Satoshi Nawata indicate that all these M5-brane surface operators can be obtained as special cases of the next construction.

The second construction consists in coupling the four-dimensional $\mathcal{N} = 2$ theory to a two-dimensional $\mathcal{N} = (2, 2)$ theory restricted to the surface. As summarized later in this section, the first part of [GLF14] (Chapter 3) exhibits such surface operators, later called M2-brane surface operators, which correspond to the insertion in the Toda CFT correlator of a degenerate vertex operator labelled by any representation of $SU(N)$. This agrees with a conjecture in [Ald+09] for the simplest degenerate vertex operator; more references are given in Chapter 3.

Finally, $\mathcal{Q}$-invariant domain walls can be constructed by letting gauge couplings vary continuously near the equator to connect two hemispheres with different gauge coupling constants. In the presence of such a domain wall, the contribution from each hemisphere to the partition function is an instanton partition function, equal to a conformal block for the Toda CFT. The domain wall changes how holomorphic and anti-holomorphic contributions are paired. This is reproduced in the Toda CFT by the insertion of a topological defect [DGG10]. When couplings on the two hemispheres are tuned so that the two theories are S-dual, one can apply S-duality to one side of the wall to get the same theory on both sides: this yields the S-duality domain wall. Since S-duality acts on conformal blocks as a modular transformation, the partition function is

$$Z = \int \mathrm{d}\alpha\, \mathcal{F}^{\sigma}_{\alpha}(z)\, \mathcal{F}^{\sigma}_{\alpha}(\bar{z}^D) = \int \mathrm{d}\alpha\, \mathrm{d}\alpha'\, \mathcal{F}^{\sigma}_{\alpha}(z)\, \mathbf{B}^{\sigma\sigma'}_{\alpha\alpha'}\, \mathcal{F}^{\sigma'}_{\alpha'}(\bar{z}) \tag{1.4.4}$$

where σ and σ' denote trinion decompositions of $C_{g,n}$ related by S-duality, $\mathcal{F}$ are conformal blocks, and $\mathbf{B}$ is the modular transformation. One-loop determinants are absorbed here in the measure $\mathrm{d}\alpha$.

The same S-duality domain wall can also be realized by coupling to the four-dimensional theory a three-dimensional $\mathcal{N} = 2$ theory on the equator. The appropriate three-dimensional theory was determined for $\mathcal{N} = 4$ SYM in [GW08a], for $\mathcal{N} = 2^*$ SYM in [HLP10], and for $N_f = 4$ $SU(2)$ SQCD in [TV12]. As argued in [DGG10], its partition function on the squashed three-sphere must be the modular transformation $\mathbf{B}_{\sigma\sigma'}$ so that coupling it to the four-dimensional theory on both hemispheres yields (1.4.4). It should thus be possible to deduce the three-dimensional theory on the S-duality domain wall for $N_f = 2N$ $SU(N)$ SQCD from the new braiding kernel (5.3.27) worked out in Section 5.3.2.

All extended operators described above can also be engineered from brane constructions in M-theory, and this helps understand their correspondence with Toda CFT observables [DGG10]. Recall that class S theories are obtained as the world-volume theory of N M5-branes wrapped on a Riemann surface Σ. One may insert transverse M5-branes with four directions in common with the N M5-branes, or M2-branes ending on a two-dimensional surface, while only breaking half of the supersymmetry. Intersections of M5-branes are labelled by a continuous (generic or partially degenerate) momentum while M2–M5 intersections are labelled by a degenerate momentum, or equivalently a representation of $SU(N)$.

From the point of view of the world-volume theory of the N M5-branes, transverse M5-branes form codimension 2 defects in the six-dimensional A_{N-1} $(2,0)$ superconformal theory. The four common directions can be distributed in various ways among Σ and space.

- A point on Σ and the whole space. This inserts a vertex operator in the CFT and alters the four-dimensional theory: with such transverse branes one engineers class S theories corresponding to Riemann surfaces $\Sigma = C_{g,n}$ with punctures.

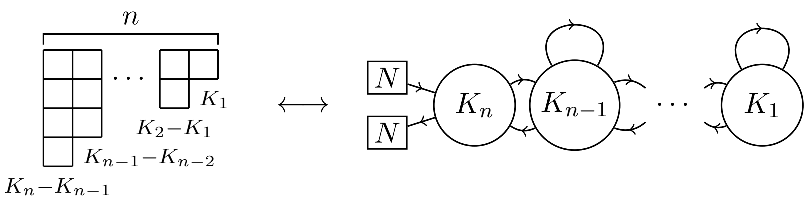
- A curve on Σ and a domain wall in four dimensions. This inserts a topological defect in the CFT correlator, corresponding to a domain wall in the class S theory. In particular the S-duality domain wall is realized by braiding a puncture along a curve.

- The whole of Σ and a surface in space. The two-dimensional CFT on Σ is altered. The M5-brane surface operator introduces non-trivial boundary conditions on the four-dimensional fields, which break the $SU(N)$ gauge group to the commutant of the continuous momentum.

Defects constructed from collections of M2-branes ending on the N M5-branes are codimension 4 defects, and can be placed in various directions.

- The whole of Σ and a point in space. This should alter the two-dimensional CFT and insert a local operator in four dimensions, but the author is not aware of quantitative results in this direction.
- A curve on Σ and a loop in four dimensions. This setup yields the correspondence between Wilson/'t Hooft loop operators in gauge theory and Verlinde loop operators in the Toda CFT, both labelled by a representation of $SU(N)$.
- A point on Σ and a surface in four dimensions. The M2-branes insert in the correlator a degenerate vertex operator labelled by a representation of $SU(N)$, and insert a surface operator in the four-dimensional theory by coupling it to a two-dimensional theory described now.

Chapter 3 [GLF14] associates a two-dimensional $\mathcal{N} = (2,2)$ gauge theory to a collection of M2-branes for any representation R of $SU(N)$:



(1.4.5)

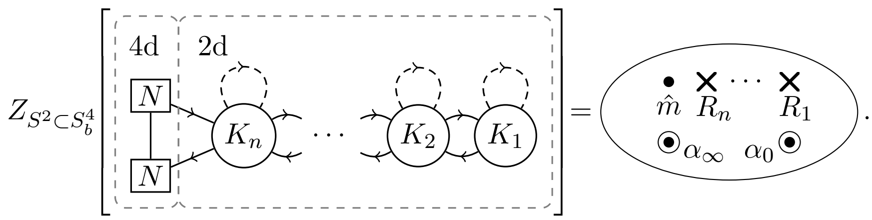
with the following notations.[3] The Young diagram of R has n columns with $K_n - K_{n-1} \geq \cdots \geq K_2 - K_1 \geq K_1 \geq 0$ boxes. The quiver on the right-hand side represents a two-dimensional $\mathcal{N} = (2,2)$ gauge theory: squares are $U(N)$ flavour groups, circles are $U(K_j)$ vector multiplets, and arrows are chiral multiplets in the bifundamental representation of groups at their end-points. Cubic superpotential terms couple each adjoint chiral multiplet (loops in the quiver) to the neighboring bifundamental chiral multiplets, and one takes all FI parameters except that of $U(K_n)$ to vanish. Note that the flavour symmetry reduces from $U(N) \times U(N)$ to $S[U(N) \times U(N)] = SU(N) \times U(1) \times SU(N)$ because the diagonal $U(1)$ is a gauge transformation.

Wrapping the brane setup on a Riemann surface Σ yields a surface operator in the class S theory defined by Σ. The two-dimensional theory is coupled by identifying its $S[U(N) \times U(N)]$ flavour symmetry with gauge or flavour symmetries of a hypermultiplet in the class S theory, then turning on a cubic superpotential for this hypermultiplet and the (anti-)fundamental chiral multiplets (the left-most arrows in the quiver). When the class S theory is placed on S^4_b, inserting the surface operator on a $U(1) \times U(1)$ invariant $S^2 \subset S^4_b$ corresponds to inserting a degenerate vertex operator labelled by R in the Toda CFT correlator.

[3] Since Chapter 3 and Chapter 4 focus on two-dimensional theories, the ranks N and K_j are denoted there by N_f and N_j.

The (complexified) FI parameter of $U(K_n)$ controls the position of the degenerate vertex operator on Σ. If the FI parameters of $U(K_j)$ with $j < n$ are taken to be non-zero instead, the degenerate puncture labelled by R is replaced by n degenerate punctures whose positions are controlled by FI parameters. The j-th puncture is labelled by the $(K_j - K_{j-1})$-th antisymmetric representation: the Young diagram (1.4.5) has split into its columns. More generally, the insertion of n punctures labelled by symmetric or antisymmetric representations can be realized by including adjoint chiral multiplets only for some of the nodes, as depicted in (1.4.6).

The correspondence is checked by considering surface operators in the simplest class S theory: N^2 free hypermultiplets. Since in this case the two-dimensional and four-dimensional theories are only coupled through flavour symmetries rather than dynamical fields, the partition function of the 4d/2d system decomposes as $Z_{S^2 \subset S^4_b} = Z_{S^4_b} Z_{S^2}$. Chapter 3 confirms that the $S^2 \subset S^4_b$ partition function of the 4d/2d system is equal to a Toda CFT $(n+3)$-point function of n degenerates ✕, a semi-degenerate •, and two generic ⦿ vertex operators:



(1.4.6)

Using explicit expressions of the S^2 and S^4_b partition functions, the equality is tested in every degeneration limit where two of the Toda CFT punctures meet, and proven for the case of a single antisymmetric degenerate vertex operator. Expressions in the main text include factors which are omitted here as they can be absorbed into a normalization of vertex operators and into ambiguities in the definition of Z_{S^2}. Table 3.1 summarizes the correspondence, and its special cases for $n = 1$ analyzed throughout Chapter 3.

The two-dimensional theory describes a $U(K_1) \times \cdots \times U(K_n)$ vector multiplet coupled to fundamental, antifundamental, adjoint, and bifundamental chiral multiplets as depicted by the quiver. For each gauge group factor, either there is an adjoint chiral multiplet and two cubic superpotential terms coupling it to neighboring bifundamental chiral multiplets, or there is a quartic superpotential term for the four neighboring chiral multiplets, and no adjoint chiral multiplet. The N fundamental and N antifundamental chiral multiplets of $U(K_n)$ are not included in such superpotentials.

The 4d/2d coupling fixes masses of the hypermultiplets, hence $Z_{S^4_b}$, in terms of twisted masses of the N fundamental and N antifundamental chiral multiplets of $U(K_n)$. These $2N$ twisted masses, denoted as m_s and $\widetilde{m}_s$ for $1 \leq s \leq N$, are redundant: a global $U(1)$ gauge transformation shifts them all. The generic momenta α_0 and α_∞ encode $2(N-1)$ twisted masses as

$$\alpha_0 = Q - \frac{1}{b}\sum_{s=1}^{N} \mathrm{i} m_s h_s \,, \qquad \alpha_\infty = Q - \frac{1}{b}\sum_{s=1}^{N} \mathrm{i}\widetilde{m}_s h_s \,, \tag{1.4.7}$$

where the weights h_s of the fundamental representation of A_{N-1} sum to zero. The remaining twisted mass appears in the semi-degenerate momentum

$$\hat{m} = (\varkappa + K_n b) h_1 \,, \qquad \varkappa = \frac{1}{b}\sum_{s=1}^{N}(1 + \mathrm{i} m_s + \mathrm{i}\widetilde{m}_s) \,. \tag{1.4.8}$$

A conformal transformation places the vertex operators with momenta α_0, α_∞ and $\hat{m}$ at 0, ∞ and 1 respectively. The gauge theory has an additional $U(1)$ flavour symmetry under which adjoints have charge ± 2 and bifundamentals ± 1. The matching requires the associated twisted mass to be $-\mathrm{i}b^2$.

Finally, degenerate vertex operators describe the remaining gauge theory information. Their n momenta $-b\Omega_j$ give gauge group ranks and the matter content. For $1 \leq j \leq n$,

$$\Omega_j = \omega_{K_j - K_{j-1}} \quad \text{or} \quad (K_j - K_{j-1}) h_1 \tag{1.4.9}$$

is the highest weight of the $(K_j - K_{j-1})$-th antisymmetric or symmetric representation of A_{N-1}. The matter content is read as follows: the factor $U(K_j)$ has an adjoint chiral multiplet if Ω_j and Ω_{j+1} are both symmetric or both antisymmetric, and otherwise it does not. Complexified FI parameters (up to a sign) are

$$\hat{z}_j = x_j / x_{j+1} \tag{1.4.10}$$

in terms of the positions x_j of degenerate vertex operators. To simplify notations above, $K_0 = 0$, Ω_{n+1} is considered to be symmetric, and $x_{n+1} = 1$.

This concludes the description of (1.4.6).

1.5 Two-dimensional $\mathcal{N} = (2,2)$ dualities

The previous section equates some surface operators in class S theories to Toda CFT correlators enriched by the insertion of degenerate vertex operators. Inspired by how Toda CFT modular invariance corresponds to S-duality, symmetries of enriched correlators are translated in Chapter 4 to dualities between pairs of two-dimensional $\mathcal{N} = (2,2)$ theories. The dualities, akin to the four-dimensional $\mathcal{N} = 1$ Seiberg duality, state that different Lagrangians flow at large distances to the same infrared theory.

Some symmetries are obvious on both sides of the equality (1.4.6). Generic vertex operators are invariant under Weyl symmetries, which permute the components of $\alpha - Q$: given (1.4.7), the corresponding twisted masses are simply permuted. Another simple symmetry is the conformal invariance under $x \mapsto x^{-1}$. It exchanges $\alpha_0 \leftrightarrow \alpha_\infty$, hence $m \leftrightarrow \widetilde{m}$, and maps all $\hat{z}_j \mapsto \hat{z}_j^{-1}$ thus corresponds to conjugating all charges in the gauge theory.

Two Toda CFT symmetries translate to interesting gauge theory dualities: conjugation of all momenta defined below, and permutations of degenerate vertex operators. For every duality, S^2 partition functions of dual theories are equal because the corresponding Toda CFT correlators (1.4.6) are equal.

Section 4.2 focuses on the simplest case of (1.4.6), namely the insertion of a single degenerate vertex operator $\widehat{V}_{-b\omega_K}$ labelled by the K-th antisymmetric representation of A_{N-1}. The corresponding gauge theory is $U(K)$ SQCD with N flavours, described by a $U(K)$ vector multiplet coupled to N fundamental and N antifundamental chiral multiplets. Its complexified FI parameter is $\hat{z} = (-1)^N z = x$, and twisted masses are fixed by (1.4.7) and (1.4.8).

The Toda CFT four-point function is invariant under conjugation of all momenta, which acts on weights as $h_s \mapsto h_s^C = -h_{N+1-s}$. Under this transformation, the degenerate momentum is mapped to $-b\omega_K^C = -b\omega_{K^D}$ with $K^D = N - K$; generic momenta are mapped as $\alpha \mapsto 2Q - \alpha$, up to an unimportant Weyl symmetry; and the semi-degenerate momentum is mapped (up to a Weyl symmetry) to $(\varkappa^D + K^D b)h_1$ with $\varkappa^D = Nb^{-1} - \varkappa$. All in all, the Toda CFT correlator is mapped to another four-point function of an antisymmetric degenerate, a semi-degenerate, and two generic vertex operators.

In gauge theory terms, momentum conjugation gives $K^D = N - K$ and

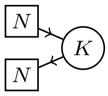
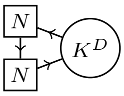
$$\begin{aligned} m_s^D &= \mathrm{i}/2 - m_s & \qquad & & m_s^D &= \mathrm{i}/2 - \widetilde{m}_s \\ \widetilde{m}_s^D &= \mathrm{i}/2 - \widetilde{m}_s & & \text{or} & \widetilde{m}_s^D &= \mathrm{i}/2 - m_s \\ \hat{z}^D &= \hat{z} & & & \hat{z}^D &= \hat{z}^{-1} \end{aligned} \tag{1.5.1}$$

where the second set of parameters results from applying the further conformal transformation $x \mapsto x^{-1}$, in other words gauge theory charge conjugation. Section 4.2 uses the first choice of parameters, while this introduction and other literature use the second one, more convenient when discussing quivers. The equality of correlators translates to

$$Z_{S_b^4} Z_{S^2}^{U(K),N}(\hat{z}, \bar{\hat{z}}, m, \widetilde{m}) = Z_{S_b^4}^D Z_{S^2}^{U(K^D),N}(\hat{z}^D, \bar{\hat{z}}^D, m^D, \widetilde{m}^D)\,. \tag{1.5.2}$$

The two S_b^4 partition functions are different because the hypermultiplets couple to two-dimensional theories with different twisted masses. Explicit computations show that their ratio is the one-loop determinant of N^2 free chiral multiplets. These multiplets have twisted masses $m_t + \widetilde{m}_s = \mathrm{i} - m_s^D - \widetilde{m}_t^D$ for $1 \leq s, t \leq N$, and transform in the bifundamental representation of the flavour group $S[U(N) \times U(N)]$. The relation between twisted masses of fundamentals q_s^D, antifundamentals $\widetilde{q}_t^D$, and free chiral multiplets M_{st}^D in the $U(K^D)$ theory can be imposed by a cubic superpotential $W = \sum_{s,t} \operatorname{Tr} q_s^D M_{st}^D \widetilde{q}_t^D$.

The conclusion is that $U(K)$ SQCD with N flavours (dubbed the electric theory) and $U(K^D)$ SQCD with N flavours and N^2 free chiral multiplets subject to a cubic superpotential (dubbed the magnetic theory) have equal S^2 partition functions. Diagramatically,

$Z_{S^2}\Bigg[$  $\Bigg|\, \hat{z}, \overline{\hat{z}}, m, \widetilde{m}\Bigg] = Z_{S^2}\Bigg[$  $\Bigg|\, \hat{z}^D, \overline{\hat{z}}^D, m^D, \widetilde{m}^D\Bigg]$. (1.5.3)

The equality of S^2 partition functions is strong evidence that the two theories are dual. This duality, Seiberg duality for two-dimensional $\mathcal{N} = (2, 2)$ theories, is a direct analogue of the well-known four-dimensional $\mathcal{N} = 1$ Seiberg duality. The dual rank $K^D = N - K$ and the cubic superpotential are the same, and in both dualities all quantum numbers of the free chiral multiplets M^D_{st} in the magnetic theory coincide with quantum numbers of mesons $\widetilde{q}_s q_t$ in the electric theory.

It is interesting to consider the Higgs branch decomposition (1.3.12) of the S^2 partition function. There is one term for each Higgs branch vacuum, that is, each solution of the BPS equations due to supersymmetric localization with a non-zero FI parameter ($|\hat{z}| \neq 1$). Assuming $|\hat{z}| < 1$ for definiteness, one finds $\binom{N}{K}$ vacua, labelled by sets of K flavours in $[\![1, N]\!]$. This number is equal to the dimension of the representation of A_{N-1} with highest weight ω_K. As a result the Higgs branch decomposition of Z_{S^2} and the s-channel decomposition of the Toda CFT four-point function have equally many terms. In fact, the two sums match term-wise: Higgs branch vacua correspond to internal momenta allowed by the fusion of $\widehat{V}_{-b\omega_K}$ with $\widehat{V}_{\alpha_0}$, one-loop determinants correspond to products of three-point functions, and classical and vortex partition functions correspond to conformal blocks. These identifications work in the same way for all cases of the matching (1.4.6), and yield new expressions for some conformal blocks.

Four-dimensional $\mathcal{N} = 1$ Seiberg duality has many generalizations, several of which hold for two-dimensional $\mathcal{N} = (2, 2)$ theories. Kutasov–Schwimmer dualities discussed next apply to SQCD enriched with adjoint chiral multiplets subject to a superpotential. For another superpotential one finds a duality referred to as the $\mathcal{N} = (2, 2)^*$-like duality. As explained later, all of these dualities apply to individual gauge group factors in quiver gauge theories.

Section 4.3 considers the next simplest case of (1.4.6), with a single degenerate vertex operator $\widehat{V}_{-Kbh_1}$ labelled by the K-th symmetric representation of A_{N-1}. The gauge theory is $U(K)$ SQCD with N flavours and an adjoint, namely a $U(K)$ vector multiplet coupled to N fundamental and N antifundamental chiral multiplets and to one adjoint chiral multiplet. Its complexified FI parameter is $\hat{z} = x$, the twisted mass m_X of the adjoint is given by $\mathrm{i}m_X = b^2$, and other twisted masses are fixed by (1.4.7) and (1.4.8).

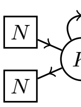
Solving the BPS equations shows that the theory has $\binom{N-1+K}{K}$ Higgs branch vacua. Contrarily to SQCD with no adjoint matter, this number is not invariant under $K \to N-K$: it grows indefinitely with K. Therefore, $U(K)$ SQCD with an adjoint cannot exhibit a duality $K \mapsto K^D$ in general. Dualities are found below by adding a superpotential term which reduces the number of vacua.

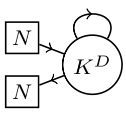
Another point of view on the absence of duality in general is seen by noting that contrarily to the antisymmetric degenerate momentum $-b\omega_K$, whose conjugate is $-b\omega_{K^D}$, the conjugate of the symmetric degenerate momentum $-Kbh_1$, namely Kbh_N, is neither symmetric nor antisymmetric. Hence, momentum conjugation does not yield a Toda CFT four-point function of the same form as the original correlator. Nevertheless, there are two cases where the four-point function has symmetries.

In the first setting (Section 4.3.1), the semi-degenerate momentum $\hat{m}$ of (1.4.8) is taken to be degenerate by setting $\varkappa = -Lb$ hence $\hat{m} = -K^D bh_1$ for $K^D = L-K$. The presence of two degenerate momenta in the Toda CFT four-point function ties the two generic momenta together: $\alpha_0 + \alpha_\infty$ may only take discrete values. The corresponding gauge theory is $U(K)$ SQCD with N fundamental chiral multiplets q_t, N antifundamentals $\widetilde{q}_t$, and an adjoint X subject to a superpotential

$$W = \sum_{t=1}^{N} \widetilde{q}_t X^{l_t} q_t \tag{1.5.4}$$

for some integers $l_t \geq 0$ with $L = \sum_t l_t$. Crossing symmetry of the two degenerate operators exchanges $K \leftrightarrow K^D = L-K$, hence $U(K)$ and $U(K^D)$ SQCD with adjoint matter and the superpotential (1.5.4) are dual. Contrarily to Seiberg duality above, external momenta are not altered hence the S_b^4 contribution does not change, and the duality does not involve additional free chiral multiplets. All in all,

$Z_{S^2}\Bigg[$ 

$\left|\begin{matrix} \hat{z}, \bar{\hat{z}}, m, \widetilde{m} \\ W = \widetilde{q} X^l q \end{matrix}\right.\Bigg] = Z_{S^2}\Bigg[$ 

$\left|\begin{matrix} \hat{z}^D, \bar{\hat{z}}^D, m^D, \widetilde{m}^D \\ W = \widetilde{q}^D (X^D)^l q^D \end{matrix}\right.\Bigg]$ (1.5.5)

with $\hat{z}^D = \hat{z}^{-1}$, $m_t^D = m_t$ and $\widetilde{m}_t^D = \widetilde{m}_t$. When all $l_t = 1$ this theory is $\mathcal{N} = (2,2)^*$ SQCD, a mass deformation of $\mathcal{N} = (4,4)$ SQCD, hence the duality is dubbed $\mathcal{N} = (2,2)^*$-like duality.

In the second setting (Section 4.3.2), $\mathrm{i}m_X = b^2 = -1/(l+1)$ for some integer $l \geq 1$. In gauge theory, this value is such that the superpotential

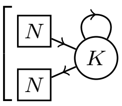
$$W = \operatorname{Tr} X^{l+1} \tag{1.5.6}$$

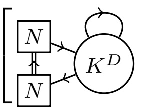
is supersymmetric. It turns out that up to a Weyl symmetry, the conjugate of a symmetric degenerate momentum $-Kbh_1$ is then also a symmetric degenerate momentum: $-K^D bh_1$ with $K^D = lN - K$. Following the same steps as for an antisymmetric degenerate momentum, conjugation of momenta translates to the Kutasov–Schwimmer duality. Parameters of the dual theory are $m_X^D = m_X = -\mathrm{i}b^2 = \mathrm{i}/(l+1)$, and

$$\begin{aligned} m_s^D &= m_X - m_s & \qquad & & m_s^D &= m_X - \widetilde{m}_s \\ \widetilde{m}_s^D &= m_X - \widetilde{m}_s & \qquad \text{or} & & \widetilde{m}_s^D &= m_X - m_s \\ \hat{z}^D &= \hat{z} & & & \hat{z}^D &= \hat{z}^{-1} \end{aligned} \tag{1.5.7}$$

where the second set of parameters is obtained by gauge theory charge conjugation. The magnetic theory also has lN^2 free chiral multiplets M_{jst}^D whose twisted masses are equal to those of mesons in the electric theory, namely $m_t + \widetilde{m}_s + jm_X$ for $0 \leq j < l$. These twisted masses are fixed in the magnetic theory by a superpotential. Diagramatically,

$$Z_{S^2}\left[\text{quiver}\;\middle|\;\begin{matrix}\hat{z}, \bar{\hat{z}}, m, \widetilde{m} \\ W = \operatorname{Tr} X^{l+1}\end{matrix}\right] = Z_{S^2}\left[\text{quiver}\;\middle|\;\begin{matrix}\hat{z}^D, \bar{\hat{z}}^D, m^D, \widetilde{m}^D \\ W = \operatorname{Tr}(X^D)^{l+1}\end{matrix}\right]. \tag{1.5.8}$$

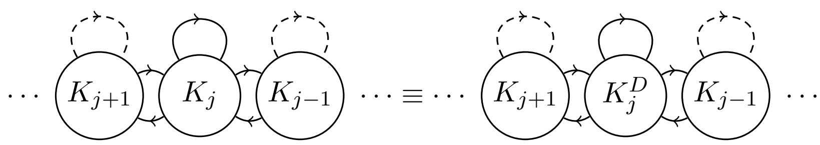




Section 4.4 describes dualities of quiver gauge theories which appear in the matching (1.4.6). It is based on the observation that Seiberg, Kutasov–Schwimmer, and $\mathcal{N}=(2,2)^*$-like dualities relating a $U(K)$ and a $U(K^D)$ theories still hold if flavour symmetries of their fundamental and antifundamental chiral multiplets are gauged. In other words, if multiplets charged under $U(K_j)$ form one of the theories for which a duality is available, then that duality can be applied.

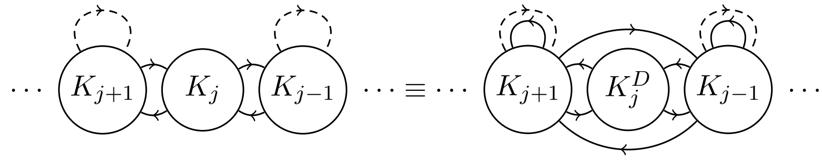
Consider a node $U(K_j)$ of the quiver (1.4.6) with $j < n$. Either there is an adjoint chiral multiplet and cubic superpotentials, or there is a quartic superpotential and no adjoint. Dualities apply in both cases.

When there is an adjoint chiral multiplet, multiplets charged under the group $U(K_j)$ form $\mathcal{N}=(2,2)^*$ $U(K_j)$ SQCD with $K_{j-1}+K_{j+1}$ flavours. The $\mathcal{N}=(2,2)^*$ dual theory has $K_j^D = K_{j-1} + K_{j+1} - K_j$ and $\hat{z}_j^D = \hat{z}_j^{-1}$, and a careful analysis shows that FI parameters of neighboring nodes are also altered: $\hat{z}_{j\pm1}^D = \hat{z}_j \hat{z}_{j\pm1}$. Twisted masses are given in the main text. Schematically,



(1.5.9)

When there is no adjoint chiral multiplet, multiplets charged under $U(K_j)$ form $\mathcal{N}=(2,2)$ SQCD with $K_{j-1}+K_{j+1}$ flavours and Seiberg duality applies. Ranks and FI parameters are mapped as in the first case. The only difference is that the dual theory has additional matter with charges equal to those of mesons of $U(K_j)$ in the original theory:



(1.5.10)

The following steps simplify the dual theory. The quartic superpotential of the original theory becomes a quadratic superpotential for the bifundamentals $M^D_{j-1,j+1}$ and $M^D_{j+1,j-1}$ of $U(K_{j-1}) \times U(K_{j+1})$ in the dual theory, and these chiral multiplets can thus be integrated out. Integrating them out combines their cubic superpotential couplings to bifundamentals of $U(K_j^D) \times U(K_{j\pm1})$, namely $\mathrm{Tr}(M^D_{j-1,j+1} q^D_{j+1,j} q^D_{j,j-1}) + \mathrm{Tr}(M^D_{j+1,j-1} q^D_{j-1,j} q^D_{j,j+1})$, into a quartic term for these bifundamentals. On the other hand, the new adjoint chiral multiplets of $U(K_{j\pm1})$ toggle between the presence and absence of adjoints for these nodes: indeed, if the original theory has no adjoint for $U(K_{j\pm1})$ then the dual theory has an adjoint, while if the original theory has an adjoint then the dual theory has two adjoints which can be integrated out thanks to a quadratic superpotential.

All in all, the dual theory has $K_j^D = K_{j-1} + K_{j+1} - K_j$ and $\hat{z}_j^D = \hat{z}_j^{-1}$ as well as $\hat{z}^D_{j\pm1} = \hat{z}_j \hat{z}_{j\pm1}$, and the presence or absence of adjoint chiral multiplets on the nodes $U(K_{j\pm1})$ is toggled if the node $U(K_j)$ has no adjoint.

This elaborate prescription translates beautifully to the exchange of two degenerate vertex operators $\widehat{V}_{-b\Omega_j}(x_j, \bar{x}_j)$ and $\widehat{V}_{-b\Omega_{j+1}}(x_{j+1}, \bar{x}_{j+1})$ in the Toda CFT. This is obtained as follows. The duality exchanges $K_j - K_{j-1}$ with $K_{j+1} - K_j$, and the exchange $x_j \leftrightarrow x_{j+1}$ reproduces the map of FI parameters. If Ω_j and Ω_{j+1} are both symmetric or both antisymmetric, exchanging them keeps the same distribution of symmetric and antisymmetric representations of A_{N-1}; accordingly, the duality does not change the matter content since the gauge theory has an adjoint. Otherwise, the exchange affects whether Ω_j and Ω_{j+1} are of the same type (symmetric/antisymmetric) as their neighbors Ω_{j-1} and Ω_{j+2}; accordingly the gauge theory has no adjoint and Seiberg duality affects the matter content of neighboring nodes.

The translation to Toda CFT immediately gives the set of dual descriptions which result from sequences of dualities on the groups $U(K_j)$ for $j < n$. These correspond to the $n!$ permutations of the n degenerate vertex operators. Dualities of the factor $U(K_n)$, explored in Section 4.4.2, are more complicated, as the fundamental and antifundamental chiral multiplets of $U(K_n)$ are subject to no superpotential coupling.

For every duality discussed in Chapter 4, S^2 partition functions of dual theories are also shown to be equal without any reference to the Toda CFT, by comparing Higgs branch decompositions. The difficult step of equating vortex partition functions is done in Appendix 4.A and Appendix 4.B. This direct proof paves the way towards tackling dualities which do not have Toda CFT translations.

Preliminary results (not included in the thesis) indicate that SQCD with two adjoints X and Y subject to the superpotential $W = \operatorname{Tr}(X^{l+1} + XY^2)$ exhibits a duality with $K^D = 3lN - K$. This is directly analoguous to the four-dimensional $\mathcal{N} = 1$ Kutasov–Schwimmer duality of type D.

Dualities of theories with orthogonal or symplectic gauge groups are also known, but the author has not tried to compare their S^2 partition functions.

The shape of the quiver can also be generalized to arbitrary graphs. Consider for definiteness $\mathcal{N} = (2,2)^*$ quivers, where each node has an adjoint chiral multiplet and cubic superpotential terms with neighboring bifundamentals. The group of dualities of such a quiver appears to be the Coxeter group whose Coxeter diagram has the same nodes as the quiver and one edge for each pair of bifundamental chiral multiplets. Thus, $\mathcal{N} = (2,2)^*$ quivers have infinitely many dual descriptions unless their shape is that of an ADE diagram. A-shaped quivers are studied above thanks to their matching with the Toda CFT, and it would be interesting to discover whether D and E-type quivers have similar realizations. Amusingly, one can realize all finite Coxeter groups ABCDEFG as duality groups by considering $\mathcal{N} = (2,2)^*$-like theories where cubic superpotentials $\tilde{q}Xq$ are replaced by $\tilde{q}X^lq$ (1.5.4) with different exponents l for every edge of the quiver. Note however that having a finite or infinite number of dual descriptions is probably only an anecdotical property of a gauge theory.

Another line of research is to consider surface operators in an interacting class S theory corresponding to a Riemann surface Σ. The $S^2 \subset S^4_b$ partition function has not yet been computed, as one needs to determine volumes of moduli spaces of combined vortices and instantons. The correspondence predicts that it will be equal to a Toda CFT correlator on Σ with the insertion of degenerate vertex operators. Permutations of the degenerate vertex operators translate to two-dimensional dualities as discussed above. Changing the trinion decomposition of the original Riemann surface Σ corresponds to S-duality of the four-dimensional theory: this may help determine how S-duality acts on surface operators. Finally, some modular transformations move degenerate punctures from one trinion of Σ to another. These translate to changing which hypermultiplet is coupled to the two-dimensional theory. The correspondence thus predicts a 4d/2d "node-hopping" duality: coupling the two-dimensional theory to different hypermultiplets describes the same surface operator.

The introduction is coming to an end. Hopefully it has elicited interest in the interplay between supersymmetric gauge theories and the Toda CFT uncovered by the AGT correspondence. In the present work, this relation yields new gauge theory dualities from explicit Toda CFT symmetries, and it provides new Toda CFT data which leads to a previously unknown braiding kernel. Fully developping consequences of the AGT correspondence will take many more years.

To decide upon their next destination, the reader is referred to the table of contents, or to the overview of the thesis on page 2. Briefly, Chapter 2 calculates S^2 partition functions, Chapter 3 matches them to Toda CFT correlators, Chapter 4 deduces gauge theory dualities from Toda symmetries, and Chapter 5 explores new Toda CFT results.

Chapter 2

Two-dimensional $\mathcal{N} = (2, 2)$ gauge theories

This is the article *Exact Results in $D = 2$ Supersymmetric Gauge Theories* by Nima Doroud, Jaume Gomis, Sungjay Lee, and the author [DGLFL12]. The text omits Section 2.6, Section 2.7, and Appendix 2.H because they are superseeded by newer results (Chapter 3 and Chapter 4), and necessary formatting changes are performed.

Abstract. *We compute exactly the partition function of two-dimensional $\mathcal{N} = (2, 2)$ gauge theories on S^2 and show that it admits two dual descriptions: either as an integral over the Coulomb branch or as a sum over vortex and anti-vortex excitations on the Higgs branches of the theory. We further demonstrate that correlation functions in two-dimensional Liouville/Toda CFT compute the S^2 partition function for a class of $\mathcal{N} = (2, 2)$ gauge theories, thereby uncovering novel modular properties in two-dimensional gauge theories. Some of these gauge theories flow in the infrared to Calabi-Yau sigma models – such as the conifold – and the topology changing flop transition is realized as crossing symmetry in Liouville/Toda CFT. Evidence for Seiberg duality in two dimensions is exhibited by demonstrating that the partition function of conjectured Seiberg dual pairs are the same.*

2.1 Introduction

It has long been recognized that many of the dynamical and quantum properties of four-dimensional gauge theories are mirrored in two-dimensional quantum field theories. This includes – among the wealth of phenomena that a four-dimensional gauge theory can exhibit – the remarkable and not yet completely understood physics of confinement and dynamical generation of a mass gap. Instantons, which mediate non-perturbative effects in four-dimensional gauge theories, are also present in two-dimensional field theories, and play a central role in determining the quantum properties of these theories. While the dynamics of two-dimensional gauge theories is tamer than in four dimensions, few exact results for correlation functions are available. In most examples, such computations heavily rely on integrability. Furthermore, given that two-dimensional theories share many of the beautiful phenomena present in four dimensions, it is a desirable goal to attain exact results in two-dimensional quantum field theories.

In this paper we obtain exact results in two-dimensional $\mathcal{N}=(2,2)$ supersymmetric gauge theories on S^2. These results are obtained using the powerful machinery of supersymmetric localization [Wit88; Wit91; Pes07]. We uncover that the partition function of these theories admit two seemingly different representations.[1] In one, the partition function is written as an integral (and discrete sum) over vector multiplet field configurations. This yields the Coulomb branch representation of the partition function

$$Z_{\text{Coulomb}}(m,\tau)=\sum_B\int_{\mathfrak{t}}\mathrm{d}a\,Z_{\text{cl}}(a,B,\tau)\,Z_{\text{one-loop}}(a,B,m)\,.$$

B is the quantized flux on S^2, a the Coulomb branch parameter, m denotes the masses of the matter fields and τ are the complexified gauge theory parameters

$$\tau=\frac{\vartheta}{2\pi}+\mathrm{i}\xi\,,$$

where ξ and ϑ are the FI (Fayet-Iliopoulos) parameter and topological angle associated to each $U(1)$ factor in the gauge group. Expressions for $Z_{\text{cl}}(a,B,\tau)$ and $Z_{\text{one-loop}}(a,B,m)$ are given in Section 2.4.

In the other representation, the path integral is given as a discrete sum over Higgs branches of the product of the vortex partition function [Sha06] at the north pole and the anti-vortex partition function at the south pole. This gives the Higgs branch representation of the partition function

$$\begin{aligned}Z_{\text{Higgs}}(m,\tau)=\sum_{v\in\text{Higgs vacua}}Z_{\text{cl}}(v,0,\tau)\,&\operatorname*{res}_{a=v}\left[Z_{\text{one-loop}}(a,0,m)\right]\\&Z_{\text{vortex}}(v,m,e^{2\pi\mathrm{i}\tau})Z_{\text{anti-vortex}}(v,m,e^{-2\pi\mathrm{i}\bar{\tau}})\,.\end{aligned}$$

In this formula the residue of the pole of $Z_{\text{one-loop}}(a,0,m)$ at the location of each Higgs branch must be taken.[2] Equivalently, this expression can be written in a holomorphically factorized form as a sum of the “norm” of the vortex partition function

$$Z_{\text{Higgs}}(m,\tau)=\sum_{v\in\text{Higgs vacua}}Z_{\text{cl}}(v,0,\tau)\,\operatorname*{res}_{a=v}\left[Z_{\text{one-loop}}(a,0,m)\right]\left|Z_{\text{vortex}}(v,m,e^{2\pi\mathrm{i}\tau})\right|^2\,.$$

[1] This can be enriched with the insertion of supersymmetric Wilson loop operators.

[2] A Higgs branch is a solution to the equation $a+m=0$.

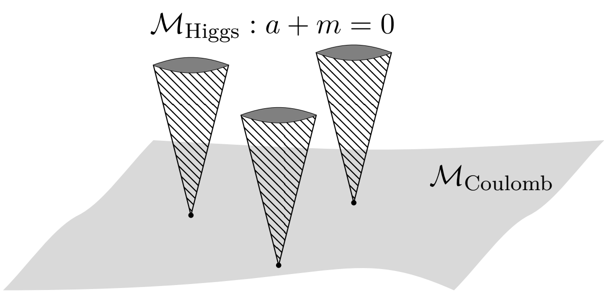


Figure 2.1: Higgs vacua. Vortices and anti-vortices on these vacua contribute to $Z_{\text{Higgs}}(m, \tau)$

Despite that the expressions for the Coulomb and Higgs branch representations are rather distinct and involve different degrees of freedom, we show that the two yield identical, dual representations of the partition function of $\mathcal{N} = (2, 2)$ gauge theories on S^2

$$Z = Z_{\text{Coulomb}} = Z_{\text{Higgs}} \, .$$

We have explicitly shown this equivalence for SQCD, with $U(N)$ gauge group and N_f fundamental and $\widetilde{N_f}$ anti-fundamental chiral multiplets. The factorization of the Coulomb branch integral is akin to the one found by Pasquetti [Pas11] and Krattenthaler et al. [KSV11] in evaluating the partition function of three-dimensional $\mathcal{N} = 2$ abelian gauge theories on the squashed S^3 [HHL11] and $S^1 \times S^2$.[3]

The fact that a correlation function in a supersymmetric gauge theory may admit multiple representations can be understood to be a consequence of the different choices of supercharge and/or deformation terms available when performing supersymmetric localization. Different choices may lead to integration over different supersymmetric configurations, but the localization argument guarantees that all (reasonable) choices must ultimately yield the same correlation function.[4] See Section 2.8 for a more detailed discussion. Our choice of localization supercharge has the elegant feature of giving rise to supersymmetry equations which interpolate between vortex equations at the north pole and anti-vortex equations at the south pole while also allowing for configurations on the Coulomb branch.

We demonstrate that the partition function of certain two-dimensional $\mathcal{N} = (2, 2)$ gauge theories on S^2 admits a dual description in terms of correlation functions in two-dimensional Liouville/Toda CFT. This is akin to the AGT correspondence [AGT09] between the partition function of

[3]Other related works on localization include [Kim09; KWY09; Jaf10; HHL10; IY11].

[4]In particular we obtain the Coulomb branch representation of the partition function using two different choices of supercharge.

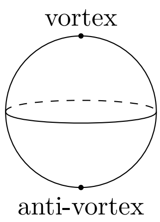


Figure 2.2: Vortex and anti-vortex configurations in the Higgs branch

four-dimensional $\mathcal{N} = 2$ gauge theories on S^4 and correlators in these two-dimensional CFTs. The key difference is that the correlators in Liouville/Toda CFT that capture the two-dimensional gauge theory partition function on S^2 involve the insertion of degenerate vertex operators of the Virasoro or W-algebra at suitable punctures on the Riemann surface. These insertions have the sought after property of restricting the sum over intermediate states to a discrete sum of conformal blocks, which precisely capture the sum over Higgs vacua in the Higgs branch representation of the partition function. Pleasingly, Z_{Higgs} exactly reproduces the sum over conformal blocks with the precise modular invariant Liouville/Toda measure by summing over vortices and anti-vortices over all Higgs vacua.

The simplest instance of this correspondence is SQED, described by a $U(1)$ vector multiplet and N_f electron and N_f positron chiral multiplets. The partition function of SQED corresponds to the A_{N_f-1} Toda CFT on the four-punctured sphere with the insertion of two non-degenerate, a semi-degenerate and a fully degenerate puncture:

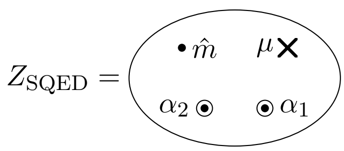


Figure 2.3: SQED partition function as Toda CFT correlator

The fact that two-dimensional $\mathcal{N} = (2,2)$ gauge theories on S^2 admit a Liouville/Toda CFT description with degenerate fields is consistent with the observation that certain half-BPS surface operators in four-dimensional $\mathcal{N} = 2$ gauge theories on S^4 are realized by the insertion of a degenerate field [Ald+09].

The correspondence we establish with Liouville/Toda CFT implies that two-dimensional $\mathcal{N} = (2,2)$ gauge theories enjoy rather interesting modular properties with respect to the complexified gauge theory parameters τ. This is a direct consequence of modular invariance, which implies that CFT correlators are independent of the choice of factorization channel (or pants decomposition) used to represent a correlator as a sum over intermediate states. The moduli of the punctured Riemann surface on which modular duality acts correspond to the vortex fugacity parameters

$$z = e^{2\pi i\tau}\,.$$

It is rather interesting that the partition function of two-dimensional $\mathcal{N} = (2,2)$ gauge theories on S^2 assembles into a modular invariant object.

Another important motivation to study two-dimensional $\mathcal{N} = (2,2)$ gauge theories is string theory. As shown in [Wit93], the Higgs branch of such a gauge theory flows in the infrared to a two-dimensional $\mathcal{N} = (2,2)$ supersymmetric non-linear sigma model with a Kähler target space. Moreover, with a suitable choice of matter content and gauge group, the gauge theory flows to an $\mathcal{N} = (2,2)$ superconformal field theory, which provides the worldsheet description of string theory on a Calabi-Yau manifold. One can hope that the exact formulae for the partition function of these gauge theories will provide a novel way to compute worldsheet instantons in the corresponding Calabi-Yau manifolds, as well as shed new light into the dynamics of these phenomenologically appealing string theory backgrounds.

The ultraviolet gauge theory description of these string theory backgrounds provides a qualitative characterization of the "phase" structure as the Kähler moduli of the Calabi-Yau manifold are changed by studying the gauge dynamics as a function of the complexified gauge theory parameters τ [Wit93]. An interesting topology changing transition – the so called flop transition – occurs in some models as the sign of the FI parameter is reversed $\xi \to -\xi$. The string dynamics in the two phases connected by a flop transition are expected to be related by analytic continuation in τ. Our exact results for the partition function of $\mathcal{N} = (2,2)$ SQED – which includes the conifold for $N_f = 2$ and higher dimensional Calabi-Yau manifolds for $N_f > 2$ – demonstrate that the results for $\xi > 0$ and $\xi < 0$ are indeed related by analytic continuation. Given the representation of the partition function of SQED in terms of a Toda CFT correlator on the four-punctured sphere, the analytic continuation describing the flop transition admits an elegant realization as crossing symmetry in Toda CFT

$$\text{flop transition} \longleftrightarrow \text{crossing symmetry}\,.$$

Furthermore, our exact results demonstrate that the geometric singularity as we move from $\xi > 0$ to $\xi < 0$ across the singular point $\xi = 0$ can be avoided by turning on a nonzero topological angle ϑ, as anticipated in [Wit93; AGM93].

Our findings are used to provide quantitative evidence for Seiberg duality [Sei94] in two dimensions by comparing the partition functions of putative dual theories in various limits and finding exact agreement. Seiberg duality in two-dimensional $\mathcal{N} = (2,2)$ gauge theories [HT06] relates theories with

$N_f > N$ fundamental chiral multiplets, trivial superpotential and gauge groups

$$SU(N) \longleftrightarrow SU(N_f - N)\,.$$

The conjectured duality was put forward in [HT06] to give a physical realization of Rødland's conjecture stating that two Calabi-Yau manifolds appear as distinct large volume limits of the same Kähler moduli space. Our results, therefore, provide further evidence for this conjecture.

The plan of the rest of the paper is as follows. In Section 2.2 we explicitly write down for gauge theories on S^2 the $\mathcal{N} = (2,2)$ supersymmetry transformations of the vector and chiral multiplet fields and the associated supersymmetric action. In Section 2.3 we specify a particular supercharge with which we perform the localization computation. We derive the partial differential equations that determine the space of supersymmetric field configurations corresponding to our choice of supercharge and show that the system of equations we get smoothly interpolates between the vortex equations at the north pole and the anti-vortex equations at the south pole. A vanishing theorem finding the most general smooth, supersymmetric solutions to our system of supersymmetry equations is proven. We find that smooth solutions are parametrized by vector multiplet fields and correspond to Coulomb phase configurations, while singular localized vortices and anti-vortices, which exist in the Higgs phase, may appear at the north and south poles of the S^2. In Section 2.4 we localize the path integral by choosing a specific deformation term and show that only Coulomb branch configurations can contribute if we consider the saddle point equations of the combined action in the limit that the coefficient of the deformation term goes to infinity. This yields the Coulomb branch representation of the partition function. Quite remarkably, the integral and sum over the Coulomb branch configurations can be carried out for arbitrary choices of gauge group G and matter representation. The resulting expression can be written as a finite sum of the product of a function with its complex conjugate. We identify this expression as the sum over Higgs vacua of the product of the vortex partition function at the north pole with the anti-vortex partition function at the south pole. In Section 2.5 we argue, by first looking at the saddle point equations for a different deformation term, that the Coulomb branch configurations are lifted and that vortex and anti-vortex configurations at the poles are the true saddle points of the path integral in this other limit. This yields the Higgs branch representation of the partition function. This way of computing the path integral gives a first principles derivation of the result obtained by brute force evaluation of the Coulomb branch representation of the partition function. The identification of the partition function of certain two-dimensional $\mathcal{N} = (2,2)$ gauge theories with Liouville/Toda correlation functions is uncovered in Section 2.6, and some of their consequences explored. In Section 2.7 we provide quantitative evidence for Seiberg duality in two dimensions by matching the partition function of Seiberg dual pairs in various limits. We conclude in Section 2.8 with a discussion of our findings and future directions. The appendices contain some detailed computations used in the bulk of the paper.

Note added: While this work was being completed, we became aware of related work [BC12], which has some overlap with this paper.

2.2 $\mathcal{N}=(2,2)$ gauge theories on S^2

In this section we explicitly construct the Lagrangian of $\mathcal{N}=(2,2)$ supersymmetric gauge theories on S^2. The basic multiplets of two-dimensional $\mathcal{N}=(2,2)$ supersymmetry are the vector multiplet and the chiral multiplet, which arise by dimensional reduction to two dimensions of the familiar four-dimensional $\mathcal{N}=1$ supersymmetry multiplets. The field content is therefore

$$\begin{aligned} \text{vector multiplet: } & (A_i,\sigma_1,\sigma_2,\lambda,\bar\lambda,\mathrm{D}) \\ \text{chiral multiplet: } & (\phi,\bar\phi,\psi,\bar\psi,F,\bar F)\,. \end{aligned} \tag{2.2.1}$$

The fields $(\lambda,\bar\lambda,\psi,\bar\psi)$ are two component complex Dirac spinors,[5] $(\phi,\bar\phi,F,\bar F)$ are complex scalar fields while $(\sigma_1,\sigma_2,\mathrm{D})$ are real scalar fields.[6] The fields in the vector multiplet transform in the adjoint representation of the gauge group G while the chiral multiplet fields transform in a representation $\mathbf{R}$ of G. The field content of an arbitrary $\mathcal{N}=(2,2)$ supersymmetric gauge theory admitting a Lagrangian description is captured by these multiplets by letting G be a product gauge group and $\mathbf{R}$ a reducible representation.

While it is well known how to construct the Lagrangian of $\mathcal{N}=(2,2)$ supersymmetric gauge theories in $\mathbb{R}^2$ (*i.e.* flat space), constructing supersymmetric theories on S^2 requires some thought, as S^2 does not admit covariantly constant spinors. Indeed, we must first characterize the $\mathcal{N}=(2,2)$ supersymmetry algebra on S^2. This is the subalgebra of the two-dimensional $\mathcal{N}=(2,2)$ superconformal algebra on S^2 that generates the isometries of S^2, but none of the conformal transformations of S^2. The $\mathcal{N}=(2,2)$ supersymmetry algebra on S^2 thus defined obeys the (anti)commutation relations of the $SU(2|1)$ superalgebra[7]

$$\begin{aligned} &[J_m,J_n]=\mathrm{i}\epsilon_{mnp}J_p \qquad [J_m,Q_\alpha]=-\frac{1}{2}\gamma_m{}^\beta{}_\alpha Q_\beta \qquad [J_m,S_\alpha]=-\frac{1}{2}\gamma_m{}^\beta{}_\alpha S_\beta \\ &\{S_\alpha,Q_\beta\}=\gamma^m_{\alpha\beta}J_m-\frac{1}{2}C_{\alpha\beta}R \qquad [R,Q_\alpha]=-Q_\alpha \qquad [R,S_\alpha]=S_\alpha\,. \end{aligned} \tag{2.2.2}$$

The supercharges Q_α and S_α are two-dimensional Dirac spinors generating the supersymmetry transformations, J_m are the $SU(2)$ charges generating the

[5] Our conventions for spinors are listed in Appendix 2.A.
[6] The reality of the auxiliary field D is altered when coupled with matter fields.
[7] See Appendix 2.B for details.

isometries of S^2 while R is a $U(1)$ R-symmetry charge. This supersymmetry algebra is the S^2 counterpart of the $\mathcal{N}=(2,2)$ super-Poincaré algebra in flat space.

Constructing a supersymmetric Lagrangian on S^2 requires finding supersymmetry transformations on the vector and chiral multiplet fields that represent the $SU(2|1)$ algebra. We construct these by restricting the $\mathcal{N}=(2,2)$ superconformal transformations to those corresponding to the $SU(2|1)$ subalgebra. The $\mathcal{N}=(2,2)$ superconformal transformations on the fields are easily obtained by combining the $\mathcal{N}=(2,2)$ super-Poincaré transformations in flat space (with the flat metric replaced by an arbitrary metric), with additional terms that are uniquely fixed by demanding that the supersymmetry transformations are covariant under Weyl transformations.[7] Given the $SU(2|1)$ supersymmetry transformations on the vector and chiral multiplet fields constructed this way and shown below, it is straightforward to construct the corresponding $SU(2|1)$ invariant Lagrangian. The supersymmetry transformations and action may equivalently be obtained by "twisted" dimensional reduction from three-dimensional $\mathcal{N}=2$ gauge theories on $S^1\times S^2$, considered in [IY11].

2.2.1 $\mathcal{N}=(2,2)$ action

Without further ado, we write down the most general renormalizable $\mathcal{N}=(2,2)$ supersymmetric action of an arbitrary gauge theory on S^2

$$S = S_{\text{v.m.}} + S_{\text{top}} + S_{\text{FI}} + S_{\text{c.m.}} + S_{\text{mass}} + S_W\,. \tag{2.2.3}$$

The vector multiplet action is given by

$$\begin{aligned} S_{\text{v.m.}} = \frac{1}{2g^2}\int \mathrm{d}^2x\,\sqrt{h}\,\mathrm{Tr}\Big\{&V_iV^i + V_3V^3 + \mathrm{D}^2 \\ &+ \mathrm{i}\lambda\Big(\not{D}\bar{\lambda} - \big[\sigma_1,\bar{\lambda}\big] - \mathrm{i}\big[\sigma_2,\gamma^{\hat{3}}\bar{\lambda}\big]\Big)\Big\}\,, \end{aligned} \tag{2.2.4}$$

where

$$\begin{aligned} V^i &= \varepsilon^{ij}D_j\sigma_2 + D^i\sigma_1\,,\\ V^3 &= \frac{1}{2}\varepsilon^{ij}F_{ij} + \mathrm{i}\,[\sigma_1,\sigma_2] + \frac{1}{r}\sigma_1\,. \end{aligned} \tag{2.2.5}$$

The bosonic part of the action can also be written as

$$\frac{1}{2g^2}\int \mathrm{d}^2x\,\sqrt{h}\;\mathrm{Tr}\left\{\left(F_{\hat{1}\hat{2}} + \frac{1}{r}\sigma_1\right)^2 + (D_i\sigma_1)^2 + (D_i\sigma_2)^2 - [\sigma_1,\sigma_2]^2 + \mathrm{D}^2\right\}\,. \tag{2.2.6}$$

In the vector multiplet action g denotes the super-renormalizable gauge coupling,[8] h is the round metric on S^2 and r is its radius.

[8]For a product gauge group, there is an independent gauge coupling for each factor in the gauge group.

For each $U(1)$ factor in G, the gauge field action in two dimensions can be enriched by the addition of the topological term

$$S_{\text{top}} = -\mathrm{i}\frac{\vartheta}{2\pi}\int \operatorname{Tr} F\,, \tag{2.2.7}$$

and of a supersymmetric FI (Fayet-Iliopoulos) D-term on S^2

$$S_{\text{FI}} = -\mathrm{i}\xi \int \mathrm{d}^2x\,\sqrt{h}\;\operatorname{Tr}\left(\mathrm{D}-\frac{\sigma_2}{r}\right)\,. \tag{2.2.8}$$

The couplings ϑ and ξ are classically marginal, and can be combined into a complex gauge coupling

$$\tau = \frac{\vartheta}{2\pi} + \mathrm{i}\xi \tag{2.2.9}$$

for each $U(1)$ factor in the gauge group. Quantum mechanically, the coupling τ depends on the energy scale, and can be traded with the dynamically generated, renormalization group invariant scale Λ.[9] We will return to this dynamical transmutation in Section 2.4.

The action for the chiral multiplet coupled to the vector multiplet is[10]

$$\begin{aligned} S_{\text{c.m.}} = \int \mathrm{d}^2x\,\sqrt{h}\Big\{&\bar{\phi}\Big(-D_i^2+\sigma_1^2+\sigma_2^2+\mathrm{i}\mathrm{D}+\mathrm{i}\frac{q-1}{r}\sigma_2-\frac{q^2-2q}{4r^2}\Big)\phi+\bar{F}F \\ &-\mathrm{i}\bar{\psi}\left(\not{D}-\sigma_1-\mathrm{i}\sigma_2\gamma^{\hat{3}}+\frac{q}{2r}\gamma^{\hat{3}}\right)\psi+\mathrm{i}\bar{\psi}\lambda\phi-\mathrm{i}\bar{\phi}\bar{\lambda}\psi\Big\}\,. \end{aligned} \tag{2.2.10}$$

Here q denotes the $U(1)$ R-charge of the chiral multiplet, which takes the value $q=0$ for the canonical chiral multiplet.[11] In a theory with flavour symmetry G_F, the $U(1)$ R-charges take values in the Cartan subalgebra of G_F (see discussion below).

In two dimensions, it is possible to turn on in a supersymmetric way twisted masses for the chiral multiplet. These supersymmetric mass terms are obtained by first weakly gauging the flavour symmetry group G_F acting on the theory, coupling the matter fields to a vector multiplet for G_F, and then turning on a supersymmetric background expectation value for the fields in that vector multiplet. For $\mathcal{N}=(2,2)$ gauge theories on S^2, unbroken $SU(2|1)$ supersymmetry (see equations (2.2.17) and (2.2.18)) implies that the mass parameters are given by a constant background expectation value

[9] The dynamical scale is given by $\Lambda^{b_0}=\mu^{b_0}e^{2\pi\mathrm{i}\tau(\mu)}$, where $\beta(\xi)\equiv\frac{b_0}{2\pi}$ and μ is the floating scale.

[10] The representation matrices of G in the representation $\mathbf{R}$, which we do not write explicitly to avoid clutter, intertwine the vector multiplet and chiral multiplet fields in the usual way.

[11] q also determines the Weyl weight of the fields in the chiral multiplet. The Weyl weight of a field can be read from the commutator of two superconformal transformations (see Appendix 2.B), which represents the two-dimensional $\mathcal{N}=(2,2)$ superconformal algebra on the fields.

for the scalar field σ_2 in the vector multiplet for G_F. This can be taken in the Cartan subalgebra of the flavour symmetry group G_F. Therefore, the supersymmetric twisted mass terms on S^2 are obtained by substituting

$$\sigma_2 \to \sigma_2 + m \tag{2.2.11}$$

in (2.2.10), with m in the Cartan subalgebra of G_F

$$S_{\text{mass}} = \int \mathrm{d}^2x\,\sqrt{h}\,\left\{\bar{\phi}\left(m^2 + 2m\sigma_2 + \mathrm{i}\frac{q-1}{r}m\right)\phi - \bar{\psi}m\gamma^{\hat{3}}\psi\right\}. \tag{2.2.12}$$

Likewise, the $U(1)$ R-charge parameters q introduced in (2.2.10) can be obtained by turning on an imaginary expectation value for the scalar field σ_2 in the vector multiplet for G_F. The corresponding supersymmetric terms in the action are obtained by shifting the action in (2.2.10) for $q=0$ by

$$\sigma_2 \to \sigma_2 + \frac{\mathrm{i}}{2r}q\,. \tag{2.2.13}$$

The flavour symmetry G_F is determined by the representation $\mathbf{R}$ under which the chiral multiplet transforms and by the choice of superpotential, as this can break the group of transformations rotating the chiral multiplets down to the actual G_F symmetry of the theory. If $\mathbf{R}$ contains N_f copies of an irreducible representation $\mathbf{r}$ and the theory has a trivial superpotential, then the theory has $U(N_f)$ as part of its flavour symmetry group and gives rise to N_f twisted mass parameters $m=(m_1,\ldots,m_{N_f})$ and N_f $U(1)$ R-charges $q=(q_1,\ldots,q_{N_f})$. Occasionally, we will find it convenient to combine these parameters into the holomorphic combination

$$\mathrm{M}_I = m_I + \frac{\mathrm{i}}{2r}q_I\,. \tag{2.2.14}$$

Finally, we can add in a supersymmetric way a superpotential for the chiral multiplet

$$S_W = \int \mathrm{d}^2x\,\sqrt{h}\,\left\{F_W + \bar{F}_{\bar{W}}\right\}, \tag{2.2.15}$$

whenever the total $U(1)$ R-charge of the superpotential is $-q_W=-2$. F_W is the gauge invariant auxiliary component of the superpotential chiral multiplet.[12] Under these conditions, the Lagrangian in (2.2.15) transforms into a total derivative under the $SU(2|1)$ supersymmetry transformations below.

A few brief remarks about the $\mathcal{N}=(2,2)$ gauge theories in S^2 thus constructed are in order. The action (and supersymmetry transformations) can be organized in a power series expansion in $1/r$, starting with the

[12] In terms of the ϕ chiral multiplet, $F_W = \frac{\partial W}{\partial\phi}F - \frac{1}{2}\frac{\partial^2 W}{\partial\phi^2}\psi\psi$. Invariance of (2.2.15) under supersymmetry when $q_W=2$ follows from equations (2.2.28) and (2.2.29).

covariantized $\mathcal{N} = (2,2)$ gauge theory action in flat space. The action is deformed by terms of order $1/r$ and $1/r^2$, with terms proportional to $1/r$ not being reflection positive. These features are consistent with the general arguments in [FS11]. The theory on S^2 breaks the classical[13] $U(1)_A$ R-symmetry of the corresponding $\mathcal{N} = (2,2)$ gauge theory in flat space. This can be observed in the asymmetry between the scalar fields σ_1 and σ_2 in the action on S^2, which are otherwise rotated into each other by the $U(1)_A$ symmetry of the flat space theory. This asymmetry is also manifested in the twisted masses m being real on S^2, while they are complex in flat space.[14] The real twisted masses m on S^2, however, combine with the $U(1)$ R-charges q into the holomorphic parameters $\mathrm{M} = m + \frac{\mathrm{i}}{2r}q$ introduced in (2.2.14).

2.2.2 Supersymmetry transformations

The gauge theory action we have written down is invariant under the $SU(2|1)$ supersymmetry algebra. The supersymmetry transformations are parametrized by conformal Killing spinors[15] ϵ and $\bar{\epsilon}$ on S^2. These can be taken to obey

$$\begin{aligned} \nabla_i \epsilon &= +\frac{1}{2r}\gamma_i \gamma^{\hat{3}} \epsilon \\ \nabla_i \bar{\epsilon} &= -\frac{1}{2r}\gamma_i \gamma^{\hat{3}} \bar{\epsilon} \,, \end{aligned} \tag{2.2.16}$$

where ϵ and $\bar{\epsilon}$ are complex Dirac spinors in two dimensions and r is the radius of the S^2. The spinors ϵ_α and $\bar{\epsilon}_\alpha$ are the supersymmetry parameters associated to the supercharges Q_α and S_α respectively. More details about the supersymmetry transformations can be found in Appendix 2.B.

As mentioned earlier, the explicit supersymmetry transformations can be found by restricting the $\mathcal{N} = (2,2)$ superconformal transformations to the $SU(2|1)$ subalgebra. The $SU(2|1)$ supersymmetry transformations of the vector multiplet fields are

$$\delta\lambda = \left(\mathrm{i}V_m \gamma^m - \mathrm{D}\right)\epsilon \tag{2.2.17}$$

$$\delta\bar{\lambda} = \left(\mathrm{i}\bar{V}_m \gamma^m + \mathrm{D}\right)\bar{\epsilon} \tag{2.2.18}$$

$$\delta A_i = -\frac{\mathrm{i}}{2}\left(\bar{\epsilon}\gamma_i\lambda + \epsilon\gamma_i\bar{\lambda}\right) \tag{2.2.19}$$

$$\delta\sigma_1 = \frac{1}{2}\left(\bar{\epsilon}\lambda - \epsilon\bar{\lambda}\right) \tag{2.2.20}$$

$$\delta\sigma_2 = -\frac{\mathrm{i}}{2}\left(\bar{\epsilon}\gamma_{\hat{3}}\lambda + \epsilon\gamma_{\hat{3}}\bar{\lambda}\right) \tag{2.2.21}$$

[13] This classical symmetry of the flat space theory, being chiral, can be anomalous.

[14] Where twisted masses correspond to background values of σ_1, σ_2 in the vector multiplet for G_F.

[15] Thus named since the defining equation $\nabla_i \epsilon = \gamma_i \tilde{\epsilon}$ is conformally invariant.

$$\begin{aligned} \delta \mathrm{D} = &-\frac{\mathrm{i}}{2}\bar{\epsilon}\left(\not{D}\lambda + [\sigma_1,\lambda] - \mathrm{i}\left[\sigma_2,\gamma^{\hat{3}}\lambda\right]\right) \\ &+\frac{\mathrm{i}}{2}\epsilon\left(\not{D}\bar{\lambda} - \left[\sigma_1,\bar{\lambda}\right] - \mathrm{i}\left[\sigma_2,\gamma^{\hat{3}}\bar{\lambda}\right]\right)\,, \end{aligned} \tag{2.2.22}$$

with V_m and $\bar{V}_m$ defined by

$$\begin{aligned} V^i &= \varepsilon^{ij}D_j\sigma_2 + D^i\sigma_1\,, \quad V^3 = \frac{1}{2}\varepsilon^{ij}F_{ij} + \mathrm{i}\,[\sigma_1,\sigma_2] + \frac{1}{r}\sigma_1 \\ \bar{V}^i &= \varepsilon^{ij}D_j\sigma_2 - D^i\sigma_1\,, \quad \bar{V}^3 = \frac{1}{2}\varepsilon^{ij}F_{ij} - \mathrm{i}\,[\sigma_1,\sigma_2] + \frac{1}{r}\sigma_1\,. \end{aligned} \tag{2.2.23}$$

The transformations of the massless chiral multiplet fields are

$$\delta\phi = \bar{\epsilon}\psi \tag{2.2.24}$$

$$\delta\bar{\phi} = \epsilon\bar{\psi} \tag{2.2.25}$$

$$\delta\psi = \mathrm{i}\left(\not{D}\phi + \sigma_1\phi - \mathrm{i}\sigma_2\phi\gamma^{\hat{3}} + \frac{q}{2r}\phi\gamma^{\hat{3}}\right)\epsilon + \bar{\epsilon}F \tag{2.2.26}$$

$$\delta\bar{\psi} = \mathrm{i}\left(\not{D}\bar{\phi} + \bar{\phi}\sigma_1 + \mathrm{i}\bar{\phi}\sigma_2\gamma^{\hat{3}} - \frac{q}{2r}\bar{\phi}\gamma^{\hat{3}}\right)\bar{\epsilon} + \epsilon\bar{F} \tag{2.2.27}$$

$$\delta F = -\mathrm{i}\left(D_i\psi\gamma^i + \sigma_1\psi - \mathrm{i}\sigma_2\psi\gamma^{\hat{3}} + \lambda\phi + \frac{q}{2r}\psi\gamma^{\hat{3}}\right)\epsilon \tag{2.2.28}$$

$$\delta\bar{F} = -\mathrm{i}\left(D_i\bar{\psi}\gamma^i + \bar{\psi}\sigma_1 + \mathrm{i}\bar{\psi}\sigma_2\gamma^{\hat{3}} - \bar{\phi}\bar{\lambda} - \frac{q}{2r}\bar{\psi}\gamma^{\hat{3}}\right)\bar{\epsilon}\,. \tag{2.2.29}$$

The supersymmetry transformations of the theory with twisted masses are obtained from equations (2.2.24–2.2.29) by shifting $\sigma_2 \to \sigma_2 + m$ as in (2.2.11).

With these transformations, the $SU(2|1)$ supersymmetry algebra (2.2.2) is realized off-shell on the vector multiplet and chiral multiplets fields. Splitting $\delta \equiv \delta_\epsilon + \delta_{\bar{\epsilon}}$, we find that this representation of $SU(2|1)$ on the fields obeys[16]

$$[\delta_\epsilon,\delta_\epsilon] = 0 \qquad\qquad [\delta_{\bar{\epsilon}},\delta_{\bar{\epsilon}}] = 0\,, \tag{2.2.30}$$

$$[\delta_\epsilon,\delta_{\bar{\epsilon}}] = \delta_{SU(2)}(\xi) + \delta_R(\alpha) + \delta_G(\Lambda) + \delta_{G_F}(\Lambda_m)\,, \tag{2.2.31}$$

thus generating an infinitesimal $SU(2)\times R\times G\times G_F$ transformation. When localizing the path integral of $\mathcal{N}=(2,2)$ gauge theories on S^2, we will choose a particular supercharge $\mathcal{Q}$ in $SU(2|1)$. The $SU(2)\times R\times G\times G_F$ transformation it generates will play an important role in our computation of the partition function.

The $SU(2)$ isometry transformation induced by the commutator of supersymmetry transformations is parametrized by the Killing vector field[17]

$$\xi^i = -\mathrm{i}\bar{\epsilon}\gamma^i\epsilon\,. \tag{2.2.32}$$

[16] The explicit form of the commutator of supersymmetry transformations on the vector multiplet and chiral multiplet fields can be found in Appendix 2.B.

[17] The fact that ξ is a Killing vector, that it obeys $\nabla^i\xi^j + \nabla^j\xi^i = 0$, is a consequence of the choice of conformal Killing spinors in (2.2.16). As desired, it does not generate conformal transformations of S^2.

It acts on the bosonic fields via the usual Lie derivative and on the fermions via the Lie-Lorentz derivative

$$\mathcal{L}_\xi \equiv \xi^i \nabla_i + \frac{1}{4} \nabla_i \xi_j \gamma^{ij} . \tag{2.2.33}$$

The $U(1)$ R-symmetry transformation generated by the commutator of the supersymmetry transformations is parametrized by

$$\alpha = -\frac{1}{2r} \bar{\epsilon} \gamma^{\hat{3}} \epsilon . \tag{2.2.34}$$

It acts on the fields by multiplication by the corresponding charge. The $U(1)$ R-symmetry charges of the various fields, supercharges and parameters are:

supersymmetry				vector multiplet					
ϵ	$\bar{\epsilon}$	Q	S	A_μ	σ_1	σ_2	λ	$\bar{\lambda}$	D
1	-1	-1	1	0	0	0	1	-1	0

chiral multiplet					
ϕ	ψ	F	$\bar{\phi}$	$\bar{\psi}$	$\bar{F}$
$-q$	$-(q-1)$	$-(q-2)$	q	$q-1$	$q-2$

Since the action of R on the fields is non-chiral, this classical symmetry is not spoiled by quantum anomalies and is an exact symmetry of the $\mathcal{N} = (2,2)$ gauge theories we have constructed.

The commutator of two supersymmetry transformations generates a field dependent gauge transformation, taking values in the Lie algebra of the gauge group G. The induced gauge transformation is labeled by the gauge parameter

$$\Lambda = (\bar{\epsilon}\epsilon)\sigma_1 - \mathrm{i}(\bar{\epsilon}\gamma^{\hat{3}}\epsilon)\sigma_2 + \xi^i A_i , \tag{2.2.35}$$

which acts on the various fields by the standard gauge redundancy transformation laws. On the gauge field it acts by

$$\delta_\Lambda A_i = D_i \Lambda \tag{2.2.36}$$

while on a field φ it acts by

$$\delta_\Lambda \varphi = \mathrm{i}\Lambda \cdot \varphi , \tag{2.2.37}$$

where Λ acts on φ in the corresponding representation of G.

Finally, in the presence of twisted masses m, a G_F flavour symmetry rotation on the chiral multiplet fields is generated by $[\delta_\epsilon, \delta_{\bar{\epsilon}}]$. The induced flavour symmetry transformation acts on the chiral multiplet fields in the fundamental representation of G_F, and is parametrized by

$$\Lambda_m = -\mathrm{i}(\bar{\epsilon}\gamma^{\hat{3}}\epsilon) m , \tag{2.2.38}$$

with m taking values in the Cartan subalgebra of G_F. It acts trivially on the vector multiplet fields.

2.3 Localization of the path integral

In this paper our goal is to perform the exact computation of the partition function of $\mathcal{N} = (2,2)$ gauge theories on S^2. The powerful tool that allow us to achieve this goal is supersymmetric localization.

The central idea of supersymmetric localization [Wit92] is that the path integral – possibly decorated with the insertion of observables or boundary conditions invariant under a supercharge $\mathcal{Q}$ – localizes to the $\mathcal{Q}$-invariant field configurations. If the orbit of $\mathcal{Q}$ in the space of fields is non-trivial,[18] then the path integral vanishes upon integrating over the associated Grassman collective coordinate. Therefore, the non-vanishing contributions to the path integral can only arise from the trivial orbits, *i.e.* the fixed points of supersymmetry. These fixed point field configurations are the solutions to the supersymmetry variation equations generated by the supercharge $\mathcal{Q}$, which we denote by

$$\delta_{\mathcal{Q}}\, \text{fermions} = 0\,. \tag{2.3.1}$$

In the path integral we must integrate over the moduli space of solutions of the partial differential equations implied by supersymmetry fixed point equations (2.3.1).

Under favorable asymptotic behavior, integration by parts implies that the result of the path integral does not depend on the deformation of the original supersymmetric Lagrangian by a $\mathcal{Q}$-exact term[19]

$$\mathcal{L} \to \mathcal{L} + t\, \mathcal{Q} \cdot V\,, \tag{2.3.2}$$

as long as V is invariant under the bosonic transformations generated by $\mathcal{Q}^2$. Obtaining a sensible path integral requires that the action is nondegenerate and that the path integral is convergent in the presence of the deformation term $\mathcal{Q} \cdot V$.

In the $t \to \infty$ limit, the semiclassical approximation with respect to $\hbar_{\text{eff}} \equiv 1/t$ is exact. In this limit, only the saddle points of $\mathcal{Q} \cdot V$ can contribute and, moreover, the path integral is dominated by the saddle points with vanishing action. However, of all the saddle points of $\mathcal{Q} \cdot V$, only the $\mathcal{Q}$-supersymmetric field configurations give a non-zero contribution. Therefore, we must integrate over the intersection of supersymmetric field configurations and saddle points of $\mathcal{Q} \cdot V$. We denote this intersection by $\mathcal{F}$.

Using the saddle point approximation, the path integral in the $t \to \infty$ limit can be calculated by restricting the original Lagrangian $\mathcal{L}$ to $\mathcal{F}$,[20] integrating out the quadratic fluctuations of all the fields in the deformation

[18] By definition of $\mathcal{Q}$-invariance of the path integral, the space of fields admits the action of $\mathcal{Q}$.

[19] $\mathcal{Q} \cdot V$ denotes the supersymmetry transformation of V generated by $\mathcal{Q}$ (see also (2.4.1)).

[20] The deformation term $\mathcal{Q} \cdot V$ vanishes on $\mathcal{F}$ since it is a linear combination of the supersymmetry equations.

$\mathcal{Q}\cdot V$ expanded around a point in $\mathcal{F}$, and integrating the combined expression over $\mathcal{F}$.[21] Of course, even though the path integral is one-loop exact with respect to t, it yields exact results with respect to the original coupling constants and parameters of the theory.

The final result of the localization computation does not depend on the choice of deformation $\mathcal{Q}\cdot V$. One may add to $\mathcal{Q}\cdot V$ another $\mathcal{Q}$-exact term, and the result of the path integral will not change as long as the new $\mathcal{Q}$-exact term is non-degenerate, and no new supersymmetric saddle points are introduced that can flow from infinity. This can be accomplished by choosing the deformation term such that it does not change the asymptotic behavior of the potential in the space of fields. We will take advantage of this freedom and choose a deformation term $\mathcal{Q}\cdot V$ that makes computations most tractable.

Since our aim is to localize the path integral of gauge theories, some care has to be taken to localize the gauge fixed theory. This requires combining in a suitable way the deformed action $\mathcal{Q}\cdot V$ and gauge fixing terms $\mathcal{L}_{\text{g.f.}}$ into a $\hat{\mathcal{Q}} = \mathcal{Q} + Q_{\text{BRST}}$ exact term $\hat{\mathcal{Q}}\cdot\hat{V}$, where $\hat{V} = V + V_{\text{ghost}}$. This refinement, while technically important, does not modify the fact that the gauge fixed path integral localizes to $\mathcal{F}$. The inclusion of the gauge fixing term, however, plays an important role in the evaluation of the one-loop determinants in the directions normal to $\mathcal{F}$.

2.3.1 Choice of supercharge

In this section we choose a particular supersymmetry generator $\mathcal{Q}$ in the $SU(2|1)$ supersymmetry algebra with which to localize the path integral of $\mathcal{N}=(2,2)$ gauge theories on S^2. We consider[22]

$$\mathcal{Q} = S_1 + Q_2\,. \tag{2.3.3}$$

This supercharge generates an $SU(1|1)$ subalgebra of $SU(2|1)$, given by

$$\mathcal{Q}^2 = J + \frac{R}{2} \qquad\qquad \left[J + \frac{R}{2}, \mathcal{Q}\right] = 0\,, \tag{2.3.4}$$

where J is the charge corresponding to a $U(1)$ subgroup of the $SU(2)$ isometry group of the S^2 while R is the R-symmetry generator in $SU(2|1)$. In terms of embedding coordinates where S^2 is parametrized by

$$X_1^2 + X_2^2 + X_3^2 = r^2\,, \tag{2.3.5}$$

[21] The original Lagrangian $\mathcal{L}$ is irrelevant for the localization one-loop analysis.

[22] In Section 2.4 we also analyze localization of the path integral with respect to both Q_1 and Q_2. The analysis leads directly to the Coulomb branch representation of the partition function. On the other hand, this other choice does not allow non-trivial field configurations in the Higgs branch, and therefore cannot give rise to the Higgs branch representation of the partition function.

J acts under an infinitesimal transformation, as follows

$$\begin{aligned} X_1 \to& X_1 - \varepsilon X_2 \\ X_2 \to& X_2 + \varepsilon X_1 \,. \end{aligned} \tag{2.3.6}$$

Geometrically, the action of J has two antipodal fixed points on S^2, which can be used to define the north and south poles of S^2. These are located at $(0,0,r)$ and $(0,0,-r)$ in the embedding coordinates (2.3.5). In terms of the coordinates of the round metric on S^2

$$\mathrm{d}s^2 = r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\varphi^2 \right) \tag{2.3.7}$$

the corresponding Killing vector is

$$\mathrm{i} \frac{\partial}{\partial \varphi} \,, \tag{2.3.8}$$

with the north and south poles corresponding to $\theta = 0$ and $\theta = \pi$ respectively. The supersymmetry algebra (2.3.4) is the same used in [Pes07] in the computation of the partition function of four-dimensional $\mathcal{N} = 2$ gauge theories on S^4.

In order to derive the supersymmetry fixed point equations (2.3.1) generated by the supercharge $\mathcal{Q}$, first we need to construct the conformal Killing spinors associated to it, which we denote by $\epsilon_{\mathcal{Q}}$ and $\bar{\epsilon}_{\mathcal{Q}}$. The conformal Killing spinors on S^2 obeying (2.2.16) are explicitly given by[23]

$$\begin{aligned} \epsilon &= \exp\left(-\frac{\mathrm{i}\theta}{2} \gamma^{\hat{2}} \right) \exp\left(\frac{\mathrm{i}\varphi}{2} \gamma^{\hat{3}} \right) \epsilon_\circ \\ \bar{\epsilon} &= \exp\left(+\frac{\mathrm{i}\theta}{2} \gamma^{\hat{2}} \right) \exp\left(\frac{\mathrm{i}\varphi}{2} \gamma^{\hat{3}} \right) \bar{\epsilon}_\circ \,, \end{aligned} \tag{2.3.9}$$

where $\epsilon_\circ$ and $\bar{\epsilon}_\circ$ are constant, complex Dirac spinors. The conformal Killing spinors $\epsilon_{\mathcal{Q}}$ and $\bar{\epsilon}_{\mathcal{Q}}$ are given by (2.3.9), with $\epsilon_\circ$ and $\bar{\epsilon}_\circ$ being chiral spinors of opposite chirality, that is

$$\begin{aligned} \gamma^{\hat{3}} \epsilon_\circ &= +\epsilon_\circ \\ \gamma^{\hat{3}} \bar{\epsilon}_\circ &= -\bar{\epsilon}_\circ \,. \end{aligned} \tag{2.3.10}$$

Therefore, explicitly

$$\begin{aligned} \epsilon_{\mathcal{Q}} &= e^{\mathrm{i}\varphi/2} \exp\left(-\frac{\mathrm{i}\theta}{2} \gamma^{\hat{2}} \right) \epsilon_\circ \\ \bar{\epsilon}_{\mathcal{Q}} &= e^{-\mathrm{i}\varphi/2} \exp\left(+\frac{\mathrm{i}\theta}{2} \gamma^{\hat{2}} \right) \bar{\epsilon}_\circ \,. \end{aligned} \tag{2.3.11}$$

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[23] In the vielbein basis $e^{\hat{1}} = r\,\mathrm{d}\theta$ and $e^{\hat{2}} = r \sin\theta \, \mathrm{d}\varphi$. For details, see Appendix 2.C.

We note that at the north and the south poles of the S^2 the conformal Killing spinors $\epsilon_{\mathcal{Q}}$ and $\bar{\epsilon}_{\mathcal{Q}}$ have definite chirality, and that the chirality at the north pole is opposite to that at the south pole

$$\begin{aligned} \gamma^{\hat{3}}\epsilon_{\mathcal{Q}}(N) &= \epsilon_{\mathcal{Q}}(N) & \gamma^{\hat{3}}\epsilon_{\mathcal{Q}}(S) &= -\epsilon_{\mathcal{Q}}(S) \\ \gamma^{\hat{3}}\bar{\epsilon}_{\mathcal{Q}}(N) &= -\bar{\epsilon}_{\mathcal{Q}}(N) & \gamma^{\hat{3}}\bar{\epsilon}_{\mathcal{Q}}(S) &= \bar{\epsilon}_{\mathcal{Q}}(S)\,. \end{aligned} \tag{2.3.12}$$

As we shall see, the fact that $\mathcal{Q}$ is chiral at the poles implies that the corresponding chiral field configurations – vortices localized at the north pole and anti-vortices at the south pole – may contribute to the partition function of $\mathcal{N} = (2,2)$ gauge theories on S^2.

We note that the circular Wilson loop operator supported on a latitude angle θ_0

$$W_{\theta_\circ} = \operatorname{Tr}\operatorname{Pexp}\oint_{\theta_\circ}\Big[-\mathrm{i}A_i\,\mathrm{d}x^i + \mathrm{i}r(\sigma_1\cos\theta_\circ - \mathrm{i}\sigma_2)\,\mathrm{d}\varphi\Big] \tag{2.3.13}$$

is invariant under the action of $\mathcal{Q}$. Therefore the expectation value of these operators can be computed when localizing with respect to the supercharge $\mathcal{Q}$.

Given our choice of supercharge $\mathcal{Q}$, we can explicitly determine the infinitesimal $J \times R \times G \times G_F$ transformation that $\mathcal{Q}^2$ generates when acting on the fields. The spinor bilinears constructed from $\epsilon_{\mathcal{Q}}$ and $\bar{\epsilon}_{\mathcal{Q}}$ in Section 2.2 evaluate to[24]

$$\begin{aligned} \bar{\epsilon}_{\mathcal{Q}}\epsilon_{\mathcal{Q}} &= \mathrm{i}\cos\theta & \xi &= -\frac{\mathrm{i}}{r}\partial_\varphi \\ \bar{\epsilon}_{\mathcal{Q}}\gamma^{\hat{3}}\epsilon_{\mathcal{Q}} &= \mathrm{i} & \alpha &= -\frac{\mathrm{i}}{2r}\,. \end{aligned} \tag{2.3.14}$$

Therefore, in view of (2.3.6), $\mathcal{Q}^2$ generates $J + R/2$, *i.e.* a simultaneous infinitesimal rotation and R-symmetry transformation with parameter

$$\varepsilon = \frac{1}{r}\,, \tag{2.3.15}$$

and a gauge transformation with gauge parameter

$$\Lambda = \mathrm{i}\cos\theta\sigma_1 + \sigma_2 - \frac{\mathrm{i}}{r}A_2\,. \tag{2.3.16}$$

On the chiral multiplet fields, $\mathcal{Q}^2$ also induces a G_F flavour symmetry rotation parametrized by the twisted masses m.

2.3.2 Localization equations

Here we present the key steps in the derivation of the set of partial differential equations that characterize the vector multiplet and chiral multiplet field

[24]By fixing the overall normalization $\bar{\epsilon}_\circ\epsilon_\circ = \mathrm{i}$.

configurations that are invariant under the action of $\mathcal{Q}$. The details of the derivation are omitted here and can be found in Appendix 2.C.

We must identify the partial differential equations implied by (2.3.1)

$$\delta_{\mathcal{Q}}\lambda = \delta_{\mathcal{Q}}\bar{\lambda} = 0 \tag{2.3.17}$$

$$\delta_{\mathcal{Q}}\psi = \delta_{\mathcal{Q}}\bar{\psi} = 0\,, \tag{2.3.18}$$

where $\delta_{\mathcal{Q}} \equiv \delta_{\epsilon_{\mathcal{Q}}} + \delta_{\bar{\epsilon}_{\mathcal{Q}}}$, from the explicit supersymmetry transformations given in equations (2.2.17, 2.2.18) and (2.2.26, 2.2.27) for the choice of conformal Killing spinors $\epsilon_{\mathcal{Q}}$ and $\bar{\epsilon}_{\mathcal{Q}}$ in (2.3.11). The moduli space of solutions to these equations, once intersected with the saddle points of our choice of $\mathcal{Q}$-exact deformation term, determines the space of field configurations that need to be integrated over in the path integral.

Given a choice of deformation term, in order for the path integral to converge we need to impose reality conditions on the fields. These reality conditions restrict the contour of path integration so that the integrand falls off sufficiently fast in the asymptotic region in the space of field configurations. The residual freedom in the choice of contour *i.e.* deformations of the contour which do not change the asymptotic behavior of the integrand, is then used to make sure that the contour of integration includes the saddle points of the deformed action.

We are interested in deformation terms that do not alter the asymptotic behavior of the original action (2.2.3). We may therefore extract the reality conditions by requiring the original path integral for some effective couplings to be convergent.

From the kinetic terms in the bosonic part of the action (2.2.3) we conclude that the scalar fields σ_1, σ_2 and the connection A_i in the vector multiplet are hermitian while the chiral multiplet complex scalars ϕ and $\bar{\phi}$ satisfy $\bar{\phi} = \phi^\dagger$. Next we note that the path integration over the chiral multiplet auxiliary fields F, $\bar{F}$ is just a Gaussian integral and we simply require $\bar{F} = F^\dagger$. For the convergence of the path integral, one should choose the contour of integration for the auxiliary field D such that $\mathrm{D} + \mathrm{i} g_{\text{eff}}^2(\phi\bar{\phi} - \xi_{\text{eff}}\mathbb{1})$ is hermitian. In other words

$$\mathrm{Im}\,\mathrm{D} + g_{\text{eff}}^2(\phi\bar{\phi} - \xi_{\text{eff}}\mathbb{1}) = 0\,, \tag{2.3.19}$$

where the explicit form of the coupling constants g_{eff}^2 and ξ_{eff} are determined by choice of $\mathcal{Q}$-exact deformation terms.

The supersymmetry fixed point equations for the vector multiplet fields (2.3.17) are given by

$$D_{\hat{2}}\sigma_1 = D_{\hat{i}}\sigma_2 = 0 \qquad D_{\hat{1}}\sigma_1 + g_{\text{eff}}^2(\phi\bar{\phi} - \xi_{\text{eff}}\mathbb{1})\sin\theta = 0 \tag{2.3.20}$$

$$\mathrm{Re}\,\mathrm{D} = [\sigma_1, \sigma_2] = 0 \qquad F_{\hat{1}\hat{2}} + \frac{\sigma_1}{r} + g_{\text{eff}}^2(\phi\bar{\phi} - \xi_{\text{eff}}\mathbb{1})\cos\theta = 0\,, \tag{2.3.21}$$

while the supersymmetry equations for the chiral multiplet fields (2.3.18) reduce to

$$\cos\frac{\theta}{2}\left(D_{\hat{1}}+\mathrm{i}D_{\hat{2}}\right)\phi+\sin\frac{\theta}{2}\left(\sigma_1-\frac{q}{2r}\right)\phi=0 \qquad F=0 \tag{2.3.22}$$

$$\sin\frac{\theta}{2}\left(D_{\hat{1}}-\mathrm{i}D_{\hat{2}}\right)\phi+\cos\frac{\theta}{2}\left(\sigma_1+\frac{q}{2r}\right)\phi=0 \qquad (\sigma_2+m)\,\phi=0\,. \tag{2.3.23}$$

These differential equations on S^2 are a supersymmetric extension of classic differential equations in physics. Our equations interpolate between BPS vortex equations at the north pole ($\theta=0$)

$$\begin{aligned} \left(D_{\hat{1}}+\mathrm{i}D_{\hat{2}}\right)\phi&=0 & D_{\hat{\imath}}\left(\sigma_1+\mathrm{i}\sigma_2\right)&=0\\ F_{\hat{1}\hat{2}}+\frac{\sigma_1}{r}+g_{\text{eff}}^2(\phi\bar{\phi}-\xi_{\text{eff}}\mathbb{1})&=0 & \mathrm{Re}\,\mathrm{D}=[\sigma_1,\sigma_2]&=0\\ \left(\sigma_1+\frac{q}{2r}\right)\phi&=0 & (\sigma_2+m)\,\phi&=0\,, \end{aligned} \tag{2.3.24}$$

and BPS anti-vortex equations at the south pole ($\theta=\pi$)

$$\begin{aligned} \left(D_{\hat{1}}-\mathrm{i}D_{\hat{2}}\right)\phi&=0 & D_{\hat{\imath}}\left(\sigma_1+\mathrm{i}\sigma_2\right)&=0\\ F_{\hat{1}\hat{2}}+\frac{\sigma_1}{r}-g_{\text{eff}}^2(\phi\bar{\phi}-\xi_{\text{eff}}\mathbb{1})&=0 & \mathrm{Re}\,\mathrm{D}=[\sigma_1,\sigma_2]&=0\\ \left(\sigma_1-\frac{q}{2r}\right)\phi&=0 & (\sigma_2+m)\,\phi&=0\,. \end{aligned} \tag{2.3.25}$$

This system of differential equations is akin to the one found in [GOP11] in the localization computation of four-dimensional $\mathcal{N}=2$ gauge theories on S^4. We return later to the study of the supersymmetry equations at the poles, which play a crucial role in our analysis, yielding the Higgs branch representation of the gauge theory partition function on S^2.

2.3.3 Vanishing theorem

As explained previously, the path integral localizes to the space $\mathcal{F}$ of supersymmetric field configurations which are also saddle points of the localizing deformation term. In this section, we consider the supersymmetry equations in the absence of effective FI parameters and we write down the most general *smooth* solutions to the supersymmetry equations for generic values of the R-charges. These solutions are parametrized by the expectation value of fields in the vector multiplet, thus, we denote this space of solutions by $\mathcal{F}_{\text{Coulomb}}$. In Section 2.4 we localize the path integral to $\mathcal{F}_{\text{Coulomb}}$ and derive the Coulomb branch representation of the partition function.

With $\xi_{\text{eff}}=0$ and for generic R-charges, the most general smooth solution to the equations (2.3.20),(2.3.21),(2.3.22) and (2.3.23) is given by[25]

$$\begin{aligned} A &= \frac{B}{2}\left(\kappa - \cos\theta\right)\mathrm{d}\varphi \qquad & \sigma_1 &= -\frac{B}{2r} \qquad & \phi &= 0 \\ \mathrm{D} &= 0 & \sigma_2 &= a & F &= 0\,, \end{aligned} \tag{2.3.26}$$

where a and B are constant commuting matrices which live in the gauge Lie algebra and its Cartan subalgebra respectively. The matrix B is further restricted by the first Chern class quantization to have integer eigenvalues. The constant κ parametrizes a pure gauge background which is necessary in any coordinate patch which includes one of the poles and can be gauged away in the coordinate patch which excludes the poles.

It is interesting to note that if the R-charge is tuned to be a negative integer or zero, then there are nontrivial solutions of the form

$$\phi = e^{\frac{\mathrm{i}}{2}(\kappa B - q)\varphi}\frac{(\sin\frac{\theta}{2})^{\frac{B-q}{2}}}{(\cos\frac{\theta}{2})^{\frac{B+q}{2}}}\phi_\circ \tag{2.3.27}$$

with $\phi_\circ$ being a constant in the kernel of $a+m$. Imposing regularity at the poles restricts the allowed value of q and B as follows: $q+|B|$ must be even and non-positive integers. In such a case, the above field configuration can be written in terms of the magnetic flux B monopole scalar harmonics $Y^{\frac{B}{2}}_{j,m}$ as

$$\phi = Y^{\frac{B}{2}}_{-\frac{q}{2},-\frac{q}{2}}\phi_\circ\,. \tag{2.3.28}$$

It is worth mentioning that these field configurations are also supersymmetric configurations in the localization computation of the partition function of three-dimensional $\mathcal{N}=2$ gauge theories on $S^1\times S^2$ [IY11], which computes the superconformal index of these theories. In our computations, we can ignore these discrete, tuned solutions to the supersymmetry equations: for theories flowing to superconformal theories in the infrared, unitarity constrains the R-charges to be non-negative. Furthermore, as will be explained in Section 2.4, these solutions are not saddle points of the localized path integral.

We note that even though our choice of $\mathcal{Q}$ breaks the $SU(2)$ symmetry of S^2, the $\mathcal{Q}$-invariant field configurations (2.3.26) are $SU(2)$ invariant. Later on, we take an alternative approach in which the Coulomb branch is lifted and the saddle point equations admit singular solutions at the poles thereby breaking the $SU(2)$ symmetry. We will consider the physics behind singular solutions localized at the north and south poles of S^2 in Section 2.5.

[25] A detailed derivation of this result is presented in Appendix 2.C.

2.4 Coulomb branch

In order to evaluate the path integral of an $\mathcal{N} = (2,2)$ gauge theory on S^2 using supersymmetric localization, we must choose a deformation of the original supersymmetric Lagrangian by a $\mathcal{Q}$-exact term (2.3.2)

$$\mathcal{L} \to \mathcal{L} + t\,\delta_{\mathcal{Q}} V\,. \tag{2.4.1}$$

The deformation term $\delta_{\mathcal{Q}} V$ defines the measure of integration through the associated one-loop determinant. In this section we calculate the contribution to the path integral due to the smooth field configurations (2.3.26). This yields the Coulomb branch representation of the path integral, as an integral over the Coulomb branch saddle points $\mathcal{F}_{\text{Coulomb}}$.

A calculation shows that the vector multiplet action (2.2.4) and the chiral multiplet action (2.2.10) are $\mathcal{Q}$-exact with respect to our choice of supercharge (2.3.3). Specifically,

$$(\bar{\epsilon}_{\mathcal{Q}}\gamma^{\hat{3}}\epsilon_{\mathcal{Q}})\, g^2\, \mathcal{L}_{\text{v.m.}} = \delta_{\mathcal{Q}}\delta_{\bar{\epsilon}_{\mathcal{Q}}}\, \text{Tr}\left(\frac{1}{2}\bar{\lambda}\gamma^{\hat{3}}\lambda - 2\text{iD}\sigma_2 + \frac{\text{i}}{r}\sigma_2^2\right)\,, \tag{2.4.2}$$

and

$$\begin{aligned} &-(\bar{\epsilon}_{\mathcal{Q}}\gamma^{\hat{3}}\epsilon_{\mathcal{Q}})\,(\mathcal{L}_{\text{c.m.}} + \mathcal{L}_{\text{mass}}) \\ &\quad = \delta_{\mathcal{Q}}\delta_{\bar{\epsilon}_{\mathcal{Q}}}\, \text{Tr}\left(\bar{\psi}\gamma^{\hat{3}}\psi - 2\bar{\phi}\left(\sigma_2 + m + \text{i}\frac{q}{2r}\right)\phi + \frac{\text{i}}{r}\bar{\phi}\phi\right)\,, \end{aligned} \tag{2.4.3}$$

where $\delta_{\mathcal{Q}} \equiv \delta_{\epsilon_{\mathcal{Q}}} + \delta_{\bar{\epsilon}_{\mathcal{Q}}}$. This implies that correlation functions of $\mathcal{Q}$-closed observables in an $\mathcal{N} = (2,2)$ gauge theory on S^2 are independent of g, the Yang-Mills coupling constant. Despite being g independent, these correlators are nontrivial functions of the renormalized FI parameter ξ_{ren} for each $U(1)$ factor in the gauge group, and of the twisted masses m.

We now turn to the choice of deformation term $\delta_{\mathcal{Q}} V$. The most canonical choice would be to take

$$V_{\text{can}} = (\delta_{\mathcal{Q}}\lambda)^\dagger\,\lambda + \left(\delta_{\mathcal{Q}}\bar{\lambda}\right)^\dagger\,\bar{\lambda} + (\delta_{\mathcal{Q}}\psi)^\dagger\,\psi + \left(\delta_{\mathcal{Q}}\bar{\psi}\right)^\dagger\,\bar{\psi}\,. \tag{2.4.4}$$

For this choice, the bosonic part of the deformation term $\delta_{\mathcal{Q}} V_{\text{can}}$ is manifestly non-negative. It is therefore guaranteed that all $\mathcal{Q}$-invariant field configurations are the saddle points of $\delta_{\mathcal{Q}} V_{\text{can}}$ with minimal (zero) action. The disadvantage of such a deformation term is that the resulting action $\delta_{\mathcal{Q}} V_{\text{can}}$ does not necessarily preserve the $SU(2)$ symmetries of S^2, thus technically complicating the computation of the one-loop determinants in the directions transverse to the $\mathcal{Q}$-invariant field configurations. But as we argued in Section 2.3, the result is largely insensitive to the choice of deformation, as long as it is non-degenerate and does not change the asymptotics of the potential in the space of fields. Therefore, we will instead use as the deformation term the technically simpler, $SU(2)$ symmetric, vector multiplet and chiral multiplet actions $\delta_{\mathcal{Q}}V = \mathcal{L}_{\text{v.m.}} + \mathcal{L}_{\text{c.m.}} + \mathcal{L}_{\text{mass}}$. Contrarily to the canonical choice $\delta_{\mathcal{Q}}V_{\text{can}}$, the saddle points of $\delta_{\mathcal{Q}}V$ do not coincide with the supersymmetric configurations and thus fully localize the path integral to the intersection.

It is straightforward to show that all Coulomb branch field configurations in $\mathcal{F}_{\text{Coulomb}}$ are saddle points of $\delta_{\mathcal{Q}}V$ and must be integrated over. However, the solutions to the vortex and anti-vortex equations we found at the poles are not saddle points of $\delta_{\mathcal{Q}}V$. This can be demonstrated using both the supersymmetry and the saddle point equations at the poles as follows.[26] Since we are taking the masses to be non-degenerate, it follows from the equations

$$(\sigma_2 + m_I)\phi_I = 0 \tag{2.4.5}$$

that any pair of distinct non-vanishing vectors ϕ_I and ϕ_J have to be independent. In addition, the above equation combined with the covariant constancy of σ_2 and its equation of motion imply

$$\sum_I (q_I - 1)\phi_I\bar{\phi}_I = 0\,, \tag{2.4.6}$$

while the equation of motion for D yields

$$\text{iD} - \sum_I \phi_I\bar{\phi}_I = 0\,. \tag{2.4.7}$$

However, since all non-vanishing ϕ_I are independent, we can conclude[27] from (2.4.6) that $\phi_I\bar{\phi}_I$ vanishes for each I. It therefore excludes the aforementioned supersymmetric solutions (2.3.28) with fine-tuned values of q from the set of saddle points. Combined with (2.4.7), it also sets $\text{D} = 0$. Plugging this result in the supersymmetry equations fixes $F = -\sigma_1/r = B/2r^2$ and $\sigma_2 = a$ and we recover the Coulomb branch field configurations spanning $\mathcal{F}_{\text{Coulomb}}$, thus eliminating the vortex and anti-vortex configurations.

The conclusion that the path integral can be written as a integral over just $\mathcal{F}_{\text{Coulomb}}$ can also be derived as follows. As we remarked earlier, the path integral does not depend on the choice of supercharge $\mathcal{Q}$ used in the localization computation. Therefore, we may instead try to localize the partition function with respect to the supercharges Q_1 and Q_2. This, however, requires finding a deformation term which is Q_1 and Q_2 exact. Such a deformation term is provided by the following terms in the action

$$\mathcal{L}_{\text{v.m.}} + \mathcal{L}_{\text{c.m.}} + \mathcal{L}_{\text{mass}} = \delta_{\epsilon_1}\delta_{\epsilon_2}V'\,, \tag{2.4.8}$$

[26] With some more effort it is possible to prove using only the equation of motion for D that the vortex and anti-vortex configurations are not saddle points of the action in the limit in which the coefficient of the deformation term $\delta_{\mathcal{Q}}V$ goes to infinity.

[27] This step requires us to assume that none of the R-charges is 1.

with $V' = 1/2\,\mathrm{Tr}(\lambda\lambda)+\bar{\phi}F$, which are exact with respect to both supercharges since $[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = 0$. In this approach the path integral localizes to the Q_1 and Q_2 invariant field configurations, which are the solutions to the equations

$$\begin{aligned} \delta_{\epsilon_1}\lambda = \delta_{\epsilon_2}\lambda = 0 \qquad & \delta_{\epsilon_1}\psi = \delta_{\epsilon_2}\psi = 0 \\ \delta_{\epsilon_1}\bar{\lambda} = \delta_{\epsilon_2}\bar{\lambda} = 0 \qquad & \delta_{\epsilon_1}\bar{\psi} = \delta_{\epsilon_2}\bar{\psi} = 0\,. \end{aligned} \tag{2.4.9}$$

These equations directly lead[28] to the Coulomb branch field configurations (2.3.26) parametrizing $\mathcal{F}_{\text{Coulomb}}$ while immediately rendering the vortex and anti-vortex configurations non-supersymmetric. Note that this conclusion is reached by considering the supersymmetry equations alone, contrary to localization with respect to $\mathcal{Q}$, where the saddle point equations of $\delta_{\mathcal{Q}}V$ also need to be invoked to show that vortex and anti-vortex configurations do not contribute. Since the saddle points and deformation term (2.4.8) are precisely the same as the one for $\mathcal{Q}$, this guarantees that we obtain the same Coulomb branch representation of the path integral. A drawback of localizing with respect to Q_1 and Q_2 is that we cannot study the expectation value of the circular Wilson loop (2.3.13) since it is not Q_1 and Q_2 invariant.

In Section 2.5 we will obtain the payoff of using the supercharge $\mathcal{Q}$. As we have shown in Section 2.3, supersymmetry leads to the vortex and anti-vortex equations at the poles. In that section, we will argue that localizing the path integral $\mathcal{Q}$ in a different limit yields the Higgs branch representation of the partition function.

2.4.1 Integral representation of the partition function

We now can write down the expression of the partition function as an integral over the Coulomb branch field configurations $\mathcal{F}_{\text{Coulomb}}$. The Coulomb branch representation of the partition function is thus given by[29]

$$Z_{\text{Coulomb}}(m,\tau) = \sum_{B}\int_{\mathfrak{t}} \mathrm{d}a\, Z_{\text{cl}}(a,B,\tau)\, Z_{\text{one-loop}}(a,B,m)\,, \tag{2.4.10}$$

where the integral over a has been reduced to the Cartan subalgebra $\mathfrak{t}$ of G. The first factor arises from evaluating the renormalized gauge theory action on the smooth supersymmetric field configurations (2.3.26)

$$Z_{\text{cl}}(a,B,\tau) = e^{-4\pi i r\xi_{\text{ren}}\,\mathrm{Tr}\,a + i\vartheta\,\mathrm{Tr}\,B}\,, \tag{2.4.11}$$

[28] Supersymmetry implies that $V_1 = V_2 = V_3 = \mathrm{D} = 0$. The fact that the solutions to these equations are the Coulomb branch field configurations (2.3.26) follows by using the equality of actions in (2.2.4) and (2.2.6), derived by integrating by parts. Non-trivial chiral multiplet configuration are manifestly non-supersymmetric.

[29] The partition function has an anomalous dependence on the radius r of the S^2 due to the conformal anomaly in two dimensions. We do not retain this factor throughout our formulae, which can be extracted from our one-loop determinants.

and the one-loop determinant $Z_{\text{one-loop}}(a, B, m)$ specifies the measure of integration over a, which is determined by the deformation term $\delta_{\mathcal{Q}} V$.

Some care has been taken to ensure that the computation, including the regularization of the one-loop determinants $Z_{\text{one-loop}}(a, B, m)$, is $\mathcal{Q}$-invariant. Even though the FI parameter ξ is classically marginal, it runs quantum mechanically according to the renormalization group equation

$$\xi(\mu) = \xi + \frac{1}{2\pi} \sum_j Q_j \ln\left(\frac{\mu}{M_{\text{UV}}}\right) = \frac{1}{2\pi} \sum_j Q_j \ln\left(\frac{\mu}{\Lambda}\right) , \qquad (2.4.12)$$

where Q_j is the charge of the j-th chiral multiplet under the $U(1)$ gauge group corresponding to ξ, M_{UV} is the ultraviolet cutoff, μ is the floating scale and Λ is the renormalization group invariant scale. A simple way of performing this renormalization in a $\mathcal{Q}$-invariant way, is to enrich the theory one is interested in with an "expectator" chiral multiplet of mass M and charge $-Q = -\sum_j Q_j$, so that in the enriched theory the FI parameter does not run. Now, to extract the result for the theory of interest, we take the answer of the finite theory in the limit where M is very large, thereby decoupling the expectator chiral multiplet. This procedure results in a $\mathcal{Q}$-invariant ultraviolet cutoff M for the theory under study. As shown in Appendix 2.E, taking M large in the one-loop determinant (2.4.16) for the expectator chiral multiplet precisely reproduces the running of the FI parameter (2.4.12) with $M_{\text{UV}} = M$ and $\mu = \varepsilon = 1/r$. That is, the renormalized coupling obtained in this way is evaluated at the inverse radius of the S^2, which is the infrared scale of S^2

$$\xi_{\text{ren}} \equiv \xi\left(\mu = 1/r\right)|_{M_{\text{UV}}=M} = \xi + \frac{1}{2\pi} \sum_i Q_i \ln\left(\frac{\varepsilon}{M}\right) . \qquad (2.4.13)$$

The one-loop factor in the localization computation $Z_{\text{one-loop}}(a, B, m)$ takes the form

$$Z_{\text{one-loop}}(a, B, m) = Z^{\text{v.m.}}_{\text{one-loop}}(a, B) \cdot Z^{\text{c.m.}}_{\text{one-loop}}(a, B, m) \cdot \mathcal{J}(a, B) , \quad (2.4.14)$$

where the Jacobian factor $\mathcal{J}(a, B)$ accounts for the reduction of the integral over all a such that $[a, B] = 0$ to an integral over the Cartan subalgebra $\mathfrak{t}$. The magnetic flux B over the S^2 breaks the gauge symmetry G down to a subgroup $H_B = \{g \in G \mid gBg^{-1} = B\}$. Therefore, the associated Jacobian factor is

$$\mathcal{J}(a, B) = \frac{1}{|\mathcal{W}(H_B)|} \prod_{\substack{\alpha \in \Delta^+ \\ \alpha \cdot B = 0}} (\alpha \cdot a)^2 , \qquad (2.4.15)$$

where $\alpha \in \Delta^+$ are positive roots of the Lie algebra of G and $|\mathcal{W}(H_B)|$ is the order of the Weyl group of H_B.

The one-loop determinants for our choice of deformation term $\delta_{\mathcal{Q}} V$, which is the sum of (2.4.2) and (2.4.3), are computed in Appendix 2.D. For a chiral multiplet in a reducible representation $\mathbf{R} = \oplus_I \mathbf{r}_I$ we obtain

$$Z^{\text{c.m.}}_{\text{one-loop}}(a,B,m) = \prod_I \prod_{w_I \in \mathbf{r}_I} (-\mathrm{i})^{w_I \cdot B} (-1)^{|w_I \cdot B|/2} \cdot \frac{\Gamma\left(\frac{q_I}{2} - \mathrm{i}r(w_I \cdot a + m_I) + \frac{|w_I \cdot B|}{2}\right)}{\Gamma\left(1 - \frac{q_I}{2} + \mathrm{i}r(w_I \cdot a + m_I) + \frac{|w_I \cdot B|}{2}\right)}, \tag{2.4.16}$$

where w_I are the weights of the representation $\mathbf{r}_I$ and $\Gamma(x)$ is the Euler gamma function. The twisted masses and R-charges m_I and q_I of the chiral multiplets, which take values in the Cartan subalgebra of the flavour symmetry G_F, combine into the holomorphic combination $M = m + \frac{\mathrm{i}}{2r} q$ introduced in (2.2.14).

For the vector multiplet contribution we obtain

$$Z^{\text{v.m.}}_{\text{one-loop}}(a,B) = \prod_{\substack{\alpha \in \Delta^+ \\ \alpha \cdot B \neq 0}} \left[\left(\frac{\alpha \cdot B}{2r} \right)^2 + (\alpha \cdot a)^2 \right]. \tag{2.4.17}$$

We note that the Jacobian factor and the vector multiplet determinant combine nicely into an unconstrained product over the positive roots of the Lie algebra

$$Z^{\text{v.m.}}_{\text{one-loop}}(a,B) \cdot J(a,B) = \frac{1}{|\mathcal{W}(H_B)|} \prod_{\alpha \in \Delta^+} \left[\left(\frac{\alpha \cdot B}{2r} \right)^2 + (\alpha \cdot a)^2 \right]. \tag{2.4.18}$$

The Coulomb branch representation of the partition function of an $\mathcal{N} = (2,2)$ gauge theory on S^2 is thus given by

$$\begin{aligned} &Z_{\text{Coulomb}}(m,\tau) \\ &= \sum_B \frac{1}{|\mathcal{W}(H_B)|} \int_{\mathfrak{t}} \mathrm{d}a \, e^{-4\pi \mathrm{i} \xi_{\text{ren}} r \operatorname{Tr} a + \mathrm{i}\vartheta \operatorname{Tr} B} \prod_{\alpha \in \Delta^+} \left[\left(\frac{\alpha \cdot B}{2r} \right)^2 + (\alpha \cdot a)^2 \right] \\ &\quad \times \prod_{I, w_I} \left[(-\mathrm{i})^{w_I \cdot B} (-1)^{|w_I \cdot B|/2} \frac{\Gamma\left(-\mathrm{i}r(w_I \cdot a + \mathrm{M}_I) + \frac{|w_I \cdot B|}{2}\right)}{\Gamma\left(1 + \mathrm{i}r(w_I \cdot a + \mathrm{M}_I) + \frac{|w_I \cdot B|}{2}\right)} \right]. \end{aligned} \tag{2.4.19}$$

The expectation value of the circular Wilson loop (2.3.13) is obtained by enriching the integrand in (2.4.19) with the insertion of

$$\operatorname{Tr} e^{2\pi a - \mathrm{i}\pi B}. \tag{2.4.20}$$

2.4.2 Factorization of the partition function

We show in this section that the Coulomb branch representation of the partition function (2.4.19) can be written as a discrete sum, whose summand factorizes into the product of two functions. A related factorization was found previously by Pasquetti [Pas11] when evaluating the partition function of three-dimensional $\mathcal{N} = 2$ abelian gauge theories on the squashed S^3.[30]

We recognize the expression we obtain as the sum over Higgs vacua of the product of the vortex partition function due to vortices at the north pole with the anti-vortex partition function due to the anti-vortices at the south pole. This result is interpreted in Section 2.5 as a direct path integral evaluation of the partition function, where the path integral is argued to localize on vortices and anti-vortices in the Higgs branch.

Let us consider for definiteness the case of two-dimensional $\mathcal{N} = (2,2)$ SQCD. This theory has $G = U(N)$ gauge group and N_f fundamental chiral multiplets and $\widetilde{N_f}$ anti-fundamental chiral multiplets. The partition function (2.4.19) of this theory is[31]

$$Z^{U(N)}_{\text{SQCD}} = \frac{1}{N!} \sum_{B \in \mathbb{Z}^N} \int \mathrm{d}^N a \left\{ e^{-4\pi \mathrm{i}\xi \operatorname{Tr} a} e^{\mathrm{i}\vartheta \operatorname{Tr} B} \prod_{i<j} \left[(a_i - a_j)^2 + \left(\frac{B_i - B_j}{2} \right)^2 \right] \right.$$
$$\left. \cdot \prod_{s=1}^{N_f} \prod_{i=1}^{N} \frac{(-1)^{\frac{|B_i|+B_i}{2}} \Gamma(-\mathrm{i}a_i - \mathrm{i}\mathrm{M}_s + \frac{|B_i|}{2})}{\Gamma(1 + \mathrm{i}a_i + \mathrm{i}\mathrm{M}_s + \frac{|B_i|}{2})} \prod_{s=1}^{\widetilde{N_f}} \prod_{i=1}^{N} \frac{(-1)^{\frac{|B_i|-B_i}{2}} \Gamma(\mathrm{i}a_i - \mathrm{i}\widetilde{\mathrm{M}}_s + \frac{|B_i|}{2})}{\Gamma(1 - \mathrm{i}a_i + \mathrm{i}\widetilde{\mathrm{M}}_s + \frac{|B_i|}{2})} \right\} . \tag{2.4.21}$$

In the large a limit, the integrand is of order $|a|^{N(N-1)+N\sum_I (q_I - 1)}$, hence this N-dimensional integral is convergent as long as

$$\sum_{s=1}^{N_f} q_s + \sum_{s=1}^{\widetilde{N_f}} \widetilde{q}_s < N_f + \widetilde{N_f} - N \,. \tag{2.4.22}$$

In the cases where $N_f > \widetilde{N_f}$, or $N_f = \widetilde{N_f}$ and $\xi > 0$, the contour can be closed towards $\mathrm{i}a_i \to +\infty$, enclosing poles of the fundamental multiplets' one-loop determinants; the contour must be chosen to enclose poles of the anti-fundamental multiplets' one-loop determinants in cases where $N_f < \widetilde{N_f}$, or $N_f = \widetilde{N_f}$ and $\xi < 0$. Assuming that all R-charges are positive, or deforming the integration contour to ensure that we enclose the same set of poles, this expresses the Coulomb branch integral as a sum of the residues at combined poles

$$\mathrm{i}a_i = -\mathrm{i}\mathrm{M}_{p_i} + n_i + \frac{|B_i|}{2} \quad \text{for all } 1 \le i \le N, \tag{2.4.23}$$

with $1 \le p_1, \ldots, p_N \le N_f$ and $n_1, \ldots, n_N \ge 0$ labelling the poles. The resulting ratios of Gamma functions in the integrand can be recast in terms

[30] The partition function of three-dimensional gauge theories on $S^2 \times S^1$ can also be factorized [Pas12].

[31] Without loss of generality we set $r = 1$ to unclutter formulas. It can easily be restored by dimensional analysis.

of Pochhammer raising factorials $(x)_n = x(x+1)\cdots(x+n-1)$ as

$$\frac{\Gamma(\mathrm{iM}_{p_i} - \mathrm{iM}_s - n_i)}{\Gamma(1 + \mathrm{iM}_s - \mathrm{iM}_{p_i} + |B_i| + n_i)} = \frac{\gamma(\mathrm{iM}_{p_i} - \mathrm{iM}_s)(-1)^{n_i}}{(1 + \mathrm{iM}_s - \mathrm{iM}_{p_i})_{n_i}(1 + \mathrm{iM}_s - \mathrm{iM}_{p_i})_{n_i+|B_i|}}\,, \tag{2.4.24}$$

where

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}\,, \tag{2.4.25}$$

and similarly for the ratios of Gamma functions coming from the anti-fundamental chiral multiplets.

The symmetry between n_i and $n_i + |B_i|$ in (2.4.24) leads us to introduce new coordinates

$$k_i^{\pm} = n_i + [B_i]^{\pm} = n_i + |B_i|/2 \pm B_i/2 \geq 0 \tag{2.4.26}$$

on the summation lattice, such that $\{n_i, n_i + |B_i|\} = \{k_i^{\pm}\}$. In Section 2.5, the N integers k_i^+ will be interpreted as labelling vortices located at the north pole, and k_i^- anti-vortices at the south pole. More precisely, $k_i^{\pm}$ measures the amount of vortex and anti-vortex charge carried by the i-th Cartan generator in $U(N)$: note that the flux $B_i = k_i^+ - k_i^-$.

This change of coordinates decouples the sums over $k^+ \geq 0$ and $k^- \geq 0$ and yields the following expression after converting signs to a shift in the theta angle

$$\begin{aligned} Z^{U(N)}_{\mathrm{SQCD}} = \frac{(2\pi)^N}{N!} \sum_{p_1,\ldots,p_N=1}^{N_f} \Bigg[& e^{4\pi\xi \sum_j \mathrm{iM}_{p_j}} \prod_{i=1}^{N} \frac{\prod_{s=1}^{\widetilde{N_f}} \gamma(-\mathrm{i}\widetilde{\mathrm{M}}_s - \mathrm{iM}_{p_i})}{\prod_{s\neq p_i}^{N_f} \gamma(1 + \mathrm{iM}_s - \mathrm{iM}_{p_i})} \\ \cdot \sum_{k_i^+ \geq 0} & \left[e^{(2\pi\mathrm{i}\tau + \mathrm{i}\pi N_f)\sum_i k_i^+} \prod_{i<j}^{N} \left(\mathrm{M}_{p_j} - \mathrm{M}_{p_i} + \mathrm{i}k_j^+ - \mathrm{i}k_i^+\right) \prod_{i=1}^{N} \frac{\prod_{s=1}^{\widetilde{N_f}} (-\mathrm{i}\widetilde{\mathrm{M}}_s - \mathrm{iM}_{p_i})_{k_i^+}}{\prod_{s=1}^{N_f} (1 + \mathrm{iM}_s - \mathrm{iM}_{p_i})_{k_i^+}} \right] \\ \cdot \sum_{k_i^- \geq 0} & \left[e^{(-2\pi\mathrm{i}\bar{\tau} + \mathrm{i}\pi\widetilde{N_f})\sum_i k_i^-} \prod_{i<j}^{N} \left(\mathrm{M}_{p_j} - \mathrm{M}_{p_i} + \mathrm{i}k_j^- - \mathrm{i}k_i^-\right) \prod_{i=1}^{N} \frac{\prod_{s=1}^{\widetilde{N_f}} (-\mathrm{i}\widetilde{\mathrm{M}}_s - \mathrm{iM}_{p_i})_{k_i^-}}{\prod_{s=1}^{N_f} (1 + \mathrm{iM}_s - \mathrm{iM}_{p_i})_{k_i^-}} \right] \Bigg]. \end{aligned} \tag{2.4.27}$$

Terms with $p_a = p_b$ for some $a \neq b \leq N$ vanish, because the sum over k^+ is then antisymmetric under the exchange of k_a^+ and k_b^+. We can thus normalize the series as

$$\begin{aligned} f(\{p_i\}, \mathrm{M}, z) &= \sum_{k_i \geq 0} \left[z^{\sum_i k_i} \prod_{i<j}^{N} \frac{\mathrm{iM}_{p_j} - \mathrm{iM}_{p_i} + k_i - k_j}{\mathrm{iM}_{p_j} - \mathrm{iM}_{p_i}} \frac{\prod_{s=1}^{\widetilde{N_f}} \prod_{i=1}^{N} (-\mathrm{i}\widetilde{\mathrm{M}}_s - \mathrm{iM}_{p_i})_{k_i}}{\prod_{s=1}^{N_f} \prod_{i=1}^{N} (1 + \mathrm{iM}_s - \mathrm{iM}_{p_i})_{k_i}} \right] \\ &= \sum_{k_i \geq 0} \left[\frac{z^{\sum_i k_i}}{\prod_i k_i!} \frac{\prod_{s=1}^{\widetilde{N_f}} \prod_{i=1}^{N} (-\mathrm{iM}_{p_i} - \mathrm{i}\widetilde{\mathrm{M}}_s)_{k_i}}{\prod_{i\neq j}^{N} (\mathrm{iM}_{p_j} - \mathrm{iM}_{p_i} - k_j)_{k_i} \prod_{s\notin\{p\}}^{N_f} \prod_{i=1}^{N} (1 + \mathrm{iM}_s - \mathrm{iM}_{p_i})_{k_i}} \right], \end{aligned} \tag{2.4.28}$$

which as we will see in the next section, corresponds to the vortex partition function studied in [Sha06], with $z = \exp(2\pi i \tau)$ playing the role of the vortex fugacity. Note that this series converges for all z (all ξ) if $N_f > \widetilde{N_f}$, and for $|z| < 1$ (that is, $\xi > 0$) if $N_f = \widetilde{N_f}$, consistent with the constraints required by our choice of contour. All in all, the partition function factorizes as

$$Z^{U(N)}_{\text{SQCD}} = \sum_{\substack{v_i = -\mathrm{M}_{p_i} \\ 1 \le p_1 < \ldots < p_N \le N_f}} Z_{\text{cl}}(v, 0, \tau) \operatorname*{res}_{a=v} Z_{\text{one-loop}}(a, 0, \mathrm{M}) \cdot f(\{p_i\}, \mathrm{M}, (-1)^{N_f} z)\, f(\{p_i\}, \mathrm{M}, (-1)^{\widetilde{N_f}} \bar{z}) \tag{2.4.29}$$

up to a constant factor, with

$$\operatorname*{res}_{a_i = -\mathrm{M}_{p_i}} Z_{\text{one-loop}}(a, 0, \mathrm{M}) = \prod_{i=1}^{N} \frac{\prod_{s=1}^{\widetilde{N_f}} \gamma(-i\widetilde{\mathrm{M}}_s - i\mathrm{M}_{p_i})}{\prod_{s \notin \{p\}}^{N_f} \gamma(1 + i\mathrm{M}_s - i\mathrm{M}_{p_i})} . \tag{2.4.30}$$

In the next section we obtain this result directly by localizing the path integral to Higgs branch configurations with vortices and anti-vortices. In the matching, some care must be taken when comparing the mass parameters of the gauge theory on the sphere with the parameters describing the theory in the Ω-background used to evaluate the vortex partition function.

The final expression we find is reminiscent of the discrete sums of the product of holomorphic and anti-holomorphic conformal blocks that appear in correlators of the A_{N_f-1} Toda CFT in the presence of completely degenerate fields. A precise matching between the partition function of $\mathcal{N} = (2,2)$ gauge theories on S^2 and correlators in Toda is provided in the abelian case in Section 2.6, and in the case of $U(N)$ in [GLF14] *(Chapter 3 of this thesis)*.

Note that this factorization result applies to any gauge group G with an abelian factor and any matter representation **R**, as shown in Appendix 2.F. This yields a representation of the path integral that can be interpreted as a sum over Higgs vacua of terms factorized into holomorphic and anti-holomorphic contributions, corresponding to vortices and anti-vortices respectively. These formulas motivate natural conjectures for the vortex partition functions corresponding to gauge theories with gauge group G. In the absence of $U(1)$ factors in the gauge group, the factorization can be carried out formally, but the two factors may be divergent series.

2.5 Higgs branch representation

The localization principle, under mild conditions, guarantees that the path integral does not depend either on the choice of supercharge $\mathcal{Q}$ or on the choice of V in the deformation term. But different choices can lead to different representations of the same path integral and therefore to non-trivial identities.

In Section 2.4 we have derived a representation of the partition function as an integral over Coulomb branch vacua. In Section 2.4.2, by explicitly evaluating the integral, we have demonstrated that the partition function also has an alternative representation as a sum – in the Higgs phase – over vortex and anti-vortex field configurations localized at the poles.

This section aims to derive from path integral localization arguments the Higgs branch representation of the partition function. This representation should have a direct derivation using localization. The appropriate choice of supercharge to use to obtain this representation is the same supercharge $\mathcal{Q}$ introduced in (2.3.3), since it has the elegant feature of giving rise to the vortex equations at the north pole

$$\begin{aligned} \left(D_{\hat{1}} + \mathrm{i}D_{\hat{2}}\right)\phi &= 0 & D_{\hat{i}}\left(\sigma_1 + \mathrm{i}\sigma_2\right) &= 0 \\ F_{\hat{1}\hat{2}} + \sigma_1 + g_{\text{eff}}^2(\phi\bar{\phi} - \xi_{\text{eff}}\mathbb{1}) &= 0 & \operatorname{Re}\mathrm{D} = [\sigma_1, \sigma_2] &= 0 \\ \left(\sigma_1 + \frac{q}{2}\right)\phi &= 0 & \left(\sigma_2 + m\right)\phi &= 0\,, \end{aligned} \tag{2.5.1}$$

and anti-vortex equations at the south pole

$$\begin{aligned} \left(D_{\hat{1}} - \mathrm{i}D_{\hat{2}}\right)\phi &= 0 & D_{\hat{i}}\left(\sigma_1 + \mathrm{i}\sigma_2\right) &= 0 \\ F_{\hat{1}\hat{2}} + \sigma_1 - g_{\text{eff}}^2(\phi\bar{\phi} - \xi_{\text{eff}}\mathbb{1}) &= 0 & \operatorname{Re}\mathrm{D} = [\sigma_1, \sigma_2] &= 0 \\ \left(\sigma_1 - \frac{q}{2}\right)\phi &= 0 & \left(\sigma_2 + m\right)\phi &= 0\,. \end{aligned} \tag{2.5.2}$$

We remark that when the effective Fayet-Iliopoulos parameters are non-vanishing, these equations admit solutions with non-vanishing ϕ. These solutions then restrict σ_2 to be a diagonal matrix with the masses of the excited chiral fields on the diagonal and the Coulomb branch configurations (2.3.26) parametrizing $\mathcal{F}_{\text{Coulomb}}$ are lifted. The $\mathcal{Q}$-invariant field configurations admitted by (2.5.1) and (2.5.2) are vortex and anti-vortex configurations at the north and south pole of the S^2. Since vortices and anti-vortices exist in the Higgs phase, we denote this space of supersymmetric field configurations that must be integrated over by $\mathcal{F}_{\text{Higgs}}$.

2.5.1 Localizing onto the Higgs branch

In this section we present a heuristic argument to introduce non-zero FI parameters in the localization computation, which as explained above yields to a representation of the path integral as a sum over vortex and anti-vortex configurations. For the purpose of this argument, we take all the R-charges to be zero.

Recall that our choice of deformation term $\delta_{\mathcal{Q}}V = \mathcal{L}_{\text{v.m.}} + \mathcal{L}_{\text{c.m.}} + \mathcal{L}_{\text{mass}}$ does not include a FI term. In Section 2.4, we performed the saddle point approximation after taking the $t \to \infty$ limit. In this limit, the effective FI parameter vanishes and the saddle point equations forbid vortices, hence the path integral localizes to $\mathcal{F}_{\text{Coulomb}}$. Instead, we assume here that there is another choice of $\mathcal{Q}$-exact deformation terms $\mathcal{Q}V'$ leading to a non-vanishing effective FI parameter $\xi_{\text{eff}} \neq 0$ in the $t \to \infty$ limit[32].

The equation of motion for the D field arising from the deformed action $S + t\delta_{\mathcal{Q}}V'$ is

$$\mathrm{i}g_{\text{eff}}^{-2}\mathrm{D} + \xi_{\text{eff}} - \sum_I \phi_I \bar{\phi}_I = 0. \tag{2.5.3}$$

On the space of $\mathcal{Q}$-supersymmetric field configurations (see Section 2.3.3), D vanishes in the bulk and we conclude that

$$\sum_I \phi_I \bar{\phi}_I = \xi_{\text{eff}} \mathbb{1}_N \,, \tag{2.5.4}$$

which, together with $(a + m_I)\phi_I = 0$ imply that the Coulomb branch is lifted, localizing instead to the Higgs branch. Moreover the supersymmetry equations at the poles yield

$$\sigma_1 \phi_I \bar{\phi}_I = -\frac{B}{2}\phi_I \bar{\phi}_I = 0 \tag{2.5.5}$$

which by virtue of (2.5.4) imply $B = \sigma_1 = 0$. This leads us directly to the vortex and anti-vortex equations at the north and the south poles.

The contribution of vortices and anti-vortices to the partition function of an $\mathcal{N} = (2,2)$ gauge theory on S^2 can be obtained as follows. Since the vortices and anti-vortices are localized at the poles, these can be studied by restricting the $\mathcal{N} = (2,2)$ gauge theory to the local $\mathbb{R}^2$ flat space near the north and south poles of S^2. Asymptotic infinity of each $\mathbb{R}^2$ is identified with a small latitude circle on S^2 close to the north and south pole respectively. Therefore, the contribution of vortices and anti-vortices is captured by the vortex/anti-vortex partition function of the gauge theory obtained by restricting our $\mathcal{N} = (2,2)$ gauge theory at the poles. As we will see in Section 2.5.2, integrating over vortex and anti-vortex configurations for all Higgs branch vacua exactly reproduces the partition function computed by integrating over the Coulomb branch found in Section 2.4.2.

2.5.2 Vortex partition function

Following the discussion in the last section, in the planes glued to the poles and in the presence of the FI parameter, the supersymmetry equations reduce to

$$(D_1 + \mathrm{i}D_2)\,\phi_I = 0\,, \qquad (\sigma_2 + m_I)\phi_I = 0\,, \qquad F_{12} + \sum_I \phi_I \bar{\phi}_I - \xi_{\text{eff}} = 0\,, \tag{2.5.6}$$

[32] See [BC12] for a choice of V'.

in the plane attached to the north pole, and

$$(D_1 - \mathrm{i}D_2)\,\phi_I = 0\,, \qquad (\sigma_2 + m_I)\phi_I = 0\,, \qquad F_{12} - \sum_I \phi_I\bar{\phi}_I + \xi_{\text{eff}} = 0\,, \tag{2.5.7}$$

in the copy of $\mathbb{R}^2$ attached to the south pole. These equations can be recognized as the differential equations describing supersymmetric vortices and anti-vortices in $\mathcal{N} = (2,2)$ supersymmetric gauge theories. Therefore, in our localization computation we must integrate over the moduli space of solutions of vortices at the north pole and anti-vortices at the south pole. For simplicity, we discuss their contribution to the partition function for $\mathcal{N} = (2,2)$ SQCD with $U(N)$ gauge group and N_f fundamental chiral multiplets and $\widetilde{N_f}$ anti-fundamental chiral multiplets.

Since the vortices and anti-vortices exist only in the Higgs phase, let us first work out the vacuum structure in the Higgs phase. We first note that vortices can only exist in vacua in which the anti-fundamental fields vanish. This follows from the known mathematical result that the vortex equations for an anti-fundamental field have no non-zero smooth solution when the background field is a connection of a bundle with positive first Chern class $c_1 = k > 0$. The vortex equations (2.5.6) and (2.5.7) then imply that exactly N chiral multiplets take non-zero values, and diagonalizing $\sigma_2 = \text{diag}(a_1, \cdots, a_N)$, one obtains that each Higgs branch of solutions to these equations is labelled by a set of distinct integers $1 \leq p_1 < \cdots < p_N \leq N_f$, with

$$a_i + m_{p_i} = 0 \qquad i = 1, \ldots, N\,, \tag{2.5.8}$$

up to permutations of integers p_i. The contribution from vortices and anti-vortices depends on the choice of Higgs branch components. In each of these components, the $U(N) \times S[U(N_f) \times U(\widetilde{N_f})]$ symmetry of the theory is broken to

$$S[U(N)_{\text{diag}} \times U(N_f - N)] \times U(1) \times SU(\widetilde{N_f})\,, \tag{2.5.9}$$

where $U(1)$ rotates fundamental and anti-fundamental chiral multiplets equally.

For a given Higgs branch component labeled by $\{p_i\}$, the familiar vortex equations (2.5.6) admit a multidimensional moduli space of solutions which we denote by $\mathcal{M}^{\{p_i\}}_{\text{vortex}}$. Since the vorticity

$$k = \frac{1}{2\pi}\int_{\mathbb{R}^2} \text{Tr}\, F \tag{2.5.10}$$

is quantized, this moduli space splits into disconnected components $\mathcal{M}^{\{p_i\},k}_{\text{vortex}}$, each of which is a Kähler manifold, of dimension $2kN_f$. Taking into account the south pole anti-vortex contributions, we find that the solutions of the localization equations on S^2 span the moduli space

$$\mathcal{F}_{\text{Higgs}} = \bigsqcup_{\{p_i\}} \left[\cup_{k=0}^{\infty}\mathcal{M}^{\{p_i\},k}_{\text{vortex}}\right] \oplus \left[\cup_{l=0}^{\infty}\mathcal{M}^{\{p_i\},l}_{\text{anti-vortex}}\right]\,. \tag{2.5.11}$$

We now argue that the vortex partition function at the poles is captured by the partition function of the $\mathcal{N}=(2,2)$ gauge theory in the Ω-background, which is a supersymmetric deformation of the $\mathcal{N}=(2,2)$ gauge theory in $\mathbb{R}^2$ by a $U(1)_\varepsilon$ equivariant rotation parameter ε. Let us recall that the supercharge with which we localize an $\mathcal{N}=(2,2)$ gauge theory on S^2 obeys

$$\mathcal{Q}^2 = J + \frac{1}{2}R\,. \tag{2.5.12}$$

The key observation is to note that (2.5.12) is precisely the supersymmetry preserved by an $\mathcal{N}=(2,2)$ gauge theory in $\mathbb{R}^2$ when placed in the Ω-background. The rotation generator in the Ω-background corresponds to $J+\frac{1}{2}R$, thus giving rise to the scalar supercharge under $U(1)_\varepsilon$ preserved by an $\mathcal{N}=(2,2)$ theory in the Ω-background. Therefore, the contribution to the partition function of an $\mathcal{N}=(2,2)$ gauge theory on S^2 due to vortices and anti-vortices localized at the poles is captured by the vortex/anti-vortex partition function of the same gauge theory placed in the Ω-background originally studied by Shadchin [Sha06] (see also [Yos11; BTZ11b; MOS11; FKNO12; KKKL12]).

The vortex partition function in the Higgs branch component $\{p_i\}$ of an $\mathcal{N}=(2,2)$ gauge theory in the Ω-background is obtained by performing the functional integral of that theory around the background field configuration of k vortices, and summing over all k. It admits an expansion

$$Z_{\text{vortex}}(\{p_i\}, M^\Omega, \widetilde{M}^\Omega, z_\Omega) = \sum_{k=0}^{\infty} z_\Omega^k Z_k(\{p_i\}, M^\Omega, \widetilde{M}^\Omega)\,, \tag{2.5.13}$$

where $z_\Omega = \exp(2\pi \mathrm{i}\tau_\Omega)$ is the vortex fugacity and $Z_k(\{p_i\}, M^\Omega, \widetilde{M}^\Omega)$ is the equivariant volume of the moduli space of k vortices. The volume is given by

$$Z_k(\{p_i\}, M^\Omega, \widetilde{M}^\Omega) = \int_{\mathcal{M}_{\text{vortex}}^{\{p_i\},k}} e^{\hat{\omega}}\,, \tag{2.5.14}$$

where $\hat{\omega}$ is the $U(1)_\varepsilon$ equivariant closed Kähler form[33] on $\mathcal{M}_{\text{vortex}}^{\{p_i\},k}$. Our computations of the supersymmetry transformations on S^2 in Section 2.3.1 imply that the equivariant rotation parameter ε for the Ω-background theory induced at the poles is given in terms of the radius of the S^2 by

$$\varepsilon = \frac{1}{r}\,. \tag{2.5.15}$$

It is pleasing that the $\mathcal{N}=(2,2)$ theory near the poles yields the Ω-deformed theory, since the integral (2.5.14) for the $\mathcal{N}=(2,2)$ theory in flat space suffers from ambiguities, such as infrared divergences. Fortunately, a closer

[33] The form $\hat{\omega}$ is also equivariant under the action of the residual symmetry of the vacuum over which vortices are considered. See (2.5.16).

inspection of the $\mathcal{N}=(2,2)$ gauge theory on S^2 near the poles cures this problem, yielding finite, unambiguous results. In fact, the Ω-deformation was first introduced to regularize otherwise infrared divergent volume integrals such as (2.5.14).

The vortex partition function of an $\mathcal{N}=(2,2)$ gauge theory in the Ω-background can be computed from the knowledge of the symplectic quotient construction of the vortex moduli space $\mathcal{M}^{\{p_i\},k}_{\text{vortex}}$ given in [HT03; Eto+05]. Some details of this construction are presented in Appendix 2.G. The volume (2.5.14) is then given by the matrix integral of a supersymmetric matrix theory action with $U(k)$ gauge group. This matrix theory can be obtained by dimensionally reducing a certain two-dimensional $\mathcal{N}=(0,2)$ $U(k)$ gauge theory to zero dimensions. This supersymmetric matrix theory inherits the supercharge $\mathcal{Q}$ of the $\mathcal{N}=(2,2)$ theory in the Ω-background as well as an equivariant

$$U(1)_\varepsilon \times S[U(N)_{\text{diag}} \times U(N_f-N)] \times U(1) \times SU(\widetilde{N_f}) \tag{2.5.16}$$

symmetry. The first factor $U(1)_\varepsilon$ is the rotational symmetry of the Ω-background while the rest is the residual symmetry of the vacuum over which vortices are studied. The integral (2.5.14) receives contributions from isolated points in the vortex moduli space $\mathcal{M}^{\{p_i\},k}_{\text{vortex}}$, corresponding to the $\mathcal{Q}$-invariant configurations. These are labeled by a partition of k into N non-negative integers

$$k=\sum_{i=1}^{N} k_i\,. \tag{2.5.17}$$

To each such partition we associate an N-component vector $\vec{k}=(k_1,\ldots,k_N)$, describing how the total vortex number k is distributed among the N Cartan generators in $U(N)$ at this point.

For the choice of Higgs branch component of the $\mathcal{N}=(2,2)$ gauge theory labelled by integers $\{p_i\}\subseteq\{1,\ldots,N_f\}$, the partition function of k-vortices admits the following contour integral representation [Sha06; DGH10] (see Appendix 2.G for details),

$$Z_k(\{p_i\},M^\Omega,\widetilde{M}^\Omega)=\oint_{\Gamma_{\{p_i\},k}}\prod_{I=1}^{k}\frac{d\varphi_I}{2\pi\mathrm{i}}\mathcal{Z}_{\text{vec}}(\varphi)\cdot\mathcal{Z}_{\text{f}}(M^\Omega,\varphi)\cdot\mathcal{Z}_{\text{af}}(\widetilde{M}^\Omega,\varphi) \tag{2.5.18}$$

$$\mathcal{Z}_{\text{vec}}(\varphi)=\frac{1}{k!\varepsilon^k}\prod_{I\neq J}^{k}\frac{\varphi_I-\varphi_J}{\varphi_I-\varphi_J-\varepsilon} \tag{2.5.19}$$

$$\mathcal{Z}_{\text{f}}(M^\Omega,\varphi)=\prod_{I=1}^{k}\prod_{s=1}^{N_f}\frac{1}{\varphi_I-M^\Omega_s} \tag{2.5.20}$$

$$\mathcal{Z}_{\text{af}}(\widetilde{M}^\Omega,\varphi)=\prod_{I=1}^{k}\prod_{t=1}^{\widetilde{N_f}}\left(\varphi_I+\widetilde{M}^\Omega_t\right)\,. \tag{2.5.21}$$

For each Higgs vacuum $\{p_i\}$ and vorticity $\vec{k}$, the integrand in (2.5.18) admits a pole at

$$\varphi_{(i,l)} = M^{\Omega}_{p_i} + (l-1)\varepsilon \qquad l = 1,2,..,k_i \qquad i = 1,\dots,N\,, \tag{2.5.22}$$

and the contour of integration $\Gamma_{\{p_i\},k}$ is carefully chosen to enclose all such poles for $\sum_{i=1}^{N} k_i = k$, and no other. The poles of (2.5.18) can be understood as the location of the fixed points under the action of $\mathcal{Q}$. Each factor in (2.5.18) reflects the contribution of the vortex collective coordinates associated to each of the $\mathcal{N} = (2,2)$ multiplets: the vector multiplet and fundamental and anti-fundamental chiral multiplets. Note here that the mass parameters in the Ω-background theory can be identified with the mass parameters of the theory on S^2,

$$M^{\Omega}_{p_i} = -\mathrm{i}m_{p_i}\,, \qquad M^{\Omega}_{s} = -\varepsilon - \mathrm{i}m_s \ (s \notin \{p_i\})\,, \qquad \widetilde{M}^{\Omega}_{s} = -\mathrm{i}\widetilde{m}_s\,. \tag{2.5.23}$$

We observe the same shift in masses as for $\mathcal{N} = 2$ gauge theories on S^4 found in [OP10]. Performing the contour integral and summing over all vortex charges $\vec{k}$, the vortex partition function for SQCD takes the following form

$$Z_{\text{vortex}}(\{p_i\}, m, \widetilde{m}, z) = \sum_{k_1+\dots+k_N=k} z^{|\vec{k}|} Z_{\vec{k}}(\{p_i\}, m, \widetilde{m})\,, \tag{2.5.24}$$

with

$$\begin{aligned} &Z_{\vec{k}}(\{p_i\}, m, \widetilde{m}) \\ &= \frac{1}{\prod_i k_i!} \frac{\prod_{s=1}^{\widetilde{N_f}} \prod_{i=1}^{N} (-\mathrm{i}rm_{p_i} - \mathrm{i}r\widetilde{m}_s)_{k_i}}{\prod_{i\neq j}(irm_{p_j} - \mathrm{i}rm_{p_i} - k_j)_{k_i} \prod_{s\notin\{p\}}^{N_f} \prod_{i=1}^{N} (1 + \mathrm{i}rm_s - \mathrm{i}rm_{p_i})_{k_i}}\,. \end{aligned} \tag{2.5.25}$$

This expression exactly agrees[34] with the expression (2.4.28) arising from factorization of the Coulomb branch representation of the partition function on S^2. Anti-vortices localized at the south pole provide an identical contribution, expanded in terms of the anti-vortex fugacity $\bar{z}$. The one loop determinant must be evaluated at the location of the Higgs branches, where there is a zero mode. Removing the zero mode amounts to taking the residue of the one-loop determinant. Summing over Higgs branch components finally leads to the Higgs branch representation of the partition function of $\mathcal{N} = (2,2)$ gauge theories on S^2

$$\begin{aligned} Z_{\text{Higgs}}(m,\tau) = \sum_{\substack{v_i=-m_{p_i} \\ \{p_i\}\subseteq[\![1,N_f,]\!]}} & Z_{\text{cl}}(v,0,\tau) \operatorname*{res}_{a=v} [Z_{\text{one-loop}}(a,0,m)] \\ & Z_{\text{vortex}}(\{p_i\}, m, (-1)^{N_f} z) Z_{\text{vortex}}(\{p_i\}, m, (-1)^{\widetilde{N_f}} \bar{z})\,. \end{aligned} \tag{2.5.26}$$

This matches with the Coulomb branch representation of the partition function computed earlier.

[34] One must analytically continue the twisted masses $m \to \mathrm{M}$ and $\widetilde{m} \to \widetilde{\mathrm{M}}$ to restore non-zero R-charges.

2.6 Gauge theory/Toda correspondence

This section is omitted here as its results are derived and extended in Chapter 3 of this thesis, based on [GLF14]. It presented the relation between the S^2 partition function of SQED and a Toda CFT four-point function.

2.7 Seiberg duality

This section is omitted here as its results are derived and extended in Chapter 4 of this thesis, based on [GLF14]. It presented the analogue of Seiberg duality for $\mathcal{N} = (2,2)$ $SU(N)$ SQCD by comparing sphere partition functions of dual theories, but only in some limits.

2.8 Discussion

In this paper we computed the exact partition function of two-dimensional $\mathcal{N} = (2,2)$ gauge theories on S^2. We have shown that there are two ways of representing the partition function. It can be either written as an integral over the Coulomb branch or as a sum over vortices and anti-vortices in the Higgs branch. By explicitly evaluating the integral representation in the Coulomb branch, we find exact agreement with the Higgs branch representation of the partition function. Quite pleasingly, despite that we are integrating over different field configurations, the two results give rise to the same partition function.

The Coulomb branch representation is found by integrating over $\mathcal{Q}$-invariant field configurations that are saddle points of the deformation action. Since our deformation term does not contain a term linear in D, the intersection of the supersymmetry fixed point equations with the saddle point equations completely lifts configurations in the Higgs branch, giving rise, as supersymmetric saddle points, to the Coulomb branch configurations $\mathcal{F}_{\text{Coulomb}}$, which we integrate over with a specific measure determined by the one-loop determinants. This implies, in particular, that the vortex and anti-vortex configurations allowed at the poles by the supersymmetry equations are forbidden. The same result can be more straightforwardly obtained by localizing the path integral with respect to different supercharges, concretely Q_1 and Q_2. In this approach, the supersymmetry equations alone forbid any non-trivial configurations in the Higgs phase while precisely reproducing the Coulomb phase field configurations $\mathcal{F}_{\text{Coulomb}}$.

The Higgs branch representation is instead found by integrating over $\mathcal{Q}$-invariant field configurations that are saddle points of a deformed action that does contain a term linear in D. In this case, the intersection of the supersymmetry equations with the equations of motion completely lifts the Coulomb branch. However, the equations now allow for non-trivial field configurations supported in the Higgs phase, which we have denoted by $\mathcal{F}_{\text{Higgs}}$. These field configurations describe vortex and anti-vortex excitations at the poles of the S^2 around each of the Higgs branches of the theory. In this Higgs branch representation, the partition function is written as a sum over Higgs branches of the product of the vortex partition function at the north pole with the anti-vortex partition function at the south pole. The deformed action that we have considered to obtain the Higgs branch representation is the same deformed action as before, but now the saddle point equations are analyzed at a large finite value of the parameter multiplying the deformation term. A more desirable and precise way to arrive at the same conclusion would be to localize the path integral with a different deformation term $\delta_Q V$ that, in the limit when the parameter multiplying it goes to infinity, yields a non-trivial linear term in D. It would be interesting to explicitly construct such a deformation term.

Conceptually, the fact that a correlation function in a supersymmetric gauge theory may admit multiple representations can be understood as follows. When computing a supersymmetric path integral by supersymmetric localization, several choices are available, including the choice of supercharge and of deformation term with which to localize (see Section 2.3 for details). Under mild conditions, the localization principle guarantees that the path integral is independent of these choices. For different choices, however, the path integral may localize to different supersymmetric field configurations and therefore provide alternative representations of the same correlation function. This general picture is behind the equivalence we find between the Coulomb and Higgs branch representation of the partition function of $\mathcal{N} = (2,2)$ gauge theories on S^2. It would be very interesting to extend this general picture to find new dual descriptions of correlation functions in supersymmetric gauge theories, as they can lead, at the very least, to novel identities or to a physical derivation of known ones.

The Higgs branch expression for the partition function shares features with the localization computation of the partition function and Wilson loops [Pes07], 't Hooft loops [GOP11] and domain walls [DGG10] in four-dimensional $\mathcal{N} = 2$ gauge theories. These correlation functions receive contributions from non-perturbative field configurations localized at the north and south poles of the corresponding sphere. In four dimensions they are due to instantons and anti-instantons, while in two dimensions the path integral is a sum over vortices at the north pole and anti-vortices at the south pole. In four dimensions the contribution of instantons and anti-instantons are captured by the instanton partition function [MNS97; Nek02], while the contribution of vortices and anti-vortices are captured by the vortex partition function [Sha06] (see also [Yos11; BTZ11b; MOS11; FKNO12; KKKL12]). An important qualitative difference, however, is that instantons and anti-instantons appear in the Coulomb phase while vortices and anti-vortices can only appear as non-trivial field configurations in the

Higgs phase. Furthermore, the four-dimensional correlation functions do not have a known dual description, while in two dimensions we find that the partition function admits a Coulomb branch representation.

Several applications and correspondences emerge from our results. A correspondence between the partition function of $\mathcal{N} = (2,2)$ gauge theories on S^2 and correlation functions in Liouville/Toda CFT has been found, extending the AGT correspondence [AGT09] (see also [Wyl09]). We have explicitly presented the A_{N_f-1} Toda representation of the partition function of SQED with N_f electrons and N_f positron chiral multiplet fields, leaving the more complete correspondence for other theories to a separate publication [GLF14]. This correspondence can be enriched by adding defects both in gauge theory and in Toda as in [Ald+09; DGOT09; DGG10] (see also [Pas10]) and it would be interesting to establish a detailed dictionary between gauge theory and Toda CFT. In fact, we have already found the effect of inserting a supersymmetric Wilson loop in (2.4.20). When the gauge group contains $U(1)$ factors, a Wilson loop insertion effectively shifts the FI parameter ξ as well as a the topological term ϑ. In the Toda CFT description, this corresponds to changing the moduli of the Riemann surface in the holomorphic sector and anti-holomorphic sector of the CFT differently. This can be realized by the insertion in Toda CFT of the complex-structure–changing topological defect operator introduced in [DGG10].

Since the correlation functions of Toda CFT are modular invariant, this correspondence implies that the gauge theories that admit a Toda CFT representation enjoy quite remarkable modularity properties in the complexified gauge theory parameters τ (2.2.9). In particular, this implies that the results from $\xi > 0$ to $\xi < 0$ are related by analytic continuation, and that the partition function in the two regimes are the same. In the example of SQED, the $\xi > 0$ regime corresponds to the factorization of the Toda CFT correlator in the s-channel, and individual Higgs vacua, labelled by masses of the fundamental chiral multiplets, match precisely with the N_f channels allowed by the fusion of the degenerate insertion with the operator which encodes the fundamental masses. The $\xi < 0$ regime is described by the u-channel factorization, and the sum over Higgs vacua – which correspond to intermediate channels in Toda – is labelled by masses of the anti-fundamental chiral multiplets.

The expansion of the partition function near $\xi = 0$ corresponds to the t-channel factorization. In this limit, the expansion in terms of vortices and anti-vortices in SQED breaks down, and it would be interesting to understand whether this expansion has an alternative description in terms of another two-dimensional gauge theory. Studying the modular properties further may lead to a picture of dualities analogous to [Gai09a]. Relatedly, it would also be interesting to study the combined dynamics of two-dimensional gauge theories on S^2 coupled to four-dimensional $\mathcal{N} = 2$ gauge theories on S^4, and their potential interpretation as surface operators. Extending the analysis to the squashed S^2 is also worth pursuing.

Our findings can also be applied to the study of $\mathcal{N} = (2,2)$ non-linear sigma models with Kähler target spaces, including Calabi-Yau manifolds. The sigma models which describe string propagation in such target spaces enjoy a rich "phase" structure as the complefixied Kähler parameters are varied. This may include the appearance of different geometries in large volume regimes as well as non-geometrical phases. Novel tools and understanding in the study of these questions were introduced in [Wit93], where these theories were given an ultraviolet definition in terms of $\mathcal{N} = (2,2)$ gauge theories. An important insight brought by the gauge theory description was the proposal that topology changing transitions – in particular the flop transition – can be described by analytic continuation in the gauge theory couplings τ. Our exact results for SQED – which include the conifold for $N_f = 2$ – quantitatively demonstrate that the two large volume regimes connected through a flop transition are indeed related by analytic continuation. Furthermore, analytic continuation in the flop transition is realized by crossing symmetry in our correspondence with Toda CFT. Our formulas further demonstrate that the physics at $\xi = 0$, while corresponding to a singular Calabi-Yau geometry, is completely regular for a non-vanishing topological angle ϑ.

Another relevant connection between $\mathcal{N} = (2,2)$ gauge theories in the ultraviolet and non-linear sigma models in the infrared is the transmutation of gauge vortices into worldsheet instantons [Wit93]. Given the exact results for the gauge theory partition function found in this paper, it would be interesting to revisit this connection, which was effectively used in [MP94] to quantitatively study worldsheet instantons.

Finally, we have used our formulas to study Seiberg duality in two dimensions, where we have demonstrated that Seiberg dual pairs have the same partition function in some limits. A very rich set of dualities relating two-dimensional $\mathcal{N} = (2,2)$ theories is mirror symmetry, which relates string theory on different mirror Calabi-Yau manifolds and in different phases. It would be very interesting to extend our results to the case of Landau-Ginzburg models and provide a detailed picture relating these models to their dual gauged linear sigma models. This requires extending our analysis by including twisted chiral multiplets and the allowing for a non-trivial Kähler potential.

Two-dimensional $\mathcal{N} = (2,2)$ non-abelian gauge theories been recently proposed to study non-toric Calabi-Yau manifolds, such as Calabi-Yau manifolds embedded in Grassmannians and determinantal Calabi-Yau varieties [Joc+12]. Due to the strong coupling dynamics of these gauge theories, these models have not been studied much. Our exact results provide a new and powerful tool to investigate the strong coupling dynamics of these $\mathcal{N} = (2,2)$ non-abelian gauged linear sigma models, which may hopefully lead to new insights into this large class of Calabi-Yau manifolds. Another direction to study further is a possible connection of our results to the physics of domain walls in three-dimensional gauge theories on S^3, generalizing the results in [DGG10; HLP10; HHL10]. Finally, our exact results may provide hints on a 4d/2d relation between the geometry of four-manifolds and two-dimensional gauge theories, resulting in a novel correspondences beyond the the 2d/4d relations of [AGT09] and 3d/3d relations of [DGG11b; DGG11a; Yam12].

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2.A Notations and conventions

We use the following conventions for indices:

$$i, j, k, \cdots = 1, 2 \qquad \text{coordinate indices on } S^2 \tag{2.A.1}$$

$$\hat{\imath}, \hat{\jmath}, \hat{k}, \cdots = \hat{1}, \hat{2} \qquad \text{tangent space indices} \tag{2.A.2}$$

$$\alpha, \beta, \gamma, \cdots = 1, 2 \qquad \text{Dirac spinor indices} \tag{2.A.3}$$

$$m, n, p = 1, 2, 3 \qquad \text{indices for } SU(2) \text{ generators} \tag{2.A.4}$$

2.A.1 S^2 conventions

We work in polar coordinates $(x^1, x^2) = (\theta, \varphi)$ where the metric on S^2 can be written as

$$\mathrm{d}s^2 = r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2\right) . \tag{2.A.5}$$

The canonical choice of orientation is

$$\varepsilon_{12} = \sqrt{h}\, \varepsilon_{\hat{1}\hat{2}} = r^2 \sin\theta\,, \tag{2.A.6}$$

with the corresponding volume-form

$$\mathrm{d}^2x\,\sqrt{h} = r^2 \sin\theta\, \mathrm{d}\theta \wedge \mathrm{d}\varphi\ . \tag{2.A.7}$$

The simplest choice of zweibein is

$$e^{\hat{1}} = r\,\mathrm{d}\theta \qquad \text{and} \qquad e^{\hat{2}} = r\sin\theta\,\mathrm{d}\varphi\,, \tag{2.A.8}$$

with the spin connection given by

$$\omega^{\hat{i}\hat{j}} = -\varepsilon^{\hat{i}\hat{j}} \cos\theta\,\mathrm{d}\varphi\ . \tag{2.A.9}$$

By D_i we denote the gauge-covariant derivative

$$D_i = \nabla_i - \mathrm{i}A_i\,, \tag{2.A.10}$$

where ∇_i is the usual covariant derivative and A_i is the gauge field. The corresponding curvature is given by

$$F_{ij} = \varepsilon_{ij} F_{\hat{1}\hat{2}} = \nabla_i A_j - \nabla_j A_i - \mathrm{i}[A_i, A_j]\,. \tag{2.A.11}$$

2.A.2 Spinors and the Clifford algebra

Our conventions for spinors are the same as in [WB92] and are listed below. Let τ_m denote the standard Pauli matrices given by

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.A.12}$$

We take our spinors to be anti-commuting Dirac spinors ϵ_α. These spinors are acted on by the γ-matrices defined by

$$(\gamma_{\hat{m}})_\alpha{}^\beta : \quad \gamma_{\hat{m}} = \tau_{\hat{m}}\ . \tag{2.A.13}$$

Evidently, the matrices $\gamma^{\hat{i}}$ satisfy the two-dimensional Clifford algebra

$$\left\{\gamma^{\hat{i}}, \gamma^{\hat{j}}\right\} = 2\delta^{\hat{i}\hat{j}}\,, \tag{2.A.14}$$

and $\gamma^{\hat{3}} = -\mathrm{i}\gamma^{\hat{1}}\gamma^{\hat{2}}$ is the two-dimensional chirality matrix.[35]

The spinor indices are raised and lowered by the (anti-symmetric) charge conjugation matrix as

$$\epsilon^\alpha = C^{\alpha\beta}\epsilon_\beta \qquad \text{and} \qquad \epsilon_\alpha = C_{\alpha\beta}\epsilon^\beta\,, \tag{2.A.15}$$

[35] In terms of the σ and $\bar{\sigma}$ matrices introduced in [WB92], the γ-matrices are given by $\gamma^m{}_\alpha{}^\beta = \frac{\mathrm{i}}{2}\epsilon^{mnp}\sigma_{n\alpha\dot{\alpha}}\bar{\sigma}_p{}^{\dot{\alpha}\beta}$.

with the consistency condition

$$C_{\alpha\gamma}C^{\gamma\beta} = \delta_\alpha^\beta\,. \tag{2.A.16}$$

More explicitly, we take $C^{12} = C_{21} = 1$ and $C^{21} = C_{12} = -1$.

We adopt the northwest-southeast convention for the implicit contraction of the spinor indices, *i.e.* for two spinors ϵ and λ we define

$$\epsilon\lambda \equiv \epsilon^\alpha\lambda_\alpha = \lambda\epsilon \qquad \text{and} \qquad \epsilon\gamma^{\hat{m}}\lambda \equiv \epsilon^\alpha(\gamma^{\hat{m}})_\alpha{}^\beta\lambda_\beta = -\lambda\gamma^{\hat{m}}\epsilon\,. \tag{2.A.17}$$

Note that the γ-matrices with both spinor indices lowered

$$(\gamma^{\hat{m}})_{\alpha\beta} \equiv C_{\beta\delta}\gamma^{\hat{m}}{}_\alpha{}^\delta\,, \tag{2.A.18}$$

are symmetric and are numerically equal to $(-\tau_3, -\mathrm{i}, \tau_1)$ for $\hat{m} = (1,2,3)$ respectively.

2.A.3 Fierz identities

Let $\bar{\epsilon}, \lambda$ and ϵ be anticommuting spinors. The following Fierz identities are used extensively in our calculations

$$(\bar{\epsilon}\lambda)\epsilon + (\lambda\epsilon)\bar{\epsilon} + (\bar{\epsilon}\epsilon)\lambda = 0\,, \tag{2.A.19}$$

$$(\bar{\epsilon}\gamma_{\hat{m}}\lambda)\gamma^{\hat{m}}\epsilon + (\bar{\epsilon}\lambda)\epsilon + 2(\bar{\epsilon}\epsilon)\lambda = 0\,. \tag{2.A.20}$$

2.B Supersymmetry transformations on S^2

The $\mathcal{N} = (2,2)$ superconformal algebra in the S^2 basis is spanned by the bosonic generators

$$J_m, K_m, R, \mathcal{A}\,, \tag{2.B.1}$$

and the supercharges

$$Q_\alpha, S_\alpha, \bar{Q}_\alpha, \bar{S}_\alpha\,. \tag{2.B.2}$$

J_m generate the $SU(2)$ isometries of S^2 while K_m generate the conformal symmetries of S^2. R and $\mathcal{A}$ are each a $U(1)$ R-symmetry generator, the first being non-chiral and the latter being chiral.

The $\mathcal{N} = (2,2)$ superconformal algebra is given by

$$\begin{aligned}
\{S_\alpha, Q_\beta\} &= \gamma^m_{\alpha\beta}J_m - \frac{1}{2}C_{\alpha\beta}R & [J_m, S^\alpha] &= -\frac{1}{2}\gamma_m^{\alpha\beta}S_\beta & [R, S_\alpha] &= +S_\alpha \\
\{\bar{S}_\alpha, \bar{Q}_\beta\} &= -\gamma^m_{\alpha\beta}J_m - \frac{1}{2}C_{\alpha\beta}R & [J_m, Q^\alpha] &= -\frac{1}{2}\gamma_m^{\alpha\beta}Q_\beta & [R, Q_\alpha] &= -Q_\alpha \\
\{Q_\alpha, \bar{Q}_\beta\} &= \gamma^m_{\alpha\beta}K_m + \frac{1}{2}C_{\alpha\beta}\mathcal{A} & [J_m, \bar{Q}^\alpha] &= -\frac{1}{2}\gamma_m^{\alpha\beta}\bar{Q}_\beta & [R, \bar{Q}_\alpha] &= +\bar{Q}_\alpha \\
\{S_\alpha, \bar{S}_\beta\} &= \gamma^m_{\alpha\beta}K_m - \frac{1}{2}C_{\alpha\beta}\mathcal{A} & [J_m, \bar{S}^\alpha] &= -\frac{1}{2}\gamma_m^{\alpha\beta}\bar{S}_\beta & [R, \bar{S}_\alpha] &= -\bar{S}_\alpha
\end{aligned}$$

$$\begin{aligned}
&[J_m, J_n] = \mathrm{i}\epsilon_{mnp}J^p && [K_m, S^\alpha] = -\frac{1}{2}\gamma_m^{\alpha\beta}\bar{Q}_\beta && [\mathcal{A}, S_\alpha] = \bar{Q}_\alpha \\
&[K_m, K_n] = -\mathrm{i}\epsilon_{mnp}J^p && [K_m, Q^\alpha] = -\frac{1}{2}\gamma_m^{\alpha\beta}\bar{S}_\beta && [\mathcal{A}, Q_\alpha] = -\bar{S}_\alpha \\
&[J_m, K_n] = \mathrm{i}\epsilon_{mnp}K^p && [K_m, \bar{Q}^\alpha] = -\frac{1}{2}\gamma_m^{\alpha\beta}S_\beta && [\mathcal{A}, \bar{Q}_\alpha] = -S_\alpha \\
& && [K_m, \bar{S}^\alpha] = -\frac{1}{2}\gamma_m^{\alpha\beta}Q_\beta && [\mathcal{A}, \bar{S}_\alpha] = Q_\alpha\,.
\end{aligned} \tag{2.B.3}$$

This algebra admits a $\mathbb{Z}_2$ automorphism, under which

$$\begin{aligned}
J_m, R, Q_\alpha, S_\alpha &\to J_m, R, Q_\alpha, S_\alpha \\
K_m, \mathcal{A}, \bar{Q}_\alpha, \bar{S}_\alpha &\to -K_m, -\mathcal{A}, -\bar{Q}_\alpha, -\bar{S}_\alpha\,.
\end{aligned} \tag{2.B.4}$$

Therefore, $\{J_m, R, S, Q\}$ generate a subalgebra. It is the $SU(2|1)$ algebra in (2.2.2), which describes the $\mathcal{N} = (2,2)$ supersymmetry algebra on S^2.

2.B.1 Realization of $SU(2|1)$ on the fields

A simple way to obtain the $SU(2|1)$ supersymmetry transformations is to first construct the $\mathcal{N} = (2,2)$ superconformal transformations and then restrict to those of the $SU(2|1)$ subalgebra. This logic applies in any dimension and gives a first principles construction of the supersymmetry transformations that does not require guesswork, at least as long as the space admits a conformal Killing spinor.

The superconformal transformations are easily obtained from the Poincaré supersymmetry transformations in flat space by demanding that once the flat metric is replaced by a curved metric, that the supersymmetry transformations are covariant under Weyl transformations. In this process, the constant supersymmetry parameters of flat space are replaced by conformal Killing spinors, which obey

$$\nabla_i\epsilon = \gamma_i\tilde{\epsilon} \qquad \nabla_i\bar{\epsilon} = \gamma_i\tilde{\bar{\epsilon}}\,. \tag{2.B.5}$$

Using that the fields and conformal Killing spinors transform with definite weight under a Weyl transformation

$$g_{ij} \to e^{2\Omega(x)}g_{ij} \tag{2.B.6}$$

we obtain the required superconformal transformations by imposing Weyl covariance. The terms that need to be modified in the vector and chiral multiplet flat space supersymmetry transformations (which can be obtained by dimensionally reducing the four-dimensional $\mathcal{N} = 1$ supersymmetry transformations in [WB92] to two dimensions) to make them Weyl covariant are[36]

$$\begin{aligned}
\bar\epsilon \not{D}\lambda &\longrightarrow \bar\epsilon \not{D}\lambda - \lambda \not{\nabla}\bar\epsilon \\
\epsilon \not{D}\bar\lambda &\longrightarrow \epsilon \not{D}\bar\lambda - \bar\lambda \not{\nabla}\epsilon \\
\not{D}\sigma_{1,2}\epsilon &\longrightarrow \not{D}\sigma_{1,2}\epsilon + \sigma_{1,2}\not{\nabla}\epsilon \\
\not{D}\sigma_{1,2}\bar\epsilon &\longrightarrow \not{D}\sigma_{1,2}\bar\epsilon + \sigma_{1,2}\not{\nabla}\bar\epsilon \\
\not{D}\phi\epsilon &\longrightarrow \not{D}\phi\epsilon + \frac{q}{2}\phi\not{\nabla}\epsilon \\
\not{D}\bar\phi\bar\epsilon &\longrightarrow \not{D}\bar\phi\bar\epsilon + \frac{q}{2}\phi\not{\nabla}\bar\epsilon \\
\not{D}\psi\epsilon &\longrightarrow \not{D}\psi\epsilon - \frac{q}{2}\psi\not{\nabla}\epsilon \\
\not{D}\bar\psi\bar\epsilon &\longrightarrow \not{D}\bar\psi\bar\epsilon - \frac{q}{2}\bar\psi\not{\nabla}\bar\epsilon\,,
\end{aligned} \tag{2.B.7}$$

where we have used the following Weyl weights w

SUSY		vector multiplet						chiral multiplet					
ϵ	$\bar\epsilon$	A_μ	σ_1	σ_2	λ	$\bar\lambda$	D	ϕ	ψ	F	$\bar\phi$	$\bar\psi$	$\bar F$
$-\frac{1}{2}$	$-\frac{1}{2}$	0	1	1	$\frac{3}{2}$	$\frac{3}{2}$	2	$\frac{q}{2}$	$\frac{q+1}{2}$	$\frac{q+2}{2}$	$\frac{q}{2}$	$\frac{q+1}{2}$	$\frac{q+2}{2}$

where w is the charge $\varphi \to e^{-w\Omega(x)}\varphi$ under the Weyl transformation (2.B.6).

In this way, we obtain the two-dimensional $\mathcal{N} = (2,2)$ superconformal transformations for the vector multiplet

$$\begin{aligned}
\delta A_i &= -\frac{\mathrm{i}}{2}\left(\bar\epsilon\gamma_i\lambda + \epsilon\gamma_i\bar\lambda\right), \\
\delta\sigma_1 &= \frac{1}{2}\left(\bar\epsilon\lambda - \epsilon\bar\lambda\right), \\
\delta\sigma_2 &= -\frac{\mathrm{i}}{2}\left(\bar\epsilon\gamma_{\hat 3}\lambda + \epsilon\gamma_{\hat 3}\bar\lambda\right), \\
\delta \mathrm{D} &= -\frac{\mathrm{i}}{2}\bar\epsilon\left(\not{D}\lambda + [\sigma_1,\lambda] - \mathrm{i}\left[\sigma_2,\gamma^{\hat 3}\lambda\right]\right) \\
&\quad + \frac{\mathrm{i}}{2}\lambda\not{\nabla}\bar\epsilon + \frac{\mathrm{i}}{2}\epsilon\left(\not{D}\bar\lambda - \left[\sigma_1,\bar\lambda\right] - \mathrm{i}\left[\sigma_2,\gamma^{\hat 3}\bar\lambda\right]\right) - \frac{\mathrm{i}}{2}\bar\lambda\not{\nabla}\epsilon, \\
\delta\lambda &= \left(\mathrm{i}\gamma^{\hat 3}F_{\hat 1\hat 2} - \gamma^{\hat 3}\not{D}\sigma_2 + \mathrm{i}\not{D}\sigma_1 - \gamma^{\hat 3}[\sigma_1,\sigma_2] - \mathrm{D}\right)\epsilon + \mathrm{i}\sigma_1\not{\nabla}\epsilon - \sigma_2\gamma^{\hat 3}\not{\nabla}\epsilon, \\
\delta\bar\lambda &= \left(\mathrm{i}\gamma^{\hat 3}F_{\hat 1\hat 2} - \gamma^{\hat 3}\not{D}\sigma_2 - \mathrm{i}\not{D}\sigma_1 + \gamma^{\hat 3}[\sigma_1,\sigma_2] + \mathrm{D}\right)\bar\epsilon - \mathrm{i}\sigma_1\not{\nabla}\bar\epsilon - \sigma_2\gamma^{\hat 3}\not{\nabla}\bar\epsilon,
\end{aligned} \tag{2.B.8}$$

and chiral multiplet

$$\begin{aligned}
\delta\phi &= \bar\epsilon\psi \\
\delta\bar\phi &= \epsilon\bar\psi
\end{aligned}$$

[36] The coefficients of the extra terms are fixed by demanding that the combination transforms *covariantly* under Weyl transformations and, in general, depend on the Weyl weight of the fields as well as the dimension of space.

$$
\begin{aligned}
\delta\psi &= \mathrm{i}\left(\not{D}\phi + \sigma_1\phi - \mathrm{i}\sigma_2\phi\gamma^{\hat{3}} + \frac{q}{2}\phi\not{\nabla}\right)\epsilon + \bar{\epsilon}F \\
\delta\bar{\psi} &= \mathrm{i}\left(\not{D}\bar{\phi} + \bar{\phi}\sigma_1 + \mathrm{i}\bar{\phi}\sigma_2\gamma^{\hat{3}} + \frac{q}{2}\bar{\phi}\not{\nabla}\right)\bar{\epsilon} + \epsilon\bar{F} \\
\delta F &= -\mathrm{i}\left(D_i\psi\gamma^i + \sigma_1\psi - \mathrm{i}\sigma_2\psi\gamma^{\hat{3}} + \lambda\phi + \frac{q}{2}\psi\not{\nabla}\right)\epsilon \\
\delta\bar{F} &= -\mathrm{i}\left(D_i\bar{\psi}\gamma^i + \bar{\psi}\sigma_1 + \mathrm{i}\bar{\psi}\sigma_2\gamma^{\hat{3}} - \bar{\phi}\bar{\lambda} + \frac{q}{2}\bar{\psi}\not{\nabla}\right)\bar{\epsilon}\,.
\end{aligned}
\tag{2.B.9}
$$

The spinors ϵ and $\bar{\epsilon}$ serve as the parameters of the superconformal transformations, such that each independent conformal Killing spinor is associated with one of the supercharges in the superconformal algebra. On S^2, we can take the conformal Killing spinors to satisfy

$$
\nabla_i\epsilon_s = \frac{s}{2r}\gamma_i\gamma^{\hat{3}}\epsilon_s \qquad \text{and} \qquad \nabla_i\bar{\epsilon}_{\bar{s}} = \frac{\bar{s}}{2r}\gamma_i\gamma^{\hat{3}}\bar{\epsilon}_{\bar{s}} \tag{2.B.10}
$$

with $s, \bar{s} = \pm$. There are four independent solutions to these equations

$$
\epsilon_s = \exp\left(-\frac{\mathrm{i}\theta}{2}\gamma_{\hat{2}}\right)\exp\left(\frac{\mathrm{i}\varphi}{2}\gamma^{\hat{3}}\right)\epsilon_{\circ}^{s}\,, \tag{2.B.11}
$$

$$
\bar{\epsilon}_{\bar{s}} = \exp\left(+\frac{\mathrm{i}\theta}{2}\gamma_{\hat{2}}\right)\exp\left(\frac{\mathrm{i}\varphi}{2}\gamma^{\hat{3}}\right)\bar{\epsilon}_{\circ}^{\bar{s}}\,. \tag{2.B.12}
$$

parametrized by four independent constant spinors $\epsilon_{\circ}^{\pm}$ and $\bar{\epsilon}_{\circ}^{\pm}$. A general superconformal transformation is then generated by a linear combination of the supercharges parametrized as follows

$$
\delta_{\epsilon_+} = \epsilon_{\circ}^{+}\tilde{\gamma}_+ Q\,, \qquad \delta_{\epsilon_-} = \epsilon_{\circ}^{-}\tilde{\gamma}_-\bar{S}\,, \qquad \bar{\delta}_{\bar{\epsilon}_+} = \bar{\epsilon}_{\circ}^{+}\tilde{\gamma}_+\bar{Q}\,, \qquad \bar{\delta}_{\bar{\epsilon}_-} = -\bar{\epsilon}_{\circ}^{-}\tilde{\gamma}_- S \tag{2.B.13}
$$

where $\tilde{\gamma}_\pm$ satisfy

$$
\begin{aligned}
\tilde{\gamma}_\pm &= \frac{1}{\sqrt{2}}\left(\mathbb{1} \pm \mathrm{i}\gamma^3\right) = \pm\mathrm{i}\gamma^3\tilde{\gamma}_\mp \\
\tilde{\gamma}_+^2 &= -\tilde{\gamma}_-^2 = \mathrm{i}\gamma^3\,, \quad \tilde{\gamma}_+\tilde{\gamma}_- = \mathbb{1}\,.
\end{aligned}
\tag{2.B.14}
$$

Using the conformal Killing spinor equations above, the superconformal algebra is realized on the vector multiplet fields as

$$
\begin{aligned}
[\delta_\epsilon, \delta_{\bar{\epsilon}}]\,\lambda &= -\mathcal{L}_\xi\lambda + \mathrm{i}\,[\Lambda, \lambda] + \mathrm{i}\frac{s-\bar{s}}{2}\alpha\lambda + \mathrm{i}\frac{s+\bar{s}}{2}\Theta\gamma^{\hat{3}}\lambda - 3\mathrm{i}\frac{s+\bar{s}}{2}\alpha\lambda\,, \\
[\delta_\epsilon, \delta_{\bar{\epsilon}}]\,\bar{\lambda} &= -\mathcal{L}_\xi\bar{\lambda} + \mathrm{i}\left[\Lambda, \bar{\lambda}\right] - \mathrm{i}\frac{s-\bar{s}}{2}\alpha\bar{\lambda} - \mathrm{i}\frac{s+\bar{s}}{2}\Theta\gamma^{\hat{3}}\bar{\lambda} - 3\mathrm{i}\frac{s+\bar{s}}{2}\alpha\bar{\lambda}\,, \\
[\delta_\epsilon, \delta_{\bar{\epsilon}}]\,A_i &= -(\mathcal{L}_\xi A)_i + D_i\Lambda\,, \\
[\delta_\epsilon, \delta_{\bar{\epsilon}}]\,\sigma_1 &= -\mathcal{L}_\xi\sigma_1 + \mathrm{i}\,[\Lambda, \sigma_1] - (s+\bar{s})\Theta\sigma_2 - \mathrm{i}(s+\bar{s})\alpha\sigma_1\,, \\
[\delta_\epsilon, \delta_{\bar{\epsilon}}]\,\sigma_2 &= -\mathcal{L}_\xi\sigma_2 + \mathrm{i}\,[\Lambda, \sigma_2] + (s+\bar{s})\Theta\sigma_1 - \mathrm{i}(s+\bar{s})\alpha\sigma_2\,, \\
[\delta_\epsilon, \delta_{\bar{\epsilon}}]\,\mathrm{D} &= -\mathcal{L}_\xi\mathrm{D} + \mathrm{i}\,[\Lambda, \mathrm{D}] - 2\mathrm{i}(s+\bar{s})\alpha\mathrm{D}\,,
\end{aligned}
\tag{2.B.15}
$$

and $[\delta_\epsilon, \delta_\epsilon] = [\delta_{\bar\epsilon}, \delta_{\bar\epsilon}] = 0$ on all the fields. Therefore $[\delta_\epsilon, \delta_{\bar\epsilon}]$ generates a space-time transformation as well as a gauge transformation, an R and $\mathcal{A}$ R-symmetry transformation and a Weyl transformation. The parameters of these transformations are given by

$$\begin{aligned} \xi^i &= -\mathrm{i}\bar\epsilon\gamma^i\epsilon\,, \\ \Lambda &= (\bar\epsilon\epsilon)\sigma_1 - \mathrm{i}(\bar\epsilon\gamma^{\hat 3}\epsilon)\sigma_2 + \xi^i A_i\,, \\ \Theta &= \frac{1}{2r}\bar\epsilon\epsilon\,, \\ \alpha &= -\frac{1}{2r}\bar\epsilon\gamma^{\hat 3}\epsilon\,, \end{aligned} \tag{2.B.16}$$

where we have omitted the subscript s and $\bar s$ on the spinors. Note that the spacetime transformation is realized by the Lie derivative on bosonic fields and by the Lie-Lorentz derivative (2.2.33) on the fermions. More explicitly, the Lie-Lorentz derivative along the vector field ξ is given by

$$\mathcal{L}_\xi = \xi^j\nabla_j - \mathrm{i}\frac{s-\bar s}{2}\Theta\gamma^{\hat 3}\,. \tag{2.B.17}$$

The superconformal algebra is realized on the chiral multiplet fields as

$$\begin{aligned} [\delta_\epsilon, \delta_{\bar\epsilon}]\,\psi &= -\mathcal{L}_\xi\psi + \mathrm{i}\Lambda\psi + \mathrm{i}\frac{s-\bar s}{2}(1-q)\alpha\psi - \mathrm{i}\frac{s+\bar s}{2}\Theta\gamma^{\hat 3}\psi - \mathrm{i}\frac{s+\bar s}{2}(q+1)\alpha\psi\,, \\ [\delta_\epsilon, \delta_{\bar\epsilon}]\,\bar\psi &= -\mathcal{L}_\xi\bar\psi - \mathrm{i}\bar\psi\Lambda + \mathrm{i}\frac{s-\bar s}{2}(q-1)\alpha\bar\psi + \mathrm{i}\frac{s+\bar s}{2}\Theta\gamma^{\hat 3}\bar\psi - \mathrm{i}\frac{s+\bar s}{2}(q+1)\alpha\bar\psi\,, \\ [\delta_\epsilon, \delta_{\bar\epsilon}]\,\phi &= -\mathcal{L}_\xi\phi + \mathrm{i}\Lambda\phi - \mathrm{i}\frac{s-\bar s}{2}q\alpha\phi - \mathrm{i}\frac{s+\bar s}{2}q\alpha\phi\,, \\ [\delta_\epsilon, \delta_{\bar\epsilon}]\,\bar\phi &= -\mathcal{L}_\xi\bar\phi - \mathrm{i}\bar\phi\Lambda + \mathrm{i}\frac{s-\bar s}{2}q\alpha\bar\phi - \mathrm{i}\frac{s+\bar s}{2}q\alpha\bar\phi\,, \\ [\delta_\epsilon, \delta_{\bar\epsilon}]\,F &= -\mathcal{L}_\xi F + \mathrm{i}\Lambda F + \mathrm{i}\frac{s-\bar s}{2}(2-q)\alpha F - \mathrm{i}\frac{s+\bar s}{2}(q+2)\alpha F\,, \\ [\delta_\epsilon, \delta_{\bar\epsilon}]\,\bar F &= -\mathcal{L}_\xi\bar F - \mathrm{i}\bar F\Lambda + \mathrm{i}\frac{s-\bar s}{2}(q-2)\alpha\bar F - \mathrm{i}\frac{s+\bar s}{2}(q+2)\alpha\bar F\,, \end{aligned} \tag{2.B.18}$$

where the parameters of the transformations are the same as those for the vector multiplet fields (2.B.16).

To obtain the $SU(2|1)$ supersymmetry transformations, we restrict the superconformal transformations (2.B.8) and (2.B.9) we have constructed to those associated with Q_α and S_α, which are parametrized by ϵ_+ and $\bar\epsilon_-$. The corresponding realization of the algebra on the fields is given by (2.B.15) and (2.B.18) with $s=1$ and $\bar s = -1$.

In the main text, we find it convenient to perform the field redefinition $\mathrm{D} \to \mathrm{D} + \sigma_2/r$, after which we obtain the supersymmetry transformations presented in Section 2.2.

2.C Supersymmetric configurations

In this appendix we present the derivation of the choice of SUSY parameters and the corresponding supersymmetric configurations.

2.C.1 Choice of supercharge

The conformal Killing spinor equations on S^2 are

$$\nabla_i \epsilon = +\frac{1}{2r}\gamma_i \gamma^{\hat{3}} \epsilon \,, \tag{2.C.1}$$

$$\nabla_i \bar{\epsilon} = -\frac{1}{2r}\gamma_i \gamma^{\hat{3}} \bar{\epsilon} \,, \tag{2.C.2}$$

with the general solutions of the form

$$\epsilon = \exp\left(-\frac{\mathrm{i}\theta}{2}\gamma_{\hat{2}}\right)\exp\left(\frac{\mathrm{i}\varphi}{2}\gamma^{\hat{3}}\right)\epsilon_\circ \,, \tag{2.C.3}$$

$$\bar{\epsilon} = \exp\left(+\frac{\mathrm{i}\theta}{2}\gamma_{\hat{2}}\right)\exp\left(\frac{\mathrm{i}\varphi}{2}\gamma^{\hat{3}}\right)\bar{\epsilon}_\circ \,. \tag{2.C.4}$$

Here, the hatted γ indices denote the tangent space (flat) indices[37]. The corresponding bilinear $\xi^i = -\mathrm{i}\bar{\epsilon}\gamma^i\epsilon$ is given by

$$\xi^1 = -\cos\varphi\left(\mathrm{i}\bar{\epsilon}_\circ\hat{\gamma}^1\epsilon_\circ\right) - \sin\varphi\left(\mathrm{i}\bar{\epsilon}_\circ\hat{\gamma}^2\epsilon_\circ\right) \,, \tag{2.C.5}$$

$$\xi^2 = -\bar{\epsilon}_\circ\epsilon_\circ + \cot\theta\sin\varphi\left(\mathrm{i}\bar{\epsilon}_\circ\hat{\gamma}^1\epsilon_\circ\right) - \cot\theta\cos\varphi\left(\mathrm{i}\bar{\epsilon}_\circ\hat{\gamma}^2\epsilon_\circ\right) \,. \tag{2.C.6}$$

We wish to find spinors such that ξ^1 vanishes while ξ^2 is a non-zero constant. The vanishing on ξ^1 for all angles φ requires $\bar{\epsilon}_\circ\gamma^1\epsilon_\circ = \bar{\epsilon}_\circ\gamma^2\epsilon_\circ = 0$. This can be achieved by choosing $\epsilon_\circ$ and $\bar{\epsilon}_\circ$ to be chiral spinors with opposite chirality. We choose the constant spinors such that

$$\gamma^{\hat{3}}\epsilon_\circ = +\epsilon_\circ \,, \tag{2.C.7}$$

$$\gamma^{\hat{3}}\bar{\epsilon}_\circ = -\bar{\epsilon}_\circ \,, \tag{2.C.8}$$

and the conformal Killing spinors reduce to

$$\epsilon = \exp\left(-\frac{\mathrm{i}\theta}{2}\gamma_{\hat{2}} + \frac{\mathrm{i}\varphi}{2}\right)\epsilon_\circ \,, \tag{2.C.9}$$

$$\bar{\epsilon} = \exp\left(+\frac{\mathrm{i}\theta}{2}\gamma_{\hat{2}} - \frac{\mathrm{i}\varphi}{2}\right)\bar{\epsilon}_\circ \,. \tag{2.C.10}$$

The spinor bilinears constructed out of these spinors take the form

$$\bar{\epsilon}\epsilon = \bar{\epsilon}_\circ\epsilon_\circ\cos\theta \,, \tag{2.C.11}$$

$$\xi = -\frac{1}{r}\bar{\epsilon}_\circ\epsilon_\circ\frac{\partial}{\partial\varphi} \,, \tag{2.C.12}$$

$$\alpha = -\frac{1}{2r}\bar{\epsilon}_\circ\epsilon_\circ \,. \tag{2.C.13}$$

[37] See Appendix 2.A.

2.C.2 Supersymmetry saddle point equations

Since after localization, only supersymmetric configurations can contribute, we write $\mathcal{Q}f = 0$ for all fermionic fields, with $\mathcal{Q}$ parametrized by the particular choice of ϵ and $\bar{\epsilon}$ we just derived. Let us fix the relative normalization of $\epsilon_\circ$ and $\bar{\epsilon}_\circ$ such that

$$\bar{\epsilon}_\circ = -\mathrm{i}\gamma^{\hat{2}}\epsilon_\circ \tag{2.C.14}$$

We thus obtain the explicit expressions

$$\epsilon = e^{\mathrm{i}\varphi/2}\left(\cos\frac{\theta}{2} - \mathrm{i}\sin\frac{\theta}{2}\gamma^{\hat{2}}\right)\epsilon_\circ \qquad \bar{\epsilon} = e^{-\mathrm{i}\varphi/2}\left(\sin\frac{\theta}{2} - \mathrm{i}\cos\frac{\theta}{2}\gamma^{\hat{2}}\right)\epsilon_\circ \tag{2.C.15}$$

$$\gamma^{\hat{1}}\epsilon = e^{\mathrm{i}\varphi/2}\left(\sin\frac{\theta}{2} - \mathrm{i}\cos\frac{\theta}{2}\gamma^{\hat{2}}\right)\epsilon_\circ \qquad \gamma^{\hat{1}}\bar{\epsilon} = e^{-\mathrm{i}\varphi/2}\left(\cos\frac{\theta}{2} - \mathrm{i}\sin\frac{\theta}{2}\gamma^{\hat{2}}\right)\epsilon_\circ \tag{2.C.16}$$

$$\gamma^{\hat{2}}\epsilon = e^{\mathrm{i}\varphi/2}\left(-\mathrm{i}\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\gamma^{\hat{2}}\right)\epsilon_\circ \quad \gamma^{\hat{2}}\bar{\epsilon} = e^{-\mathrm{i}\varphi/2}\left(-\mathrm{i}\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\gamma^{\hat{2}}\right)\epsilon_\circ \tag{2.C.17}$$

$$\gamma^{\hat{3}}\epsilon = e^{\mathrm{i}\varphi/2}\left(\cos\frac{\theta}{2} + \mathrm{i}\sin\frac{\theta}{2}\gamma^{\hat{2}}\right)\epsilon_\circ \qquad \gamma^{\hat{3}}\bar{\epsilon} = e^{-\mathrm{i}\varphi/2}\left(\sin\frac{\theta}{2} + \mathrm{i}\cos\frac{\theta}{2}\gamma^{\hat{2}}\right)\epsilon_\circ \tag{2.C.18}$$

Thanks to those expressions for various gamma matrices acting on our conformal Killing spinors, $\delta\lambda = 0$ and $\delta\bar{\lambda} = 0$ may be written as

$$\begin{aligned} 0 = \delta\lambda = &\left[\sin\frac{\theta}{2}\left(\mathrm{i}V_{\hat{1}} + V_{\hat{2}}\right) + \mathrm{i}\cos\frac{\theta}{2}\left(V_{\hat{3}} + \mathrm{i}\mathrm{D}\right)\right] e^{\mathrm{i}\frac{\varphi}{2}}\epsilon_0 \\ &+ \left[\cos\frac{\theta}{2}\left(V_{\hat{1}} + \mathrm{i}V_{\hat{2}}\right) - \sin\frac{\theta}{2}\left(V_{\hat{3}} - \mathrm{i}\mathrm{D}\right)\right] e^{\mathrm{i}\frac{\varphi}{2}}\gamma^{\hat{2}}\epsilon_0 \end{aligned} \tag{2.C.19}$$

$$\begin{aligned} 0 = \delta\bar{\lambda} = &\left[\cos\frac{\theta}{2}\left(\mathrm{i}\bar{V}_{\hat{1}} + \bar{V}_{\hat{2}}\right) + \mathrm{i}\sin\frac{\theta}{2}\left(\bar{V}_{\hat{3}} - \mathrm{i}\mathrm{D}\right)\right] e^{-\mathrm{i}\frac{\varphi}{2}}\epsilon_0 \\ &+ \left[\sin\frac{\theta}{2}\left(\bar{V}_{\hat{1}} + \mathrm{i}\bar{V}_{\hat{2}}\right) - \cos\frac{\theta}{2}\left(\bar{V}_{\hat{3}} + \mathrm{i}\mathrm{D}\right)\right] e^{-\mathrm{i}\frac{\varphi}{2}}\gamma^{\hat{2}}\epsilon_0. \end{aligned} \tag{2.C.20}$$

while $\delta\psi = 0$ and $\delta\bar{\psi} = 0$ yields

$$\begin{aligned} 0 = \delta\psi = \mathrm{i}&\left[\sin\frac{\theta}{2}\left(D_-\phi - \mathrm{i}e^{-\mathrm{i}\varphi}F\right) + \cos\frac{\theta}{2}\left(\sigma_1 - \mathrm{i}\sigma_2 + \frac{q}{2r}\right)\phi\right] e^{\mathrm{i}\frac{\varphi}{2}}\epsilon_\circ \\ &+ \left[\cos\frac{\theta}{2}\left(D_+\phi - \mathrm{i}e^{-\mathrm{i}\varphi}F\right) + \sin\frac{\theta}{2}\left(\sigma_1 + \mathrm{i}\sigma_2 - \frac{q}{2r}\right)\phi\right] e^{\mathrm{i}\frac{\varphi}{2}}\gamma^{\hat{2}}\epsilon_\circ, \end{aligned} \tag{2.C.21}$$

$$\begin{aligned} 0 = \delta\bar{\psi} = \mathrm{i}&\left[\cos\frac{\theta}{2}\left(D_-\bar{\phi} - \mathrm{i}e^{\mathrm{i}\varphi}\bar{F}\right) + \sin\frac{\theta}{2}\bar{\phi}\left(\sigma_1 + \mathrm{i}\sigma_2 + \frac{q}{2r}\right)\right] e^{-\mathrm{i}\frac{\varphi}{2}}\epsilon_\circ \\ &+ \left[\sin\frac{\theta}{2}\left(D_+\bar{\phi} - \mathrm{i}e^{\mathrm{i}\varphi}\bar{F}\right) + \cos\frac{\theta}{2}\bar{\phi}\left(\sigma_1 - \mathrm{i}\sigma_2 + \frac{q}{2r}\right)\right] e^{-\mathrm{i}\frac{\varphi}{2}}\gamma^{\hat{2}}\epsilon_\circ\,. \end{aligned} \tag{2.C.22}$$

Here $D_\pm = D_{\hat{1}} \pm \mathrm{i} D_{\hat{2}}$ and for future reference, we define $\sigma_\pm = \sigma_1 \pm \mathrm{i}\sigma_2$. Since $\epsilon_\circ$ and $\gamma^{\hat{2}}\epsilon_\circ$ are linearly independent, each square bracket must vanish separately. Using the reality conditions

$$A_i^\dagger = A_i \qquad \bar{\phi}^\dagger = \phi \qquad \sigma_\pm^\dagger = \sigma_\mp \qquad \bar{F}^\dagger = F \tag{2.C.23}$$

we can write the equations as

$$\begin{aligned} \sin\frac{\theta}{2} D_\pm \sigma_+ + \cos\frac{\theta}{2}\left(F_{\hat{1}\hat{2}} + \frac{\sigma_1}{r} + \mathrm{iD} \mp \mathrm{i}\,[\sigma_1, \sigma_2]\right) &= 0 \\ \cos\frac{\theta}{2} D_\pm \sigma_- - \sin\frac{\theta}{2}\left(F_{\hat{1}\hat{2}} + \frac{\sigma_1}{r} - \mathrm{iD} \pm \mathrm{i}\,[\sigma_1, \sigma_2]\right) &= 0 \end{aligned} \tag{2.C.24}$$

$$\begin{aligned} \sin\frac{\theta}{2}\left(D_-\phi \pm \mathrm{i}e^{-\mathrm{i}\varphi} F\right) + \cos\frac{\theta}{2}\left(\sigma_\mp + \frac{q}{2r}\right)\phi &= 0 \\ \cos\frac{\theta}{2}\left(D_+\phi \pm \mathrm{i}e^{-\mathrm{i}\varphi} F\right) + \sin\frac{\theta}{2}\left(\sigma_\pm - \frac{q}{2r}\right)\phi &= 0\,. \end{aligned} \tag{2.C.25}$$

Taking linear combinations of each set of these equations and using the reality conditions, we obtain the desired SUSY equations

$$\begin{aligned} D_{\hat{2}}\sigma_1 = D_{\hat{2}}\sigma_2 = D_{\hat{1}}\sigma_2 = 0 \qquad & \mathrm{Re}\,\mathrm{D} = [\sigma_1, \sigma_2] = 0 \\ D_{\hat{1}}\sigma_1 - \mathrm{Im}\,\mathrm{D}\sin\theta = 0 \qquad & F_{\hat{1}\hat{2}} + \frac{\sigma_1}{r} - \mathrm{Im}\,\mathrm{D}\cos\theta = 0\,, \end{aligned} \tag{2.C.26}$$

$$\begin{aligned} \cos\frac{\theta}{2} D_+\phi + \sin\frac{\theta}{2}\left(\sigma_1 - \frac{q}{2r}\right)\phi = 0 \qquad & \sigma_2\phi = 0 \\ \sin\frac{\theta}{2} D_-\phi + \cos\frac{\theta}{2}\left(\sigma_1 + \frac{q}{2r}\right)\phi = 0 \qquad & F = 0\,. \end{aligned} \tag{2.C.27}$$

2.C.3 $\mathcal{Q}$-supersymmetric field configurations

To compute the path integral using localization on supersymmetric configurations, we need to find the space of solutions of equations (2.C.26) and (2.C.27).

Let us first analyze the vector multiplet field equations.

For concreteness, we choose the coordinate patch $0 < \theta < \pi$, where we can gauge away the dθ-component of the gauge field[38]. The general solution to (2.C.26) takes the form

$$A = r\sigma_1 \cos\theta\, \mathrm{d}\varphi, \qquad \sigma_1 = \sigma_1(\theta), \qquad \sigma_2 = \sigma_2(\varphi)\,. \tag{2.C.28}$$

Imposing the chiral multiplet supersymmetry equations (2.C.27) and plugging in the above form for the vector multiplet fields we obtain

$$\begin{aligned} \left(\sin\theta\,\partial_\theta + \frac{q}{2}\cos\theta + \sigma_1\right)\phi = 0 \qquad & F = 0 \\ \left(\partial_\varphi + \mathrm{i}\frac{q}{2}\right)\phi = 0 \qquad & (\sigma_2 + m)\phi = 0 \end{aligned} \tag{2.C.29}$$

[38] Every 1-form $w = w_\theta\, \mathrm{d}\theta$ on S^2 is, up to dφ terms, closed and therefore exact – since the $H^1(S^2) = 0$.

where we have also included the mass term which, as explained in Section 2.2 is just a shift in σ_2 by a diagonal matrix valued in the flavor symmetry group. For generic values of R-charges q, the only solution of the above equations which is periodic in φ is

$$\phi = 0 \,. \tag{2.C.30}$$

Consequently, in the absence of *effective* Fayet-Iliopoulos parameters[39], the reality conditions necessary for having a convergent path integral constrain the vector multiplet auxiliary field to vanish, *i.e.*

$$\mathrm{Im}\,\mathrm{D} = -g^2\phi\bar{\phi} = 0 \,. \tag{2.C.31}$$

The vanishing of the auxiliary field in turn forces σ_1 to be a constant and the general solution to the supersymmetry equations (2.C.26) and (2.C.27) takes the form

$$\begin{aligned} A &= \frac{B}{2}(\kappa - \cos\theta)\,\mathrm{d}\varphi & \sigma_1 &= -\frac{B}{2r} \\ \sigma_2 &= a & \mathrm{D} &= 0 \\ \phi = \bar{\phi}^\dagger &= 0 & F = \bar{F}^\dagger &= 0 \end{aligned} \tag{2.C.32}$$

where $\delta A = \frac{\kappa B}{2}\,\mathrm{d}\varphi$ is the appropriate gauge transformation to extend the solution to the coordinate patches including the north pole (with $\kappa = 1$) or the south pole (where $\kappa = -1$). We conclude that for general R-charge assignments, $\mathcal{F}_0$ – the space of smooth solutions to the supersymmetry fixed point equations – is parametrized by two constant matrices, a and B, where B is further constrained by the first Chern class quantization to take integer values.

We note in passing that for special values of the R-charges, there exist non-trivial solutions to the chiral multiplet supersymmetry equations which take the form

$$\phi = e^{\frac{\mathrm{i}}{2}(\kappa B - q)\varphi}\frac{(\sin\frac{\theta}{2})^{\frac{B-q}{2}}}{(\cos\frac{\theta}{2})^{\frac{B+q}{2}}}\phi_\circ, \qquad \text{subject to} \qquad (a+m)\phi_\circ = 0 \,. \tag{2.C.33}$$

2.D One-loop determinants

Here we present the computation of the one-loop determinants in the localization computation of the partition function. Our starting point is the quadratic part of the vector and chiral multiplet actions (2.2.4) and (2.2.10)

[39] To localize the path integral, we need to add to the action a $\mathcal{Q}$-exact deformation term with an arbitrary parameter t which we then take to ∞. The effective FI parameters are then ξ/t which vanish in the $t \to \infty$ limit.

in the background (2.3.26) with the addition of the gauge fixing ghosts $\bar{c}, c$ and the Lagrange multiplier b. The various terms are

$$S_{\mathrm{b}}^{\mathrm{v.m.}} = \int \mathrm{d}^2x\,\sqrt{h}\,\mathrm{Tr}\left\{A^i\left(\mathrm{M}^2 + \frac{1}{r^2}\right)A_i + \frac{\mathrm{i}}{2r^2}\varepsilon_{ij}A^i[B,A^j] + \frac{2}{r}\sigma_1\varepsilon^{ij}D_iA_j\right.$$
$$\left. + \sigma_1\left(\mathrm{M}^2 + \frac{1}{r^2}\right)\sigma_1 + \sigma_2\mathrm{M}^2\sigma_2 + \mathrm{D}^2 - \mathcal{G}^2\right\},$$

$$S_{\mathrm{f}}^{\mathrm{v.m.}} = \int \mathrm{d}^2x\,\sqrt{h}\,\mathrm{Tr}\left\{\bar{\lambda}\left(\mathrm{i}\not{D} - \frac{\mathrm{i}}{2r}[B,\cdot\,] + \gamma^{\hat{3}}[a,\cdot\,]\right)\lambda\right\}, \tag{2.D.1}$$

$$S_{\mathrm{ghost}} = \int \mathrm{d}^2x\,\sqrt{h}\,\mathrm{Tr}\left\{\bar{c}\mathrm{M}^2c - b\mathcal{G}(A_i,\sigma_1,\sigma_2)\right\}, \tag{2.D.2}$$

$$S_{\mathrm{b}}^{\mathrm{c.m.}} = \int \mathrm{d}^2x\ \sqrt{h}\ \left\{\bar{\phi}\left(\mathrm{M}^2 + \mathrm{i}\frac{q-1}{r}a - \frac{q^2-2q}{4r^2}\right)\phi + \bar{F}F\right\}, \tag{2.D.3}$$

$$S_{\mathrm{f}}^{\mathrm{c.m.}} = \int \mathrm{d}^2x\ \sqrt{h}\ \left\{\bar{\psi}\left(-\mathrm{i}\not{D} - \frac{\mathrm{i}}{2r}B - \left(a + \frac{\mathrm{i}q}{2r}\right)\gamma^{\hat{3}}\right)\psi\right\}, \tag{2.D.4}$$

where $\mathcal{G}$ is the gauge fixing condition corresponding to the choice of gauge

$$\mathcal{G}(A_i,\sigma_1,\sigma_2) = D_iA^i + \frac{\mathrm{i}}{2r}[B,\sigma_1] - \mathrm{i}[a,\sigma_2] = 0, \tag{2.D.5}$$

and M^2 is given by

$$\mathrm{M}^2 = -D_i^2 + \frac{1}{4r^2}B^2 + a^2\,, \tag{2.D.6}$$

where a and B act in the appropriate representations. We note that (2.D.5) is the background gauge field choice $D_MA^M = 0$ in four dimensions dimensionally reduced to two dimensions. This choice simplifies computations considerably.

The integral over b imposes the background field gauge (2.D.5) while integrating out the auxiliary fields D and F yields a trivial factor. We now analyze the rest.

2.D.1 Dirac operator in monopole background

Before computing the one-loop determinant contribution of fermionic fields, let us first derive the spectrum of the Dirac operator in the background (2.3.26). Since the index of the Dirac operator, acting in the representation $\mathbf{R}$ of the gauge algebra, is given by

$$\mathrm{ind}(\not{D}) = \frac{1}{2\pi}\int_{S^2}\mathrm{Tr}\,F = \mathrm{Tr}\,B\,, \tag{2.D.7}$$

we anticipate $|\,\mathrm{Tr}\,B|$ zero-modes. Excluding these modes, we may diagonalize the Dirac operator using spinor monopole harmonics. For each weight w

of the representation $\mathbf{R}$ and each mode (J, m) such that $J > |B_w|/2$ and $-J \leq m \leq J$ we have

$$(\mathrm{i}\not{D})_{J,m} = \begin{pmatrix} \lambda_{J,m} & 0 \\ 0 & -\lambda_{J,m} \end{pmatrix} \tag{2.D.8}$$

since $\mathrm{i}\not{D}$ is traceless. The spectrum of $\mathrm{i}\not{D}$ can easily be derived from the spectrum of $-\not{D}^2$ when expressed in terms of the scalar Laplacian

$$(\mathrm{i}\not{D})^2 = \begin{pmatrix} -(D_i^-)^2 + \frac{1-B_w}{2r^2} & 0 \\ 0 & -(D_i^+)^2 + \frac{1+B_w}{2r^2} \end{pmatrix}. \tag{2.D.9}$$

Here $(D_i^\pm)^2 \equiv (\partial_i - \mathrm{i}\frac{B_w \pm 1}{2}\omega_i)^2$ denotes the scalar Laplacian in the monopole background with monopole charge $\frac{B_w \pm 1}{2}$. The connection ω_i is expressed in terms of the spin connection (2.A.9) as $\omega_i = \omega_i^{\hat{1}\hat{2}}$. In the rest of this subsection, we drop the subscript in B_w to avoid cluttering the notation.

The eigen-value of the scalar Laplacian in the (J, m) mode is given by

$$-(D_i^\pm)^2_{J,m} = \frac{J(J+1)}{r^2} - \frac{(B \pm 1)^2}{4r^2}, \tag{2.D.10}$$

where J runs from $\frac{|B \pm 1|}{2}$ to ∞ in integer steps and the multiplicity in each mode is $2J + 1$. Using this expression for the eigenvalues and the relation between the eigenvalues of the scalar Laplacian, which can be easily read off from (2.D.8) and (2.D.9), we conclude that the spectrum of the Dirac operator consists of

$$0, \quad \text{with multiplicity } |B|, \text{ and} \tag{2.D.11}$$

$$+\sqrt{\frac{(J+\frac{1}{2})^2 - (\frac{B}{2})^2}{r^2}}, \quad \text{with multiplicity } 2J+1, \tag{2.D.12}$$

$$-\sqrt{\frac{(J+\frac{1}{2})^2 - (\frac{B}{2})^2}{r^2}}, \quad \text{with multiplicity } 2J+1 \tag{2.D.13}$$

for $J = \frac{|B|+1}{2}, \ldots$ We also note that the fermonic zero-modes are spinors of a definite chirality, which depends on the sign of B.

2.D.2 Chiral multiplet determinant

Using the spectrum of the Dirac operator we just derived, we can easily compute the fermionic determinant of the chiral multiplet. First, note that $\gamma^{\hat{3}}$ anticommutes with $\not{D}$, hence, a shift in $\not{D}$ by $\gamma^{\hat{3}}$ results in a shift in the square of the eigenvalues. Therefore, we have

$$
\begin{aligned}
\det \Delta_{\mathrm{f}}^{\text{c.m.}} &= \det \left[-\mathrm{i}\not{D} - \frac{\mathrm{i}B}{2r} - \left(a + \frac{\mathrm{i}q}{2r}\right)\gamma^{\hat{3}} \right] \qquad (2.\mathrm{D}.14)\\
&= \prod_w (-\mathrm{i})^{B_w} \left(\frac{q+|B_w|}{2r} - \mathrm{i}a_w \right)^{|B_w|} \\
&\times \prod_{J=\frac{|B_w|+1}{2}}^{\infty} \left[-\left(\frac{B_w}{2r}\right)^2 - \left(\frac{(J+\frac{1}{2})^2 - (\frac{B_w}{2})^2}{r^2} + \left(a_w + \frac{\mathrm{i}q}{2r}\right)^2 \right) \right]^{2J+1} \\
&= \prod_w (-\mathrm{i})^{B_w} \prod_{J=0}^{\infty} \Bigg[\left(\frac{J}{r} + \frac{|B_w|+q}{2r} - \mathrm{i}a_w \right)^{2J+|B_w|} \\
&\qquad \times (-1)^{|B_w|} \left(\frac{J+1}{r} + \frac{|B_w|-q}{2r} + \mathrm{i}a_w \right)^{2J+|B_w|+2} \Bigg].
\end{aligned}
$$

Here we have used the notation $x_w \equiv x \cdot w$, where w are the weights of the representation $\mathbf{R}$ under which the chiral multiplet transforms.

The bosonic determinant may be written as

$$
\begin{aligned}
(\det \Delta_{\mathrm{b}}^{\text{c.m.}})^{\frac{1}{2}} &= \prod_w \prod_{J=\frac{|B_w|}{2}}^{\infty} \left[\left(\frac{J+\frac{1}{2}}{r} \right)^2 + \left(a_w + \mathrm{i}\frac{q-1}{2r} \right)^2 \right]^{2J+1} \qquad (2.\mathrm{D}.15)\\
&= \prod_w \prod_{J=0}^{\infty} \left[\left(\frac{J}{r} + \frac{|B_w|+q}{2r} - \mathrm{i}a_w \right) \cdot \left(\frac{J+1}{r} + \frac{|B_w|-q}{2r} + \mathrm{i}a_w \right) \right]^{2J+|B_w|+1}.
\end{aligned}
$$

Putting the two together we have the one-loop contribution from the chiral multiplet fields:

$$
Z_{\text{one-loop}}^{\text{c.m.}}(a,B,m) = \prod_{w\in\mathbf{R}} (-\mathrm{i})^{B_w} \prod_{J=0}^{\infty} (-1)^{|B_w|} \left[\frac{J+1+\frac{|B_w|-q}{2} + \mathrm{i}ra_w}{J+\frac{|B_w|+q}{2} - \mathrm{i}ra_w} \right] \qquad (2.\mathrm{D}.16)
$$

These infinite products can be regularized using Euler's gamma function

$$
\frac{1}{\Gamma(z)} = \left[\prod_{J=0}^{\infty} (z+J) \right]_{\text{reg}} \qquad (2.\mathrm{D}.17)
$$

to yield, in the presence of a twisted mass m introduced by shifting $a \to a+m$

$$
Z_{\text{one-loop}}^{\text{c.m.}}(a,B,m) = \prod_{w\in\mathbf{R}} (-\mathrm{i})^{B_w} (-1)^{|B_w|/2} \frac{\Gamma\left(\frac{q}{2} - \mathrm{i}r(a_w+m) + \frac{|B_w|}{2}\right)}{\Gamma\left(1-\frac{q}{2} + \mathrm{i}r(a_w+m) + \frac{|B_w|}{2}\right)}. \qquad (2.\mathrm{D}.18)
$$

The chiral multiplet determinant has a pole when $a+m$ has a zero and q is a non-positive integer. More precisely, there is a pole whenever $|B| \leq -q$ with

$B-q$ even when acting on ϕ. These poles are due to the zero modes found in (2.3.28), which exist precisely under these conditions. In evaluating the determinant for these tuned values of q, the zero modes must be excluded, thus yielding a finite result.

2.D.3 Vector multiplet determinant

The fermion contribution to the vector multiplet one-loop determinant is the same as that of a chiral multiplet in the adjoint representation with R-charge $q=0$. It is given by

$$\begin{aligned}\det \Delta_{\mathrm{f}}^{\mathrm{v.m.}} &= \prod_{\alpha\in\Delta} (-\mathrm{i})^{B_\alpha} \prod_{J=0}^{\infty} (-1)^{|B_\alpha|} \Bigg[\left(\frac{J}{r} + \frac{|B_\alpha|}{2r} - \mathrm{i}a_\alpha\right)^{2J+|B_\alpha|} \\ &\qquad \left(\frac{J+1}{r} + \frac{|B_\alpha|}{2r} + \mathrm{i}a_\alpha\right)^{2J+|B_\alpha|+2} \Bigg] \\ &= \prod_{\alpha\in\Delta_+} \prod_{J=0}^{\infty} \left\{ \left[\left(\frac{J}{r} + \frac{|B_\alpha|}{2r}\right)^2 + a_\alpha^2 \right]^{2J+|B_\alpha|} \left[\left(\frac{J+1}{r} + \frac{|B_\alpha|}{2r}\right)^2 + a_\alpha^2 \right]^{2J+|B_\alpha|+2} \right\}.\end{aligned} \tag{2.D.19}$$

where $\alpha\in\Delta_+$ are the positive roots of the Lie algebra of G.

In order to compute the contribution from the bosonic fields, we need to write down the mode expansion of the fields. For the scalars fields σ_1 and σ_2, we may use the expansion in the standard scalar monopole harmonics

$$\sigma_s^\alpha = \sum_{J=\frac{|B_\alpha|}{2}}^{\infty} \sum_{m=-J}^{J} \frac{1}{r} \sigma_{s,J,m}^{\alpha} Y_{J,m}^{\frac{|B.\alpha|}{2}} \tag{2.D.20}$$

where we have introduced a factor of $\frac{1}{r}$ for normalization and $s=1,2$. As for the gauge field, the mode expansion is much more subtle. A basis of monopole vector spherical harmonics is given in [Wei93]. Expanding the gauge field in this basis we find

$$A_i^\alpha = \sum_{\lambda=\pm} \sum_{J=J_0^\lambda}^{\infty} \sum_{m=-J}^{J} A_{J,m}^{\alpha,\lambda} \left(C_{J,m}^{\lambda,\frac{B_\alpha}{2}}\right)_i , \tag{2.D.21}$$

where $J_0^\pm = \frac{|B_\alpha|}{2} \mp 1$ for $\frac{|B_\alpha|}{2} \geq 1$ and $J_0^\pm = \frac{|B_\alpha|+1}{2} \mp \frac{1}{2}$ otherwise. The reality condition on the gauge field then implies $A_{-\alpha} = A_\alpha^*$ and for scalars $\sigma_{s,-\alpha} = \sigma_{s,\alpha}^*$. The explicit form of $\left(C_{J,m}^{\lambda,\frac{B_\alpha}{2}}\right)_i$ is not necessary for our computation and will be omitted here. All we need are some basic properties of the basis elements which are

$$\delta_{\lambda}^{\lambda'} \delta_J^{J'} \delta_m^{m'} = \int \mathrm{d}^2 x \sqrt{h} \left(C_{J',m'}^{\lambda',\frac{B_\alpha}{2}} \right)^*_i \left(C_{J,m}^{\lambda,\frac{B_\alpha}{2}} \right)^i , \tag{2.D.22}$$

$$-D_j^2 \left(C_{J,m}^{\lambda,\frac{B_\alpha}{2}} \right)^i = \frac{1}{r^2} \left[J(J+1) - \left(\frac{|B_\alpha|}{2} - \lambda \right)^2 \right] \left(C_{J,m}^{\lambda,\frac{B_\alpha}{2}} \right)^i , \tag{2.D.23}$$

$$D_i \left(C_{J,m}^{\lambda,\frac{B_\alpha}{2}} \right)^i = -\frac{1}{\sqrt{2} r^2} \sqrt{J(J+1) - \frac{|B_\alpha|}{2} \left(\frac{|B_\alpha|}{2} - \lambda \right)} Y_{J,m}^{\frac{|B_\alpha|}{2}} , \tag{2.D.24}$$

$$\mathrm{i}\varepsilon_{ij} \left(C_{J,m}^{\lambda,\frac{B_\alpha}{2}} \right)^j = -\lambda \left(C_{J,m}^{\lambda,\frac{B_\alpha}{2}} \right)_i . \tag{2.D.25}$$

Using the above expansion for the gauge field and the scalars and performing the integral over S^2, the bosonic part of the vector multiplet action in (2.D.1) can be written as

$$\begin{aligned} S_{\mathrm{b}}^{\mathrm{v.m.}} \simeq & \sum_{\lambda=\pm} \sum_{J=J_0^\lambda}^{\infty} \sum_{m=-J}^{J} A_{J,m}^{-\alpha,\lambda} \left[\frac{J(J+1)}{r^2} + a_\alpha^2 + \lambda \frac{B_\alpha}{2r^2} \right] A_{J,m}^{\alpha,\lambda} \\ & - \sum_{\lambda=\pm} \sum_{J=\frac{|B_\alpha|}{2}}^{\infty} \sum_{m=-J}^{J} \sigma_{1,J,m}^{-\alpha} \mathrm{i}\lambda\sqrt{2} \, \frac{\sqrt{J(J+1) - \frac{|B_\alpha|}{2}\left(\frac{|B_\alpha|}{2} - \lambda\right)}}{r^2} A_{J,m}^{\alpha,\lambda} \\ & + \sum_{s=1,2} \sum_{J=\frac{|B_\alpha|}{2}}^{\infty} \sum_{m=-J}^{J} \sigma_{s,J,m}^{-\alpha} \left[\frac{J(J+1)}{r^2} + a_\alpha^2 + \frac{2-s}{r^2} \right] \sigma_{s,J,m}^{\alpha} , \end{aligned}$$

where there is an implicit summation over all roots $\alpha \in \Delta$.

In order to compute the determinant, it is best to break it down into three factors. The first one isolates the $J = \frac{|B_\alpha|}{2} - 1$ contribution, which is only non-trivial when $\frac{|B_\alpha|}{2} - 1$ is non-negative. In this case we have

$$\det(\Delta_{\mathrm{b},1}^{\mathrm{v.m.}}) = \prod_{\alpha\in\Delta, |B_\alpha|\geq 2} \left[\left(\frac{B_\alpha}{2r} \right)^2 + a_\alpha^2 \right]^{|B_\alpha|-1} . \tag{2.D.26}$$

The second factor is

$$\det(\Delta_{\mathrm{b},2}^{\mathrm{v.m.}}) = \frac{\det(\mathrm{M}^2)}{\prod_{\alpha\in\Delta} \left[\left(\frac{B_\alpha}{2r} \right)^2 + a_\alpha^2 \right]^{|B_\alpha|+1}} \tag{2.D.27}$$

where the numerator is just the contribution of σ_2 and the denominator is a factor that we have included to shift the lowest mode of A^- (which has $J = |B_\alpha|/2 + 1$). With this shift, the rest of the determinant is $\det(\Delta_{\mathrm{b},3}^{\mathrm{v.m.}})$

given by the following where $\aleph^{\pm} = \frac{\mathrm{i}}{r^2}\sqrt{\frac{1}{2}\big(J(J+1) - \frac{|B_\alpha|}{2}\big[\frac{|B_\alpha|}{2} \pm 1\big]\big)}$

$$
\begin{aligned}
&\prod_{\alpha} \prod_{J=\frac{|B_\alpha|}{2}}^{\infty} \begin{vmatrix} \frac{J(J+1)-\frac{|B_\alpha|}{2}}{r^2} + a_\alpha^2 & 0 & -\aleph^+ \\ 0 & \frac{J(J+1)+\frac{|B_\alpha|}{2}}{r^2} + a_\alpha^2 & \aleph^- \\ \aleph^+ & -\aleph^- & \frac{J(J+1)+1}{r^2} + a_\alpha^2 \end{vmatrix}^{2J+1} \\
&= \prod_{\alpha\in\Delta} \prod_{J=\frac{|B_\alpha|}{2}}^{\infty} \left[\left(\frac{J(J+1)}{r^2} + a_\alpha^2\right)\left(\frac{J^2}{r^2} + a_\alpha^2\right)\left(\left(\frac{J+1}{r}\right)^2 + a_\alpha^2\right)\right]^{2J+1} \\
&= \det(\mathrm{M}^2) \prod_{\alpha\in\Delta} \prod_{J=0}^{\infty} \left[\left(\left(\frac{J}{r} + \frac{|B_\alpha|}{2r}\right)^2 + a_\alpha^2\right)\left(\left(\frac{J+1}{r} + \frac{|B_\alpha|}{2r}\right)^2 + a_\alpha^2\right)\right]^{2J+|B_\alpha|+1}
\end{aligned}
$$

where

$$
\det(\mathrm{M}^2) = \prod_{\alpha\in\Delta} \prod_{J=\frac{|B_\alpha|}{2}}^{\infty} \left[\frac{J(J+1)}{r^2} + a_\alpha^2\right]^{2J+1} . \tag{2.D.28}
$$

Note the shift in the lowest mode of A^- at the top left component in the matrix. As we mentioned earlier, this a factor that we multiply and divide by hand to avoid isolating the $J = \frac{|B_\alpha|}{2}$ mode. Note also that in this case the off-diagonal terms $(1,3)$ and $(3,1)$ vanish. Including the contribution from the ghosts – which is $\det(\mathrm{M}^2)$ – the one-loop partition function of the vector-multiplet becomes

$$
\begin{aligned}
&\det(\Delta_{\mathrm{b}}^{\text{v.m.}})^{\frac{1}{2}} \;/\; \det(\mathrm{M}^2) \\
&= \frac{\prod_{\alpha\in\Delta_+}\prod_{J=0}^{\infty}\left[\left(\left(\frac{J}{r} + \frac{|B_\alpha|}{2r}\right)^2 + a_\alpha^2\right)\left(\left(\frac{J+1}{r} + \frac{|B_\alpha|}{2r}\right)^2 + a_\alpha^2\right)\right]^{2J+|B_\alpha|+1}}{\prod_{\alpha\in\Delta_+}\left[\left(\frac{B_\alpha}{2r}\right)^2 + a_\alpha^2\right]^{|B_\alpha|+1} \prod_{\alpha\in\Delta_+,|B_\alpha|\geq 2}\left[\left(\frac{B_\alpha}{2r}\right)^2 + a_\alpha^2\right]^{-|B_\alpha|+1}} \\
&= \det(\Delta_{\mathrm{f}}^{\text{v.m.}}) \cdot \prod_{\alpha\in\Delta_+}\left[\frac{1}{\left(\frac{B_\alpha}{2r}\right)^2 + a_\alpha^2}\right]^{|B_\alpha|} \prod_{\alpha\in\Delta_+,|B_\alpha|\geq 2}\left[\frac{1}{\left(\frac{B_\alpha}{2r}\right)^2 + a_\alpha^2}\right]^{1-|B_\alpha|} .
\end{aligned} \tag{2.D.29}
$$

Therefore, we find that

$$
Z_{\text{one-loop}}^{\text{v.m.}}(a, B) = \prod_{\substack{\alpha\in\Delta^+ \\ B_\alpha\neq 0}} \left[\left(\frac{B_\alpha}{2r}\right)^2 + a_\alpha^2\right] . \tag{2.D.30}
$$

2.E One-loop running of FI parameter

Consider a two-dimensional $\mathcal{N} = (2,2)$ gauge theory with a $U(1)$ gauge group factor in the presence of an FI parameter ξ. When the sum of the

$U(1)$ charges of the chiral multiplets $Q = \sum_i Q_i$ is non-vanishing, the FI parameter gets renormalized according to

$$\xi(\mu) = \xi + \frac{1}{2\pi}\sum_j Q_j \ln\left(\frac{\mu}{M_{\text{UV}}}\right). \tag{2.E.1}$$

In our localization computation, some care has been taken to regularize the theory in a $\mathcal{Q}$-invariant way. We accomplish this by introducing an "expectator" chiral multiplet of charge $-Q$, mass M, and R-charge $q = 0$. In this enriched theory the FI parameter does not run. However, we recover the original theory by decoupling the expectator chiral multiplet by taking its mass M to be large. We now demonstrate by analyzing the one-loop determinant of the expectator chiral multiplet that this yields the running of the FI parameter with $M_{\text{UV}} = M$ and $\mu = 1/r$.

The relevant one-loop determinant of the expectator chiral multiplet is

$$\ln Z^{\text{c.m.}}_{\text{one-loop}}(a, B, M) = \ln\left[\frac{\Gamma\left(\frac{QB+q}{2} + \mathrm{i}rQa - \mathrm{i}rM\right)}{\Gamma\left(1 + \frac{QB-q}{2} - \mathrm{i}rQa + \mathrm{i}rM\right)}\right] + O(1)\,. \tag{2.E.2}$$

The asymptotic expansion of $\Gamma(z)$ with large imaginary argument is given by

$$\ln\Gamma(z) = \left(z - \frac{1}{2}\right)\ln z - z + O(1) \tag{2.E.3}$$

where the terms of order 1 depend on the sign of $\operatorname{Im} z$ but are irrelevant for renormalization of ξ. Using this asymptotic form for large mass M in (2.E.2) yields

$$\begin{aligned}\ln Z^{\text{c.m.}}_{\text{one-loop}}(a, B, M) &\underset{rM\gg 1}{\simeq} 2\mathrm{i}rM\,(1 - \ln rM) + (q-1)\ln rM + 2\mathrm{i}rQa\ln rM \\ &= 2\mathrm{i}rM\,(1 - \ln rM) + (q-1)\ln rM + 4\pi\mathrm{i}ra\frac{1}{2\pi}Q\ln\left(\frac{M}{\varepsilon}\right),\end{aligned} \tag{2.E.4}$$

where $\varepsilon = \frac{1}{r}$. Note that the first two terms do not have any physical effect since they just rescale the partition function by an a-independent factor. The last term, however, combines with the on-shell classical piece of the action

$$\ln Z_0 \simeq -4\pi\mathrm{i}ra\xi \tag{2.E.5}$$

to account for the running of the FI parameter

$$\ln Z_0 \cdot Z^{\text{c.m.}}_{\text{one-loop}}(a, B, M) \simeq -4\pi\mathrm{i}ra\xi_{\text{ren}}\,, \tag{2.E.6}$$

with

$$\xi_{\text{ren}} = \xi + \frac{1}{2\pi}\sum_i Q_i \ln\left(\frac{\varepsilon}{M}\right). \tag{2.E.7}$$

2.F Factorization for any $\mathcal{N} = (2,2)$ gauge theory

We repeat in this appendix in full generality the proof of Section 2.4.2 that the partition function can be written as a finite sum of terms, each of which is a product of a holomorphic and an antiholomorphic functions of the complex parameter τ associated to each $U(1)$ gauge factor. We start from (2.4.19) with arbitrary gauge group G and matter representation R, which we recall in a more compact form below as (2.F.6). The vector multiplet one-loop determinant in the original expression can be recast in terms of the one-loop determinant of an adjoint chiral multiplet with $\mathrm{iM} = -1$ (in this appendix we take $r = 1$),

$$\begin{aligned}\prod_{\alpha\in\Delta^+} &\left[(\alpha\cdot a)^2 + \left(\frac{\alpha\cdot B}{2}\right)^2\right] \\ &= \prod_{\alpha\in\Delta^+} \frac{\Gamma(1-\mathrm{i}\alpha\cdot a+|\alpha\cdot B|/2)}{\Gamma(-\mathrm{i}\alpha\cdot a+|\alpha\cdot B|/2)}\frac{\Gamma(1+\mathrm{i}\alpha\cdot a+|\alpha\cdot B|/2)}{\Gamma(\mathrm{i}\alpha\cdot a+|\alpha\cdot B|/2)} \\ &= (-1)^{2\rho\cdot B}\prod_{\alpha\in\Delta}(-1)^{(\alpha\cdot B)^+}\frac{\Gamma(1-\mathrm{i}\alpha\cdot a+|\alpha\cdot B|/2)}{\Gamma(\mathrm{i}\alpha\cdot a+|\alpha\cdot B|/2)}\,,\end{aligned} \tag{2.F.1}$$

where we introduced the Weyl element $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$ (signs cancel thanks to $\alpha\cdot B$ being integers and $\Delta = \Delta^+ \cup -\Delta^+$). The extra sign can be combined with the classical factor as

$$\prod_{\substack{\text{abelian}\\ \text{factors}}} e^{-4\pi\mathrm{i}\xi\,\mathrm{Tr}\,a+\mathrm{i}\vartheta\,\mathrm{Tr}\,B} \prod_{\substack{\text{non-abelian}\\ \text{factors}}} e^{2\pi\mathrm{i}\rho\cdot B} = e^{2\pi\mathrm{i}\mathtt{t}\cdot(\mathrm{i}a+B/2)}e^{-2\pi\mathrm{i}\bar{\mathtt{t}}\cdot(\mathrm{i}a-B/2)} \tag{2.F.2}$$

where the (non-integer) weight $\mathtt{t}$ depends holomorphically on the complexified parameters $\tau = \vartheta/(2\pi) + \mathrm{i}\xi$ for each abelian factor in G:

$$\mathtt{t} = \sum_{\substack{\text{abelian}\\ \text{factors}}} \left(\frac{\vartheta}{2\pi} + \mathrm{i}\xi\right)\mathrm{Tr} + \sum_{\substack{\text{non-abelian}\\ \text{factors}}} \rho\,. \tag{2.F.3}$$

Next we show that in the factor corresponding to one weight w_I of the representation of a chiral multiplet I, the sign can be absorbed by modifying the arguments of Gamma functions,

$$(-1)^{(w_I\cdot B)^+}\frac{\Gamma\left(-\mathrm{iM}_I - \mathrm{i}w_I\cdot a + |w_I\cdot B|/2\right)}{\Gamma\left(1+\mathrm{iM}_I + \mathrm{i}w_I\cdot a + |w_I\cdot B|/2\right)} = \frac{\Gamma\left(-\mathrm{iM}_I - \mathrm{i}w_I\cdot a - w_I\cdot B/2\right)}{\Gamma\left(1+\mathrm{iM}_I + \mathrm{i}w_I\cdot a - w_I\cdot B/2\right)}\,. \tag{2.F.4}$$

When $w_I\cdot B$ is negative, this identity is trivial, while for positive (integer) $w_I\cdot B$ it results from Euler's identity $\Gamma(x)\Gamma(1-x) = \pi/[\sin\pi x]$ and anti-periodicity of the sine function,

$$\begin{aligned}(-1)^{w_I\cdot B}&\pi/\left[\sin\pi\left(-\mathrm{iM}_I - \mathrm{i}w_I\cdot a + w_I\cdot B/2\right)\right] \\ &= \pi/\left[\sin\pi\left(-\mathrm{iM}_I - \mathrm{i}w_I\cdot a - w_I\cdot B/2\right)\right].\end{aligned} \tag{2.F.5}$$

From this we deduce

$$\begin{aligned} Z_{\text{Coulomb}}(\mathrm{M},\mathtt{t},\bar{\mathtt{t}}) = \frac{1}{|\mathcal{W}(G)|} \sum_B \int_{\mathfrak{t}} \mathrm{d}a\, e^{2\pi \mathrm{i}\mathtt{t}\cdot(\mathrm{i}a+B/2)} e^{-2\pi\mathrm{i}\bar{\mathtt{t}}\cdot(\mathrm{i}a-B/2)} \\ \times \prod_{I,w_I} \frac{\Gamma(-\mathrm{i}\mathrm{M}_I - w_I\cdot(\mathrm{i}a+B/2))}{\Gamma(1+\mathrm{i}\mathrm{M}_I + w_I\cdot(\mathrm{i}a-B/2))}\,, \end{aligned} \tag{2.F.6}$$

with a sum ranging over all GNO-quantized B (including gauge equivalent values), an integral ranging over the Cartan subalgebra $\mathfrak{t}$, and a product over weights of the representation R in which the chiral multiplets transform, as well as weights of an additional adjoint representation for the vector multiplet determinant.

Just as we did in Section 2.4.2 for the case of SQCD, we close each of the integration contours in a direction that depends on the matter content and the sign of ξ for each abelian gauge factor. Each factor in the integrand of Z has poles whenever the numerator Gamma function has a non-positive integer argument while the denominator one does not, namely when

$$\mathrm{i}w_I\cdot a = -\mathrm{i}\mathrm{M}_I + |w_I\cdot B|/2 + n \tag{2.F.7}$$

for some non-negative integer n. Evaluating the $N=\operatorname{rank}(\mathfrak{g})$ integrals in (2.F.6) yields a sum over common poles obeying (2.F.7) for N different choices of a flavor I and a weight w_I, such that the chosen w_I span weight space[40]. Explicitly,

$$\mathrm{i}w_j\cdot a = -\mathrm{i}\mathrm{M}_{p_j} + n_j + |w_j\cdot B|/2\,, \qquad \text{for all } 1\le j\le N\,. \tag{2.F.8}$$

Note that the contours do not enclose all such combined poles. The combinations of flavors p_j and weights w_j over which we sum thus obey further constraints, such as restricting p_j to (anti)fundamental flavors in the case of SQCD. Those constraints are complicated to obtain in general, hence preventing this analysis from providing a fully explicit factorized expression of the partition function. However, they do not affect any of the analysis proving that factorization does indeed occur.

We introduce the dual basis to w_j, given by elements λ_j of the Cartan subalgebra such that $w_j\cdot\lambda_k=\delta_{jk}$. For every weight w that appears in the Coulomb branch expression, all $w\cdot\lambda_j$ are rational, and

$$w = \sum_{j=1}^N (w\cdot\lambda_j) w_j\,. \tag{2.F.9}$$

The partition function is expressed in terms of

$$w\cdot(\mathrm{i}a\pm B/2) = \sum_{j=1}^N (w\cdot\lambda_j)(-\mathrm{i}\mathrm{M}_{p_j} + n_j + (w_j\cdot B)^{\pm})\,, \tag{2.F.10}$$

[40]If the chosen w_I did not span weight space, the conditions (2.F.8) would not constrain a to a given element in the Cartan subalgebra.

where we use the notation $(x)^{\pm}=(|x|\pm x)/2$. Contrarily to the SQCD case where all $w\cdot\lambda_j$ are 0 or ± 1, the integers n_j and $(w_j\cdot B)^{\pm}$ may not lead to integer shifts of $w\cdot(\mathrm{i}a\pm B/2)$ hence of the Gamma function arguments. This was a key ingredient in Section 2.4.2 to extract the Pochhammer symbols in terms of which the partition function factorizes. We recover integer shifts by splitting the sums over n_j and $w_j\cdot B$ depending on residues modulo the lowest common denominator μ_j of all $w\cdot\lambda_j$. Namely, for each $1\leq j\leq N$ we use Euclidean division to write

$$n_j+(w_j\cdot B)^{\pm}=k_j^{\pm}\mu_j+d_j^{\pm}\,, \tag{2.F.11}$$

with a quotient $0\leq k_j^{\pm}$ and a remainder $0\leq d_j^{\pm}<\mu_j$. Clearly, each choice of integers $k_j^{\pm}$ and $d_j^{\pm}$ in those ranges corresponds to integers n_j and a vector B in the Cartan subalgebra, determined by

$$n_j=\min(k_j^{+}\mu_j+d_j^{+},k_j^{-}\mu_j+d_j^{-}) \tag{2.F.12}$$

$$w_j\cdot B=k_j^{+}\mu_j+d_j^{+}-k_j^{-}\mu_j-d_j^{-}\,. \tag{2.F.13}$$

However, the element B thus constructed may not obey GNO quantization, which requires that for every weight w,

$$w\cdot B=\sum_{j=1}^{N}(w\cdot\lambda_j)(w_j\cdot B)=\sum_{j=1}^{N}(w\cdot\lambda_j)(k_j^{+}\mu_j+d_j^{+}-k_j^{-}\mu_j-d_j^{-}) \tag{2.F.14}$$

is an integer. Since all $\mu_j(w\cdot\lambda_j)$ are integers, (2.F.14) reduces to a condition on $d_j^{\pm}$, only, with no restriction on $k_j^{\pm}\geq 0$.

Hence, the sums over n and B split into a sum over (allowed combinations of) degeneracy parameters $d_j^{\pm}$, and a sum over vortex parameters $k_j^{\pm}$. We have thus expressed the partition function as

$$\begin{aligned}Z(m,\mathtt{t},\bar{\mathtt{t}})=\sum_{\{(p_j,w_j)\}}\sum_{\{d_j^{\pm}\}}\sum_{k_j^{\pm}\geq 0}\operatorname{res}\Bigg[\\ e^{2\pi\mathrm{i}\sum_{j=1}^{N}(\mathtt{t}\cdot\lambda_j)(-\mathrm{iM}_{p_j}+d_j^{+}+k_j^{+}\mu_j)}e^{-2\pi\mathrm{i}\sum_{j=1}^{N}(\bar{\mathtt{t}}\cdot\lambda_j)(-\mathrm{iM}_{p_j}+d_j^{-}+k_j^{-}\mu_j)}\\ \times\prod_{I,w_I}\frac{\Gamma(-\mathrm{iM}_I-\sum_{j=1}^{N}(w_I\cdot\lambda_j)(-\mathrm{iM}_{p_j}+d_j^{+}+k_j^{+}\mu_j))}{\Gamma(1+\mathrm{iM}_I+\sum_{j=1}^{N}(w_I\cdot\lambda_j)(-\mathrm{iM}_{p_j}+d_j^{-}+k_j^{-}\mu_j))}\Bigg]\,,\end{aligned} \tag{2.F.15}$$

up to constant factors, and replacing the N singular Gamma functions by their residue at that pole. The vorticities $k_j^{\pm}$ introduce integer shifts in the arguments of Gamma functions, indeed, by construction of μ_j, all $\mu_j(w\cdot\lambda_j)$ are integers. This enables us to extract from the summand the factors that only depends on the choice of flavors, weights, and degeneracy parameters,

p_j, w_j, and $d_j^\pm$,

$$Z_{\text{cl}} = e^{2\pi\mathrm{i}\sum_{j=1}^N (\mathtt{t}\cdot\lambda_j)(-\mathrm{i}\mathrm{M}_{p_j}+d_j^+)} e^{-2\pi\mathrm{i}\sum_{j=1}^N (\bar{\mathtt{t}}\cdot\lambda_j)(-\mathrm{i}\mathrm{M}_{p_j}+d_j^-)} \tag{2.F.16}$$

$$\operatorname{res} Z_{\text{one-loop}} = \operatorname{res} \prod_{I,w_I} \gamma\Big(-\mathrm{i}\mathrm{M}_I - \sum_{j=1}^N (w_I\cdot\lambda_j)(-\mathrm{i}\mathrm{M}_{p_j}+d_j^+)\Big)\,, \tag{2.F.17}$$

where, once more, gamma functions should be replaced by their residue when appropriate. After removing these $k_j^\pm$-independent factors, we are left with

$$\sum_{k_j^\pm\geq 0}\Bigg[e^{2\pi\mathrm{i}\sum_{j=1}^N \mu_j(\mathtt{t}\cdot\lambda_j)k_j^+} e^{-2\pi\mathrm{i}\sum_{j=1}^N \mu_j(\bar{\mathtt{t}}\cdot\lambda_j)k_j^-} \\ \times \prod_{I,w_I} \frac{\big(-\mathrm{i}\mathrm{M}_I - \sum_{j=1}^N (w_I\cdot\lambda_j)(-\mathrm{i}\mathrm{M}_{p_j}+d_j^+)\big)_{-\sum_{j=1}^N \mu_j(w_I\cdot\lambda_j)k_j^+}}{\big(1+\mathrm{i}\mathrm{M}_I + \sum_{j=1}^N (w_I\cdot\lambda_j)(-\mathrm{i}\mathrm{M}_{p_j}+d_j^-)\big)_{\sum_{j=1}^N \mu_j(w_I\cdot\lambda_j)k_j^-}}\Bigg] \tag{2.F.18}$$

$$= \sum_{k_j^-\geq 0} \frac{e^{-2\pi\mathrm{i}\sum_{j=1}^N \mu_j(\bar{\mathtt{t}}\cdot\lambda_j)k_j^-}}{\prod_{I,w_I}\big(1+\mathrm{i}\mathrm{M}_I + \sum_{j=1}^N (w_I\cdot\lambda_j)(-\mathrm{i}\mathrm{M}_{p_j}+d_j^-)\big)_{\sum_{j=1}^N \mu_j(w_I\cdot\lambda_j)k_j^-}} \\ \times \sum_{k_j^+\geq 0} \frac{e^{2\pi\mathrm{i}\sum_{j=1}^N \mu_j(\mathtt{t}\cdot\lambda_j)k_j^+}\prod_{I,w_I}(-1)^{\sum_{j=1}^N \mu_j(w_I\cdot\lambda_j)k_j^+}}{\prod_{I,w_I}\big(1+\mathrm{i}\mathrm{M}_I + \sum_{j=1}^N (w_I\cdot\lambda_j)(-\mathrm{i}\mathrm{M}_{p_j}+d_j^+)\big)_{\sum_{j=1}^N \mu_j(w_I\cdot\lambda_j)k_j^+}}\,. \tag{2.F.19}$$

The partition function reduces to a finite sum of factorized terms,

$$Z(\mathtt{t},\bar{\mathtt{t}},\mathrm{M}) = \sum_{\{(p_j,w_j)\}}\sum_{\{d_j^\pm\}} Z_{\text{cl}}(\mathtt{t},\bar{\mathtt{t}},\mathrm{M}) \operatorname{res} Z_{\text{one-loop}}(\mathrm{M}) \\ Z_{\text{vortex}}(\mathtt{t},\mathrm{M}) Z_{\text{anti-vortex}}(\bar{\mathtt{t}},\mathrm{M})\,, \tag{2.F.20}$$

where each of the factors additionally depends on the choice of vacuum $\{p_j, w_j, d_j^\pm\}$. This extends the result of Section 2.4.2 to a general gauge group G and a general chiral multiplet representation R of G.

2.G Vortex partition function

We describe in this appendix the procedure used to evaluate the contribution from vortex (and anti-vortex) configurations. For simplicity, we only consider the case of SQCD, the two-dimensional $\mathcal{N}=(2,2)$ $U(N)$ supersymmetric gauge theory with $N_f \geq N$ fundamental chiral multiplets of masses $(M_1,\ldots,M_{N_f})$ and $\widetilde{N}_f \leq N_f$ anti-fundamental chiral multiplets of masses $(\widetilde{M}_1,\ldots,\widetilde{M}_{N_f})$. The flavour group is $U(1)_{\text{anti-diag}} \times SU(N_f)\times SU(\widetilde{N}_f)$, hence $\sum_{s=1}^{N_f} M_s = \sum_{s=1}^{\widetilde{N}_f} \widetilde{M}_s$.

As we show in Section 2.5, the presence of vortex/anti-vortex solutions requires the scalar field σ_2 to take specific values, labelled by a choice of N masses $M_{p_1}, \ldots, M_{p_N}$. For such a choice of Higgs vacuum, the moduli space of solutions to the vortex equations (2.5.1) splits into discrete components $\mathcal{M}_{\text{vortex}}^{\{p_i\},k}$, where the vorticity k is defined by

$$k = \frac{1}{2\pi}\int_{\mathbb{R}^2} \operatorname{Tr} F\,. \tag{2.G.1}$$

The equivariant volume of the moduli space $\mathcal{M}_{\text{vortex}}$ can be expressed as a finite dimensional integral [Sha06]. We denote by $\hat{\mathbf{M}}$ the diagonal $N \times N$ matrix with eigenvalues M_{p_i}, by $\check{\mathbf{M}}$ the diagonal matrix whose eigenvalues are masses of the other $N_f - N$ (non-excited) fundamental chiral multiplets, and by $\widetilde{\mathbf{M}}$ the matrix of anti-fundamental masses.

2.G.1 Vortex matrix model

The moduli space $\mathcal{M}_{\text{vortex}}^{\{p_i\},k}$ of configurations with k vortices admits an ADHM-like construction, which can be understood as the supersymmetric vacua of a certain gauged matrix model preserving two supercharges [Yos11; HT03; KKKL12]. The relevant representations of the supersymmetry algebra can be obtained from the dimensional reduction of $N = (2, 0)$ supersymmetry in two dimensions. This gauged matrix model involves one $U(k)$ vector multiplet $(\varphi, \lambda, \bar{\lambda}, D)$, and is coupled to one adjoint chiral multiplet (X, χ), N fundamental chiral multiplets (I, μ), $N_f - N$ anti-fundamental chiral multiplet (J, ν) and $\widetilde{N_f}$ fundamental Fermi multiplets (ξ, G). The matrix model preserves three global symmetry groups $U(1)_R$, $U(1)_J$ and $U(1)_A$, which can be identified as the R-symmetry group, the rotational symmetry group J and the axial R-symmetry group of the given two-dimensional theory, respectively. As mentioned before, $U(1)_A$ may suffers from an axial anomaly. Under these three $U(1)$ symmetry groups, the supercharges Q and $\bar{Q}$ have charges $(-1, +1, -1)$ and $(+1, -1, -1)$. For later convenience, we summarize global and gauge charges of the matrix model variables in the table below.

	X	χ	I	μ	J	ν	ξ	$\bar{\varphi}$	λ	$\bar{\lambda}$
$U(1)_R$	0	-1	0	-1	0	-1	-1	0	-1	$+1$
$U(1)_{2J}$	-2	-1	0	$+1$	0	$+1$	$+1$	0	$+1$	-1
$U(1)_A$	0	-1	0	-1	0	-1	$+1$	$+2$	$+1$	$+1$
$U(1)_\varepsilon$	-2	-2	0	0	0	0	0	0	0	0
$U(k)$	**adj**		**k**		$\bar{\mathbf{k}}$		**k**	**adj**		

Here the $U(1)_\varepsilon$ symmetry group can be identified as a twisted rotational symmetry group $J + R/2$ of the two-dimensional theory. Note that the complex scalar field X represents the position of the k vortices while I and

J represent orientation modes. The supersymmetric vacuum equation with a positive FI parameter $r \sim 1/g^2 > 0$ is given by

$$\begin{aligned} &[X, X^\dagger] + II^\dagger - J^\dagger J = r\mathbb{1}_k \\ &\qquad \varphi I - I\hat{\mathbf{M}} = 0 \qquad [\varphi, \bar{\varphi}] = 0 \\ &\qquad J\varphi - \check{\mathbf{M}}J = 0 \qquad [\varphi, X] = 0\,, \end{aligned} \tag{2.G.2}$$

where X, I and J denote $k\times k$, $k \times N$ and $(N_f - N)\times k$ matrices. The choice of Higgs vacuum in the original two-dimensional gauge theory is encoded in the matrices $\hat{\mathbf{M}}$ and $\check{\mathbf{M}}$. The solutions of (2.G.2) describe the moduli space $\mathcal{M}^{\{p_i\},k}_{\text{vortex}}$ of k vortices, and the volume of the moduli space can be identified as the partition function of this matrix model.

2.G.2 Vortex partition function

Since the matrix model describing moduli space of vortices in $\mathbb{R}^2$ has an infinite volume, it must be modified by turning on a chemical potential associated to the twisted rotational symmetry group $U(1)_\varepsilon$. The chemical potential ε can be understood as the Omega deformation parameter in the given two-dimensional theory, which is the inverse radius of the sphere S^2.

In the context of the matrix model, the chemical potential can be introduced by weakly gauging $U(1)_\varepsilon$, hence modifying (2.G.2) to the deformed supersymmetry vacuum equation

$$\begin{aligned} &[X, X^\dagger] + II^\dagger - J^\dagger J = r\mathbb{1}_k \\ &\qquad \varphi I - I\hat{\mathbf{M}} = 0 \qquad [\varphi, \bar{\varphi}] = 0 \\ &\qquad J\varphi - \check{\mathbf{M}}J = 0 \qquad [\varphi, X] = \varepsilon X\,, \end{aligned} \tag{2.G.3}$$

and adding a new (deformed) fermion equation

$$\varphi\xi + \xi\widetilde{\mathbf{M}} = 0\,. \tag{2.G.4}$$

Due to the chemical potential ε, the space of vacua is reduced to isolated points, fixed points of supersymmetry.

We explain how to characterize such fixed points. Suppose without loss of generality that ε is positive definite. One can show from the deformed supersymmetry vacuum equations that $J = 0$ and the N chiral multiplets I are each an eigenvector of the operator φ. More specifically, denoting by $|\alpha\rangle$ an eigenvector of the operator φ with eigenvalue α,

$$I = |M_{p_1}\rangle \oplus \cdots \oplus |M_{p_N}\rangle\,. \tag{2.G.5}$$

Then, the vector space of dimension k on which φ acts can be spanned by generators constructed by successive actions of X on $|M_{p_i}\rangle$

$$|M_{p_i} + l\varepsilon\rangle \overset{\text{def}}{\propto} X^l |M_{p_i}\rangle \qquad (l = 0, 1, .., k_i - 1)\,, \tag{2.G.6}$$

with $\sum_{i=1}^{N} k_i = k$. As a consequence, the fixed points are characterized by N one-dimensional Young diagrams. The number of boxes k_i of the i-th 1-d Young diagram determines the vorticity of the i-th $U(1)$ factor in the Cartan subalgebra of $U(N)$. The matrix components of X are then determined using the first relation of (2.G.3).

The partition function of the matrix model can be reduced to a Gaussian integral around such fixed points. The results are nicely expressed as the following contour-integral expression [Sha06; DGH10]

$$Z_{\vec{k}}(\{p_i\}, M, \widetilde{M}) = \oint_{\Gamma_{\{p_i\},k}} \prod_{I=1}^{k} \frac{d\varphi_I}{2\pi \mathrm{i}} \, \mathcal{Z}_{\text{vec}}(\varphi) \cdot \mathcal{Z}_{\text{fund}}(M, \varphi) \cdot \mathcal{Z}_{\text{anti-fund}}(\widetilde{M}, \varphi) \tag{2.G.7}$$

$$\mathcal{Z}_{\text{vec}}(\varphi) = \frac{1}{k!\,\varepsilon^k} \prod_{I \neq J}^{k} \frac{\varphi_I - \varphi_J}{\varphi_I - \varphi_J - \varepsilon} \tag{2.G.8}$$

$$\mathcal{Z}_{\text{fund}}(M, \varphi) = \prod_{I=1}^{k} \prod_{s=1}^{N_f} \frac{1}{\varphi_I - M_s} \tag{2.G.9}$$

$$\mathcal{Z}_{\text{anti-fund}}(\widetilde{M}, \varphi) = \prod_{I=1}^{k} \prod_{s=1}^{\widetilde{N_f}} \left(\varphi_I + \widetilde{M}_s\right) , \tag{2.G.10}$$

where the contour $\Gamma_{\{p_i\},k}$ is chosen such that it encircles poles at

$$\varphi_I = \varphi_{(i,l)} = M_{p_i} + (l-1)\varepsilon \qquad (l = 1, 2, .., k_i) , \tag{2.G.11}$$

which can be understood as the fixed points (2.G.6). The vortex partition function of the two-dimensional gauge theory in a specific choice of Higgs branch component $\{p_i\}$ thus takes the form

$$Z_{\text{vortex}}(\{p_i\}, M, \widetilde{M}, z) = \sum_{k_1+\cdots+k_N=k} z^{|\vec{k}|} Z_{\vec{k}}(\{p_i\}, M, \widetilde{M}) . \tag{2.G.12}$$

The residues of (2.G.7) can be expressed as Pochhammer raising factorials $(x)_n = x(x+1)\cdots(x+n-1)$ and the full vortex partition function of SQCD in the Higgs vacuum labelled by $\{p_i\}$ is

$$Z^{\text{SQCD}}_{\text{vortex}} = \sum_{\vec{k}} \frac{\left(z^{|\vec{k}|}/(k_1!\cdots k_N!)\right) \prod_{i=1}^{N} \prod_{s=1}^{\widetilde{N_f}} \left(\frac{1}{\varepsilon}(M_{p_i} + \widetilde{M}_s)\right)_{k_i}}{\prod_{i\neq j}^{N} \left(\frac{1}{\varepsilon}(M_{p_i} - M_{p_j}) - k_j\right)_{k_j} \prod_{i=1}^{N} \prod_{s\notin\{p_j\}}^{N_f} \left(\frac{1}{\varepsilon}(M_{p_i} - M_s)\right)_{k_i}} . \tag{2.G.13}$$

2.H $SU(N)$ partition function in various limits

This appendix is omitted in the thesis as it only checked Seiberg duality in very specific limits. Newer results are given in Chapter 4.

Chapter 3

AGT for surface operators

This is the first part (half of Section 1, and Section 2) of the article *M2-brane surface operators and gauge theory dualities in Toda* by Jaume Gomis and the author [GLF14]. Chapter 4 contains the second part of the article (half of Section 1, Section 3 and Appendix B), and Chapter 5 extends its Appendix A. Only minor changes are made.

Abstract. *We give a microscopic two-dimensional $\mathcal{N} = (2, 2)$ gauge theory description of arbitrary M2-branes ending on N_f M5-branes wrapping a punctured Riemann surface. These realize surface operators in four-dimensional $\mathcal{N} = 2$ field theories. We show that the expectation value of these surface operators on the sphere is captured by a Toda CFT correlation function in the presence of an additional degenerate vertex operator labelled by a representation $\mathcal{R}$ of $SU(N_f)$, which also labels M2-branes ending on M5-branes. We prove that symmetries of Toda CFT correlators provide a geometric realization of dualities between two-dimensional gauge theories, including $\mathcal{N} = (2, 2)$ analogues of Seiberg and Kutasov–Schwimmer dualities. As a bonus, we find new explicit conformal blocks, braiding matrices, and fusion rules in Toda CFT.*

3.1 Introduction and conclusions

The traditional order parameters for the phases of four-dimensional gauge theories are the Wilson [Wil74] and 't Hooft [Hoo78] operators. In recent years, the construction of nonlocal surface operators [GW06], which insert probe strings, have enlarged the space of order parameters of gauge theories. Indeed, surface operators can distinguish phases which are otherwise indistinguishable using the Wilson–'t Hooft criteria [GK13].

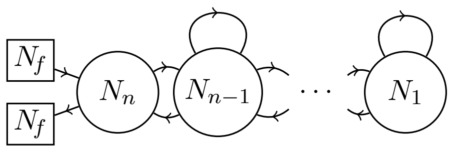
A surface operator can be defined either by specifying a codimension-two singularity for the elementary fields or by coupling a two-dimensional field theory to the bulk four-dimensional one [GW06]. The couplings between bulk and defect degrees of freedom can result in rich dynamics for the combined system, arising from the synergy of two-dimensional and four-dimensional strong coupling dynamics. For a sample of early references on surface operators see [GM07; Guk07; Wit07; BGP07; BHV08; GW08b; DGM08; Gai09b].

Surface operators also play a fundamental role in the six-dimensional $\mathcal{N} = (2,0)$ supersymmetric field theory living on the worldvolume of a collection of N_f coincident and flat M5-branes. A class of surface operators in this theory are labeled by a representation $\mathcal{R}$ of A_{N_f-1} and admit an M-theory realization as a collection of M2-branes ending on the M5-branes along the domain of support of the surface operator.

In this chapter we give a microscopic two-dimensional gauge theory description of all such surface operators when the M5-branes wrap a punctured Riemann surface C [Gai09a]. This realizes a surface operator in a four-dimensional $\mathcal{N} = 2$ gauge theory.

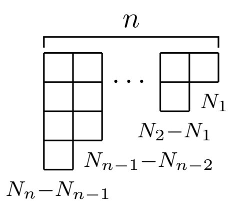
					C		
M5	0	1	2	3	4	5	
M2	0	1					6

The surface operator associated to a collection of M2-branes labeled by a representation $\mathcal{R}$ of A_{N_f-1} corresponds to the following two-dimensional $\mathcal{N} = (2,2)$ gauge theory

$\mathcal{R} \longleftrightarrow$ 

(3.1.1)

coupled to the bulk theory. A cubic superpotential couples each adjoint chiral multiplet to the neighboring bifundamental chiral multiplets. The FI parameters associated to $U(N_j)$ for $j < n$ vanish. The ranks N_j encode the representation $\mathcal{R}$ whose Young diagram



(3.1.2)

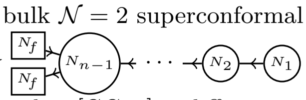
has n columns with $N_n - N_{n-1} \geq N_{n-1} - N_{n-2} \geq \cdots \geq N_2 - N_1 \geq N_1 \geq 0$ boxes.[1]

[1] The highest weight of $\mathcal{R}$ is $\Omega = \sum_{j=1}^{n} \omega_{N_j - N_{j-1}}$ in terms of the fundamental weights ω_K of A_{N_f-1}.

Advances in the computation of supersymmetric partition functions of four-dimensional $\mathcal{N} = 2$ gauge theories on the squashed four-sphere S_b^4 [Pes07; HH12] have resulted in exact formulae for the expectation value of Wilson [Pes07] and 't Hooft operators [GOP11] as functions of the gauge couplings and masses of the hypermultiplets. The gauge theory computation of the expectation value of surface operators supported on a squashed $S^2 \subset S_b^4$ are not yet available. However, recent results in the exact computation of the two-sphere partition function of $\mathcal{N} = (2, 2)$ supersymmetric field theories [BC12; DGLFL12; GL12; DG13], when suitably coupled to those in [Pes07; HH12], provide a concrete avenue of investigation of the expectation value of half-BPS surface operators in four-dimensional $\mathcal{N} = 2$ theories on S_b^4 using Feynman path integrals.

For the four-dimensional $\mathcal{N} = 2$ theories obtained by wrapping M5-branes on punctured Riemann surfaces, also known as class S theories [Gai09a], the S_b^4 partition function in [Pes07; HH12] admits an elegant representation [AGT09] (see also [Wyl09]) in terms of two-dimensional Toda CFT correlation functions. In the correspondence between four-dimensional $\mathcal{N} = 2$ theories and Toda CFT, the expectation value of Wilson and 't Hooft operators on S_b^4 are realized as Toda CFT correlators in the presence of loops operators and topological webs [Ald+09; DGOT09; DGG10] (see also [Pas10; GLF10; Bul13]). Degenerate vertex operators in A_{N_f-1} Toda CFT are conjectured to realize the insertion of a supersymmetric surface operator [Ald+09] (see also [DGH10; Tak10; BTZ11b; BTZ11a; DGLFL12]).

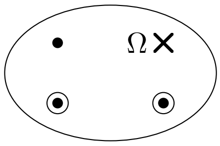
In this chapter we identify the two-dimensional $\mathcal{N} = (2, 2)$ gauge theory that realizes an arbitrary degenerate operator in Toda CFT, which in turn corresponds to an arbitrary M2-brane configuration ending on wrapped M5-branes.[2] A degenerate operator with Toda momentum $\alpha = -b\Omega$, where Ω is the highest weight vector of a representation $\mathcal{R}(\Omega)$ of A_{N_f-1}, corresponds to the quiver gauge theory (3.1.1). The complexified FI parameter associated to the $U(N_n)$ gauge group encodes the position of the degenerate puncture (the other FI parameters must vanish in this correspondence). The surface operator is supported on an $S^2 \subset S_b^4$ invariant under the $U(1) \times U(1)$ isometries of S_b^4.[3]

[2] Another class of surface operators can be realized by M5-branes, and are labeled by a partition ρ of N_f. It was conjectured in [BFFR10] that the instanton partition function of four-dimensional $\mathcal{N} = 2$ $SU(N_f)$ SYM in the presence of such an M5-defect labeled by ρ is the norm of a Whittaker vector in the W-algebra W_ρ. Some checks of this conjecture and generalizations have appeared in [AT10; KPPW10; Wyl10b; Wyl10a; Tac11; KT11; Bel12; Tan13]. We propose that the surface operator associated to an M5-defect labeled by ρ, with $N_f = K_1 + \cdots + K_n$, corresponds to coupling the bulk $\mathcal{N} = 2$ superconformal field theory to the two-dimensional $\mathcal{N} = (2, 2)$ gauge theory  with $N_j = K_1 + \cdots + K_j$. The two-dimensional theory proposed in [GG13] is different.

[3] Degenerate operators with momentum $\alpha = -\Omega/b$ correspond to the same quiver gauge

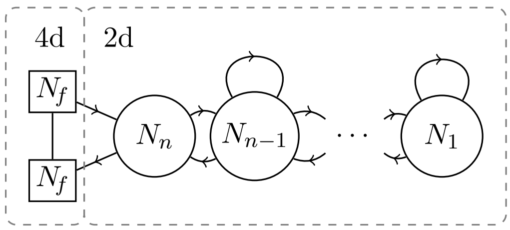
The quiver gauge theory (3.1.1) can be used to construct a surface operator in any four-dimensional $\mathcal{N} = 2$ gauge theory that contains an $SU(N_f) \times SU(N_f) \times U(1)$ flavour or gauge symmetry group. This is the flavour symmetry of the chiral multiplets charged only under the $U(N_n)$ gauge group factor in (3.1.1). A surface operator is constructed by identifying the common $SU(N_f) \times SU(N_f) \times U(1)$ symmetry groups of the four-dimensional and two-dimensional theories.

The simplest four-dimensional $\mathcal{N} = 2$ class S theory in which we can include a surface defect is the theory of N_f^2 hypermultiplets. This theory is realized by wrapping N_f M5-branes on a trinion with two full and one simple puncture. We explicitly show that the partition function of this theory in the presence of the surface operator labelled by a representation $\mathcal{R}(\Omega)$ is given by the Toda four-point function[4] obtained by adding to the trinion a degenerate field with momentum $\alpha = -b\Omega$

$$Z^{\mathcal{R}(\Omega)}_{S^2 \subset S^4_b} = $$ 

. (3.1.3)

The two-dimensional quiver gauge theory (3.1.1) is coupled to the four-dimensional field theory by (weakly) gauging the $SU(N_f) \times SU(N_f) \times U(1)$ flavour symmetry associated to the trinion. The coupling can also be described by a cubic superpotential between the bulk hypermultiplets and the fundamental and antifundamental chiral multiplets on the defect. The combined 4d/2d quiver diagram describing the insertion of the surface operator in this four-dimensional theory is



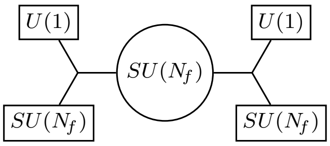
(3.1.4)

This construction can be enriched by allowing one (or both) of the $SU(N_f)$ flavour symmetry groups of (3.1.1) to be coupled to one (or two) $SU(N_f)$ gauge group factors of a four-dimensional theory. An interesting theory where such surface operators can be inserted is four-dimensional $\mathcal{N} = 2$

theory but now supported on the other $U(1) \times U(1)$ invariant $S^2 \subset S^4_b$. The most general degenerate momentum $\alpha = -b\Omega - \Omega'/b$ corresponds to the insertion of the associated surface operators on both S^2's, but with a non-trivial coupling at their intersection points, namely the poles of S^4_b.

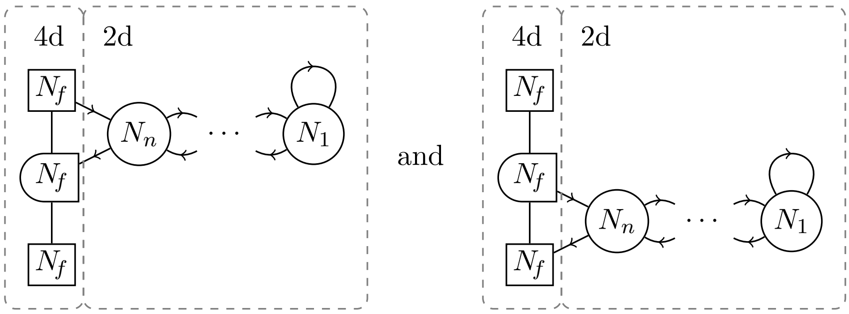
[4] The four point function in (3.1.3) contains full ⊙, simple •, and degenerate ✕ punctures.

superconformal SQCD. The SQCD quiver description



(3.1.5)

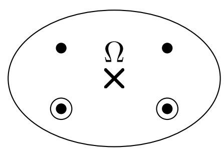
makes an $U(N_f)^2$ flavour symmetry manifest. Both sides of the quiver represent a hypermultiplet transforming in the bifundamental representation of the $SU(N_f)$ gauge group and a $U(N_f)$ flavour group. The two-dimensional gauge theory (3.1.1) can now be coupled to SQCD by identifying the two-dimensional flavour symmetry with the $U(N_f)$ flavour symmetry of either of these hypermultiplets and the $SU(N_f)$ gauge group. The two resulting surface operators in SQCD are realized by the 4d/2d quiver diagrams



(3.1.6)

Here we introduce the hybrid node ▢ to denote a four-dimensional gauge group which gauges a two-dimensional flavour symmetry.

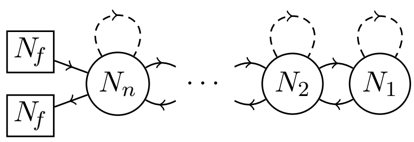
The correspondence we propose between these surface operators and Toda CFT correlators predicts a duality between the two coupled 4d/2d theories in (3.1.6), since $SU(N_f)$ SQCD is the theory on N_f M5-branes wrapping a sphere with two full and two simple punctures. The weakly coupled regime of SQCD corresponds to a pants decomposition where the two simple punctures belong to distinct trinions, which are joined by a thin tube. In this framework, coupling the two-dimensional theory (3.1.1) to either of the two hypermultiplets in SQCD correspond to inserting a degenerate operator with momentum $\alpha = -b\Omega$ in either trinion. The partition functions of the two surface operators in SQCD are thus both realized as the same five-point function of two full, two simple, and an additional degenerate puncture:

$Z[(3.1.6)] =$  . (3.1.7)

In this language, the two 4d/2d quiver diagrams (3.1.6) correspond to two different degeneration limits of the five-point function. It is important to note that this "node-hopping" duality[5] of the 4d/2d theory is distinct from the usual S-duality of four-dimensional $\mathcal{N} = 2$ SQCD.

More generally, the surface operator (3.1.1) can be inserted in an arbitrary class S theory whenever the corresponding Riemann surface has at least one simple puncture.[6] The generalized S-duality symmetry groupoid of a class S theory, which is realized as the Moore-Seiberg groupoid of the punctured Riemann surface, is enriched in the presence of surface operators. The addition of a degenerate puncture to the Riemann surface allows for further pants decomposition of the enriched Riemann surface, and thereby more duality transformations, that go beyond the dualities of the purely four-dimensional theory (see Table 4.2 on page 156). The node-hopping duality (3.1.6) provides an example of a new duality of the 4d/2d system.

In Chapter 4 we "geometrize" dualities of two-dimensional $\mathcal{N} = (2, 2)$ quiver gauge theories in terms of symmetries of Toda CFT correlation functions. The quiver gauge theories we consider are



(3.1.8)

where an adjoint chiral multiplet can be added to any gauge group factor. Each adjoint chiral multiplet is coupled to the neighboring bifundamental chiral multiplets through a cubic superpotential, while nodes without an adjoint chiral multiplet have a quartic superpotential for the neighboring bifundamental chiral multiplets. Finally, the N_f fundamental and antifundamental chiral multiplets have no superpotential coupling.

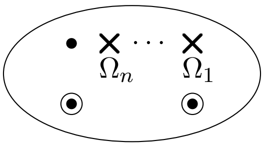
We show that surface operators obtained by coupling the two-dimensional gauge theories (3.1.8) to class S theories have a Toda CFT realization. The quiver with n gauge nodes corresponds to the insertion of n degenerate fields labeled by either symmetric or antisymmetric representations of A_{N_f-1}. The n complexified FI parameters encode the position of the n degenerate punctures. We now build the representations labelling degenerate punctures recursively from the matter content of (3.1.8). If the $U(N_n)$ factor has an adjoint chiral multiplet, then the representation carried by the n-th puncture

[5] The node-hopping duality was first observed in the superconformal index of some 4d/2d theories in [GG13], whose 4d/2d quiver notation we have borrowed. The superconformal index with surface operators has been considered in [Nak11; GRR12; ABFH13; BFHR14]. This observable does not depend on either the 4d or the 2d coupling constants.

[6] Inserting multiple degenerate punctures near distinct simple punctures corresponds to including multiple surface operators built using distinct $SU(N_f) \times SU(N_f) \times U(1)$ groups of the four-dimensional theory. In a pants decomposition where the degenerate punctures are all inserted near the same simple puncture, the surface operator describes a single two-dimensional gauge theory coupled through a given $SU(N_f) \times SU(N_f) \times U(1)$ symmetry group.

is of symmetric type, and otherwise of antisymmetric type. Then sequentially for each gauge group factor $U(N_j)$ from $j = n-1$ to 1, the j-th puncture is labelled by a representation of the same type as the $(j+1)$-th puncture if there is an adjoint chiral multiplet, and otherwise by a representation of the other type. The Young diagram labelling the j-th puncture has $N_j - N_{j-1}$ boxes for $1 \leq j \leq n$, where $N_0 = 0$. See Table 3.1 for useful special cases and Figure 3.1 for a concrete example. The sphere partition function of the surface operator inserted by (3.1.8) in the trinion theory of free hypermultiplets is the Toda CFT correlator

$$Z^{(3.1.8)}_{S^2\subset S^4_b} =$$



. (3.1.9)

We also identify the gauge theory corresponding to multiple degenerate punctures labelled by arbitrary representations of A_{N_f-1}.

The present chapter is devoted to the correspondence between surface operators labeled by two-dimensional quiver gauge theories and Toda CFT degenerate operators. We derive the identification by coupling the two-dimensional theories to the trinion theory of free hypermultiplets, as this choice of a free four-dimensional theory lets us concentrate on the two-dimensional theories. The $S^2 \subset S^4_b$ partition function of these surface operators corresponds to Toda CFT correlators involving one simple, two full, and additional degenerate operators.

After describing our gauge theory setup, and recalling explicit expressions for the S^4_b and S^2 contributions, we proceed to expand S^2 partition functions in various limits and compare them with Toda CFT results. First, we review the case of SQED in some detail in Section 3.3: this $U(1)$ gauge theory corresponds to the insertion of the simplest Toda CFT degenerate vertex operator, labelled by the fundamental representation of A_{N_f-1} [DGLFL12]. Then, we move on to $U(N)$ SQCD in Section 3.4, which corresponds to inserting a degenerate operator labeled by an antisymmetric representation of A_{N_f-1}. Using new braiding matrices derived in Section 5.2, we prove that the Toda CFT correlator and the partition function of the 4d/2d theory are equal. We then describe in Section 3.4.3 how one can decouple some free hypermultiplets from the four-dimensional theory and chiral multiplets from the two-dimensional theory: the procedure translates to a collision limit where two Toda CFT vertex operators combine into an irregular puncture (see also Section 5.6). In Section 3.5, we add adjoint matter to SQCD to get SQCDA, and find that it corresponds to a degenerate operator labelled by a symmetric representation. We then consider SQCDA with different superpotentials in Section 3.5.3 and give their Toda CFT interpretation.

Table 3.1: Correspondence between surface operators defined by $\mathcal{N} = (2, 2)$ gauge theories and degenerate operators labelled by representations of A_{N_f-1}. In the last line, $\hat{z}_j$ is a combination of the FI parameter and theta angle for the group $U(N_j)$.

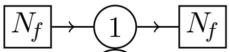
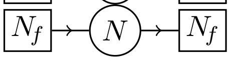
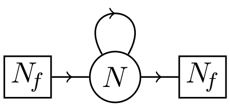
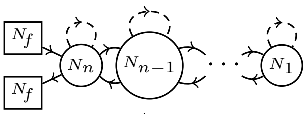
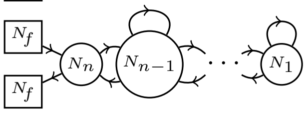
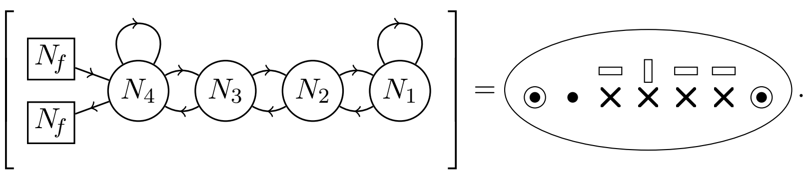
2d Gauge theory	Field content	Representation	Equation
SQED		Fundamental	(3.3.1) p.115
SQCD		Antisymmetric	(3.4.1) p.123
SQCDA		Symmetric	(3.5.1) p.132
with $W = \sum_t \widetilde{q}_t X^{l_t} q_t$		Two symmetrics	(3.5.19) p.137
with $W = \operatorname{Tr} X^{l+1}$		Quasi-rectangular	(3.5.24) p.138
$\prod_j U(N_j)$ quiver with some adjoints		Antisymmetrics and symmetrics	(3.6.1) p.141
$\prod_j U(N_j)$ quiver $\hat{z}_1 = \cdots = \hat{z}_{n-1} = 1$		Arbitrary	(3.6.33) p.150

Figure 3.1: Example of mapping between multiple Toda CFT degenerate punctures and a quiver gauge theory.

The $U(N_1) \times \cdots \times U(N_4)$ linear quiver given below has adjoint chiral multiplets for $U(N_1)$ and $U(N_4)$, hence two cubic superpotential terms coupling these to neighboring bifundamental multiplets. It also has two quartic superpotential terms coupling bifundamentals charged under $U(N_2)$, and those charged under $U(N_3)$. The partition function of the surface operator inserted by coupling the theory to N_f^2 hypermultiplets is captured by a Toda CFT correlator with degenerate punctures:

$Z_{S^2\subset S_b^4}$



The degenerate punctures are labelled by the $(N_4 - N_3)$-th symmetric, the $(N_3 - N_2)$-th antisymmetric, the $(N_2 - N_1)$-th symmetric, and the N_1-th symmetric representations, depicted by cartoons of their Young diagrams. Whenever two neighboring punctures have a different type of representation the corresponding gauge theory node has no adjoint, while neighbors of the same type yield an adjoint. The end node $U(N_4)$ is special and has an adjoint because the first puncture is symmetric.

Finally, in Section 3.6, we show that the previous results arise as special cases of surface operators described by the quivers (3.1.8), which correspond to the insertion of several (symmetric and antisymmetric) degenerate operators. We briefly discuss a brane diagram interpretation of the dictionary. By fusing representations, we deduce in Section 3.6.2 which surface operator corresponds to an arbitrary degenerate operator. All cases are summarized in Table 3.1.

3.2 Surface operators as Toda degenerates

In the next few sections, we consider half-BPS surface operators obtained by coupling two-dimensional $\mathcal{N} = (2,2)$ gauge theories to four-dimensional $\mathcal{N} = 2$ theories of class S. We enrich the dictionary between class S theories and Riemann surfaces by identifying surface operators which correspond to the insertion of arbitrary degenerate punctures.

To make the two-dimensional features most visible, we restrict ourselves to surface operators in the simplest class S theory, which is the theory on N_f M5-branes wrapping a sphere with two full and one simple puncture, namely the theory of N_f^2 free hypermultiplets Φ^{4d}. The M-theory description makes an $SU(N_f)\times SU(N_f)\times U(1)$ flavour symmetry manifest, and the hypermultiplets transform in the trifundamental representation of this group. All two-dimensional theories we study contain N_f fundamental chiral multiplets q and N_f antifundamental chiral multiplets $\widetilde{q}$ of a $U(N)$ gauge group factor. The 4d/2d coupling takes the form of a superpotential term $\sum_{s,t} \widetilde{q}_t q_s (\Phi^{\text{4d}}_{st}|_{\text{2d}})$ in two dimensions, which identifies the flavour symmetries $S[U(N_f)\times U(N_f)]$ of these chiral multiplets[7] and of the hypermultiplets. To write the superpotential term explicitly, the four-dimensional $\mathcal{N} = 2$ hypermultiplets should be decomposed into two-dimensional $\mathcal{N} = (2,2)$ components. Weakly gauging the common flavour group then gives twisted masses to the two-dimensional chiral multiplets and masses to the four-dimensional hypermultiplets, related by (3.2.17).

We place the four-dimensional theory on a squashed sphere S^4_b

$$\frac{x_0^2}{r} + \frac{x_1^2 + x_2^2}{\ell^2} + \frac{x_3^2 + x_4^2}{\tilde{\ell}^2} = 1 \tag{3.2.1}$$

where $b^2 = \ell/\tilde{\ell}$, and we place surface operators at $x_3 = x_4 = 0$,[8] hence on

[7]The full flavour symmetry of the two-dimensional quiver gauge theories we consider also contains a $U(1)$ factor, under which adjoint chiral multiplets have charge ± 2 and bifundamental chiral multiplets have charge ∓ 1.

[8]Inserting the surface operators at $x_1 = x_2 = 0$ instead would exchange $\ell \leftrightarrow \tilde{\ell}$: we would find degenerate operators with momenta $-\frac{1}{b}\Omega$ instead of $-b\Omega$, where Ω is a highest weight of A_{N_f-1}.

the squashed two-sphere

$$\frac{x_0^2}{r} + \frac{x_1^2 + x_2^2}{\ell^2} = 1\,. \tag{3.2.2}$$

The full partition function of the 4d/2d theory is then the product

$$Z_{S^2 \subset S_b^4} = Z_{S_b^4}^{\text{free}} Z_{S^2} \tag{3.2.3}$$

of the partition functions of the free hypermultiplets on S_b^4 [HH12] and of the two-dimensional gauge theory on the squashed two-sphere [BC12; DGLFL12; GL12]. The two factors do not dependent on r, but only on the equatorial radii ℓ and $\tilde{\ell}$.

The S_b^4 partition function of a single free hypermultiplet of mass m only depends on the dimensionless mass[9] $m = \sqrt{\ell\tilde{\ell}}\,\mathrm{m}$. It reads [HH12][10]

$$Z_{S_b^4}^{\text{free}}(\mathrm{m}) = \frac{1}{\Upsilon(\frac{b}{2} + \frac{1}{2b} - \mathrm{i}m)}\,. \tag{3.2.4}$$

The S_b^4 partition function of the four-dimensional theory is the product of N_f^2 such inverses of Upsilon functions. The complexified masses m_{st} of the N_f^2 hypermultiplets in this class S theory arise from weakly gauging a $S[U(N_f) \times U(N_f)]$ subgroup of the full flavour symmetry, made manifest in the description as M5-branes wrapping a trinion. With such masses, the S_b^4 partition function is then equal to a Toda CFT correlator with one simple and two full punctures. Inserting one or more degenerate punctures in the correlator corresponds to including the associated surface operator in the theory of N_f^2 hypermultiplets: for given degenerate punctures, we will find the gauge theory description of the associated surface operator by comparing the enriched Toda CFT correlator with the partition function of the 4d/2d theory on $S^2 \subset S_b^4$.

The second contribution to the partition function of the $S^2 \subset S_b^4$ system is the partition function of the two-dimensional theory. We recall now the data defining an $\mathcal{N} = (2,2)$ theory of vector and chiral multiplets, and expressions for its partition function on S^2. Besides the gauge group G (throughout the paper, $G = U(N)$ or a product of such factors) and the representation R of G in which matter multiplets transform, the S^2 partition function depends on a (real) twisted mass m and a $U(1)$ R-charge q for each chiral multiplet, that is, for each irreducible factor in R. Those are conveniently combined as the dimensionless complexified twisted mass

$$m = \ell\mathrm{m} + \frac{\mathrm{i}q}{2}\,, \tag{3.2.5}$$

[9] In our correspondence m also has an imaginary part, which is linked to the $U(1)$ R-charges of the two-dimensional chiral multiplets.

[10] The sign of m is irrelevant since the Upsilon function (5.4.5) obeys $\Upsilon(b+\frac{1}{b}-x) = \Upsilon(x)$.

where ℓ is the equatorial radius of the squashed S^2. Furthermore, for each $U(1)$ factor of G, an FI parameter ξ and a theta angle ϑ can be turned on. It will be practical to consider the complex combination

$$z = e^{-2\pi\xi + \mathrm{i}\vartheta} \tag{3.2.6}$$

for each $U(1)$ gauge group factor. Unless stated otherwise, the parameters m and z are generic. We also assume that R-charges are small and positive, $0 < \mathrm{Re}(-2\mathrm{i}m) < 1$, and otherwise define the partition function by analytic continuation.

For a choice of supercharge $\mathcal{Q}$ in the supersymmetry algebra, and of a $\mathcal{Q}$-exact deformation term $\mathcal{Q}V$ such that $\mathcal{Q}^2 V = 0$, supersymmetric localization reduces the partition function to an integral over saddle points of $\mathcal{Q}V$. When $\mathcal{Q}V$ is chosen appropriately, in particular with a positive semidefinite bosonic part, the integral is finite dimensional and more tractable than the original path integral.

One choice of deformation term leads to an expression of the partition function as an integral over the Coulomb branch [BC12; DGLFL12]:[11]

$$Z = \sum_{B\in\mathfrak{h}_{\mathbb{Z}}} \int_{\mathfrak{h}} \frac{\mathrm{d}\sigma\, Z_{\mathrm{cl}}}{(2\pi)^{\dim\mathfrak{h}}\mathcal{W}} \prod_{\alpha>0}\left[(\alpha\sigma)^2 + \frac{(\alpha B)^2}{4}\right] \prod_{w\in R}\left[\frac{\Gamma(-w(\mathrm{i}m + \mathrm{i}\sigma + \frac{B}{2}))}{\Gamma(1 + w(\mathrm{i}m + \mathrm{i}\sigma - \frac{B}{2}))}\right]. \tag{3.2.7}$$

Here, $\mathcal{W}$ is the order of the Weyl group of G, the sum is restricted to GNO quantized fluxes $B \in \mathfrak{h}$, and the integral over the lowest component σ of the vector multiplet ranges in the Cartan algebra $\mathfrak{h}$ of G. When $G = U(N_1) \times \cdots \times U(N_n)$, the classical contribution is

$$Z_{\mathrm{cl}} = z^{\mathrm{i}\sigma + \frac{B}{2}} \bar{z}^{\mathrm{i}\sigma - \frac{B}{2}} = \prod_{i=1}^{n}\left[z_i^{\mathrm{Tr}\left(\mathrm{i}\sigma_i + \frac{B_i}{2}\right)} \bar{z}_i^{\mathrm{Tr}\left(\mathrm{i}\sigma_i - \frac{B_i}{2}\right)}\right] = \prod_{i=1}^{n} e^{-4\pi\xi\,\mathrm{Tr}(\mathrm{i}\sigma_i) + \mathrm{i}\vartheta\,\mathrm{Tr}(B_i)} \tag{3.2.8}$$

and is invariant under $\vartheta \to \vartheta + 2\pi$ since B_i is an $N_i \times N_i$ (diagonal) matrix of integers. The vector multiplet one-loop determinant is a product over all positive roots α of G, and the chiral multiplet one-loop determinant, a product over all weights w of R, involves the complexified twisted mass $w \cdot m$ of the irreducible factor of R containing w.[12]

A different choice of deformation term [BC12; DGLFL12] localizes the path integral to the Higgs branch of the theory rather than its Coulomb branch, yielding a finite sum

$$Z = \sum_{v\in\{\text{Higgs vacua}\}} (z\bar{z})^{\mathrm{i}v} \operatorname*{res}_{\sigma=v}\left[\prod_{\alpha}(\mathrm{i}\alpha\sigma) \prod_{w\in R}\gamma(-w(\mathrm{i}m + \mathrm{i}\sigma))\right] Z_{\mathrm{v}}(v, m, z) Z_{\bar{\mathrm{v}}}(v, m, \bar{z}) \tag{3.2.9}$$

[11] Our normalization differs by $(2\pi)^{\dim\mathfrak{h}}$ from [DGLFL12] as this will simplify the expression of dualities.

[12] Roots and weights are linear forms on $\mathfrak{h}$, and we use the notation $\alpha\sigma = \alpha(\sigma) \in \mathbb{R}$.

which includes a vortex contribution Z_{v} depending holomorphically on z and an antivortex contribution depending on $\bar{z}$. Here, $\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}$, and factors other than Z_{v} and $Z_{\bar{\mathrm{v}}}$ are obtained as the residue at $\sigma = v$ and $B = 0$ of the Coulomb branch integrand. Higgs branch vacua are defined as having non-zero vevs for the lowest component ϕ of some chiral fields. They are labelled by solutions (σ, ϕ) of the D-term equation $\phi\phi^\dagger = \xi$ and of $(\sigma + m)\phi = 0$, where σ and m act on ϕ through the action of G and of the flavour symmetry group G_f. The set of values of σ for which the D-term equation has a solution depends on signs of the FI parameters ξ_j for each $U(1)$ factor in G: each choice of signs leads to a different expansion (3.2.9). Even after solving these equations, one must in principle evaluate Z_{v} as the volume of a moduli space of vortices. However, the Coulomb branch representation provides a convenient short-cut: closing the $\mathrm{d}\sigma$ integrals (3.2.7) towards $\sigma \to \pm i\infty$ depending on the matter content and on signs of FI parameters exppresses the partition function as a sum over poles, which is then rewritten as a finite sum of factorized terms (3.2.9). The manipulations are most easily done on specific examples, as we will see, but work for an arbitrary gauge group and matter representation (see Appendix 2.F).

In the coming sections we associate a two-dimensional $\mathcal{N} = (2,2)$ gauge theory, hence a surface operator, to each choice of representation $\mathcal{R}(\Omega)$ of A_{N_f-1}. We work out equalities of the form

$$Z^{(\Omega)}_{S^2 \subset S^4_b} = A|x|^{2\gamma_0}|1-x|^{2\gamma_1} \left\langle \widehat{V}_{\alpha_\infty}(\infty) \widehat{V}_{\hat{m}}(1) \widehat{V}_{\alpha_0}(0) \widehat{V}_{-b\Omega}(x, \bar{x}) \right\rangle \qquad (3.2.10)$$

between the partition function on $S^2 \subset S^4_b$ of the 4d/2d system associated to a given representation $\mathcal{R}(\Omega)$ and Toda CFT correlators with two full punctures at 0 and ∞, one simple at 1, and one degenerate.[13] The position x of the degenerate puncture is related to a complexified FI parameter z. The two-dimensional theories we consider involve N_f fundamental and N_f anti-fundamental chiral multiplets of a gauge group factor $U(N_n)$, whose twisted masses we denote by m_t and $\widetilde{m}_t$.

Let us first explain how the factor $A|x|^{2\gamma_0}|1-x|^{2\gamma_1}$ can be absorbed into the partition function (specifically the S^2 contribution). In the coming sections it will be easier to manipulate explicit expressions of partition functions and correlators, hence we will keep the factor explicitly, with the understanding that it has no physical content. In terms of gauge theory data,

[13] Toda CFT notations are reviewed in Chapter 5. Vertex operators $\widehat{V}_\alpha$ are labelled by their momentum α, a linear combination of the weights h_s ($1 \leq s \leq N_f$) of the fundamental representation of A_{N_f-1}. They are primary operators for the W_{N_f} chiral algebra. Generic momenta depend on $N_f - 1$ parameters and label full punctures. Semi-degenerate vertex operators, with momentum $\varkappa h_1$ (or its conjugate $-\varkappa h_{N_f}$), have null descendants under W_{N_f} and label simple punctures. Degenerate vertex operators have momentum $-b\Omega - \Omega'/b$ for a pair of highest weights Ω and Ω' of representations of A_{N_f-1}.

it turns out that we can split

$$\gamma_0 = \gamma_0^\circ(\Omega, b) - \frac{N_n}{N_f}\sum_{t=1}^{N_f} \mathrm{i}m_t \quad \gamma_1 = \gamma_1^\circ(\Omega, b) + \frac{N_n}{N_f}\sum_{t=1}^{N_f}(\mathrm{i}m_t + \mathrm{i}\widetilde{m}_t)\,, \quad (3.2.11)$$

and $A = A^\circ(\Omega, b)\, b^{-2N_n \sum_t (\mathrm{i}m_t + \mathrm{i}\widetilde{m}_t)}$, where A°, γ_0° and γ_1° depend only on b and Ω. The factor decomposes as

$$A|x|^{2\gamma_0}|1-x|^{2\gamma_1} = \frac{A^\circ |x|^{2\gamma_0^\circ}|1-x|^{2\gamma_1^\circ}}{|x|^{2(N_n/N_f)\sum_t \mathrm{i}m_t}}\left[\frac{|1-x|^{2N_n}}{b^{2N_n N_f}}\right]^{\frac{\sum_t(\mathrm{i}m_t+\mathrm{i}\widetilde{m}_t)}{N_f}} \quad (3.2.12)$$

and can be absorbed in the partition function through three different mechanisms. Firstly, the two-sphere partition function is subject to certain ambiguities [GL12] (see also [CC14]). These are captured by local supergravity counterterms [GGK14]. A change in the renormalization scheme changes the partition function by

$$Z \to f(z)\bar{f}(\bar{z})Z\,, \quad (3.2.13)$$

where f is a holomorphic function of the complexified FI parameter z. This lets us absorb the first factor in (3.2.12) as a renormalization ambiguity of $Z_{S^2\subset S^4_b}$. Secondly, a constant $U(1)$ gauge transformation lets us shift the partition function by any power of $|z|^2 = |x|^2$ hence absorb the denominator in (3.2.12). Finally, the last factor can be absorbed through a complexified FI parameter $z_{\mathrm{fl}} = b^{2N_n N_f}/(1-x)^{2N_n}$ for the $U(1)$ subgroup of the flavour group $S[U(N_f)\times U(N_f)]$ which acts on the fundamental and antifundamental chiral multiplets. Indeed, such an FI parameter multiplies the partition function by $(z_{\mathrm{fl}}\bar{z}_{\mathrm{fl}})^{\mathrm{i}\sigma_{\mathrm{fl}}}$, where σ_{fl} is the bottom component of the vector multiplet used to weakly gauge the $U(1)$ flavour symmetry, that is, $\sigma_{\mathrm{fl}} = \sum_t(m_t + \widetilde{m}_t)/(2N_f)$.

We are ready to discuss how we will derive equalities of the form (3.2.10), or more generally for a set of highest weights Ω_j of A_{N_f-1}:

$$Z^{\{\Omega_j\}}_{S^2\subset S^4_b} = \left\langle \widehat{V}_{\alpha_\infty}(\infty)\widehat{V}_{\hat{m}}(1)\widehat{V}_{\alpha_0}(0)\prod_{j=1}^{n}\widehat{V}_{-b\Omega_j}(x_j,\bar{x}_j)\right\rangle \quad (3.2.14)$$

where α_0 and α_∞ are generic, $\hat{m}$ is semi-degenerate, and we have omitted the factors which can be absorbed into the partition function. The dictionary between gauge theory and Toda CFT data identifies the momenta α_0, α_∞, and $\hat{m}$ to the three factors of the flavour symmetry group $SU(N_f)\times SU(N_f)\times U(1)$ acting on fundamental and antifundamental chiral multiplets:

$$\begin{aligned} \alpha_0 &= Q - \frac{1}{b}\sum_{s=1}^{N_f}\mathrm{i}m_s h_s & \hat{m} &= (\varkappa + Nb)h_1 \\ \alpha_\infty &= Q - \frac{1}{b}\sum_{s=1}^{N_f}\mathrm{i}\widetilde{m}_s h_s & \varkappa &= \frac{1}{b}\sum_{s=1}^{N_f}(1 + \mathrm{i}m_s + \mathrm{i}\widetilde{m}_s) \end{aligned} \quad (3.2.15)$$

with Toda CFT notations given in Chapter 5. The degenerate operators encode the choice of gauge groups and matter content of the gauge theory.

As explained below, we will start in each case by matching the dependence of the S^2 partition function on FI parameters z_j with the dependence of Toda CFT correlators on the position of degenerate operators x_j. Once this is done, there remains a universal relative factor between the S^2 partition function and the Toda CFT correlator, which turns out to be a Toda CFT three-point function of two generic and one semi-degenerate vertex operators[14]

$$\widehat{C}(\alpha_0, \alpha_\infty, \varkappa h_1) = \prod_{s=1}^{N_f} \prod_{t=1}^{N_f} \frac{1}{\Upsilon\big(\frac{1}{b}(1 + \mathrm{i}m_s + \mathrm{i}\widetilde{m}_t)\big)} . \tag{3.2.16}$$

These Upsilon functions are precisely reproduced by the S_b^4 partition function (3.2.4) of N_f^2 free hypermultiplets with (dimensionless) masses

$$m_{st} = \mathrm{i}\frac{1-b^2}{2b} - \frac{1}{b}(m_s + \widetilde{m}_t) . \tag{3.2.17}$$

The dimensionful masses $(\ell\tilde{\ell})^{\frac{-1}{2}} m_{st}$ and twisted masses $\ell^{-1} m_s$ and $\ell^{-1}\widetilde{m}_t$ both originate from weakly gauging the common flavour symmetry group $SU(N_f) \times SU(N_f) \times U(1)$, and indeed, the relation between dimensionful masses has no relative factor of b:

$$\left[\frac{m_{st}}{\sqrt{\ell\tilde{\ell}}} + \frac{\mathrm{i}}{2\ell} + \frac{\mathrm{i}}{2\tilde{\ell}}\right] + \frac{m_s + \widetilde{m}_t}{\ell} = \frac{\mathrm{i}}{\ell} \tag{3.2.18}$$

The masses m_{st} can also be found by requiring that the two-dimensional superpotential $\sum_{s,t} \widetilde{q}_t q_s \Phi_{st}^{4\mathrm{d}}|_{2\mathrm{d}}$ is supersymmetric hence has R-charge 2 (complexified twisted mass i). From this perspective, the shift in the four-dimensional masses likely arises from mixing the $U(1)_R$ symmetry with geometrical symmetries.

In Section 3.3 and Section 3.4, we identify degenerate vertex operators labelled by the fundamental (resp. antisymmetric) representation of A_{N_f-1} to SQED (resp. SQCD). The Toda CFT correlator is a four-point function which depends on a single cross-ratio x, while the two-dimensional theory has a single $U(1)$ gauge group factor hence a single complexified FI parameter $z = e^{-2\pi\xi + \mathrm{i}\vartheta}$. We prove as follows that the Toda correlator is equal to the $S^2 \subset S_b^4$ partition function. First, we write the Higgs branch expressions of the S^2 partition function in the regions $\xi > 0$ and $\xi < 0$, that is, $|z| < 1$ and $|z| > 1$. The two expressions match with expansions of the Toda CFT

[14]Note the shift between $\varkappa h_1$ in the three-point function (3.2.16) and $\hat{m}$ in the $(n+3)$-point function (3.2.14). The insertion of degenerate operators near a simple puncture thus shifts the dictionary between the semi-degenerate momentum of the puncture and the corresponding hypermultiplet mass. As a result, the node-hopping duality relates surface operators in four-dimensional theories which differ by shifts in complexified masses of hypermultiplets.

Table 3.2: Relation between parts of the Higgs branch decomposition of the S^2 partition function, and the s-channel decomposition of corresponding Toda CFT correlators. Explicit expressions differ by $A|x|^{2\gamma_0}|1-x|^{2\gamma_1}$, an ambiguity in Z.

	Gauge theory	Toda CFT
Terms in the sum	Higgs vacua	Internal momenta
Asymptotics at 0	Classical contribution $(z\bar{z})^{\mathrm{i}v}$	$(x\bar{x})^{\Delta(\alpha_0-bh)-\Delta(\alpha_0)-\Delta(-b\omega)}$
Leading coefficient	One-loop determinant Z_{1l}	Three-point functions
Holomorphic series	Vortex partition function Z_{v}	Conformal blocks (normalized)

correlator in the s-channel $|x| < 1$ and u-channel $|x| > 1$ as described in Table 3.2: the Higgs branch vacua correspond to choices of internal momenta and we match the leading powers of $|z|^2 = |x|^2$. On the gauge theory side, the exponents of $|z|^2$ are read from the classical contribution, while on the Toda CFT side the exponents of $|x|^2$ are sums of dimensions of vertex operators. We then derive the braiding matrices which relate s-channel and u-channel conformal blocks and show that they are equal to the corresponding gauge theory data. These braiding matrices let us express the monodromy around ∞ as a matrix in the basis of s-channel conformal blocks (the monodromy around 0 is diagonal in this basis). Finally, we prove that the S^2 partition function has only one branch point besides $z = 0$ and $z = \infty$, and identify gauge theory exponents with those in the t-channel $x \to 1$ of the Toda correlator. Therefore the monodromy matrix around 1 is simply the inverse product of the monodromies around 0 and ∞. Since their monodromy matrices around all three branch points coincide, the S^2 partition function and Toda CFT four-point function must be equal up to a factor with no monodromy. Since in expansions around 0, 1 and ∞ the exponents match, the factor has no pole on the sphere hence is a constant: it is precisely given by the S_b^4 contribution (3.2.16) of N_f^2 hypermultiplets.

When the FI parameter ξ is changed continuously from $\xi < 0$ to $\xi > 0$, the two-dimensional gauge theory experiences a flop transition between vortices carried by fundamental matter and vortices carried by antifundamental matter. The flop transition is realized in the Toda CFT as crossing symmetry from the s-channel to the u-channel [DGLFL12]. This geometric approach implies that the resuls for $\xi < 0$ and $\xi > 0$ are related by analytic continuation. There is no Higgs branch expansion as $\xi \to 0$: instead, we build a decomposition of the Coulomb branch integral in this limit. It would be interesting to provide a gauge theory interpretation of this "t-channel" decomposition, and of the braiding matrices relating $\xi > 0$ and $\xi < 0$ vortex partition functions.

In Section 3.5, we identify degenerate vertex operators labelled by symmetric representations of A_{N_f-1} to SQCD with an additional adjoint chiral multiplet (SQCDA). The discussion is very similar to the previous cases, but braiding matrices are not available.[15] Instead, we check that the leading coefficients and powers of $|x|^2$ coincide, both in the s-channel and in the u-channel. We then check that the S^2 partition function has a branch point corresponding to the t-channel, and that the leading powers of $|1-x|^2$ coincide. As before, the Toda CFT four-point function is equal to the S^2 partition function up to a constant, which is the S_b^4 partition function of N_f^2 free hypermultiplets.

In Section 3.6 we identify the quiver gauge theory which corresponds to sets of degenerate operators labelled by symmetric or antisymmetric representations of A_{N_f-1}. The identification is checked by comparing the expansion of the $S^2 \subset S_b^4$ partition function and of the Toda CFT correlator in various limits. Seiberg-like dualities let us probe further limits: as seen in Section 4.4.1, permutations of the n degenerate vertex operators relate dual gauge theories. First, we equate exponents and leading coefficients in the channel where degenerate punctures are at $0 < |x_1| < \cdots < |x_n| < 1$. Thanks to dualities, exponents and leading coefficients also match for all other orderings of the n degenerate punctures. By symmetry, the gauge theory and Toda CFT exponents and leading coefficients also match in all channels with $1 < |x_1|, \ldots, |x_n| < \infty$. In each of the $2(N!)$ channels the decompositions involve many exponents and factors, and all match. We then equate exponents which appear in any of the limits $x_n \to 1$ or $x_j \to x_{j+1}$, hence also in the limits $x_j \to 1$ or $x_j \to x_k$ thanks to dualities.

Building upon the identification of the quiver which corresponds to the insertion of any number of antisymmetric degenerate vertex operators, we show in Section 3.6.2 that bringing all punctures $x_j = x$ to the same position yields a degenerate vertex operator labelled by an arbitrary representation of A_{N_f-1}: all other terms in the fusion of antisymmetric degenerate vertex operators appear with higher powers of some $|x_j - x_k|^2$ hence are suppressed. This determines the quiver gauge theory corresponding to an arbitrary degenerate vertex operator $\widehat{V}_{-b\Omega}$.

The surface operators we consider are constructed by coupling N_f fundamental and N_f antifundamental chiral multiplets of an $\mathcal{N} = (2,2)$ theory to N_f^2 hypermultiplets. Making some antifundamental chiral multiplets and some hypermultiplets massive yields surface operators described by $\mathcal{N} = (2,2)$ theories with N_f fundamental and $\widetilde{N_f} < N_f$ antifundamental chiral multiplets, coupled to $N_f\widetilde{N_f}$ free hypermultiplets. The limit corresponds to a collision limit of the punctures $\widehat{V}_{\hat{m}}$ and $\widehat{V}_{\alpha_\infty}$ in (3.2.14), which builds an irregular

[15] It is technically difficult to write down braiding matrices in this case. On the gauge theory side, the Mellin–Barnes integral (used for SQED and SQCD to interpolate between $|z| \lessgtr 1$ expansions) is much more involved. On the Toda CFT side, recursion relations for the braiding matrices contain many more terms than for the antisymmetric case.

puncture (see Section 5.6 and for $N_f = 2$ see [GT12]). We only study this limit for SQCD (see Section 3.4.3), but the discussion applies to all our surface operators.

3.3 SQED and Toda fundamental degenerate

We review in this section the case of $\mathcal{N} = (2,2)$ SQED on S^2, namely a $U(1)$ vector multiplet coupled to N_f fundamental and N_f antifundamental chiral multiplets, whose twisted masses (plus R-charges) we denote by m_s and $\widetilde{m}_s$ for $1 \leq s \leq N_f$. It was shown [DGLFL12] that the S^2 partition function of SQED matches an A_{N_f-1} Toda CFT four-point function, up to a constant. We find that the constant factor reproduces the S_b^4 partition function of N_f^2 free hypermultiplets with masses (3.2.17), hence the Toda correlator in fact captures the partition function of the surface operator inserted in this free four-dimensional theory. The precise relation is[16]

$$Z^{\text{SQED}}_{S^2\subset S^4_b}(m,\widetilde{m},z,\bar{z}) = A|x|^{2\gamma_0}|1-x|^{2\gamma_1}\left\langle \widehat{V}_{\alpha_\infty}(\infty)\widehat{V}_{\hat{m}}(1)\widehat{V}_{-bh_1}(x,\bar{x})\widehat{V}_{\alpha_0}(0)\right\rangle . \tag{3.3.1}$$

The Toda CFT correlator (see Chapter 5 for conventions) features a degenerate field $\widehat{V}_{-bh_1}$ inserted at $x = (-1)^{N_f}z$ and labelled by the fundamental representation $\mathcal{R}(h_1)$ of A_{N_f-1}, a semi-degenerate field $\widehat{V}_{\hat{m}}$ at 1, and two generic fields $\widehat{V}_{\alpha_0}$ and $\widehat{V}_{\alpha_\infty}$. Momenta are related to twisted masses through

$$\begin{aligned} \alpha_0 &= Q - \frac{1}{b}\sum_{s=1}^{N_f} \mathrm{i}m_s h_s & \hat{m} &= (\varkappa + b)h_1 \\ \alpha_\infty &= Q - \frac{1}{b}\sum_{s=1}^{N_f} \mathrm{i}\widetilde{m}_s h_s & \varkappa &= \frac{1}{b}\sum_{s=1}^{N_f}(1 + \mathrm{i}m_s + \mathrm{i}\widetilde{m}_s)\,, \end{aligned} \tag{3.3.2}$$

and the exponents and coefficient are

$$\gamma_0 = -\frac{1}{N_f}\sum_{s=1}^{N_f}\mathrm{i}m_s - \frac{N_f-1}{2}(b^2+1) \tag{3.3.3}$$

$$\gamma_1 = -\frac{N_f-1}{N_f}b^2 + \frac{1}{N_f}\sum_{s=1}^{N_f}(\mathrm{i}m_s + \mathrm{i}\widetilde{m}_s) \tag{3.3.4}$$

$$A = b^{N_f(1+b^2)-b^2-2b\varkappa}\,. \tag{3.3.5}$$

Permuting the m_s, or the $\widetilde{m}_s$, does not affect the partition function. This is reproduced in the Toda CFT by the invariance of $\widehat{V}_{\alpha_0}$ and $\widehat{V}_{\alpha_\infty}$ under Weyl

[16] As explained below (3.2.10), the factor $A|x|^{2\gamma_0}|1-x|^{2\gamma_1}$ can be absorbed into the partition function. To compare gauge theory and Toda CFT results it is best to keep the factor explicitly.

transformations (the normalization is chosen to cancel reflexion amplitudes). The similarity between α_0 and α_∞ is also expected, as swapping them and changing x to its inverse amounts in gauge theory to charge conjugation, which swaps m_s and $\widetilde{m}_s$, and changes z to its inverse. Under this transformation, Z_{S^2} is invariant, while the Toda correlator receives a small shift controlled by the dimension $\Delta(-bh_1)$ of the degenerate insertion: this shift is absorbed by the factor $|x|^{2\gamma_0}|1-x|^{2\gamma_1}$.

In [DGLFL12], the equality was shown directly thanks to known expressions [FL07] for the W_{N_f} conformal block involved. The approach does not generalize, because conformal blocks with higher degenerate insertions were not previously known.[17] Instead, we prove the correspondence for SQED (treated here) and SQCD (see Section 3.4) by comparing monodromy matrices around branch points. In the main text, we find expansions around all branch points and compare leading terms, as this is enough to fix uniquely the dictionary between gauge theory and Toda CFT parameters. In Section 5.2 we derive the braiding matrices relating s-channel and u-channel expansions of the Toda CFT correlator, and their gauge theory analogues. The braiding matrices match. From this we deduce the matching of monodromy matrices around all branch points, expressed in a single basis, and not only of their eigenvalues compared in the main text. These results suffice to prove that the partition function and the correlator are equal.

To prepare for the somewhat technical computations ahead, we first go through the various steps here in the well-controlled case of SQED and Toda CFT fundamental degenerate fields. The expansions near $z = 0$ and $z = \infty$ follow [DGLFL12] closely, while the expansion near $z = (-1)^{N_f}$ is new. All three play an important role in later sections.

3.3.1 Expanding the SQED partition function

The Coulomb branch expression for the partition function of SQED is

$$Z_{S^2}^{\text{SQED}} = \sum_{B\in\mathbb{Z}} \int_{\mathbb{R}} \frac{d\sigma}{2\pi} z^{i\sigma+\frac{B}{2}} \bar{z}^{i\sigma-\frac{B}{2}} \prod_{s=1}^{N_f} \left[\frac{\Gamma(-im_s - i\sigma - \frac{B}{2})}{\Gamma(1+im_s+i\sigma-\frac{B}{2})} \frac{\Gamma(-i\widetilde{m}_s + i\sigma + \frac{B}{2})}{\Gamma(1+i\widetilde{m}_s - i\sigma + \frac{B}{2})} \right] . \tag{3.3.6}$$

As we will see shortly, the contour of integration for σ can be closed in the lower or upper half plane depending on whether $|z| \lessgtr 1$, leading to distinct expressions of Z as a sum over poles lying in either half plane. We will match the resulting expressions with the s-channel and u-channel decompositions of the Toda CFT four-point correlator.

To find out which half-plane the contour should enclose, we study the asymptotic behaviour of the integrand. First, rewrite the ratios of Gamma

[17] We derive such explicit conformal blocks from our matchings in Section 5.4.4.

functions so that the numerator and denominator have no common poles,

$$\frac{\Gamma(-\upsilon \pm \frac{B}{2})}{\Gamma(1+\upsilon \pm \frac{B}{2})} = (-1)^{\frac{B\mp|B|}{2}} \frac{\Gamma(-\upsilon + \frac{|B|}{2})}{\Gamma(1+\upsilon+\frac{|B|}{2})}\,, \tag{3.3.7}$$

and absorb the resulting sign $(-1)^{N_f B}$ by introducing

$$x = (-1)^{N_f} z\,, \qquad \bar{x} = (-1)^{N_f}\bar{z}\,. \tag{3.3.8}$$

Thanks to $\frac{\Gamma(\upsilon+a)}{\Gamma(\upsilon+b)} \sim \upsilon^{a-b}$, valid when $|\upsilon| \to \infty$ away from the negative real axis, the integrand is

$$\begin{aligned} &x^{\mathrm{i}\sigma+\frac{B}{2}}\bar{x}^{\mathrm{i}\sigma-\frac{B}{2}} \prod_{s=1}^{N_f}\left[\frac{\Gamma(-\mathrm{i}m_s - \mathrm{i}\sigma + \frac{|B|}{2})}{\Gamma(1+\mathrm{i}m_s+\mathrm{i}\sigma+\frac{|B|}{2})}\frac{\Gamma(-\mathrm{i}\widetilde{m}_s + \mathrm{i}\sigma + \frac{|B|}{2})}{\Gamma(1+\mathrm{i}\widetilde{m}_s - \mathrm{i}\sigma + \frac{|B|}{2})}\right] \\ &\sim x^{\mathrm{i}\sigma+\frac{B}{2}}\bar{x}^{\mathrm{i}\sigma-\frac{B}{2}}\left(\sigma^2+\frac{B^2}{4}\right)^{-\sum_{s=1}^{N_f}(1+\mathrm{i}m_s+\mathrm{i}\widetilde{m}_s)} \end{aligned} \tag{3.3.9}$$

as $|\mathrm{i}\sigma \pm \frac{B}{2}| \to \infty$.

As long as we keep $\sigma \in \mathbb{R}$ on the integration contour, the factor $x^{\mathrm{i}\sigma+\frac{B}{2}}\bar{x}^{\mathrm{i}\sigma-\frac{B}{2}}$ is simply a phase. If $|x| = |z| < 1$, this factor decays exponentially towards $\sigma \to -\mathrm{i}\infty$, hence the contour of integration can be closed in this half-plane. On the other hand, for $|x| = |z| > 1$, the integrand decays exponentially in the $\sigma \to \mathrm{i}\infty$ half-plane.

The integrand (3.3.9) has poles whenever one of $-\mathrm{i}m_p - \mathrm{i}\sigma + \frac{|B|}{2}$ or $-\mathrm{i}\widetilde{m}_p + \mathrm{i}\sigma + \frac{|B|}{2}$ is a non-positive integer, that is, at

$$\mathrm{i}\sigma = -\mathrm{i}m_p + k + \frac{|B|}{2} \quad \text{or} \quad \mathrm{i}\widetilde{m}_p - k - \frac{|B|}{2} \tag{3.3.10}$$

for a fundamental or antifundamental flavour $1 \le p \le N_f$ and an integer $k \ge 0$. Since R-charges are positive, $-\mathrm{i}m_p$ has a positive real part and $\mathrm{i}\widetilde{m}_p$ a negative real part, hence the poles of the fundamental multiplets' one-loop determinants lie in the half-plane towards $\sigma \to -\mathrm{i}\infty$, while the other half plane contains those of antifundamental multiplets.

Let us focus on the case $|x| = |z| < 1$. We then sum residues of the integrand of (3.3.6) over poles (3.3.10) where $\mathrm{i}\sigma$ has a positive real part. This yields

$$\begin{aligned} Z = \sum_{p=1}^{N_f}\sum_{k\ge 0}\sum_{B\in\mathbb{Z}} \Bigg\{ & z^{-\mathrm{i}m_p+k+\frac{|B|+B}{2}}\bar{z}^{-\mathrm{i}m_p+k+\frac{|B|-B}{2}} \\ & \cdot \prod_{s=1}^{N_f}{}' \left[\frac{\Gamma(-\mathrm{i}m_s+\mathrm{i}m_p-k-\frac{|B|+B}{2})}{\Gamma(1+\mathrm{i}m_s-\mathrm{i}m_p+k+\frac{|B|-B}{2})}\frac{\Gamma(-\mathrm{i}\widetilde{m}_s-\mathrm{i}m_p+k+\frac{|B|+B}{2})}{\Gamma(1+\mathrm{i}\widetilde{m}_s+\mathrm{i}m_p-k-\frac{|B|-B}{2})}\right]\Bigg\}\,, \end{aligned} \tag{3.3.11}$$

where the singular factor $\Gamma(-k-\frac{|B|+B}{2})$ appearing for $s=p$ should be replaced by its residue $(-1)^{k+\frac{|B|+B}{2}}/\,\Gamma(1+k+\frac{|B|+B}{2})$. Note that k and B appear as the combinations $k^\pm = k+\frac{|B|\pm B}{2}$ only, and that the sums over $k\geq 0$ and $B\in\mathbb{Z}$ are equivalent to sums over $k^+\geq 0$ and $k^-\geq 0$. Hence,

$$Z=\sum_{p=1}^{N_f}\sum_{k^\pm\geq 0}(z\bar{z})^{-\mathrm{i}m_p}z^{k^+}\bar{z}^{k^-}\prod_{s=1}^{N_f}{}'\left[\frac{\Gamma(-\mathrm{i}m_s+\mathrm{i}m_p-k^+)}{\Gamma(1+\mathrm{i}m_s-\mathrm{i}m_p+k^-)}\frac{\Gamma(-\mathrm{i}\widetilde{m}_s-\mathrm{i}m_p+k^+)}{\Gamma(1+\mathrm{i}\widetilde{m}_s+\mathrm{i}m_p-k^-)}\right], \tag{3.3.12}$$

with the same caveat as above, namely, $\Gamma(-k^+)\to(-1)^{k^+}/\Gamma(1+k^+)$. Since each Gamma function argument depends on only one of k^+ and k^-, the contribution from each flavour p factorizes as the product of two series in (positive) powers of z and of $\bar{z}$. We extract from the series a normalization factor (the value at $k^\pm=0$), by writing the Gamma functions in terms of Pochhammer symbols $(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}$ and of $\gamma(x)=\frac{\Gamma(x)}{\Gamma(1-x)}$,

$$\frac{\Gamma(-\mathrm{i}m_s+\mathrm{i}m_p-k^+)}{\Gamma(1+\mathrm{i}m_s-\mathrm{i}m_p+k^-)}=\frac{(-1)^{k^+}\gamma(-\mathrm{i}m_s+\mathrm{i}m_p)}{(1+\mathrm{i}m_s-\mathrm{i}m_p)_{k^+}(1+\mathrm{i}m_s-\mathrm{i}m_p)_{k^-}}. \tag{3.3.13}$$

We deduce the partition function for $|x|=|z|<1$ in terms of "s-channel" vortex partition functions

$$Z=\sum_{p=1}^{N_f}\left\{(x\bar{x})^{-\mathrm{i}m_p}\frac{\prod_{s\neq p}^{N_f}\gamma(-\mathrm{i}m_s+\mathrm{i}m_p)}{\prod_{s=1}^{N_f}\gamma(1+\mathrm{i}\widetilde{m}_s+\mathrm{i}m_p)}f_p^{(\mathrm{s})}(m,\widetilde{m},x)f_p^{(\mathrm{s})}(m,\widetilde{m},\bar{x})\right\} \tag{3.3.14}$$

$$f_p^{(\mathrm{s})}(m,\widetilde{m},x)=\sum_{k\geq 0}x^k\prod_{s=1}^{N_f}\frac{(-\mathrm{i}\widetilde{m}_s-\mathrm{i}m_p)_k}{(1+\mathrm{i}m_s-\mathrm{i}m_p)_k}=F\left(\begin{matrix}(-\mathrm{i}\widetilde{m}_s-\mathrm{i}m_p),\,1\leq s\leq N_f\\(1+\mathrm{i}m_s-\mathrm{i}m_p),\,s\neq p\end{matrix}\middle|x\right). \tag{3.3.15}$$

The $f_p^{(\mathrm{s})}$ are hypergeometric functions, related later on to s-channel conformal blocks in the Toda CFT. Similar computations for $|x|=|z|>1$ convert the sum over poles at $\mathrm{i}\sigma=\mathrm{i}\widetilde{m}_p-\cdots$ to a factorized form, related to the u-channel decomposition of a Toda CFT correlator,

$$Z=\sum_{p=1}^{N_f}\left\{(x\bar{x})^{\mathrm{i}\widetilde{m}_p}\frac{\prod_{s\neq p}^{N_f}\gamma(-\mathrm{i}\widetilde{m}_s+\mathrm{i}\widetilde{m}_p)}{\prod_{s=1}^{N_f}\gamma(1+\mathrm{i}m_s+\mathrm{i}\widetilde{m}_p)}f_p^{(\mathrm{u})}(m,\widetilde{m},x)f_p^{(\mathrm{u})}(m,\widetilde{m},\bar{x})\right\} \tag{3.3.16}$$

$$f_p^{(\mathrm{u})}(m,\widetilde{m},x)=\sum_{k\geq 0}\frac{1}{x^k}\prod_{s=1}^{N_f}\frac{(-\mathrm{i}m_s-\mathrm{i}\widetilde{m}_p)_k}{(1+\mathrm{i}\widetilde{m}_s-\mathrm{i}\widetilde{m}_p)_k}=F\left(\begin{matrix}(-\mathrm{i}m_s-\mathrm{i}\widetilde{m}_p),\,1\leq s\leq N_f\\(1+\mathrm{i}\widetilde{m}_s-\mathrm{i}\widetilde{m}_p),\,s\neq p\end{matrix}\middle|\frac{1}{x}\right). \tag{3.3.17}$$

The factorized (3.3.14) and (3.3.16) reproduce the general form (3.2.9)

$$Z = \sum_{\text{vacua}} \operatorname{res}\Big[Z_{\text{cl}}(\sigma, 0, z, \bar{z}) Z_{1l}(m, \sigma, 0)\Big] Z_{\text{v},p}(m, z) Z_{\bar{\text{v}},p}(m, \bar{z}) \tag{3.3.18}$$

obtained when localizing to the Higgs branch of the theory for positive and for negative FI parameter ξ, respectively. Indeed, Higgs branch vacua are labelled by solutions of

$$\sum_{s=1}^{N_f} \Big(|q_s|^2 - |\widetilde{q}_s|^2\Big) = \xi = -\frac{1}{2\pi} \ln|z| \tag{3.3.19}$$

and $(\sigma + m_s) q_s = 0 = (\sigma - \widetilde{m}_s)\widetilde{q}_s$ for all s. For $|z| < 1$, that is, $\xi > 0$, at least one of the positively charged fields q_s is non-zero, thus $\sigma = -m_s$. For $|z| > 1$, that is, $\xi < 0$, one of the negatively charged fields is non-zero, and $\sigma = \widetilde{m}_s$. One easily checks that evaluating the classical contribution, and the residue of the one-loop contribution (which is the integrand of the Coulomb branch representation (3.3.6)) at those values of σ and at $B = 0$ yields the relevant factors in (3.3.14) and (3.3.16). The hypergeometric functions $f_p^{(\text{s})}$ and $f_p^{(\text{u})}$ obtained from factorization also match with known vortex and anti-vortex partition functions (see [BC12; DGLFL12]). For more general theories, factorization always yields explicit expressions for the vortex partition functions, while earlier methods soon become intractable.

The s-channel factors in (3.3.14) also have a Mellin–Barnes integral representation

$$\begin{aligned}(-x)^{-\mathrm{i}m_p} f_p^{(\text{s})}(x) = \prod_{s=1}^{N_f} &\left[\frac{\Gamma(1 + \mathrm{i}m_s - \mathrm{i}m_p)}{\Gamma(-\mathrm{i}\widetilde{m}_s - \mathrm{i}m_p)}\right] \\ &\cdot \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{d}\kappa}{2\pi\mathrm{i}} \frac{\prod_{s=1}^{N_f} \Gamma(-\mathrm{i}\widetilde{m}_s + \kappa)}{\prod_{s\neq p}^{N_f} \Gamma(1 + \mathrm{i}m_s + \kappa)} \Gamma(-\kappa - \mathrm{i}m_p)(-x)^{\kappa}\end{aligned} \tag{3.3.20}$$

which converges for $|\arg(-x)| < \pi$, that is, away from the positive real axis. On the other hand, the s- and u-channel expansions found above imply that the partition function has branch points at 0 and ∞, but is otherwise smooth away from the unit circle. Hence, the partition function can only have branch points at $x \in \{0, 1, \infty\}$.

We have already given expansions near 0 and ∞, so we now focus on powers of $|1 - x|^2$ as $x \to 1$. The Higgs branch localization has no analogue at $x = 1$, because the FI parameter $\xi = -\frac{1}{2\pi} \ln|z|$ vanishes and the manifold of solutions of $\sum_{s=1}^{N_f} \big(|q_s|^2 - |\widetilde{q}_s|^2\big) = \xi$ experiences a flop transition. Instead, we find an explicit decomposition starting from the Coulomb branch integral.

As $x \to 1$, split the Coulomb branch representation (3.3.6) into the two regions, $|\mathrm{i}\sigma + \frac{B}{2}| \lessgtr |\ln x|^{-1}$. In the first, $x^{\mathrm{i}\sigma + \frac{B}{2}} \bar{x}^{\mathrm{i}\sigma - \frac{B}{2}}$ is given by a convergent series in integer powers of $\ln x$ and $\ln \bar{x}$ thanks to

$$x^{\mathrm{i}\sigma+\frac{B}{2}} = \sum_{k\geq 0} \frac{(\mathrm{i}\sigma + \frac{B}{2})^k}{k!} (\ln x)^k \,. \tag{3.3.21}$$

In the second, the product of Gamma functions in the integrand can be approximated as (3.3.9) through Stirling's approximation, and the sum over B can be replaced by a continuous integral, leading to a contribution

$$\int \mathrm{d}B \, \frac{\mathrm{d}\sigma}{2\pi} e^{(\mathrm{i}\sigma+\frac{B}{2})\ln x} e^{(\mathrm{i}\sigma-\frac{B}{2})\ln \bar{x}} \Big(\sigma^2 + \frac{B^2}{4}\Big)^{-\Sigma} = \frac{1}{\pi} \int \mathrm{d}\rho \, \rho \, \mathrm{d}\theta \, e^{2\mathrm{i}\rho\cos\theta|\ln x|} \rho^{-2\Sigma} \,, \tag{3.3.22}$$

where $\Sigma = \sum_{s=1}^{N_f} (1 + \mathrm{i}m_s + \mathrm{i}\widetilde{m}_s)$ and we applied the change of variables $\rho e^{i\theta}|\ln x| = (\sigma - \mathrm{i}\frac{B}{2})\ln x$. Rescaling then ρ by $|\ln x|$, we find that the contribution behaves as

$$|\ln x|^{2\Sigma-2} \sim |1 - x|^{2\left[-1+\sum_{s=1}^{N_f}(1+\mathrm{i}m_s+\mathrm{i}\widetilde{m}_s)\right]} \,, \tag{3.3.23}$$

as $x \to 1$, multiplied by a series in powers of $(1 - x)$ and $(1 - \bar{x})$. We thus find

$$Z = |1 - x|^0 G(1 - x, 1 - \bar{x}) + |1 - x|^{2\left[-1+\sum_{s=1}^{N_f}(1+\mathrm{i}m_s+\mathrm{i}\widetilde{m}_s)\right]} H(1 - x, 1 - \bar{x}) \tag{3.3.24}$$

for some series G and H in positive integer powers of $1 - x$ and $1 - \bar{x}$. Since the N_f terms of the Higgs branch expansions around $x = 0$ and ∞ are linearly independent, the series G and H cannot both factorize. When studying the gauge theory analogue of the braiding matrix relating the s- and u-channel expansions in Section 5.2, we find that H factorizes as $h(1 - x)\bar{h}(1 - \bar{x})$, while G is a sum of $N_f - 1$ such factorized terms, with no preferred choice of splitting. We can expect the factorization of H because in the limit $|\mathrm{i}\sigma \pm \frac{B}{2}| \to \infty$ the integrand (3.3.22) factorizes into functions of $\mathrm{i}\sigma \pm \frac{B}{2}$.

3.3.2 Matching parameters for SQED

We wish to equate the expansions of Z obtained so far with an A_{N_f-1} Toda CFT correlator. Since the S^2 partition function has branch points at $(-1)^{N_f} z \in \{0, 1, \infty\}$, and factorizes when expanded around each of those points, the Toda correlator must be a four-point function with insertions at 0, 1, ∞, and $x = (-1)^{N_f} z$. The expansions near branch points have finitely many terms, hence the operator inserted at x must be a degenerate operator $\widehat{V}_{-b\omega}$ (labelled by the highest weight ω of a representation $\mathcal{R}(\omega)$ of A_{N_f-1}), and the correlator has the form

$$\Big\langle \widehat{V}_{\alpha_\infty}(\infty) \widehat{V}_{\hat{m}}(1) \widehat{V}_{-b\omega}(x, \bar{x}) \widehat{V}_{\alpha_0}(0) \Big\rangle \,. \tag{3.3.25}$$

The number of internal momenta allowed by the fusion rule for $\widehat{V}_{-b\omega}$ with a generic operator is equal to the dimension of $\mathcal{R}(\omega)$, hence $\mathcal{R}(\omega)$ must be the fundamental or antifundamental representation, to match the number of terms in (3.3.14) and (3.3.16). Without loss of generality (we can at this point conjugate all momenta), we choose the operator $\widehat{V}_{-bh_1}$, where h_1 is the highest weight of the fundamental representation. The momenta α_0, $\hat{m}$ and α_∞ can then be obtained by comparing dimensions of Toda CFT operators with the powers of $|x|^2$ and of $|1-x|^2$ appearing in the expansions of Z around $x=0$, $x=1$, and $x=\infty$.

The s-channel decomposition of the Toda correlator is a sum over internal momenta $\alpha_0 - bh_p$ labelling W_{N_f} primary operators:

$$\begin{aligned}&\left\langle \widehat{V}_{\alpha_\infty}(\infty)\widehat{V}_{\hat{m}}(1)\widehat{V}_{-bh_1}(x,\bar{x})\widehat{V}_{\alpha_0}(0)\right\rangle \\ &= \sum_{p=1}^{N_f} \widehat{C}(\alpha_\infty,\hat{m},\alpha_0-bh_p)\widehat{C}^{\alpha_0-bh_p}_{-bh_1,\alpha_0}\left|\mathcal{F}^{(\mathrm{s})}_{\alpha_0-bh_p}\begin{bmatrix}\hat{m} & -bh_1\\ \alpha_\infty & \alpha_0\end{bmatrix}(x)\right|^2 ,\end{aligned} \tag{3.3.26}$$

where $\widehat{C}$ denote three-point functions, $\mathcal{F}^{(\mathrm{s})}_{\alpha_0-bh_p}(x)$ are W_{N_f} conformal blocks, and $|\mathcal{F}(x)|^2 = \mathcal{F}(x)\,\mathcal{F}\bar{x}$ with otherwise identical parameters. Conformal invariance fixes $\mathcal{F}^{(\mathrm{s})}_{\alpha_0-bh_p}(x) = x^{\Delta(\alpha_0-bh_p)-\Delta(\alpha_0)-\Delta(-bh_1)}(1+\cdots)$, with a series $(1+\cdots)$ in positive integer powers of x. We compute

$$\Delta(\alpha_0-bh_p)-\Delta(\alpha_0)-\Delta(-bh_1) = b\langle\alpha_0-Q,h_p\rangle + \frac{N_f-1}{2}(b^2+1)\,. \tag{3.3.27}$$

This should be compared with the powers $x^{-\mathrm{i}m_p}$ appearing in (3.3.14). Since the weights h_p sum to zero, $\sum_p\langle\alpha-Q,h_p\rangle=0$, and we must allow for an overall shift by x^{γ_0} between the partition function and the correlator. Power matching then dictates

$$b\langle\alpha_0-Q,h_p\rangle + \frac{N_f-1}{2}(b^2+1)+\gamma_0 = -\mathrm{i}m_p\,, \tag{3.3.28}$$

up to permutations, from which we deduce α_0 and γ_0 given in (3.3.2) and (3.3.3). Permuting the m_p is equivalent to permuting the components of α_0-Q, a Weyl reflexion under which the primary operator $\widehat{V}_{\alpha_0}$ is invariant.

Next, the u-channel decomposition is a sum over the internal momenta $\alpha_\infty - bh_p$. Conformal invariance fixes the leading behaviour $\mathcal{F}^{(\mathrm{u})}_{\alpha_\infty-bh_p}(x) = x^{\Delta(\alpha_\infty)-\Delta(\alpha_\infty-bh_p)-\Delta(-bh_1)}(1+\cdots)$, with a series $(1+\cdots)$ in negative integer powers of x. We compute

$$\Delta(\alpha_\infty)-\Delta(\alpha_\infty-bh_p)-\Delta(-bh_1) = \frac{N_f-1}{2}(b^2+1)+\frac{N_f-1}{N_f}b^2-b\langle\alpha_\infty-Q,h_p\rangle\,, \tag{3.3.29}$$

which should be compared with $x^{\mathrm{i}\widetilde{m}_p-\gamma_0}$. Once more, we must allow for an overall ambiguity: besides x^{γ_0}, the only other factor that can appear is

$(1-x)^{\gamma_1}$, since the Toda correlator is only singular at 0, 1, and ∞. This factor does not alter powers at $x=0$, and the power matching at $x=\infty$ reads

$$-b\langle\alpha_\infty - Q, h_p\rangle + \frac{N_f-1}{2}(b^2+1) + \frac{N_f-1}{N_f}b^2 + \gamma_0 + \gamma_1 = \mathrm{i}\widetilde{m}_p\,, \quad (3.3.30)$$

up to permutations: this fixes α_∞ and γ_1 to (3.3.2) and (3.3.4).

Finally, the expansion of Z near $x=1$ involves the leading powers $(1-x)^0$ with multiplicity N_f-1 and $(1-x)^{-1+\sum_{p=1}^{N_f}(1+\mathrm{i}m_p+\mathrm{i}\widetilde{m}_p)}$ with no multiplicity. On the Toda CFT side, the exponents that can appear in the t-channel are

$$\begin{aligned}&\Delta(\alpha_1-bh_p)-\Delta(\alpha_1)-\Delta(-bh_1)+\gamma_1\\&=b\langle\alpha_1-Q,h_p\rangle+\frac{N_f-1}{2}(b^2+1)-\frac{N_f-1}{N_f}b^2+\frac{1}{N_f}\sum_{p=1}^{N_f}(\mathrm{i}m_p+\mathrm{i}\widetilde{m}_p)\,.\end{aligned} \quad (3.3.31)$$

If α_1 were generic, all shifts $-bh_p$ would be allowed by the fusion, but summing the powers (3.3.31) for $1\leq p\leq N_f$ does not yield the similar gauge theory sum $-1+\sum_{p=1}^{N_f}(1+\mathrm{i}m_p+\mathrm{i}\widetilde{m}_p)$. Instead, we take $\alpha_1=\hat{m}=(\varkappa+b)h_1$ to be a semi-degenerate momentum (with a shift by b to simplify expressions), so that the fusion rule only allows shifts to $\hat{m}-bh_2$ and $\hat{m}-bh_1$. Setting the exponent for a shift $\hat{m}-bh_2$ to 0 fixes $\varkappa$ to (3.3.2), and the second power matches (setting $\hat{m}-bh_1$ to 0 instead would fail to match the second power). The $SU(N_f)\times SU(N_f)\times U(1)$ flavour symmetry of the gauge theory is reproduced by the two generic and one semi-degenerate operators in the correlator, allowing us to package the twisted masses of fundamental chiral multiplets into α_0, those of antifundamental multiplets into α_∞, and the axial mass into $\hat{m}$.

Finally, the overall constant A is fixed in Section 5.4.4 by comparing gauge theory one-loop determinants and Toda three-point functions: for A given by (3.3.5),

$$Z_{S_b^4}^{\text{free}}\frac{\prod_{s\neq p}^{N_f}\gamma(\mathrm{i}m_p-\mathrm{i}m_s)}{\prod_{t=1}^{N_f}\gamma(1+\mathrm{i}m_p+\mathrm{i}\widetilde{m}_t)} = A\widehat{C}(\alpha_\infty,(\varkappa+b)h_1,\alpha_0-bh_p)\widehat{C}^{\alpha_0-bh_p}_{-bh_1,\alpha_0}\,. \quad (3.3.32)$$

The same relation holds for u-channel constant factors (with an identical value of A), as we can obtain most readily thanks to the invariance of Z under $m_p\leftrightarrow\widetilde{m}_p$ and $z\leftrightarrow\frac{1}{z}$ (gauge theory charge conjugation) and equivalently of the Toda correlator (up to a shift in exponents) under $\alpha_0\leftrightarrow\alpha_\infty$ and $x\leftrightarrow\frac{1}{x}$.

We have thus fixed how gauge theory and Toda CFT parameters match. One way to prove the matching is to directly equate gauge theory factors with conformal blocks as done in [DGLFL12], but this approach does not generalize. Instead, we show in Section 5.2.1 that the matrix to change basis from s-channel factors $x^{-\mathrm{i}m_p}f_p^{(\mathrm{s})}(x)$ to u-channel factors is identical to the appropriate braiding matrix in the Toda CFT. Since the eigenvalues of monodromies around 0 and ∞ also match up to shifts by the γ_i as we just saw, the monodromy matrices themselves agree. The last monodromy matrix, around $x = 1$, thus also matches. Therefore, the partition function and the correlator differ by a factor with no monodromy. Since the precise exponents match, the relative factor is in fact constant, and comparing constant coefficients establishes the matching (3.3.1).

3.4 SQCD and Toda antisymmetric degenerate

We now extend the matching to the case of $\mathcal{N} = (2,2)$ SQCD, that is, a $U(N)$ vector multiplet coupled to N_f fundamental and N_f antifundamental chiral multiplets, with twisted masses (plus R-charges) m_s and $\widetilde{m}_s$. The partition function of the S^2 surface operator defined by this theory coupled to N_f^2 hypermultiplets with masses (3.2.17) on S_b^4 is captured by a Toda CFT four-point function with a degenerate operator $\widehat{V}_{-b\omega_N}$ labelled by the N-th antisymmetric representation of A_{N_f-1}. Explicitly, we prove that[18]

$$Z^{U(N)\ \mathrm{SQCD}}_{S^2\subset S^4_b}(m,\widetilde{m},z,\bar{z}) = A|x|^{2\gamma_0}|1-x|^{2\gamma_1}\left\langle \widehat{V}_{\alpha_\infty}(\infty)\widehat{V}_{\hat{m}}(1)\widehat{V}_{-b\omega_N}(x,\bar{x})\widehat{V}_{\alpha_0}(0)\right\rangle \tag{3.4.1}$$

with $x = (-1)^{N_f} z$, momenta

$$\begin{aligned} \alpha_0 &= Q - \frac{1}{b}\sum_{s=1}^{N_f} \mathrm{i}m_s h_s & \hat{m} &= (\varkappa + Nb)h_1 \\ \alpha_\infty &= Q - \frac{1}{b}\sum_{s=1}^{N_f} \mathrm{i}\widetilde{m}_s h_s & \varkappa &= \frac{1}{b}\sum_{s=1}^{N_f}(1+\mathrm{i}m_s+\mathrm{i}\widetilde{m}_s)\,, \end{aligned} \tag{3.4.2}$$

and coefficients

$$\gamma_0 = -\frac{N}{N_f}\sum_{s=1}^{N_f}\mathrm{i}m_s - \frac{N(N_f-N)}{2}(b^2+1) \tag{3.4.3}$$

$$\gamma_1 = -\frac{N(N_f-N)}{N_f}b^2 + \frac{N}{N_f}\sum_{s=1}^{N_f}(\mathrm{i}m_s+\mathrm{i}\widetilde{m}_s) \tag{3.4.4}$$

$$A = b^{NN_f(1+b^2)-N^2b^2-2Nb\varkappa}\,. \tag{3.4.5}$$

Setting $N = 1$ in (3.4.1) reproduces the SQED matching (3.3.1). We recognize the same symmetries as SQED. Permuting twisted masses m_s or $\widetilde{m}_s$ amounts to a Weyl transformation of α_0 or α_∞. Gauge theory charge conjugation,

[18] As explained below (3.2.10), the factor $A|x|^{2\gamma_0}|1-x|^{2\gamma_1}$ can be absorbed into the partition function. To compare gauge theory and Toda CFT results it is best to keep the factor explicitly.

which swaps $m_s \leftrightarrow \widetilde{m}_s$ and $z \leftrightarrow \frac{1}{z}$, corresponds to the conformal map $(\infty, 1, x, 0) \to (0, 1, \frac{1}{x}, \infty)$, which exchanges $\alpha_0 \leftrightarrow \alpha_\infty$ and $x \leftrightarrow \frac{1}{x}$ in the Toda CFT correlator.

We start the analysis from the Coulomb branch representation

$$Z_{S^2}^{\text{SQCD}} = \frac{1}{N!} \sum_{B \in \mathbb{Z}^N} \int_{\mathbb{R}^N} \frac{\mathrm{d}^N \sigma}{(2\pi)^N} \Bigg\{ z^{\operatorname{Tr}(\mathrm{i}\sigma + \frac{B}{2})} \bar{z}^{\operatorname{Tr}(\mathrm{i}\sigma - \frac{B}{2})} \prod_{i<j} \left[(\sigma_i - \sigma_j)^2 + \frac{(B_i - B_j)^2}{4} \right] \cdot \prod_{j=1}^{N} \prod_{s=1}^{N_f} \left[\frac{\Gamma(-\mathrm{i}m_s - \mathrm{i}\sigma_j - \frac{B_j}{2})}{\Gamma(1 + \mathrm{i}m_s + \mathrm{i}\sigma_j - \frac{B_j}{2})} \frac{\Gamma(-\mathrm{i}\widetilde{m}_s + \mathrm{i}\sigma_j + \frac{B_j}{2})}{\Gamma(1 + \mathrm{i}\widetilde{m}_s - \mathrm{i}\sigma_j + \frac{B_j}{2})} \right] \Bigg\} . \tag{3.4.6}$$

The partition function can be studied in the same way as that of SQED, by closing the integration contours towards either half-plane depending on whether $|z| \lessgtr 1$, thus obtaining an s-channel and a u-channel decompositions akin to (3.3.14) and (3.3.16). Interestingly, there is a shortcut, as the SQCD partition function can be expressed as a differential operator acting on the product of N copies of the SQED partition function:

$$Z_{S^2}^{\text{SQCD}}(m, \widetilde{m}, z, \bar{z}) = \frac{1}{N!} \left[\prod_{i<j} \left[-(z_i \partial_{z_i} - z_j \partial_{z_j})(\bar{z}_i \partial_{\bar{z}_i} - \bar{z}_j \partial_{\bar{z}_j}) \right] \prod_{j=1}^{N} Z_{S^2}^{\text{SQED}}(m, \widetilde{m}, z_j, \bar{z}_j) \right]_{\substack{z_j = z \\ \bar{z}_j = \bar{z}}} . \tag{3.4.7}$$

Since the differential operator cannot introduce branch points, the SQCD partition function has the same branch points $z \in \{0, (-1)^{N_f}, \infty\}$ as the SQED partition function, and we switch to using the coordinate $x = (-1)^{N_f} z$.

3.4.1 Expanding the SQCD partition function

Using the s-channel decomposition (3.3.14) of Z^{SQED} in the above yields a sum over flavours $1 \leq p_1, \ldots, p_N \leq N_f$. The summand factorizes, since both the differential operator and the terms in Z^{SQED} are products of a holomorphic and an antiholomorphic parts. The holomorphic and the antiholomorphic factors are each totally antisymmetric in the p_j, hence reducing the sum to $1 \leq p_1 < \cdots < p_N \leq N_f$. Explicitly,

$$Z = \sum_{1 \leq p_1 < \cdots < p_N \leq N_f} \left[(x\bar{x})^{-\sum_{j=1}^{N} \mathrm{i}m_{p_j}} \prod_{j=1}^{N} \frac{\prod_{s \notin \{p\}}^{N_f} \gamma(-\mathrm{i}m_s + \mathrm{i}m_{p_j})}{\prod_{s=1}^{N_f} \gamma(1 + \mathrm{i}\widetilde{m}_s + \mathrm{i}m_{p_j})} f_{\{p\}}^{(\mathrm{s})}(x) f_{\{p\}}^{(\mathrm{s})}(\bar{x}) \right] \tag{3.4.8}$$

where we have cancelled $\prod_{i \neq j} \gamma(-\mathrm{i}m_{p_i} + \mathrm{i}m_{p_j}) = \prod_{i \neq j} (\mathrm{i}m_{p_i} - \mathrm{i}m_{p_j})^{-1}$ and defined

$$f_{\{p\}}^{(\mathrm{s})}(x) = \left[\prod_{i<j} \frac{-\mathrm{i}m_{p_i} + \mathrm{i}m_{p_j} + x_i \partial_{x_i} - x_j \partial_{x_j}}{-\mathrm{i}m_{p_i} + \mathrm{i}m_{p_j}} \prod_{j=1}^{N} f_{p_j}^{(\mathrm{s})}(x_j) \right]_{x_j = x} \tag{3.4.9}$$

$$= \sum_{k_1,\dots,k_N\geq 0} \prod_{j=1}^{N} \frac{(x^{k_j}/k_j!)\prod_{s=1}^{N_f}(-\mathrm{i}\widetilde{m}_s - \mathrm{i}m_{p_j})_{k_j}}{\prod_{i\neq j}^{N}(\mathrm{i}m_{p_i} - \mathrm{i}m_{p_j} - k_i)_{k_j}\prod_{s\notin\{p\}}^{N_f}(1+\mathrm{i}m_s - \mathrm{i}m_{p_j})_{k_j}}\,, \tag{3.4.10}$$

a series in positive integer powers of x, with radius of convegence 1, and whose first term is normalized to be 1. Similarly, the u-channel expansion near $x = \infty$ reads

$$Z = \sum_{1\leq p_1<\cdots<p_N\leq N_f}\left[(x\bar{x})^{\sum_{j=1}^{N}\mathrm{i}\widetilde{m}_{p_j}}\frac{\prod_{j=1}^{N}\prod_{s\notin\{p\}}^{N_f}\gamma(-\mathrm{i}\widetilde{m}_s+\mathrm{i}\widetilde{m}_{p_j})}{\prod_{j=1}^{N}\prod_{s=1}^{N_f}\gamma(1+\mathrm{i}m_s+\mathrm{i}\widetilde{m}_{p_j})}f^{(\mathrm{u})}_{\{p\}}(x)f^{(\mathrm{u})}_{\{p\}}(\bar{x})\right] \tag{3.4.11}$$

where

$$f^{(\mathrm{u})}_{\{p\}}(x) = \sum_{k_1,\dots,k_N\geq 0}\prod_{j=1}^{N}\frac{1/(k_j!x^{k_j})\prod_{s=1}^{N_f}(-\mathrm{i}m_s - \mathrm{i}\widetilde{m}_{p_j})_{k_j}}{\prod_{i\neq j}^{N}(\mathrm{i}\widetilde{m}_{p_i} - \mathrm{i}\widetilde{m}_{p_j} - k_i)_{k_j}\prod_{s\notin\{p\}}^{N_f}(1+\mathrm{i}\widetilde{m}_s - \mathrm{i}\widetilde{m}_{p_j})_{k_j}} \tag{3.4.12}$$

are series in negative integer powers of x.

The s- and u-channel decompositions above can also be obtained by localizing to the Higgs branch of the theory, with a positive or a negative FI parameter. In this setting, they arise as sums over Higgs branch vacua, labelled by solutions $(\sigma, q_s, \widetilde{q}_s)$ of

$$\begin{aligned}(\sigma + m_s)q_s &= 0\\ (-\sigma + \widetilde{m}_s)\widetilde{q}_s &= 0\end{aligned}\qquad \sum_{s=1}^{N_f}(q_s q_s^\dagger - \widetilde{q}_s^\dagger\widetilde{q}_s) = \xi\,\mathrm{id}_N\,, \tag{3.4.13}$$

up to gauge transformations. In the region $|x| = |z| < 1$, that is, $\xi > 0$, the D-term equation (3.4.13) can be rewritten as

$$\sum_{s=1}^{N_f} q_s q_s^\dagger = \xi\,\mathrm{id}_N + \sum_{s=1}^{N_f}\widetilde{q}_s^\dagger\widetilde{q}_s\,, \tag{3.4.14}$$

which is positive definite, hence has full rank N. Therefore, the non-zero vectors q_s, which are eigenvectors of σ, span $\mathbb{C}^{N_f}$. The eigenvalues of σ are thus completely fixed to be $-m_{p_j}$ for a choice of N distinct flavours p_j. On the contrary, for $|x| = |z| > 1$, that is, $\xi < 0$, the antifundamental chiral fields $\widetilde{q}_s$ span $\mathbb{C}^{N_f}$, and σ has eigenvalues $\widetilde{m}_{p_j}$. The classical and one-loop contributions derived for each of those vacua is equal to those appearing in (3.4.8) and (3.4.11). More tediously, one checks that the vortex partition functions are indeed given by $f^{(\mathrm{s})}_{\{p\}}(x)$ and $f^{(\mathrm{u})}_{\{p\}}(x)$.

Once more, the t-channel is the most troublesome. We know from (3.3.24) the expansion of the SQED partition function near $x = 1$, leading to

$$Z^{\mathrm{SQED}} = G(1-x, 1-\bar{x}) + |1-x|^{2(\gamma-1)}h(1-x)\bar{h}(1-\bar{x}) \tag{3.4.15}$$

where

$$\gamma = \sum_{s=1}^{N_f} (1 + \mathrm{i}m_s + \mathrm{i}\widetilde{m}_s) \,. \tag{3.4.16}$$

The functions G and $h\bar{h}$ are series in positive integer powers of $1-x$ and $1-\bar{x}$, and G does not factorize because the eigenvalue 1 of the monodromy has multiplicity $N_f - 1$. Plug this t-channel expansion into (3.4.7):

$$Z^{\mathrm{SQCD}}(z,\bar{z}) = \frac{1}{N!} \Bigg[\prod_{i<j} \Big[-(x_i\partial_{x_i} - x_j\partial_{x_j})(\bar{x}_i\partial_{\bar{x}_i} - \bar{x}_j\partial_{\bar{x}_j}) \Big] \cdot \prod_{j=1}^{N} \Big\{ G(1-x_j, 1-\bar{x}_j) + |1-x_j|^{2\gamma-2} h(1-x_j)\bar{h}(1-\bar{x}_j) \Big\} \Bigg]_{\substack{x_j = x \\ \bar{x}_j = \bar{x}}} . \tag{3.4.17}$$

Among the 2^N terms in the product of SQED partition functions, any which contains the factor $|1-x_j|^{2\gamma-2} h(1-x_j)\bar{h}(1-\bar{x}_j)$ for two indices i and j is annihilated by $x_i\partial_{x_i} - x_j\partial_{x_j}$, hence does not contribute. The annihilation does not take place when $G(1-x_j, 1-\bar{x}_j)$ appears twice, as it relies on separating the holomorphic and antiholomorphic parts. Thus, $1+N$ terms remain, and we can replace the product by

$$\prod_{j=1}^{N} G(1-x_j, 1-\bar{x}_j) + \sum_{j=1}^{N} |1-x_j|^{2\gamma-2} h(1-x_j)\bar{h}(1-\bar{x}_j) \prod_{i\neq j}^{N} G(1-x_i, 1-\bar{x}_i) \,. \tag{3.4.18}$$

Derivatives acting on G, h and $\bar{h}$ yield other series in positive integer powers of $1-x_j$ and $1-\bar{x}_j$, hence for the purpose of finding exponents for $|1-x|^2$ we only need to keep track of $|1-x_j|^{2\gamma-2}$. At most $(N-1)$ x_j derivatives can affect it, hence the SQCD partition function takes the form

$$Z^{\mathrm{SQCD}}(z,\bar{z}) = G'(1-x, 1-\bar{x}) + |1-x|^{2(\gamma-N)} H'(1-x, 1-\bar{x}) \,, \tag{3.4.19}$$

for some series G' and H'. The two terms correspond to eigenvalues 1 and $e^{2\pi\mathrm{i}(\gamma-N)}$ of the monodromy around $x=1$. We find out the multiplicities with which the powers appear by doing a finer expansion: split $G(1-x_j, 1-\bar{x}_j) = \sum_{i=1}^{N_f-1} g_i(1-x_j)\bar{g}_i(1-\bar{x}_j)$ non-canonically. Antisymmetry restricts the sum of N_f^N terms to $\binom{N_f}{N}$, each of which is a product of N distinct terms of Z^{SQED} among $h\bar{h}$ and the $g_i\bar{g}_i$. The exponent for a given combination is $2(\gamma-N)$ if $h\bar{h}$ appears, and 0 otherwise. The multiplicity of $|1-x|^0$ is thus $\binom{N_f-1}{N}$, and that of $|1-x|^{2(\gamma-N)}$ is $\binom{N_f-1}{N-1}$.

3.4.2 Matching parameters for SQCD

We are at last ready to match SQCD and Toda CFT parameters. The partition function depends on a single parameter x encoded as the position of a puncture, hence we expect a four-point function on the Toda side. The s-channel and u-channel decompositions involve $\binom{N_f}{N}$ terms, hence the Toda degenerate operator is labelled by the N-th antisymmetric representation $\mathcal{R}(\omega_N)$ of A_{N_f-1}, which has the correct dimension. The highest weight of this representation is $\omega_N = h_1+\cdots+h_N$, and its weights are $h_{\{p\}} = h_{p_1}+\cdots+h_{p_N}$, labelled by N-element sets $1 \leq p_1 < \cdots < p_N \leq N_f$.

The s-channel Toda exponents

$$\begin{aligned}&\Delta(\alpha_0 - bh_{\{p\}}) - \Delta(\alpha_0) - \Delta(-b\omega_N) + \gamma_0 \\ &= b\sum_{j=1}^{N}\langle\alpha_0 - Q, h_{p_j}\rangle + \frac{N(N_f-N)}{2}(b^2+1) + \gamma_0\end{aligned} \tag{3.4.20}$$

must be equal to $-\sum_{j=1}^{N} \mathrm{i}m_{p_j}$ from gauge theory (up to permutations): this constraint fixes α_0 and γ_0 as given in (3.4.2) and (3.4.3). Matching powers in the u-channel,

$$\begin{aligned}\sum_{j=1}^{N}\mathrm{i}\widetilde{m}_{p_j} &= \Delta(\alpha_\infty) - \Delta(\alpha_\infty - bh_{\{p\}}) - \Delta(-b\omega_N) + \gamma_0 + \gamma_1 \\ &= -b\sum_{j=1}^{N}\langle\alpha_\infty - Q, h_{p_j}\rangle + \frac{N(N_f-N)}{2}(b^2+1) + \frac{N(N_f-N)}{N_f}b^2 + \gamma_0 + \gamma_1\end{aligned} \tag{3.4.21}$$

fixes α_∞ and γ_1.

We finally match powers in the t-channel. From our SQED experience, we expect the momentum at 1 to be the semi-degenerate $\hat{m} = (\varkappa + Nb)h_1$ (the shift by Nb simplifies expressions). We compute the exponents

$$\begin{aligned}&\Delta\big((\varkappa + Nb)h_1 - bh_{\{p\}}\big) - \Delta\big((\varkappa + Nb)h_1\big) - \Delta(-b\omega_N) + \gamma_1 \\ &= (b\varkappa + Nb^2)\langle h_1, h_{\{p\}}\rangle + \frac{N}{N_f}\Bigg[\sum_{s=1}^{N_f}(1 + \mathrm{i}m_s + \mathrm{i}\widetilde{m}_s) + Nb^2\Bigg] + (1+b^2)\sum_{j=1}^{N}(p_j - j - 1)\,,\end{aligned} \tag{3.4.22}$$

where $\langle h_1, h_{\{p\}}\rangle = \delta_{1\in\{p\}} - N/N_f$. Two different sets $\{p\}$ must reproduce the gauge theory exponents 0 and $-N + \sum_{s=1}^{N_f}(1 + \mathrm{i}m_s + \mathrm{i}\widetilde{m}_s)$. One set must contain 1 and the other not, since the exponents would otherwise only differ by an integer multiple of $1+b^2$: this fixes $\varkappa = \pm\sum_{s=1}^{N_f}(1+\mathrm{i}m_s+\mathrm{i}\widetilde{m}_s)+n(b+\frac{1}{b})$ for some integer n. Comparing the coefficients of $\sum_{s=1}^{N_f}(1+\mathrm{i}m_s+\mathrm{i}\widetilde{m}_s)$ selects the positive sign, and also implies that the exponent 0 corresponds to a case where $1 \notin \{p\}$ while the other exponent has $1 \in \{p'\}$. Comparing the coefficients of b^2+1, the Toda CFT and gauge exponents match if

$$-\frac{N}{N_f}n + \sum_{j=1}(p_j - j - 1) = 0 \qquad \frac{N_f - N}{N_f}n + \sum_{j=1}(p'_j - j - 1) = -N \tag{3.4.23}$$

for the choices of $\{p\}$ and $\{p'\}$ corresponding to the two exponents. Since $1 \notin \{p\}$, $p_j \geq j+1$ and the first relation implies $n \leq 0$. Since $p'_j \geq j$, the second implies $n \geq 0$, and we conclude that $\varkappa$ is given by (3.4.2), that $\{p\} = [\![2, N+1]\!]$ and that $\{p'\} = [\![1, N]\!]$. After we show independently that the partition function and Toda correlator are equal, we deduce that the fusion of $\widehat{V}_{-b\omega_N}$ with $\widehat{V}_{\varkappa' h_1}$ allows the momenta $\varkappa' h_1 - b\omega_N$ and $\varkappa' h_1 + bh_1 - b\omega_{N+1}$. This is consistent with the case $\varkappa' = -kb$ for which the semi-degenerate insertion becomes a degenerate field labelled by the k-th antisymmetric representation: the tensor product of this representation with the N-th antisymmetric splits as a sum of two irreducible representations of A_{N_f-1}, with highest weights $kh_1 + \omega_N$ and $(k-1)h_1 + \omega_{N+1}$. We discuss such fusion rules further in Section 5.5.

Last, we fix the constant A. We check in Section 5.4.4 that the one-loop determinant and the three-point functions appearing in the s-channel decompositions of Z^{SQCD} and of the Toda correlator match, for A given in (3.4.5):

$$Z^{\text{free}}_{S^4_b} \frac{\prod_{s\notin\{p\}}^{N_f} \prod_{t\in\{p\}} \gamma(\mathrm{i}m_t - \mathrm{i}m_s)}{\prod_{s=1}^{N_f} \prod_{t\in\{p\}} \gamma(1 + \mathrm{i}m_t + \mathrm{i}\widetilde{m}_s)} = A\widehat{C}(\alpha_\infty, (\varkappa + Nb)h_1, \alpha_0 - bh_{\{p\}})\widehat{C}^{\alpha_0 - bh_{\{p\}}}_{-b\omega_N, \alpha_0} . \tag{3.4.24}$$

Having settled the dictionary above, we know that gauge theory and Toda CFT monodromy matrices around each of 0, 1 and ∞ have matching eigenvalues. In Section 5.2.2, we compute the braiding matrix of $\widehat{V}_{-b\omega_N}$ and $\widehat{V}_{\hat{m}}$ by combining the fusion of N operators $\widehat{V}_{-bh_1}$ into $\widehat{V}_{-b\omega_N}$ with the braiding matrices for each individual $\widehat{V}_{-bh_1}$ with $\widehat{V}_{\hat{m}}$. The result agrees with the analogue for SQCD, an antisymmetric combination of the matrix for SQED, worked out in the same section. Therefore, the monodromy matrices around 0 and around ∞ are equal for SQCD and the Toda CFT. Monodromy matrices around 1 then also match, hence the Toda CFT correlator and gauge theory partition function are equal up to a factor with no monodromy, which is constant since the precise exponents at 0, 1 and ∞ match. The constant factors work out, thereby concluding the proof of the matching (3.4.1).

3.4.3 Decoupled multiplets and irregular puncture

In this section, we give large twisted masses to $N_f - \widetilde{N_f}$ of the N_f antifundamental chiral multiplets of the SQCD surface operator, hence to $N_f(N_f - \widetilde{N_f})$ of the four-dimensional hypermultiplets. The massive multiplets decouple, and we obtain in this limit (3.4.26) a surface operator described by a $U(N)$ vector multiplet, N_f fundamental and $\widetilde{N_f} < N_f$ antifundamental chiral multiplets, coupled to the remaining $N_f\widetilde{N_f}$ free hypermultiplets. On the Toda CFT side of the matching (3.4.1), the limit amounts to building a Toda CFT irregular puncture from the collision of two vertex operators. We give the precise matching (3.4.33) in the case $\widetilde{N_f} = N_f - 1$, and claim that further limits for $\widetilde{N_f} \leq N_f - 2$ also lead to well-defined irregular punctures.

In a two-dimensional $\mathcal{N} = (2,2)$ gauge theory, whenever the total charge $Q = \sum_i Q_i$ of all chiral multiplets under a given $U(1)$ gauge group factor is non-zero (in our case, $Q = N_f - \widetilde{N_f} > 0$), the corresponding FI parameter runs logarithmically, and the theta angle is shifted. An ultraviolet cutoff can be introduced supersymmetrically by enriching the theory with a single "expectator" chiral multiplet of large twisted mass[19] $\Lambda \in \mathbb{R}$ and $U(1)$ charge $-Q$, or with Q antifundamental expectator chiral multiplets of twisted masses Λ. We take the latter approach, as the resulting enriched theory is simply SQCD with N_f fundamental and N_f antifundamental chiral multiplets. Each expectator chiral multiplet brings a one-loop contribution

$$\prod_{j=1}^{N} \frac{\Gamma(-\mathrm{i}\Lambda + \mathrm{i}\sigma_j + \frac{B_j}{2})}{\Gamma(1 + \mathrm{i}\Lambda - \mathrm{i}\sigma_j + \frac{B_j}{2})} \overset{\Lambda\to\infty}{\sim} \prod_{j=1}^{N} \left(\frac{\Gamma(-\mathrm{i}\Lambda)}{\Gamma(1+\mathrm{i}\Lambda)} (-\mathrm{i}\Lambda)^{\mathrm{i}\sigma_j + B_j/2} (\mathrm{i}\Lambda)^{\mathrm{i}\sigma_j - B_j/2} \right) \tag{3.4.25}$$

to the Coulomb branch expression for the enriched theory. The original partition function is thus a limit of the enriched partition function,

$$Z(m, \widetilde{m}, z, \bar{z}) = \lim_{\Lambda\to\infty} \left[\frac{1}{\gamma(-\mathrm{i}\Lambda)^{N(N_f - \widetilde{N_f})}} Z_{\mathrm{enr}}(m, \{\widetilde{m}, \Lambda\}, z_{\mathrm{bare}}, \bar{z}_{\mathrm{bare}}) \right], \tag{3.4.26}$$

where the factor $\gamma(-\mathrm{i}\Lambda)^{-N(N_f - \widetilde{N_f})}$ has no physical effect, and the bare parameter z_{bare} appearing in the enriched theory is related to the renormalized $z = z_{\mathrm{ren}}$ (at the scale ℓ given by the equatorial radius of the squashed sphere) via

$$z_{\mathrm{bare}} = \frac{z_{\mathrm{ren}}}{(-\mathrm{i}\Lambda)^{N_f - \widetilde{N_f}}}, \quad \text{and} \quad \bar{z}_{\mathrm{bare}} = \frac{\bar{z}_{\mathrm{ren}}}{(\mathrm{i}\Lambda)^{N_f - \widetilde{N_f}}}. \tag{3.4.27}$$

In particular, the FI parameter runs logarithmically, and the theta angle is shifted:

$$\xi_{\mathrm{ren}} = \xi_{\mathrm{bare}} - \frac{1}{2\pi}(N_f - \widetilde{N_f}) \ln \Lambda, \quad \text{and} \quad \vartheta_{\mathrm{ren}} = \vartheta_{\mathrm{bare}} + \frac{\pi}{2}(N_f - \widetilde{N_f}). \tag{3.4.28}$$

Since the Coulomb branch representation involves an integral over arbitrarily large values of $\sigma \pm \mathrm{i}\frac{B}{2}$, our derivation of (3.4.26) above is not rigorous. However, one can split the integral into a region $\left|\sigma \pm \mathrm{i}\frac{B}{2}\right| \ll \Lambda$ and its complement, and check that the contribution from large $\sigma \pm \mathrm{i}\frac{B}{2}$ becomes negligible as $\Lambda \to \infty$. It is more convenient to perform such steps on the Higgs branch decomposition (3.4.8) of Z_{enr} near 0. Regardless of the value

[19]The dimensionful cutoff is Λ/ℓ in terms of the equatorial radius ℓ of the squashed two-sphere.

of z, the series expansions of vortex partition functions converges for Λ large enough that $|z_{\text{bare}}| = |z|/\Lambda^{N_f-\widetilde{N_f}} < 1$. Then each term in the series for the enriched theory converges to the appropriate term for the $\widetilde{N_f} < N_f$ theory. Since the sum of terms with $\sum_{j=1}^{N} k_j > K$ decreases exponentially with K in both series, $Z_{\text{v,enr}}(z_{\text{bare}}) \to Z_{\text{v}}(z)$. Other factors work out as for the Coulomb branch representation.

In the limit above, $N_f(N_f - \widetilde{N_f})$ of the N_f^2 free hypermultiplets on S_b^4 become infinitely massive, and the corresponding factors must be removed from the enriched partition function to retain a finite result. The partition function of the surface operator with $\widetilde{N_f} < N_f$ in a theory of $N_f\widetilde{N_f}$ free hypermultiplets of masses (3.2.17) is thus the limit

$$Z^{U(N)}_{S^2\subset S_b^4}(z,\bar{z}) = \lim_{\Lambda\to\infty}\left[\left[\frac{\prod_{s=1}^{N_f}\Upsilon\left(\frac{1}{b}(1+\mathrm{i}m_s+\mathrm{i}\Lambda)\right)}{\gamma(-\mathrm{i}\Lambda)^N}\right]^{N_f-\widetilde{N_f}} Z^{U(N)}_{S^2\subset S_b^4,\text{enr}}(z_{\text{bare}},\bar{z}_{\text{bare}})\right]. \tag{3.4.29}$$

We now provide a Toda CFT interpretation of the limit for $N_f - \widetilde{N_f} = 1$. For simplicity, label antifundamental multiplets of the enriched theory starting with the expectator multiplet, so that $\widetilde{m}_1 = \Lambda \to \infty$. Replace the partition function of the enriched defect in (3.4.29) by its corresponding Toda CFT four-point function through the matching (3.4.1). After a conformal transformation which maps $(\infty, 1, x/(-\mathrm{i}\Lambda), 0)$ to $(0, x/(-\mathrm{i}\Lambda), 1, \infty)$,

$$\begin{aligned} Z^{U(N)}_{S^2\subset S_b^4}(z,\bar{z}) = \lim_{\Lambda\to\infty}\Bigg[&A_{\text{enr}}\left|\frac{x}{\Lambda}\right|^{2\gamma_{0,\text{enr}}-2\Delta(\alpha_0)-2\Delta(-b\omega_N)+2\Delta(\alpha_\infty)+2\Delta(\hat{m})}\left|1-\frac{x}{-\mathrm{i}\Lambda}\right|^{2\gamma_{1,\text{enr}}} \\ &\cdot\frac{\prod_{s=1}^{N_f}\Upsilon\left(\frac{1}{b}(1+\mathrm{i}m_s+\mathrm{i}\Lambda)\right)}{\gamma(-\mathrm{i}\Lambda)^N}\left\langle \widehat{V}_{\alpha_0}(\infty)\widehat{V}_{-b\omega_N}(1)\widehat{V}_{\hat{m}}\left(\frac{x}{-\mathrm{i}\Lambda},\frac{\bar{x}}{\mathrm{i}\Lambda}\right)\widehat{V}_{\alpha_\infty}(0)\right\rangle\Bigg] \end{aligned} \tag{3.4.30}$$

with $x = (-1)^{N_f}z$, and parameters α_0, $\hat{m} = (\varkappa + Nb)h_1$, α_∞, A_{enr}, $\gamma_{0,\text{enr}}$ and $\gamma_{1,\text{enr}}$ given below (3.4.1). In the limit $\Lambda \to \infty$, the exponent $\gamma_{1,\text{enr}} \sim \frac{N}{N_f}\mathrm{i}\Lambda$, thus $|1 - x/(-\mathrm{i}\Lambda)|^{2\gamma_{1,\text{enr}}} \to e^{(N/N_f)(x+\bar{x})}$.

In the same limit, the punctures $\widehat{V}_{\hat{m}}$ and $\widehat{V}_{\alpha_\infty}$ collide, with momenta growing as the inverse of the distance, keeping a constant sum $c_0 + Nbh_1 = (\varkappa+Nb)h_1+\alpha_\infty$ given in (3.4.34). We study such collision limits in Section 5.6 and define (5.6.33)

$$\begin{aligned} \widehat{\mathbb{V}}_{c_0+Nbh_1;-(x/b)h_1,(\bar{x}/b)h_1}(0) = \lim_{\Lambda\to\infty}\Bigg[&\Upsilon\left(\varkappa+Nb+\langle Q-c_0-Nbh_1,h_1\rangle\right)^{N_f} \\ &\cdot\left[\frac{\Lambda}{b}\right]^{\langle Q,Q\rangle-2\Delta(c_0+Nbh_1)}\left|\frac{x}{\Lambda}\right|^{2\langle(\varkappa+Nb)h_1,c_0-\varkappa h_1\rangle}\widehat{V}_{(\varkappa+Nb)h_1}\left(\frac{x}{-\mathrm{i}\Lambda},\frac{\bar{x}}{\mathrm{i}\Lambda}\right)\widehat{V}_{c_0-\varkappa h_1}(0)\Bigg]_{\varkappa\sim\mathrm{i}\Lambda/b}. \end{aligned} \tag{3.4.31}$$

The Upsilon functions and gamma functions in (3.4.30) and (3.4.31) can be recast in the same form through the asymptotics (5.4.7), (5.4.6), and

$\gamma(1+\mathrm{i}\Lambda+a) \sim \gamma(1+\mathrm{i}\Lambda)\Lambda^{2a}$. Let $\mathrm{i}\underline{m} = \frac{1}{N_f}\sum_{s=1}^{N_f} \mathrm{i}m_s$. Then,

$$\begin{aligned} &\gamma(1+\mathrm{i}\Lambda)^N \prod_{s=1}^{N_f} \Upsilon\big(\tfrac{1}{b}(1+\mathrm{i}\Lambda+\mathrm{i}m_s)\big) \\ &\quad= \gamma(1+\mathrm{i}\Lambda)^N \prod_{s=1}^{N_f} \Upsilon\big(\tfrac{1}{b}(1+\mathrm{i}\Lambda+\mathrm{i}\underline{m}) + \langle Q-\alpha_0, h_s\rangle\big) \\ &\quad\sim \gamma(1+\mathrm{i}\Lambda)^N \Upsilon\big(\tfrac{1}{b}(1+\mathrm{i}\Lambda+\mathrm{i}\underline{m})\big)^{N_f} [\Lambda/b]^{\langle Q,Q\rangle - 2\Delta(\alpha_0)} \\ &\quad\sim \Upsilon\big(\tfrac{1}{b}(1+\mathrm{i}\Lambda+\mathrm{i}\underline{m}) + bN/N_f\big)^{N_f} b^{N+2N\mathrm{i}\Lambda} [\Lambda/b]^{\langle Q,Q\rangle - 2\Delta(\alpha_0) - 2N\mathrm{i}\underline{m} + N(N_f-N)b^2/N_f} \end{aligned} \tag{3.4.32}$$

The last Upsilon functions are precisely those appearing in (3.4.31). Plugging back into (3.4.30), all powers of Λ and b^Λ cancel, and we can drop the limit.

All in all, the partition function of a surface operator describing a $U(N)$ vector multiplet with N_f fundamental and $\widetilde{N_f} = N_f - 1$ antifundamental chiral multiplets, coupled to $N_f(N_f-1)$ hypermultiplets on S_b^4 is equal to a Toda CFT correlator with an antisymmetric degenerate insertion and a rank 1 irregular puncture:[20]

$$Z_{S^2\subset S_b^4}^{U(N),N_f,N_f-1}(z,\bar z) = A|x|^{2\gamma_0} e^{\frac{N}{N_f}(x+\bar x)} \Big\langle \widehat{V}_{\alpha_0}(\infty)\widehat{V}_{-b\omega_N}(1)\widehat{\mathbb{V}}_{c_0+Nbh_1;c_1,\bar c_1}(0)\Big\rangle\,. \tag{3.4.33}$$

As before, $x = (-1)^{N_f} z$. The irregular puncture $\widehat{\mathbb{V}}$ is defined above and in Section 5.6. The momenta c_0, c_1, $\bar c_1$, and α_0 are

$$\begin{aligned} &c_0 = Q + \frac{1}{b}\sum_{s=1}^{N_f}(1+\mathrm{i}m_s)h_1 + \frac{1}{b}\sum_{s=2}^{N_f}\mathrm{i}\widetilde{m}_s(h_1-h_s) \\ &c_1 = -\frac{x}{b}h_1 \qquad \bar c_1 = \frac{\bar x}{b}h_1 \qquad \alpha_0 = Q - \frac{1}{b}\sum_{s=1}^{N_f}\mathrm{i}m_s h_s \end{aligned} \tag{3.4.34}$$

and the constant A and exponent γ_0 are[21]

$$A = b^{N(N_f-1)(b^2+1)+2\Delta(\alpha_0)-2\Delta(c_0)} \tag{3.4.35}$$

$$\gamma_0 = \Delta(c_0) - \Delta(\alpha_0) - N\sum_{s=1}^{N_f}\mathrm{i}m_s - N\sum_{s=2}^{N_f}(1+\mathrm{i}\widetilde{m}_s) - \frac{N(N-1)}{2}b^2\,. \tag{3.4.36}$$

[20] Following the argument below (3.2.10), the factor $A|x|^{2\gamma_0}e^{(N/N_f)(x+\bar x)}$ can be absorbed in the S^2 partition function. We keep the factor explicitly to compare gauge theory and Toda CFT results.

[21] Mapping $\{0,1,\infty\}$ to $\{\infty,x,0\}$ gives a closer analogue of the $\widetilde{N_f}=N_f$ matching. This replaces γ_0 by the simpler $\gamma_0 - \Delta(c_0+Nbh_1) + \Delta(\alpha_0) + \Delta(-b\omega_N) = -\frac{N}{N_f}\sum_{s=1}^{N_f}\mathrm{i}m_s - \frac{N(N_f-N)}{2}(b^2+1)$. However, the transformation properties (5.6.18) of rank 1 irregular punctures would make the parameters c_1 and $\bar c_1$ infinite. The best convention to cancel this infinity is not clear.

As we have seen, it is natural from the gauge theory point of view to decouple further antifundamental chiral multiplets by making them massive. Specifically, from (3.4.29) we know that the partition function of a surface operator described by a $U(N)$ vector multiplet coupled to N_f fundamental and $\widetilde{N_f} = N_f - k \leq N_f - 2$ antifundamental chiral multiplets is a limit of $Z^{U(N),N_f,N_f-1}_{S^2\subset S^4_b}(z/(-\mathrm{i}\Lambda)^{k-1}, \bar{z}/(\mathrm{i}\Lambda)^{k-1})$ with twisted masses $\widetilde{m}_2 = \cdots = \widetilde{m}_k = \Lambda$, multiplied by some factor. On the Toda CFT side of the matching (3.4.33), the limit amounts to taking $\langle c_0, h_s\rangle \sim \mathrm{i}\Lambda/b$ for $2 \leq s \leq k$ and letting c_1 and $\bar{c}_1$ decrease as $\Lambda^{-(k-1)}$. Such a limit does not fit in the framework described in Section 5.6, since the parameter c_0 blows up. However, translating the gauge theory factors to the Toda CFT and setting $N = 0$ for simplicity, we find that the two-point function of a generic vertex operator $\widehat{V}_{\alpha_0}$ with

$$\left[|\nu|^{2\Delta(c_0)-\langle Q,Q\rangle} \prod_{t=2}^{k} \Upsilon\big(\langle Q - c_0, h_t\rangle\big)^{N_f} \widehat{\mathbb{V}}_{c_0;-\nu h_1,\bar{\nu} h_1}\right]_{\substack{\nu = x/[b(-\mathrm{i}\Lambda)^{k-1}] \\ c_0 \sim \frac{\mathrm{i}\Lambda}{b}(kh_1 - \omega_k)}} \tag{3.4.37}$$

remains finite as $\Lambda \to \infty$. This suggests that the operator (3.4.37) itself has a limit. Additionally, the OPE (5.6.13) of the stress-energy tensor with a rank 1 puncture includes a term $\Delta(c_0) + \langle c_1, \partial_{c_1}\rangle$, and the normalization factor $|\nu|^{2\Delta(c_0)}$ ensures that the singular term $\Delta(c_0)$ is absorbed in $\langle c_1, \partial_{c_1}\rangle$. Unfortunately, it is difficult to go further, as the OPE with higher currents of the W_{N_f} algebra contain many singular terms, and all must be carefully cancelled by the choice of normalization before taking the limit.

Having dissected the partition function of theories with fundamental and antifundamental matter, we consider next theories with an adjoint chiral multiplet.

3.5 SQCDA and Toda symmetric degenerate

We focus in this section on $\mathcal{N} = (2,2)$ SQCDA: a $U(N)$ vector multiplet coupled to an adjoint chiral multiplet X and N_f fundamental and N_f antifundamental chiral multiplets. Twisted masses (plus R-charges) are m_X, m_s, and $\widetilde{m}_s$. This theory, coupled to N_f^2 hypermultiplets with masses given by (3.2.17), defines a surface operator. We equate the $S^2 \subset S^4_b$ partition function of the 4d/2d system to a Toda CFT correlator with a degenerate field $\widehat{V}_{-Nbh_1}$ labelled by the N-th symmetric representation of A_{N_f-1}. Namely, we check that[22]

$$Z^{U(N)\text{ SQCDA}}_{S^2\subset S^4_b}(m, \widetilde{m}, m_X, z, \bar{z}) = A|y|^{2\gamma_0}|1-y|^{2\gamma_1}\left\langle \widehat{V}_{\alpha_\infty}(\infty)\widehat{V}_{\hat{m}}(1)\widehat{V}_{-Nbh_1}(y,\bar{y})\widehat{V}_{\alpha_0}(0)\right\rangle \tag{3.5.1}$$

[22]As explained below (3.2.10), the factor $A|x|^{2\gamma_0}|1-x|^{2\gamma_1}$ can be absorbed into the partition function. To compare gauge theory and Toda CFT results it is best to keep the factor explicitly.

with $y = (-1)^{N_f+N-1}z$ and[23] $b^2 = \mathrm{i}m_X$, momenta

$$\begin{aligned} \alpha_0 &= Q - \frac{1}{b}\sum_{s=1}^{N_f} \mathrm{i}m_s h_s & \hat{m} &= (\varkappa + Nb)h_1 \\ \alpha_\infty &= Q - \frac{1}{b}\sum_{s=1}^{N_f} \mathrm{i}\widetilde{m}_s h_s & \varkappa &= \frac{1}{b}\sum_{s=1}^{N_f}(1 + \mathrm{i}m_s + \mathrm{i}\widetilde{m}_s)\,, \end{aligned} \tag{3.5.2}$$

and coefficients

$$\gamma_0 = -\frac{N}{N_f}\sum_{s=1}^{N_f} \mathrm{i}m_s - \frac{N(N_f-1)}{2}(b^2+1) - \frac{N(N-1)}{2}b^2 \tag{3.5.3}$$

$$\gamma_1 = -\frac{N(N_f-N)}{N_f}b^2 + \frac{N}{N_f}\sum_{s=1}^{N_f}(\mathrm{i}m_s + \mathrm{i}\widetilde{m}_s) \tag{3.5.4}$$

$$A = b^{NN_f(1+b^2)-N^2b^2-2Nb\varkappa}\prod_{\nu=1}^{N}\gamma(-\nu b^2)\,. \tag{3.5.5}$$

We recognize the same symmetries as for SQED and SQCD, under permutations of the m_s or the $\widetilde{m}_s$, and under $z \leftrightarrow \frac{1}{z}$ and exchanging those two sets of masses. Setting $N = 1$ reproduces the matching (3.3.1) of SQED, but A has an additional factor of $\gamma(-b^2) = \gamma(-\mathrm{i}m_X)$: this is the one-loop determinant of the adjoint chiral multiplet, which decouples in an abelian theory.

Given the geometrical origin of the deformation parameter, one has $b^2 > 0$. On the other hand, the S^2 partition function is defined with positive R-charges $\mathrm{Re}(-2\mathrm{i}m)$. The two requirements are incompatible with $b^2 = \mathrm{i}m_X$, hence one of those two parameters must be continued beyond its usual range. For now, we analytically continue the R-charge: it is easier because the partition function depends holomorphically on $\mathrm{i}m_X$, as deduced from explicit expressions. However, we will encounter in Section 3.5.3 a setting where $b^2 = \mathrm{i}m_X$ is fixed to a real negative value. Given that the Upsilon function which appears in $Z^{\text{free}}_{S^4_b}$ and in Toda correlators cannot be continued to negative b^2, we will have to first recast the relation (3.5.1) in the form $Z_{S^2} = \langle\cdots\rangle / Z^{\text{free}}_{S^4_b}$ for the analytic continuation in b to make sense.

Once more, we fix the dictionary and demonstrate the equality by comparing exponents in the s-, t- and u-channels. The equality of Toda CFT three-point functions and gauge theory one-loop determinants (for the s- and u-channels) is checked in Section 5.4.4, and the expression of A is found there.

---

[23] The full flavour group of SQCDA is $U(1) \times S[U(N_f) \times U(N_f)]$, where the factors act on the adjoint, fundamental, and antifundamental chiral multiplets. The relation $b^2 = \mathrm{i}m_X$ identifies the first $U(1)$ flavour symmetry with rotations transverse to the surface operator.

The Coulomb branch representation reads

$$
\begin{aligned}
&Z^{\text{SQCDA}} \\
&= \frac{1}{N!} \sum_{B\in\mathbb{Z}^N} \int_{\mathbb{R}^N} \frac{\mathrm{d}^N\sigma}{(2\pi)^N} \biggl\{ z^{\operatorname{Tr}(\mathrm{i}\sigma+\frac{B}{2})} \bar{z}^{\operatorname{Tr}(\mathrm{i}\sigma-\frac{B}{2})} \prod_{i<j} \biggl[(\sigma_i-\sigma_j)^2 + \frac{(B_i-B_j)^2}{4} \biggr] \\
&\cdot \prod_{j=1}^{N} \prod_{s=1}^{N_f} \biggl[\frac{\Gamma(-\mathrm{i}m_s - \mathrm{i}\sigma_j - \frac{B_j}{2})}{\Gamma(1+\mathrm{i}m_s+\mathrm{i}\sigma_j-\frac{B_j}{2})} \frac{\Gamma(-\mathrm{i}\widetilde{m}_s + \mathrm{i}\sigma_j + \frac{B_j}{2})}{\Gamma(1+\mathrm{i}\widetilde{m}_s - \mathrm{i}\sigma_j + \frac{B_j}{2})} \biggr] \\
&\cdot \prod_{i=1}^{N} \prod_{j=1}^{N} \biggl[\frac{\Gamma(-\mathrm{i}m_X - \mathrm{i}\sigma_i + \mathrm{i}\sigma_j - \frac{B_i-B_j}{2})}{\Gamma(1+\mathrm{i}m_X+\mathrm{i}\sigma_i-\mathrm{i}\sigma_j - \frac{B_i-B_j}{2})} \biggr] \biggr\} .
\end{aligned}
\tag{3.5.6}
$$

We will expand this partition function around the points 0, $(-1)^{N_f+N-1}$ and ∞, where, as a function of z, it has branch points. This follows the path we traced for SQED: the behaviours near $z=0$ and ∞ are probed by closing integration contours towards $\pm\mathrm{i}\infty$. The partition function is then expressed as a sum over poles of the integrand, which are characterized up to integers by the set of Gamma functions which are singular for those values of $\mathrm{i}\sigma$. The behaviour near $(-1)^{N_f+N-1}$ is found by splitting the Coulomb branch integral depending on whether each $|\sigma_j \pm \frac{\mathrm{i}B_j}{2}| \lesseqgtr \ln|z|$.

3.5.1 Expanding the SQCDA partition function

We start with the s-channel expansion for $|z|<1$. Ignoring for a moment the magnetic flux B, and integer shifts due to the infinite set of poles of the Gamma function, we find that poles enclosed by the contour must be such that each $\mathrm{i}\sigma_j$ is either $-\mathrm{i}m_s$ for some flavour s, or $\mathrm{i}\sigma_i - \mathrm{i}m_X$ for some other color i. As in the case of SQCD, the vector multiplet one-loop determinant enforces $\mathrm{i}\sigma_i \neq \mathrm{i}\sigma_j$ for any two distinct colors, hence $\{\mathrm{i}\sigma_j\}$ is $\{-\mathrm{i}m_s - \mu\mathrm{i}m_X \mid 1\leq s\leq N_f, 0\leq\mu<n_s\}$ for some choice of integers n_s with $n_1+\cdots+n_{N_f}=N$. It is convenient to label colors with indices (s,μ) instead of $j\in[\![1,N]\!]$, and denote $I=\{(s,\mu)\}$. The sums over B and over poles of Gamma functions introduce shifts, in the form of sums over $2N$ integers $k^{\pm}_{s\mu}\geq 0$, and poles are

$$
\mathrm{i}\sigma_{s\mu} \pm \frac{B_{s\mu}}{2} = -\mathrm{i}m_s - \mu\mathrm{i}m_X + k^{\pm}_{s\mu} \tag{3.5.7}
$$

for $(s,\mu)\in I$. The partition function can then be recast as a sum over residues at those values of $\mathrm{i}\sigma \pm \frac{B}{2}$. It turns out that the residues vanish unless $k^{\pm}_{s\mu} \leq k^{\pm}_{s(\mu+1)}$ for every $(s,\mu)\in I$ and sign $\pm$: this indicates that (3.5.7) also labels some points which are not poles; thankfully, the residue formula is robust against such overcounting.

Since every factor in the Coulomb branch formula depends only on $\mathrm{i}\sigma+\frac{B}{2}$ hence on k^+, or on $\mathrm{i}\sigma-\frac{B}{2}$ hence on k^-, the series over k^+ and over k^-

decouple, and Z^{SQCDA} splits into a sum of factorized terms labelled by the choice of $\{n_s\}$,

$$Z_{S^2}^{\text{SQCDA}} = \sum_{n_1+\cdots+n_{N_f}=N} \Big\{ (z\bar{z})^{\sum_{(s,\mu)\in I}(-\mathrm{i}m_s-\mu\mathrm{i}m_X)} Z_{1\mathrm{l},\{n\}} Z_{\mathrm{v},\{n\}}(z) Z_{\mathrm{v},\{n\}}(\bar{z}) \Big\} \tag{3.5.8}$$

where the one-loop contribution, obtained by setting $k^\pm = 0$, simplifies to

$$Z_{1\mathrm{l},\{n\}} = \prod_{(s,\mu)\in I} \prod_{t=1}^{N_f} \frac{\gamma(-\mathrm{i}m_t - n_t\mathrm{i}m_X + \mathrm{i}m_s + \mu\mathrm{i}m_X)}{\gamma(1+\mathrm{i}\widetilde{m}_t + \mathrm{i}m_s + \mu\mathrm{i}m_X)}\,, \tag{3.5.9}$$

and the vortex partition function is

$$\begin{aligned} &Z_{\mathrm{v},\{n\}}(z) \\ &= \sum_{k:I\to\mathbb{Z}_{\geq 0}} \prod_{(s,\mu)\in I} \Bigg[\big[(-1)^{N_f+N-1}z\big]^{k_{s\mu}} \prod_{t=1}^{N_f} \frac{(-\mathrm{i}\widetilde{m}_t - \mathrm{i}m_s - \mu\mathrm{i}m_X)_{k_{s\mu}}}{(1+\mathrm{i}m_t - \mathrm{i}m_s + (n_t-\mu)\mathrm{i}m_X)_{k_{s\mu}}} \\ &\quad \cdot \frac{\prod_{t=1}^{N_f}(1+\mathrm{i}m_t - \mathrm{i}m_s + (n_t-\mu)\mathrm{i}m_X + k_{s\mu} - k_{t(n_t-1)})_{k_{t(n_t-1)}}}{\prod_{(t,\nu)\in I}(1+\mathrm{i}m_t - \mathrm{i}m_s + (\nu-\mu)\mathrm{i}m_X + k_{s\mu} - k_{t\nu})_{k_{t\nu}-k_{t(\nu-1)}}} \Bigg] \end{aligned} \tag{3.5.10}$$

where we define $k_{t,-1} = 0$ for convenience. Carrying through the same procedure for $|z| > 1$ yields a u-channel decomposition similar to the s-channel decomposition (3.5.8), with $m_s \leftrightarrow \widetilde{m}_s$, $y \to y^{-1}$ and $\bar{y} \to \bar{y}^{-1}$.

Having found powers of $|z|$ in the s-channel and u-channel decompositions of Z^{SQCDA}, we now expand the Coulomb branch integral in the t-channel. The first step is to use the identity $\frac{\Gamma(-\mathrm{i}a-B/2)}{\Gamma(1+\mathrm{i}a-B/2)} = (-1)^B \frac{\Gamma(-\mathrm{i}a+B/2)}{\Gamma(1+\mathrm{i}a+B/2)}$ on the one-loop determinants of fundamental chiral multiplets, and on half of the Gamma functions stemming from the adjoint chiral multiplet, and absorb the resulting signs into

$$y = (-1)^{N_f+N-1}z \qquad \text{and} \qquad \bar{y} = (-1)^{N_f+N-1}\bar{z}\,. \tag{3.5.11}$$

The integrand resulting from this operation can be recast as

$$\begin{aligned} &y^{\mathrm{Tr}(\mathrm{i}\sigma+\frac{B}{2})}\bar{y}^{\mathrm{Tr}(\mathrm{i}\sigma-\frac{B}{2})} \prod_{j=1}^{N}\prod_{s=1}^{N_f}\Bigg[\frac{\Gamma(-\mathrm{i}m_s - \mathrm{i}\sigma_j + \frac{B_j}{2})}{\Gamma(1+\mathrm{i}\widetilde{m}_s - \mathrm{i}\sigma_j + \frac{B_j}{2})}\frac{\Gamma(-\mathrm{i}\widetilde{m}_s + \mathrm{i}\sigma_j + \frac{B_j}{2})}{\Gamma(1+\mathrm{i}m_s + \mathrm{i}\sigma_j + \frac{B_j}{2})}\Bigg] \\ &\gamma(-\mathrm{i}m_X)^N \prod_{\pm}\prod_{i<j}^{N}\Bigg[\frac{(\pm(\mathrm{i}\sigma_i - \mathrm{i}\sigma_j) + \frac{B_i-B_j}{2})\Gamma(-\mathrm{i}m_X \pm (\mathrm{i}\sigma_i - \mathrm{i}\sigma_j) + \frac{B_i-B_j}{2})}{\Gamma(1+\mathrm{i}m_X \pm (\mathrm{i}\sigma_i - \mathrm{i}\sigma_j) + \frac{B_i-B_j}{2})}\Bigg] \end{aligned} \tag{3.5.12}$$

by writing the vector multiplet one-loop determinant as a product of $\pm(\mathrm{i}\sigma_i - \mathrm{i}\sigma_j) + \frac{B_i-B_j}{2}$. We now split the sums and integrals in the same way as for SQED on page 119, one pair (σ_j, B_j) at a time. For $|\mathrm{i}\sigma_j + \frac{B_j}{2}| < |\ln y|^{-1}$, we expand the classical contribution $y^{\mathrm{i}\sigma_j+\frac{B_j}{2}}\bar{y}^{\mathrm{i}\sigma_j-\frac{B_j}{2}}$ as a series in $\ln y$ and $\ln\bar{y}$; the integral and sum only contributes a constant factor. For $|\mathrm{i}\sigma_j+\frac{B_j}{2}|>|\ln y|^{-1}$, the sum over B_j is well approximated by an integral, and we expand the Gamma functions which involve this particular combination as a power of $|\mathrm{i}\sigma_j+\frac{B_j}{2}|$ times a power series in $(\mathrm{i}\sigma_j\pm\frac{B_j}{2})^{-1}$. Rescaling $\mathrm{i}\sigma_j+\frac{B_j}{2}$ by $\ln y$ makes the classical contribution independent of y, and extracting a power of $|\ln y|$ leaves a series in $\ln y$ and $\ln\bar{y}$ as the sole dependence in y. After performing this procedure for all pairs (σ_j, B_j), we obtain 2^N contributions, labelled by the set $K\subseteq\{1,\ldots,N\}$ of colors j such that $|\mathrm{i}\sigma_j+\frac{B_j}{2}|>|\ln y|^{-1}$ is large. The contribution for a given set K behaves as

$$Z_K\sim|1-y|^{-2k+2k\sum_{s=1}^{N_f}(1+\mathrm{i}m_s+\mathrm{i}\widetilde{m}_s)+2k[2N-k-1]\mathrm{i}m_X}\,, \tag{3.5.13}$$

multiplied by a constant and by a series in powers of $1-y$ and $1-\bar{y}$, where $k=\#K$ is the number of elements in K and we used $(\ln y)^\alpha=(1-y)^\alpha\cdot(\text{series})$. There are $N+1$ distinct exponents, corresponding to values $k\in[\![0,N]\!]$. This approach does not seem amenable to finding multiplicities attached to each power of $1-y$, hence we will not be able to probe that aspect of the correspondence.

3.5.2 Matching parameters for SQCDA

We are now ready to match the gauge theory data to Toda CFT data. The s- and u-channel decompositions of Z^{SQCDA} have

$$\binom{N_f+N-1}{N}=\dim(\mathcal{R}(Nh_1)) \tag{3.5.14}$$

terms, which is the dimension of the N-th symmetric representation $\mathcal{R}(Nh_1)$ of A_{N_f-1}, with highest weight Nh_1. Thus, in analogy with SQCD, we expect Z^{SQCDA} to match a Toda four-point correlation function involving the degenerate operator $\widehat{V}_{-Nbh_1}$. The fusion rule then allows shifts of generic momenta by $-bh=-b\sum_{s=1}^{N_f}n_sh_s$ for a choice of integers $n_1+\cdots+n_{N_f}=N$. We thus wish to match the s-channel exponents

$$\Delta(\alpha_0-bh)-\Delta(\alpha_0)-\Delta(-Nbh_1)+\gamma_0=-\sum_{s=1}^{N_f}\Big[n_s\mathrm{i}m_s+\frac{n_s(n_s-1)}{2}\mathrm{i}m_X\Big]\,. \tag{3.5.15}$$

This equality holds if $\mathrm{i}m_X=b^2$, and α_0 and γ_0 are as given in (3.5.2) and (3.5.3). The u-channel powers are similar,

$$\Delta(\alpha_\infty)-\Delta(\alpha_\infty-bh)-\Delta(-Nbh_1)+\gamma_0+\gamma_1=\sum_{s=1}^{N_f}\Big[n_s\mathrm{i}\widetilde{m}_s+\frac{n_s(n_s-1)}{2}\mathrm{i}m_X\Big]\,, \tag{3.5.16}$$

and the equality holds for values of α_∞ and γ_1 in (3.5.2) and (3.5.4).

We find in Section 5.5 that the fusion of $(\varkappa + Nb)h_1$ with $-Nbh_1$ allows the t-channel internal momenta $(\varkappa + nb)h_1 - nbh_2$ for $0 \leq n \leq N$. This fusion rule (5.5.15) provides the powers of $1-y$ for the t-channel of the Toda correlator, and power matching then requires

$$\begin{aligned} &\Delta((\varkappa + nb)h_1 - nbh_2) - \Delta((\varkappa + Nb)h_1) - \Delta(-Nbh_1) + \gamma_1 \\ &\quad = k\Big[\sum_{s=1}^{N_f}(\mathrm{i}m_s + \mathrm{i}\widetilde{m}_s) + (N_f - 1) + (2N - k - 1)\mathrm{i}m_X\Big] . \end{aligned} \tag{3.5.17}$$

The exponents are equal if $n = N - k$, and $\varkappa$ is as given in (3.5.2).

Finally, as checked in Section 5.4.4, the Toda CFT three-point functions which appear in the s-channel decomposition of the correlator reproduce the corresponding one-loop determinants in (3.5.8), provided A is as given in (3.5.5). For any given N, the techniques of Section 5.2.2 can yield the Toda CFT braiding matrix of $\widehat{V}_{-Nbh_1}$ with $\widehat{V}_{\hat{m}}$. However, we did not find a closed form of those matrices or their gauge theory analogues to provide a proof of the matching (3.5.1). *Note added in the thesis: see* (5.3.19) *for an expression of this braiding matrix found since.*

3.5.3 Adding a superpotential to SQCDA

We now discuss the effect of adding to SQCDA a superpotential term of the form $W = \sum_{t=1}^{N_f} \widetilde{q}_t X^{l_t} q_t$ or $W = \operatorname{Tr} X^{l+1}$, where q_t, $\widetilde{q}_t$, and X denote the fundamental, antifundamental, and adjoint chiral multiplets, and l_t and l are non-negative integers.

The deformation term which localizes to the Higgs branch of the theory with no superpotential can still be used in the presence of a superpotential, and it yields the same decomposition into vortex and anti-vortex partition functions. Hence, the only effect of the superpotential on the partition function is to constrain the (complexified) twisted masses of chiral multiplets. On the other hand, the superpotential term is in fact $\mathcal{Q}$-exact for the choice of localization supercharge $\mathcal{Q}$, thus one can include it into the deformation term. This lifts some vacua of the deformation term through F-term constraints, thus removes some terms from the sum over Higgs branch vacua. The two deformation terms must yield equal results for the partition function. Therefore, the terms forbidden by F-term constraints must vanish in the larger sum: they must have zero one-loop determinant. As a result, we can either solve D-term and F-term equations to find vacua of the enhanced deformation term, or remove vacua of the original deformation term whose one-loop determinant vanishes when imposing the superpotential constraint on R-charges.

First, we focus on a generalization of the superpotential $\widetilde{q}Xq$ of $\mathcal{N} =$

$(2,2)^*$ SQCD,[24]

$$W = \sum_{t=1}^{N_f} \widetilde{q}_t X^{l_t} q_t \,, \tag{3.5.18}$$

where $l_t \geq 0$ is an integer for each flavour $1 \leq t \leq N_f$. We let $L = \sum_{t=1}^{N_f} l_t$. The superpotential must have a total R-charge of 2 and a vanishing twisted mass, hence $\mathrm{i}\widetilde{m}_t + l_t \mathrm{i}m_X + \mathrm{i}m_t = -1$ for each $1 \leq t \leq N_f$. The one-loop determinant (3.5.9) then contains a vanishing factor $1/\gamma(1 + \mathrm{i}\widetilde{m}_t + \mathrm{i}m_t + l_t \mathrm{i}m_X) = 0$ whenever any $n_t > l_t$, thus those terms do not contribute to the partition function. An equivalent point of view is that the corresponding Higgs branch vacua have $X^{n_t-1}q_t \neq 0$ and are forbidden by the F-term equation $X^{l_t}q_t = 0$. Terms in the Higgs branch representation of the partition function are thus labelled by integers $0 \leq n_t \leq l_t$ with $\sum_{t=1}^{N_f} n_t = N$. Note that $n_t \leq l_t$ implies $N \leq L$, analoguous to the condition $N \leq N_f$ for SQCD.

The constraint on (complexified) twisted masses translates to a constraint on the momenta of operators appearing in the corresponding Toda CFT correlator. The semi-degenerate operator becomes degenerate:

$$\hat{m} = \left[\frac{1}{b}\sum_{t=1}^{N_f}(1 + \mathrm{i}m_t + \mathrm{i}\widetilde{m}_t) + Nb\right] h_1 = -(L-N)bh_1 \,, \tag{3.5.19}$$

where we used $\mathrm{i}m_X = b^2$. Thus, the outgoing momentum $2Q - \alpha_\infty$ must take the form $\alpha_0 - bh - bh'$, where $h = \sum_t n_t h_t$ is a weight of $\mathcal{R}(Nh_1)$ and $h' = \sum_t n'_t h_t$ is a weight of $\mathcal{R}((L-N)h_1)$. The superpotential ensures that this is the case:

$$2Q - \alpha_\infty = Q + \frac{1}{b}\sum_{t=1}^{N_f} \mathrm{i}\widetilde{m}_t h_t = Q - \frac{1}{b}\sum_{t=1}^{N_f}(\mathrm{i}m_t + l_t b^2 + 1)h_t = \alpha_0 - b\sum_{t=1}^{N_f} l_t h_t \,. \tag{3.5.20}$$

The conformal block decomposition contains one term for each way of splitting $\sum_t l_t h_t$ into a sum $h + h'$ of weights of $\mathcal{R}(Nh_1)$ and $\mathcal{R}((L-N)h_1)$, that is, each set of integers $0 \leq n_t \leq l_t$ with $\sum_t n_t = N$. In Section 4.3.1, we note that the vertex operators $\widehat{V}_{-(L-N)bh_1}$ and $\widehat{V}_{-Nbh_1}$ have the same form with $N \leftrightarrow L - N$, and deduce a duality between theories with gauge groups $U(N)$ and $U(L-N)$. This duality reduces when all $l_t = 1$ to an $\mathcal{N} = (2,2)^*$ analogue of Seiberg duality.

Our second example of superpotential only involves the adjoint chiral multiplet, and constrains its complexified twisted mass:

$$W = \operatorname{Tr} X^{l+1} \,, \qquad b^2 = \mathrm{i}m_X = \frac{-1}{l+1} \tag{3.5.21}$$

where $l \geq 1$. The superpotential constraint sets b to an imaginary value, for which S_b^4 does not make sense. Instead of a surface operator on $S^2 \subset S_b^4$

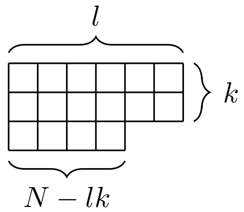
[24] $\mathcal{N} = (2,2)^*$ SQCD is the mass deformation of $\mathcal{N} = (4,4)$ SQCD.

we must thus manipulate the two-dimensional theory on S^2 only. Correspondingly, the matching (3.5.24) with the Toda CFT is written in the form $Z_{S^2} = \big[\langle\cdots\rangle / Z^{\text{free}}_{S^4_b}\big]_{b^2=-1/(l+1)}$, where the right-hand side is analytically continued after taking the ratio.[25]

For $\mathrm{i}m_X = \frac{-1}{l+1}$, the one-loop determinant (3.5.9) vanishes whenever any $n_s > l$: the numerator factor for $t = s$ and $\mu = n_s - l - 1$ is $\gamma(\mathrm{i}m_s - \mathrm{i}m_s + (n_s - l - 1 - n_s)\mathrm{i}m_X) = \gamma(1) = 0$. Equivalently, Higgs branch vacua have $X^{n_s-1}q_s \neq 0$ and are forbidden if $n_s > l$ by the F-term equation $X^l = 0$. The S^2 partition function in the presence of $W = \operatorname{Tr} X^{l+1}$ is thus a sum over choices of integers $0 \le n_s \le l$ with $\sum_{s=1}^{N_f} n_s = N$.

We see that introducing the superpotential $W = \operatorname{Tr} X^{l+1}$ replaces the sum over weights $\sum_{s=1}^{N_f} n_s h_s$ of the symmetric representation $\mathcal{R}(Nh_1)$ by a sum over a restricted set of weights, with $0 \le n_s \le l$. Those are precisely the weights of the representation with highest weight

$$\omega_{N,l} = l\omega_k + (N - lk)h_{k+1} \quad \text{and Young diagram}$$



, (3.5.22)

where k is defined by $kl \le N < (k+1)l$. The “quasi-rectangular” Young diagram is obtained by placing N boxes into as many l-box rows as possible followed by a row with any remaining box. For $l \ge N$, none of the one-loop determinants vanish, and the Young diagram is that of the N-th symmetric representation: this is the same as for SQCDA. For $l = 1$, the Young diagram becomes a column, hence we sum over weights of the N-th antisymmetric representation, as for SQCD with no adjoint: correspondingly, the superpotential $W = \operatorname{Tr} X^2$ lets us integrate out the adjoint chiral multiplet.

From our experience with SQCD and SQCDA, we expect the sum over weights of $\mathcal{R}(\omega_{N,l})$ to have a Toda CFT analogue involving the degenerate operator $\widehat{V}_{-b\omega_{N,l}}$. This is confirmed by the observation that the momenta $-Nbh_1$ and $-b\omega_{N,l}$ are Weyl conjugate when $b^2 = \frac{-1}{l+1}$ since

$$\begin{aligned}
&\big\{\tfrac{1}{b}\langle -Nbh_1 - Q, h_p\rangle \bigm| 1 \le p \le N_f\big\} \\
&= \Big\{\frac{N}{N_f} + \frac{N_f - 1}{2}l - N\Big\} \cup \Big\{\frac{N}{N_f} + \frac{N_f - 1}{2}l - kl \Bigm| 1 \le k \le N_f - 1\Big\} \\
&= \big\{\tfrac{1}{b}\langle -b\omega_{N,l} - Q, h_p\rangle \bigm| 1 \le p \le N_f\big\}\,.
\end{aligned} \tag{3.5.23}$$

[25]The central charge $c = (N_f - 1)\big[1 + N_f(N_f + 1)(b^2 + 2 + b^{-2})\big] = -(N_f - 1)(N_f l - 1)(N_f l + l + 1)/(l + 1)$ is negative for the value $b^2 = -1/(l + 1)$ we consider.

Therefore, $\widehat{V}_{-Nbh_1}$ and $\widehat{V}_{-b\omega_{N,l}}$ are equal up to a scalar factor for this value of b^2. This assertion should be handled with care, as the Toda CFT is ill defined for $b^2 < 0$.

Trusting the assertion leads us to the proposal[26]

$$\begin{aligned} &Z_{S^2}^{U(N)\ \text{SQCDA},W=\operatorname{Tr}X^{l+1}}\left(m,\widetilde{m},m_X=\frac{\mathrm{i}}{l+1},z,\bar{z}\right) \\ &\quad = A|y|^{2\gamma_0}|1-y|^{2\gamma_1}\left[\frac{\left\langle \widehat{V}_{\alpha_\infty}(\infty)\widehat{V}_{(\varkappa+Nb)h_1}(1)\widehat{V}_{-b\omega_{N,l}}(y,\bar{y})\widehat{V}_{\alpha_0}(0)\right\rangle}{\left\langle \widehat{V}_{\alpha_\infty}(\infty)\widehat{V}_{\varkappa h_1}(1)\widehat{V}_{\alpha_0}(0)\right\rangle}\right]_{b^2\to\frac{-1}{l+1}} \end{aligned} \tag{3.5.24}$$

for some A, and with other parameters given below the SQCDA matching (3.5.1). Importantly, we have moved the S_b^4 partition function of $\mathcal{N}=2$ free hypermultiplets to the right-hand side (in the form of a Toda CFT three-point function), and we only set $b^2 = \frac{-1}{l+1}$ after evaluating the ratio of Toda CFT correlators. We can thus expect Upsilon functions in the numerator and denominator to cancel, leaving a product of gamma functions which can be analytically continued to $b^2 = \frac{-1}{l+1}$ and should reproduce one-loop determinants in the left-hand side.

When $l \geq N$, (3.5.24) is simply the SQCDA matching (3.5.1) at $\mathrm{i}m_X = b^2 = \frac{-1}{l+1}$, with the same value of A. When $l=1$, we expect the claim to reproduce the SQCD result (3.4.1), and indeed the SQCDA parameters which appear in (3.5.24) are equal for $\mathrm{i}m_X = b^2 = \frac{-1}{2}$ to the corresponding SQCD parameters, with the exception of A.

It is difficult to find A in general, because three-point functions involving $\widehat{V}_{-b\omega_{N,l}}$ take complicated forms for $1 < l < N$. Using [FL07, equations (1.53) and (1.56)], we tested the proposal (3.5.24) for $N_f = N = 3$ and $l = 2$, which corresponds to the adjoint representation of $SU(3)$. Three-point functions $\widehat{C}^{\alpha-bh}_{-b(h_1-h_3),\alpha}$ associated to non-zero weights h of the adjoint representation are ratios of Gamma functions, and yield the expected one-loop determinants when $b^2 = \frac{-1}{l+1} = \frac{-1}{3}$. For general b, the three-point function $\widehat{C}^{\alpha}_{-b(h_1-h_3),\alpha}$ associated to the zero weight is expressed in terms of hypergeometric functions evaluated at 1, but at the point $b^2 = \frac{-1}{3}$ the value agrees numerically with the Gamma functions expected from gauge theory.

More generally, a Toda CFT four-point function with a fully degenerate vertex operator other than $\widehat{V}_{-b\omega_N}$ or $\widehat{V}_{-Nbh_1}$ (and the usual two generic and one semi-degenerate vertex operators) cannot coincide with the partition function of a surface operator described by a single $\mathcal{N}=(2,2)$ $U(N)$ vector multiplet coupled to some chiral multiplets, except for special values of b as is the case here. Indeed, as described by Fateev and Litvinov [FL07], the Toda three-point function $\widehat{C}^{\alpha-bh}_{-b\omega,\alpha}$ only takes the form of a ratio of Gamma

[26] As explained below (3.2.10), the factor $A|x|^{2\gamma_0}|1-x|^{2\gamma_1}$ can be absorbed into the partition function. To compare gauge theory and Toda CFT results it is best to keep the factor explicitly.

functions if the weight h appears with no multiplicity in $\mathcal{R}(\omega)$. Since one-loop determinants are always such ratios, they can only reproduce Toda CFT three-point functions for general b if weights have no multiplicities.

However, higher degenerate fields can be obtained by considering the collision limit of simpler degenerate fields. For instance, the three-point function $\widehat{C}^{\alpha}_{-b(h_1-h_3),\alpha}$ mentioned above is equal to a four-point function involving a fundamental and an antifundamental degenerate fields, in the limit where the two punctures collide. In the next section, we match Toda CFT correlators involving more than one (symmetric or antisymmetric) degenerate vertex operator with S^2 partition functions of quiver gauge theories. Colliding antisymmetric degenerate operators, we obtain expressions for Toda CFT correlators of arbitrary degenerate operators $\widehat{V}_{-b\Omega}$ with two generic and one semi-degenerate vertex operators, for any b.

3.6 Quivers and multiple Toda degenerates

We have focused so far on surface operators described by $U(N)$ gauge theories, which have a single FI parameter. Those correspond to Toda CFT four-point functions, which involve a single anharmonic ratio x. Here, we equate the partition function of surface operators described by certain $U(N_1)\times\cdots\times U(N_n)$ quiver gauge theories and $(n+3)$-point functions with n symmetric or antisymmetric degenerate operators. In detail,[27]

$$Z_{S^2\subset S^4_b}^{\prod_j U(N_j),W_\eta}(m,z,\bar{z}) = Aa(x)a(\bar{x})\left\langle \widehat{V}_{\alpha_\infty}(\infty)\widehat{V}_{\hat{m}}(1)\prod_{j=1}^{n}\widehat{V}_{-b\Omega(K_j,\epsilon_j)}(x_j,\bar{x}_j)\widehat{V}_{\alpha_0}(0)\right\rangle . \tag{3.6.1}$$

The matching gives a detailed description of the moduli space parametrized by the z_j. We describe notations below, then consider several limits to fix all parameters of the matching in Section 3.6.1. Fine-tuning FI parameters such that degenerate punctures collide on the Toda CFT side, we deduce in Section 3.6.2 the microscopic description of the surface operator which corresponds to arbitrary degenerate punctures in the Toda CFT. Brane diagrams (see Figure 3.2) clarify some aspects of the correspondence.

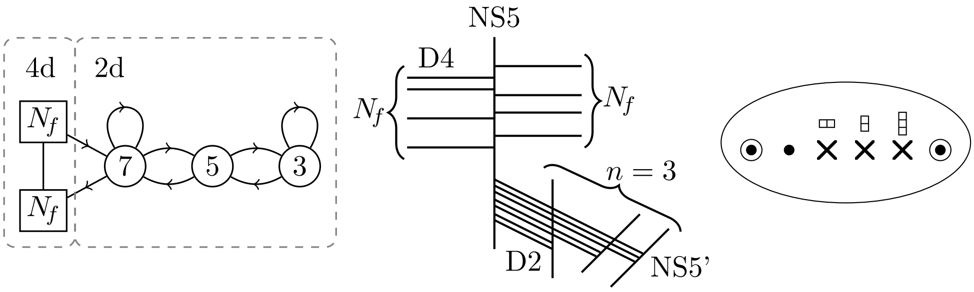
The surface operator depends on a choice of n signs $\eta_j=\pm1$ and integer parameters $N_n\geq\cdots\geq N_1\geq 0$. It also depends on n FI and theta parameters combined as

$$z_j = e^{-2\pi\xi_j+\mathrm{i}\vartheta_j} \quad\text{and}\quad \hat{z}_j=(-1)^{N_{j-1}+N_{j+1}+N_j-1}z_j \tag{3.6.2}$$

for $1\leq j\leq n$, where $N_0=0$, $N_{n+1}=N_f$, and the sign is chosen for later

[27]Following the arguments below (3.2.10), the factor $Aa(x)a(\bar{x})$ can be absorbed into the partition function. To compare gauge theory and Toda CFT results it is best to keep the factor explicitly.

Figure 3.2: Example of a 4d/2d quiver, its corresponding brane diagram, and Toda CFT correlator.

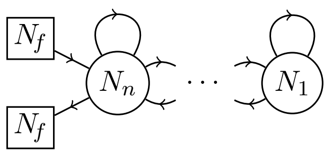


N_f semi-infinite D4 branes ending on each side of a single NS5 brane engineer at low energies the theory of N_f^2 free hypermultiplets on their four-dimensional intersection. Adding D2 branes stretched between the NS5 brane and n additional NS5 branes inserts a surface operator with support on the boundary of the added D2 branes. Rotating some NS5 branes (rotated branes are denoted as NS5' and are all parallel) alters the surface operator, which is then precisely the one discussed in the main text.

The ranks $N_n \geq \cdots \geq N_1$ are the numbers of D2 branes between consecutive NS5/NS5' branes. When these are parallel (both NS5 or both NS5'), the corresponding $U(N_j)$ group has an adjoint chiral multiplet ($\eta_j = +1$), otherwise not ($\eta_j = -1$). Equivalently, the j-th brane is an NS5 if $\epsilon_j = \prod_{i=j}^{n} \eta_i$ is 1 and otherwise it is an NS5'. The Toda CFT data appears by turning on FI parameters, as this separates the NS5/NS5' branes along the D4 brane direction. Then $K_j = (N_j - N_{j-1})$ D2 branes stretch between the original NS5 brane and the j-th NS5/NS5' brane, corresponding to the K_j-th symmetric (or antisymmetric if $\epsilon_j = -1$) degenerate operator.

We will see in Section 4.4 that permuting the (ϵ_j, K_j) or equivalently the NS5/NS5' branes is a (Seiberg-like) duality of the surface operator.

convenience. The operator is defined by the $\mathcal{N}=(2,2)$ quiver



(3.6.3)

which describes a $U(N_1)\times\cdots\times U(N_n)$ vector multiplet coupled to various chiral multiplets. First, N_f fundamentals q_t and N_f antifundamentals $\widetilde{q}_t$ of $U(N_n)$. Next, for each $1\leq j\leq n-1$, one pair of bifundamentals of $U(N_j)\times U(N_{j+1})$: $\phi_{j(j+1)}$ in the representation $N_j\otimes\overline{N}_{j+1}$ and $\phi_{(j+1)j}$ in the representation $\overline{N}_j\otimes N_{j+1}$. Finally, for each $1\leq j\leq n$, one adjoint X_j. The (complexified) twisted masses m_t, $\widetilde{m}_t$, $m_{j(j+1)}$, $m_{(j+1)j}$ and m_{jj} of these fields are constrained by a superpotential coupling W_η.

The superpotential has the following terms, whose (non-zero) coefficients cannot be determined using our methods.

$$\begin{cases}\operatorname{Tr}\bigl(X_j^2\bigr) & \text{for } 1\leq j\leq n \text{ if } \eta_j=-1\\ \operatorname{Tr}\bigl(\phi_{j(j+1)}\phi_{(j+1)j}\phi_{j(j-1)}\phi_{(j-1)j}\bigr) & \text{for } 1<j<n \text{ if } \eta_j=-1\\ \operatorname{Tr}\bigl(X_j\phi_{j(j+1)}\phi_{(j+1)j}\bigr) & \text{for } 1\leq j<n \text{ if } \eta_j=1\\ \operatorname{Tr}\bigl(X_j\phi_{j(j-1)}\phi_{(j-1)j}\bigr) & \text{for } 1<j\leq n \text{ if } \eta_j=1\,.\end{cases} \tag{3.6.4}$$

In other words the adjoint multiplets of nodes with $\eta_j=1$ have a cubic coupling to neighboring bifundamental multiplets, while nodes with $\eta_j=-1$ entail a quartic coupling of neighboring bifundamental multiplets. The $\operatorname{Tr}(X_j^2)$ term for $\eta_j=-1$ gives a mass to the adjoint multiplet X_j, hence the theory (3.6.3) is equivalent in the low-energy to the analoguous theory (3.1.8) from the introduction, which omits these X_j. Here, we include adjoint multiplets for all nodes to simplify signs in the definition (3.6.2) of $\hat{z}_j$. Indeed, integrating out X_j when $\eta_j=-1$ shifts the corresponding theta angle $z_j\to(-1)^{N_j-1}z_j$, thus complicating (3.6.2) to keep $\hat{z}_j$ fixed.

The superpotential W_η must have R-charge 2 (twisted mass i) to be supersymmetric. This fixes twisted masses of bifundamental and adjoint multiplets in terms of the signs η and a single continuous parameter,[28] which will match with b^2 in the Toda CFT. To ease the comparison with the Toda CFT correlator, we define signs $\epsilon_j=\prod_{i=j}^n\eta_i$ for $1\leq j\leq n+1$ and find

$$\mathrm{i}m_{jj}=\begin{cases}-1-b^2 & \text{if } \epsilon_{j+1}=\epsilon_j=-1\\ -1/2 & \text{if } \epsilon_{j+1}\neq\epsilon_j\\ b^2 & \text{if } \epsilon_{j+1}=\epsilon_j=+1\end{cases} \tag{3.6.5}$$

[28] The full flavour symmetry of the two-dimensional theory is $S[U(N_f)\times U(N_f)]\times U(1)$, where the first factor acts on fundamental and antifundamental chiral multiplets. Under the $U(1)$ factor, the adjoint chiral multiplet X_j has charge $\epsilon_j+\epsilon_{j+1}$ and the bifundamental multiplets $\phi_{(j-1)j}$ and $\phi_{j(j-1)}$ have charge $-\epsilon_j$, where $\epsilon_j=\prod_{i=j}^n\eta_i$.

$$\mathrm{i}m_{(j-1)j} + \mathrm{i}m_{j(j-1)} = \begin{cases} b^2 & \text{if } \epsilon_j = -1 \\ -1-b^2 & \text{if } \epsilon_j = +1 \,. \end{cases}$$

Equivalently, W_η could be defined as containing all gauge invariant combinations of the fields which have total R-charge 2 (twisted mass i), given the mass assignment (3.6.5). As always, the twisted masses and R-charges of fundamental and antifundamental chiral multiplets are unconstrained.

On the other hand, the Toda CFT $(n+3)$-point function involves two generic and one semi-degenerate vertex operators $\widehat{V}_{\alpha_\infty}(\infty)$, $\widehat{V}_{\hat{m}}(1)$, and $\widehat{V}_{\alpha_0}(0)$ with momenta

$$\begin{aligned} \alpha_0 &= Q - \frac{1}{b}\sum_{s=1}^{N_f} \mathrm{i}m_s h_s & \hat{m} &= (\varkappa + N_n b) h_1 \\ \alpha_\infty &= Q - \frac{1}{b}\sum_{s=1}^{N_f} \mathrm{i}\widetilde{m}_s h_s & \varkappa &= \frac{1}{b}\sum_{s=1}^{N_f}(1+\mathrm{i}m_s+\mathrm{i}\widetilde{m}_s) \end{aligned} \tag{3.6.6}$$

which coincide with those of earlier sections. It also involves n fully degenerate vertex operators $\widehat{V}_{-b\Omega(K_j,\epsilon_j)}(x_j,\bar{x}_j)$ at

$$x_j = \prod_{i=j}^{n} \hat{z}_i \quad \text{for } 1 \le j \le n\,. \tag{3.6.7}$$

Each degenerate operator is labelled by the highest weight $\Omega(K,+1) = Kh_1$ of a symmetric representation or $\Omega(K,-1) = \omega_K$ of an antisymmetric representation of A_{N_f-1}, depending on the signs $\epsilon_j = \prod_{i=j}^n \eta_i$ and the integers

$$K_1 = N_1\,, \quad \text{and} \quad K_j = N_j - N_{j-1} \quad \text{for } 1 < j \le n\,. \tag{3.6.8}$$

Finally, the factors A and a are

$$A = b^{N_n N_f(1+b^2) - N_n^2 b^2 - 2N_n b\varkappa} \prod_{j|\epsilon_j=+1} \prod_{1\le\nu\le K_j} \gamma(-\nu b^2) \tag{3.6.9}$$

$$a(x)a(\bar{x}) = \prod_{j=1}^{n} |x_j|^{2\beta_j} \prod_{j=1}^{n} |1-x_j|^{2\gamma_j} \prod_{i<j}^{n} |x_j - x_i|^{2\gamma_{ij}} \tag{3.6.10}$$

with the exponents

$$\begin{aligned} \beta_j = {} & -\frac{K_j}{N_f}\sum_{t=1}^{N_f} \mathrm{i}m_t + \Delta\big(-b\Omega(K_j,\epsilon_j)\big) + \frac{K_j(N_f-K_j)}{2N_f} b^2 \\ & - N_{j-1}\mathrm{i}m_{j(j-1)} - K_j \sum_{i=j+1}^{n} \mathrm{i}m_{(i-1)i} \end{aligned} \tag{3.6.11}$$

$$\gamma_j = (-1-b^2)K_j + b(\varkappa + N_n b)K_j/N_f \tag{3.6.12}$$

$$\gamma_{ij} = \begin{cases} b^2 K_i - b^2 K_i K_j / N_f & \text{if } \epsilon_j = -1 \\ (-1-b^2)K_i - b^2 K_i K_j/N_f & \text{if } \epsilon_j = +1 \end{cases} \tag{3.6.13}$$

for $1 \leq i < j \leq n$. When $n = 1$, the matching (3.6.1) reproduces the known cases of SQCD ($\eta_1 = -1$) and SQCD with an adjoint ($\eta_1 = 1$). Also, for $n > 1$ setting $N_1 = 0$ reduces the matching to the case $n \to n - 1$.

As a preliminary check of the equality (3.6.1), we can recognize a few symmetries. Permuting the flavours of fundamental quarks q_t, hence their twisted masses m_t, does not alter the partition function. This is translated on the Toda CFT side into a Weyl transformation of the momentum α_0, which permutes the $\langle \alpha_0 - Q, h_t \rangle$. Similarly, permuting the $\widetilde{m}_t$ amounts to a Weyl transformation of α_∞. Next, performing charge conjugation on all gauge group factors maps $\hat{z}_j \to \hat{z}_j^{-1}$, $m_t \leftrightarrow \widetilde{m}_t$, and $m_{j(j+1)} \leftrightarrow m_{(j+1)j}$: this corresponds on the Toda CFT side to the conformal map $x \to x^{-1}$, which swaps $\alpha_0 \leftrightarrow \alpha_\infty$ and maps $x_j \to x_j^{-1}$. The transformation of $a(x)a(\bar{x})$ compensates exactly the conformal factor $|x_j|^{-4\Delta(-b\Omega(K_j,\epsilon_j))}$ for each j. Finally, shifting the twisted masses of bifundamentals while keeping the sums $m_{j(j+1)} + m_{(j+1)j}$ constant amounts to a constant gauge transformation, whose sole effect on the partition function is in overall powers of $|x_j|^2$: on the Toda CFT side of (3.6.1), only the exponents β_j change.

3.6.1 Matching parameters for quivers

We first expand the partition function and the correlator in the s-channel, that is, the region where $0 < |x_1| < \cdots < |x_n| < 1$ or equivalently where all FI parameters are positive: $|\hat{z}_j| < 1$. We map vacua of the gauge theory to choices of internal momenta in the correlator. The classical and one-loop contributions match as expected with the exponents and three-point functions, while the vortex partition functions give predictions for Toda CFT conformal blocks (see Section 5.4.4). This check fixes $\{K_j, \epsilon_j\}$, the momentum α_0, the overall constant factor A and the exponents $\beta_j + \sum_{i<j} \gamma_{ij}$. The momentum α_∞ is fixed by the symmetry under charge conjugation discussed earlier. Then, we justify the relation between the gauge theory data $\{\eta_j, \hat{z}_j\}$ and the Toda CFT data $\{\epsilon_j, x_j\}$ by counting distinct exponents in the limit where two neighboring punctures collide. Comparing the exponents only fixes the momentum $\hat{m}$ and the exponents γ_n and $\gamma_{(j-1)j}$. The remaining exponents γ_j and γ_{ij} are fixed thanks to Seiberg dualities which translates in this setting to permutations of the n punctures (see Section 4.4).

It is easiest to find Higgs branch vacua of the gauge theory by solving the D-term and F-term equations, assuming as before that the twisted masses m_s of fundamental chiral multiplets are generic. Schematically, the derivation goes as follows. Diagonalize all σ_j. Introduce $\mathrm{i}\sigma_{n+1} = \operatorname{diag}(-\mathrm{i}m_1, \ldots, -\mathrm{i}m_{N_f})$, $N_{n+1} = N_f$, and $N_0 = 0$ to simplify the discussion. Integrate out all X_j which have twisted mass $m_{jj} = \mathrm{i}/2$, that is, $\eta_j = -1$. The D-term equations (for $|\hat{z}_j| < 1$) impose that the images of X_j, $\phi_{j(j+1)}$ and $\phi_{j(j-1)}$ span $\mathbb{C}^{N_j}$, hence all eigenvalues of σ_j are constrained to be equal to another eigenvalue of σ_j or of $\sigma_{j\pm1}$, minus a twisted mass. As a result, all eigenvalues of $\mathrm{i}\sigma_j$ take the form $\mathrm{i}\sigma_{j,\mathrm{a}} = -\mathrm{i}m_s - \sum_{i=j+1}^{n} \mathrm{i}m_{(i-1)i} + \mu(1+b^2) - \nu b^2$ where $\mu,\nu \in \mathbb{Z}_{\geq 0}$. Using the F-term constraint, one can then bound the multiplicity of such an eigenvalue by the multiplicity of the eigenvalue $\mathrm{i}\sigma_{j,\mathrm{a}} - \mathrm{i}m_{jk}$ of $\mathrm{i}\sigma_k$, for $k \in \{j, j\pm 1\}$ (only $k \in \{j\pm 1\}$ if X_j was integrated out). Since each eigenvalue $-\mathrm{i}m_s$ of $\mathrm{i}\sigma_{n+1}$ has multiplicity 1, we deduce by induction on $n+1-j$, μ, and ν that all eigenvalues have multiplicity 1. The statement is in fact stronger: for any eigenvalue $\mathrm{i}\sigma_{j,\mathrm{a}}$ of $\mathrm{i}\sigma_j$, and for $k \in \{j, j\pm 1\}$ (or only $k \in \{j \pm 1\}$), $\mathrm{i}\sigma_{j,\mathrm{a}} - \mathrm{i}m_{jk}$ is an eigenvalue of $\mathrm{i}\sigma_k$, and the relevant component of ϕ_{jk} is non-zero. Solving the F-term constraints then becomes a combinatorical problem, whose details depend on the superpotential W_η.

At the end of the day, one finds that vacua obey

$$\mathrm{i}\sigma_j = \operatorname{diag}\left(-\mathrm{i}m_s - \sum_{i=j+1}^{n} (\mathrm{i}m_{(i-1)i}) - \nu b^2 \;\middle|\; 0 \leq \nu < n_s^j, 1 \leq s \leq N_f\right) \tag{3.6.14}$$

for $1 \leq j \leq n$, where $n_s^j \geq 0$ are integers such that $\sum_{s=1}^{N_f} n_s^j = N_j$ and

$$\begin{cases} n_s^{j-1} \leq n_s^j \leq n_s^{j-1} + 1 & \text{if } \epsilon_j = -1 \\ n_s^{j-1} \leq n_s^j & \text{if } \epsilon_j = +1 \end{cases} \tag{3.6.15}$$

where $n_s^0 = 0$. These conditions are equivalent to requiring that for each $1 \leq j \leq n$ the difference $h_{[n^j]} - h_{[n^{j-1}]} = \sum_{s=1}^{N_f} (n_s^j - n_s^{j-1}) h_s$ is a weight of the symmetric or antisymmetric representation $\mathcal{R}(\Omega(K_j, \epsilon_j))$ of rank $K_j = N_j - N_{j-1}$. The S^2 partition function is then a sum

$$Z_{S^2} = \sum_{\{n_s^j\}} Z_{\mathrm{cl}} Z_{11} Z_{\mathrm{v}} Z_{\bar{\mathrm{v}}} \tag{3.6.16}$$

over choices of $\{n_s^j\}$ consistent with the constraints above. Terms of this sum are in a natural bijection with terms of the s-channel decomposition of the Toda CFT correlator in (3.6.1): the internal momenta are $\alpha_0 - bh_{[n^j]}$ for $1 \leq j \leq n$. Thus, counting terms fixes the degenerate momenta $-b\Omega(K_j, \epsilon_j)$ in terms of the N_j and η_j.

Since the Higgs branch and Coulomb branch representations of S^2 partition functions coincide, $Z_{\mathrm{cl}} Z_{11}$ is the residue at the pole (3.6.14) of the Coulomb branch integrand, and $Z_{\mathrm{v}} Z_{\bar{\mathrm{v}}}$ is the additional contribution from poles for which $\mathrm{i}\sigma_j^{\pm}$ is (3.6.14) plus integers. We find in particular that the classical contribution reproduces the powers of x_j expected from the Toda CFT up to shifts by $\beta_j + \sum_{i=1}^{j-1} \gamma_{ij}$,

$$\begin{aligned} Z_{\mathrm{cl}} &= \prod_{j=1}^{n} |z_j|^{2\operatorname{Tr}\mathrm{i}\sigma_j} = \prod_{j=1}^{n} |z_j|^{2\left[-\sum_{s=1}^{N_f}(n_s^j \mathrm{i}m_s) - N_j \sum_{i=j+1}^{n}(\mathrm{i}m_{(i-1)i}) - \sum_{s=1}^{N_f}\sum_{\nu=0}^{n_s^j-1} \nu b^2\right]} \\ &= \prod_{j=1}^{n} |x_j|^{2\left[\beta_j + \sum_{i=1}^{j-1}(\gamma_{ij}) + \Delta(\alpha_0 - bh_{[n^j]}) - \Delta(\alpha_0 - bh_{[n^{j-1}]}) - \Delta(-b\Omega(K_j, \epsilon_j))\right]}, \end{aligned} \tag{3.6.17}$$

provided that α_0 is as given in (3.6.6), and $\beta_j + \sum_{i=1}^{j-1} \gamma_{ij}$ as in (3.6.11) and (3.6.13). By symmetry, α_∞ is as given in (3.6.6). Similarly, a tedious calculation shows that for each term the one-loop determinant Z_{1l} matches with the product of Toda CFT three-point functions, up to precisely the constant A given in (3.6.9).

Next, let us probe the collision of two neighboring punctures, starting again from the s-channel $0 < |x_1| < \cdots < |x_n| < 1$. The Coulomb branch representation of the S^2 partition function of interest has the form

$$Z_{S^2}^{\prod_j U(N_j), W_\eta} = \prod_{j=1}^{n} \left[\frac{1}{N_j!} \sum_{B_j \in \mathbb{Z}^{N_j}} \int \frac{\mathrm{d}^{N_j} \sigma_j}{(2\pi)^{N_j}} \right] \prod_{j=1}^{n} \left[z_j^{\operatorname{Tr} \mathrm{i}\sigma_j^+} \bar{z}_j^{\operatorname{Tr} \mathrm{i}\sigma_j^-} \right] Z_{1l,\text{v.m.}} Z_{1l,\text{c.m.}} \tag{3.6.18}$$

where $\mathrm{i}\sigma_j^\pm = \mathrm{i}\sigma_j \pm B_j/2$, $Z_{1l,\text{v.m.}}$ is the one-loop determinant of vector multiplets, a Vandermonde factor, and $Z_{1l,\text{c.m.}}$ is the one-loop determinant of chiral multiplets, a product of Gamma functions. Collecting all factors which depend on $\sigma_k^\pm$ for a given $k < n$ yields the integral

$$\begin{aligned} Z_k = \sum_{B_k \in \mathbb{Z}^{N_k}} & \int \frac{\mathrm{d}^{N_k} \sigma_k \, z_k^{\operatorname{Tr} \mathrm{i}\sigma_k^+} \bar{z}_k^{\operatorname{Tr} \mathrm{i}\sigma_k^-}}{N_k! (2\pi)^{N_k}} \\ & \cdot \prod_{i<j}^{N_k} \left[- \prod_{\pm} (\mathrm{i}\sigma_{ki}^\pm - \mathrm{i}\sigma_{kj}^\pm) \right] \prod_{i,j=1}^{N_k} \frac{\Gamma(-\mathrm{i}m_{kk} - \mathrm{i}\sigma_{ki}^+ + \mathrm{i}\sigma_{kj}^+)}{\Gamma(1 + \mathrm{i}m_{kk} + \mathrm{i}\sigma_{ki}^- - \mathrm{i}\sigma_{kj}^-)} \\ & \cdot \prod_{l \in \{k \pm 1\}} \prod_{i=1}^{N_k} \prod_{j=1}^{N_l} \left[\frac{\Gamma(-\mathrm{i}m_{kl} + \mathrm{i}\sigma_{lj}^+ - \mathrm{i}\sigma_{ki}^+)}{\Gamma(1 + \mathrm{i}m_{kl} - \mathrm{i}\sigma_{lj}^- + \mathrm{i}\sigma_{ki}^-)} \frac{\Gamma(-\mathrm{i}m_{lk} - \mathrm{i}\sigma_{lj}^+ + \mathrm{i}\sigma_{ki}^+)}{\Gamma(1 + \mathrm{i}m_{lk} + \mathrm{i}\sigma_{lj}^- - \mathrm{i}\sigma_{ki}^-)} \right], \end{aligned} \tag{3.6.19}$$

which resembles the S^2 partition function of SQCDA with N_k colors and $N_{k-1} + N_{k+1}$ flavours, with twisted masses $m_{kl} - \sigma_{lj}$ and $m_{lk} + \sigma_{lj}$. The shifts of σ_{lj} by $\pm B_{lj}/2$ cannot be incorporated in such twisted masses, as the ratios of Gamma functions involve both σ_{lj}^+ and σ_{lj}^-.

However, we can still apply the same techniques as in Section 3.5, and close the $\mathrm{i}\sigma_k$ integration contours towards $\pm\infty$ depending on whether $|z_k| \lessgtr 1$. The sum over poles factorizes as in the case of SQCDA, and the resulting vortex and antivortex partition functions are those of SQCDA with twisted masses $m_{kl} - \sigma_{lj}^+$ and $m_{lk} + \sigma_{lj}^+$ for vortices, and $m_{kl} - \sigma_{lj}^-$ and $m_{lk} + \sigma_{lj}^-$ for antivortices. As we saw in Section 3.5, those vortex partition functions have branch points when $\hat{z}_k = (-1)^{N_{k-1} + N_{k+1} + N_k - 1} z_k$ is 1 or ∞. We now prove that the powers of $1 - \hat{z}_k$ which appear in an expansion of Z_v near $\hat{z}_k = 1$ coincide with the powers of $x_{k+1} - x_k$ obtained in the fusion of the punctures at x_k and x_{k+1}. This implies that $x_k = x_{k+1} \hat{z}_k$, as announced, and fixes $\gamma_{k(k+1)}$. To proceed further, we need to distinguish the cases $\eta_k = \pm 1$.

If $\eta_k = -1$, then $\mathrm{i}m_{kk} = -1/2$, and the adjoint chiral multiplet one-loop determinant is simply a sign. Thus, the vortex partition functions are those of SQCD. From (3.4.19), the exponents of $1-\hat{z}_k$ which occur in an expansion near 1 are 0 and

$$\begin{aligned} &N_{k-1}(1+\mathrm{i}m_{k(k-1)}+\mathrm{i}m_{(k-1)k})+N_{k+1}(1+\mathrm{i}m_{k(k+1)}+\mathrm{i}m_{(k+1)k})-N_k \\ &=\begin{cases} -K_k(1+b^2)-K_{k+1}b^2 & \text{if } \epsilon_k=-\epsilon_{k+1}=-1 \\ K_kb^2+K_{k+1}(1+b^2) & \text{if } \epsilon_k=-\epsilon_{k+1}=+1\,. \end{cases} \end{aligned} \tag{3.6.20}$$

The analoguous limit in the Toda CFT correlator is $x_k \to x_{k+1}$. The channel where the punctures at x_k and x_{k+1} are fused allows two internal momenta. Indeed, $\epsilon_k = -\epsilon_{k+1}$, hence one of the punctures is labelled by a symmetric representation and the other one by an antisymmetric representation. The fusion of two such representations is the direct sum of two irreducible representations:

$$\mathcal{R}(Kh_1)\otimes\mathcal{R}(\omega_L)=\mathcal{R}(Kh_1+\omega_L)\oplus\mathcal{R}((K-1)h_1+\omega_{L+1}) \tag{3.6.21}$$

assuming $K, L \geq 1$. The corresponding exponents of $x_{k+1}-x_k$ are

$$\Delta(-Kbh_1-b\omega_L)-\Delta(-Kbh_1)-\Delta(-b\omega_L)=-Kb^2+\frac{KL}{N_f}b^2 \tag{3.6.22}$$

$$\Delta(-(K-1)bh_1-b\omega_{L+1})-\Delta(-Kbh_1)-\Delta(-b\omega_L)=L(1+b^2)+\frac{KL}{N_f}b^2\,, \tag{3.6.23}$$

and match with the gauge theory exponents up to precisely $\gamma_{k(k+1)}$ given in (3.6.13). Indeed, if $\epsilon_k = -\epsilon_{k+1} = -1$, then K and L above are K_{k+1} and K_k, the first Toda CFT exponent corresponds to the gauge theory exponent (3.6.20), and the second to 0. If $\epsilon_k = -\epsilon_{k+1} = 1$, then $K = K_k$ and $L = K_{k+1}$, the first Toda CFT exponent corresponds to 0 and the second to (3.6.20).

If instead $\eta_k = +1$, then the adjoint chiral multiplet remains, and the vortex partition functions involve more powers of $1-\hat{z}_k$, given in (3.5.13). Namely,

$$\begin{aligned} &\big(1-\hat{z}_k\big)^{-\nu+\nu N_{k-1}(1+\mathrm{i}m_{k(k-1)}+\mathrm{i}m_{(k-1)k})+\nu N_{k+1}(1+\mathrm{i}m_{k(k+1)}+\mathrm{i}m_{(k+1)k})+\nu[2N_k-\nu-1]\mathrm{i}m_{kk}} \\ &=\big(1-\hat{z}_k\big)^{-\nu(1+\mathrm{i}m_{kk})+\nu[K_k-K_{k+1}-\nu]\mathrm{i}m_{kk}} \end{aligned} \tag{3.6.24}$$

for $0 \leq \nu \leq N_k$. The remaining σ_j integrals ($j \neq k$) do not affect these exponents. From the derivation of (3.5.13), we know that the contribution for a given ν comes from the region where ν components $\sigma_{k,\mathrm{a}}$ of σ_k are large. The corresponding Gamma functions in the Coulomb branch integral are expanded as powers of $\mathrm{i}\sigma^{\pm}_{k,\mathrm{a}}$. Afterwards, one can close contours of all σ_j for $j < k$ as we have done to find the s-channel expansion. The Gamma functions which were expanded in powers of $\mathrm{i}\sigma^{\pm}_{k,\mathrm{a}}$ do not contribute poles, hence the sum over poles is non-empty only if $N_k - \nu \geq N_{k-1} \geq \cdots \geq N_1$. As a result, $\nu \leq N_k - N_{k-1} = K_k$. Changing variables to $\mu = K_k - \nu$, we deduce

$$Z = |1-\hat{z}_k|^{2[-K_k(1+\mathrm{i}m_{kk})]} \sum_{\mu=0}^{K_k} |1-\hat{z}_k|^{2[\mu(1+\mathrm{i}m_{kk})-(K_k-\mu)(K_{k+1}-\mu)\mathrm{i}m_{kk}]} (\text{series}) \tag{3.6.25}$$

where (series) denote series in non-negative integer powers of $1-\hat{z}_k$ and $\overline{1-\hat{z}_k}$. In Section 4.4, we relate the S^2 partition function of the quiver gauge theory we are studying to another such partition function, with in particular $K_k \leftrightarrow K_{k+1}$. This restricts the sum over μ to $0 \leq \mu \leq \min(K_k, K_{k+1})$. On the Toda CFT side, the limit is $x_k \to x_{k+1}$, and we are interested in the fusion of two degenerate punctures, labelled by two symmetric or two antisymmetric representations since $\epsilon_k = \epsilon_{k+1}$. Given that

$$\begin{aligned} \mathcal{R}(\omega_K) \otimes \mathcal{R}(\omega_L) &= \bigoplus_{\mu=0}^{\min(K,L)} \mathcal{R}(\omega_{K+L-\mu} + \omega_\mu) \\ \mathcal{R}(Kh_1) \otimes \mathcal{R}(Lh_1) &= \bigoplus_{\mu=0}^{\min(K,L)} \mathcal{R}\big((K+L-\mu)h_1 + \mu h_2\big)\,, \end{aligned} \tag{3.6.26}$$

the Toda CFT exponents of $x_{k+1} - x_k$ are

$$\begin{aligned} &\Delta\big(-b\omega_{K+L-\mu} - b\omega_\mu\big) - \Delta(-b\omega_K) - \Delta(-b\omega_L) \\ &\quad = \frac{KL}{N_f} b^2 - \mu b^2 + (K-\mu)(L-\mu)(b^2+1) \qquad \text{if } \epsilon_k = \epsilon_{k+1} = -1 \end{aligned} \tag{3.6.27}$$

$$\begin{aligned} &\Delta\big(-(K+L-\mu)bh_1 - \mu bh_2\big) - \Delta(-Kbh_1) - \Delta(-Lbh_1) \\ &\quad = \frac{KL}{N_f} b^2 + \mu(b^2+1) - (K-\mu)(L-\mu)b^2 \qquad \text{if } \epsilon_k = \epsilon_{k+1} = +1 \end{aligned} \tag{3.6.28}$$

where K and L are K_k and K_{k+1}. Again, the Toda CFT exponents match with the gauge theory exponents up to precisely $\gamma_{k(k+1)}$ given in (3.6.13).

Note that matching the number of distinct powers of $1-\hat{z}_k$ in gauge theory with the number of internal momenta in the fusion of punctures at x_{k-1} and x_k is enough to fix the relation between the signs $\{\eta_j\}$ and $\{\epsilon_j\}$. When the adjoint X_j can be integrated out ($\eta_j = -1$), the gauge theory involves two exponents only, and correspondingly the two neighboring punctures are labelled by different types of representations (one is symmetric and the other antisymmetric), whose fusion has two terms. When the adjoint X_j remains ($\eta_j = +1$), the gauge theory involves many exponents, and the two punctures have the same type, hence a fusion with many terms.

The situation is very similar in the limit $x_n = \hat{z}_n \to 1$. The gauge theory involves two exponents if $\eta_n = -1$, and $N_n - N_{n-1}$ if $\eta_n = +1$. On the Toda CFT side, the fusion of the semidegenerate momentum $\hat{m}$ with the degenerate $-b\Omega(K_n, \epsilon_n)$ gives two momenta if $\epsilon_n = -1$, and K_n if $\epsilon_n = +1$. Hence $\epsilon_n = \eta_n$ and $K_n = N_n - N_{n-1}$. Calculating the exponents and comparing them fixes $\hat{m}$ to (3.6.6) and γ_n to (3.6.12).

All other exponents γ_{ij} and γ_j are fixed thanks to the identification of permutations of degenerate punctures with gauge theory dualities found in Section 4.4.1.

3.6.2 Arbitrary Toda degenerates

We now consider the matching (3.6.1) in the case where $K_{j+1} \geq K_j$ for $1 \leq j \leq n-1$, and $\epsilon_j = -1$ for all $1 \leq j \leq n$, that is, $\eta_n = -1$ and $\eta_j = +1$ for all $1 \leq j \leq n-1$. In the course of fixing parameters for the matching, we have found that the expansion near $x_k = x_{k+1}$ involves the $\min(K_k, K_{k+1}) = K_k$ powers (3.6.25) of $x_{k+1} - x_k = x_{k+1}(1 - \hat{z}_k)$, for $1 \leq k \leq n-1$. Given our assumptions, these exponents all have a non-negative real part (the vortex partition functions contribute integer exponents $\nu \geq 0$):

$$\operatorname{Re}\big((K_k - \mu)b^2 + (K_k - \mu)(K_{k+1} - \mu)(1 + b^2) + \nu\big) \geq 0\,. \tag{3.6.29}$$

The real part vanishes if and only if $\mu = K_k$ and $\nu = 0$. As $\hat{z}_k \to 1$, only the term with $\mu = K_k$ and $\nu = 0$ remains. On the Toda CFT side, this limit selects the fusion

$$\mathcal{R}(\omega_{K_{k+1}}) \otimes \mathcal{R}(\omega_{K_k}) \longrightarrow \mathcal{R}(\omega_{K_{k+1}} + \omega_{K_k})\,. \tag{3.6.30}$$

We can carry this process further and take the fusion of arbitrarily many antisymmetric degenerate operators. For definiteness, let us send $x_k \to x_n$ for k going from $n-1$ to 1, in this order. At a given step $x_k \to x_n$, the Littlewood–Richardson rule gives

$$\mathcal{R}(\Omega) \otimes \mathcal{R}(\omega_{K_k}) = \sideset{}{'}\bigoplus_{h \in \mathcal{R}(\omega_{K_k})} \mathcal{R}(\Omega + h) \tag{3.6.31}$$

with a sum running over weights h of $\mathcal{R}(\omega_{K_k})$ such that $\Omega + h$ is a dominant weight. In our setting, $\Omega = \omega_{K_n} + \cdots + \omega_{K_{k+1}}$. The power of $x_n - x_k$ for a weight h is

$$\Delta(-b\Omega - bh) - \Delta(-b\Omega) - \Delta(-b\omega_{K_k}) + \sum_{l=k+1}^{n} \gamma_{kl} = b\langle Q, \omega_{K_k} - h\rangle + b^2\langle \Omega, \omega_{K_k} - h\rangle\,, \tag{3.6.32}$$

which has a positive real part unless $h = \omega_{K_k}$, in which case it vanishes. Thus, setting $x_k = x_n$ selects precisely the fusion of $-b\Omega$ and $-b\omega_{K_k}$ into $-b\Omega - b\omega_{K_k}$.

Any dominant weight Ω is a sum of fundamental weights, hence the four-point function of two generic and one semi-degenerate vertex operators with an arbitrary degenerate vertex operator $\widehat{V}_{-b\Omega}$ is equal to the partition function of an S^2 surface operator built from a certain quiver on S_b^4, with some fine-tuned FI parameters and theta angles. Namely, decomposing $\Omega = \omega_{K_n} + \cdots + \omega_{K_1}$ with $K_n \geq \cdots \geq K_1$, we find

$$Z_{S^2\subset S_b^4}^{\prod_k U(N_k),W_\eta}(m,z,\bar{z}) = Aa(x)a(\bar{x})\Big\langle \widehat{V}_{\alpha_\infty}(\infty)\widehat{V}_{\hat{m}}(1)\widehat{V}_{-b\Omega}(x,\bar{x})\widehat{V}_{\alpha_0}(0)\Big\rangle\,, \tag{3.6.33}$$

where[29] $N_k = \sum_{j=1}^k K_j$ for $1 \leq k \leq n$,

$$\begin{aligned} &\eta_n = -1 \quad \text{and} \quad \hat{z}_n = x\,, \\ &\eta_k = +1 \quad \text{and} \quad \hat{z}_k = 1 \quad \text{for } 1 \leq k \leq n-1\,, \end{aligned} \tag{3.6.34}$$

and the momenta α_0, α_∞, and $\hat{m}$ are given by (3.6.6). The factor

$$a(x)a(\bar{x}) = |x|^{2\beta}|1-x|^{2\gamma} \tag{3.6.35}$$

differs from (3.6.10) and has the exponents

$$\beta = \langle Q, -b\Omega\rangle - \frac{N_n}{N_f}\sum_{t=1}^{N_f} \mathrm{i}m_t - \sum_{j=1}^{n-1} N_j b^2 \tag{3.6.36}$$

$$\gamma = -b^2\frac{N_n(N_f-N_n)}{N_f} + \frac{N_n}{N_f}\sum_t(\mathrm{i}m_t + \mathrm{i}\widetilde{m}_t)\,. \tag{3.6.37}$$

Finally, the overall constant A is identical to the constant in (3.6.1), given by (3.6.9), because the three-point functions $\widehat{C}_{-b\omega_K,-b\Omega}^{-b(\Omega+\omega_K)}$ are in fact all equal to 1. Incidentally, in the case $\Omega = Nh_1$, the factor $Aa(x)a(\bar{x})$ coincides with the factor we found in the matching between the same Toda CFT correlator and the SQCDA surface operator. Thus, SQCDA and the $U(N)\times\cdots\times U(1)$ theory which appears in this matching have equal S^2 partition functions. The relation between these theories may run deeper.

Since the partition function in (3.6.33) is known explicitly, the matching gives an explicit expression for the Toda CFT four-point function of two full, one simple, and a degenerate operator $\widehat{V}_{-b\Omega}$. The Higgs branch expansion of Z provides conformal blocks as explicit series. From the Coulomb branch representation of Z for $\hat{m} = 0$ one can extract integral expressions for the three-point function of a degenerate operator $\widehat{V}_{-b\Omega}$ with generic vertex operators. These expressions typically involve fewer integrals than expressions obtained form the Coulomb gas formalism, but we have not investigated this direction further.

[29] As explained below (3.2.10), the factor $Aa(x)a(\bar{x})$ can be absorbed into the partition function.

More generally, any Toda CFT $(p+3)$-point function with two generic and one semi-degenerate operators at 0, ∞ and 1, and p arbitrary degenerate operators $\widehat{V}_{-b\Omega_l}(x_l,\bar{x}_l)$ is equal to the partition function of a surface operator describing a certain quiver gauge theory. This matching directly derives from the matching (3.6.1), with only antisymmetric degenerate operators, and taking all but p of the $\hat{z}$ equal to 1. Concretely, we express each highest weight as

$$\Omega_l = \sum_{j=1}^{c_l} \omega_{K_{l,j}} \,, \tag{3.6.38}$$

where c_l is the number of columns in the Young diagram of Ω_l and $K_{l,c_l} \geq \cdots \geq K_{l,2} \geq K_{l,1} \geq 0$ are the number of boxes in each column. We then define an order on the pairs $\bigl\{(l,j) \bigm| 1 \leq l \leq p, 1 \leq j \leq c_l\bigr\}$ by $(k,i) \leq (l,j)$ if $k < l$ or if $k = l$ and $i \leq j$. The gauge group is then

$$\prod_{l=1}^{p}\prod_{j=1}^{c_l} U(N_{l,j}) \quad \text{where} \quad N_{l,j} = \sum_{(k,i)\leq(l,j)} K_{k,i} \,. \tag{3.6.39}$$

The matter content of the theory consists as usual of pairs of bifundamental chiral multiplets between neighboring nodes, namely $(k,i) \leftrightarrow (k,i+1)$ and $(k,c_k) \leftrightarrow (k+1,1)$, of an adjoint chiral multiplet for every node except $U(N_{p,c_p})$, and of N_f fundamental and N_f antifundamental chiral multiplets for this last node $U(N_{p,c_p})$. Complexified FI parameters associated to each node $U(N_{l,j})$ are given by

$$\hat{z}_{l,j} = \begin{cases} 1 & \text{if } 1 \leq j < c_l \\ x_l/x_{l+1} & \text{if } j = c_l \,, \end{cases} \tag{3.6.40}$$

where $x_{p+1} = 1$. Detailed factors can be read from the matching (3.6.1) using this gauge theory data.

All in all, we have identified the $\mathcal{N} = (2,2)$ surface operator corresponding to the insertion of an arbitrary set of degenerate vertex operators in a Toda CFT three-point function. It would be interesting to calculate the expectation values of such surface operators in an interacting four-dimensional theory of class S.

Chapter 4

Two-dimensional gauge theory dualities

This is the second part of the article *M2-brane surface operators and gauge theory dualities in Toda* by Jaume Gomis and the author [GLF14].

4.1 Introduction

This chapter describes dualities of two-dimensional $\mathcal{N} = (2,2)$ gauge theories which can be obtained as manifest Toda CFT symmetries. The dualities relate the IR limits of these theories, and we probe them by comparing the S^2 partition functions of the dual theories. The contribution of free hypermultiplets to the partition function of the 4d/2d theory plays little role. We find several Seiberg-like dualities (generalizing the duality found by Hori and Tong [HT06]) relating theories with similar matter content but different gauge groups (summarized in Table 4.1). The dualities are most clearly seen through their Toda CFT interpretation as conjugation of all momenta or as the crossing symmetry exchanging two degenerate operators. Nevertheless, we also show directly in Appendix 4.A and Appendix 4.B that the S^2 partition functions [BC12; DGLFL12] of dual theories are equal.[1] This completes the dictionary between symmetries of Toda CFT correlators and dualities of 4d/2d gauge theories (see Table 4.2).

We start in Section 4.2 with the two-dimensional analogue of Seiberg duality [Sei94], between $\mathcal{N} = (2,2)$ $U(N)$ SQCD with N_f flavours, and $U(N_f - N)$ SQCD with N_f flavours. The corresponding Toda CFT correlators are simply related by conjugating all momenta, a symmetry under which the fundamental weights of A_{N_f-1} transform as $(\omega_N)^C = \omega_{N_f-N}$. This operation

[1] This was shown previously for SQCD with N_f fundamental and $\widetilde{N_f} \leq N_f - 2$ antifundamental chiral multiplets [BC12], and generalized very recently to arbitrary $\widetilde{N_f}$ in [BPZ14]. Our proofs follow the same logic but also apply to theories with an adjoint chiral multiplet and a superpotential.

provides us with the precise map of parameters: $N^D = N_f - N$, $z^D = z$, and $m^D = \mathrm{i}/2 - m$ for the complexified twisted masses of every chiral multiplet.[2] In addition to fundamental and antifundamental chiral multiplets, the $U(N_f - N)$ theory involves a free chiral multiplet transforming in the bifundamental representation of the flavour symmetry group $S[U(N_f) \times U(N_f)]$. These free chiral multiplets couple to the charged multiplets through a cubic superpotential, which must have total R-charge 2 (complexified twisted mass i) to be supersymmetric. As was also observed recently in [BPZ14], the theories differ by a shift in the FI parameter associated to the $U(1)$ flavour symmetry. In Section 4.2.2, we deduce Seiberg duality relations between theories with N_f fundamental and $\widetilde{N_f} < N_f$ antifundamental chiral multiplets. For this, we let some of the twisted masses of antifundamental multiplets go to infinity and take into account the renormalization of the FI parameter: this limit precisely corresponds to merging the Toda CFT operators inserted at ∞ and 1 into an irregular puncture [GT12].

We then move on in Section 4.3 to dualities of $U(N)$ SQCDA, which has fundamental, antifundamental, and adjoint chiral multiplets. Without further restriction, the theory features no duality. We find two choices of superpotentials for which the theory has a dual description: both dualities appear to be new in two-dimensional $\mathcal{N} = (2, 2)$ theories.

In Section 4.3.1, we consider SQCDA with the superpotential

$$W = \sum_{t=1}^{N_f} \widetilde{q}_t X^{l_t} q_t \,, \tag{4.1.1}$$

described by a choice of N_f integers $l_t \geq 0$, where q_t, $\widetilde{q}_t$ and X are the fundamental, antifundamental, and adjoint chiral multiplets. The theory is a simple generalization of $\mathcal{N} = (2, 2)^*$ SQCD.[3] The constraint on R-charges due to the superpotential translates to a very natural constraint in the Toda CFT language. Namely, the momentum labelling the simple puncture gets fine-tuned to become a degenerate operator, labelled by a symmetric representation of A_{N_f-1}. The crossing symmetry exchanging these two degenerate vertex operators thus provides us with a duality between two-dimensional SQCDA theories with the superpotential (4.1.1). The $U(N^D)$ dual theory features the same chiral multiplets and superpotential as the $U(N)$ theory, with $m^D = m$, $N^D = \sum_t l_t - N$, and $z^D = z^{-1}$.

In Section 4.3.2, we consider SQCDA with the superpotential

$$W = \operatorname{Tr} X^{l+1} \tag{4.1.2}$$

[2] By weakly gauging the flavour symmetry and turning on a constant background for the resulting vector multiplet, chiral multiplets can be given twisted masses and R-charges, which combine into a complex parameter m for each chiral multiplet.

[3] $\mathcal{N} = (2, 2)^*$ SQCD is the mass deformation of the $\mathcal{N} = (4, 4)$ theory of a $U(N)$ vector multiplet coupled to N_f fundamental hypermultiplets. Its cubic superpotential $W = \sum_t \widetilde{q}_t X q_t$ corresponds to taking all $l_t = 1$.

for some integer $l \geq 0$, where X is the adjoint chiral multiplet. We find a direct analogue of the four-dimensional Kutasov–Schwimmer duality [Kut95; KS95]. It turns out that given the superpotential constraint, conjugation maps the (symmetric) degenerate operator describing $U(N)$ SQCDA to the degenerate operator describing $U(N^D)$ SQCDA. The dual gauge theory has $N^D = lN_f - N$, $z^D = z$, $m_t^D = m_X - m_t$, $\widetilde{m}_t^D = m_X - \widetilde{m}_t$, and $m_X^D = m_X$. As in four dimensions [Kut95; KS95], the dual theory features l additional free chiral multiplets in the bifundamental representation of $S[U(N_f) \times U(N_f)]$, which correspond to mesons of the electric theory. As for SQCD, the limit where twisted masses of some chiral multiplets are very large yields similar dualities between theories with a different number of fundamental and antifundamental chiral multiplets.

Lastly, we describe dualities of quiver gauge theories in Section 4.4. We consider the $U(N_1) \times \cdots \times U(N_n)$ quiver theories (3.1.8) which correspond in the Toda CFT to the insertion of n degenerate vertex operators. Dualities of other $\mathcal{N} = (2,2)$ quiver gauge theories were considered recently in [BPZ14].

In Section 4.4.1 we apply Seiberg duality or the $\mathcal{N} = (2,2)^*$ duality (depending on the presence or absence of an adjoint chiral multiplet) to gauge group factors $U(N_j)$ with $j < n$. We show that the duality translates to the exchange of degenerate punctures numbered j and $j+1$ in the Toda CFT. Each permutation of the n degenerate punctures is thus realized as a combination of such Seiberg and $\mathcal{N} = (2,2)^*$ dualities.

Based on this geometric realization of dualities for $j < n$, we construct in Section 4.4.2 the full set of dual theories obtained through Seiberg and $\mathcal{N} = (2,2)^*$ dualities acting on any gauge group. We find no Toda CFT description of the duality acting on $U(N_n)$, except when all degenerate vertex operators are labelled by antisymmetric representations of A_{N_f-1}. Then, conjugating all Toda CFT momenta yields a different set of degenerate operators of the same type, and it turns out that the corresponding dual gauge theories are related by a sequence of Seiberg and $\mathcal{N} = (2,2)^*$ dualities on all nodes. A particular case is the quiver (3.1.1) which corresponds to a single degenerate vertex operator labelled by an arbitrary representation $\mathcal{R}$: applying the same sequence of Seiberg and $\mathcal{N} = (2,2)^*$ dualities corresponds to conjugating $\mathcal{R}$ and all Toda CFT momenta. This result concludes the description of dualities of two-dimensional $\mathcal{N} = (2,2)$ gauge theories which correspond to manifest symmetries of the Toda CFT.

We check all dualities by proving that the S^2 partition functions of dual theories are equal up to simple ambiguous factors. In all cases, the factors can be absorbed in either one of the dual partition functions through the ambiguities described below (3.2.10), namely a renormalization scheme ambiguity, a global gauge transformation, and a flavour FI parameter.

Analytic proofs that vortex partition of dual theories are equal are relegated to appendices: Appendix 4.A for Seiberg duality and Appendix 4.B for dualities of SQCD with an adjoint.

Table 4.1: Dualities of $\mathcal{N}=(2,2)$ quiver gauge theories realized as symmetries in the Toda CFT. Chiral multiplets are denoted by q_t (fundamentals), $\widetilde{q}_t$ (antifundamentals), and X (adjoint). For Seiberg and Kutasov–Schwimmer dualities, the magnetic theory contains extra free chiral multiplets whose charges are identical to those of mesons in the electric theory.

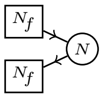
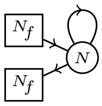
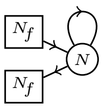
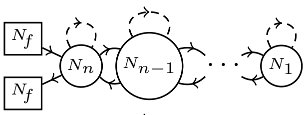
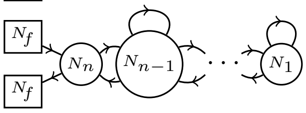
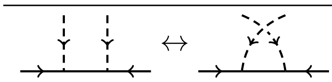
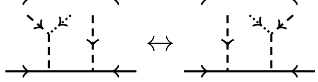
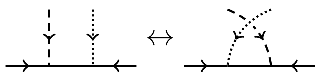
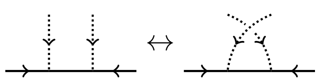
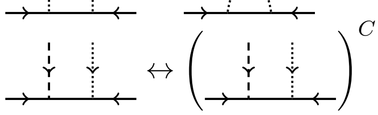
Duality	Quiver	W	Dual parameters	Toda symmetry
Seiberg		0	$N^D = N_f - N$ $z^D = z,\ m^D = \frac{\mathrm{i}}{2} - m$	Conjugation p. 157 $(-b\omega_N)^C = -b\omega_{N^D}$
$(2,2)^*$-like		$\sum_t \widetilde{q}_t X^{l_t} q_t$	$N^D = \sum_t l_t - N$ $z^D = z^{-1},\ m^D = m$	Crossing p. 162 simple $\to$ degenerate
Kutasov–Schwimmer		$\operatorname{Tr} X^{l+1}$	$N^D = lN_f - N$ $z^D = z,\ m^D = \frac{\mathrm{i}}{l+1} - m$	Conjugation p. 166 $(-Nbh_1)^C \equiv -N^D bh_1$
Quiver			$N_j^D = N_{j-1} + N_{j+1} - N_j$ $z_j^D = z_j^{-1},\ z_{j\pm 1}^D = z_j z_{j\pm 1}$	Crossing p. 171 $\omega_{N_j - N_{j-1}} \leftrightarrow \omega_{N_{j+1} - N_j}$
Quiver			$N_j^D = jN_f - N_j\ \forall j$ $m^D = \frac{\mathrm{i}}{2} - m$	Conjugation p. 174 $\omega^C_{N_j - N_{j-1}} = \omega_{N_j^D - N_{j-1}^D}$

Table 4.2: The effect of a few Toda CFT moves on the corresponding 4d/2d gauge theory. Besides the symmetry under changing trinion decomposition, Toda CFT correlators are also invariant under conjugation of all momenta. Full punctures are drawn as solid lines, simple punctures as dashed lines, and degenerate punctures as dotted lines.

Toda CFT move	$\Longleftrightarrow$	Gauge theory duality
	$\Longleftrightarrow$	4d S-duality
	$\Longleftrightarrow$	4d/2d node-hopping
	$\Longleftrightarrow$	2d flop transition
	$\Longleftrightarrow$	2d Seiberg and $(2,2)^*$ dualities for quivers
	$\Longleftrightarrow$	2d Seiberg and Kutasov–Schwimmer dualities

4.2 Seiberg duality as momentum conjugation

Seiberg duality relates theories with different gauge groups but the same flavour symmetry. In our two-dimensional $\mathcal{N} = (2,2)$ context, it is expected that $U(N)$ SQCD with N_f fundamental and $\widetilde{N_f} \leq N_f$ antifundamental chiral multiplets is dual to $U(N_f - N)$ SQCD with the same number of chiral multiplets and $N_f\widetilde{N_f}$ additional free mesons, for an appropriate choice of twisted masses. In the case $\widetilde{N_f} \leq N_f - 2$, the series giving vortex partition functions were proven term by term to be equal in [BC12], and the relation for S^2 partition functions was deduced. For $\widetilde{N_f} = N_f - 1$ or $\widetilde{N_f} = N_f$, vortex partition functions differ by a non-trivial factor, found numerically in [HO13, Appendix F]. Our direct proof in Appendix 4.A (similar to that of [BPZ14] found independently) is technical and by itself provides no insight on the factor. In contrast, the factor appears naturally in the proof we give here via the Toda CFT.

We denote by m_s and $\widetilde{m}_s$ the twisted masses (with R-charges) in the electric theory, and by m_s^D and $\widetilde{m}_s^D$ those in the dual magnetic theory. We shall prove that

$$\begin{aligned} &Z_{S^2}^{U(N),N_f,\widetilde{N_f}}(z,\bar{z},m,\widetilde{m}) \\ &\quad = a(z,\bar{z}) \prod_{s=1}^{N_f} \prod_{t=1}^{\widetilde{N_f}} \Big[\gamma(-\mathrm{i}m_s - \mathrm{i}\widetilde{m}_t)\Big] Z_{S^2}^{U(N^D),N_f,\widetilde{N_f}}(z^D,\bar{z}^D,m^D,\widetilde{m}^D) \end{aligned} \tag{4.2.1}$$

where z and z^D are renormalized values at the scale ℓ of the sphere, and dual parameters are $N^D = N_f - N$, $z^D = (-1)^{N_f - \widetilde{N_f}} z$, $\bar{z}^D = (-1)^{N_f - \widetilde{N_f}} \bar{z}$, $m_s^D = \frac{\mathrm{i}}{2} - m_s$, and $\widetilde{m}_s^D = \frac{\mathrm{i}}{2} - \widetilde{m}_s$. The factor

$$a(z,\bar{z}) = \begin{cases} |z|^{2\delta_0} & \text{if } \widetilde{N_f} \leq N_f - 2 \\ |z|^{2\delta_0} e^{(-1)^{N_f}(z-\bar{z})} & \text{if } \widetilde{N_f} = N_f - 1 \\ |z|^{2\delta_0} \Big|1 - (-1)^{N_f} z\Big|^{2\delta_1} & \text{if } \widetilde{N_f} = N_f \end{cases} \tag{4.2.2}$$

is given in terms of the exponents

$$\delta_0 = \gamma_0 - \gamma_0^D = -\frac{N_f - N}{2} - \sum_{s=1}^{N_f} \mathrm{i}m_s \tag{4.2.3}$$

$$\delta_1 = \gamma_1 - \gamma_1^D = N_f - N + \sum_{s=1}^{N_f} (\mathrm{i}m_s + \mathrm{i}\widetilde{m}_s) \tag{4.2.4}$$

which we will obtain from the exponents γ_i in the matching (3.4.1), and their duals γ_i^D. The factor $a(z,\bar{z})$ could be absorbed into Z in three parts as discussed below (3.2.10). First, a renormalization scheme ambiguity absorbs any factor independent of twisted masses. Next, a global gauge transformation shifts the partition function by any power of $|z|^2$. A last factor (present only for $\widetilde{N_f} = N_f$) can be absorbed by turning on an FI parameter for the $U(1)$ flavour group under which fundamental and antifundamental chiral multiplets have the same charge.

The product of gamma functions in (4.2.1) is the (one-loop determinant) contribution from $N_f \widetilde{N_f}$ free mesons with twisted masses $m_s + \widetilde{m}_t = \mathrm{i} - m_s^D - \widetilde{m}_t^D$. These twisted masses are equal to those of the mesons $\widetilde{q}_t q_s$, where q_s and $\widetilde{q}_t$ are fundamental and antifundamental quarks of the electric theory. In the magnetic theory, the twisted masses derive from the superpotential coupling $W = \widetilde{q}^D M q^D$, which has total R-charge 2, hence total (complexified) twisted mass $\widetilde{m}_t^D + (\mathrm{i} - m_s^D - \widetilde{m}_t^D) + m_s^D = \mathrm{i}$.

Applied twice, the duality (4.2.1) yields the original theory, since parameters are mapped back to those of the electric theory. An immediate consistency check is thus

$$\gamma(-\mathrm{i}m_s - \mathrm{i}\widetilde{m}_t)\gamma(-\mathrm{i}m_s^D - \mathrm{i}\widetilde{m}_t^D) = \frac{\Gamma(-\mathrm{i}m_s - \mathrm{i}\widetilde{m}_t)}{\Gamma(1 + \mathrm{i}m_s + \mathrm{i}\widetilde{m}_t)} \frac{\Gamma(1 + \mathrm{i}m_s + \mathrm{i}\widetilde{m}_t)}{\Gamma(-\mathrm{i}m_s - \mathrm{i}\widetilde{m}_t)} = 1 \tag{4.2.5}$$

and that the $a(z, \bar{z})$ factors cancel thanks to

$$\delta_0^D = -\frac{N_f - N^D}{2} - \sum_{s=1}^{N_f} \mathrm{i}m_s^D = -\frac{N}{2} + \sum_{s=1}^{N_f} \mathrm{i}m_s + \frac{N_f}{2} = -\delta_0 \tag{4.2.6}$$

$$\delta_1^D = N_f - N^D + \sum_{s=1}^{N_f} (\mathrm{i}m_s^D + \mathrm{i}\widetilde{m}_s^D) = N - \sum_{s=1}^{N_f} (\mathrm{i}m_s + \mathrm{i}\widetilde{m}_s) - N_f = -\delta_1 \tag{4.2.7}$$

and, for $\widetilde{N_f} = N_f - 1$, $(-1)^{N_f}(z^D - \bar{z}^D) = -(-1)^{N_f}(z - \bar{z})$. A second consistency check, in the case $\widetilde{N_f} = N_f$, is the symmetry under charge conjugation $z \leftrightarrow 1/z$, $\bar{z} \leftrightarrow 1/\bar{z}$, and $\mathrm{i}m_s \leftrightarrow \mathrm{i}\widetilde{m}_s$. We find that δ_1 is left unchanged, and that δ_0 is mapped to $-\delta_0 - \delta_1$, consistent with $a(1/z, 1/\bar{z}) = |z|^{-2\delta_0 - 2\delta_1} |1 - (-1)^{N_f} z|^{2\delta_1}$.

4.2.1 Momentum conjugation for $\widetilde{N_f} = N_f$

To derive the Seiberg duality relation (4.2.1) for $\widetilde{N_f} = N_f$, we rely on the matching (3.4.1) relating the S^2 partition function of $U(N)$ SQCD to a Toda CFT four-point function:

$$Z_{S^2 \subset S_b^4}^{U(N), \widetilde{N_f} = N_f}(m, \widetilde{m}, z, \bar{z}) = A|x|^{2\gamma_0}|1 - x|^{2\gamma_1} \left\langle \widehat{V}_{\alpha_\infty}(\infty) \widehat{V}_{\hat{m}}(1) \widehat{V}_{-b\omega_N}(x, \bar{x}) \widehat{V}_{\alpha_0}(0) \right\rangle . \tag{4.2.8}$$

The four-point function features two generic operators $\widehat{V}_\alpha$, a semi-degenerate operator $\widehat{V}_{\hat{m}}$, and the degenerate operator $\widehat{V}_{-b\omega_N}$ inserted at $x = (-1)^{N_f} z$ and labelled by the highest weight ω_N of the N-th antisymmetric representation of A_{N_f-1}. The relation between gauge theory twisted masses m and $\widetilde{m}$, and Toda CFT momenta α_0, α_∞, and $\hat{m}$ is given in Section 3.4.

Toda CFT correlators are invariant under changing all momenta to their conjugate, that is, applying the $\mathbb{C}$-linear transformation $h_s \to h_s^C = -h_{N_f+1-s}$ which maps weights of a representation of A_{N_f-1} to weights of the conjugate representation. This transformation maps the degenerate momentum $-b\omega_N$ to[4]

$$(-b\omega_N)^C = -\sum_{s=1}^{N} b h_s^C = \sum_{s=N_f-N+1}^{N_f} b h_s = -\sum_{s=1}^{N_f-N} b h_s = -b\omega_{N_f-N}\,, \qquad (4.2.9)$$

which is precisely the degenerate momentum appearing in the description of the Seiberg dual SQCD theory. The semi-degenerate momentum $\hat{m} = (\varkappa + Nb)h_1$ becomes $\hat{m}^C = -(\varkappa + Nb)h_{N_f}$, which is not along h_1. However, the Weyl reflexion defined by the cyclic permutation of $[\![1, N_f]\!]$ maps $\hat{m}^C$ to

$$\left[N_f\left(b + \frac{1}{b}\right) - \varkappa - Nb\right] h_1 = (\varkappa^D + N^D b)h_1 = \hat{m}^D\,, \qquad (4.2.10)$$

where $\varkappa^D = N_f/b - \varkappa$. Indeed, $\langle \hat{m}^C - Q, h_s\rangle = \langle \hat{m}^D - Q, h_{s+1}\rangle$ for all $1 \leq s \leq N_f - 1$, and $\langle \hat{m}^C - Q, h_{N_f}\rangle = \langle \hat{m}^D - Q, h_1\rangle$. Finally, the generic momenta α_0 and α_∞ remain generic after conjugation, and we have

$$\langle \alpha^C - Q, h_p\rangle = \langle \alpha - Q^C, h_p^C\rangle = -\langle \alpha - Q, h_{N_f+1-p}\rangle\,, \qquad (4.2.11)$$

where we used that $\langle \alpha_1, \alpha_2\rangle = \langle \alpha_1^C, \alpha_2^C\rangle$ and that $Q = Q^C$. A Weyl reflexion then permutes $\langle \alpha - Q, h_{N_f+1-p}\rangle \to \langle \alpha - Q, h_p\rangle$, hence conjugation followed by this reflexion simply changes $\alpha \to 2Q - \alpha$.

We thus find that conjugation of all momenta (with subsequent Weyl reflexions) relates two correlators which correspond to SQCD with $U(N)$ and $U(N_f - N)$ gauge groups. Converting the relation between momenta to gauge theory variables, we find $m_s^D = \frac{\mathrm{i}}{2} - m_s$ and $\widetilde{m}_s^D = \frac{\mathrm{i}}{2} - \widetilde{m}_s$, as we claimed below (4.2.1).[5]

In our normalization, both generic and non-degenerate operators are Weyl reflexion invariant, without reflexion amplitudes. The two Toda CFT correlators are thus equal, and we divide the factor relating the S^2 partition functions and Toda CFT correlator for one theory by the factor for the other

[4] The third step uses that the weights h_s of the fundamental representation of A_{N_f-1} sum to zero.

[5] A global $U(1)$ gauge transformation is identical to the flavour symmetry which shifts $\mathrm{i}m_s$ and $-\mathrm{i}\widetilde{m}_s$ by the same amount. This has no physical effect: the Toda correlator is invariant, and the partition function is multiplied by a power of $|z|^2$. Dual twisted masses are only defined up to such a shift, which also alters δ_0.

theory to find (for $\widetilde{N_f} = N_f$)

$$Z_{S^2}^{U(N)}(z,\bar{z},m,\widetilde{m}) = \frac{Z_{S_b^4}^{\text{free},D} A\, |z|^{2\gamma_0} \left|1-(-1)^{N_f} z\right|^{2\gamma_1}}{Z_{S_b^4}^{\text{free}} A^D\, |z|^{2\gamma_0^D} \left|1-(-1)^{N_f} z\right|^{2\gamma_1^D}} Z_{S^2}^{U(N^D)}(z^D,\bar{z}^D,m^D,\widetilde{m}^D)\,. \tag{4.2.12}$$

We recognize the factor $a(z,\bar{z}) = |z|^{2\gamma_0-2\gamma_0^D} |1-(-1)^{N_f} z|^{2\gamma_1-2\gamma_1^D}$ announced in (4.2.2). The ratio A/A^D simplifies:

$$\frac{A}{A^D} = \frac{b^{NN_f(1+b^2)-N^2b^2-2N\sum_{s=1}^{N_f}(1+\mathrm{i}m_s+\mathrm{i}\widetilde{m}_s)}}{b^{N^DN_f(1+b^2)-(N^D)^2b^2-2N^D\sum_{s=1}^{N_f}(1+\mathrm{i}m_s^D+\mathrm{i}\widetilde{m}_s^D)}} = b^{-N_f\sum_{s=1}^{N_f}(1+2\mathrm{i}m_s+2\mathrm{i}\widetilde{m}_s)}\,. \tag{4.2.13}$$

The hypermultiplets masses (3.2.17) involved in the S_b^4 partition functions (3.2.4) associated to the electric and magnetic theories are

$$m_{st} = \mathrm{i}\frac{1-b^2}{2b} - \frac{1}{b}(m_s+\widetilde{m}_t) \tag{4.2.14}$$

$$m_{st}^D = \mathrm{i}\frac{1-b^2}{2b} - \frac{1}{b}(\mathrm{i}-m_s-\widetilde{m}_t) = -\mathrm{i}b - m_{st}\,, \tag{4.2.15}$$

thus the constant factor is

$$\begin{aligned}\frac{Z_{S_b^4}^{\text{free},D} A}{Z_{S_b^4}^{\text{free}} A^D} &= \frac{A}{A^D}\prod_{s,t}\frac{\Upsilon(\frac{b}{2}+\frac{1}{2b}-\mathrm{i}m_{st})}{\Upsilon(\frac{b}{2}+\frac{1}{2b}-\mathrm{i}m_{st}^D)} = \frac{A}{A^D}\prod_{s,t}\frac{\Upsilon(\frac{b}{2}+\frac{1}{2b}+\mathrm{i}m_{st})}{\Upsilon(\frac{b}{2}+\frac{1}{2b}+\mathrm{i}m_{st}-b)}\\ &= b^{-N_f\sum_{s=1}^{N_f}(1+2\mathrm{i}m_s+2\mathrm{i}\widetilde{m}_s)}\prod_{s,t}\left[b^{b^2-2b\mathrm{i}m_{st}}\gamma\left(\frac{1-b^2}{2}+b\mathrm{i}m_{st}\right)\right] = \prod_{s,t}\gamma(-\mathrm{i}m_s-\mathrm{i}\widetilde{m}_t)\,.\end{aligned} \tag{4.2.16}$$

The one-loop determinants of free mesons appear here thanks to the shift by b in $\mathrm{i}m_{st}^D = b - \mathrm{i}m_{st}$, which relies on the shift between m_{st} and $\frac{-1}{b}(m_s+\widetilde{m}_t)$ in (3.2.17). We obtain this constant factor more directly for any $\widetilde{N_f} \le N_f$ in the next section.

4.2.2 Decoupling multiplets towards $\widetilde{N_f} < N_f$

We could find analoguous Seiberg duality relations for $\widetilde{N_f} < N_f$ via the matching of Section 3.4.3 with Toda irregular punctures, but those cases are also easily accessed by taking some twisted masses of antifundamental multiplets to be very large in the $\widetilde{N_f} = N_f$ duality. The reverse process, which decreases $N_f > \widetilde{N_f}$ by giving some fundamental multiplets large twisted masses, is more difficult, and must be significantly altered to reach the case $N_f = \widetilde{N_f}$ in Appendix 4.A.

Our starting point to prove (4.2.1) is the Higgs branch decomposition of the S^2 partition function of interest [BC12; DGLFL12]:

$$Z^{U(N),N_f,\widetilde{N_f}} = \sum_{1\leq p_1<\cdots<p_N\leq N_f} \left[(z\bar{z})^{-\sum_{j=1}^N \mathrm{i}m_{p_j}} Z_{11,\{p\}}^{N_f,\widetilde{N_f}} f_{\{p\}}^{(\mathrm{s}),N_f,\widetilde{N_f}}((-1)^{N_f}z) f_{\{p\}}^{(\mathrm{s}),N_f,\widetilde{N_f}}((-1)^{\widetilde{N_f}}\bar{z}) \right]$$

$$Z_{11,\{p\}}^{N_f,\widetilde{N_f}} = \prod_{j=1}^N \frac{\prod_{s\notin\{p\}}^{N_f} \gamma(-\mathrm{i}m_s + \mathrm{i}m_{p_j})}{\prod_{s=1}^{\widetilde{N_f}} \gamma(1+\mathrm{i}\widetilde{m}_s + \mathrm{i}m_{p_j})} \qquad f_{\{p\}}^{(\mathrm{s}),N_f,\widetilde{N_f}}(x) = \sum_{k=0}^\infty \frac{x^k}{k!} f_{\{p\},k}^{(\mathrm{s}),N_f,\widetilde{N_f}} \tag{4.2.17}$$

$$f_{\{p\},k}^{(\mathrm{s}),N_f,\widetilde{N_f}} = k! \sum_{k_1+\cdots+k_N=k} \prod_{j=1}^N \left[\frac{\prod_{s=1}^{\widetilde{N_f}} (-\mathrm{i}\widetilde{m}_s - \mathrm{i}m_{p_j})_{k_j}}{k_j! \prod_{i\neq j}^N (\mathrm{i}m_{p_i} - \mathrm{i}m_{p_j} - k_i)_{k_j} \prod_{s\notin\{p\}}^{N_f} (1+\mathrm{i}m_s - \mathrm{i}m_{p_j})_{k_j}} \right],$$

which generalizes (3.4.8) to $\widetilde{N_f} < N_f$. The series $f_{\{p\}}^{(\mathrm{s}),N_f,\widetilde{N_f}}(x)$ converge on the unit disc if $\widetilde{N_f} = N_f$, and on the whole complex plane if $\widetilde{N_f} < N_f$. We shall equate the term of (4.2.17) labelled by $\{p\} \subseteq [\![1, N_f]\!]$ with the term labelled by the complement $\{p\}^\complement$ for the dual theory. The powers of $|z|^2$ match:

$$-\sum_{j=1}^N \mathrm{i}m_{p_j} = -\sum_{s=1}^{N_f} \mathrm{i}m_s + \sum_{s\in\{p\}^\complement} \mathrm{i}m_s = -\sum_{s=1}^{N_f} \mathrm{i}m_s - \frac{N_f - N}{2} - \sum_{s\in\{p\}^\complement} \mathrm{i}m_s^D = \delta_0 - \sum_{s\in\{p\}^\complement} \mathrm{i}m_s^D . \tag{4.2.18}$$

The constant is fixed as the ratio of one-loop determinants Z_{11}

$$\frac{Z_{11,\{p\}}^{N_f,\widetilde{N_f}}}{Z_{11,\{p\}^\complement}^{N_f,\widetilde{N_f},D}} = \prod_{s=1}^{\widetilde{N_f}} \frac{\prod_{t\in\{p\}^\complement} \gamma(1+\mathrm{i}\widetilde{m}_s^D + \mathrm{i}m_t^D)}{\prod_{t\in\{p\}} \gamma(1+\mathrm{i}\widetilde{m}_s + \mathrm{i}m_t)} = \prod_{s=1}^{\widetilde{N_f}} \prod_{t=1}^{N_f} \gamma(-\mathrm{i}\widetilde{m}_s - \mathrm{i}m_t), \tag{4.2.19}$$

which is independent of $\{p\}$. There remains to match vortex partition functions,

$$f_{\{p\}}^{(\mathrm{s}),N_f,\widetilde{N_f}}(x) = a(x) f_{\{p\}^\complement}^{(\mathrm{s}),N_f,\widetilde{N_f},D}(x^D) \tag{4.2.20}$$

where

$$a(x) = \begin{cases} (1-x)^{\delta_1} & \text{if } \widetilde{N_f} = N_f \\ e^x & \text{if } \widetilde{N_f} = N_f - 1 \\ 1 & \text{if } \widetilde{N_f} \leq N_f - 2 \end{cases} \tag{4.2.21}$$

and $x^D = (-1)^{N_f - \widetilde{N_f}} x$. From the case $\widetilde{N_f} = N_f$ studied in the previous section, we now derive the relations for $\widetilde{N_f} < N_f$ by taking a limit where $N_f - \widetilde{N_f}$ antifundamental chiral multiplets are given large twisted masses. We give a proof independent of the Toda CFT in Appendix 4.A.

Let us expand $f^{(\mathrm{s}),N_f,\widetilde{N_f}}_{\{p\},k}$, for some $\widetilde{N_f} \leq N_f$, in the limit $\widetilde{m}_{\widetilde{N_f}} = \Lambda \to +\infty$. This relies on the asymptotic behaviour $(\rho+a)_k \sim \rho^k$ of Pochhammer symbols when $|\rho| \to \infty$:

$$f^{(\mathrm{s}),N_f,\widetilde{N_f}}_{\{p\},k} \sim (-\mathrm{i}\Lambda)^k f^{(\mathrm{s}),N_f,\widetilde{N_f}-1}_{\{p\},k} . \tag{4.2.22}$$

Summing over $k \geq 0$,

$$f^{(\mathrm{s}),N_f,\widetilde{N_f}}_{\{p\}}\left(\frac{x}{-\mathrm{i}\Lambda}\right) \to f^{(\mathrm{s}),N_f,\widetilde{N_f}-1}_{\{p\}}(x)\,, \tag{4.2.23}$$

as $\Lambda \to \infty$, and for a fixed x. We then apply this limit to (4.2.20) for $\widetilde{N_f} = N_f$ after changing $x \to x/(-\mathrm{i}\Lambda)$. Since $\delta_1 \sim \mathrm{i}\Lambda$, we get $a(\widetilde{N_f} = N_f, \mathrm{i}x/\Lambda) = e^{\delta_1 \ln(1-\mathrm{i}x/\Lambda)} \sim e^x$, which is the exponential factor for $\widetilde{N_f} = N_f - 1$. In the limit where further twisted masses become very large while keeping the appropriate combination $-\mathrm{i}\Lambda x$ fixed, the exponential factor becomes $e^{x/(-\mathrm{i}\Lambda)} \to 1$, yielding $a(x) = 1$ for $\widetilde{N_f} \leq N_f - 2$. The relative sign between x and x^D is due to the sign difference $\mathrm{i}\widetilde{m}^D \sim -\mathrm{i}\widetilde{m}$ for each of the $N_f - \widetilde{N_f}$ antifundamental multiplets which we decouple.

This concludes the proof of the Seiberg duality relation (4.2.1) for all $\widetilde{N_f} \leq N_f$ as limits of the case $\widetilde{N_f} = N_f$, itself derived from the correspondence with the Toda CFT.

4.3 SQCDA dualities: crossing and conjugation

In this section, we find two new Seiberg-like dualities between pairs of $\mathcal{N} = (2,2)$ theories with adjoint matter and a superpotential. The first is realized in the Toda CFT as crossing symmetry, and contains as a special case a duality between $\mathcal{N} = (2,2)^*$ theories, proposed in [OR10] for particular twisted masses, and recently in [BPZ14]. The second is realized as conjugation symmetry, and is a direct two-dimensional analogue of the four-dimensional Kutasov–Schwimmer duality [Kut95; KS95]. We test both dualities by comparing S^2 partition functions using the matching with Toda CFT correlators. We also provide direct proofs that the S^2 partition functions of dual theories are equal, without relying on the Toda CFT. Namely, classical and one-loop contributions are compared in the main text, and vortex partition functions in Appendix 4.B. It would be interesting to work out the mapping of chiral rings of dual theories.

Each duality relates theories with $U(N)$ and $U(N^D)$ vector multiplets coupled to one adjoint, N_f fundamental, and $\widetilde{N_f}$ antifundamental chiral multiplets. We assume by symmetry that $\widetilde{N_f} \leq N_f$. As for the Seiberg duality, the magnetic theory includes additional free matter. In the electric theory, chiral multiplets are denoted by X, q_t, and $\widetilde{q}_t$, and their (complexified) twisted masses by m_X, m_t, and $\widetilde{m}_t$ respectively. The FI parameters and theta angles (renormalized, at the scale ℓ of the sphere) are combined as usual into a complex parameter z. We use the notations X^D, q_t^D, $\widetilde{q}_t^D$, m_X^D, m_t^D, $\widetilde{m}_t^D$, and z^D for the magnetic theory.

Recall that when $\widetilde{N_f} = N_f$ we have the matching

$$Z_{S^2\subset S_b^4}^{U(N)\text{ SQCDA}}(m,\widetilde{m},m_X,z,\bar{z}) = A|y|^{2\gamma_0}|1-y|^{2\gamma_1}\left\langle \widehat{V}_{\alpha_\infty}(\infty)\widehat{V}_{\hat{m}}(1)\widehat{V}_{-Nbh_1}(y,\bar{y})\widehat{V}_{\alpha_0}(0)\right\rangle \tag{4.3.1}$$

for $y = (-1)^{N_f+N-1}z$, $b^2 = \mathrm{i}m_X$, with other parameters given below (3.5.1). The four-point function can exhibit two types of symmetries. If the semi-degenerate momentum $\hat{m} = (\varkappa+Nb)h_1$ is in fact degenerate ($\hat{m} = -N^D bh_1$), then crossing symmetry exchanges the two degenerate operators via $N \leftrightarrow N^D$ and $y \to y^{-1}$. This yields the duality in Section 4.3.1. On the other hand, it turns out that for fined-tuned values $\mathrm{i}m_X = b^2 = \frac{-1}{l+1}$ the degenerate operator $\widehat{V}_{-Nbh_1}$ is conjugate to another degenerate operator, $\widehat{V}_{-N^Dbh_1}$. This leads to the Kutasov–Schwimmer duality in Section 4.3.2, which we then extend to $\widetilde{N_f} < N_f$ as we did for the Seiberg duality.

4.3.1 $(2,2)^*$-like duality as a braiding move

Let us describe the first duality more precisely. With notations as above, the electric and magnetic theories are $\mathcal{N} = (2,2)$ SQCDA theories with N and N^D colors and the same matter content and superpotential

$$W = \sum_{t=1}^{N_f} \widetilde{q}_t X^{l_t} q_t \qquad \text{hence} \quad 1 + \mathrm{i}m_t + \mathrm{i}\widetilde{m}_t + l_t \mathrm{i}m_X = 0\,, \tag{4.3.2}$$

where $l_t \geq 0$ are integers, and $\widetilde{N_f} = N_f$. We will find that $N^D = L - N$ with $L = \sum_{t=1}^{N_f} l_t$, that twisted masses are the same in the two theories, and that $z^D = (-1)^L z^{-1}$ and $\bar{z}^D = (-1)^L \bar{z}^{-1}$.

In particular, when all $l_t = 1$ the theories are $\mathcal{N} = (2,2)^*$ SQCD with gauge groups $U(N)$ and $U(N_f - N)$, and the duality is an $\mathcal{N} = (2,2)^*$ analogue of the $\mathcal{N} = (2,2)$ Seiberg duality found earlier. In the special case $m_X = \mathrm{i}/2$, the two dualities agree after charge conjugation, which exchanges $m_s^D \leftrightarrow \widetilde{m}_s^D$ and maps $z^D \to (z^D)^{-1}$. The agreement is expected since the superpotential term $W = \operatorname{Tr} X^2$ is then supersymmetric and X can be integrated out, shifting the theta angle by $(N-1)\pi$ in the process: this leads to a sign difference in the maps $z \to z^D$ of the two dualities.

We test the duality by proving that S^2 partition functions match:

$$Z_{S^2}^{U(N)}(z,\bar{z}) = |z|^{2\delta_0}\left|1-(-1)^{N_f+N-1}z\right|^{2\delta_1} Z_{S^2}^{U(N^D)}(z^D,\bar{z}^D) \tag{4.3.3}$$

for $W=\sum_{t=1}^{N_f}\widetilde{q}_t X^{l_t} q_t$, with dual parameters given above, and the exponents

$$\delta_0 = -(L-N)(1+\mathrm{i}m_X) - \sum_{t=1}^{N_f}\sum_{\nu=0}^{l_t-1}(\mathrm{i}m_t+\nu\mathrm{i}m_X) \tag{4.3.4}$$

$$\delta_1 = (L-2N)(1+\mathrm{i}m_X)\,. \tag{4.3.5}$$

As discussed below (4.2.2) for the Seiberg duality, the powers of $|z|^2$ and $|1-z|^2$ can be absorbed as ambiguities of the S^2 partition function.[6] The same consistency checks as for the Seiberg duality apply. Repeating the duality yields the original parameters, and the factors cancel thanks to $\delta_0^D=\delta_0+\delta_1$ and $\delta_1^D=-\delta_1$. The relation is also invariant under charge conjugation, which exchanges twisted masses of fundamental and antifundamental chiral multiplets, since δ_1 is unchanged and $\delta_0\to-\delta_0-\delta_1$. We first derive dual parameters from the matching of SQCDA partition functions to Toda CFT correlators. For completeness, we then prove the relation by comparing classical, one-loop and vortex contributions of the two theories.

Recall the matching (3.5.1) between the partition function of a sphere surface operator describing $U(N)$ SQCDA and a Toda CFT correlator with the symmetric degenerate operator $\widehat{V}_{-Nbh_1}$, a semi-degenerate operator $\widehat{V}_{\hat{m}}$, and two generic operators. We find in Section 3.5.3 that the superpotential $W=\sum_t\widetilde{q}_t X^{l_t} q_t$ constrains twisted masses in such a way that $\hat{m}=-(L-N)bh_1=-N^Dbh_1$. The S^2 partition function of the electric theory we are studying is thus

$$Z_{S^2}^{U(N)}(z,\bar{z}) = \widetilde{A}|y|^{2\gamma_0}|1-y|^{2\gamma_1}\Big\langle \widehat{V}_{\alpha_\infty}(\infty)\widehat{V}_{-N^Dbh_1}(1)\widehat{V}_{-Nbh_1}(y,\bar{y})\widehat{V}_{\alpha_0}(0)\Big\rangle \tag{4.3.6}$$

where $y=(-1)^{N_f+N-1}z$, $b^2=\mathrm{i}m_X$, momenta and exponents are given below (3.5.1), and we have absorbed in $\widetilde{A}$ the contributions from the S_b^4 hypermultiplets and from the differing normalization of semidegenerate and degenerate operators. The Toda CFT correlator is invariant under $N\to N^D$, $y\to y^D=y^{-1}$, and the conformal map $(\infty,1,y^{-1},0)\to(\infty,y,1,0)$. This implies that

$$Z_{S^2}^{U(N)}(z,\bar{z}) = \frac{\widetilde{A}|y|^{2\gamma_0}|1-y|^{2\gamma_1}}{\widetilde{A}^D|y^D|^{2\gamma_0^D}|1-y^D|^{2\gamma_1^D}} |y|^{\Delta(\alpha_\infty)-\Delta(-N^Dbh_1)-\Delta(-Nbh_1)-\Delta(\alpha_0)} Z_{S^2}^{U(N^D)}(z^D,\bar{z}^D)\,. \tag{4.3.7}$$

We deduce the exponents (4.3.4) and (4.3.5) by computing $\delta_1=\gamma_1-\gamma_1^D$ and

$$\delta_0=\gamma_0+\Delta(\alpha_\infty)-\Delta(-N^Dbh_1)-\Delta(-Nbh_1)-\Delta(\alpha_0)+\gamma_0^D+\gamma_1^D\,. \tag{4.3.8}$$

[6]For $\mathcal{N}=(2,2)^*$ theories, the power of $1-z$ relating dual vortex partition functions was found numerically by Honda and Okuda [HO13].

We also obtain $z^D = (-1)^{N_f+N^D-1} y^D = (-1)^{N_f+N^D-1} y^{-1} = (-1)^{N^D-N} z^{-1}$ and $N^D = L - N$ as announced.

There remains to fix the overall constant factor, since $\widetilde{A}/\widetilde{A}^D$ is difficult to evaluate ($\widetilde{A}$ and $\widetilde{A}^D$ are singular for our choice of twisted masses). This is done by comparing the s-channel decomposition (as $z \to 0$) of the electric theory with the u-channel decomposition (as $z^D \to \infty$) of the magnetic theory. Recall from Section 3.5.3 that the s-channel Higgs branch vacua of the electric theory are labelled by ordered partitions $\sum_{t=1}^{N_f} n_t = N$ with $0 \leq n_t \leq l_t$. The classical and one-loop contributions (3.5.9) are

$$Z^{(\mathrm{s})}_{\mathrm{cl},\{n_t\}}(z,\bar{z}) = (z\bar{z})^{\sum_{s=1}^{N_f}\sum_{\mu=0}^{n_s-1}(-\mathrm{i}m_s-\mu\mathrm{i}m_X)} \tag{4.3.9}$$

$$Z^{(\mathrm{s})}_{1\mathrm{l},\{n_t\}} = \prod_{s=1}^{N_f}\prod_{\mu=0}^{n_s-1}\prod_{t=1}^{N_f} \frac{\gamma(\mathrm{i}m_s - \mathrm{i}m_t + (\mu - n_t)\mathrm{i}m_X)}{\gamma(\mathrm{i}m_s - \mathrm{i}m_t + (\mu - l_t)\mathrm{i}m_X)}\,. \tag{4.3.10}$$

Similarly, u-channel Higgs branch vacua of the magnetic theory are labelled by partitions $\sum_{t=1}^{N_f} n_t^D = N^D$ with $0 \leq n_t^D \leq l_t$, and are in a natural bijection with those of the electric theory through $n_t^D = l_t - n_t$. The classical contributions match up to $|z|^{2\delta_0}$:

$$\sum_{s=1}^{N_f}\sum_{\mu=0}^{n_s-1}(-\mathrm{i}m_s - \mu\mathrm{i}m_X) = \delta_0 - \sum_{s=1}^{N_f}\sum_{\mu=0}^{l_s-n_s-1}(\mathrm{i}\widetilde{m}_s + \mu\mathrm{i}m_X)\,. \tag{4.3.11}$$

The one-loop contributions are equal, with no relative constant factor, since

$$Z^{(\mathrm{s})}_{1\mathrm{l},\{n_t\}} = \prod_{s=1}^{N_f}\prod_{\mu=0}^{n_s-1}\prod_{t=1}^{N_f}\prod_{\nu=0}^{l_t-n_t-1} \frac{\gamma(1 + \mathrm{i}m_s + \mathrm{i}\widetilde{m}_t + (\mu+\nu+1)\mathrm{i}m_X)}{\gamma(1 + \mathrm{i}m_s + \mathrm{i}\widetilde{m}_t + (\mu+\nu)\mathrm{i}m_X)} \tag{4.3.12}$$

is invariant under $m \leftrightarrow \widetilde{m}$ and $n \to l - n$. We prove in Appendix 4.B that the vortex partition functions match up to $(1-y)^{\delta_1}$. This establishes the duality relation (4.3.3).

From the duality we can extract information about powers of $|1-y|^2$ which appear in the expansion of Z near $y = 1$. In the electric theory, the powers are given by (3.5.13), valid for all SQCDA theories: replacing k by $N-k$ there,

$$Z^{U(N)}_{S^2}(z,\bar{z}) = |1-y|^{-2N(1+\mathrm{i}m_X)} \sum_{k=0}^{N}\Big[|1-y|^{2[k(1+\mathrm{i}m_X)-(N-k)(N^D-k)\mathrm{i}m_X]}(\cdots)\Big] \tag{4.3.13}$$

for some series $(\cdots)$ in non-negative powers of $(1-y)$ and $(1-\bar{y})$. The magnetic theory has a similar expansion with $N \leftrightarrow N^D$. Since the two must match, we deduce that the expansion (4.3.13) holds, with a sum restricted to $0 \leq k \leq \min(N, N^D)$. This list of exponents is useful to identify the correct relation between quiver gauge theories and correlators in Section 3.6.

4.3.2 Kutasov–Schwimmer duality as conjugation

The Kutasov–Schwimmer duality [Kut95; KS95], initially proposed between four-dimensional theories, is similar to the Seiberg duality, with an additional adjoint matter multiplet X subject to the superpotential coupling $W = \operatorname{Tr} X^{l+1}$. Through the matching found in Section 3.5.3, the duality is realized as conjugation of momenta in the Toda CFT when $\widetilde{N_f} = N_f$. Theories with $\widetilde{N_f} < N_f$ are obtained by decoupling chiral multiplets. For $l = 1$, integrating out X reproduces the Seiberg duality between SQCD theories.

The electric and magnetic theories are $\mathcal{N} = (2,2)$ SQCDA theories with gauge groups $U(N)$ and $U(N^D)$ and the superpotential coupling

$$W = \operatorname{Tr} X^{l+1} \qquad \text{hence} \quad \mathrm{i}m_X^D = \mathrm{i}m_X = \frac{-1}{l+1} \tag{4.3.14}$$

for some integer $l \geq 1$. As we will see, $z^D = (-1)^{(l+1)N_f - \widetilde{N_f}} z$, $\bar{z}^D = (-1)^{(l+1)N_f - \widetilde{N_f}} \bar{z}$ (in terms of renormalized parameters at the scale ℓ of the sphere), $N^D = lN_f - N$, $m_t^D = m_X - m_t$, $\widetilde{m}_t^D = m_X - \widetilde{m}_t$, $m_X^D = m_X$, and the magnetic theory also features $lN_f\widetilde{N_f}$ free mesons M_{jst}^D with twisted masses $m_{jst}^D = m_s + \widetilde{m}_t + jm_X$ for $0 \leq j < l$, $1 \leq s \leq N_f$, $1 \leq t \leq \widetilde{N_f}$. We assume that $l \leq N \leq lN_f - l$.

We test the duality by comparing S^2 partition functions. Namely, we prove that for $W = \operatorname{Tr} X^{l+1}$

$$Z_{S^2}^{U(N),N_f,\widetilde{N_f}}(m; z, \bar{z}) = a(z,\bar{z}) \prod_{j,s,t} \gamma(-\mathrm{i}m_{jst}^D) Z_{S^2}^{U(N^D),N_f,\widetilde{N_f}}(m^D; z^D, \bar{z}^D) \tag{4.3.15}$$

with dual parameters given above. The constant factor in (4.3.15) is the one-loop determinant of free mesons M_{jst}^D whose twisted masses $m_{jst}^D = m_s + \widetilde{m}_t + jm_X$ are fixed by the full superpotential coupling

$$W = \operatorname{Tr}\big[(X^D)^{l+1}\big] + \sum_{s=1}^{N_f} \sum_{t=1}^{\widetilde{N_f}} \sum_{j=0}^{l-1} M_{jst}^D \, [\widetilde{q}_t^D (X^D)^{l-1-j} q_s^D] \,. \tag{4.3.16}$$

Relative coefficients are irrelevant, as the superpotential only affects the S^2 partition function by fixing complexified twisted masses. The electric theory features mesons $\widetilde{q}_t X^j q_s$ which have the same twisted masses $m_s + \widetilde{m}_t + jm_X$. The factor $a(z, \bar{z})$ is

$$a(z,\bar{z}) = \begin{cases} |z|^{2\delta_0} & \text{if } \widetilde{N_f} \leq N_f - 2 \\ |z|^{2\delta_0} e^{l(-1)^{N_f+N-1}(z-\bar{z})} & \text{if } \widetilde{N_f} = N_f - 1 \\ |z|^{2\delta_0} \Big|1 - (-1)^{N_f+N-1} z\Big|^{2\delta_1} & \text{if } \widetilde{N_f} = N_f \end{cases} \tag{4.3.17}$$

$$\delta_0 = -\frac{lN_f}{2} + \frac{lN}{l+1} - l\sum_{s=1}^{N_f} \mathrm{i}m_s\,, \qquad \delta_1 = lN_f - \frac{2lN}{l+1} + l\sum_{s=1}^{N_f}(\mathrm{i}m_s + \mathrm{i}\widetilde{m}_s)\,. \tag{4.3.18}$$

As discussed below (4.2.2) for the Seiberg duality, this factor can be absorbed as an ambiguity of the S^2 partition function.

The same consistency checks as for the Seiberg duality apply. Repeating the duality yields the original parameters, and the factors $a(z,\bar{z})$ and products of gamma functions cancel. Charge conjugation leaves the relation invariant in the case $\widetilde{N_f} = N_f$.

Let us first derive (4.3.15) for $\widetilde{N_f} = N_f$ from Toda CFT conjugation. Recall (3.5.24), which expresses the partition functions of interest as $b^2 \to \frac{-1}{l+1}$ limits of Toda CFT four-point functions. The relevant correlator is $\langle \widehat{V}_{\alpha_\infty}(\infty)\widehat{V}_{\hat{m}}(1)\widehat{V}_{-b\omega_{N,l}}(x,\bar{x})\widehat{V}_{\alpha_0}(0)\rangle$. Here, $\omega_{N,l} = l\omega_k + (N-lk)h_{k+1}$ with k defined by $kl \leq N < (k+1)l$, and its conjugate weight is $\omega_{N,l}^C = \omega_{N^D,l}$ with $N^D = lN_f - N$. As for the Seiberg duality, we follow the conjugation of $\hat{m} = (\varkappa + Nb)h_1$ by a Weyl reflexion to get a momentum along h_1,

$$\hat{m}^D = (\varkappa^D + N^D b)h_1 = \left[N_f\left(b + \frac{1}{b}\right) - \varkappa - Nb\right]h_1\,. \tag{4.3.19}$$

Thus, $\varkappa^D = \frac{1}{b}N_f(1-(l-1)b^2) - \varkappa$, which is $\frac{2}{b}N_f(1+b^2) - \varkappa$ when $b^2 = \frac{-1}{l+1}$. Finally, the generic momenta α_0 and α_∞ are mapped as $\alpha \to 2Q - \alpha$ under a conjugation followed by the maximal Weyl reflexion. Translating to gauge theory parameters thanks to the dictionary (3.5.2) yields the values of N^D, m_X^D, m_s^D, and $\widetilde{m}_s^D$ claimed earlier. The position $y = (-1)^{N_f+N-1}z$ of the degenerate operator is not affected by conjugation, hence $y^D = y$ and $z^D = (-1)^{lN_f}z$. The factor $a(z,\bar{z})$ given in (4.3.17) is the ratio of factors $|y|^{2\gamma_0}|1-y|^{2\gamma_1}$ for the electric and magnetic theories.

Since the constant factor A which appears in the matching (3.5.24) is not known, we cannot deduce the presence of free mesons in the magnetic theory through conjugation of momenta. Instead, we use the Higgs branch decomposition (3.5.8), which expresses both partition functions as sums over choices of $0 \leq n_s \leq l$ with $n_1 + \cdots + n_{N_f} = N$. The classical contribution for the term labelled by $\{n_s\}$ in the electric theory is $|z|^{2\delta_0}$ times the classical contribution for the term labelled by $\{n_s^D = l - n_s\}$ in the magnetic theory. We then compare the one-loop determinants (3.5.9) for those vacua,

$$\begin{aligned}\frac{Z_{11,\{n\}}}{Z^D_{11,\{l-n\}}} &= \prod_{s=1}^{N_f}\prod_{t=1}^{N_f}\left[\prod_{\mu=0}^{n_s-1}\frac{\gamma(-\mathrm{i}m_t + \mathrm{i}m_s + \frac{n_t-\mu}{l+1})}{\gamma(1+\mathrm{i}\widetilde{m}_t + \mathrm{i}m_s - \frac{\mu}{l+1})}\prod_{\mu=0}^{l-n_s-1}\frac{\gamma(1+\mathrm{i}\widetilde{m}_t^D + \mathrm{i}m_s^D - \frac{\mu}{l+1})}{\gamma(-\mathrm{i}m_t^D + \mathrm{i}m_s^D + \frac{n_t^D-\mu}{l+1})}\right]\\ &= \prod_{s=1}^{N_f}\prod_{t=1}^{N_f}\prod_{j=0}^{l-1}\gamma\left(-\mathrm{i}\widetilde{m}_t - \mathrm{i}m_s + \frac{j}{l+1}\right)\end{aligned} \tag{4.3.20}$$

and find the one-loop determinants of mesons with twisted masses m^D_{jst} as announced. Finally, we prove in Appendix 4.B that vortex partition functions of dual theories are equal up to the factor $(1-y)^{\delta_1}$, hence establishing the relation (4.3.15) for $\widetilde{N_f} = N_f$.

For completeness, we compare exponents which appear when expanding the S^2 partition functions of the dual theories near $y=1$. Those exponents are given by (3.5.13): for a general $\mathcal{N}=(2,2)$ SQCDA theory with N colors there are $N+1$ exponents labelled by an integer $0 \leq k \leq N$. The set of exponents thus does not match for the $U(N)$ and $U(N^D)$ theories we consider here. In fact, it turns out that only the subset labelled by $0 \leq k \leq l$ matches (assuming that $l \leq N \leq lN_f - l$): the coefficients of all other exponents must thus vanish when $\mathrm{i}m_X = \frac{-1}{l+1}$.

We now take the limit $\mathrm{i}\widetilde{m}_t = \mathrm{i}\Lambda \to \pm\mathrm{i}\infty$ for $N_f - \widetilde{N_f}$ antifundamental flavours $\widetilde{N_f} < t \leq N_f$. The multiplets $\widetilde{q}_t$ decouple, the FI parameter is renormalized, and we will be left with the Kutasov–Schwimmer duality (4.3.15) for $\widetilde{N_f} < N_f$.

The twisted mass Λ appears in the Coulomb branch expansion (3.5.6) through the one-loop determinant of antifundamental chiral multiplets: for fixed σ and B

$$\prod_{j=1}^{N} \prod_{t=\widetilde{N_f}+1}^{N_f} \frac{\Gamma(-\mathrm{i}\widetilde{m}_t + \mathrm{i}\sigma_j + \frac{B_j}{2})}{\Gamma(1+\mathrm{i}\widetilde{m}_t - \mathrm{i}\sigma_j + \frac{B_j}{2})} \sim \left[\gamma(-\mathrm{i}\Lambda)^N (-\mathrm{i}\Lambda)^{\mathrm{Tr}(\mathrm{i}\sigma+\frac{B}{2})} (\mathrm{i}\Lambda)^{\mathrm{Tr}(\mathrm{i}\sigma-\frac{B}{2})}\right]^{N_f-\widetilde{N_f}} . \tag{4.3.21}$$

The powers of $\pm\mathrm{i}\Lambda$ combine nicely with the classical contributions $z_{\text{bare}}^{\mathrm{Tr}(\mathrm{i}\sigma+B/2)}$ and $\bar{z}_{\text{bare}}^{\mathrm{Tr}(\mathrm{i}\sigma-B/2)}$, and we get the integrand of the Coulomb branch representation for the theory with $\widetilde{N_f} < N_f$ and $z = z_{\text{ren}} = (-\mathrm{i}\Lambda)^{N_f-\widetilde{N_f}} z_{\text{bare}}$. A careful treatment shows that the limit $\Lambda \to \pm\infty$ and the integration commute, because the contribution for large values of σ and B falls off fast enough at infinity. As mentionned for a similar limit of the SQCD theory in Section 3.4.3, it is easier to work out this convergence issue in the Higgs branch decomposition where terms decrease exponentially in the vorticity k. Either way yields

$$\begin{aligned} &Z_{S^2}^{N_f,N_f}\left(\frac{z}{(-\mathrm{i}\Lambda)^{N_f-\widetilde{N_f}}}, \frac{\bar{z}}{(\mathrm{i}\Lambda)^{N_f-\widetilde{N_f}}}, \{m_s\}, \{\widetilde{m}_s, \Lambda\}\right) \\ &\qquad \sim \gamma(-\mathrm{i}\Lambda)^{N(N_f-\widetilde{N_f})} Z_{S^2}^{N_f,\widetilde{N_f}}(\{m_s\}, \{\widetilde{m}_s\}, z, \bar{z}) \,. \end{aligned} \tag{4.3.22}$$

Given the form of (4.3.22), the next step is to consider the duality (4.3.15) with the replacement $\widetilde{N_f} \to N_f$, $z \to z/(-\mathrm{i}\Lambda)^{N_f-\widetilde{N_f}}$ and $\bar{z} \to \bar{z}/(\mathrm{i}\Lambda)^{N_f-\widetilde{N_f}}$, in the limit where $\Lambda \to \pm\infty$. The factor $a(z,\bar{z}) = |z|^{2\delta_0}|1-(-1)^{N_f+N-1}z|^{2\delta_1}$

with $\delta_1 \sim \mathrm{i}\Lambda l$ becomes

$$\begin{aligned} a_{N_f,N_f}\left(\frac{z}{(-\mathrm{i}\Lambda)^{N_f-\widetilde{N_f}}}, \frac{\bar{z}}{(\mathrm{i}\Lambda)^{N_f-\widetilde{N_f}}}\right) &\sim \Lambda^{-2(N_f-\widetilde{N_f})\delta_0} a_{N_f,\widetilde{N_f}}(z,\bar{z}) \\ &\sim \begin{cases} \Lambda^{-2(N_f-\widetilde{N_f})\delta_0}|z|^{2\delta_0} & \text{if } \widetilde{N_f} \leq N_f-2 \\ \Lambda^{-2(N_f-\widetilde{N_f})\delta_0}|z|^{2\delta_0} e^{l(-1)^{N_f+N-1}(z-\bar{z})} & \text{if } \widetilde{N_f} = N_f-1. \end{cases} \end{aligned} \tag{4.3.23}$$

The gamma functions in (4.3.15) become those for $\widetilde{N_f} < N_f$, multiplied by

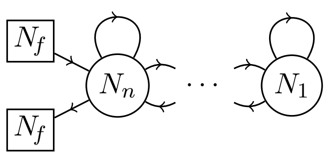
$$\begin{aligned} \prod_{s=1}^{N_f}\prod_{j=0}^{l-1}\prod_{t=\widetilde{N_f}+1}^{N_f} \gamma\Big(-\mathrm{i}\Lambda - \mathrm{i}m_s + \frac{j}{l+1}\Big) &\sim \prod_{s=1}^{N_f}\prod_{j=0}^{l-1}\Big[\gamma(-\mathrm{i}\Lambda)\,\Lambda^{2(-\mathrm{i}m_s+\frac{j}{l+1})}\Big]^{N_f-\widetilde{N_f}} \\ \sim \Big[\gamma(-\mathrm{i}\Lambda)^{lN_f}\Lambda^{2(\delta_0+N^D\frac{l}{l+1})}\Big]^{N_f-\widetilde{N_f}} &\sim \Big[\Lambda^{2\delta_0}\gamma(-\mathrm{i}\Lambda)^N\gamma(-\mathrm{i}\Lambda^D)^{-N^D}\Big]^{N_f-\widetilde{N_f}}, \end{aligned} \tag{4.3.24}$$

where we used $\gamma(\mathrm{i}x+a) \sim \gamma(\mathrm{i}x)|x|^{2a}$ as $x \to \pm\infty$, and $\Lambda^{\frac{2l}{l+1}}\gamma(-\mathrm{i}\Lambda) \sim \gamma(-\mathrm{i}\Lambda^D)^{-1}$. Combining (4.3.23) and (4.3.24) with the power of $\gamma(-\mathrm{i}\Lambda)$ from (4.3.22) and the power of $\gamma(-\mathrm{i}\Lambda^D)$ for the dual theory establishes the Kutasov–Schwimmer duality relation (4.3.15) for all $\widetilde{N_f} \leq N_f$.

4.4 Dualities for quivers

We revisit here the $\mathcal{N} = (2,2)$ quivers of Section 3.6 and express some Seiberg and $\mathcal{N} = (2,2)^*$ dualities as permutations of Toda CFT punctures in Section 4.4.1. This lets us construct in Section 4.4.2 the full set of theories obtained through Seiberg and $\mathcal{N} = (2,2)^*$ dualities. For a particular choice of matter content, a certain combination of dualities is realized as conjugation of momenta in the Toda CFT.

The gauge theories depend on ranks $N_j \geq 0$, signs η_j, and complexified FI parameters $(\hat{z}_j, \overline{\hat{z}_j})$ for $1 \leq j \leq n$, as well as twisted masses. They are described by the quiver



(4.4.1)

The theories consist of a $U(N_1)\times\cdots\times U(N_n)$ vector multiplet coupled to chiral multiplets which transform in the following representations: N_f fundamentals and N_f antifundamentals of $U(N_n)$, two bifundamentals of $U(N_j)\times U(N_{j-1})$ for each $2 \leq j \leq n$, and one adjoint of $U(N_j)$ for each $1 \leq j \leq n$. Let

$\epsilon_j = \prod_{i=j}^{n} \eta_i$. The twisted masses m_t, $\widetilde{m}_t$, $m_{j(j-1)}$, $m_{(j-1)j}$, and m_{jj} of those chiral multiplets obey (3.6.5), that is,

$$\mathrm{i}m_{j(j-1)} + \mathrm{i}m_{(j-1)j} = -1 - 2q_j \quad \text{and} \quad \mathrm{i}m_{jj} = q_j + q_{j+1}\,, \tag{4.4.2}$$

where $q_j = b^2/2$ if $\epsilon_j = 1$ and $q_j = -(1+b^2)/2$ otherwise for some parameter b^2. The twisted masses are such that a given superpotential W_η has R-charge 2 (twisted mass i). Whenever $\eta_j = -1$, the superpotential term $\mathrm{Tr}(X_j^2)$ lets us integrate out X_j.

We gave evidence in Section 3.6 that the partition function on S_b^4 of the S^2 surface operator obtained by coupling such a theory to free hypermultiplets is equal to a Toda CFT $(n+3)$-point function, namely the correlator of two generic, one semi-degenerate, and n degenerate vertex operators. The momenta of the first three operators encode the twisted masses m_t and $\widetilde{m}_t$. The degenerate operators are inserted at positions $x_j = \prod_{i=j}^{n} \hat{z}_i$, and have momenta $-b\Omega_j = -b\Omega(K_j, \epsilon_j)$, where $K_j = N_j - N_{j-1}$, $\epsilon_j = \prod_{i=j}^{n} \eta_i$, and $\Omega(K,+1) = Kh_1$ and $\Omega(K,-1) = \omega_K$.

Crossing symmetry of the Toda CFT correlator states that the labelling of degenerate operators by integers $1 \leq j \leq n$ is irrelevant. Therefore, the $n!$ gauge theories which correspond to each labelling of the degenerate punctures should all have identical S^2 partition functions, up to simple factors. It turns out that each transposition $k \leftrightarrow k+1$ (for $k < n$) corresponds to a duality acting on the node $U(N_k)$ of the quiver gauge theory: Seiberg duality (see Section 4.2) if $\eta_k = -1$, or the $\mathcal{N} = (2,2)^*$ duality (see Section 4.3.1) if $\eta_k = +1$. In Section 4.4.1 we work out details and make sure that transpositions correctly reproduce the mapping of parameters for such dualities. As a result, the groupoid generated by Seiberg and $\mathcal{N} = (2,2)^*$ dualities acting on nodes with $k < n$ is realized as permutations of punctures in the Toda CFT.

In Section 4.4.2, we extend the groupoid by including the action of Seiberg duality on the node $U(N_n)$ when it is applicable ($\eta_n = -1$): the $\mathcal{N} = (2,2)^*$ duality never applies, since the N_f fundamental and antifundamental chiral multiplets are not constrained by a superpotential. The result of acting with Seiberg duality on $U(N_n)$ is not a quiver of the same type, hence is not given a Toda CFT interpretation in our work. However, for a specific choice of matter content which corresponds to the case where all degenerate vertex operators are labelled by antisymmetric representations of A_{N_f-1}, applying Seiberg duality in turn to all the nodes yields a quiver of the original form. This combination of dualities corresponds to conjugating Toda CFT momenta.

All our results extend to theories with any number $\widetilde{N_f} \leq N_f$ of antifundamental chiral multiplets following the discussion for Seiberg duality of SQCD in Section 4.2.2. We focus on $\widetilde{N_f} = N_f$ because the matching between partition functions and Toda CFT correlators was only derived in this case in Section 3.6: for $\widetilde{N_f} < N_f$, the correlator contains an irregular puncture as described for SQCD in Section 3.4.3.

4.4.1 Seiberg dualities from braiding moves

We now prove that the action of Seiberg duality or the $\mathcal{N}=(2,2)^*$ duality (depending on η_k) on the node $U(N_k)$ translates to the transposition $(x_k,\epsilon_k,K_k)\leftrightarrow(x_{k+1},\epsilon_{k+1},K_{k+1})$ of two degenerate punctures, for $1\leq k\leq n-1$. Specifically, we show that the S^2 partition functions of the theories described by the Toda CFT data before and after the transposition are equal. Most gauge theory parameters describing the electric and magnetic theories are the same, with the following changes: $\eta^D_{k\pm1}=\eta_{k\pm1}\eta_k$, $N^D_k=N_{k+1}+N_{k-1}-N_k$, $\hat{z}^D_{k\pm1}=\hat{z}_{k\pm1}\hat{z}_k$ and $\hat{z}^D_k=\hat{z}^{-1}_k$.

The multiplets which interact with the $U(N_k)$ vector multiplet of the electric theory are those of $\mathcal{N}=(2,2)$ SQCDA with N_k colors and $N_{k-1}+N_{k+1}$ flavours. If $\eta_k=-1$, then $\mathrm{i}m_{kk}=-1/2$ and W_η contains the term $\operatorname{Tr}(X_k^2)$ which lets us integrate out the adjoint chiral multiplet X_k, leaving $\mathcal{N}=(2,2)$ SQCD. If instead $\eta_k=+1$, then $\mathrm{i}m_{kk}+\mathrm{i}m_{k(k\pm1)}+\mathrm{i}m_{(k\pm1)k}=-1$ and W_η contains the terms $\operatorname{Tr}(\phi_{(k\pm1)k}X_k\phi_{k(k\pm1)})$: this is $\mathcal{N}=(2,2)^*$ SQCD. In both settings, the theory admits a dual description with $N_{k+1}+N_{k-1}-N_k$ colors, and some mesons if $\eta_k=-1$ (see Section 4.2 and Section 4.3.1). As we will now see, parameters map precisely as expected from the Toda CFT.

In the Coulomb branch representation of the S^2 partition function of the electric theory, we collect all factors which depend on the scalar σ_k of the $U(N_k)$ vector multiplet. This yields an integral Z_k (3.6.19) very similar to the partition function of $\mathcal{N}=(2,2)$ SQCDA with N_k colors and $N_{k-1}+N_{k+1}$ flavours. The usual contour techniques apply and yield a factorized expression for Z_k in the region $|\hat{z}_k|<1$, namely

$$\begin{aligned}Z_k=\sum_{\text{Higgs vacuum } v^\pm}\ &\hat{z}_k^{\operatorname{Tr}\mathrm{i}v^+}\bar{\hat{z}}_k^{\operatorname{Tr}\mathrm{i}v^-}Z_{1\text{l},\{v^\pm\}}\big(\{m_{kl}-\sigma^\pm_{lj}\},\{m_{lk}+\sigma^\pm_{lj}\}\big)\\ &\cdot Z_{\text{v},v^+}\big(\{m_{kl}-\sigma^+_{lj}\},\{m_{lk}+\sigma^+_{lj}\};\hat{z}_k\big)\\ &\cdot Z_{\bar{\text{v}},v^-}\big(\{m_{kl}-\sigma^-_{lj}\},\{m_{lk}+\sigma^-_{lj}\};\bar{\hat{z}}_k\big)\,.\end{aligned}\tag{4.4.3}$$

As discussed above, the superpotential W_η reduces SQCDA to $\mathcal{N}=(2,2)$ SQCD or $\mathcal{N}=(2,2)^*$ SQCD depending on η_k. In both cases, Higgs branch vacua are labelled by sets of N_k "flavours" among

$$L_k=\big\{(l,j)\mid l=k\pm1,1\leq j\leq N_l\big\}\,,\tag{4.4.4}$$

and the eigenvalues of $v^\pm$ for a given N_k-element subset $E\subset L_k$ are

$$v^\pm_{(l,j)}=-m_{kl}+\sigma^\pm_{lj}\quad\text{for } (l,j)\in E\,.\tag{4.4.5}$$

The vortex partition functions in (4.4.3) are those of the relevant $\mathcal{N}=(2,2)$ SQCD or $\mathcal{N}=(2,2)^*$ SQCD theory with $N_{k+1}+N_{k-1}$ fundamental multiplets of twisted masses $\{m_{kl}-\sigma^+_{lj}\}$ and the same number of antifundamental multiplets of twisted masses $\{m_{lk}+\sigma^+_{lj}\}$, in the Higgs branch vacuum v^+.

The antivortex partition functions are obtained by replacing σ^+_{lj} by σ^-_{lj} and v^+ by v^-. The one-loop determinant for the vacuum labelled by $E \subset L_k$ is

$$Z_{1l,E} = \prod_{(l,j)\in E} \prod_{(l',j')\in L_k} \left[\frac{\Gamma(-\mathrm{i}m_{kl'} - \delta_{l'j'\in E}\mathrm{i}m_{kk} + \mathrm{i}m_{kl} + \mathrm{i}\sigma^+_{l'j'} - \mathrm{i}\sigma^+_{lj})}{\Gamma(1 + \mathrm{i}m_{kl'} + \delta_{l'j'\in E}\mathrm{i}m_{kk} - \mathrm{i}m_{kl} - \mathrm{i}\sigma^-_{l'j'} + \mathrm{i}\sigma^-_{lj})} \cdot \frac{\Gamma(-\mathrm{i}m_{l'k} - \mathrm{i}m_{kl} - \mathrm{i}\sigma^+_{l'j'} + \mathrm{i}\sigma^+_{lj})}{\Gamma(1 + \mathrm{i}m_{l'k} + \mathrm{i}m_{kl} + \mathrm{i}\sigma^-_{l'j'} - \mathrm{i}\sigma^-_{lj})} \right]. \tag{4.4.6}$$

We now need to distinguish $\eta_k = \pm 1$ because explicit expressions differ. We will bring the results together at the end of this section.

Focus first on the case $\eta_k = +1$. Since $1 + \mathrm{i}m_{kk} + \mathrm{i}m_{k(k\pm1)} + \mathrm{i}m_{(k\pm1)k} = 0$, the factors with $(l, j) \in E$ and $(l', j') \in E$ in (4.4.6) cancel. The remaining factors are invariant under the exchanges $E \to E^{\complement}$ and $m_{kl} - \sigma^\pm_{lj} \leftrightarrow m_{lk} + \sigma^\pm_{lj}$. As a result, the one-loop determinant for the s-channel vacuum E of the $U(N_k)$ theory is equal to the one-loop determinant for the u-channel vacuum $E^{\complement}$ of a theory with identical twisted masses but $N^D_k = \#E^{\complement} = N_{k-1} + N_{k+1} - N_k$ colors. As discussed in Section 4.3.1 and shown directly in Appendix 4.B, the vortex partition functions of the $U(N_k)$ theory in the s-channel vacuum E and of the $U(N^D_k)$ theory in the u-channel vacuum $E^{\complement}$ are equal up to a factor (4.B.30)

$$Z^{U(N_k)}_{\mathrm{v},E}(\hat z_k) = (1 - \hat z_k)^{-\delta_1} Z^{U(N^D_k)}_{\mathrm{v},E^{\complement}}((\hat z^D_k)^{-1}) \tag{4.4.7}$$

with $\delta_1 = (N_k - N^D_k)(1 + \mathrm{i}m_{kk})$, provided $\hat z^D_k = \hat z^{-1}_k$ as expected from the Toda CFT symmetry. Finally, the classical contribution transforms as follows:

$$\prod_{(l,j)\in E} \hat z_k^{-\mathrm{i}m_{kl}+\mathrm{i}\sigma^+_{lj}} = \hat z_k^{-\delta_0 + \operatorname{Tr}\mathrm{i}\sigma^+_{k-1} + \operatorname{Tr}\mathrm{i}\sigma^+_{k+1}} \prod_{(l,j)\in E^{\complement}} (\hat z^D_k)^{\mathrm{i}m_{lk}+\mathrm{i}\sigma^+_{lj}} \tag{4.4.8}$$

with $\delta_0 = N_{k-1}\mathrm{i}m_{k(k-1)} + N_{k+1}\mathrm{i}m_{k(k+1)} + (1 + \mathrm{i}m_{kk})N^D_k$. All in all,

$$Z^{U(N_k)}_k(z_k, \bar z_k) = |\hat z_k|^{-2\delta_0} |1 - \hat z_k|^{-2\delta_1} \hat z_k^{\operatorname{Tr}\mathrm{i}\sigma^+_{k-1} + \operatorname{Tr}\mathrm{i}\sigma^+_{k+1}} \bar{\hat z}_k^{\operatorname{Tr}\mathrm{i}\sigma^-_{k-1} + \operatorname{Tr}\mathrm{i}\sigma^-_{k+1}} Z^{U(N^D_k)}_k(z^D_k, \bar z^D_k)\,. \tag{4.4.9}$$

In the full S^2 partition function of the quiver theory, the powers of $\hat z_k$ and $\bar{\hat z}_k$ combine with the classical contribution for the gauge group factors $U(N_{k\pm1})$ and yield

$$|\hat z_k|^{-2\delta_0} |1 - \hat z_k|^{-2\delta_1} \prod_{l=k\pm1} (\hat z_l \hat z_k)^{\operatorname{Tr}\mathrm{i}\sigma^+_l} \left(\overline{\hat z_l \hat z_k}\right)^{\operatorname{Tr}\mathrm{i}\sigma^-_l}\,. \tag{4.4.10}$$

Therefore, the S^2 partition functions of the $U(N_1) \times \cdots \times U(N_n)$ theory and of the theory with $N^D_k = N_{k-1} + N_{k+1} - N_k$, $\hat z^D_k = \hat z^{-1}_k$, and $\hat z^D_{k\pm1} = \hat z_{k\pm1}\hat z_k$ are equal up to $|\hat z_k|^{-2\delta_0}|1 - \hat z_k|^{-2\delta_1}$. On the Toda CFT side, this factor is due to differences in powers of $|x_k|^2$, $|x_{k+1}|^2$ and $|x_{k+1}-x_k|^2$ which appear in the correspondences for the electric and magnetic theories. In gauge theory, the factor can be absorbed into the partition function: since δ_1 only depends on b, the N_j, and the matter content, $|1-\hat{z}_k|^{-2\delta_1}$ is a renormalization scheme ambiguity, while $|\hat{z}_k|^{-2\delta_0}$ can be absorbed by a global $U(1) \subset U(N_k)$ gauge transformation. Such ambiguities are described below (3.2.10).

The case $\eta_k = -1$ follows the same ideas, but is more intricate. The Higgs branch decomposition (4.4.6) involves vortex partition functions of $\mathcal{N} = (2,2)$ SQCD. As for the previous case, those are equal up to a power of $(1-\hat{z}_k)$ to vortex partition functions of a dual theory with N_k^D colors and twisted masses $m^D = \mathrm{i}/2 - m$. Explicitly,

$$\begin{aligned} &Z_{\mathrm{v},E}(\{m_{kl} - \sigma^+_{lj}\}, \{m_{lk} + \sigma^+_{lj}\}; \hat{z}_k) \\ &\quad = (1-\hat{z}_k)^{-N_k - 2q_k N_{k-1} - 2q_{k+1}N_{k+1}} \\ &\qquad Z_{\mathrm{v},E^{\complement}}(\{\mathrm{i}/2 - m_{kl} + \sigma^+_{lj}\}, \{\mathrm{i}/2 - m_{lk} - \sigma^+_{lj}\}; \hat{z}_k)\,. \end{aligned} \tag{4.4.11}$$

The signs with which σ^+_{lj} appears in the right-hand side are inconvenient, as it implies that chiral multiplets which transform under the fundamental representation of $U(N_k^D)$ also transform in the fundamental representation of $U(N_l^D)$, and not the antifundamental representation. This is fixed by conjugating all $U(N_k^D)$ charges: $\hat{z}_k \to \hat{z}_k^{-1}$ and the vortex partition function becomes a u-channel ($|\hat{z}_k^{-1}| \to \infty$) vortex partition function of SQCD with N_k^D colors, $N_{k-1} + N_{k+1}$ flavours, and $\hat{z}_k^D = \hat{z}_k^{-1}$. Once this is understood, the classical contributions (of the electric s-channel vacuum labelled by E and the magnetic u-channel vacuum labelled by $E^{\complement}$) are equal up to powers of $|\hat{z}_k|^2$, provided $\hat{z}_k^D = \hat{z}_k^{-1}$ and $\hat{z}_{k\pm1}^D = \hat{z}_{k\pm1}\hat{z}_k$. This is precisely the map described by the exchange of Toda CFT punctures.

The one-loop determinants, on the other hand, transform non-trivially. This is expected from the study of Seiberg duality for $\mathcal{N} = (2,2)$ SQCD: the magnetic theory includes mesons whose one-loop determinants appear in the S^2 partition function. There, the mesons are realized as $M_{ts} = \widetilde{q}_t q_s$ in terms of the electric quarks and antiquarks q_s and $\widetilde{q}_t$, and couple to the magnetic multiplets through a superpotential term $\widetilde{q}_t^D M_{ts} q_s^D$. In our current setting, the mesons are the four combinations $M_{ll'} = \phi_{lk}\phi_{kl'}$ in the electric theory, and couple to the magnetic multiplets through the superpotential $\mathrm{Tr}(M_{ll'}\phi^D_{l'k}\phi^D_{kl})$. The mesons $M_{(k\pm1)(k\pm1)}$ transform in the adjoint representations of $U(N_{k\pm1})$, and the mesons $M_{(k\pm1)(k\mp1)}$ in bifundamental representations of $U(N_{k+1}) \times U(N_{k-1})$. Since the (electric) superpotential features the term $\mathrm{Tr}(M_{(k-1)(k+1)}M_{(k+1)(k-1)})$ for $\eta_k = -1$, these two mesons can be integrated out, leaving the term $\mathrm{Tr}(\phi^D_{(k-1)k}\phi^D_{k(k+1)}\phi^D_{(k+1)k}\phi^D_{k(k-1)})$ in the superpotential of the magnetic theory. This term is expected from $\eta_k^D = -1$.

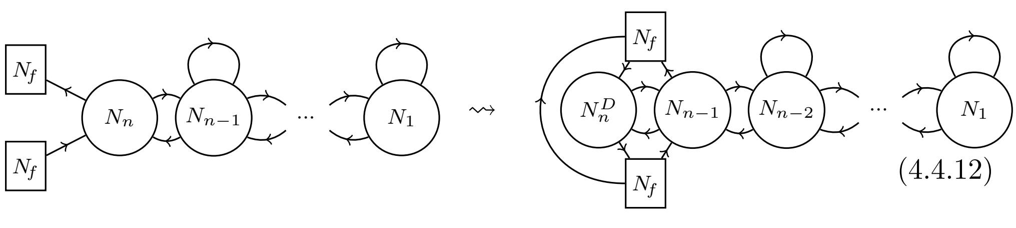
Next, for each of $l = k \pm 1$ there are two cases. If $\eta_{k\pm1} = +1$ then the superpotential term $\mathrm{Tr}(X_{k\pm1}M_{(k\pm1)(k\pm1)})$ lets us integrate out both $X_{k\pm1}$

and the meson $M_{(k\pm1)(k\pm1)}$, leaving $\operatorname{Tr}(\phi_{(k\pm1)(k\pm2)}\phi_{(k\pm2)(k\pm1)}\phi^D_{(k\pm1)k}\phi^D_{k(k\pm1)})$ as a superpotential term in the magnetic theory. This is expected from $\eta^D_{k\pm1} = -1$ (multiplets $\phi_{ll'}$ with $l, l' \neq k$ are not affected by the duality). If instead $\eta_{k\pm1} = -1$, then we integrate out $X_{k\pm1}$, and note the presence of magnetic superpotential terms $\operatorname{Tr}(\phi_{(k\pm1)(k\pm2)}\phi_{(k\pm2)(k\pm1)}M_{(k\pm1)(k\pm1)})$ and $\operatorname{Tr}(M_{(k\pm1)(k\pm1)}\phi^D_{(k\pm1)k}\phi^D_{k(k\pm1)})$. Those are expected from $\eta^D_{k\pm1} = +1$. In both cases, the change in matter content between the electric and magnetic theories and the mapping of twisted masses are encoded in the map $\eta^D_{k\pm1} = \eta_{k\pm1}\eta_k$ implied by the exchange $\epsilon_{k-1} \leftrightarrow \epsilon_k$.

Combining the classical, one-loop, and vortex contributions yields the equality of S^2 partition functions up to powers of $|\hat{z}_k|^2$ and $|1-\hat{z}_k|^2$ when $\eta_k = -1$. As for $\eta_k = +1$, the powers of $|1-\hat{z}_k|^2$ and $|\hat{z}_k|^2$ are an ambiguity of the S^2 partition function. This concludes the proof (for arbitrary η) that applying Seiberg duality or the $\mathcal{N} = (2,2)^*$ duality to the gauge group factor $U(N_k)$ with $1 \leq k < n$ corresponds to transposing the punctures k and $k+1$ in the Toda CFT correlator. Therefore, permutations of Toda CFT degenerate punctures encapsulate the mapping of parameters for arbitrary combinations of dualities which act on the nodes with $1 \leq k < n$.

4.4.2 Seiberg dualities from momentum conjugation

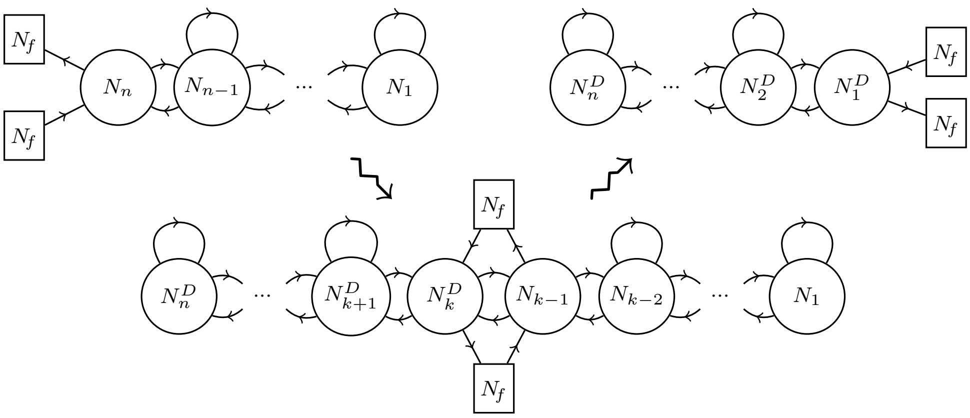
We now find all theories obtained through dualities acting on any node, including $U(N_n)$. For simplicity, we first consider the theory with $\eta_n = -1$ and $\eta_k = +1$ for $k < n$, which corresponds to a Toda CFT correlator where all degenerate punctures are labelled by antisymmetric representations of A_{N_f-1} (all $\epsilon_k = -1$). Since $\eta_n = -1$, the superpotential includes a term $\operatorname{Tr} X_n^2$ which lets us integrate out the adjoint chiral multiplet X_n. Therefore, the chiral multiplets which couple to the $U(N_n)$ vector multiplet are those of $\mathcal{N} = (2,2)$ SQCD with N_n colors and $N_f + N_{n-1}$ flavours. Applying Seiberg duality and charge conjugation to the node $U(N_n)$ yields a similar quiver gauge theory with N_n replaced by $N_n^D = N_f + N_{n-1} - N_n$. Recall that the Seiberg dual of a theory includes additional multiplets with charges identical to mesons of the original theory. Here, these are N_f^2 free chiral multiplets, and N_f fundamental, N_f antifundamental, and one adjoint of $U(N_{n-1})$. The magnetic theory thus has two adjoints of $U(N_{n-1})$. Given the cubic superpotential which links the bifundamentals of $U(N_n)\times U(N_{n-1})$ and the adjoint of $U(N_{n-1})$ in the electric theory, the two adjoints of $U(N_{n-1})$ couple through a quadratic superpotential term and can thus be integrated out. Therefore, the two dual theories are given by the quivers



(4.4.12)

Figure 4.1: Sequence of Seiberg dualities on the quiver with all $\epsilon_k = -1$.

The quiver with $\eta_n = -1$ and $\eta_k = +1$ for $k < n$ corresponds to a Toda CFT correlator with only antisymmetric degenerate operators. Acting with Seiberg dualities successively on all nodes from $U(N_n)$ to $U(N_1)$ yields a quiver of the same form, which corresponds to the Toda CFT correlator with all momenta conjugated. The original quiver, the quiver after acting on the k-th node, and the final quiver are drawn here without free mesons transforming in the bifundamental representation of the flavour group $S[U(N_f) \times U(N_f)]$ to avoid clutter. After acting on the k-th node, the complexified FI parameters are given by $(\hat{z}_{n-1}, \dots, \hat{z}_k, (\hat{z}_n \hat{z}_{n-1} \cdots \hat{z}_k)^{-1}, (\hat{z}_n \cdots \hat{z}_{k-1}), \hat{z}_{k-2}, \dots, \hat{z}_1)$ in this order. Dual ranks are $N_j^D = (n + 1 - j)N_f + N_{j-1} - N_n$.



where the superpotential is the sum of all gauge (and flavour) invariant cubic combinations of bifundamental and adjoint chiral multiplets. The complexified FI parameters of the magnetic theory are $\hat{z}_n^D = \hat{z}_n^{-1}$, $\hat{z}_{n-1}^D = \hat{z}_n \hat{z}_{n-1}$, and $\hat{z}_k^D = \hat{z}_k$ for $k \leq n-2$.

The absence of adjoint chiral multiplet of $U(N_{n-1})$ in the second theory lets us apply Seiberg duality (and charge conjugation) to this node of the second quiver. Once more, the resulting quiver contains additional matter, including an adjoint of $U(N_{n-2})$ which cancels the already present adjoint because of a quadratic superpotential. The procedure can thus be continued by acting on successive nodes from $U(N_n)$ to $U(N_1)$. The resulting quivers are given in Figure 4.1.

We note in particular that the last quiver, obtained after applying Seiberg duality to all the nodes, has the same form as the original quiver: only one gauge group factor features fundamental and antifundamental chiral multiplets. This quiver gauge theory, or rather the $\mathcal{N} = (2,2)$ surface operator it defines in any class S theory, has a Toda CFT interpretation as the insertion of some degenerate vertex operators. Given the matter content of the gauge theory, all n degenerate vertex operators are labelled by antisymmetric representations of A_{N_f-1}. The ranks of these representations are obtained from the number of colors in the dual theory:

$$K_j^D = N_j^D - N_{j+1}^D = N_f - (N_j - N_{j-1}) = N_f - K_j \tag{4.4.13}$$

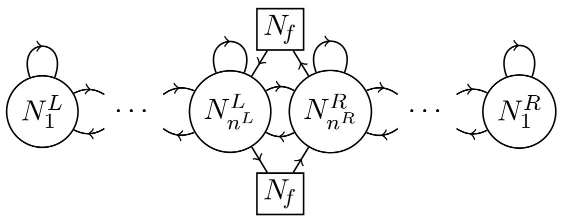
for $1 \leq j \leq n$, where $N_{n+1}^D = N_0 = 0$. The positions of punctures are obtained from the FI parameters:

$$x_j^D = \prod_{i=1}^{j} \hat{z}_i^D = \left[\prod_{i=j}^{n} \hat{z}_i\right]^{-1} = x_j^{-1}\,. \tag{4.4.14}$$

Both of these maps are reproduced by conjugating all Toda CFT momenta and applying the conformal transformation $x \to x^{-1}$ to the correlator. This conformal transformation could be avoided by applying charge conjugation to all nodes of the quiver, mapping all complexified FI parameters to their inverse in the process.

All in all, Toda CFT conjugation translates to a combination of Seiberg dualities and charge conjugations. Here, the precise choice of matter content of the gauge theory is essential. On the gauge theory side, it ensures the absence of adjoint chiral multiplet at each step hence allows Seiberg duality to be applied. On the Toda CFT side, the conjugate of a symmetric representation is neither symmetric nor antisymmetric, thus momentum conjugation only yields symmetric or antisymmetric representations if the original representations were all antisymmetric. It should be noted that this choice of signs is identical to that made in Section 3.6.2 to fuse degenerate punctures into a degenerate puncture labelled by an arbitrary representation, hence conjugating this representation corresponds to a set of Seiberg dualities on the gauge theory quiver.

We now go back to a quiver given by arbitrary signs η_k, and determine all dual descriptions obtained through Seiberg and $\mathcal{N} = (2,2)^*$ dualities. Inspired by the quivers which appeared when all $\epsilon_k = -1$, we consider the more general class of quivers



(4.4.15)

The multiplets described by this quiver are subject to a superpotential which depends on some signs η_k^L for $1 \leq k \leq n^L$ and η_k^R for $1 \leq k \leq n^R$. Namely, the superpotential is a sum of W_{η^L} defined as in (3.6.4) for fields charged under the $U(N_k^L)$, W_{η^R} for fields charged under the $U(N_k^R)$, and two cubic terms coupling each bifundamental of $U(N^L_{n^L})\times U(N^R_{n^R})$ to multiplets charged under the flavour groups. For $n^L=0$ or $n^R=0$ we retrieve the quivers studied throughout this paper. Whenever $\eta^L_k=-1$, the superpotential contains a quadratic term $\operatorname{Tr}\bigl((X^L_k)^2\bigr)$ which lets us integrate out the adjoint chiral multiplet X^L_k of $U(N^L_k)$, and similarly $\eta^R_k=-1$ lets us integrate out X^R_k.

Even though we have not given a Toda CFT interpretation for this class of quivers, we find analogues of the Toda CFT parameters (x_k,ϵ_k,K_k) which are simply transposed under dualities. Let $x^L_{n^L+1}=x^R_{n^R+1}=\epsilon^L_{n^L+1}=\epsilon^R_{n^R+1}=1$ and

$$x^L_j=\prod_{i=j}^{n^L}\hat{z}^L_i\,,\quad \epsilon^L_j=\prod_{i=j}^{n^L}\eta^L_i\,,\quad K^L_j=N^L_j-N^L_{j-1}\quad \text{for } 1\leq j\leq n^L\,,\tag{4.4.16}$$

$$x^R_j=\prod_{i=j}^{n^R}\hat{z}^R_i\,,\quad \epsilon^R_j=\prod_{i=j}^{n^R}\eta^R_i\,,\quad K^R_j=N^R_j-N^R_{j-1}\quad \text{for } 1\leq j\leq n^R\,,\tag{4.4.17}$$

where $N^L_0=N^R_0=0$.

Acting with Seiberg or the $\mathcal{N}=(2,2)^*$ duality (depending on η^L_k) on a node $U(N^L_k)$ with $k<n^L$ exchanges $(x^L_k,\epsilon^L_k,K^L_k)\leftrightarrow(x^L_{k+1},\epsilon^L_{k+1},K^L_{k+1})$. This is proven through the same calculations as for the case $n^R=0$ treated in Section 4.4.1. Similarly, acting with a duality on $U(N^R_k)$ with $k<n^R$ exchanges $(x^R_k,\epsilon^R_k,K^R_k)\leftrightarrow(x^R_{k+1},\epsilon^R_{k+1},K^R_{k+1})$.

Let us now understand how dualities act on $U(N^R_{n^R})$. If $\epsilon^R_{n^R}=\eta^R_{n^R}=+1$, the fields which couple to the gauge group factor $U(N^R_{n^R})$ are those of $\mathcal{N}=(2,2)$ SQCDA, no simplification occurs, and neither Seiberg nor the $\mathcal{N}=(2,2)^*$ duality applies. However, if $\epsilon^R_{n^R}=\eta^R_{n^R}=-1$, we can integrate out the adjoint chiral multiplet to obtain SQCD with $N^R_{n^R}$ colors and $N_f+N^L_{n^L}+N^R_{n^R-1}$ flavours, and Seiberg duality yields a theory with $(N^R_{n^R})^D=N_f+N^L_{n^L}+N^R_{n^R-1}-N^R_{n^R}$ colors. The magnetic theory has the same form (4.4.15) as the electric theory, but it features fundamental and antifundamental chiral multiplets of $U\bigl((N^R_{n^R})^D\bigr)$ and $U(N^R_{n^R-1})$ rather than $U(N^L_{n^L})$ and $U(N^R_{n^R})$: in other words, $n^L\to n^L+1$ and $n^R\to n^R-1$. Due to the additional mesons after Seiberg duality, both $\eta^L_{n^L}$ and $\eta^R_{n^R-1}$ change signs, thus toggling between the presence or absence of an adjoint chiral multiplet. From our previous work on the action of Seiberg duality on quivers, we also know that FI parameters map as $\hat{z}^L_{n^L}\to\hat{z}^L_{n^L}\hat{z}^R_{n^R}$, $\hat{z}^R_{n^R}\to(\hat{z}^R_{n^R})^{-1}$ and $\hat{z}^R_{n^R-1}\to\hat{z}^R_{n^R-1}\hat{z}^R_{n^R}$. Translating to the parameters (x,ϵ,K), we find that the set

$$\bigl\{\bigl((x^L_j)^{-1},\epsilon^L_j,N_f-K^L_j\bigr)\bigm| 1\leq j\leq n^L\bigr\}\cup\bigl\{(x^R_j,\epsilon^R_j,K^R_j)\bigm| 1\leq j\leq n^R\bigr\}\tag{4.4.18}$$

is unchanged: the triplet $(x^R_{n^R},\epsilon^R_{n^R},K^R_{n^R})$ is simply moved from the second part of the set (on the right of flavour nodes) to the first part (on the left). By symmetry, the discussion applies to the node $U(N^L_{n^L})$: if $\eta^L_{n^L} = +1$ there is no duality, while if $\eta^L_{n^L} = -1$ Seiberg duality moves $\big((x^L_{n^L})^{-1}, \epsilon^L_{n^L}, N_f - K^L_{n^L}\big)$ from the left part to the right part of (4.4.18).

All in all, Seiberg and $\mathcal{N} = (2,2)^*$ dualities acting on any of the nodes of (4.4.15) correspond to transpositions of $\big((x^L_1)^{-1}, \epsilon^L_1, N_f - K^L_1\big)$, $\dots$, $\big((x^L_{n^L})^{-1}, \epsilon^L_{n^L}, N_f - K^L_{n^L}\big)$, $\diamond$, $(x^R_{n^R}, \epsilon^R_{n^R}, K^R_{n^R})$, $\dots$, $(x^R_1, \epsilon^R_1, K^R_1)$. The position of $\diamond$ indicates the position of the flavour nodes in the quiver. Only triplets (x, ϵ, K) with $\epsilon = -1$ can be exchanged with $\diamond$. Therefore, combinations of dualities correspond to all permutations which leave triplets with $\epsilon^L_j = +1$ to the left of $\diamond$ and those with $\epsilon^R_j = +1$ to the right of $\diamond$. Denoting by n^L_+ and n^R_+ the number of such triplets, and by n_- the total number of triplets with $\epsilon = -1$, we conclude that the number of dual descriptions of the theory (4.4.15) is

$$\sum_{k=0}^{n_-} \binom{n_-}{k} (n^L_+ + k)!(n^R_+ + n_- - k)! = \frac{n^L_+! n^R_+! (n^L_+ + n^R_+ + n_- + 1)!}{(n^L_+ + n^R_+ + 1)!} . \quad (4.4.19)$$

As a last comment, we propose that the partition function of the S^2 surface operator defined by coupling (4.4.15) to N_f^2 free hypermultiplets on S^4_b should be equal to

$$Z^{(4.4.15)}_{S^2 \subset S^4_b} = \left\langle \widehat{V}_{\alpha_\infty}(\infty) \widehat{V}_{\hat{m}}(1) \widehat{V}_{\alpha_0}(0) \prod_{j=1}^{n^L} \widehat{V}_{-b\Omega(K^L_j, \epsilon^L_j)^C}\big((x^L_j)^{-1}\big) \prod_{j=1}^{n^R} \widehat{V}_{-b\Omega(K^R_j, \epsilon^R_j)}(x^R_j) \right\rangle \quad (4.4.20)$$

up to factors that can be absorbed in Z. Here, $\Omega(K, +1) = K h_1$ is the highest weight of a symmetric representation while $\Omega(K, -1) = \omega_K$ is the highest weight of an antisymmetric representation. The proposal is consistent with the action of dualities as permutations of (x, ϵ, K) triplets described above: in particular, an antisymmetric representation with highest weight ω_K can be seen either as part of the left product ($\omega_K = \Omega(N_f - K, -1)^C$) or as part of the right product ($\omega_K = \Omega(K, -1)$), and this choice reproduces the Seiberg duality map. On the contrary, the conjugate of a symmetric representation is neither symmetric nor antisymmetric, so punctures with $\epsilon = +1$ belong to a given product and cannot be moved to the other one. We have not explored this proposal further, as fusions of antisymmetric representations are enough to obtain arbitrary representations.

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4.A SQCD vortex partition functions

In this appendix and the next, we prove that the vortex partition functions of some dual theories are equal up to simple factors. The equalities are most easily seen through the matching with Toda CFT correlators, as done in the main text. However, the matching is not proven in all cases, so we proceed to establish the equalities directly using integral representations of the vortex partition functions. We cover the case of Seiberg duality for $\mathcal{N} = (2,2)$ SQCD in this appendix. We then add adjoint matter and a superpotential in Appendix 4.B: this includes as special cases the Seiberg duality for $\mathcal{N} = (2,2)^*$ SQCD, and the Kutasov–Schwimmer duality. The two appendices use similar ideas but are independent.

We focus first on the S^2 partition function of an $\mathcal{N} = (2,2)$ theory of a $U(N)$ vector multiplet coupled to N_f fundamental and $\widetilde{N_f}$ antifundamental chiral multiplets. Its expression can be decomposed as (3.4.8) into vortex partition functions [BC12; DGLFL12]. By symmetry we can assume that $\widetilde{N_f} < N_f$, or that $\widetilde{N_f} = N_f$ and $|z| < 1$. The relevant vortex partition functions are then labelled by N-element subsets of $[\![1, N_f]\!]$ and take the form

$$Z_{\mathrm{v},\{p\}}(m, \widetilde{m}, z) = \sum_{k=0}^{\infty} \left[(-1)^{N_f} z\right]^k Z_{k,\{p\}}(m, \widetilde{m}) , \tag{4.A.1}$$

where the k-vortex partition function is

$$Z_{k,\{p\}}(m, \widetilde{m}) = \sum_{k_1+\cdots+k_N=k} \prod_{j=1}^{N} \left[\frac{(1/k_j!) \prod_{s=1}^{\widetilde{N_f}} (-\mathrm{i}\widetilde{m}_s - \mathrm{i}m_{p_j})_{k_j}}{\prod_{i\neq j}^{N} (\mathrm{i}m_{p_i} - \mathrm{i}m_{p_j} - k_i)_{k_j} \prod_{s\notin\{p\}}^{N_f} (1 + \mathrm{i}m_s - \mathrm{i}m_{p_j})_{k_j}} \right] . \tag{4.A.2}$$

We prove that the vortex partition function is invariant under the Seiberg duality map $N^D = N_f - N$, $\{p\}^D \to \{p\}^\complement$ (the set complement), $m_s^D = \frac{\mathrm{i}}{2} - m_s$, $\widetilde{m}_s^D = \frac{\mathrm{i}}{2} - \widetilde{m}_s$, $z^D = (-1)^{\widetilde{N_f}+N_f} z$, up to a simple overall factor. This is based on the proof [BC12] that, for $\widetilde{N_f} \leq N_f - 2$, the k-vortex partition function is invariant. Since $Z_{k,\{p\}}$ depends analytically on the m_s and $\widetilde{m}_s$, we only need to prove the equality when R-charges $\mathrm{Re}(-2\mathrm{i}m_s)$ and $\mathrm{Re}(-2\mathrm{i}\widetilde{m}_s)$ are between 0 and 1; the same is then true of the R-charges in the dual theory.

Consider a closed contour C_k^+ which lies in the half-plane $\mathrm{Re}(\varphi) > -\frac{1}{2}$ and surrounds with a positive orientation all points $-\mathrm{i}m_s + \nu$ and $\frac{1}{2} + \mathrm{i}m_s + \nu$ for $1 \leq s \leq N_f$ and integer $0 \leq \nu < \kappa$. This set of points, which all have positive real part, is invariant under the duality map $-\mathrm{i}m_s^D = \frac{1}{2} + \mathrm{i}m_s$. The contour $C_k^- = -\frac{1}{2} - C_k^+$ lies in the half-plane $\mathrm{Re}(\varphi) < 0$ and surrounds with a positive orientation all points $-1 - \mathrm{i}m_s - \nu$ and $-\frac{1}{2} + \mathrm{i}\widetilde{m}_s - \nu$ for $1 \leq s \leq N_f$ and integer $0 \leq \nu < \kappa$. Define the contour integrals

$$I^{\pm}_{k,\{p\}}(m,\widetilde{m}) = \frac{1}{k!}\int_{(C^{\pm}_k)^k} \frac{\mathrm{d}^k\varphi}{(2\pi\mathrm{i})^k} \prod_{\kappa\neq\lambda}^{k} \frac{\varphi_\kappa - \varphi_\lambda}{\varphi_\kappa - \varphi_\lambda - 1} \prod_{\kappa=1}^{k} \frac{\prod_{s=1}^{\widetilde{N}_f}(\varphi_\kappa - \mathrm{i}\widetilde{m}_s)}{\prod_{s=1}^{N_f}(\varphi_\kappa + \mathrm{i}m_s + \delta_{s\notin\{p\}})}\,. \tag{4.A.3}$$

As we will see shortly, $I^{\pm}$ are essentially k-vortex partition functions of Seiberg dual theories (4.A.7). Given our choice of contours, the change of variables $\varphi \to \varphi^D = -\frac{1}{2} - \varphi$ maps $C^{\pm}_k \to C^{\mp}_k$, and we find

$$I^{\pm}_{k,\{p\}^{\complement}}(m^D,\widetilde{m}^D) = (-1)^{(1+\widetilde{N}_f+N_f)k} I^{\mp}_{k,\{p\}}(m,\widetilde{m})\,, \tag{4.A.4}$$

where the sign comes from $\mathrm{d}\varphi^D = -\mathrm{d}\varphi$, $\varphi^D - \mathrm{i}\widetilde{m}^D = -(\varphi - \mathrm{i}\widetilde{m})$, and $\varphi^D + \mathrm{i}m^D_s + \delta_{s\in\{p\}} = -(\varphi + \mathrm{i}m_s + \delta_{s\notin\{p\}})$.

Poles of the integrand for which all $\mathrm{Re}(\varphi_\kappa) > -\frac{1}{2}$ are labelled by choices of N integers $k_s \geq 0$ with $\sum_{s\in\{p\}} k_s = k$, such that the φ_κ are given in some order by

$$\{\varphi_\kappa\} = \{-\mathrm{i}m_s + \nu \mid s \in \{p\}, 0 \leq \nu < k_s\}\,, \tag{4.A.5}$$

hence $(C^{+}_k)^k$ surrounds precisely those poles. Similarly, poles with $\mathrm{Re}(\varphi_\kappa) < 0$ are

$$\{\varphi_\kappa\} = \{-1 - \mathrm{i}m_s - \nu \mid s \notin \{p\}, 0 \leq \nu < k_s\}\,, \tag{4.A.6}$$

labelled by $N_f - N$ integers $k_s \geq 0$ for $s \notin \{p\}$, summing to k, and $(C^{-}_k)^k$ surrounds precisely those poles. For a given choice of $k_1 + \cdots + k_N = k$, the residue at each of the $k!$ points $\{\varphi_\kappa\} = \{-\mathrm{i}m_{p_j} + \nu \mid 1 \leq j \leq N, 0 \leq \nu < k_j\}$ reproduces the corresponding term in the k-vortex partition function (the factor $1/k!$ cancels the choice of ordering of φ_κ), hence the k-vortex partition functions are

$$\begin{aligned} Z_{k,\{p\}}(m,\widetilde{m}) &= I^{+}_{k,\{p\}}(m,\widetilde{m}) \\ Z_{k,\{p\}^{\complement}}(m^D,\widetilde{m}^D) &= (-1)^{(1+\widetilde{N}_f+N_f)k} I^{-}_{k,\{p\}}(m,\widetilde{m})\,, \end{aligned} \tag{4.A.7}$$

where the dual relation derives from (4.A.4) or from summing residues at poles surrounded by $(C^{-}_k)^k$.

4.A.1 SQCD with $\widetilde{N}_f < N_f$

As long as $\widetilde{N}_f \leq N_f - 2$, the integrand in (4.A.3) is regular at infinity, hence we can choose C^+ along $-\frac{1}{4} + \mathrm{i}\mathbb{R}$, from $\mathrm{i}\infty$ to $-\mathrm{i}\infty$: then $C^- = -\frac{1}{2} - C^+$ has the opposite orientation, and $I^{-}_{k,\{p\}}(m,\widetilde{m}) = (-1)^k I^{+}_{k,\{p\}}(m,\widetilde{m})$. Therefore

$$Z_{k,\{p\}^{\complement}}(m^D,\widetilde{m}^D) = (-1)^{(\widetilde{N}_f+N_f)k} Z_{k,\{p\}}(m,\widetilde{m}) \tag{4.A.8}$$

hence vortex partition functions are equal:

$$Z_{\mathrm{v},\{p\}^{\complement}}(m^D,\widetilde{m}^D,z^D) = Z_{\mathrm{v},\{p\}}(m,\widetilde{m},z)\,, \tag{4.A.9}$$

where $z^D = (-1)^{\widetilde{N_f}+N_f} z$. This result strongly relies on our ability to reverse contours, that is, on the absence of poles at infinity for $\widetilde{N_f} \leq N_f - 2$. For $\widetilde{N_f} = N_f - 1$ or $\widetilde{N_f} = N_f$, we must take into account the contribution from infinity.

Consider first the case $\widetilde{N_f} = N_f - 1$. We shift the pole at infinity to a finite position through the regulating factor $\mathrm{i}M/(\varphi_\kappa + \mathrm{i}M)$. This is equivalent to adding a fundamental chiral multiplet with a twisted mass M, which we demand to lie in the strip $0 < \mathrm{Re}(-2\mathrm{i}M) < 1$. In the limit $|M| \to \infty$, the contours $(C_k^{\pm})^k$ only surround poles of the original integral, which are independent of M, and the regulator does not affect residues. Therefore,

$$I^{\pm}_{k,\{p\}}(m,\widetilde{m}) = \lim_{|M|\to\infty} \frac{1}{k!}\int_{(C_k^{\pm})^k} \frac{\mathrm{d}^k\varphi}{(2\pi\mathrm{i})^k}\Bigg\{\prod_{\kappa\neq\lambda}^{k} \frac{\varphi_\kappa-\varphi_\lambda}{\varphi_\kappa-\varphi_\lambda-1} \cdot \prod_{\kappa=1}^{k}\left[\frac{\prod_{s=1}^{N_f-1}(\varphi_\kappa - \mathrm{i}\widetilde{m}_s)}{\prod_{s=1}^{N_f}(\varphi_\kappa + \mathrm{i}m_s + \delta_{s\notin\{p\}})}\frac{\mathrm{i}M}{\varphi_\kappa+\mathrm{i}M}\right]\Bigg\}. \tag{4.A.10}$$

Poles of the integrand above with all $\mathrm{Re}(\varphi_\kappa) < 0$ are identical to those of the non-regulated integral, hence integrating along the contour $-\frac{1}{4} + \mathrm{i}\mathbb{R}$ yields $I^{-}_{k,\{p\}}(m,\widetilde{m})$ by closing the contour towards $-\infty$. Closing the contour instead towards $+\infty$ surrounds poles at

$$\{\varphi_\kappa\} = \{-\mathrm{i}m_s + \nu \mid s\in\{p\}, 0\leq\nu<k_s\} \cup \{-\mathrm{i}M + \nu \mid 0 \leq \nu < l\}\,, \tag{4.A.11}$$

for each choice of non-negative integers k_s for $s \in \{p\}$, and l, such that $l + \sum_{s\in\{p\}} k_s = k$. The residue at such a point is (factors of $\mathrm{i}M$ cancel out)

$$\frac{(-1)^l}{l!} \operatorname*{res}_{\{\varphi_\kappa|1\leq\kappa\leq k-l\}}\left[\prod_{\kappa\neq\lambda}^{k-l}\frac{\varphi_\kappa-\varphi_\lambda}{\varphi_\kappa-\varphi_\lambda-1}\prod_{\kappa=1}^{k-l}\frac{\prod_{s=1}^{N_f-1}(\varphi_\kappa-\mathrm{i}\widetilde{m}_s)}{\prod_{s=1}^{N_f}(\varphi_\kappa+\mathrm{i}m_s+\delta_{s\notin\{p\}})}\right], \tag{4.A.12}$$

where the residue is precisely one of the contributions to $I^{+}_{k-l,\{p\}}(m,\widetilde{m})$. The contributions for a fixed l combine into the full $(k-l)$-vortex partition function. All in all, using (4.A.7) $I^{-}_{k,\{p\}}(m,\widetilde{m}) = Z_{k,\{p\}^{\complement}}(m^D,\widetilde{m}^D)$ and $I^{+}_{k,\{p\}} = Z_{k,\{p\}}$ when $\widetilde{N_f} = N_f - 1$,

$$Z_{k,\{p\}^{\complement}}(m^D,\widetilde{m}^D) = (-1)^k\sum_{l=0}^{k}\frac{(-1)^l}{l!}Z_{k-l,\{p\}}(m,\widetilde{m}) \tag{4.A.13}$$

$$Z_{\{p\}^{\complement}}(m^D,\widetilde{m}^D,z^D) = e^{-z}Z_{\{p\}}(m,\widetilde{m},z)\,. \tag{4.A.14}$$

Alternatively, the factor e^z can be obtained from the case $N_f = \widetilde{N_f} + 2$ (where there is no factor) by decoupling one of the fundamental chiral multiplets through the limit $|m_{N_f}| \to \infty$. For an arbitrary $N_f > \widetilde{N_f}$,

$$Z^{N_f,\widetilde{N_f}}_{k,\{p\}} \sim \begin{cases} (\mathrm{i}m_{N_f})^{-k} \sum_{l=0}^{k} (-1)^l \binom{k}{l} (-\mathrm{i}m_{N_f})^{l(\widetilde{N_f}+2-N_f)} Z^{N_f-1,\widetilde{N_f}}_{k-l,\{p\}} & \text{if } N_f \in \{p\}, \\ (\mathrm{i}m_{N_f})^{-k} Z^{N_f-1,\widetilde{N_f}}_{k,\{p\}} & \text{if } N_f \notin \{p\}. \end{cases} \tag{4.A.15}$$

If $N_f \geq \widetilde{N_f} + 3$, terms other than $l = 0$ in the sum are of a lower order, thus $Z^{N_f,\widetilde{N_f}}_{k,\{p\}} \sim (\mathrm{i}m_{N_f})^{-k} Z^{N_f-1,\widetilde{N_f}}_{k,\{p\}}$, consistent with the equality (4.A.8) of Seiberg-dual vortex partition functions in those cases. If $N_f = \widetilde{N_f} + 2$, we find

$$Z^{\widetilde{N_f}+2,\widetilde{N_f}}_{\{p\}}(\mathrm{i}m_{N_f} x) \sim e^{-x\delta_{N_f\in\{p\}}} Z^{\widetilde{N_f}+1,\widetilde{N_f}}_{\{p\}}(x)\,. \tag{4.A.16}$$

Exactly one of two Seiberg-dual vortex partition functions exhibit this exponential factor, and with opposite signs since $\mathrm{i}m^D_{N_f} \sim -\mathrm{i}m_{N_f}$. Starting from the Seiberg duality relation (4.A.9) for $N_f \geq \widetilde{N_f} + 2$, we thus obtain the exponential factor in (4.A.14) for $N_f = \widetilde{N_f} + 1$. Unfortunately, the same technique fails to reach the case $N_f = \widetilde{N_f}$, because terms beyond (4.A.15) contribute to the limit $|m_{N_f}| \to \infty$ (with x/m_{N_f} kept constant). We avoid this issue in the contour integral approach by introducing different parameters for each occurence of m_{N_f}, as we now see.

4.A.2 SQCD with $\widetilde{N_f} = N_f$

When $\widetilde{N_f} = N_f$, we regulate using $\prod_{\kappa=1}^{k}\left[-(\mathrm{i}M_\kappa)^2 \,/\, (\varphi_\kappa^2 - (\mathrm{i}M_\kappa)^2)\right]$ with M_κ real for simplicity. This factor is similar to the contribution from two fundamental chiral multiplets with opposite twisted masses, but importantly we let the parameter M_κ depend on κ. In fact, we will consider the limit where masses have different scales, $1 \ll |M_1| \ll \cdots \ll |M_k|$, as this simplifies the expansion of residues. For large enough $|M_\kappa|$, the additional poles lie outside the contours $(C_k^{\pm})^k$, and the regulating factor tends to 1 when evaluated on the contour (or at poles it encloses), thus

$$I^{\pm}_{k,\{p\}}(m,\widetilde{m}) = \lim_{|M_\kappa|\to\infty} \frac{1}{k!} \int_{(C_k^{\pm})^k} \frac{\mathrm{d}^k\varphi}{(2\pi\mathrm{i})^k} \Bigg\{ \prod_{\kappa=1}^{k} \left[\frac{-(\mathrm{i}M_\kappa)^2}{\varphi_\kappa^2 - (\mathrm{i}M_\kappa)^2}\right] \cdot \prod_{\kappa\neq\lambda}^{k} \left[\frac{\varphi_\kappa - \varphi_\lambda}{\varphi_\kappa - \varphi_\lambda - 1}\right] \prod_{\kappa=1}^{k}\prod_{s=1}^{N_f} \left[\frac{\varphi_\kappa - \mathrm{i}\widetilde{m}_s}{\varphi_\kappa + \mathrm{i}m_s + \delta_{s\notin\{p\}}}\right] \Bigg\}. \tag{4.A.17}$$

Poles of the integrand above with all $\operatorname{Re}(\varphi_\kappa) \leq -\frac{1}{4}$ are identical to those of the non-regulated integral, hence integrating along the contour $-\frac{1}{4} + \mathrm{i}\mathbb{R}$ yields $Z^-_{k,\{p\}}(m, \widetilde{m})$ by closing the contour towards $-\infty$.

Closing the contour instead towards $+\infty$ surrounds numerous poles:

$$\{\varphi_\kappa\} = \{-\mathrm{i}m_s + \mu \mid s \in \{p\}, 0 \leq \mu < k_s\} \cup \{\epsilon_\kappa \mathrm{i}M_\kappa + \nu \mid \kappa \in K, 0 \leq \nu < l_\kappa\}\,, \tag{4.A.18}$$

where K is the set of $1 \leq \kappa \leq k$ such that $\varphi_\kappa = \epsilon_\kappa \mathrm{i}M_\kappa$ for some sign $\epsilon_\kappa = \pm 1$, and where the integers $k_s \geq 0$ for $s \in \{p\}$ and $l_\kappa > 0$ for $\kappa \in K$ sum to k. To specify a pole completely, one needs to know $\{K, \epsilon_\kappa, l_\kappa, k_s\}$, but also which component of φ is equal to each $-\mathrm{i}m_s + \mu$ and each $\epsilon_\kappa \mathrm{i}M_\kappa + \nu$. This is encoded in maps σ and τ such that

$$\varphi_{\sigma(s,\mu)} = -\mathrm{i}m_s + \mu \quad \text{and} \quad \varphi_{\tau(\kappa,\nu)} = \epsilon_\kappa \mathrm{i}M_\kappa + \nu\,. \tag{4.A.19}$$

Note that $\tau(\kappa, \nu) = \kappa$ if and only if $\nu = 0$.

We expand the residue at the pole defined by $\{K, \epsilon_\kappa, l_\kappa, k_s, \sigma, \tau\}$ in the limit $1 \ll |M_1| \ll \cdots \ll |M_k|$:

$$\begin{aligned}
&\frac{1}{k!} \prod_{\kappa\in K} \left[\frac{-\epsilon_\kappa \mathrm{i}M_\kappa}{2} \prod_{\nu=1}^{l_\kappa-1} \left[\frac{-(\mathrm{i}M_{\tau(\kappa,\nu)})^2}{(\epsilon_\kappa \mathrm{i}M_\kappa + \nu)^2 - (\mathrm{i}M_{\tau(\kappa,\nu)})^2}\right]\right] \prod_{s\in\{p\}} \prod_{\mu=0}^{k_s-1} \left[1 + O\left(\frac{1}{M^2_{\sigma(s,\mu)}}\right)\right] \\
&\cdot \prod_{\kappa\in K} \left[1 - \frac{l_\kappa \Sigma}{\epsilon_\kappa \mathrm{i}M_\kappa} + O\left(\frac{1}{M_\kappa^2}\right)\right] \prod_{\kappa\in K} \left[1 + O\left(\frac{1}{M_\kappa^2}\right)\right] \prod_{\kappa\in K} \left[\frac{1}{l_\kappa}\right] \\
&\cdot \operatorname*{res}_{\varphi_{\sigma(s,\mu)} = -\mathrm{i}m_s + \mu} \left\{ \prod_{\kappa\neq\lambda\in\{\sigma(s,\mu)\}} \left[\frac{\varphi_\kappa - \varphi_\lambda}{\varphi_\kappa - \varphi_\lambda - 1}\right] \prod_{\kappa\in\{\sigma(s,\mu)\}} \prod_{t=1}^{N_f} \left[\frac{\varphi_\kappa - \mathrm{i}\widetilde{m}_t}{\varphi_\kappa + \mathrm{i}m_t + \delta_{t\notin\{p\}}}\right] \right\}.
\end{aligned} \tag{4.A.20}$$

The first line consists of all factors coming from the regulator; the next factor comes from $(\varphi_{\tau(\dots)} - \mathrm{i}\widetilde{m}_s)/(\varphi_{\tau(\dots)} + \mathrm{i}m_s + \delta_{s\notin\{p\}})$, and involves

$$\Sigma = \sum_{s=1}^{N_f} (\mathrm{i}\widetilde{m}_s + \mathrm{i}m_s + \delta_{s\notin\{p\}})\,; \tag{4.A.21}$$

the following two factors come from the ratio $(\varphi - \varphi)/(\varphi - \varphi - 1)$ where either one or both components of φ take the form $\varphi_{\tau(\kappa,\mu)}$; the last line consists of all finite factors, independent of the M_κ, which organize themselves into a residue along the components $\varphi_{\sigma(s,\mu)}$. A useful simplification is

$$\frac{-(\mathrm{i}M_{\tau(\kappa,\nu)})^2}{(\epsilon_\kappa \mathrm{i}M_\kappa + \nu)^2 - (\mathrm{i}M_{\tau(\kappa,\nu)})^2} \sim \begin{cases} -M^2_{\tau(\kappa,\nu)} M_\kappa^{-2} & \text{if } \tau(\kappa,\nu) < \kappa, \\ 1 & \text{if } \tau(\kappa,\nu) > \kappa. \end{cases} \tag{4.A.22}$$

On its own, the residue (4.A.20) grows like $\prod_\kappa (-\epsilon_\kappa \mathrm{i}M_\kappa)$, but we will see that the sum over all possible choices of the signs ϵ_κ (keeping $\{K, l_\kappa, k_s, \sigma, \tau\}$

fixed) has a finite limit. More precisely, starting from $\lambda = k$, and all the way down to $\lambda = 1$, we sum over $\epsilon_\lambda = \pm 1$ (if $\lambda \in K$) and take the limit $|M_\lambda| \to \infty$. At each step there are three cases. If $\lambda = \sigma(s,\mu)$, the twisted mass appears only in a factor $1 + O(1/M_\lambda^2)$, which thus drops out. If $\lambda = \tau(\kappa,\nu) > \kappa$, then the factor (4.A.22) containing M_λ drops out. The case $\lambda = \tau(\kappa,\nu) < \kappa$ does not appear, as we see shortly. Finally, if $\lambda \in K$, several factors contain M_λ:

$$\frac{-\epsilon_\lambda \mathrm{i} M_\lambda}{2} \prod_{\substack{1\leq\nu<l_\lambda \\ \tau(\lambda,\nu)<\lambda}} \left[\frac{-(\mathrm{i}M_{\tau(\lambda,\nu)})^2}{(\epsilon_\lambda \mathrm{i}M_\lambda+\nu)^2-(\mathrm{i}M_{\tau(\lambda,\nu)})^2}\right]\left[1-\frac{l_\lambda\Sigma}{\epsilon_\lambda \mathrm{i}M_\lambda}+O\Big(\frac{1}{M_\lambda^2}\Big)\right]\left[1+O\Big(\frac{1}{M_\lambda^2}\Big)\right]. \tag{4.A.23}$$

This expression vanishes in the limit $|M_\lambda| \to \infty$ if any $\tau(\lambda,\nu) < \lambda$, thus only poles for which all $\tau(\lambda,\nu) \geq \lambda$ contribute in the limit we consider. Otherwise, the expression above is $\frac{1}{2}\big(-\epsilon_\lambda \mathrm{i}M_\lambda + l_\lambda\Sigma + O(1/M_\lambda)\big)$, whose sum over $\epsilon_\lambda = \pm 1$ is the finite result $l_\lambda\Sigma$. All in all, the sum over all choices of signs ϵ of the residue at the pole defined by $\{K, \epsilon_\kappa, l_\kappa, k_s, \sigma, \tau\}$ has a finite limit

$$\frac{1}{k!}\Sigma^{\#K} \operatorname*{res}_{\varphi_{\sigma(s,\mu)}=-\mathrm{i}m_s+\mu}\left\{\prod_{\kappa\neq\lambda\in\{\sigma(s,\mu)\}}\left[\frac{\varphi_\kappa-\varphi_\lambda}{\varphi_\kappa-\varphi_\lambda-1}\right]\prod_{\kappa\in\{\sigma(s,\mu)\}}\prod_{t=1}^{N_f}\left[\frac{\varphi_\kappa-\mathrm{i}\widetilde{m}_t}{\varphi_\kappa+\mathrm{i}m_t+\delta_{t\not\in\{p\}}}\right]\right\}, \tag{4.A.24}$$

which turns out to only depends on the number $\#K$ of elements in K and on the k_s.

We must now sum this expression over all choices of sets K, of integers $l_\kappa > 0$ and $k_s \geq 0$, and of indices $\sigma(s,\mu)$ and $\tau(\kappa,\nu) > \kappa$. The choice of $\{K, l_\kappa, k_s, \sigma, \tau\}$ can be split into a choice of $\{K, l_\kappa, \tau\}$ followed by a choice of integers $k_s \geq 0$ summing to $k - l$, where $l = \sum_{\kappa\in K} l_\kappa$, and finally a choice of σ labelling the complement of $T = \{\tau(\cdot,\cdot)\}$ by pairs (s,μ). This last choice does not affect the residue, hence contributes a factor of $(k-l)!$. The sum over $\{k_s\}$ (summing to $k-l$) of the residue in (4.A.24) yields the $(k-l)$-vortex partition function. Thus,

$$Z^-_{k,\{p\}}(m,\widetilde{m}) = (-1)^k \sum_{l=0}^{\infty}\left[\frac{(k-l)!}{k!}\sum_{T|\#T=l}\sum_{K\subseteq T}\sum_{\{l_\kappa\geq 1\}}\sum_{\tau}[\Sigma^{\#K}]Z^+_{k-l,\{p\}}(m,\widetilde{m})\right]. \tag{4.A.25}$$

The number of choices of $\{K, l_\kappa, \tau\}$ with a given $\#K$ only depends on the size $l = \#T$, thus the choice of T contributes a factor $k!/[l!(k-l)!]$. At this point, we could conclude by noting that we expressed $Z^-_{k,\{p\}}(m,\widetilde{m})$ in terms of the $Z^+_{k-l,\{p\}}(m,\widetilde{m})$ with coefficients depending only on l and the combination Σ of twisted masses, and neither on N_f nor on N. The coefficients can thus be obtained through the special case $\widetilde{N_f} = N_f = 1$, $N = 0$, for which computations are elementary, leading to a Seiberg duality relation valid for arbitrary $\widetilde{N_f} = N_f$ and N.

For completeness, we go through the combinatorical exercise. Since only $l = \#T$ affects the counting, we can fix $T = [\![1, l]\!]$ to simplify the discussion. Define the map $\upsilon : T \to T$ such that for each $\kappa \in K$, $\upsilon(\kappa) = \kappa$ and $\upsilon(\tau(\kappa,\nu)) = \max\{\tau(\kappa,\mu) \mid 0 \leq \mu < \nu, \tau(\kappa,\mu) < \tau(\kappa,\nu)\}$ for $\nu > 0$. The data of $K \subseteq T = [\![1,l]\!]$ and $\upsilon : T \to T$ with $\upsilon(\kappa) = \kappa$ for $\kappa \in K$ and $\upsilon(\lambda) < \lambda$ for $\lambda \in T \setminus K$ is in fact equivalent to that of $\{K, l_\kappa, \tau\}$. There are $\prod_{\lambda \in T\setminus K}(\lambda - 1)$ maps υ, hence

$$\begin{aligned} Z^-_{k,\{p\}}(m,\widetilde{m}) &= (-1)^k \sum_{l=0}^{\infty} \left\{ \frac{1}{l!} \left(\sum_{K\subseteq[\![1,l]\!]} \Sigma^{\#K} \prod_{\lambda\in[\![1,l]\!]\setminus K} (\lambda - 1) \right) Z^+_{k-l,\{p\}}(m,\widetilde{m}) \right\} \\ &= (-1)^k \sum_{l=0}^{\infty} \frac{(\Sigma)_l}{l!} Z^+_{k-l,\{p\}}(m,\widetilde{m}) \,, \end{aligned} \tag{4.A.26}$$

where $(\Sigma)_l = \Sigma \cdots (\Sigma + l - 1)$ is the Pochhammer symbol. From this, we can finally deduce the Seiberg duality relation

$$Z_{\{p\}^\complement}(m^D, \widetilde{m}^D, z^D) = (1-z)^{-\Sigma} Z_{\{p\}}(m, \widetilde{m}, z) \,, \tag{4.A.27}$$

with $z^D = z$, and where we recall $\Sigma = \sum_{s=1}^{N_f}(\mathrm{i}\widetilde{m}_s + \mathrm{i}m_s) + N_f - N$. This relation precisely matches that obtained in the main text as Toda conjugation, in particular the exponent (4.2.4).

4.B SQCDA vortex partition functions

We now adapt the proof to $\mathcal{N} = (2,2)$ SQCDA theories with a superpotential. The field content consists of a vector multiplet coupled to one adjoint chiral multiplet X, N_f fundamental chiral multiplets q_s, and N_f antifundamental chiral multiplets $\widetilde{q}_s$. As in Section 3.5.3 we consider two cases: the superpotential $W = \sum_{t=1}^{N_f} \widetilde{q}_t X^{l_t} q_t$ and the superpotential $W = \operatorname{Tr} X^{l+1}$ for integers $l_t, l \geq 0$. Both choices exhibit common features, with l_t replaced by l for the second superpotential.

In Section 4.3.1 and Section 4.3.2, we find that pairs of such theories with gauge groups $U(N)$ and $U(N^D)$ are dual, using symmetries of Toda CFT correlators. Parameters are mapped as follows: $m^D_X = m_X$, $N^D = L - N$ with $L = \sum_{t=1}^{N_f} l_t$, and

$$m^D_t = m_t, \qquad \widetilde{m}^D_t = m^D_t, \qquad z^D = (-1)^L z^{-1} \quad \text{for } W = \sum_{t=1}^{N_f} \widetilde{q}_t X^{l_t} q_t \tag{4.B.1}$$

$$m^D_t = m_X - m_t, \quad \widetilde{m}^D_t = m_X - \widetilde{m}_t, \quad z^D = (-1)^L z \qquad \text{for } W = \operatorname{Tr} X^{l+1} \,. \tag{4.B.2}$$

Higgs branch vacua of the $U(N)$ theory are labelled by integers $0 \leq n_t \leq l_t$ with sum N. Those are in a natural bijection $n^D_t = l_t - n_t$ to integers

$0 \leq n_t^D \leq l_t$ with sum $L - N$, which label Higgs branch vacua of the dual theory. We check the dualities at the level of classical and one-loop contributions in Section 4.3.1 and Section 4.3.2. We now prove the relations (4.B.30) and (4.B.31) between the vortex partition functions of the $U(N)$ theory in the vacuum $\{n_t\}$ and of the $U(L-N)$ theory in the vacuum $\{l_t - n_t\}$.

4.B.1 Preliminary result for $N_f = 1$

Later on, we prove that dual vortex partition functions are equal up to some factor which only depends on very little data. To fix the factor, we will use the simple case of $N_f = 1$ SQCDA with $1 + \mathrm{i}m_1 + \mathrm{i}\widetilde{m}_1 + N\mathrm{i}m_X = 0$, which we consider now. Its unique vacuum has $n_1 = N$, and we prove that $Z_{\mathrm{v},\{N\}}(y) = (1-y)^{-N(1+\mathrm{i}m_X)}$. By analyticity, it is enough to show this when $\mathrm{Re}(\mathrm{i}\widetilde{m}_1) < 0 < \mathrm{Re}(-\mathrm{i}m_1) < \mathrm{Re}(-\mathrm{i}m_X)$.

The vortex partition function, given by the series (3.5.10), has a Mellin–Barnes integral representation

$$Z_{\mathrm{v},\{N\}}(y) = \sum_{\{k_\mu \geq 0\}} y^{\sum k_\mu} \prod_{\mu,\nu=0}^{N-1} \frac{((\nu-\mu-1)\mathrm{i}m_X - k_\nu)_{k_\mu}}{((\nu-\mu)\mathrm{i}m_X - k_\nu)_{k_\mu}} \tag{4.B.3}$$

$$= (-y)^{N\mathrm{i}m_1 + \frac{1}{2}(N-1)N\mathrm{i}m_X} \prod_{\mu=1}^{N} \frac{\sin\pi(-\mu\mathrm{i}m_X)}{\pi} \tag{4.B.4}$$

$$\cdot \frac{1}{N!}\int_{\mathbb{R}^N} \frac{\mathrm{d}^N\sigma}{(2\pi)^N} (-y)^{\mathrm{Tr}\,\mathrm{i}\sigma} \prod_{j=1}^{N} \Big[\Gamma(-\mathrm{i}m_1 - \mathrm{i}\sigma_j)\Gamma(-\mathrm{i}\widetilde{m}_1 + \mathrm{i}\sigma_j)\Big]$$

$$\frac{\prod_{i,j=1}^{N} \Gamma(\mathrm{i}\sigma_i - \mathrm{i}\sigma_j - \mathrm{i}m_X)}{\prod_{i\neq j}^{N} \Gamma(\mathrm{i}\sigma_i - \mathrm{i}\sigma_j)}$$

which analytically continues $Z_{\mathrm{v},\{N\}}(y)$ from the unit disc to $y \notin \mathbb{R}_{\geq 0}$. Closing contours towards $\mathrm{i}\infty$ yields a similar relation for $|y| > 1$, with $m_1 \leftrightarrow \widetilde{m}_1$ and $y \to y^{-1}$. Hence, the analytic continuations obey

$$Z_{\mathrm{v},\{N\}}(y) = (-y)^{-N(1+\mathrm{i}m_X)} Z_{\mathrm{v},\{N\}}(y^{-1})\,. \tag{4.B.5}$$

The function $(1-y)^{N(1+\mathrm{i}m_X)} Z_{\mathrm{v},\{N\}}(y)$ is thus analytic on the Riemann sphere away from $y = 1$. Furthermore, we can bound it by a power of $|1-y|$ in two pairs of angular sectors centered at $y = 1$, whose union is a neighborhood of $y = 1$.

The first angular sector is defined by $|1-y| < M(1-|y|)$ for some $M > 0$ and is contained in the open unit disc. The coefficients in the series (4.B.3) grow at most polynomially in the exponent $\sum_\mu k_\mu$ of y, and the number of terms contributing for a given power of y also grows polynomially. Hence,

$$|Z_{\mathrm{v},\{N\}}(y)| \leq \sum_{k\geq 0} C_1 (k+1)_{C_2} |y|^k = C_2! C_1 (1-|y|)^{-1-C_2} \tag{4.B.6}$$

for some $C_1, C_2 > 0$ which do not depend on y. Thus $|1-y|^{1+C_2} Z_{\mathrm{v},\{N\}}(y)$ is bounded in each sector $|1-y| < M(1-|y|)$. By the symmetry $y \to y^{-1}$, the function is also bounded in a similar sector $|y-1| < M(|y|-1)$. We have thus probed the function away from the unit circle.

The next pair of sectors is probed using the Mellin–Barnes representation (4.B.4), which converges away from the real axis. Set $y = re^{\epsilon \mathrm{i}\theta}$ with $\frac{1}{2} < r < 2$ (to avoid $\{0, \infty\}$), $\epsilon = \pm 1$, and $0 < \theta < \pi$ (that is, $y \notin \mathbb{R}$). Then

$$|(-y)^{\mathrm{i}\sigma_j}| = e^{\epsilon(\pi-\theta)\sigma_j} \leq e^{(\pi-\theta)|\sigma|} . \tag{4.B.7}$$

For some large enough $C_1, C_2 > 0$ which depend on the twisted masses, we have

$$\begin{aligned} \left|\frac{\Gamma(-\mathrm{i}m_1 - \mathrm{i}\sigma_j)}{\Gamma(1+\mathrm{i}\widetilde{m}_1 - \mathrm{i}\sigma_j)}\right| &< C_1 \big(|\sigma|+1\big)^{N \operatorname{Re}(\mathrm{i}m_X)} \\ \left|\frac{\Gamma(\mathrm{i}\sigma_i - \mathrm{i}\sigma_j - \mathrm{i}m_X)}{\Gamma(\mathrm{i}\sigma_i - \mathrm{i}\sigma_j)}\right| &< C_2 \big(|\sigma|+1\big)^{\operatorname{Re}(-\mathrm{i}m_X)} \end{aligned} \tag{4.B.8}$$

for all σ, where $|\sigma| = \big(\sum_{i=1}^N |\sigma_i|^2\big)^{1/2}$ is larger than all $|\sigma_j|$ and all $|\sigma_i - \sigma_j|$. The inequalities rely on the asymptotics $\Gamma(a+\mathrm{i}v)/\Gamma(b+\mathrm{i}v) \sim (\mathrm{i}v)^{a-b}$ as $v \to \pm\infty$, and the continuity of both ratios of Gamma functions. Since $0 < \operatorname{Im}(\widetilde{m}_1) < 1$, we also have

$$|\Gamma(1+\mathrm{i}\widetilde{m}_1 - \mathrm{i}\sigma_j)\Gamma(-\mathrm{i}\widetilde{m}_1 + \mathrm{i}\sigma_j)| \leq \frac{2\pi e^{-\pi|\sigma_j - \operatorname{Re}(\widetilde{m}_1)|}}{|\sin(\pi \operatorname{Im}(\widetilde{m}_1))|} < C_3 e^{-\pi|\sigma|} \tag{4.B.9}$$

for some $\widetilde{m}_1$-dependent $C_3 > 0$. Combining the bounds into (4.B.4) yields

$$|Z_{\mathrm{v},\{N\}}(y)| \leq C_4 \int_{\mathbb{R}^N} \mathrm{d}^N\sigma \, e^{-N\theta|\sigma|} \big(|\sigma|+1\big)^{N \operatorname{Re}(\mathrm{i}m_X)} \tag{4.B.10}$$

for some $C_4 > 0$. Switching to polar coordinates, letting $\tau = \theta(|\sigma|+1)$, and bounding $(\tau-\theta)^{N-1} < \tau^{N-1}$ leads to

$$|\theta^{N(1+\mathrm{i}m_X)} Z_{\mathrm{v},\{N\}}(y)| \leq C_5 \int_\theta^\infty \mathrm{d}\tau \, e^{-N\tau} \tau^{N \operatorname{Re}(\mathrm{i}m_X)+N-1} \leq C_6 \tag{4.B.11}$$

for some $C_5, C_6 > 0$. In any angular sector centered at $y = 1$ and away from the real axis, $|1-y|$ is bounded by some multiple of $\theta = \arg(y)$, hence $(1-y)^{N(1+\mathrm{i}m_X)} Z_{\mathrm{v},\{N\}}(y)$ is bounded both above and below the real axis.

We have bounded the function $(1-y)^{N(1+\mathrm{i}m_X)} Z_{\mathrm{v},\{N\}}(y)$ by a power of $|1-y|$ in a neighborhood of $y = 1$. Since the function is analytic away from 1, it takes the form $P(y)/(1-y)^n$, where $P(y)$ is a polynomial of degree at most $n \geq 0$. In the second pair of sectors, we found that the function is bounded as $y \to 1$, thus $n = 0$ and the function is the constant $(1-y)^{N(1+\mathrm{i}m_X)} Z_{\mathrm{v},\{N\}}(y) = Z_{\mathrm{v},\{N\}}(0) = 1$.

4.B.2 Proof for SQCDA

Let us move on to the proof per se. We start with the vortex partition function (3.5.10) of the $U(N)$ SQCDA theory in a given Higgs branch vacuum $\{n_s\}$. The terms of this series in powers of $y = (-1)^{N_f+N-1}z$ are labelled by integer vorticities $k_{s\mu} \geq 0$ for $1 \leq s \leq N_f$ and $0 \leq \mu < n_s$:

$$Z_{\mathrm{v},\{n_s\}}(y) = \sum_{k\geq 0} y^k Z_{\mathrm{v},\{n_s\},k} = \sum_{k\geq 0} y^k \sum_{\sum k_{s\mu}=k} V_{\{n_s\}}^{\{k_{s\mu}\}} . \tag{4.B.12}$$

The contribution $V_{\{n_s\}}^{\{k_{s\mu}\}}$ for a given choice of vorticities is a ratio of Pochhammer symbols, which we massage using $(1-x-k)_{k-j} = (-1)^{k-j}(x)_k/(x)_j$ into

$$V_{\{n_s\}}^{\{k_{s\mu}\}} = \prod_{(s,\mu)\in I}\left[\prod_{t=1}^{N_f} \frac{(-\mathrm{i}\widetilde{m}_t - \mathrm{i}m_{s\mu})_{k_{s\mu}}}{(1+\mathrm{i}m_t + n_t \mathrm{i}m_X - \mathrm{i}m_{s\mu})_{k_{s\mu}}} \prod_{(t,\nu)\in I} \frac{(\mathrm{i}m_{t\nu} - \mathrm{i}m_{s\mu} - \mathrm{i}m_X - k_{t\nu})_{k_{s\mu}}}{(\mathrm{i}m_{t\nu} - \mathrm{i}m_{s\mu} - k_{t\nu})_{k_{s\mu}}}\right] . \tag{4.B.13}$$

Here, m_s, $\widetilde{m}_s$, and m_X are complexified twisted masses of the chiral multiplets, we denote $m_{s\mu} = m_s + \mu m_X$, the products range over $I = \{(s,\mu) \mid 1 \leq s \leq N_f, 0 \leq \mu < n_s\}$, and we have swapped $(s,\mu) \leftrightarrow (t,\nu)$ compared to (3.5.10). Using that

$$\prod_{(s,\mu)\in I}\prod_{(t,\nu)\in I} \frac{(\mathrm{i}m_{t\nu} - \mathrm{i}m_{s\mu} - A - k_{t\nu})_{k_{s\mu}}}{(\mathrm{i}m_{t\nu} - \mathrm{i}m_{s\mu} - A)_{k_{s\mu}}} = \prod_{\substack{(s,\mu)\in I\\ 0\leq i<k_{s\mu}}} \prod_{\substack{(t,\nu)\in I\\ 0\leq j<k_{t\nu}}} \frac{\mathrm{i}m_{t\nu} - j - \mathrm{i}m_{s\mu} + i - A - 1}{\mathrm{i}m_{t\nu} - j - \mathrm{i}m_{s\mu} + i - A} \tag{4.B.14}$$

for a generic $A \in \mathbb{C}$, we can express $V_{\{n_s\}}^{\{k_{s\mu}\}}$ in terms of the combinations $-\mathrm{i}m_{s\mu} + i$ for $(s,\mu) \in I$ and $0 \leq i < k_{s\mu}$. We find that $(-1)^k V_{\{n_s\}}^{\{k_{s\mu}\}}$ is the residue at $\{\varphi_\kappa\} = \{-\mathrm{i}m_{s\mu} + i \mid 0 \leq i < k_{s\mu}\}$ of the integrand in (4.B.15) below, after $|M_\kappa| \to \infty$.

The discussion above leads us to the contour integral ($I_0 = 1$ is an empty product)

$$\begin{aligned} I_k = \lim_{|M_1|\to\infty} \cdots \lim_{|M_k|\to\infty} \frac{1}{k!} \prod_{\kappa=1}^{k} \left[\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{d}\varphi_\kappa}{2\pi\mathrm{i}}\right] \Bigg\{ \prod_{\kappa=1}^{k} \frac{-(\mathrm{i}M_\kappa)^2}{(\varphi_\kappa - \frac{1}{2})^2 - (\mathrm{i}M_\kappa)^2} \prod_{\kappa\neq\lambda}^{k} \frac{\varphi_\kappa - \varphi_\lambda}{\varphi_\kappa - \varphi_\lambda - 1} \\ \cdot \prod_{\kappa,\lambda=1}^{k} \frac{\varphi_\kappa - \varphi_\lambda - 1 - \mathrm{i}m_X}{\varphi_\kappa - \varphi_\lambda - \mathrm{i}m_X} \prod_{\kappa=1}^{k}\prod_{s=1}^{N_f} \left[\frac{\varphi_\kappa - \mathrm{i}\widetilde{m}_s}{\varphi_\kappa + 1 + \mathrm{i}m_s + n_s \mathrm{i}m_X} \frac{\varphi_\kappa + \mathrm{i}m_s - \mathrm{i}m_X}{\varphi_\kappa + \mathrm{i}m_s + (n_s-1)\mathrm{i}m_X}\right] \Bigg\} \end{aligned} \tag{4.B.15}$$

whose residues include all contributions to the k-vortex partition function $Z_{\mathrm{v},\{n_s\},k}$. As in the SQCD case, we move the pole at infinity to a finite value through a regulating factor, which depends on large real parameters with $1 \ll |M_1| \ll \cdots \ll |M_k|$. The small shift by $\frac{1}{2}$ moves poles away from the imaginary axis. We assume that the complex parameters m_s and m_X are in the ranges

$$0 < \mathrm{Re}(\mathrm{i}m_X) < 1 \qquad (n_s - 1)\,\mathrm{Re}(\mathrm{i}m_X) < \mathrm{Re}(-\mathrm{i}m_s) < n_s\,\mathrm{Re}(\mathrm{i}m_X)\,. \tag{4.B.16}$$

This constraint is eventually lifted since the relation we will deduce between vortex partition functions is analytic in m_s and m_X.

Close the contours of (4.B.15) towards $+\infty$ first. Because of the factors $1/(\varphi_\kappa - \varphi_\lambda - 1)$ and $1/(\varphi_\kappa - \varphi_\lambda - \mathrm{i}m_X)$, the surrounded poles are such that the φ_λ are organized into groups of components with related values:

$$\{\varphi_\lambda \mid \lambda \in S\} = \coprod_{1\leq s\leq N_f} \{-\mathrm{i}m_s + (1 - n_s + \mu)\mathrm{i}m_X + i \mid 0 \leq \mu < n'_s, 0 \leq i < k'_{s\mu}\} \tag{4.B.17}$$

$$\{\varphi_\lambda \mid \lambda \in T\} = \coprod_{\kappa\in K} \{\tfrac{1}{2} + \epsilon_\kappa \mathrm{i}M_\kappa + \nu\mathrm{i}m_X + j \mid 0 \leq \nu < \hat{n}_\kappa, 0 \leq j < \hat{k}_{\kappa\nu}\} \tag{4.B.18}$$

where K is the set of indices for which $\varphi_\kappa = \frac{1}{2} + \epsilon_\kappa \mathrm{i}M_\kappa$, and $[\![1, k]\!] = S \sqcup T$. Note that all $n'_s \leq n_s$, otherwise the numerator factor $\prod_\lambda(\varphi_\lambda + \mathrm{i}m_s - \mathrm{i}m_X)$ would vanish. Introducing if necessary $k'_{sn'_s} = \cdots = k'_{s(n_s-1)} = 0$, we set $n'_s = n_s$, then define $k_{s\mu} = k'_{s(n_s-1-\mu)}$.

The pole is uniquely determined by the partition $[\![1, k]\!] = S \sqcup T$, the set $K \subseteq T$, the signs $\epsilon_\kappa = \pm 1$, the non-negative integers n_s (fixed when defining I_k), $k_{s\mu}$, $\hat{n}_\kappa$, and $\hat{k}_{\kappa\nu}$, and the maps σ and τ defined by

$$\varphi_{\sigma(s,\mu,i)} = -\mathrm{i}m_s - \mu\mathrm{i}m_X + i \quad \text{and} \quad \varphi_{\tau(\kappa,\nu,j)} = \frac{1}{2} + \epsilon_\kappa \mathrm{i}M_\kappa + \nu\mathrm{i}m_X + j \tag{4.B.19}$$

for $1 \leq s \leq N_f$, $0 \leq \mu < n_s$, $0 \leq i < k_{s\mu}$, and for $\kappa \in K$, $0 \leq \mu < \hat{n}_\kappa$, $0 \leq j < \hat{k}_{\kappa\mu}$. This data is constrained: σ is a bijection from $\{(s, \mu, i) \mid 0 \leq i < k_{s\mu}\}$ to S, hence $\sum k_{s\mu} = \#S$, and τ is a bijection from $\{(\kappa, \nu, j) \mid 0 \leq j < \hat{k}_{\kappa\nu}\}$ to T, hence $\sum \hat{k}_{\kappa\nu} = \#T$. Also, $\tau(\kappa, 0, 0) = \kappa$ for all $\kappa \in K$.

Let $t = \#T$. It is convenient to parametrize poles in terms of the data t, T, $(K, \hat{n}_\kappa, \hat{k}_{\kappa\nu}, \tau)$, $(k_{s\mu}, \sigma)$, and ϵ_κ. When summing residues of I_k at such poles, we will first sum over choices of signs ϵ_κ and take the limits $|M_\kappa| \to \infty$. The result is independent of σ, which thus contributes only a combinatorical factor. Then follows a sum over choices of $k_{s\mu}$, whose only constraint is $\sum k_{s\mu} = k - t$. Since the residue of I_k involves the vortex contribution $V^{\{k_{s\mu}\}}_{\{n_s\}}$, the sum over $k_{s\mu}$ yields the $(k - t)$-vortex partition function. Summing over the remaining data, we find that I_k is a linear combination of $(k - t)$-vortex partition functions for $0 \leq t \leq k$, whose coefficients only depend on t, $\mathrm{i}m_X$, and a single combination Σ of the twisted masses. This allows us to fix the coefficients by considering a simple case.

Let us proceed. The residue at (4.B.19) of (4.B.15) has the following asymptotics:

$$\prod_{\kappa=1}^{k}\left[1+O\left(\frac{1}{M_\kappa^2}\right)\right]\prod_{\kappa<\lambda}^{k}\left[1+O\left(\frac{M_\kappa^2}{M_\lambda^2}\right)\right]\prod_{\tau(\kappa,\mu,j)<\kappa}\left[O\left(\frac{M^2_{\tau(\kappa,\mu,j)}}{M_\kappa^2}\right)\right] \tag{4.B.20}$$

$$\cdot\frac{(-1)^k}{k!}\prod_{\kappa\in K}f_{\{\hat{n}_\kappa\},\{\hat{k}_{\kappa\nu}\}}(\mathrm{i}m_X)\prod_{\kappa\in K}\left[\frac{-\epsilon_\kappa \mathrm{i}M_\kappa}{2}+\frac{\Sigma}{2}\sum_{\nu=0}^{\hat{n}_\kappa-1}\hat{k}_{\kappa\nu}+O\left(\frac{1}{M_\kappa}\right)\right]V^{\{k_{s\mu}\}}_{\{n_s\}}$$

where f is a rational function of $\mathrm{i}m_X$ with integer coefficients, and

$$\Sigma = 2N\mathrm{i}m_X+\sum_{s=1}^{N_f}(1+\mathrm{i}m_s+\mathrm{i}\widetilde{m}_s)\,. \tag{4.B.21}$$

We expect the divergent piece $-\epsilon_\kappa \mathrm{i}M_\kappa/2$ of the residue to cancel when summing over signs ϵ_κ. Let us take limits $|M_\lambda|\to\infty$ from $\lambda=k$ down to $\lambda=1$ carefully. At each step there are two cases. If $\lambda\in K$, then the limit vanishes whenever any $\tau(\lambda,\mu,j)<\lambda$. Hence, only poles with all $\tau(\lambda,\mu,j)\geq\lambda$ contribute and we can focus on those. The M_λ-dependent terms are then of the form $-\epsilon_\lambda \mathrm{i}M_\lambda/2$ plus a finite part. Summing over $\epsilon_\lambda=\pm1$ only leaves the finite part. On the other hand, if $\lambda\notin K$, then taking the limit $|M_\lambda|\to\infty$ is trivial as M_λ only appears in factors $[1+O(1/M_\lambda^2)]$ and $[1+O(M_\kappa^2/M_\lambda^2)]$ for $\kappa<\lambda$ (importantly, we have already taken the limits $|M_\kappa|\to\infty$ for all $\kappa>\lambda$).

All in all, we are left with a non-divergent expression for I:

$$I_k=\frac{1}{k!}\sum_{t,T}\sum_{K,\{\hat{n}_\kappa\},\{\hat{k}_{\kappa\nu}\},\tau}\sum_{\{k_{s\mu}\},\sigma}\Sigma^{\#K}\prod_{\kappa\in K}\left[f_{\{\hat{n}_\kappa\},\{\hat{k}_{\kappa\nu}\}}(\mathrm{i}m_X)\sum_{\nu=0}^{\hat{n}_\kappa-1}\hat{k}_{\kappa\nu}\right]V^{\{k_{s\mu}\}}_{\{n_s\}}\,. \tag{4.B.22}$$

The summand is independent of σ, and there are $(k-t)!$ maps σ. Summing $V^{\{k_{s\mu}\}}_{\{n_s\}}$ over $k_{s\mu}$ with $\sum k_{s\mu}=k-t$ yields $Z_{\mathrm{v},\{n_s\},k-t}$. The sum over K, $\hat{n}_\kappa$, $\hat{k}_{\kappa\nu}$, τ does not depend on the precise set T, but only on $t=\#T$. The choice of T thus simply contributes a factor $k!/[t!(k-t)!]$, which cancels the overall $1/k!$, and $(k-t)!$ coming from the choice of σ. For a fixed $j=\#K$, the remaining sums yield a rational function of $\mathrm{i}m_X$ which can only depend on the two integers $0\leq j\leq t\leq k$:

$$I_k=\sum_{t=0}^{k}\sum_{j=0}^{t}f_{tj}(\mathrm{i}m_X)\Sigma^j Z_{\mathrm{v},\{n_s\},k-t}\,. \tag{4.B.23}$$

Since the f_{tj} do not depend on k, summing over k yields

$$\sum_{k\geq0}y^kI_k=\sum_{t\geq0}\sum_{j=0}^{t}[y^tf_{tj}(\mathrm{i}m_X)\Sigma^j]Z_{\mathrm{v},\{n_s\}}(y)=f(\mathrm{i}m_X,\Sigma;y)Z_{\mathrm{v},\{n_s\}}(y)\,. \tag{4.B.24}$$

In Section 4.B.1, we consider the case $N_f = 1$, $n_1 = N$, $1 + im_1 + i\widetilde{m}_1 + Nim_X = 0$, for which $\Sigma = Nim_X$, and find that

$$Z_{v,\{N\}}(y)\big|_{1+im_1+i\widetilde{m}_1+Nim_X=0} = (1-y)^{-N(im_X+1)} = (1-y)^{-[1+1/(im_X)]\Sigma} . \tag{4.B.25}$$

On the other hand, since the factors $\varphi_\kappa - i\widetilde{m}_1$ and $\varphi_\kappa + 1 + im_1 + n_1 im_X$ in (4.B.15) cancel, the integrand of I_k has no pole with $\mathrm{Re}(\varphi_\kappa) < 0$, thus $I_k = \delta_{k0}$. As a result,

$$f(im_X, \Sigma; y) = (1-y)^{[1+1/(im_X)]\Sigma} \tag{4.B.26}$$

for all $\Sigma = Nim_X$. This fixes each polynomial $\sum_{j=0}^{t} f_{tj}(im_X)\Sigma^j$ at an infinite set of values, hence determines f completely.

At last, we are ready to wrap up, by showing that I_k is the k-vortex partition function of the dual theory. Close contours of (4.B.15) towards $-\infty$. The surrounded poles are labelled by non-negative integers $n'_t \geq 0$ and $k'_{t\nu} \geq 0$ for $1 \leq t \leq N_f$ and $0 \leq \nu < n'_t$:

$$\{\varphi_\kappa\} = \{-1 - im_t - n_t im_X - \nu im_X - j \mid 0 \leq \nu < n'_t, 0 \leq j < k'_{t\nu}\} . \tag{4.B.27}$$

For the choice of superpotential $W = \sum_{t=1}^{N_f} \widetilde{q}_t X^{l_t} q_t$, the constraint $1 + im_t + i\widetilde{m}_t + l_t im_X = 0$ implies that the numerator factor $\prod_\kappa(\varphi_\kappa - i\widetilde{m}_t)$ vanishes unless all $k'_{t\nu} = 0$ for $\nu \geq l_t - n_t$. For the choice of superpotential $W = \mathrm{Tr}\, X^{l+1}$, the constraint $1 + (l+1)im_X = 0$ implies that $\prod_\kappa(\varphi_\kappa + im_t - im_X)$ vanishes unless all $k'_{t\nu} = 0$ for $\nu \geq l - n_t$. We can thus take $n'_t = l_t - n_t$ in both cases, and let $k_{t\nu} = k'_{t(l_t-n_t-1-\nu)}$ so that

$$\{\varphi_\kappa\} = \{-1 - im_t - (l_t - 1 - \nu)im_X - j \mid 0 \leq \nu < l_t - n_t, 0 \leq j < k_{t\nu}\} . \tag{4.B.28}$$

Summing over residues yields, after some massaging,

$$I_k = \sum_{\{k_{t\nu}\geq 0 | 0 \leq \nu < l_t - n_t\}} \prod_{(s,\mu)} \Bigg[\prod_{(t,\nu)} \frac{(im_s - im_t + (l_s - l_t + \nu - \mu - 1)im_X - k_{t\nu})_{k_{s\mu}}}{(im_s - im_t + (l_s - l_t + \nu - \mu)im_X - k_{t\nu})_{k_{s\mu}}} \cdot \prod_{t=1}^{N_f} \Bigg[\frac{(1 + i\widetilde{m}_t + im_s + (l_s - 1 - \mu)im_X)_{k_{s\mu}}}{(im_s - im_t + (l_s - l_t - 1 - \mu)im_X)_{k_{s\mu}}} \frac{(im_s - im_t + (l_s - \mu)im_X + 1)_{k_{s\mu}}}{(im_s - im_t + (l_s - n_t - \mu)im_X + 1)_{k_{s\mu}}} \Bigg] \Bigg] . \tag{4.B.29}$$

For $W = \sum_{t=1}^{N_f} \widetilde{q}_t X^{l_t} q_t$, the summand takes the general form (4.B.13) of $V^{\{k\}}_{\{l_t - n_t\}}$, with $m_t \leftrightarrow \widetilde{m}_t$ since $1 + i\widetilde{m}_t + im_s + (l_s - 1 - \mu)im_X = im_s - im_t + (l_s - l_t - 1 - \mu)im_X$. Thus, I_k is the k-vortex partition function of the SQCDA theory with N_f flavour, $L - N$ colors, the superpotential $W = \sum_t \widetilde{q}_t X^{l_t} q_t$, interchanged twisted masses $m_t \leftrightarrow \widetilde{m}_t$ compared to the $U(N)$ theory, and the same value of y. Charge conjugation maps twisted masses back to those of the $U(N)$ theory, and maps $y \to y^D = y^{-1}$. Summing $y^k I_k$ then yields the u-channel vortex partition function of the $U(L-N)$ theory (that is, a series in powers of $(y^D)^{-1}$). We finally combine the relation (4.B.24) and the explicit factor (4.B.26) with $\Sigma = (2N-L)\mathrm{i}m_X$ to get

$$Z^{U(L-N)}_{\mathrm{v},\{l_s-n_s\}}((y^D)^{-1}) = (1-y)^{(2N-L)(1+\mathrm{i}m_X)} Z^{U(N)}_{\mathrm{v},\{n_s\}}(y) \quad \text{for } W = \sum_{t=1}^{N_f} \widetilde{q}_t X^{l_t} q_t \,. \tag{4.B.30}$$

For $W = \operatorname{Tr} X^{l+1}$, we have $\mathrm{i}m_s - \mathrm{i}m_t + (l_s - l_t - 1 - \mu)\mathrm{i}m_X = \mathrm{i}m_s - \mathrm{i}m_t + (l_s - \mu)\mathrm{i}m_X + 1$, and again the summand takes the form of $V^{\{k\}}_{\{l_t-n_t\}}$, with $\mathrm{i}m_s \to \mathrm{i}m_s^D = \mathrm{i}m_X - \mathrm{i}m_s$ and $\mathrm{i}\widetilde{m}_t \to \mathrm{i}\widetilde{m}_t^D = \mathrm{i}m_X - \mathrm{i}\widetilde{m}_t$. Combining the relation (4.B.24) and the explicit factor (4.B.26) with $\Sigma = 2N\mathrm{i}m_X + \sum_{t=1}^{N_f}(1 + \mathrm{i}m_t + \mathrm{i}\widetilde{m}_t)$ and $1 + 1/(\mathrm{i}m_X) = -l$ yields the Kutasov–Schwimmer duality relation

$$Z^{U(lN_f-N)}_{\mathrm{v},\{l-n_s\}}(m_t^D, \widetilde{m}_t^D; y^D) = (1-y)^{-\delta_1} Z^{U(N)}_{\mathrm{v},\{n_s\}}(m_t, \widetilde{m}_t; y) \quad \text{for } W = \operatorname{Tr} X^{l+1} \tag{4.B.31}$$

with $\delta_1 = -\frac{2l}{l+1}N + l\sum_{t=1}^{N_f}(1 + \mathrm{i}m_t + \mathrm{i}\widetilde{m}_t)$, as obtained in (4.3.18) through the relation with conjugation of momenta in the Toda CFT. Since $y^D = y$, $z^D = (-1)^{N_f l} z$. In Section 4.3.2, we extend the Kutasov–Schwimmer duality to theories with a different number of fundamental and antifundamental chiral multiplets.

Chapter 5

Toda conformal field theory

Two-dimensional field theories with conformal symmetries are very constrained, as the conformal symmetry algebra (two copies of the Virasoro algebra) is infinite-dimensional. We[1] shall assume that the reader is familiar with 2d CFT (conformal field theory) and otherwise refer them to Section 1.2 or the review [Rib14]. As initiated by A.B. Zamolodchikov and Fateev [Zam85; FZ87], one can consider 2d CFTs with additional symmetries beyond the Virasoro algebra. One such extension of the Virasoro algebra is the W-algebra W_N associated to the Dynkin diagram A_{N-1} (see the review [BS92]), which includes some higher-spin currents $W^{(k)}(z)$ for $k = 2, \ldots, N$ beyond the usual stress-energy tensor $T(z) = W^{(2)}(z)$. The W_N-invariant analogues of minimal models were constructed in [FL88]. These theories are rational: their spectrum decomposes into a finite number of irreducible representations of W_N.

This chapter explores the A_{N-1} Toda CFT (the standard recent reference is [FL07]), arguably the simplest non-rational 2d CFT with extended symmetry W_N, which reduces to the well-known Liouville CFT for $N = 2$. The spectrum of these theories is continuous and involves in some sense exactly one copy of each unitary representation of their symmetry algebra W_N (or of the Virasoro algebra W_2). This choice of spectrum is already very constraining and little additional data is needed to derive many correlators. Throughout this chapter we try to clarify what properties are specific to the Toda CFT and which ones only depend on the W_N symmetry algebra.

Section 5.1 is devoted to the W_N algebra. We present its free-field realization, introduce W_N primary operators (vertex operators), evaluate their quantum numbers, construct null-vectors using screening charges, define semi-degenerate and degenerate primary operators, and state a few fusion rules. The two following sections work their way up to the braiding kernel for two semi-degenerate primary operators, related by the AGT correspondence

[1] The force of habit leads me to use the formal “we” even though this chapter, contrarily to others, is not an article written in collaboration with other authors.

to S-duality walls for 4d $\mathcal{N} = 2$ $SU(N)$ SQCD. Section 5.2 derives the braiding of a semi-degenerate operator and a degenerate operator labelled by an antisymmetric representation of A_{N-1}, while Section 5.3 covers symmetric representations and deduces the braiding of two semi-degenerate operators.

We move on in Section 5.4 to Toda CFT per se. We write down its Lagrangian, then use the Coulomb gas formalism to deduce two-point functions, and some three-point functions with one degenerate operator, confirming the fusion rules stated earlier. From the braiding matrices we deduce shift relations which uniquely determine the Toda three-point function with one semi-degenerate vertex operators as done in [FL07]. We then collect explicit expressions for various W_N conformal blocks involving degenerate vertex operators, and for the associated products of Toda three-point functions. In Section 5.5 we derive some more fusion rules. We wrap up with Section 5.6 on obtaining irregular punctures as collision limits of products of vertex operators, as these appear in some cases of the AGT correspondence.

This chapter is based on Appendix A of my paper [GLF14] with Jaume Gomis and on a paper in preparation alone. Section 5.2.2, Section 5.4.1, and Section 5.4.4 incorporate the contents of A.3, A.1, and a combination of A.2 and A.5 of [GLF14], respectively, while Section 5.5 is A.4 and Section 5.6 is A.6.

5.1 W algebra

As mentionned in the chapter introduction, the W_N algebra extends the Virasoro algebra by adding some currents besides the stress-energy tensor $T(z)$. One can choose various bases for these currents, and we will use the basis coming from the Miura transformation (Section 5.1.1): the currents are then $W^{(k)}(z)$ for integers $2 \leq k \leq N$, with $W^{(2)}(z) = T(z)$. Other bases are obtained by adding to $W^{(k)}(z)$ some derivatives and products of $W^{(j)}(z)$ with $j < k$. We construct highest weight representations (Section 5.1.2) and null-vectors in some of them (Section 5.1.3). We then find out which non-zero two-point functions are allowed by the W_N symmetry (Section 5.1.4) and discuss a few fusion rules (Section 5.1.5).

5.1.1 Miura transformation

Here, we give a free-field realization of the W_N algebra, obtained through the Miura transformation.

Let us introduce some notations. Denote by $h_1, \ldots, h_N$ the weights of the fundamental representation of A_{N-1}, which obey $\sum_{s=1}^{N} h_s = 0$ hence span an $(N-1)$-dimensional space. Let $\omega_k = h_1 + \cdots + h_k$ be the fundamental weights, that is, the highest weights of antisymmetric representations of A_{N-1}. Among the roots $e = h_i - h_j$ for $1 \leq i \neq j \leq N$ of A_{N-1}, those with $i < j$ are positive roots, and the simple roots are $e_j = h_j - h_{j+1}$ for $1 \leq j \leq N-1$.

The Weyl vector is half the sum of all positive roots: $\rho = \frac{1}{2}\sum_{e>0} e$, and we set $Q = q\rho$. The Killing bilinear form $\langle\ ,\ \rangle$ obeys

$$\begin{aligned}
\langle h_s, h_t\rangle &= \delta_{st} - \frac{1}{N} & \langle h_s, \omega_k\rangle &= \delta_{s\le k} - \frac{k}{N} \\
\langle h_s, e_j\rangle &= \delta_{sj} - \delta_{s(j+1)} & \langle \omega_j, \omega_k\rangle &= \min(j,k) - \frac{jk}{N} \\
\langle \omega_j, e_k\rangle &= \delta_{jk} & \langle e_j, e_k\rangle &= -\delta_{j(k-1)} + 2\delta_{jk} - \delta_{j(k+1)} \\
\langle \rho, h_s\rangle &= \frac{N+1}{2} - s & \langle \rho, \omega_k\rangle &= \frac{k(N-k)}{2} \\
\langle \rho, e_k\rangle &= 1 & \langle \rho, \rho\rangle &= \frac{N(N^2-1)}{12}\,.
\end{aligned} \tag{5.1.1}$$

Finally, $\varphi(z,\bar{z})$ will be an $(N-1)$ component scalar field in this space, whose coefficients have the OPE (operator product expansion)

$$\langle \varphi(z,\bar{z}), h_s\rangle\langle\varphi(0,0), h_t\rangle \sim -\langle h_s, h_t\rangle \log|z|^2 + O(1)\,. \tag{5.1.2}$$

The Miura transformation consists in rewriting a product of N linear differential operators $q\partial_z + \langle h_s, \partial_z\varphi\rangle$ as a degree N differential operator expressed in powers of $q\partial_z$:

$$\prod_{s=N}^{1}\left(q\partial_z + \langle h_s, \partial_z\varphi(z)\rangle\right) = \sum_{k=0}^{N} W^{(k)}(z)(q\partial_z)^{N-k}\,. \tag{5.1.3}$$

The first few operators $W^{(k)}(z)$ obtained in this way are

$$W^{(0)}(z) = 1 \tag{5.1.4}$$

$$W^{(1)}(z) = \sum_{s=N}^{1}\langle h_s, \partial_z\varphi\rangle = 0 \tag{5.1.5}$$

$$\begin{aligned}
W^{(2)}(z) &= \sum_{s=N}^{1}(N-s)q\partial_z\langle h_s, \partial_z\varphi\rangle + \sum_{1\le s<t\le N}\langle h_s, \partial_z\varphi\rangle\langle h_t, \partial_z\varphi\rangle \\
&= \langle q\rho, \partial_z^2\varphi\rangle - \frac{1}{2}\langle\partial_z\varphi, \partial_z\varphi\rangle\,,
\end{aligned} \tag{5.1.6}$$

where $\rho = \sum_{s=N}^{1}(N-s)h_s$ is the Weyl vector of A_{N-1} (half the sum of the positive roots). Using the OPE (5.1.2) it is straightforward to check that $W^{(2)}(z)$ follows the standard OPE for a stress-energy tensor $T(z)$,

$$T(z)T(x) = \frac{c}{2(z-x)^4} + \frac{2T(x)}{(z-x)^2} + \frac{\partial_x T(x)}{z-x} + O(1) \tag{5.1.7}$$

with the central charge $c = (N-1)(1+N(N+1)q^2)$. Other OPEs $W^{(j)}(z)W^{(k)}(x)$ are more intricate but are computable (see [BS92] equation (6.52)).

It will be useful to know two terms in the OPE $T(z)W^{(k)}(x)$, namely those multiplying $1/(z-x)^2$ and $1/(z-x)$. To this end, evaluate the OPE of $T(z) = \langle q\rho, \partial_z^2\varphi\rangle - \frac{1}{2}\langle\partial_z\varphi, \partial_z\varphi\rangle$ and products $P(x)$ of $\langle\partial_x^{1+j}\varphi(x), h_s\rangle$ for $j \geq 0$ and $1 \leq s \leq N$, keeping only terms of order $1/(z-x)^2$ and $1/(z-x)$. Contracting $\partial_z^2\varphi(z)$ with such a product gives terms of order at least $1/(z-x)^3$, which we can drop. Double contractions of both $\partial_z\varphi$ give terms of order at least $1/(z-w)^4$, also ignored. For single contractions we can consider each factor $\langle\partial_x^{1+j}\varphi(x), h_s\rangle$ independently and get

$$-\langle\partial_z\varphi, \partial_z\overbrace{\varphi\rangle\langle\partial_x^{1+j}\varphi}(x), h_s\rangle = \frac{(j+1)!\langle\partial_z\varphi, h_s\rangle}{(z-x)^{j+2}} = \sum_{i\geq 0} \frac{(j+1)!}{i!}\frac{\langle\partial_x^{1+i}\varphi, h_s\rangle}{(z-x)^{j-i+2}} . \tag{5.1.8}$$

The term of order $1/(z-x)^2$ is $i=j$ hence $(j+1)\langle\partial_x^{1+j}\varphi, h_s\rangle$, while the term of order $1/(z-x)$ is $i=j+1$ hence $\partial_x\langle\partial_x^{1+j}\varphi, h_s\rangle$. More generally, the term of order $1/(z-x)^2$ is equal to $P(x)$ times the number of derivatives ∂_x in $P(x)$, while the term of order $1/(z-x)$ is equal to $\partial_x P(x)$. Counting derivatives in $W^{(k)}(x)$ is straightforward, and we deduce that

$$T(z)W^{(k)}(x) = \cdots + \frac{kW^{(k)}(x)}{(z-x)^2} + \frac{\partial_x W^{(k)}(x)}{z-x} + O(1) . \tag{5.1.9}$$

By adding to $W^{(k)}(z)$ some derivatives and products of the $W^{(j)}(z)$ with $j<k$ one can go from the Miura basis to a basis $\mathbf{W}^{(k)}(z)$ of operators with definite spins. The operators $\mathbf{W}^{(k)}(z)$ for $k \geq 3$ are primary with dimension k, namely

$$T(z)\mathbf{W}^{(k)}(w) = \frac{k\mathbf{W}^{(k)}(w)}{(z-w)^2} + \frac{\partial_w \mathbf{W}^{(k)}(w)}{z-w} + O(1) . \tag{5.1.10}$$

This simple OPE is balanced by the fact that other OPEs $\mathbf{W}^{(k)}(z)\mathbf{W}^{(j)}(w)$ are very complicated and not known in general. Furthermore, there seems to be no proof that such a basis of primary operators $\mathbf{W}^{(k)}(z)$ exists fo all N.

From an OPE $W^{(j)}(z)W^{(k)}(w)$ one can extract commutation relations for the Fourier modes defined by

$$W^{(k)}(z) = \sum_{n\in\mathbb{Z}} W_n^{(k)} z^{-n-k} \tag{5.1.11}$$

and similarly for the $\mathbf{W}$ basis. For instance, the OPE of $T(z)$ and $T(w)$ yields the Virasoro commutation relations $[L_n, L_m] = \frac{c}{12}(n^3-n)\delta_{n+m} + (n-m)L_{n+m}$. In principle this approach gives all commutation relations in the W_N algebra, for instance, the OPE (5.1.10) yields $[L_n, \mathbf{W}_m^{(k)}] = ((k-1)n-m)\mathbf{W}_{n+m}^{(k)}$ for $k \geq 3$. Similarly, the (partial) OPE (5.1.9) yields $[L_0, W_m^{(k)}] = -mW_m^{(k)}$ for $k \geq 2$. However, writing explicit commutators is only feasible for small N (see [Zam85] for $N=3$), as the OPEs are not all known in general.

5.1.2 Vertex operators

We exhibit a set of highest weight representations of W_N, namely representations for which the spectrum of L_0 (energy) is bounded below, and which have a unique[2] "highest weight" state $|\alpha\rangle$ of lowest energy. Since $[L_0, W_m^{(k)}] = -mW_m^{(k)}$, one has

$$W_n^{(k)}\,|\alpha\rangle = 0 \quad \text{for } n > 0 \tag{5.1.12}$$

$$W_0^{(k)}\,|\alpha\rangle = w^{(k)}(\alpha)\,|\alpha\rangle \tag{5.1.13}$$

for some quantum numbers $w^{(k)}(\alpha)$ which characterize the representation ($w^{(2)}(\alpha)$ is the dimension). This is equivalent to stating that the operator V_α corresponding to $|\alpha\rangle$ through the state-operator correspondence is a W_N primary, namely

$$W^{(k)}(z)V_\alpha(w) = \frac{w^{(k)}(\alpha)V_\alpha(w)}{(z-w)^k} + \frac{W_{-1}^{(k)}V_\alpha(w)}{(z-w)^{k-1}} + \cdots + \frac{W_{-k+1}^{(k)}V_\alpha(w)}{z-w} + O(1)\,, \tag{5.1.14}$$

where $W_{-j}^{(k)}V_\alpha$ denotes the descendant of V_α obtained by acting with the W_N algebra generator.

We construct such W_N primary operators in the free-field formalism as vertex operators $V_\alpha(z) = {:}e^{\langle\alpha,\varphi(z,\bar z)\rangle}{:}$, labelled by an $(N-1)$-component momentum $\alpha = \sum_{s=1}^N \alpha_s h_s$. The OPE

$$\partial\varphi(z)V_\alpha(w) = {:}\Big(\partial\varphi(z) - \frac{\alpha}{z-w}\Big)V_\alpha(w){:} \tag{5.1.15}$$

lets us in principle calculate the OPE of every $W^{(k)}(z)$ with $V_\alpha(w)$. Indeed,

$$\sum_{k=0}^{N} W^{(k)}(z)V_\alpha(w)(q\partial_z)^{N-k} = {:}\prod_{s=N}^{1}\big(q\partial_z + \langle h_s, \partial_z\varphi(z)\rangle\big){:}V_\alpha(w) \tag{5.1.16}$$

$$= {:}\prod_{s=N}^{1}\left(q\partial_z + \left\langle h_s, \partial_z\varphi(z) - \frac{\alpha}{z-w}\right\rangle\right)V_\alpha(w){:}\,. \tag{5.1.17}$$

For each k the most singular term is of order $1/(z-w)^k$ and we reproduce the OPE (5.1.14) with

$$w^{(k)}(\alpha) = (-1)^k \sum_{1\le s_1<\cdots<s_k\le N}\prod_{j=1}^{k}\big(\langle\alpha, h_{s_j}\rangle + (j-1)q\big)\,. \tag{5.1.18}$$

[2]It is possible to show that zero-modes commute $[W_0^{(j)}, W_0^{(k)}] = 0$ hence any representation of W_N with L_0 bounded below decomposes into such highest weight representations. We ignore convergence issues entirely.

Therefore, V_α is a W_N primary as announced, and generates a highest weight state (5.1.12) when acting on the vacuum. The first few quantum numbers are $w^{(0)} = 1$ and $w^{(1)} = \sum_s \langle \alpha, h_s \rangle = 0$, and the conformal dimension is

$$\Delta(\alpha) = w^{(2)}(\alpha) = \sum_{1 \leq s < t \leq N} \langle \alpha, h_s \rangle (\langle \alpha, h_t \rangle + q) \tag{5.1.19}$$

$$= -\frac{1}{2} \sum_s \langle \alpha, h_s \rangle^2 + q \langle \alpha, \rho \rangle = \frac{1}{2} \langle \alpha, 2Q - \alpha \rangle \,. \tag{5.1.20}$$

Interestingly, the $w^{(k)}$ are invariant under (shifted) Weyl symmetries, namely permutations of the components $\langle \alpha - Q, h_s \rangle$ of $\alpha - Q$. To prove this, focus on one transposition $\langle \alpha - Q, h_t \rangle \leftrightarrow \langle \alpha - Q, h_{t+1} \rangle$, and denote by α' the resulting momentum. We claim that the term labelled by any set $S = \{s_1 < \cdots < s_k\} \subseteq \{1, \ldots, N\}$ in $w^{(k)}(\alpha)$ is present in $w^{(k)}(\alpha')$ as the term labelled by the set $S_{t \leftrightarrow t+1}$. If neither t nor $t+1$ appear among the s_j then $S_{t \leftrightarrow t+1} = S$ and the terms are obviously equal. If only one of them appears, say $t = s_j$ for some j, then the matching term in $w^{(k)}(\alpha')$ is the one labelled by indices $s_1 < \ldots < s_k$ with s_j replaced by $t+1$ (if initially $t+1 = s_j$ then replace $s_j \to t$ instead). Finally, if both appear, they must be contiguous, so $t = s_j$ and $t+1 = s_{j+1}$. The relevant factors in the product are

$$\langle \alpha, h_t \rangle + (j-1)q = \langle \alpha - Q, h_t \rangle + \big((N-1)/2 + j - t\big)q \tag{5.1.21}$$

$$\langle \alpha, h_{t+1} \rangle + jq = \langle \alpha - Q, h_{t+1} \rangle + \big((N-1)/2 + j - t\big)q \,, \tag{5.1.22}$$

and are exchanged by $\langle \alpha - Q, h_t \rangle \leftrightarrow \langle \alpha - Q, h_{t+1} \rangle$, hence the term in $w^{(k)}(\alpha')$ with the same choice of $s_1 < \cdots < s_k$ is equal to that in $w^{(k)}(\alpha)$. As a result, primary operators with momenta related by (shifted) Weyl symmetries have the same values $w^{(k)}(\alpha)$. All operators $W_n^{(k)}$ with $n \geq 0$ act in the same way on the states $|\alpha\rangle$ and $|\alpha'\rangle$, hence the highest weight representations that they generate are isomorphic. In fact, Weyl symmetries are the only redundancies in the description of the $w^{(k)}$ in terms of α. Indeed, requiring that $w^{(k)}(\alpha)$ takes a prescribed value gives a degree k polynomial equation on components of α. This generically lets us express one component as one of k different roots in terms of the other components, and iterating gives $2 \cdots N = N!$ solutions, exactly the number provided by Weyl symmetries.

All in all, highest weigh representations of W_N are precisely labelled by a momentum α up to Weyl symmetries. As a slight abuse of notation, we will use V_α to denote both the free-field vertex operator specifically and more generally any W_N primary operator which generates a representation isomorphic to it.

Another interesting transformation is conjugation of momenta (we also call it Toda CFT conjugation), the $\mathbb{C}$-linear map

$$h_s \to (h_s)^C = -h_{N+1-s} \,. \tag{5.1.23}$$

Conjugation maps the highest weight of a representation to the highest weight of the conjugate representation, hence its name. The Killing form is invariant since $\langle \alpha^C, \beta^C \rangle = \langle \alpha, \beta \rangle$, and $\rho^C = \rho$ thus $Q^C = Q$. The numbers $w^{(k)}$ transform non-trivially, but it turns out that their analogues $\mathbf{w}^{(k)}$ obtained in the basis $\mathbf{W}^{(k)}$ of primary operators transform as $\mathbf{w}^{(k)}(\alpha^C) = (-1)^k \mathbf{w}^{(k)}(\alpha)$. Note also that $\alpha \to 2Q - \alpha^C$ is a Weyl symmetry since $\langle 2Q - \alpha^C - Q, h_s \rangle = \langle Q - \alpha, h_s^C \rangle = \langle \alpha - Q, h_{N+1-s} \rangle$.

5.1.3 Screening charges give null descendants

Screening charges are operators which commute with the symmetry algebra W_N.

In the free field realization, some screening charges can be constructed from vertex operators V_α with $\alpha = b^{\pm 1} e_j$, where e_j is a simple root and b is defined by $q = b + 1/b$. These vertex operators have dimension (5.1.19)

$$\Delta(b^{\pm 1} e_j) = \frac{1}{2} \langle b^{\pm 1} e_j, 2Q - b^{\pm 1} e_j \rangle = 1 \,, \tag{5.1.24}$$

and a straightforward computation based on the Miura transformation shows that the OPEs $W^{(k)}(z) V_{b^{\pm 1} e_j}(w)$ are total derivatives. This implies that any integral $Q_j^\pm = \oint \mathrm{d}z\, V_{b^{\pm 1} e_j}(z)$ on a closed contour has a trivial OPE with the $W^{(k)}(z)$, and is a screening charge.

We now use screening charges to detect null vectors, namely descendants of primary operators which are themselves primary. This relies on expressing the OPE of a screening charge $Q_j^\pm$ with a vertex operator V_β as a combination of descendants of some other vertex operator $V_{\beta'}$. Since $Q_j^\pm$ commutes with the W_N algebra generators, the resulting descendant transforms exactly like V_β, hence is itself a primary. The OPE (5.1.2) implies

$$V_\alpha(z) V_\beta(0) = |z|^{-2\langle \alpha, \beta \rangle} {:}e^{\langle \alpha, \varphi(z,\bar{z}) \rangle} e^{\langle \beta, \varphi(0) \rangle}{:} = |z|^{-2\langle \alpha, \beta \rangle} \sum_{m,n \geq 0} z^m \bar{z}^n [V_{\alpha+\beta}(0)] \,, \tag{5.1.25}$$

where the brackets denote descendants of the vertex operator $V_{\alpha+\beta}$. Replacing (α, β) by $(b^{\pm 1} e_j, 2Q - \alpha^C - b^{\pm 1} e_j)$ yields

$$Q_j^\pm V_{2Q - \alpha^C - b^{\pm 1} e_j} = \oint \mathrm{d}z |z|^{-2\langle b^{\pm 1} e_j, 2Q - \alpha^C - b^{\pm 1} e_j \rangle} \sum_{m,n \geq 0} z^m \bar{z}^n [V_{2Q - \alpha^C}] \,. \tag{5.1.26}$$

When the exponent of $|z|^{-2}$ is an integer we can integrate z along a circle around 0, getting a non-zero result if the exponent

$$\langle b^{\pm 1} e_j, 2Q - \alpha^C - b^{\pm 1} e_j \rangle = 2(1 + b^{\pm 2}) - \langle b^{\pm 1} e_j, \alpha^C \rangle - 2b^{\pm 2} = 2 - b^{\pm 1} \langle \alpha^C, e_j \rangle \tag{5.1.27}$$

is positive. The result is a descendant of $V_{2Q - \alpha^C}$ but also a primary operator with quantum numbers equal to those of $V_{2Q - \alpha^C - b^{\pm 1} e_j}$: this is the definition of a null-vector. One subtlety is that if the exponent (5.1.27) is 1 then the integral picks up the $m = n = 0$ term in (5.1.26), hence exactly $V_{2Q-\alpha^C}$ and not a strict descendant. For all other values of the exponent, we obtain a genuine null-vector. Therefore, whenever $b^{\pm 1}\langle \alpha^C, e_j\rangle = -n$ for some integer $n \geq 0$, some sign $\pm$, and some j, the representation of W_N generated by $V_{2Q-\alpha^C}$ contains a null-vector. Comparing dimensions yields that the null-vector is at level $1 + n$.

This claim can be simplified a bit: the set of simple roots is invariant under conjugation, and we can replace $V_{2Q-\alpha^C}$ by any primary with the same quantum numbers, such as V_α. We deduce that V_α (or rather any primary operator with the same quantum numbers) has a null-vector descendant whenever any $\langle \alpha, e_j\rangle = -nb^{\pm 1}$ with $n \geq 0$. Using multiple screening charges, it is possible to show that V_α also has a null-vector descendant when $\langle \alpha, e_j\rangle = -n_1 b - n_2/b$ for $n_1, n_2 \geq 0$. Considering special cases leads to a conjecture: V_α has one descendant of level $n_1 n_2$ for each positive root $e = h_i - h_j$ $(i < j)$ such that $\langle \alpha - Q, e\rangle = -n_1 b - n_2/b$ with integers $n_1, n_2 > 0$.

A momentum (and representation of W_N) for which all $\langle \alpha, e_j\rangle \in -b\mathbb{Z}_{\geq 0} - b^{-1}\mathbb{Z}_{\geq 0}$ is called degenerate (or fully degenerate, for emphasis), and it takes the form $\alpha = -b\omega - \omega'/b$ for some dominant weights ω and ω' (highest weights of some representations of A_{N-1}).

A semi-degenerate momentum $\alpha = \varkappa h_1$ or $-\varkappa h_N$ is such that all $\langle \alpha, e_j\rangle$ vanish except a single one which is not constrained. The remarks below (5.1.27) show that this representation of W_N has $N - 2$ null vectors at level 1, and it turns out that it has $\frac{1}{2}(N-1)(N-2)$ independent null vectors. The momentum $-\varkappa h_N$ is in fact mapped by the Weyl reflexion defined by the permutation $(1\,2 \cdots N)$ to

$$\varkappa^D h_1 = \left[N\left(b + \frac{1}{b}\right) - \varkappa\right] h_1\,, \tag{5.1.28}$$

thus without loss of generality semi-degenerate momenta are $\varkappa h_1$.

Finally, a generic momentum is a momentum such that the representation of W_N contains no null vector; we will often parametrize such momenta as $\alpha = Q + \mathrm{i}a$, to make Weyl symmetry more manifest. One could distinguish many more types of momenta depending on which $\langle \alpha, e_j\rangle \in -b\mathbb{Z}_{\geq 0} - b^{-1}\mathbb{Z}_{\geq 0}$, but that is not useful in this work.

Another set of words to describe the same thing is that a "full puncture" is a place where a generic primary operator is inserted, a "simple puncture" corresponds to a semi-degenerate momentum, and a "degenerate puncture" to a degenerate momentum.

5.1.4 Two-point function

We determine in this section which two-point functions of W_N primaries may be non-zero. In 2d CFT the Virasoro symmetry imposes that two-point functions of primary operators can only be non-zero if the two dimensions coincide. In a W_N-invariant theory, we show that symmetry restricts possible non-zero two-point functions further, fixing all quantum numbers as described in (5.1.34). This is proven as follows (see [FL07] for $N = 3$), assuming that W_N admits a basis of currents $\mathbf{W}^{(k)}(z)$ which are primary for $k \geq 3$.

Under the conformal transformation $x = 1/z$ the primary operator $\mathbf{W}^{(k)}(x)$ maps to $(\partial z/\partial x)^k \mathbf{W}^{(k)}(z) = (-1)^k z^{2k} \mathbf{W}^{(k)}(z)$. Regularity at $x = 0$ requires $\mathbf{W}^{(k)}(z) = O(z^{-2k})$ at $z \to \infty$. Now the OPE (5.1.14) between $W^{(k)}$ and V_α, or rather its analogue in the basis $\mathbf{W}^{(k)}$, leads to the Ward identity

$$\begin{aligned}
&\langle \mathbf{W}^{(k)}(z) V_\alpha(x) V_\beta(0)\rangle \\
&= \frac{\mathbf{w}^{(k)}(\alpha)\langle V_\alpha(x)V_\beta(0)\rangle}{(z-x)^k} + \sum_{j=1}^{k-1} \frac{\langle \mathbf{W}^{(k)}_{-j} V_\alpha(x)\, V_\beta(0)\rangle}{(z-x)^{k-j}} \\
&\quad + \frac{\mathbf{w}^{(k)}(\beta)\langle V_\alpha(x)V_\beta(0)\rangle}{z^k} + \sum_{j=1}^{k-1} \frac{\langle V_\alpha(x)\, \mathbf{W}^{(k)}_{-j} V_\beta(0)\rangle}{z^{k-j}} \,.
\end{aligned} \tag{5.1.29}$$

Expanding in powers of z as $z \to \infty$ and setting the coefficients of $z^{-1}, \ldots, z^{-(2k-1)}$ to zero, we obtain $2k-1$ linear equations for the $2k-1$ two-point functions $\langle V_\alpha(x)V_\beta(0)\rangle$, $\langle \mathbf{W}^{(k)}_{-j} V_\alpha(x)\, V_\beta(0)\rangle$ and $\langle V_\alpha(x)\, \mathbf{W}^{(k)}_{-j} V_\beta(0)\rangle$ for $1 \leq j < k$. This has a non-zero solution, in other words, the two-point function $\langle V_\alpha V_\beta\rangle$ can be non-zero, only if the determinant of the $(2k-1)\times(2k-1)$ matrix vanishes for all x (see [FL07] equation (2.7) for $N = 3$).

In detail, the first $k-1$ equations (coefficients of z^{-1} to $z^{-(k-1)}$) express each $\langle V_\alpha(x)\, \mathbf{W}^{(k)}_{-j} V_\beta(0)\rangle$ in terms of other two-point functions, so we can focus on the other k equations. The $z^{-(k+j)}$ equation with $0 < j < k$ is

$$0 = \mathbf{w}^{(k)}(\alpha)\langle V_\alpha(x)V_\beta(0)\rangle \binom{k+j-1}{j} + \sum_{i=1}^{k-1} \langle \mathbf{W}^{(k)}_{-i} V_\alpha(x)\, V_\beta(0)\rangle x^i \binom{k+j-1}{j+i} \tag{5.1.30}$$

while the one for $j = 0$ has an additional term depending on $\mathbf{w}^{(k)}(\beta)$:

$$0 = \left[\mathbf{w}^{(k)}(\alpha) + \mathbf{w}^{(k)}(\beta)\right] \langle V_\alpha(x)V_\beta(0)\rangle + \sum_{i=1}^{k-1} \langle \mathbf{W}^{(k)}_{-i} V_\alpha(x)\, V_\beta(0)\rangle x^i \binom{k-1}{i} . \tag{5.1.31}$$

Summing $(-1)^j \binom{k-1}{j}$ times these equations for $0 \leq j \leq k-1$ yields

$$\begin{aligned}
&- \mathbf{w}^{(k)}(\beta)\langle V_\alpha(x)V_\beta(0)\rangle \\
&= \sum_{0\leq j<k} (-1)^j \binom{k-1}{j} \sum_{i=0}^{k-1} \langle \mathbf{W}^{(k)}_{-i} V_\alpha(x)\, V_\beta(0)\rangle x^i \binom{k+j-1}{j+i} \\
&= (-1)^{k-1} \mathbf{w}^{(k)}(\alpha)\langle V_\alpha(x)V_\beta(0)\rangle \,.
\end{aligned} \tag{5.1.32}$$

The second equality here is due to binomial identities found by extracting the coefficient of X^i, $0 < i < k$ in

$$\begin{aligned}
&\sum_{0\leq j<k}\left[(-1)^j\binom{k-1}{j}\sum_{-j\leq i<k}\binom{k+j-1}{j+i}X^i\right] \\
&\quad = \sum_{0\leq j<k}\left[(-1)^j\binom{k-1}{j}(1+X)^{k+j-1}/X^j\right] \\
&\quad = (1+X)^{k-1}(1-(1+X)/X)^{k-1} = (-1-1/X)^{k-1} \,.
\end{aligned} \tag{5.1.33}$$

From (5.1.32) we deduce that $\langle V_\alpha V_\beta\rangle = 0$ unless

$$\mathbf{w}^{(k)}(\beta) = (-1)^k \mathbf{w}^{(k)}(\alpha) \quad \text{for all } 2 \leq k \leq N. \tag{5.1.34}$$

This generalizes the usual constraint that $\Delta(\beta) = \Delta(\alpha)$ due to Virasoro symmetry.

We are not yet done, because explicit expressions of $\mathbf{w}^{(k)}(\alpha)$ are not known. We will show in Section 5.4.2 that the two-point functions $\langle V_{2Q-\alpha}V_\alpha\rangle$ are non-zero in the Toda CFT, hence $\beta = 2Q - \alpha$ must obey (5.1.34). The constraint is thus equivalent to $\mathbf{w}^{(k)}(\beta) = \mathbf{w}^{(k)}(2Q-\alpha)$ for all $2 \leq k \leq N$. This in turn means that the representations of W_N generated by the highest-weight states $|\beta\rangle$ and $|2Q-\alpha\rangle$ are isomorphic, hence β and $2Q-\alpha$ are equal up to Weyl symmetries. All in all, the only non-zero two-point functions are $\langle V_{2Q-\alpha}V_\alpha\rangle$ up to Weyl symmetry.

5.1.5 Some fusion rules

We state here some fusion rules involving degenerate W_N pimary operators. Unfortunately, we were unable to find proofs that hold for general N using only W_N symmetry. We perform a few checks in Section 5.4.2 for the Toda CFT using the Coulomb gas formalism.

When a Virasoro primary operator is degenerate (has a null-vector descendant), correlation functions involving this primary obey a differential equation. On the other hand, three-point functions of Virasoro primaries depend on the position of the operators in a way that is fixed by conformal dimensions. Plugging the three-point function dependence into the null-descendant differential equation then yields a constraint on the dimensions of the three operators, which translates to a linear relation on the momenta. This linear equation has a natural generalization to W_N, which we state without proof:

$$\begin{aligned}
&\langle V_\alpha V_\beta V_{-b\omega-\omega'/b}\rangle \neq 0 \\
&\quad\Longrightarrow\ \alpha+\beta = 2Q + bh + h'/b \text{ for some } h \in \mathcal{R}(\omega) \text{ and } h' \in \mathcal{R}(\omega'),
\end{aligned} \tag{5.1.35}$$

up to a Weyl symmetry of β (or α). Here, ω and ω' are the highest weights of the representations $\mathcal{R}(\omega)$ and $\mathcal{R}(\omega')$ of A_{N-1}, and the possible values for $\alpha+\beta$ are characterized by pairs (h,h') of weights of these representations. In the Coulomb gas formalism applicable to the Toda CFT, the condition on $\alpha+\beta$ arises in part as a screening condition on the sum of all momenta: $\alpha+\beta-b\omega-\omega'/b = 2Q-\sum_{j=1}^{N-1}(m_j b+n_j/b)e_j$ for integers $m_j, n_j \geq 0$, where we recognize $b^{\pm 1}e_j$ as the momenta used in the construction of screening charges (see Section 5.4.2).

The constraint (5.1.35) translates to a fusion rule, namely a restriction on which conformal families may appear in the OPE of $V_{-b\omega-\omega'/b}$ with V_α. Indeed, if the OPE contains a term $V_{\alpha'}$ then the three-point function (5.1.35) with $\beta = 2Q-\alpha'$ is non-zero hence $\alpha+2Q-\alpha' = 2Q+bh+h'/b$, that is, $\alpha' = \alpha-bh-h'/b$. Thus, the OPE of a degenerate and an arbitrary primary operators may only contain a finite set of conformal families (brackets denote contributions from descendants)

$$V_{-b\omega-\omega'/b}\times V_\alpha = \sum_{h\in\mathcal{R}(\omega)}\sum_{h'\in\mathcal{R}(\omega')}[V_{\alpha-bh-h'/b}]\,. \qquad (5.1.36)$$

In the simplest non-trivial case of a degenerate operator labelled by the fundamental representation ($\omega = h_1$, $\omega' = 0$) this sum has N terms,

$$V_{-bh_1}\times V_\alpha = \sum_{s=1}^{N}[V_{\alpha-bh_s}]\,. \qquad (5.1.37)$$

This last fusion rule can be derived for $N = 3$ by writing down a differential equation for the three-point function $\langle V_{-bh_1}V_\alpha V_\beta\rangle$ using explicit null-vectors [FL07], but that is unfeasible for $N > 3$. It would be interesting to bypass the need for null-vectors; note that the Coulomb gas formalism is not a solution, as it relies on the specific Lagrangian of the Toda CFT rather than only on symmetry considerations.

When the arbitrary momentum α is replaced by a semi-degenerate momentum $\varkappa h_1$, the null-vectors of $\varkappa h_1$ restrict the fusion rule (5.1.37) further to

$$V_{-bh_1}\times V_{\varkappa h_1} = [V_{\varkappa h_1-bh_1}]+[V_{\varkappa h_1-bh_2}]\,. \qquad (5.1.38)$$

This can be shown for $N = 3$ using explicit differential equations as for the case of a generic momentum α. We will show it in the Toda CFT by checking that the structure constants associated to the other momenta $\varkappa h_1 - bh_s$ for $s\geq 3$ vanish. This fusion rule is generalized in Section 5.5.2 to a proposal for the fusion of $V_{-b\omega-\omega'/b}$ with $V_{\varkappa h_1}$.

Another special case of (5.1.36) is the fusion of two degenerate operators. As observed in the study of W_N minimal models [FL88], it parallels the Clebsch-Gordan decomposition of tensor products of representations of A_{N-1}

(more precisely of $A_{N-1} \times A_{N-1}$):

$$V_{-b\Omega_1-\Omega_2/b} \times V_{-b\Omega'_1-\Omega'_2/b} = \sum_{\mathcal{R}(\Omega''_i)\subseteq\mathcal{R}(\Omega_i)\otimes\mathcal{R}(\Omega'_i)} [V_{-b\Omega''_1-\Omega''_2/b}] . \qquad (5.1.39)$$

Once more, this fusion rule extends the case of Virasoro degenerate primaries (for $N = 2$).

Let us conclude this section by noting that the three-point function $\langle V_{\varkappa h_1} V_\alpha V_\beta \rangle$ of one semi-degenerate and two arbitrary primary operators determines all three-point functions of their descendants. The position-dependence of the correlator of primaries is known and determines how the Virasoro generators $L_n = W^{(2)}_n$ act. We focus on the $3(N+1)(N-2)/2$ correlators $\langle W^{(k)}_{-j} V_{\varkappa h_1} V_\alpha V_\beta \rangle$ with $1 \leq j \leq k-1$ and $3 \leq k \leq N$, and where $W^{(k)}_{-j}$ acts on any of the three primary operators. Following the same idea as for the two-point function in Section 5.1.4, we can write a Ward identity for $\mathbf{W}^{(k)}$ and use the asymptotics $\mathbf{W}^{(k)}(z) = O(z^{-2k})$ as $z \to \infty$ to obtain $2k-1$ linear equations concerning these correlators. The $\frac{1}{2}(N-1)(N-2)$ independent null vectors of $V_{\varkappa h_1}$ each imply one additional linear equation. All in all, we have

$$\sum_{k=3}^{N} (2k-1) + \frac{1}{2}(N-1)(N-2) = \frac{3}{2}(N+1)(N-2) \qquad (5.1.40)$$

linear equations for these correlators and for $\langle V_{\varkappa h_1} V_\alpha V_\beta \rangle$. Generically, the system can be solved, and the correlators $\langle W^{(k)}_{-j} V_{\varkappa h_1} V_\alpha V_\beta \rangle$ are equal to some number depending on momenta, on k, and j, times the three-point function of primaries. The procedure can be repeated to yield all correlators of descendants. In a full W_N symmetric 2d CFT, the symmetry algebra is actually the product of a left-moving W_N algebra with generators $W^{(k)}_n$ and a right-moving W_N algebra with generators $\overline{W}^{(k)}_n$. In this discussion, the left-moving and right-moving sectors are independent, hence the three-point function of arbitrary descendants (obtained by acting with W and $\overline{W}$ generators) factorizes as the product of $\langle V_{\varkappa h_1} V_\alpha V_\beta \rangle$, a factor depending on the W generators, and a factor depending on the $\overline{W}$ generators. The factorization will play an important (technical) role when determining some braiding matrices in the next section.

5.2 Braiding matrices

This section and the next are devoted to determining some braiding matrices and braiding kernels of W_N primaries.

We first consider four-point conformal blocks of one degenerate primary operator with momentum $-bh_1$, one semi-degenerate, and two generic primary operators (Section 5.2.1). As found by Fateev and Litvinov [FL07], this is a hypergeometric function; our approach, based on monodromies, does not rely on the differential equation which can only be derived explicitly for $N = 3$. We deduce the braiding matrix relating s-channel and u-channel conformal blocks.

We then repeat the analysis (in Section 5.2.2) with a degenerate momentum $-b\omega_K$ labelled by an antisymmetric representation $\mathcal{R}(\omega_K)$ of A_{N-1} instead of the fundamental representation $\mathcal{R}(h_1)$. We obtain the relevant braiding matrices as a combination of K copies of the braiding matrices from the present section, and deduce explicit expressions for the conformal blocks. More precisely, we confirm that the vortex partition functions of $\mathcal{N} = (2, 2)$ SQCD with gauge group $U(K)$ and N matter fields are equal to conformal blocks (up to simple factors) by computing braiding matrices for the two sets of objects. This relation fits into the AGT correspondence and is explored further in [GLF14] (Chapter 3).

5.2.1 Semi-degenerate and fundamental degenerate

We focus here on $\langle VVVV\rangle = \langle V_{\alpha_\infty}(\infty)V_{(\varkappa+b)h_1}(1)V_{-bh_1}(x,\bar{x})V_{\alpha_0}(0)\rangle$, the four-point correlation function of two generic momenta α_∞ and α_0, one semi-degenerate $(\varkappa + b)h_1$ and a degenerate $-bh_1$, labelled by the highest weight h_1 of the fundamental representation of A_{N-1}. The shift by b in $\varkappa$ simplifies some expressions. This correlator can be expressed in terms of hypergeometric series. This was initially obtained by solving the null-vector differential equations for $N = 3$ and writing the natural generalization of these results [FL07]. Here, we bypass the differential equation (null-vectors are not tractable for $N > 3$) and use monodromy properties of conformal blocks. We rely on the fusion rules (5.1.37) and (5.1.38)

$$V_{-bh_1} \times V_\alpha = \sum_{s=1}^{N} [V_{\alpha-bh_s}] \tag{5.2.1}$$

$$V_{-bh_1} \times V_{(\varkappa+b)h_1} = [V_{\varkappa h_1}] + [V_{(\varkappa+b)h_1-bh_2}] \,, \tag{5.2.2}$$

where each $V_{\dots}$ stands for any W_N primary operator with the same quantum numbers, and brackets denote contributions from descendants.

As usual in 2d CFT, the four-point function can be expanded in three different channels by inserting a complete set of states separating two pairs of primary operators. Let us first focus on the s-channel, where we pair V_{-bh_1} with V_{α_0}. The sum over states can be grouped according to the underlying primary operator V^i_α (the index i distinguishes primary operators with the same momentum α), and which combinations W_{-I} and $\overline{W}_{-\bar{I}}$ of left-moving and right-moving generators of W_N act on V^i_α to generate a given descendant:

$$\langle VVVV\rangle = \sum_{\alpha,i,j} \sum_{W_{-I},\overline{W}_{-\bar{I}}} \sum_{W_{-J},\overline{W}_{-\bar{J}}} \begin{aligned}[t] &\langle V_{\alpha_\infty} V_{(\varkappa+b)h_1} W_{-I}\overline{W}_{-\bar{I}} V^i_\alpha\rangle \\ &(K^{-1}_\alpha)_{iI\bar{I},jJ\bar{J}} \langle W_{-J}\overline{W}_{-\bar{J}} V^j_{2Q-\alpha} V_{-bh_1} V_{\alpha_0}\rangle \,. \end{aligned} \tag{5.2.3}$$

This formula involves the inverse of the matrix K_α of two-point functions of descendants $W_{-I}\overline{W}_{-\bar{I}}V_\alpha^i$ with descendants $W_{-J}\overline{W}_{-\bar{J}}V_{2Q-\alpha}^j$. As we saw at the end of Section 5.1.5, the three-point functions in (5.2.3) are uniquely determined in terms of the three-point functions of primaries, thanks to the presence of a semi-degenerate (or a fully degenerate) operator in each of them. Furthermore, the factors due to W_{-I} and to $\overline{W}_{-\bar{I}}$ factorize, and do not depend on the multiplicity indices i, j. Thus, the four-point function takes the form

$$\langle VVVV\rangle = \sum_\alpha \sum_{i,j} \Big[\langle V_{\alpha_\infty} V_{(\varkappa+b)h_1} V_\alpha^i\rangle (K_\alpha^{-1})_{i,j} \langle V_{2Q-\alpha}^j V_{-bh_1} V_{\alpha_0}\rangle \\ \mathbb{F}(\alpha_\infty, (\varkappa+b)h_1, \alpha, -bh_1, \alpha_0|x,\bar{x})\Big] \tag{5.2.4}$$

where the conformal blocks $\mathbb{F}$ factorize holomorphically as $\mathcal{F}(x)\,\mathcal{F}(\bar{x})$ and do not depend on the multiplicity indices i, j. Both of these properties would fail in the absence of the semi-degenerate operator, as the contribution from descendants may then depend on the multiplicity indices i, j and the coordinate dependence is then a sum of holomorphically factorized combinations $\mathcal{F}(x)\,\mathcal{F}(\bar{x})$ for each i, j. One last property of $\mathbb{F}$ is its $x, \bar{x} \to 0$ expansion

$$\mathbb{F}(x,\bar{x}) = |x|^{2[\Delta(\alpha)-\Delta(\alpha_0)-\Delta(-bh_1)]}(1+\cdots) \tag{5.2.5}$$

where $(1+\cdots)$ is a series in non-negative integer powers of x and $\bar{x}$.

The fusion rule (5.1.37) restricts the internal momentum α to be $\alpha_0 - bh_s$ for $1 \le s \le N$. Hence,

$$\langle VVVV\rangle = \sum_{p=1}^N C_p^{(\mathrm{s})}\,\mathcal{F}_p^{(\mathrm{s})}(x)\,\mathcal{F}_p^{(\mathrm{s})}(\bar{x}) \tag{5.2.6}$$

for some constants $C_p^{(\mathrm{s})}$ and holomorphic/antiholomorphic functions $\mathcal{F}_p^{(\mathrm{s})}$ with the $x \to 0$ expansion

$$\begin{aligned}\mathcal{F}_p^{(\mathrm{s})}(x) &= x^{\Delta(\alpha_0-bh_p)-\Delta(\alpha_0)-\Delta(-bh_1)}(1+\cdots) \\ &= x^{b\langle\alpha_0-Q,h_p\rangle+\frac{N-1}{2}(b^2+1)}(1+\cdots)\end{aligned} \tag{5.2.7}$$

where $1+\cdots$ is a series in non-negative integer powers of x, and similarly for $\mathcal{F}_p^{(\mathrm{s})}(\bar{x})$. The superscript $^{(\mathrm{s})}$ indicates that this expansion comes from the s-channel decomposition of the four-point function. Because of radial ordering, the functions $\mathcal{F}_p^{(\mathrm{s})}$ are a priori only defined on the unit disc (with a branch point at 0), but since $\langle VVVV\rangle$ is smooth away from 0, 1, and ∞ the functions can be analytically continued to any simply connected domain avoiding these points. Two natural choices that we will use at times are the complex plane minus cuts on $(-\infty,0]\cup[1,\infty)$, and the complex plane minus cuts on $[0,1]\cup[1,\infty)$.

From the u-channel decomposition we get

$$\langle VVVV\rangle = \sum_{p=1}^{N} C_p^{(\mathrm{u})}\, \mathcal{F}_p^{(\mathrm{u})}(x)\, \mathcal{F}_p^{(\mathrm{u})}(\bar{x}) \tag{5.2.8}$$

with an $x \to \infty$ expansion in terms of a series $1 + \cdots$ with non-negative integer powers of $1/x$:

$$\begin{aligned} \mathcal{F}_p^{(\mathrm{u})}(x) &= x^{\Delta(\alpha_\infty)-\Delta(\alpha_\infty - bh_p)-\Delta(-bh_1)}(1+\cdots) \\ &= x^{-b\langle \alpha_\infty - Q, h_p\rangle + \frac{N-1}{2}(b^2+1)+\frac{N-1}{N}b^2}(1+\cdots)\,. \end{aligned} \tag{5.2.9}$$

Again, $\mathcal{F}_p^{(\mathrm{u})}(x)$ can be extended to the complex plane minus some cuts, for instance along $[0,1]\cup[1,\infty)$.

The t-channel is more subtle, as it involves three-point functions of V_{α_∞}, V_{α_0}, and a descendant of the primary $V_{\varkappa h_1}$ or $V_{(\varkappa+b)h_1 - bh_2}$. Contributions from primaries with momentum $\varkappa h_1$ factorize and do not depend on the multiplicity index because this momentum is semidegenerate. Contributions from primaries with momentum $(\varkappa+b)h_1 - bh_2$ need not factorize (and for $N>2$ they do not). We deduce that

$$\langle VVVV\rangle = C_1^{(\mathrm{t})}\, \mathcal{F}_1^{(\mathrm{t})}(x)\, \mathcal{F}_1^{(\mathrm{t})}(\bar{x}) + C_2^{(\mathrm{t})} \mathbb{F}_2^{(\mathrm{t})}(x,\bar{x}) \tag{5.2.10}$$

with the following $x \to 1$ expansions, where $1+\cdots$ denote series in non-negative integer powers of $(1-x)$ and $(1-\bar{x})$,

$$\begin{aligned} \mathcal{F}_1^{(\mathrm{t})}(x)\, \mathcal{F}_1^{(\mathrm{t})}(\bar{x}) &= |1-x|^{2[\Delta(\varkappa h_1)-\Delta((\varkappa+b)h_1)-\Delta(-bh_1)]}(1+\cdots) \\ &= |1-x|^{2b(\varkappa+b)(N-1)/N}(1+\cdots) \end{aligned} \tag{5.2.11}$$

$$\begin{aligned} \mathbb{F}_2^{(\mathrm{t})}(x,\bar{x}) &= |1-x|^{2[\Delta((\varkappa+b)h_1 - bh_2)-\Delta((\varkappa+b)h_1)-\Delta(-bh_1)]}(1+\cdots) \\ &= |1-x|^{2[-b(\varkappa+b)/N + b^2+1]}(1+\cdots)\,. \end{aligned} \tag{5.2.12}$$

Away from the cuts, the equality $\sum_p C_p^{(\mathrm{s})}\left|\mathcal{F}_p^{(\mathrm{s})}\right|^2 = \sum_p C_p^{(\mathrm{u})}\left|\mathcal{F}_p^{(\mathrm{u})}\right|^2$ implies that $\mathcal{F}_p^{(\mathrm{u})}$ is a linear combination of the $\mathcal{F}_s^{(\mathrm{s})}$. Similarly, $\mathcal{F}_1^{(\mathrm{t})}$ is a linear combination of the $\mathcal{F}_s^{(\mathrm{s})}$, and the non-factorized function $\mathbb{F}_2^{(\mathrm{t})}$ is (non-canonically) a sum of $N-1$ terms which are factorized. The expansions of $\mathcal{F}^{(\mathrm{s})}$, $\mathcal{F}^{(\mathrm{u})}$ and $\mathbb{F}^{(\mathrm{t})}$ imply certain monodromy properties when analytically continuing the functions through cuts. The monodromy $M_{(0)}$ around $x=0$ is diagonal in the basis $\mathcal{F}^{(\mathrm{s})}$ and eigenvalues can be read off from the expansion (5.2.7). We can read off eigenvalues of the monodromy $M_{(1)}$ around $x=1$ from (5.2.11) and (5.2.12), the latter having multiplicity $N-1$. Finally, the monodromy $M_{(\infty)}$ has N eigenvalues known from (5.2.9). Additionally, $M_{(\infty)} = M_{(1)}M_{(0)}$ since $x\in\{0,1,\infty\}$ are the only singular points.

We thus want to find triplets of $N\times N$ unitary matrices $X = YZ$, where $Z = M_{(0)}$ is the diagonal matrix $\mathrm{diag}(z_1,\ldots,z_N)$ (in the basis of s-channel conformal blocks), $Y = M_{(1)}$ has eigenvalues $(y_1, y_2, \dots, y_2)$ and $X = M_{(\infty)}$ has eigenvalues $(x_1, \dots, x_N)$. Of course, all $|z_j|^2 = |y_j|^2 = |x_j|^2 = 1$. In terms of a normalized eigenvector $v \in \mathbb{C}^N$ of Y with eigenvalue y_1 and $|v|^2 = 1$, the unitary matrix Y is

$$Y_{ps} = y_2 \delta_{ps} + (y_1 - y_2) v_p (v_s)^* . \tag{5.2.13}$$

For any $\lambda \in \mathbb{C}$ the matrix $Y - \lambda Z^{-1}$ is the sum of a rank 1 part $(y_1 - y_2) v_p (v_s)^*$ and a diagonal part. Its determinant is

$$\det(Y - \lambda Z^{-1}) = \left[1 + \sum_{s=1}^{N} \frac{y_1 - y_2}{y_2 - \lambda z_s^{-1}} |v_s|^2\right] \prod_{s=1}^{N} (y_2 - \lambda z_s^{-1}) , \tag{5.2.14}$$

and we want to find v such that this matches with the characteristic polynomial of X

$$\det(Y - \lambda Z^{-1}) = \det(X - \lambda \mathbb{1}) \det(Z)^{-1} = \prod_{s=1}^{N} \frac{x_s - \lambda}{z_s} \tag{5.2.15}$$

Assuming that all z_p are distinct (this genericity condition could be removed with some work), these polynomials in λ match if and only if their values at each point $\lambda = y_2 z_p$ are equal. We thus want

$$\prod_{s=1}^{N} (x_s - y_2 z_p) = (y_1 - y_2) z_p |v_p|^2 \prod_{s \neq p}^{N} (y_2 z_s - y_2 z_p) . \tag{5.2.16}$$

This fixes all $|v_s|^2$. The vector v is thus fixed up to phase rotations of its components, in other words phase rotations of the basis vectors (this does not affect Z). All in all, the constraints on $X = M_{(\infty)}$, $Y = M_{(1)}$ and $Z = M_{(0)}$ fix the monodromy matrices completely, up to a choice of basis.

We now write down the expressions proposed in [FL07] for s-channel conformal blocks:

$$\begin{aligned} \mathcal{F}_p^{(\mathrm{s})}(x) = {}& x^{b\langle \alpha_0 - Q, h_p\rangle + \frac{N-1}{2}(b^2+1)} (1-x)^{b^2+1-b(\varkappa+b)/N} \\ & {}_N F_{N-1}\left(\begin{smallmatrix} 1 - b\varkappa/N + b\langle \alpha_0 - Q, h_p\rangle + b\langle \alpha_\infty - Q, h_s\rangle,\, 1 \le s \le N \\ 1 + b\langle \alpha_0 - Q, h_p - h_s\rangle,\, s \neq p \end{smallmatrix} \middle| x \right) , \end{aligned} \tag{5.2.17}$$

where the hypergeometric function ${}_N F_{N-1}$ is defined in terms of Pochhammer symbols $(a)_k = \Gamma(a+k)/\Gamma(a)$ by the series

$$F\left(\begin{smallmatrix} a_1 & \cdots & a_N \\ b_1 & \cdots & b_{N-1} \end{smallmatrix} \middle| x \right) = \sum_{k \ge 0} \frac{x^k}{k!} \frac{(a_1)_k \cdots (a_N)_k}{(b_1)_k \cdots (b_{N-1})_k} . \tag{5.2.18}$$

The functions $\mathcal{F}_p^{(\mathrm{s})}(x)$ have the expected monodromies and asymptotic behaviour around 0. Standard properties of the hypergeometric function (see [Nør55]) ensure that they can be analytically continued with branch points at $x \in \{0, 1, \infty\}$. They can be written as a linear combination of functions $\mathcal{F}^{(\mathrm{u})}_s(x)$ which have the expected monodromies and asymptotic behaviour around ∞, and similarly (non-uniquely) functions $\mathcal{F}^{(\mathrm{t})}$ which have the expected monodromies and asymptotic behaviour around 1.

Since the monodromies around $\{0, 1, \infty\}$ have the correct eigenvalues, and given our discussion above, the monodromy matrices themselves of the proposed (5.2.17) are equal to the monodromy matrices of actual conformal blocks. Therefore, the proposed (5.2.17) are the correct conformal blocks, possibly multiplied by a meromorphic function with no branch point. Given that the powers of x at 0 and ∞ and of $1-x$ at 1 also match, the meromorphic function has no pole throughout the Riemann sphere, hence is just a normalization constant. In our convention, conformal blocks have a leading coefficient 1, hence (5.2.17) is correct.

For completeness, we give here the braiding matrix $\mathbf{B}^\epsilon$ relating s-channel and u-channel conformal blocks in the half-planes $\epsilon \operatorname{Im}(x) > 0$, which is defined by $\mathcal{F}^{(\mathrm{s})}_p(x) = \sum_s \mathbf{B}^\epsilon_{ps} \mathcal{F}^{(\mathrm{u})}_s$. The components (5.2.31) are computed in the next section to be

$$\mathbf{B}^\epsilon_{ps}\begin{bmatrix}(\varkappa+b)h_1 & -bh_1\\ \alpha_\infty & \alpha_0\end{bmatrix} = \frac{e^{\mathrm{i}\pi\epsilon[\frac{b\varkappa}{N}-\frac{N-1}{N}b^2-1]}\prod_{t\neq p}^N \Gamma(1+b\langle Q-\alpha_0, h_t-h_p\rangle)}{\prod_{u=1}^N \Gamma(1-\frac{b\varkappa}{N}-b\langle Q-\alpha_0,h_p\rangle - b\langle Q-\alpha_\infty,h_u\rangle)} \\ \frac{\pi e^{\mathrm{i}\pi\epsilon[1-\frac{b\varkappa}{N}-b\langle Q-\alpha_0,h_p\rangle-b\langle Q-\alpha_\infty,h_s\rangle]}}{\sin\pi(1-\frac{b\varkappa}{N}-b\langle Q-\alpha_0,h_p\rangle-b\langle Q-\alpha_\infty,h_s\rangle)} \\ \frac{\prod_{u\neq s}^N \Gamma(b\langle Q-\alpha_\infty, h_s-h_u\rangle)}{\prod_{t=1}^N \Gamma(\frac{b\varkappa}{N}+b\langle Q-\alpha_0,h_t\rangle + b\langle Q-\alpha_\infty,h_s\rangle)} . \tag{5.2.19}$$

5.2.2 Semi-degenerate and antisymmetric degenerate

This section originated as Appendix A.3 of [GLF14]. We move on to four-point functions involving one degenerate primary operator with momentum $-b\omega_K$ labelled by the K-th antisymmetric representation $\mathcal{R}(\omega_K)$ of A_{N-1}. We prove the following s-channel decomposition. Let $\gamma(x) = \Gamma(x)/\Gamma(1-x)$ and write $\alpha_\bullet = Q - \mathrm{i}a_\bullet$. The four-point function decomposes as a sum over weights $h_{\{p\}} = h_{p_1} + \cdots + h_{p_K}$ of $\mathcal{R}(\omega_K)$ with $1 \le p_1 < \cdots < p_K \le N$, namely

$$\begin{aligned}&\langle V_{\alpha_\infty}(\infty) V_{(\varkappa+Kb)h_1}(1) V_{-b\omega_K}(x,\bar{x}) V_{\alpha_0}(0)\rangle \\ &\quad = C\sum_{\{p\}}\Bigg[\prod_{j=1}^K \frac{\prod_{s\notin\{p\}}^N \gamma(b\langle \mathrm{i}a_0, h_{p_j}-h_s\rangle)}{\prod_{s=1}^N \gamma(b\varkappa/N + b\langle \mathrm{i}a_0, h_{p_j}\rangle + b\langle \mathrm{i}a_\infty, h_s\rangle)}\,\mathcal{F}^{(\mathrm{s})}_{\{p\}}(x)\,\mathcal{F}^{(\mathrm{s})}_{\{p\}}(\bar{x})\Bigg] ,\end{aligned} \tag{5.2.20}$$

with conformal blocks $\mathcal{F}^{(\mathrm{s})}_{\{p\}}$ given by generalizations of hypergeometric series:

$$\mathcal{F}^{(\mathrm{s})}_{\{p\}}(x) = x^{b\langle -\mathrm{i}a_0, h_{\{p\}}\rangle + \frac{K(N-K)}{2}(b^2+1)}(1-x)^{K(b^2+1)-Kb(\varkappa+Kb)/N} \cdot$$
$$\cdot \sum_{k_1,\ldots,k_K\geq 0} \prod_{j=1}^{K} \frac{(x^{k_j}/k_j!)\prod_{s=1}^{N}(1-b\varkappa/N - b\langle \mathrm{i}a_0, h_{p_j}\rangle - b\langle \mathrm{i}a_\infty, h_s\rangle)_{k_j}}{\prod_{i\neq j}^{K}(b\langle \mathrm{i}a_0, h_{p_i}-h_{p_j}\rangle - k_i)_{k_j}\prod_{s\notin\{p\}}^{N}(1+b\langle \mathrm{i}a_0, h_s - h_{p_j}\rangle)_{k_j}} . \tag{5.2.21}$$

Of course, the four-point function has a u-channel decomposition completely analoguous to this s-channel decomposition. The expression (5.2.20) for the four-point function in fact comes from its relation with the partition function of a supersymmetric surface operator on the sphere. We explore this relation and how it fits into the AGT correspondence in Chapter 3.

The proof goes as follows. We find an Mellin–Barnes integral representation which analytically continues the expression for $\mathcal{F}^{(\mathrm{s})}_{\{p\}}(x)$ to the upper or lower half-plane, and we expand it near $x=\infty$ in terms of eigenfunctions of the monodromy near ∞. These eigenfunctions $\mathcal{F}^{(\mathrm{u})}_{\{p\}}(x)$ each take the form $x^{\cdots}(1+\cdots)$ for some exponent and some seriees in non-negative integer powers of $1/x$. The decomposition of $\mathcal{F}^{(\mathrm{s})}$ as a linear combination of $\mathcal{F}^{(\mathrm{u})}$ in each half-plane yields braiding matrices $\mathbf{B}^{\pm}$:

$$\mathcal{F}^{(\mathrm{s})}_{\{p\}}(x) \overset{\text{cont}}{=} \sum_{\{s\}} \mathbf{B}^{\epsilon}_{\{p\}\{s\}}\, \mathcal{F}^{(\mathrm{u})}_{\{s\}}(x) , \tag{5.2.22}$$

where ϵ is the sign of $\mathrm{Im}(x)$. We then compute braiding matrices via a 2d CFT calculation and check that they are identical to $\mathbf{B}^{\epsilon}$. The monodromy matrix around 1 can be expressed in terms of braiding matrices (namely, $M_{(1)} = \mathbf{B}^+[\mathbf{B}^-]^{-1}$), hence the functions $\mathcal{F}^{(\mathrm{s})}_{\{p\}}(x)$ have the same monodromy matrices around 0 and around 1 as the actual conformal blocks. Given that there are only three branch points, the last monodromy matrix, around ∞, is also correct. The equality of monodromy matrices then implies that $\mathcal{F}^{(\mathrm{s})}_{\{p\}}$ are the correct conformal blocks up to some rational function, and this function is a constant because it has no pole.

Mellin–Barnes calculation

To compute the braiding matrix for $\mathcal{F}^{(\mathrm{s})}_{\{p\}}$ it is convenient to introduce notations. Let $\mathrm{i}m_p$ and $\mathrm{i}\widetilde{m}_p$ be $2N$ complex numbers such that $\alpha_0 = Q - \frac{1}{b}\sum_{p=1}^{N}\mathrm{i}m_p h_p$ and $\alpha_\infty = Q - \frac{1}{b}\sum_{p=1}^{N}\mathrm{i}\widetilde{m}_p h_p$ and $\varkappa = \frac{1}{b}\sum_{p=1}^{N}(1+\mathrm{i}m_p+\mathrm{i}\widetilde{m}_p)$. This parametrization is redundant, namely shifting all $\mathrm{i}m_p$ and $-\mathrm{i}\widetilde{m}_p$ by the same amount does not affect the momenta α_0, α_∞ and $(\varkappa+Kb)h_1$, but this will not be important.

With these notations, equation (5.2.21) reads

$$\mathcal{F}^{(\mathrm{s})}_{\{p\}}(x) = (1-x)^{-\gamma_1}x^{-\gamma_0-\sum_{j=1}^{K}\mathrm{i}m_{p_j}} f^{(\mathrm{s})}_{\{p\}}(x) \tag{5.2.23}$$

with exponents

$$\gamma_0 = -\frac{K(N-K)}{2}(b^2+1) - \frac{K}{N}\sum_{s=1}^{N} \mathrm{i}m_s \qquad (5.2.24)$$

$$\gamma_1 = -\frac{K(N-K)}{N}b^2 + \frac{K}{N}\sum_{s=1}^{N}(\mathrm{i}m_s + \mathrm{i}\widetilde{m}_s) \qquad (5.2.25)$$

and the hypergeometric-like series reads

$$f^{(\mathrm{s})}_{\{p\}}(x) = \sum_{k_1,\dots,k_K\geq 0}\prod_{j=1}^{K}\frac{(x^{k_j}/k_j!)\prod_{s=1}^{N}(-\mathrm{i}m_{p_j}-\mathrm{i}\widetilde{m}_s)_{k_j}}{\prod_{i\neq j}^{K}(\mathrm{i}m_{p_i}-\mathrm{i}m_{p_j}-k_i)_{k_j}\prod_{s\notin\{p\}}^{N}(1+\mathrm{i}m_s-\mathrm{i}m_{p_j})_{k_j}}. \qquad (5.2.26)$$

We shall work with the s-channel factors $(-x)^{-\sum_{j=1}^{K}\mathrm{i}m_{p_j}} f^{(\mathrm{s})}_{\{p\}}(x)$, then convert results to $\mathcal{F}^{(\mathrm{s})}$ by including some phases.

Consider first the case $K=1$. In this case, the explicit expressions $\mathcal{F}^{(\mathrm{s})}_p(x)$ in terms of hypergeometric functions are known to be correct (see Section 5.2.1). We now derive the braiding matrix announced in (5.2.19).

The s-channel factor $(-x)^{-\mathrm{i}m_p} f^{(\mathrm{s})}_p$ can be expressed as the Mellin–Barnes integral (5.2.27) given below, which converges away from the positive real axis. For $|x| \lessgtr 1$ we can close the contour integral towards $\kappa \to \pm\infty$, enclosing either the poles at $\kappa + \mathrm{i}m_p \in \mathbb{Z}_{\geq 0}$ or the N families of poles at $\kappa - \mathrm{i}\widetilde{m}_s \in \mathbb{Z}_{\leq 0}$ labelled by $1 \leq s \leq N$. The first choice yields a single s-channel factor, while the second yields a sum of N u-channel factors:

$$\begin{aligned}(-x)^{-\mathrm{i}m_p} f^{(\mathrm{s})}_p(x) &\overset{\text{cont}}{=} D_p \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\mathrm{d}\kappa}{2\pi\mathrm{i}}\frac{\prod_{s=1}^{N}\Gamma(-\mathrm{i}\widetilde{m}_s+\kappa)}{\prod_{s\neq p}^{N}\Gamma(1+\mathrm{i}m_s+\kappa)}\Gamma(-\kappa-\mathrm{i}m_p)(-x)^{\kappa}\\ &\overset{\text{cont}}{=} \sum_{s=1}^{N} D_p \check{B}^0_{ps}\widetilde{D}_s(-x)^{\mathrm{i}\widetilde{m}_s} f^{(\mathrm{u})}_s(x)\,. \end{aligned} \qquad (5.2.27)$$

The coefficients D, $\check{B}^0$ and $\widetilde{D}$ are given in (5.2.29) below. There is no need to write down the explicit expression for the series $f^{(\mathrm{u})}_s(x) = 1 + \cdots$ in non-positive integer powers of x.

It is also convenient to work with the s-channel factors $x^{-\mathrm{i}m_p} f^{(\mathrm{s})}_p(x)$, analytically continued with branch cuts on $(-\infty,0]\cup[1,+\infty)$, and the u-channel factors $x^{\mathrm{i}\widetilde{m}_s} f^{(\mathrm{u})}_s(x)$, with branch cuts along $(-\infty,0]\cup[0,1]$, rather than with the factors appearing in (5.2.27), which all have branch cuts along the positive real axis $[0,1]\cup[1,+\infty)$. Using $(-x)^{\lambda} = e^{-\mathrm{i}\pi\epsilon\lambda}x^{\lambda}$ for $\epsilon = \operatorname{sign}(\operatorname{Im} x)$, we obtain

$$x^{-\mathrm{i}m_p} f^{(\mathrm{s})}_p(x) \overset{\text{cont}}{=} \sum_{s=1}^{N} D_p \check{B}^{\epsilon}_{ps}\widetilde{D}_s x^{\mathrm{i}\widetilde{m}_s} f^{(\mathrm{u})}_s(x)\,. \qquad (5.2.28)$$

This braiding only differs from (5.2.27) by a phase in $\check{B}^\epsilon$:

$$\check{B}^\epsilon_{ps} = \frac{\pi e^{\pi\epsilon(m_p+\widetilde{m}_s)}}{\sin\pi(-\mathrm{i}\widetilde{m}_s - \mathrm{i}m_p)} \tag{5.2.29}$$

$$D_p = \prod_{t=1}^{N} \frac{\Gamma(1+\mathrm{i}m_t - \mathrm{i}m_p)}{\Gamma(-\mathrm{i}\widetilde{m}_t - \mathrm{i}m_p)} \quad \widetilde{D}_s = \frac{\prod_{t\neq s}^{N}\Gamma(-\mathrm{i}\widetilde{m}_t + \mathrm{i}\widetilde{m}_s)}{\prod_{t=1}^{N}\Gamma(1+\mathrm{i}m_t+\mathrm{i}\widetilde{m}_s)}\,. \tag{5.2.30}$$

We now convert the braiding to the functions $\mathcal{F}^{(\mathrm{s})}_p(x)$ (5.2.23). Upon analytic continuation and expansion near $x\to\infty$ the relative factor $(1-x)^{-\gamma_1}x^{-\gamma_0}$ becomes $e^{\mathrm{i}\pi\epsilon\gamma_1}(1-1/x)^{-\gamma_1}x^{-\gamma_0-\gamma_1}$. Therefore,

$$\begin{aligned}\mathcal{F}^{(\mathrm{s})}_p(x) &= (1-x)^{-\gamma_1}x^{-\gamma_0}x^{-\mathrm{i}m_p}f^{(\mathrm{s})}_p(x)\\ &\overset{\text{cont}}{=}\sum_{s=1}^{N}\mathbf{B}^\epsilon_{ps}(1-1/x)^{-\gamma_1}x^{-\gamma_0-\gamma_1+\mathrm{i}\widetilde{m}_s}f^{(\mathrm{u})}_s(x) = \sum_{s=1}^{N}\mathbf{B}^\epsilon_{ps}\,\mathcal{F}^{(\mathrm{u})}_s(x)\end{aligned} \tag{5.2.31}$$

with a braiding matrix $\mathbf{B}^\epsilon_{ps} = e^{\mathrm{i}\pi\epsilon\gamma_1}D_p\check{B}^\epsilon_{ps}\widetilde{D}_s$ in terms of (5.2.29). We gave this braiding matrix in terms of momenta instead of the $\{\mathrm{i}m_s, I\widetilde{m}_t\}$ notation in equation (5.2.19).

Next, go back to general $K\geq 1$. The s-channel factors $f^{(\mathrm{s})}_{\{p\}}(x)$ can be written in terms of derivatives of a product of s-channel factors $f^{(\mathrm{s})}_{p_j}(x)$ of the $K=1$ case. We can then analytically continue each such factor using (5.2.28):

$$x^{-\sum_{j=1}^{K}\mathrm{i}m_{p_j}}f^{(\mathrm{s})}_{\{p\}}(x) = \left[\prod_{i<j}\frac{x_i\partial_{x_i}-x_j\partial_{x_j}}{-\mathrm{i}m_{p_i}+\mathrm{i}m_{p_j}}\prod_{j=1}^{K}\left[x_j^{-\mathrm{i}m_{p_j}}f^{(\mathrm{s})}_{p_j}(x_j)\right]\right]_{x_j=x} \tag{5.2.32}$$

$$\overset{\text{cont}}{=}\left[\prod_{i<j}\frac{x_i\partial_{x_i}-x_j\partial_{x_j}}{-\mathrm{i}m_{p_i}+\mathrm{i}m_{p_j}}\prod_{j=1}^{K}\sum_{s_j=1}^{N}\left[D_{p_j}\check{B}^\epsilon_{p_js_j}\widetilde{D}_{s_j}\,x_j^{\mathrm{i}\widetilde{m}_{s_j}}f^{(\mathrm{u})}_{s_j}(x_j)\right]\right]_{x_j=x} \tag{5.2.33}$$

$$= \sum_{s_1\neq\cdots\neq s_K}\left[\prod_{j=1}^{K}\left[D_{p_j}\check{B}^\epsilon_{p_js_j}\widetilde{D}_{s_j}\right]\prod_{i<j}\left[\frac{\mathrm{i}\widetilde{m}_{s_i}-\mathrm{i}\widetilde{m}_{s_j}}{-\mathrm{i}m_{p_i}+\mathrm{i}m_{p_j}}\right]x^{\sum_{j=1}^{K}\mathrm{i}\widetilde{m}_{s_j}}f^{(\mathrm{u})}_{\{s\}}(x)\right] \tag{5.2.34}$$

$$= \sum_{1\leq s_1<\cdots<s_K\leq N} D_{\{p\}}\check{B}^\epsilon_{\{p\}\{s\}}\widetilde{D}_{\{s\}}x^{\sum_{j=1}^{K}\mathrm{i}\widetilde{m}_{s_j}}f^{(\mathrm{u})}_{\{s\}}(x)\,. \tag{5.2.35}$$

To get (5.2.34), we note that if $s_i = s_j$ for some $i\neq j$, the differential operators $x_i\partial_{x_i}$ and $x_j\partial_{x_j}$ act identically on the product of factors $x_j^{\mathrm{i}\widetilde{m}_{s_j}}f^{(\mathrm{u})}_{s_j}(x_j)$, once x_i and x_j are set to x, hence the term does not contribute. After restricting ourselves to terms with all s_i distinct, we can safely extract the product of $\mathrm{i}\widetilde{m}_{s_i} - \mathrm{i}\widetilde{m}_{s_j}$ to normalize $f^{(\mathrm{u})}_{\{s\}}(x)$. The last step sums over permutations of the s_i, to collect terms with the same factor, labelled by the set $\{s\}$. The resulting ingredients are two diagonal matrices,

$$D_{\{p\}} = \frac{\prod_{j=1}^K D_{p_j}}{\prod_{i<j}(-\mathrm{i}m_{p_i} + \mathrm{i}m_{p_j})} , \quad \widetilde{D}_{\{s\}} = \prod_{i<j}(\mathrm{i}\widetilde{m}_{s_i} - \mathrm{i}\widetilde{m}_{s_j}) \prod_{j=1}^K \widetilde{D}_{s_j} , \quad (5.2.36)$$

and the K-th wedge power $\check{B}^{\epsilon}_{\{p\}\{s\}}$ of the $K = 1$ matrix $\check{B}^{\epsilon}_{ps}$:

$$\check{B}^{\epsilon}_{\{p\}\{s\}} = \sum_{\sigma\in S_K} (-1)^\sigma \prod_{j=1}^K \check{B}^{\epsilon}_{p_j s_{\sigma(j)}} = \sum_{\sigma\in S_K} (-1)^\sigma \prod_{j=1}^K \frac{\pi e^{\pi\epsilon(m_{p_j} + \widetilde{m}_{s_{\sigma(j)}})}}{\sin\pi(-\mathrm{i}\widetilde{m}_{s_{\sigma(j)}} - \mathrm{i}m_{p_j})} \quad (5.2.37)$$

$$= \int \frac{\mathrm{d}^K\kappa_1}{(2\mathrm{i})^K} \frac{\prod_{i<j}\sin\pi(\kappa_i - \kappa_j)\sin\pi(\mathrm{i}\widetilde{m}_{s_i} - \mathrm{i}\widetilde{m}_{s_j})}{\prod_{i,j=1}^K \sin\pi(\kappa_j + \mathrm{i}\widetilde{m}_{s_i})} \prod_{j=1}^K \frac{\pi e^{\pi\epsilon(m_{p_j} + \widetilde{m}_{s_j})}}{\sin\pi(\kappa_j - \mathrm{i}m_{p_j})} \quad (5.2.38)$$

$$= \frac{\pi^K e^{\pi\epsilon\sum_{j=1}^K(m_{p_j} + \widetilde{m}_{s_j})} \prod_{i<j}\sin\pi(\mathrm{i}\widetilde{m}_{s_i} - \mathrm{i}\widetilde{m}_{s_j}) \prod_{i<j}\sin\pi(\mathrm{i}m_{p_i} - \mathrm{i}m_{p_j})}{\prod_{i,j}\sin\pi(-\mathrm{i}\widetilde{m}_{s_i} - \mathrm{i}m_{p_j})} . \quad (5.2.39)$$

Each of the $\mathrm{d}\kappa_j$ contours in (5.2.38) is a pair of vertical lines $\frac{1}{2} - \mathrm{i}\infty \to \frac{1}{2} + \mathrm{i}\infty$ and $\mathrm{i}\infty \to -\mathrm{i}\infty$, surrounding poles at $\kappa_j = -\mathrm{i}\widetilde{m}_{s_{\sigma(j)}}$. Convergence is guaranteed since the integrand decreases exponentially as $\operatorname{Im}\kappa \to \pm\infty$ (for $-1 \leq \epsilon \leq 1$). If two $\sigma(j)$ are equal, the numerator sines lead to a vanishing residue. Otherwise, the first fraction completely cancels and we retrieve (5.2.37). Next, we note that the integrand has period 1, hence the contour can be replaced by $-\frac{1}{2} - \mathrm{i}\infty \to -\frac{1}{2} + \mathrm{i}\infty$ and $\mathrm{i}\infty \to -\mathrm{i}\infty$, which surrounds poles at $\kappa_j = \mathrm{i}m_{p_j}$, with a factor of $(-1)^K$ to account for the orientation of the contour. This yields the last expression.

As in the case $K = 1$ (see equation (5.2.31)), the braiding matrix for $\mathcal{F}^{(\mathrm{s})}_{\{p\}}(x)$ is obtained by including a phase $e^{\mathrm{i}\pi\epsilon\gamma_1}$. Namely,

$$\mathcal{F}^{(\mathrm{s})}_{\{p\}}(x) = \sum_{\{s\}} \mathbf{B}^{\epsilon}_{\{p\}\{s\}} \mathcal{F}^{(\mathrm{u})}_{\{s\}}(x) , \qquad \mathbf{B}^{\epsilon}_{\{p\}\{s\}} = e^{\mathrm{i}\pi\epsilon\gamma_1} D_{\{p\}} \check{B}^{\epsilon}_{\{p\}\{s\}} \widetilde{D}_{\{s\}} . \quad (5.2.40)$$

CFT calculation

So far we have found the braiding matrix (5.2.40) for the functions $\mathcal{F}^{(\mathrm{s})}_{\{p\}}$ which should be s-channel conformal blocks. Here, we show through a CFT calculation that (5.2.40) is indeed the correct braiding matrix from s-channel to u-channel conformal blocks. We use a standard notation for conformal blocks, writing the four external momenta in a 2×2 table, and the internal momentum separately.

The braiding matrix $\mathbf{B}^{\epsilon}_{PS}$ is defined by

$$\mathcal{F}^{(\mathrm{s})}_{\alpha_0 - bh_P}\begin{bmatrix}\hat{m} & -b\omega_K \\ \alpha_\infty & \alpha_0\end{bmatrix} = \sum_{\substack{S\subseteq[\![1,N]\!] \\ \#S=K}} \mathbf{B}^{\epsilon}_{PS}\begin{bmatrix}\hat{m} & -b\omega_K \\ \alpha_\infty & \alpha_0\end{bmatrix}\mathcal{F}^{(\mathrm{u})}_{\alpha_\infty - bh_S}\begin{bmatrix}\hat{m} & -b\omega_K \\ \alpha_\infty & \alpha_0\end{bmatrix} \tag{5.2.41}$$

where $\hat{m} = (\varkappa + Kb)h_1$, and we will often decompose $\alpha_0 = Q - \mathrm{i}a_0$ and $\alpha_\infty = Q - \mathrm{i}a_\infty$. Using the dictionary between $\{\mathrm{i}m_s, \mathrm{i}\widetilde{m}_t\}$ and momenta, we wish to prove that

$$\begin{aligned}\mathbf{B}^{\epsilon}_{PS}&\begin{bmatrix}(\varkappa + Kb)h_1 & -b\omega_K \\ Q - \mathrm{i}a_\infty & Q - \mathrm{i}a_0\end{bmatrix} \\ &= e^{-\mathrm{i}\pi\epsilon\frac{K(N-K)}{N}b^2}\prod_{p\in P}\frac{e^{\pi\epsilon b\langle a_0, h_p\rangle}\prod_{t\notin P}^{N}\Gamma(1 + b\langle \mathrm{i}a_0, h_t - h_p\rangle)}{\prod_{u\notin S}^{N}\Gamma(1 - \frac{b\varkappa}{N} - b\langle \mathrm{i}a_0, h_p\rangle - b\langle \mathrm{i}a_\infty, h_u\rangle)} \\ &\qquad\cdot\prod_{s\in S}\frac{e^{\pi\epsilon b\langle a_\infty, h_s\rangle}\prod_{u\notin S}^{N}\Gamma(b\langle \mathrm{i}a_\infty, h_s - h_u\rangle)}{\prod_{t\notin P}^{N}\Gamma(\frac{b\varkappa}{N} + b\langle \mathrm{i}a_0, h_t\rangle + b\langle \mathrm{i}a_\infty, h_s\rangle)}.\end{aligned} \tag{5.2.42}$$

We proceed by induction on K. The result holds for $K = 1$, because conformal blocks are proven in Section 5.2.1 to be equal to the $\mathcal{F}^{(\mathrm{s})}_p$. From here on, we assume (5.2.42) for a given K. In particular, the s-channel conformal blocks are given for that value of K by the factors $\mathcal{F}^{(\mathrm{s})}_P$.

The t-channel block for the fusion of V_{-bh_1} and $V_{-b\omega_K}$ into $V_{-b\omega_{K+1}}$ is the linear combination

$$\begin{aligned}\mathcal{F}^{(\mathrm{t})}_{-b\omega_{K+1}}&\begin{bmatrix}-bh_1 & -b\omega_K \\ 2Q - \alpha_0 + bh_P & \alpha_0\end{bmatrix} \\ &= \sum_{p\in P}\mathbf{F}_{p,P}[\alpha_0]\,\mathcal{F}^{(\mathrm{s})}_{\alpha_0 - bh_{P\setminus\{p\}}}\begin{bmatrix}-bh_1 & -b\omega_K \\ 2Q - \alpha_0 + bh_P & \alpha_0\end{bmatrix}\end{aligned} \tag{5.2.43}$$

whose monodromy is $e^{2\pi\mathrm{i}[\Delta(-b\omega_K)+\Delta(-bh_1)-\Delta(-b\omega_{K+1})]} = e^{-2\pi\mathrm{i}[K(b^2+1)+b^2K/N]}$ around $x = 1$. We shall prove that the fusion coefficients

$$\mathbf{F}_{p,P}[\alpha_0] = \frac{\Gamma((K+1)(1+b^2))}{\Gamma(1+b^2)}\prod_{t\in P\setminus\{p\}}\left[\frac{\Gamma(b\langle Q - \alpha_0, h_t - h_p\rangle)}{\Gamma(1 + b^2 + b\langle Q - \alpha_0, h_t - h_p\rangle)}\right] \tag{5.2.44}$$

give this monodromy, and are normalized so that the dominant power of $1 - x$ has a coefficient 1.

Braid $V_{-b\omega_K}$ and V_{-bh_1} in the right-hand side of (5.2.43) using (5.2.42) with $P \to P \setminus \{p\}$, $\varkappa \to -(K+1)b$, $\mathrm{i}a_\infty \to -\mathrm{i}a_0 - bh_P$ and $S \to P \setminus \{s\}$ for some $s \in P$ ($h_P - h_S$ must be a weight of the fundamental representation, because of V_{-bh_1}):

$$\sum_{p\in P} \mathbf{F}_{p,P}[\alpha_0]\mathbf{B}^{\epsilon}_{P\backslash\{p\},P\backslash\{s\}}\begin{bmatrix} -bh_1 & -b\omega_K \\ 2Q-\alpha_0+bh_P & \alpha_0 \end{bmatrix}$$
$$= e^{-\mathrm{i}\pi\epsilon\frac{K}{N}b^2}\sum_{p\in P} e^{\pi\epsilon b\langle a_0,h_s-h_p\rangle}\frac{\prod_{t\in P\backslash\{s\}}\sin\pi(1+b^2+b\langle \mathrm{i}a_0,h_t-h_p\rangle)}{\prod_{t\in P\backslash\{p\}}\sin\pi(b\langle \mathrm{i}a_0,h_t-h_p\rangle)}$$
$$\cdot\frac{\Gamma((K+1)(1+b^2))}{\Gamma(1+b^2)}\prod_{t\in P\backslash\{s\}}\left[\frac{\Gamma(b\langle \mathrm{i}a_0,h_s-h_t\rangle)}{\Gamma(1+b^2+b\langle \mathrm{i}a_0,h_s-h_t\rangle)}\right] \tag{5.2.45}$$
$$= e^{-\mathrm{i}\pi\epsilon\left[\frac{K}{N}b^2+K(1+b^2)\right]}\mathbf{F}_{s,P}[2Q-\alpha_0+bh_P]\,. \tag{5.2.46}$$

We have used

$$\sum_{p\in P} e^{\pi\epsilon b\langle a_0,h_s-h_p\rangle}\frac{\prod_{t\in P\backslash\{s\}}\sin\pi(1+b^2+b\langle \mathrm{i}a_0,h_t-h_p\rangle)}{\prod_{t\in P\backslash\{p\}}\sin\pi(b\langle \mathrm{i}a_0,h_t-h_p\rangle)}$$
$$=\int\frac{\mathrm{d}\kappa}{2\mathrm{i}}\frac{\prod_{t\in P\backslash\{s\}}\sin\pi(1+b^2+b\langle \mathrm{i}a_0,h_t\rangle+\kappa)}{e^{\pi\epsilon(-b\langle a_0,h_s\rangle+\mathrm{i}\kappa)}\prod_{t\in P}\sin\pi(b\langle \mathrm{i}a_0,h_t\rangle+\kappa)} = e^{-\mathrm{i}\pi\epsilon K(1+b^2)}\,, \tag{5.2.47}$$

where the contour surrounds the rectangle $\operatorname{Re}\kappa\in[0,1]$, $\operatorname{Im}\kappa\in(-\infty,\infty)$. Summing over poles yields the sum over $p\in P$ in the first line. The integrals over the lines $1-\mathrm{i}\infty\to 1+\mathrm{i}\infty$ and $\mathrm{i}\infty\to-\mathrm{i}\infty$ cancel because the integrand is 1-periodic, and the integrals over $1+\mathrm{i}\infty\to\mathrm{i}\infty$ and $-\mathrm{i}\infty\to 1-\mathrm{i}\infty$ yield 0 and $e^{-\mathrm{i}\pi\epsilon K(1+b^2)}$ in some order.

In (5.2.46), we have only done one braiding move, not a full monodromy (two braiding moves). However, the combination of u-channel conformal blocks is identical to (5.2.43) after changing $\mathrm{i}a_0\to\mathrm{i}a_\infty=-\mathrm{i}a_0-bh_P$, thus, by symmetry, braiding once more to reach the s-channel yields the same phase factor. Therefore, (5.2.43) has the announced monodromy around $x=1$.

There remains to fix the normalization. We evaluate at $x=1$ the explicit expression (5.4.37) of s-channel conformal blocks which appear in (5.2.43), after removing a power of $(1-x)$,

$$\left[(1-x)^{-K(b^2+1)-\frac{K}{N}b^2}\mathcal{F}^{(\mathrm{s})}_{\alpha_0-bh_{P\backslash\{p\}}}\begin{bmatrix} -bh_1 & -b\omega_K \\ 2Q-\alpha_0+bh_P & \alpha_0 \end{bmatrix}(x)\right]_{x=1}$$
$$=\sum_{\substack{k:\ P\to\mathbb{Z}_{\geq 0}\\ k_p=0}}(-1)^{\sum_{s\in P}k_s}\prod_{s,t\in P}\frac{(1+b^2+b\langle \mathrm{i}a_0,h_t-h_s\rangle)_{k_s}}{(b\langle \mathrm{i}a_0,h_t-h_s\rangle-k_t+\delta_{tp})_{k_s}}\,. \tag{5.2.48}$$

This only depends on the $\langle \mathrm{i}a_0,h_t\rangle$ with $t\in P$, and does not depend on N. We can thus take $N=K+1$, in which case $-b\omega_K=bh_N$ and the fusion is a special case of equation (B.14) of [GLF10], where the normalization is known to be (5.2.44).

We are now ready to find the braiding matrix of $V_{-b\omega_{K+1}}$ with $V_{\hat{m}}$ (where $\hat{m} = (\varkappa + (K+1)b)h_1$). This braiding, followed by writing $V_{-b\omega_{K+1}}$ as the fusion of V_{-bh_1} and $V_{-b\omega_K}$, is equivalent to performing the fusion step first, then braiding each of V_{-bh_1} and $V_{-b\omega_K}$ in turn around the semi-degenerate operator. The equivalence is encoded as a pentagon identity: for any $(K+1)$-element sets of flavours P and S, and for $s \in S$,

$$\mathbf{B}^{\epsilon}_{PS}\begin{bmatrix}\hat{m} & -b\omega_{K+1}\\ \alpha_\infty & \alpha_0\end{bmatrix}\mathbf{F}_{s,S}[2Q-\alpha_\infty]$$
$$= \sum_{p\in P}\mathbf{F}_{p,P}[\alpha_0]\mathbf{B}^{\epsilon}_{ps}\begin{bmatrix}\hat{m} & -bh_1\\ \alpha_\infty & \alpha_0 - bh_{P\setminus\{p\}}\end{bmatrix}\mathbf{B}^{\epsilon}_{P\setminus\{p\},S\setminus\{s\}}\begin{bmatrix}\hat{m} & -b\omega_K\\ \alpha_\infty - bh_s & \alpha_0\end{bmatrix}. \tag{5.2.49}$$

As a consistency check, we compute a slightly more general right-hand side, with $S\setminus\{s\}$ replaced by any K-element subset S' of $[\![1,N]\!]$. This altered right-hand side must vanish whenever $s \in S'$. After extracting factors independent of p in (5.2.51) below, we will obtain a sum over p of products of sines, which is a sum of residues:

$$\sum_{p\in P}\frac{\prod_{u\in S'}\frac{1}{\pi}\sin\pi(\frac{b\varkappa}{N}+b\langle \mathrm{i}a_0,h_p\rangle+b\langle \mathrm{i}a_\infty,h_u\rangle+b^2\delta_{us})}{\frac{1}{\pi}\sin\pi(\frac{b\varkappa}{N}+b^2+b\langle \mathrm{i}a_0,h_p\rangle+b\langle \mathrm{i}a_\infty,h_s\rangle)\prod_{t\in P\setminus\{p\}}\frac{1}{\pi}\sin\pi(b\langle \mathrm{i}a_0,h_t-h_p\rangle)}$$
$$=\sum_{p\in P}\underset{\kappa=b\langle \mathrm{i}a_0,h_p\rangle}{\mathrm{res}}\frac{-\prod_{u\in S'}\frac{1}{\pi}\sin\pi(\frac{b\varkappa}{N}+b\langle \mathrm{i}a_\infty,h_u\rangle+b^2\delta_{us}+\kappa)}{\frac{1}{\pi}\sin\pi(\frac{b\varkappa}{N}+b^2+b\langle \mathrm{i}a_\infty,h_s\rangle+\kappa)\prod_{t\in P}\frac{1}{\pi}\sin\pi(b\langle \mathrm{i}a_0,h_t\rangle-\kappa)}$$
$$=\frac{\prod_{u\in S'}\frac{1}{\pi}\sin\pi(b^2\delta_{us}+b\langle \mathrm{i}a_\infty,h_u-h_s\rangle-b^2)}{\prod_{t\in P}\frac{1}{\pi}\sin\pi(\frac{b\varkappa}{N}+b\langle \mathrm{i}a_0,h_t\rangle+b\langle \mathrm{i}a_\infty,h_s\rangle+b^2)}. \tag{5.2.50}$$

This sum of residues is the residue at $\kappa = -\frac{b\varkappa}{N} - b^2 - b\langle \mathrm{i}a_\infty, h_s\rangle$ in the last line, because the function of κ is 1-periodic and vanishes at $\kappa \to \pm\mathrm{i}\infty$, hence the integral over the boundary of $[0,1]\times(-\infty,\infty)$ vanishes. As expected, the result is 0 when $s \in S'$ (take $u = s$). It is otherwise a product of sines, and we get in that case the last equality below (with $S = S' \cup \{s\}$):

$$\sum_{p\in P}\mathbf{F}_{p,P}[\alpha_0]\mathbf{B}^{\epsilon}_{ps}\begin{bmatrix}\hat{m} & -bh_1\\ \alpha_\infty & \alpha_0 - bh_{P\setminus\{p\}}\end{bmatrix}\mathbf{B}^{\epsilon}_{P\setminus\{p\},S'}\begin{bmatrix}\hat{m} & -b\omega_K\\ \alpha_\infty - bh_s & \alpha_0\end{bmatrix}$$
$$=\frac{e^{-\mathrm{i}\pi\epsilon\delta_{s\in S'}b^2+\pi\epsilon b\langle a_0,h_P\rangle+\pi\epsilon b\langle a_\infty,h_s+h_{S'}\rangle}\prod_{u\neq s}^{N}\Gamma(b\langle \mathrm{i}a_\infty,h_s-h_u\rangle)}{\prod_{t=1}^{N}\Gamma(\frac{b\varkappa}{N}+b^2\delta_{t\in P}+b\langle \mathrm{i}a_0,h_t\rangle+b\langle \mathrm{i}a_\infty,h_s\rangle)}$$
$$\cdot\frac{e^{-\mathrm{i}\pi\epsilon(K+1)(N-K-1)b^2/N}\prod_{t\in P}\prod_{v\notin P}^{N}\Gamma(1+b\langle \mathrm{i}a_0,h_v-h_t\rangle)}{\prod_{t\in P}\prod_{w\notin S'}^{N}\Gamma(1-\frac{b\varkappa}{N}-b\langle \mathrm{i}a_0,h_t\rangle-b\langle \mathrm{i}a_\infty,h_w\rangle-b^2\delta_{sw})}\times$$

$$\cdot \frac{\Gamma\big((K+1)(1+b^2)\big)}{\Gamma(1+b^2)} \prod_{u\in S'} \frac{\prod_{w\notin S'}^N \Gamma(b\langle \mathrm{i}a_\infty, h_u - h_w\rangle + b^2\delta_{su} - b^2\delta_{sw})}{\prod_{v\notin P}^N \Gamma(\frac{b\varkappa}{N} + b\langle \mathrm{i}a_0, h_v\rangle + b\langle \mathrm{i}a_\infty, h_u\rangle + b^2\delta_{us})}$$
$$\cdot \sum_{p\in P} \frac{\prod_{u\in S'} \frac{1}{\pi}\sin\pi(\frac{b\varkappa}{N} + b\langle \mathrm{i}a_0, h_p\rangle + b\langle \mathrm{i}a_\infty, h_u\rangle + b^2\delta_{us})}{\frac{1}{\pi}\sin\pi(\frac{b\varkappa}{N} + b^2 + b\langle \mathrm{i}a_0, h_p\rangle + b\langle \mathrm{i}a_\infty, h_s\rangle) \prod_{t\in P\setminus\{p\}} \frac{1}{\pi}\sin\pi(b\langle \mathrm{i}a_0, h_t - h_p\rangle)} \tag{5.2.51}$$

$$\overset{s\notin S'}{=} e^{-\mathrm{i}\pi\epsilon\frac{(K+1)(N-K-1)}{N}b^2} \prod_{t\in P} \frac{e^{\pi\epsilon b\langle a_0, h_t\rangle} \prod_{v\notin P}^N \Gamma(1 + b\langle \mathrm{i}a_0, h_v - h_t\rangle)}{\prod_{w\notin S}^N \Gamma(1 - \frac{b\varkappa}{N} - b\langle \mathrm{i}a_0, h_t\rangle - b\langle \mathrm{i}a_\infty, h_w\rangle)}$$
$$\cdot \prod_{u\in S} \frac{e^{\pi\epsilon b\langle a_\infty, h_u\rangle} \prod_{w\notin S}^N \Gamma(b\langle \mathrm{i}a_\infty, h_u - h_w\rangle)}{\prod_{v\notin P}^N \Gamma(\frac{b\varkappa}{N} + b\langle \mathrm{i}a_0, h_v\rangle + b\langle \mathrm{i}a_\infty, h_u\rangle)}$$
$$\cdot \frac{\Gamma\big((K+1)(1+b^2)\big)}{\Gamma(1+b^2)} \prod_{u\in S\setminus\{s\}} \frac{\Gamma(b\langle \mathrm{i}a_\infty, h_s - h_u\rangle)}{\Gamma(1 + b^2 + b\langle \mathrm{i}a_\infty, h_s - h_u\rangle)} \,. \tag{5.2.52}$$

We recognize in the last line the fusion coefficient $\mathbf{F}_{s,S}[2Q - \alpha_\infty]$. What remains is the braiding matrix of $V_{-b\omega_{K+1}}$ with $V_{\hat{m}}$, which we check to be (5.2.41) with $K \to K+1$. This concludes the induction.

5.3 Braiding kernel

The previous section provides the braiding matrix of a semi-degenerate vertex operator around a degenerate vertex operator $V_{-b\omega_K}$. We move on in Section 5.3.1 to the case of a degenerate momentum $-Kbh_1$ labelled by a symmetric representation $\mathcal{R}(Kh_1)$ of A_{N-1} instead of the fundamental $\mathcal{R}(h_1)$. We write down conformal blocks predicted by the matching with $\mathcal{N} = (2,2)$ gauge theory and compute their braiding matrix. In principle we could prove that these expressions are correct by computing the braiding matrix in terms of K copies of the fundamental case, but we will not attempt this very tedious calculation, which turns out to be unnecessary.

Braiding matrices with a general $K \in \mathbb{Z}_{\geq 0}$ are used as an inspiration (in Section 5.3.2) for the braiding kernel of two semi-degenerate and two generic primary operators, where we promote the discrete $-Kbh_1$ to a continuous $\varkappa h_1$. The braiding kernel generalizes the braiding of Virasoro conformal blocks [PT99], and it obeys some shift relations. More work is needed to determine whether these shift relations fix the braiding kernel uniquely; in any case they are enough to prove that the braidings of Section 5.3.1 are correct, hence the conformal blocks as well.

5.3.1 Semi-degenerate and symmetric degenerate

Results in this section are new. They concern four-point functions involving primary operators with momenta α_0 and α_∞ (generic), $(\varkappa + Kb)h_1$

(semi-degenerate), and a degenerate $-Kbh_1$ labelled by the symmetric representation $\mathcal{R}(Kh_1)$ of A_{N-1}. The AGT correspondence for surface operators, explained in Chapter 3, provides an explicit expression for the correlator. From this expression we deduce the braiding matrix for the four-point conformal blocks.

We switch to the notations $\mathrm{i}m_p$ and $\mathrm{i}\widetilde{m}_p$ introduced in the last section: they obey $\alpha_0 = Q - \frac{1}{b}\sum_{p=1}^N \mathrm{i}m_p h_p$ and $\alpha_\infty = Q - \frac{1}{b}\sum_{p=1}^N \mathrm{i}\widetilde{m}_p h_p$, as well as $\varkappa = \frac{1}{b}\sum_{p=1}^N (1 + \mathrm{i}m_p + \mathrm{i}\widetilde{m}_p)$, and an overall shift of all $\mathrm{i}m_p$ and $-\mathrm{i}\widetilde{m}_p$ leaves the momenta α_0, α_∞ and $(\varkappa + Kb)h_1$ invariant. With these notations, the four-point function is expressed as a sum over weights $h_{[n]} = \sum_{s=1}^N n_s h_s$ of the symmetric representation $\mathcal{R}(Kh_1)$ of A_{N-1} (see (3.5.8)):

$$\begin{aligned}&\langle V_{\alpha_\infty}(\infty) V_{(\varkappa+Kb)h_1}(1) V_{-Kbh_1}(x,\bar{x}) V_{\alpha_0}(0)\rangle \\ &\quad = C \sum_{n_1+\cdots+n_N=K} \left[\prod_{(s,\mu)}^{[n]} \prod_{t=1}^{N} \frac{\gamma(\mathrm{i}m_{s\mu} - \mathrm{i}m_{tn_t})}{\gamma(1 + \mathrm{i}\widetilde{m}_t + \mathrm{i}m_{s\mu})} \mathcal{F}^{(\mathrm{s})}_{[n]}(x)\, \mathcal{F}^{(\mathrm{s})}_{[n]}(\bar{x}) \right], \end{aligned} \tag{5.3.1}$$

where we introduced the notations $\prod_{(s,\mu)}^{[n]} = \prod_{s=1}^N \prod_{\mu=0}^{n_s-1}$ and $\mathrm{i}m_{s\mu} = \mathrm{i}m_s + \mu b^2$. The conformal blocks $\mathcal{F}^{(\mathrm{s})}_{[n]}$ are given by

$$\mathcal{F}^{(\mathrm{s})}_{[n]}(x) = (1-x)^{-\gamma_1} x^{-\gamma_0 - \sum_{(s,\mu)}^{[n]} \mathrm{i}m_{s\mu}} f^{(\mathrm{s})}_{[n]}(x) \tag{5.3.2}$$

with exponents

$$\gamma_0 = -\frac{K(K-1)}{2} b^2 - \frac{K(N-1)}{2}(b^2+1) - \frac{K}{N}\sum_{s=1}^N \mathrm{i}m_s \tag{5.3.3}$$

$$\gamma_1 = -\frac{K(N-K)}{N} b^2 + \frac{K}{N}\sum_{s=1}^N (\mathrm{i}m_s + \mathrm{i}\widetilde{m}_s) \tag{5.3.4}$$

and series

$$f^{(\mathrm{s})}_{[n]}(x) = \sum_{k_{\dots}\geq 0} \prod_{(s,\mu)}^{[n]} \left[x^{k_{s\mu}} \prod_{t=1}^{N} \frac{(-\mathrm{i}\widetilde{m}_t - \mathrm{i}m_{s\mu})_{k_{s\mu}}}{(1 + \mathrm{i}m_{tn_t} - \mathrm{i}m_{s\mu})_{k_{s\mu}}} \right. \tag{5.3.5}$$

$$\left. \cdot \frac{\prod_{t=1}^N (1 + \mathrm{i}m_{tn_t} - \mathrm{i}m_{s\mu} + k_{s\mu} - k_{t(n_t-1)})_{k_{t(n_t-1)}}}{\prod_{(t,\nu)}^{[n]} (1 + \mathrm{i}m_{t\nu} - \mathrm{i}m_{s\mu} + k_{s\mu} - k_{t\nu})_{k_{t\nu} - k_{t(\nu-1)}}} \right]. \tag{5.3.6}$$

For a given weight $h_{[n]}$ of $\mathcal{R}(Kh_1)$, and a choice of $1 \leq p \leq N$ we define

$$\begin{aligned} I^p_{[n]} = \frac{\Gamma(-b^2)^K}{K!} \prod_{j=1}^K \left[\int_{-\infty}^{\infty} \frac{\mathrm{d}\sigma_j}{2\pi} \right] \Bigg\{ \prod_{i\neq j}^K \frac{\Gamma(\mathrm{i}\sigma_i - \mathrm{i}\sigma_j - b^2)}{\Gamma(\mathrm{i}\sigma_i - \mathrm{i}\sigma_j)} \\ \cdot \prod_{j=1}^K \left[(-x)^{\mathrm{i}\sigma_j} \frac{\prod_{s=1}^N [\Gamma(-\mathrm{i}\widetilde{m}_s + \mathrm{i}\sigma_j)\Gamma(-\mathrm{i}m_s - \mathrm{i}\sigma_j)]}{\prod_{s\neq p}^N [\Gamma(1 + \mathrm{i}m_{sn_s} + \mathrm{i}\sigma_j)\Gamma(-\mathrm{i}m_{sn_s} - \mathrm{i}\sigma_j)]} \right] \Bigg\}. \end{aligned} \tag{5.3.7}$$

The contours lie between poles of all the $\Gamma(-\mathrm{i}\widetilde{m}_s + \mathrm{i}\sigma_j)$ and poles of all the $\Gamma(-\mathrm{i}m_s - \mathrm{i}\sigma_j)$. The integral converges for x away from the positive real axis. The omission of some Γ functions (those with $s = p$) in the denominator is crucial for convergence, but this arbitrary choice of p will complicate calculations.

For $|x| \lessgtr 1$ we can close contours towards $\mathrm{i}\sigma_j \to \mp\mathrm{i}\infty$, enclosing some poles. The first case yields a linear combination of s-channel factors:

$$I^p_{[n]} = \sum_{[k]} T^p_{[n][k]} (-x)^{-\sum_{(s,\mu)}^{[k]} \mathrm{i}m_{s\mu}} f^{(\mathrm{s})}_{[k]} \tag{5.3.8}$$

$$T^p_{[n][k]} = \prod_{(s,\mu)}^{[k]} \frac{\prod_{t=1}^N [\Gamma(-\mathrm{i}\widetilde{m}_t - \mathrm{i}m_{s\mu})\Gamma(\mathrm{i}m_{s\mu} - \mathrm{i}m_{tk_t})]}{\prod_{t\neq p}^N [\Gamma(1 + \mathrm{i}m_{tn_t} - \mathrm{i}m_{s\mu})\Gamma(\mathrm{i}m_{s\mu} - \mathrm{i}m_{tn_t})]} \,. \tag{5.3.9}$$

The sum ranges over weights of $\mathcal{R}(Kh_1)$, but only some components of the matrix T^p are non-zero, namely those for which $k_s \leq n_s$ for all $s \neq p$. The second case yields a linear combination of u-channel factors

$$I^p_{[n]} = \sum_{[\widetilde{n}]} U^p_{[n][\widetilde{n}]} (-x)^{\sum_{(s,\mu)}^{[\widetilde{n}]} \mathrm{i}\widetilde{m}_{s\mu}} f^{(\mathrm{u})}_{[\widetilde{n}]} \tag{5.3.10}$$

$$U^p_{[n][\widetilde{n}]} = \prod_{(s,\mu)}^{[\widetilde{n}]} \frac{\prod_{t=1}^N [\Gamma(\mathrm{i}\widetilde{m}_{s\mu} - \mathrm{i}\widetilde{m}_{t\widetilde{n}_t})\Gamma(-\mathrm{i}\widetilde{m}_{s\mu} - \mathrm{i}m_t)]}{\prod_{t\neq p}^N [\Gamma(1 + \mathrm{i}m_{tn_t} + \mathrm{i}\widetilde{m}_{s\mu})\Gamma(-\mathrm{i}\widetilde{m}_{s\mu} - \mathrm{i}m_{tn_t})]} \,. \tag{5.3.11}$$

This leads to the braiding

$$(-x)^{-\sum_{(s,\mu)}^{[n]} \mathrm{i}m_{s\mu}} f^{(\mathrm{s})}_{[n]} = \sum_{[\widetilde{n}]} ((T^p)^{-1} U^p)_{[n][\widetilde{n}]} (-x)^{\sum_{(s,\mu)}^{[\widetilde{n}]} \mathrm{i}\widetilde{m}_{s\mu}} f^{(\mathrm{u})}_{[\widetilde{n}]} \,. \tag{5.3.12}$$

We thus need to invert the matrix T^p then multiply the result by U^p. A consistency check is that $(T^p)^{-1}U^p$ should not depend on p.

Split $T^p = \check{T}^p D^p$ with

$$\check{T}^p_{[n][k]} = \prod_{(s,\mu)}^{[k]} \prod_{t\neq p}^{N} \frac{1}{\pi} \sin \pi(\mathrm{i}m_{s\mu} - \mathrm{i}m_{tn_t}) \tag{5.3.13}$$

$$D^p_{[k][l]} = \delta_{[k][l]} \prod_{(s,\mu)}^{[k]} \prod_{t=1}^{N} [\Gamma(-\mathrm{i}\widetilde{m}_t - \mathrm{i}m_{s\mu})\Gamma(\mathrm{i}m_{s\mu} - \mathrm{i}m_{tk_t})] \,. \tag{5.3.14}$$

After some trial and error, one finds that $\left((\check{T}^p)^{-1}\right)_{[n][k]}$ vanishes if $n_t < k_t$ for any $t \neq p$, and otherwise is

$$\frac{\prod_{s<t}^N \left[\frac{1}{\pi} \sin \pi(\mathrm{i}m_{sk_s} - \mathrm{i}m_{tk_t}) \frac{1}{\pi} \sin \pi(\mathrm{i}m_{tn_t} - \mathrm{i}m_{sn_s})\right]}{\prod_{t\neq p}^N \left[\frac{1}{\pi} \sin \pi(\mathrm{i}m_{tk_t} - \mathrm{i}m_{pk_p}) \prod_{\substack{1\leq s\leq N, 0\leq\mu\leq n_s \\ (s,\mu)\neq(t,k_t)}} \frac{1}{\pi} \sin \pi(\mathrm{i}m_{s\mu} - \mathrm{i}m_{tk_t})\right]} \tag{5.3.15}$$

We must check that $\sum_{[k]}\bigl((\check{T}^p)^{-1}\bigr)_{[n][k]}(\check{T}^p)_{[k][j]} = \delta_{[n][j]}$. Both matrices are triangular in the sense that $(\check{T}^p)_{[k][j]} = 0$ if any $k_t < j_t$ for $t \neq p$, and similarly for $(\check{T}^p)^{-1}$. Their product is thus also triangular in the same sense: it vanishes whenever any $n_t < j_t$ for $t \neq p$. It is straightforward to compute the diagonal coefficients $\bigl((\check{T}^p)^{-1}\check{T}^p\bigr)_{[n][n]} = \bigl((\check{T}^p)^{-1}\bigr)_{[n][n]}(\check{T}^p)_{[n][n]} = 1$. There remains to show that coefficients $[n][j]$ of the product with $j_s \leq n_s$ for all $s \neq p$, and with $j_p > n_p$ (equivalently $[j] \neq [n]$) vanish. Cancelling functions $\frac{1}{\pi}\sin\pi(\dots)$ in the numerator and denominator as much as possible yields

$$\sum_{[k]}\bigl((\check{T}^p)^{-1}\bigr)_{[n][k]}(\check{T}^p)_{[k][j]} \tag{5.3.16}$$

$$= \sum_{[k], j_s\leq k_s\leq n_s\ \forall s\neq p}\ \prod_{\substack{s<t\\ s,t\neq p}}^{N}\left[\frac{1}{\pi}\sin\pi(\mathrm{i}m_{sk_s}-\mathrm{i}m_{tk_t})\frac{1}{\pi}\sin\pi(\mathrm{i}m_{tn_t}-\mathrm{i}m_{sn_s})\right]$$

$$\cdot\prod_{t\neq p}^{N}\frac{\frac{1}{\pi}\sin\pi(\mathrm{i}m_{pn_p}-\mathrm{i}m_{tn_t})\prod_{\mu=n_p+1}^{j_p-1}\frac{1}{\pi}\sin\pi(\mathrm{i}m_{p\mu}-\mathrm{i}m_{tk_t})}{\prod_{\substack{s\neq p, j_s\leq\mu\leq n_s\\ (s,\mu)\neq(t,k_t)}}\frac{1}{\pi}\sin\pi(\mathrm{i}m_{s\mu}-\mathrm{i}m_{tk_t})}.$$

This is the sum of residues of

$$\prod_{\substack{s<t\\ s,t\neq p}}^{N}\left[\frac{1}{\pi}\sin\pi(\tau_t-\tau_s)\frac{1}{\pi}\sin\pi(\mathrm{i}m_{tn_t}-\mathrm{i}m_{sn_s})\right]$$
$$\cdot\prod_{t\neq p}^{N}\frac{\frac{1}{\pi}\sin\pi(\mathrm{i}m_{pn_p}-\mathrm{i}m_{tn_t})\prod_{\mu=n_p+1}^{j_p-1}\frac{1}{\pi}\sin\pi(\mathrm{i}m_{p\mu}+\tau_t)}{\prod_{s\neq p}^{N}\prod_{\mu=j_s}^{n_s}\frac{1}{\pi}\sin\pi(\mathrm{i}m_{s\mu}+\tau_t)} \tag{5.3.17}$$

at $\tau_t = -\mathrm{i}m_{tk_t}$. Each τ_t appears in $N-2+j_p-n_p-1$ sines in the numerator, and $\sum_{s\neq p}(1+n_s-j_s) = N-1+j_p-n_p$ in the denominator, *i.e.*, two more. Thus the function is 1-periodic in each variable τ_t, and decays like $1/e^{2\pi\mathrm{i}|\tau_t|}$ as $\tau_t\to\pm\infty$. The sum of residues thus vanishes, because it is the sum of all residues in a fundamental domain of the periodicity, and there is no contribution from infinity. This establishes (5.3.15).

We can compute the braiding matrix as $(T^p)^{-1}U^p = (D^p)^{-1}(\check{T}^p)^{-1}U^p$, and write the resulting sum as a sum of residues of some function of $N-1$ variables τ_t for $t\neq p$. Relabelling the variables τ_t using a permutation of $[\![1,N]\!]$ so that they are numbered from 1 to $N-1$ and $\phi(N)=p$, we obtain

$$\bigl((T^p)^{-1}U^p\bigr)_{[n][\widetilde{n}]} \tag{5.3.18}$$

$$= (-1)^{\phi}\prod_{t=1}^{N}\frac{\prod_{(s,\mu)}^{[\widetilde{n}]}\Gamma(\mathrm{i}\widetilde{m}_{s\mu}-\mathrm{i}\widetilde{m}_{t\widetilde{n}_t})\Gamma(-\mathrm{i}\widetilde{m}_{s\mu}-\mathrm{i}m_t)}{\prod_{(s,\mu)}^{[n]}\Gamma(-\mathrm{i}\widetilde{m}_t-\mathrm{i}m_{s\mu})\Gamma(\mathrm{i}m_{s\mu}-\mathrm{i}m_{tn_t})}\prod_{s<t}^{N}\frac{\sin\pi(\mathrm{i}m_{tn_t}-\mathrm{i}m_{sn_s})}{\pi}$$

$$\prod_{j=1}^{N-1}\left[\sum_{k_j=0}^{n_{\phi(j)}}\operatorname*{res}_{\tau_j=-\mathrm{i}m_{\phi(j)k_j}}\frac{\prod_{(s,\mu)}^{[\widetilde{n}]}\frac{1}{\pi}\sin\pi(-\mathrm{i}\widetilde{m}_{s\mu}+\tau_j)}{\prod_{s=1}^{N}\prod_{\mu=0}^{n_s}\frac{1}{\pi}\sin\pi(\mathrm{i}m_{s\mu}+\tau_j)}\prod_{i=0}^{j-1}\frac{1}{\pi}\sin\pi(\tau_j-\tau_i)\right]$$

where $(-1)^\phi$ is the signature of ϕ. This expression does not change if we replace ϕ by another permutation such that $\phi(N)=p$ and we permute the τ_j accordingly: indeed, the sign coming from $\sin\pi(\tau_j-\tau_i)$ is compensated by the change in $(-1)^\phi$. Since the braiding matrix must not depend on p (see comments near (5.3.12)), we expect that (5.3.18) is in fact completely independent of ϕ.

To show independence on p, choose an index $1\leq j\leq N-1$. The variable τ_j appears in $N-2+K$ sines in the numerator and $N+K$ sines in the denominator of (5.3.18). We thus have $\tau_j\to\tau_j+1$ periodicity, and no residue at infinity, hence the sum of residues at $\tau_j=-\mathrm{i}m_{\phi(j)k_j}$ is equal to minus the sum of all other residues in a strip of width 1. This yields a sum over $\tau_j=-\mathrm{i}m_{\phi(i)k}$ for all $1\leq i\leq N$ with $i\neq j$ and $0\leq k\leq n_{\phi(i)}$. The contribution from a given i with $i<N$ (and $i\neq j$) vanishes by antisymmetry under the exchange $\tau_i\leftrightarrow\tau_j$, thus only the poles at $-\mathrm{i}m_{\phi(N)k}=-\mathrm{i}m_{pk}$ contribute. All in all, we obtain the same expression as (5.3.18), with $\phi(j)$ and $\phi(N)$ exchanged. The sign coming from reversing the contour is absorbed into a change of the signature $(-1)^\phi$.

As before, the braiding matrix for $\mathcal{F}^{(\mathrm{s})}_{[n]}(x)$ is obtained by including a phase $e^{\mathrm{i}\pi\epsilon\gamma_1}$, and another phase comes from using factors $x^{\cdots}$ instead of $(-x)^{\cdots}$. Putting everything together yields

$$\mathcal{F}^{(\mathrm{s})}_{[n]}(x)=\sum_{[\widetilde{n}]}\mathbf{B}^{\epsilon}_{[n][\widetilde{n}]}\,\mathcal{F}^{(\mathrm{u})}_{[\widetilde{n}]}(x) \tag{5.3.19}$$

with

$$\mathbf{B}^{\epsilon}_{[n][\widetilde{n}]}=e^{\mathrm{i}\pi\epsilon\gamma_1}e^{\mathrm{i}\pi\epsilon\sum_{(s,\mu)}^{[n]}(-\mathrm{i}m_{s,\mu})}\big((T^p)^{-1}U^p\big)_{[n][\widetilde{n}]}e^{-\mathrm{i}\pi\epsilon\sum_{(s,\mu)}^{[\widetilde{n}]}\mathrm{i}\widetilde{m}_{s,\mu}}\,. \tag{5.3.20}$$

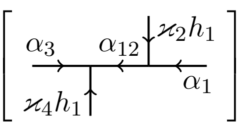
The explicit expression of $(T^p)^{-1}U^p$ involves a permutation ϕ, but as shown above it is independent of the permutation and of p. We do not translate this expression back from the $\{\mathrm{i}m,\mathrm{i}\widetilde{m}\}$ notation to momenta: one simply has to replace $\mathrm{i}m_{s\mu}=b\langle Q-\alpha_0,h_s\rangle+\mu b^2+\frac{1}{N}\sum_{t=1}^N\mathrm{i}m_t$ and $\mathrm{i}\widetilde{m}_{s\mu}=\frac{b\varkappa}{N}+b\langle Q-\alpha_\infty,h_s\rangle+\mu b^2-1-\frac{1}{N}\sum_{t=1}^N\mathrm{i}m_t$, then shift the variables τ_j to absorb $\frac{1}{N}\sum_{t=1}^N\mathrm{i}m_t$.

5.3.2 Braiding kernel of two semi-degenerates

Results in this section are new and will form the core of an upcoming publication. We find the braiding kernel of two semi-degenerate primary operators, which generalizes the braiding/fusion kernel for Virasoro ($N=2$) conformal blocks [PT99]. In the context of the AGT relation, this kernel corresponds to the partition function of a domain wall between S-dual 4d $\mathcal{N}=2$ $SU(N)$ SQCD with $2N$ flavours, but we did not have time to explore this in the thesis.

The four-point function $\langle V_{\alpha_3}(\infty)V_{\alpha_4}(1)V_{\alpha_2}(x,\bar{x})V_{\alpha_1}(0)\rangle$ with two generic momenta α_1, α_3 and two semi-degenerate momenta $\alpha_2 = \varkappa_2 h_1$, $\alpha_4 = \varkappa_4 h_1$ has an s-channel decomposition

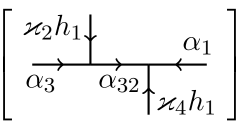
$$\langle V_{\alpha_3}(\infty)V_{\varkappa_4 h_1}(1)V_{\varkappa_2 h_1}(x,\bar{x})V_{\alpha_1}(0)\rangle$$
$$= \int \mathrm{d}\alpha_{12}\, C(\alpha_3, \varkappa_4 h_1, 2Q-\alpha_{12})C(\alpha_{12}, \varkappa_2 h_1, \alpha_1)\left|\mathcal{F}\left[\ \cdots\ \right](x)\right|^2 . \quad (5.3.21)$$



Note that the internal momentum α_{12} is continuous rather than discrete because there is no fully degenerate vertex operator. The s-channel conformal blocks are in principle fixed by W_N symmetry. In practice, closed forms are only known thanks to the AGT relation with instanton partition functions, and we will not need them. In this section we normalize conformal blocks as $\mathcal{F}^{(s)}_{\alpha_{12}}(x) = (-x)^{\Delta(\alpha_{12})-\Delta(\alpha_1)-\Delta(\varkappa_2 h_1)}(1+\cdots)$: the use of $-x$ instead of x avoids uninteresting phases.

The s-channel decomposition (5.3.21) has a u-channel counterpart with $\varkappa_2 \leftrightarrow \varkappa_4$:

$$\langle V_{\alpha_3}(\infty)V_{\varkappa_4 h_1}(1)V_{\varkappa_2 h_1}(x,\bar{x})V_{\alpha_1}(0)\rangle$$
$$= \int \mathrm{d}\alpha_{32}\, C(\alpha_3, \varkappa_2 h_1, \alpha_{32})C(2Q-\alpha_{32}, \varkappa_4 h_1, \alpha_1)\left|\mathcal{F}\left[\ \cdots\ \right](x)\right|^2 . \quad (5.3.22)$$



Again, we normalize these u-channel conformal blocks so that their leading term is a power of $(-x)$, namely $\mathcal{F}^{(u)}_{\alpha_{32}}(x) \sim x^{\Delta(\alpha_3)-\Delta(\alpha_{32})-\Delta(\varkappa_2 h_1)}$. Both sets of conformal blocks are analytic on $\mathbb{C}\setminus\mathbb{R}_{\geq 0}$. The two decompositions are related by a braiding transformation

$$\mathcal{F}^{(s)}_{\alpha_{12}}(x) = \int \mathrm{d}\alpha_{32}\, \mathbf{B}_{\alpha_{12}\alpha_{32}}\, \mathcal{F}^{(u)}_{\alpha_{32}}(x) \ . \quad (5.3.23)$$

Our goal is to find the braiding kernel $\mathbf{B}_{\alpha_{12}\alpha_{32}}$.

From Section 5.3.1 we know this braiding kernel in the limit $\varkappa_2 h_1 \to -Kbh_1$, in other words when one of the semi-degenerate operators turns into a degenerate operator. Then it takes the form of a sum of residues (hence an integral) of a product of sines. This product of sines can be recast in terms of a special function S_b which we introduce below. The braiding kernel for generic $\varkappa_2$ should thus be an integral of some S_b functions, and it indeed is (5.3.27).

The Barnes double-Gamma function Γ_b obeys

$$\Gamma_b(x+b) = \Gamma_b(x)\sqrt{2\pi}b^{xb-1/2}/\Gamma(xb) \quad (5.3.24)$$

and $\Gamma_b(x) = \Gamma_{1/b}(x)$. The function is analytic in x except for poles at $x = -mb - n/b$ for integers $m, n \geq 0$. From Γ_b one constructs the double sine function $S_b(x)$ and the Upsilon function $\Upsilon(x)$:

$$S_b(x) = \Gamma_b(x)/\Gamma_b(q-x) \qquad \text{and} \qquad \Upsilon(x) = 1/\left[\Gamma_b(x)\Gamma_b(q-x)\right] . \quad (5.3.25)$$

S_b has poles at $x = -mb - n/b$ and zeros at $x = (1+m)b + (1+n)/b$ for integers $m, n \geq 0$, while Υ has zeros at all these points and has no pole. Finally, the two functions obey shift relations deduced from (5.3.24):

$$S_b(x+b) = 2\sin \pi bx S_b(x) \qquad \text{and} \qquad \Upsilon(x+b) = b^{1-2bx}\gamma(bx)\Upsilon(x) \quad (5.3.26)$$

with $\gamma(x) = \Gamma(x)/\Gamma(1-x)$.

In terms of these functions, and writing all $\alpha_\bullet = Q - \mathrm{i}a_\bullet$, the claim is

$$\begin{aligned}
&\mathbf{B}_{\alpha_{12}\alpha_{32}}\begin{bmatrix} \varkappa_4 h_1 & \varkappa_2 h_1 \\ \alpha_3 & \alpha_1 \end{bmatrix} \\
&= \prod_{s,t=1}^{N}\left[\frac{\Gamma_b(\frac{\varkappa_2}{N} + \langle \mathrm{i}a_3, h_s\rangle - \langle \mathrm{i}a_{32}, h_t\rangle)\Gamma_b(q - \frac{\varkappa_4}{N} - \langle \mathrm{i}a_1, h_s\rangle - \langle \mathrm{i}a_{32}, h_t\rangle)}{\Gamma_b(\frac{\varkappa_2}{N} + \langle \mathrm{i}a_1, h_s\rangle - \langle \mathrm{i}a_{12}, h_t\rangle)\Gamma_b(q - \frac{\varkappa_4}{N} - \langle \mathrm{i}a_3, h_s\rangle - \langle \mathrm{i}a_{12}, h_t\rangle)}\right] \\
&\quad \prod_{s\neq t}^{N}\left[\frac{\Gamma_b(q + \langle \mathrm{i}a_{12}, h_s - h_t\rangle)}{\Gamma_b(\langle \mathrm{i}a_{32}, h_s - h_t\rangle)}\right]\prod_{j=1}^{N-1}\left[\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}\tau_j\right]\left\{\prod_{i\neq j}^{N-1}\frac{1}{S_b(\tau_i - \tau_j)}\right. \\
&\quad \left.\prod_{j=1}^{N-1}\prod_{s=1}^{N}\left[\frac{S_b(-\langle \mathrm{i}a_3, h_s\rangle + \tau_j)S_b(\frac{\varkappa_2}{N} + \frac{\varkappa_4}{N} - q + \langle \mathrm{i}a_1, h_s\rangle + \tau_j)}{S_b(\frac{\varkappa_2}{N} - \langle \mathrm{i}a_{32}, h_s\rangle + \tau_j)S_b(\frac{\varkappa_4}{N} + \langle \mathrm{i}a_{12}, h_s\rangle + \tau_j)}\right]\right\}
\end{aligned} \tag{5.3.27}$$

up to a constant factor that does not depend on any momentum. It would be interesting to recast the integral over τ as an integral over a momentum, and give a physical interpretation of it. The integration contours go from $-\mathrm{i}\infty$ to $\mathrm{i}\infty$ with poles of the numerator S_b functions to the left of the contours, and poles of the denominator to the right. For instance, if all components $\langle \mathrm{i}a_1, h_s\rangle$ and $\langle \mathrm{i}a_3, h_2\rangle$ are purely imaginary and $0 < \operatorname{Re}\frac{\varkappa_i}{N} < q$, then contours can be taken to be vertical lines with $\max(0, q - \frac{\varkappa_2}{N} - \frac{\varkappa_4}{N}) < \operatorname{Re}(\tau_j) < \min(q - \frac{\varkappa_2}{N}, q - \frac{\varkappa_4}{N})$. For other values of momenta, the contour is deformed to keep the same set of poles on each side. Another remark is that $\prod_{i\neq j}^{N-1}\frac{1}{S_b(\tau_i-\tau_j)}$ simplifies to a product of sines and has no pole.

In a normalization of conformal blocks where the leading term is a power of x, the braiding kernel is

$$\mathbf{B}^{\epsilon}_{\alpha_{12}\alpha_{32}}\begin{bmatrix} \varkappa_4 h_1 & \varkappa_2 h_1 \\ \alpha_3 & \alpha_1 \end{bmatrix} = e^{\mathrm{i}\pi\epsilon[\Delta(\alpha_{12})-\Delta(\alpha_1)+\Delta(\alpha_{32})-\Delta(\alpha_3)]}\mathbf{B}_{\alpha_{12}\alpha_{32}}\begin{bmatrix} \varkappa_4 h_1 & \varkappa_2 h_1 \\ \alpha_3 & \alpha_1 \end{bmatrix} \tag{5.3.28}$$

where ϵ is the sign of $\operatorname{Im} x$.

A preliminary check of (5.3.27) is that it reproduces known results [PT99] for the Liouville theory ($N = 2$). In their equation (48) replace their Q by q to avoid confusion, shift the integration variable $s \to s - \alpha_{21} + \alpha_4 - q/2$, then map $\alpha_2 \to q - \alpha_2$ (for $N = 2$ this is a Weyl symmetry). The factors with $U_{3,4}$ become $S_b(\pm(\alpha_3 - q/2) + s)$. The factors with $U_{1,2}$ become $S_b(\alpha_4 + \alpha_2 - q \pm (\alpha_1 - q/2) + s)$. The denominator factors with $V_{1,2}$ become $S_b(\alpha_2 \pm (\alpha_{32} - q/2) + s)$. The denominator factors with $V_{3,4}$ become

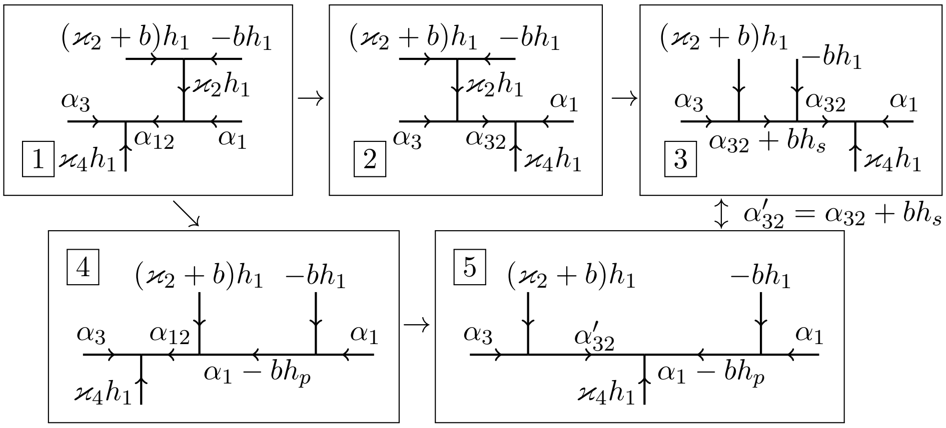


Figure 5.1: Pentagon identity. $1\to 2$ and $4\to 5$ are braidings of two semi-degenerates, $2\to 3$ and $1\to 4$ are known fusions of V_{-bh_1} and a semi-degenerate, $5\leftrightarrow 3$ is a known braiding of V_{-bh_1} and a semi-degenerate.

$S_b\big(\alpha_4\pm(\alpha_{21}-q/2)+s\big)$. Thus, the integrand from [PT99] coincides with that of (5.3.27) for $N=2$. It is straightforward to check that prefactors also coincide.

To confirm (5.3.27) we check that the braiding kernel obeys some expected shift relations. We then describe how to take the limit $\varkappa_2 h_1\to -Kbh_1$ to retrieve the sum of residues from Section 5.3.1.

Shift relations

Braiding and fusion kernels (or matrices) obey pentagon and hexagon relations. Here we consider the pentagon relation shown in Figure 5.1. Going through the moves $1\to 2\to 3$ we find

$$\mathcal{F}[1]=\int \mathrm{d}\alpha_{32}\,\mathbf{B}_{\alpha_{12}\alpha_{32}}\begin{bmatrix}\varkappa_4 h_1 & \varkappa_2 h_1\\ \alpha_3 & \alpha_1\end{bmatrix}\mathcal{F}[2] \tag{5.3.29}$$

$$=\int \mathrm{d}\alpha_{32}\sum_{s=1}^{N}\mathbf{B}_{\alpha_{12}\alpha_{32}}\begin{bmatrix}\varkappa_4 h_1 & \varkappa_2 h_1\\ \alpha_3 & \alpha_1\end{bmatrix}\mathbf{F}_s\begin{bmatrix}(\varkappa_2+b)h_1 & -bh_1\\ \alpha_3 & 2Q-\alpha_{32}\end{bmatrix}\mathcal{F}[3]\ . \tag{5.3.30}$$

On the other hand, going through the moves $1\to 4\to 5\to 3$ yields

$$\mathcal{F}[1]=\sum_{p=1}^{N}\mathbf{F}_p\begin{bmatrix}(\varkappa_2+b)h_1 & -bh_1\\ 2Q-\alpha_{12} & \alpha_1\end{bmatrix}\mathcal{F}[4] \tag{5.3.31}$$

$$=\sum_{p=1}^{N}\int \mathrm{d}\alpha'_{32}\,\mathbf{F}_p\begin{bmatrix}(\varkappa_2+b)h_1 & -bh_1\\ 2Q-\alpha_{12} & \alpha_1\end{bmatrix}\mathbf{B}_{\alpha_{12}\alpha'_{32}}\begin{bmatrix}\varkappa_4 h_1 & (\varkappa_2+b)h_1\\ \alpha_3 & \alpha_1-bh_p\end{bmatrix}\mathcal{F}[5] \tag{5.3.32}$$

$$= \sum_{p,s=1}^{N} \int \mathrm{d}\alpha'_{32} \mathbf{F}_p \begin{bmatrix} (\varkappa_2+b)h_1 & -bh_1 \\ 2Q-\alpha_{12} & \alpha_1 \end{bmatrix} \cdot \mathbf{B}_{\alpha_{12}\alpha'_{32}} \begin{bmatrix} \varkappa_4 h_1 & (\varkappa_2+b)h_1 \\ \alpha_3 & \alpha_1 - bh_p \end{bmatrix} \cdot \mathbf{B}_{ps} \begin{bmatrix} \varkappa_4 h_1 & -bh_1 \\ \alpha'_{32} & \alpha_1 \end{bmatrix} \mathcal{F}[3, \alpha_{32} = \alpha'_{32} - bh_s] \; . \tag{5.3.33}$$

The coefficients of each conformal block of the form $\mathcal{F}[3]$ (these are labelled by the choice of α_{32} and $1 \le s \le N$) must be the same in (5.3.30) and (5.3.33). Given that we know braiding and fusion matrices for V_{-bh_1}, $V_{(\varkappa_2+b)h_1}$ and two generic vertex operators, this pentagon relation expresses the braiding kernel $\mathbf{B}_{\alpha_{12}\alpha_{32}}$ as a sum over p of braiding kernels with $\varkappa_2 \to \varkappa_2 + b$, $\alpha_1 \to \alpha_1 - bh_p$ and $\alpha_{32} \to \alpha'_{32} = \alpha_{32} + bh_s$. Thus, if the braiding kernel is known for some value $\varkappa_2 = \lambda$, it can be deduced for $\varkappa_2 = \lambda - Kb$ for integer $K \ge 0$. Repeating very similar computations for the pentagon identity $(1 \to 2 \to 3 \to 5) = (1 \to 4 \to 5)$ allows the opposite shifts: from the $\varkappa_2 = \lambda$ braiding kernel one gets the $\varkappa_2 = \lambda + Kb$ braiding kernel. By symmetry, identical shift relations exist with $b \to \frac{1}{b}$, thus fixing braiding kernels for $\varkappa_2 = \lambda + Kb + L/b$ for all integers K, L. For generic real b^2, continuity then determines the braiding kernel uniquely. The point that is not clear to the author is whether $\varkappa_2 = 0$ (or any $\varkappa_2 = -Kb$) is a valid starting point for this reasoning, as the braiding kernel is singular there.

To check that the proposal (5.3.27) obeys the pentagon identity, we will need the braiding matrix obtained from (5.2.19) using $\alpha_1 = Q - \mathrm{i}a_1$ and $\alpha'_{32} = Q - \mathrm{i}a_{32} + bh_s$:

$$\mathbf{B}_{ps} \begin{bmatrix} \varkappa_4 h_1 & -bh_1 \\ \alpha'_{32} & \alpha_1 \end{bmatrix} = \prod_{t \neq p}^{N} \frac{\Gamma(1 + b\langle \mathrm{i}a_1, h_t - h_p \rangle)}{\Gamma(\frac{b\varkappa_4}{N} + b\langle \mathrm{i}a_1, h_t\rangle + b\langle \mathrm{i}a_{32}, h_s\rangle - b^2)} \prod_{u \neq s}^{N} \frac{\Gamma(b\langle \mathrm{i}a_{32}, h_s - h_u\rangle - b^2)}{\Gamma(1 - \frac{b\varkappa_4}{N} - b\langle \mathrm{i}a_1, h_p\rangle - b\langle \mathrm{i}a_{32}, h_u\rangle)} \, . \tag{5.3.34}$$

We will also need coefficients of the fusion of $(\varkappa_2 + b)h_1$ and $-bh_1$ into $\varkappa_2 h_1$, which can be deduced from the braiding matrix (5.2.19), as done in [GLF10] (see equation (B.14) there).

$$\mathbf{F}_p \begin{bmatrix} (\varkappa_2+b)h_1 & -bh_1 \\ Q - \mathrm{i}a' & Q - \mathrm{i}a \end{bmatrix} = \Gamma(b\varkappa_2) \frac{\prod_{t\neq p}^{N} \Gamma(b\langle \mathrm{i}a, h_p - h_t\rangle)}{\prod_{t=1}^{N} \Gamma(\frac{b\varkappa_2}{N} + b\langle \mathrm{i}a, h_p\rangle + b\langle \mathrm{i}a', h_t\rangle)} \, . \tag{5.3.35}$$

We now write down (5.3.33) explicitly for a fixed choice of α_{32} and of $1 \le s \le N$, and simplify it in order to find (5.3.30). All generic momenta are written as $\alpha = Q - \mathrm{i}a$ and we denote $\mathrm{i}a^u = \langle \mathrm{i}a, h_u\rangle$ for conciseness. Note that $\alpha'_{32} = Q - \mathrm{i}a_{32} + bh_s$. Let us go!

$$\begin{aligned}
&\sum_{p=1}^{N} \mathbf{F}_p \begin{bmatrix} (\varkappa_2+b)h_1 & -bh_1 \\ 2Q-\alpha_{12} & \alpha_1 \end{bmatrix} \mathbf{B}_{\alpha_{12}\alpha'_{32}} \begin{bmatrix} \varkappa_4 h_1 & (\varkappa_2+b)h_1 \\ \alpha_3 & \alpha_1 - bh_p \end{bmatrix} \mathbf{B}_{ps} \begin{bmatrix} \varkappa_4 h_1 & -bh_1 \\ \alpha'_{32} & \alpha_1 \end{bmatrix} \\
&= \sum_{p=1}^{N} \Bigg(\Gamma(b\varkappa_2) \frac{\prod_{t\neq p}^{N} \Gamma(b\mathrm{i}a_1^p - b\mathrm{i}a_1^t)}{\prod_{t=1}^{N} \Gamma(\frac{b\varkappa_2}{N} + b\mathrm{i}a_1^p - b\mathrm{i}a_{12}^t)} \\
&\prod_{t,u=1}^{N} \left[\frac{\Gamma_b(\frac{\varkappa_2}{N} + \mathrm{i}a_3^t - \mathrm{i}a_{32}^u + b\delta_{s,u}) \Gamma_b(q - \frac{\varkappa_4}{N} - \mathrm{i}a_1^t - \mathrm{i}a_{32}^u - b\delta_{p,t} + b\delta_{s,u})}{\Gamma_b(\frac{\varkappa_2}{N} + \mathrm{i}a_1^t - \mathrm{i}a_{12}^u + b\delta_{p,t}) \Gamma_b(q - \frac{\varkappa_4}{N} - \mathrm{i}a_3^t - \mathrm{i}a_{12}^u)} \right] \\
&\prod_{t\neq u}^{N} \left[\frac{\Gamma_b(q + \mathrm{i}a_{12}^t - \mathrm{i}a_{12}^u)}{\Gamma_b(\mathrm{i}a_{32}^t - \mathrm{i}a_{32}^u - b\delta_{s,t} + b\delta_{s,u})} \right] \prod_{j=1}^{N-1} \left[\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}\tau_j \right] \Bigg\{ \prod_{i\neq j}^{N-1} \frac{1}{S_b(\tau_i - \tau_j)} \\
&\prod_{j=1}^{N-1} \prod_{t=1}^{N} \left[\frac{S_b(-\mathrm{i}a_3^t + \tau_j) S_b(\frac{\varkappa_2}{N} + \frac{\varkappa_4}{N} - q + \mathrm{i}a_1^t + \tau_j + b\delta_{p,t})}{S_b(\frac{\varkappa_2}{N} - \mathrm{i}a_{32}^t + \tau_j + b\delta_{s,t}) S_b(\frac{\varkappa_4}{N} + \mathrm{i}a_{12}^t + \tau_j)} \right] \Bigg\} \\
&\prod_{t\neq p}^{N} \frac{\Gamma(1 + b\mathrm{i}a_1^t - b\mathrm{i}a_1^p)}{\Gamma(\frac{b\varkappa_4}{N} + b\mathrm{i}a_1^t + b\mathrm{i}a_{32}^s - b^2)} \prod_{u\neq s}^{N} \frac{\Gamma(b\mathrm{i}a_{32}^s - b\mathrm{i}a_{32}^u - b^2)}{\Gamma(1 - \frac{b\varkappa_4}{N} - b\mathrm{i}a_1^p - b\mathrm{i}a_{32}^u)} \Bigg) .
\end{aligned} \tag{5.3.36}$$

We collect factors which do not depend on p, s using (5.3.24) and (5.3.25). Factors of $\sqrt{2\pi}$ and powers of b cancel, many Gamma functions cancel, and others combine through $\Gamma(x)\Gamma(1-x) = \pi/\sin\pi x$.

$$\begin{aligned}
&= \prod_{t,u=1}^{N} \left[\frac{\Gamma_b(\frac{\varkappa_2}{N} + \mathrm{i}a_3^t - \mathrm{i}a_{32}^u) \Gamma_b(q - \frac{\varkappa_4}{N} - \mathrm{i}a_1^t - \mathrm{i}a_{32}^u)}{\Gamma_b(\frac{\varkappa_2}{N} + \mathrm{i}a_1^t - \mathrm{i}a_{12}^u) \Gamma_b(q - \frac{\varkappa_4}{N} - \mathrm{i}a_3^t - \mathrm{i}a_{12}^u)} \right] \prod_{t\neq u}^{N} \left[\frac{\Gamma_b(q + \mathrm{i}a_{12}^t - \mathrm{i}a_{12}^u)}{\Gamma_b(\mathrm{i}a_{32}^t - \mathrm{i}a_{32}^u)} \right] \\
&\int \Bigg\{ \frac{\prod_{j=1}^{N-1} \mathrm{d}\tau_j}{\prod_{i\neq j} S_b(\tau_i - \tau_j)} \prod_{j=1}^{N-1} \prod_{t=1}^{N} \left[\frac{S_b(-\mathrm{i}a_3^t + \tau_j) S_b(\frac{\varkappa_2}{N} + \frac{\varkappa_4}{N} - q + \mathrm{i}a_1^t + \tau_j)}{S_b(\frac{\varkappa_2}{N} - \mathrm{i}a_{32}^t + \tau_j) S_b(\frac{\varkappa_4}{N} + \mathrm{i}a_{12}^t + \tau_j)} \right] \\
&\qquad \frac{\Gamma(b\varkappa_2) \prod_{u\neq s}^{N} \Gamma(b\mathrm{i}a_{32}^u - b\mathrm{i}a_{32}^s) \prod_{t=1}^{N} \frac{1}{\pi} \sin\pi(\frac{b\varkappa_4}{N} + b\mathrm{i}a_1^t + b\mathrm{i}a_{32}^s - b^2)}{\prod_{j=1}^{N-1} \frac{1}{\pi} \sin\pi b(\frac{\varkappa_2}{N} - \mathrm{i}a_{32}^s + \tau_j) \prod_{t=1}^{N} \Gamma(\frac{b\varkappa_2}{N} + b\mathrm{i}a_3^t - b\mathrm{i}a_{32}^s)} \\
&\qquad \sum_{p=1}^{N} \frac{\prod_{j=1}^{N-1} \frac{1}{\pi} \sin\pi b(\frac{\varkappa_2}{N} + \frac{\varkappa_4}{N} - q + \mathrm{i}a_1^p + \tau_j)}{\frac{1}{\pi} \sin\pi(\frac{b\varkappa_4}{N} + b\mathrm{i}a_1^p + b\mathrm{i}a_{32}^s - b^2) \prod_{t\neq p}^{N} \frac{1}{\pi} \sin\pi(b\mathrm{i}a_1^p - b\mathrm{i}a_1^t)} \Bigg\}
\end{aligned} \tag{5.3.37}$$

The last line is a sum of residues of $\prod_{j=1}^{N-1} \left[\frac{1}{\pi} \sin\pi(\frac{b\varkappa_2}{N} + \frac{b\varkappa_4}{N} - bq + \upsilon + b\tau_j)\right] / \left[\frac{1}{\pi} \sin\pi(\frac{b\varkappa_4}{N} + \upsilon + b\mathrm{i}a_{32}^s - b^2) \prod_{t=1}^{N} \frac{1}{\pi} \sin\pi(\upsilon - b\mathrm{i}a_1^t)\right]$ at $\upsilon = b\mathrm{i}a_1^p$, which is equal to minus its residue at $\upsilon = b^2 - \frac{b\varkappa_4}{N} - b\mathrm{i}a_{32}^s$. That residue turns out to cancel most of the second to last line. Together, these last two lines of (5.3.37) are equal to

$$\frac{\Gamma(b\varkappa_2) \prod_{u\neq s}^{N} \Gamma(b\mathrm{i}a_{32}^u - b\mathrm{i}a_{32}^s)}{\prod_{t=1}^{N} \Gamma(\frac{b\varkappa_2}{N} + b\mathrm{i}a_3^t - b\mathrm{i}a_{32}^s)} = \mathbf{F}_s \begin{bmatrix} (\varkappa_2+b)h_1 & -bh_1 \\ \alpha_3 & 2Q-\alpha_{32} \end{bmatrix} . \tag{5.3.38}$$

In particular, this does not depend on τ_j and can be pulled out of the integral. The first two lines of (5.3.37) then reproduce precisely the braiding matrix (5.3.27) of two semi-degenerate vertex operators. This concludes our check of the pentagon relation (5.3.30) = (5.3.33).

Next, we describe the limit $\varkappa_2 h_1 \to -Kbh_1$.

From semi-degenerate to degenerate

When a contour integral is pinched by poles getting close together from the two sides of the contour, the integral is singular. Indeed, if $f(z)$ is holomorphic in a neighborhood of a, and a_L and a_R are points in this neighborhood, then

$$\int_{\text{between}} \mathrm{d}z \, \frac{f(z)}{(z-a_L)(z-a_R)} = 2\pi\mathrm{i}\frac{f(a_L)}{a_L - a_R} + \int_{\text{left}} \mathrm{d}z \, \frac{f(z)}{(z-a_L)(z-a_R)} \tag{5.3.39}$$

where the initial contour goes between the two points, with a_L on its left and a_R on its right, and where the second contour is moved through a_L. The singular part as $a_L, a_R \to a$ is obtained by taking the limit $f(z)/(z-a)^2$ of the original function and considering its second residue (multiplying it by $z-a$ then take the residue). We shall denote this operation of taking the second residue as res^2.

The integrand in (5.3.27) has poles at

$$\tau_j = \begin{cases} \langle \mathrm{i}a_3, h_s\rangle - mb - n/b \\ q - \frac{\varkappa_2}{N} - \frac{\varkappa_4}{N} - \langle \mathrm{i}a_1, h_s\rangle - mb - n/b \end{cases} \tag{5.3.40}$$

or

$$\tau_j = \begin{cases} q - \frac{\varkappa_2}{N} + \langle \mathrm{i}a_{32}, h_s\rangle + mb + n/b \\ q - \frac{\varkappa_4}{N} - \langle \mathrm{i}a_{12}, h_s\rangle + mb + n/b \end{cases} \tag{5.3.41}$$

for integers $m, n \geq 0$. As mentionned before, the contour is chosen with poles (5.3.40) on the left and poles (5.3.41) on the right. This is possible as long as the two sets of poles are disjoint. Otherwise, the contour is pinched between the two sets and the integral diverges. Whenever one of the (5.3.40) is equal to $q - \frac{\varkappa_4}{N} - \langle \mathrm{i}a_{12}, h_s\rangle + mb + n/b$, the contour is pinched, but the prefactors in (5.3.27) (more precisely, the denominator Γ_b functions) cancel the singularity. On the other hand, if one of the (5.3.40) is equal to $q - \frac{\varkappa_2}{N} + \langle \mathrm{i}a_{32}, h_s\rangle + mb + n/b$, then prefactors do not cancel the singularity, and the braiding kernel is genuinely singular. These singularity, together with those of numerator Γ_b functions in (5.3.27), precisely reproduce singularities of the u-channel three-point functions, at least for the Toda CFT:

$$C(\alpha_3, \varkappa_2 h_1, 2Q - \alpha_{32}) C(\alpha_{32}, \varkappa_4 h_1, \alpha_1) = \frac{\cdots}{\prod_{t,u} \left[\Upsilon(\frac{\varkappa_2}{N} + \langle \mathrm{i}a_3, h_t\rangle - \langle \mathrm{i}a_{32}, h_u\rangle)\Upsilon(\frac{\varkappa_4}{N} + \langle \mathrm{i}a_1, h_t\rangle + \langle \mathrm{i}a_{32}, h_u\rangle)\right]} \,. \tag{5.3.42}$$

It may be interesting to pursue further the analysis by considering multiple singularities, keeping in mind the constraints $\sum_t \langle \mathrm{i}a_j, h_t\rangle = 0$ for each momentum.

We now focus on the limit $\varkappa_2 = -Kb + N\mathrm{i}\varepsilon$ for $\varepsilon \to 0$ (and $\varepsilon > 0$). The OPE of V_{-Kbh_1} with a generic vertex operator constrains α_{12} and α_{32}, so we further focus on $\alpha_{21} = \alpha_1 - bh_{[n]}$ and $\alpha_{32} = \alpha_3 - bh_{[\widetilde{n}]}$ for some weights $h_{[n]}$ and $h_{[\widetilde{n}]}$ in $\mathcal{R}(Kh_1)$. To simplify some later expressions we set $\varkappa_4 = \varkappa + Kb$. The poles (5.3.40) are now at

$$\tau_j = \begin{cases} \langle \mathrm{i}a_3, h_s\rangle - mb - n/b \\ q - \frac{\varkappa}{N} - \langle \mathrm{i}a_1, h_s\rangle - mb - n/b - \mathrm{i}\varepsilon \end{cases} \tag{5.3.43}$$

and the poles (5.3.41) are now at

$$\tau_j = \begin{cases} q + \langle \mathrm{i}a_3, h_s\rangle + b\widetilde{n}_s + mb + n/b - \mathrm{i}\varepsilon \\ q - \frac{\varkappa}{N} - \langle \mathrm{i}a_1, h_s\rangle - bn_s + mb + n/b\,. \end{cases} \tag{5.3.44}$$

As $\varepsilon \to 0$, the contour is thus pinched whenever $\tau_j = q - \frac{\varkappa}{N} - \langle \mathrm{i}a_1, h_s\rangle - bl$ for any $1 \leq j \leq N-1$, $1 \leq s \leq N$ and $0 \leq l \leq n_s$. The most singular contribution, of order $1/\varepsilon^{N-1}$, comes from values of τ where all τ_j take this form.

We will only describe the contour integral part of the braiding matrix (5.3.27), as prefactors only make computations more tedious. The term of order $1/\varepsilon^{N-1}$ in this integral is

$$\begin{aligned} I = \prod_{j=1}^{N-1} \Bigg[\sum_{p_j=1}^{N} \sum_{k_j=0}^{n_{p_j}} \operatorname*{res}^2_{\tau_j = q - \frac{\varkappa}{N} - \langle \mathrm{i}a_1, h_{p_j}\rangle - k_j b} \Bigg] \Bigg\{ \prod_{i\neq j}^{N-1} \frac{1}{S_b(\tau_i - \tau_j)} \\ \prod_{j=1}^{N-1} \prod_{s=1}^{N} \Bigg[\frac{S_b(-\langle \mathrm{i}a_3, h_s\rangle + \tau_j)}{S_b(-\langle \mathrm{i}a_3, h_s\rangle - \widetilde{n}_s b + \tau_j)} \frac{S_b(\frac{\varkappa}{N} - q + \langle \mathrm{i}a_1, h_s\rangle + \tau_j)}{S_b(\frac{\varkappa}{N} + \langle \mathrm{i}a_1, h_s\rangle + n_s b + \tau_j)} \Bigg] \Bigg\}. \end{aligned} \tag{5.3.45}$$

Note that

$$\prod_{i\neq j}^{N-1} \frac{1}{S_b(\tau_i - \tau_j)} = \prod_{i<j}^{N-1} \Big(-4b^2 \sin \pi b(\tau_i - \tau_j) \sin \frac{\pi}{b}(\tau_i - \tau_j) \Big)\,. \tag{5.3.46}$$

The shift relations for Γ_b and S_b yield

$$\begin{aligned} I = \prod_{j=1}^{N-1} \Bigg[\sum_{p_j=1}^{N} \sum_{k_j=0}^{n_{p_j}} \operatorname*{res}^2_{\tau_j = q - \frac{\varkappa}{N} - \langle \mathrm{i}a_1, h_{p_j}\rangle - k_j b} \Bigg] \Bigg\{ \\ \frac{\prod_{i<j}^{N-1} [-4b^2 \sin \pi b(\tau_i - \tau_j) \sin \frac{\pi}{b}(\tau_i - \tau_j)]}{\prod_{j=1}^{N-1} \prod_{s=1}^{N} [2 \sin \frac{\pi}{b}(\frac{\varkappa}{N} - q + \langle \mathrm{i}a_1, h_s\rangle + \tau_j)]} \\ \prod_{j=1}^{N-1} \prod_{s=1}^{N} \Bigg[\frac{\prod_{\mu=0}^{\widetilde{n}_s - 1} [2 \sin \pi(-b^2 - \mu b^2 - b\langle \mathrm{i}a_3, h_s\rangle + b\tau_j)]}{\prod_{\mu=0}^{n_s} [2 \sin \pi(-b^2 + \mu b^2 + \frac{b\varkappa}{N} + b\langle \mathrm{i}a_1, h_s\rangle + b\tau_j)]} \Bigg] \Bigg\}. \end{aligned} \tag{5.3.47}$$

This expression differs from the desired sum of residues (5.3.18) in the following respects: $\tau_j \to b\tau_j + \frac{b\varkappa}{N} - b^2 - 1$, a sum over choices of the p_j, and additional factors of the form $\sin\frac{\pi}{b}(\cdots)$. These factors are independent of the k_j except for a sign. After extracting a sign and taking the residue, these factors are

$$\frac{\prod_{i<j}^{N-1} \sin\frac{\pi}{b}(\langle \mathrm{i}a_1, h_{p_i} - h_{p_j}\rangle)}{\prod_{j=1}^{N-1}\prod_{s\neq p_j}^{N} \sin\frac{\pi}{b}(\langle \mathrm{i}a_1, h_s - h_{p_j}\rangle)}, \tag{5.3.48}$$

and vanish unless all p_i are distinct. Let p_N be the only element of $[\![1,N]\!]\setminus\{p_i \mid i < N\}$ so that p is a permutation of $[\![1,N]\!]$. Then the above factors are

$$\prod_{j=1}^{N-1} \frac{\prod_{i=1}^{j-1} \sin\frac{\pi}{b}(\langle \mathrm{i}a_1, h_{p_i} - h_{p_j}\rangle)}{\prod_{i\neq j}^{N} \sin\frac{\pi}{b}(\langle \mathrm{i}a_1, h_{p_i} - h_{p_j}\rangle)} = \frac{1}{\prod_{j=1}^{N-1}\prod_{i=j+1}^{N} \sin\frac{\pi}{b}(\langle \mathrm{i}a_1, h_{p_i} - h_{p_j}\rangle)}, \tag{5.3.49}$$

which is independent of the permutation p, except for a sign: the signature of p. For each permutation p we get a sum of residues times the signature of p, and this structure coincides with that of (5.3.18). Below that equation we prove that it is independent of the permutation, hence summing over permutation simply introduces a trivial factor. We have thus reproduced qualitatively the structure of the braiding matrix of V_{-Kbh_1} by taking the appropriate limit of the braiding kernel. This is confirmed by a more detailed calculation.

5.4 Toda CFT correlators

After having spent many pages on general properties of theories with W_N symmetry, it is time for us to focus on one particular such theory, the A_{N-1} Toda CFT. We follow in part the standard reference [FL07]. We present the Lagrangian of Toda CFT (Section 5.4.1), then use it with the Coulomb gas technique to evaluate some three-point functions (Section 5.4.2). This yields a few fusion rules that we were unable to prove on general symmetry grounds alone. We then derive the correlator of a semi-degenerate and two generic pimary operators, based on shift relations (Section 5.4.3). Finally, we collect expressions for some conformal blocks and correlators involving degenerate operators (Section 5.4.4).

5.4.1 Basics of Toda CFT

The Toda field theory is a unitary W_N-invariant 2d CFT at (generic) $c > 1$ with at most one primary operator for each choice of quantum numbers. Microscopically, the Toda field theory describes a scalar field φ in the Cartan subalgebra of A_{N-1}, minimally coupled to the metric, with an exponential potential term. We refer to Section 5.1 for a description of the W_N algebra, its representations, and some Lie algebra conventions. The Lagrangian action

$$\mathcal{A}_{\text{Toda}} = \int \mathrm{d}z\,\mathrm{d}\bar{z}\,\sqrt{g}\left[\frac{1}{8\pi}g^{ab}\langle\partial_a\varphi,\partial_b\varphi\rangle + \langle Q,\varphi\rangle 4\pi R + \mu\sum_{k=1}^{N-1} e^{b\langle e_k,\varphi\rangle}\right] \quad (5.4.1)$$

depends on a coupling constant b, and a cosmological constant μ and involves the Ricci scalar R of the background metric g^{ab}. If $Q = (b+1/b)\rho$, this Lagrangian defines a 2d CFT with (two copies of) the W_N symmetry algebra. Indeed, each potential term $\int \mathrm{d}z\,\mathrm{d}\bar{z}\, e^{b\langle e_k,\varphi(z,\bar{z})\rangle}$ is a screening charge (see Section 5.1.3), which means that it is W_N invariant. The theory is (non-manifestly) invariant under $(b,\hat{\mu}) \to (1/b,\hat{\mu})$ with

$$\hat{\mu} = \left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{1/b}. \quad (5.4.2)$$

Vertex operators $V_\alpha = e^{\langle\alpha,\varphi\rangle}$, labelled by their momentum α in the Cartan subalgebra, are primary operators for the W_N symmetry algebra. We show in Section 5.1.2 that the quantum numbers (5.1.18) of V_α (eigenvalues of zero-modes $W_0^{(k)}$) are invariant under the Weyl group, which acts by permuting the N components $\langle\alpha - Q, h_s\rangle$. In particular, the dimension is simply $\Delta(\alpha) = \frac{1}{2}\langle 2Q-\alpha,\alpha\rangle$. Assuming that the spectrum of Toda CFT is as small as possible, it has no degeneracy, and we deduce that vertex operators V_α are invariant up to a constant factor (reflexion amplitude) under Weyl symmetries. Later on, we absorb reflexion amplitudes into the normalization (5.4.3). Besides Weyl symmetries, which apply to individual momenta, correlators of vertex operators are invariant under conjugating all momenta $\alpha_i \to \alpha_i^C$ (conjugation is defined in (5.1.23)), because the action (5.4.1) is invariant.

Recall from Section 5.1.3 that fully degenerate operators $V_{-b\omega-\frac{1}{b}\omega'}$ are labelled by pairs (ω,ω') of highest weights of A_{N-1} representations. We only consider in this work degenerate momenta of the form $\alpha = -b\omega$, and mapping $b \to \frac{1}{b}$ would probe degenerate momenta $\alpha = -\frac{1}{b}\omega$, but the mixed case with non-zero ω and ω' is harder to access. We denote the representation of A_{N-1} with highest weight ω by $\mathcal{R}(\omega)$.

Semi-degenerate momenta are $\varkappa h_1$, up to Weyl symmetry. Since $\varkappa h_1^C$ is mapped by a Weyl symmetry to a momentum $\varkappa^D h_1$ along h_1, the vertex operators $V_{\varkappa h_1^C}$ and $V_{\varkappa^D h_1}$ are equal up to a reflexion amplitude, which we absorb in the normalization (5.4.4). This equality is crucial to obtain some dualities in Chapter 4 as conjugation of momenta.

In view of the matching of parameters with gauge theory in the AGT correspondence, we often write generic momenta as $\alpha = Q - \mathrm{i}a$. The dimension is $\Delta(Q-\mathrm{i}a) = \frac{1}{2}\langle Q,Q\rangle - \frac{1}{2}\langle \mathrm{i}a,\mathrm{i}a\rangle$. Weyl reflexions act by permuting the $\langle a,h_s\rangle$. In terms of the Upsilon function (5.4.5) below, we introduce the normalization

$$\widehat{V}_{Q-\mathrm{i}a} = \frac{\hat{\mu}^{-\langle\mathrm{i}a,\rho\rangle}}{\prod_{s<t}^{N}\Upsilon(\langle\mathrm{i}a,h_s-h_t\rangle)}V_{Q-\mathrm{i}a}\,, \quad (5.4.3)$$

where the product ranges over positive roots $e = h_s - h_t$. The normalization factor is invariant under conjugation, hence does not spoil this symmetry of Toda CFT correlators involving generic operators $\widehat{V}_\alpha$. As we will see, the three-point function $\langle \widehat{V}_\alpha \widehat{V}_{\alpha'} \widehat{V}_{\varkappa h_1}\rangle$ given in (5.4.28) is invariant under Weyl reflexions permuting the $\langle a, h_s\rangle$, hence the normalized operator $\widehat{V}_\alpha$ is Weyl invariant with no reflexion amplitude. We cannot use the same normalization for semi-degenerate and fully degenerate operators, as it is singular. Instead, we let

$$\widehat{V}_{\varkappa h_1} = \frac{\hat{\mu}^{\langle \varkappa h_1,\rho\rangle}}{\big(\Upsilon(b)\big)^{N-1}\Upsilon(\varkappa)} V_{\varkappa h_1}\,, \qquad \widehat{V}_{-b\omega} = \big[\hat{\mu} b^{2(b+\frac{1}{b})}\big]^{\langle -b\omega,\rho\rangle} V_{-b\omega}\,. \tag{5.4.4}$$

The normalizations of generic and semi-degenerate operators are invariant under $b \to \frac{1}{b}$.

The Upsilon function appearing above depends implicitly on the coupling constant b (it is invariant under $b \to \frac{1}{b}$), and for generic real b it is a holomorphic function, uniquely determined by its normalization $\Upsilon\big(\frac{1}{2}(b+\frac{1}{b})\big) = 1$ and by shift relations

$$\Upsilon(x+b) = \gamma(bx) b^{1-2bx}\Upsilon(x)\,, \qquad \Upsilon(x+1/b) = \gamma(x/b) b^{2x/b-1}\Upsilon(x)\,. \tag{5.4.5}$$

Also, $\Upsilon(b+\frac{1}{b}-x) = \Upsilon(x)$ and the function has zeros at $-mb - n\frac{1}{b}$ and $(m+1)b + (n+1)\frac{1}{b}$ for integers $m, n \geq 0$, and no poles. As $x \to \pm i\infty$, one has

$$\frac{\Upsilon(x+a)}{\Upsilon(x)} \sim \left(\frac{-x^2}{e^2}\right)^{ax} |x|^{a(a-b-1/b)} \sim \big(\gamma(bx) b^{1-2bx}\big)^{a/b} |x|^{a(a-b)} \tag{5.4.6}$$

$$\prod_{s=1}^{N} \frac{\Upsilon(x+\langle\alpha, h_s\rangle)}{\Upsilon(x)} \sim |x|^{\langle\alpha,\alpha\rangle} \tag{5.4.7}$$

for any a and any momentum α. The gamma function $\gamma(x) = \Gamma(x)/\Gamma(1-x)$ obeys by construction $\gamma(1-x) = 1/\gamma(x)$ and will also appear in three-point functions with a degenerate momentum. Conformal blocks involve Pochhammer symbols

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = (-1)^k \frac{\Gamma(1-x)}{\Gamma(1-x-k)} = \frac{(-1)^k}{(1-x)_{-k}}\,. \tag{5.4.8}$$

This equality is shown using the Euler identity $\Gamma(x)\Gamma(1-x) = \pi/\sin \pi x$.

5.4.2 Fusion coefficients

We evaluate some three-point functions of a degenerate vertex operator with two generic vertex operators using the Coulomb gas formalism, and deduce fusion rules and coefficients.

In the path integral expression for a Toda CFT correlator $\langle V_{\alpha_1}\cdots V_{\alpha_l}\rangle$ split the scalar field $\varphi = \varphi_0 + \widetilde{\varphi}$ into a constant zero-mode φ_0 and a piece with vanishing integral. Performing the φ_0 integral naively yields

$$\langle V_{\alpha_1}\cdots V_{\alpha_l}\rangle = \frac{1}{b^{N-1}}\langle V_{\alpha_1}\cdots V_{\alpha_l}\prod_{k=1}^{N-1}\Bigl(\Gamma(-s_k)(\mu Q_k^+)^{s_k}\Bigr)\rangle_{\mu=0} \tag{5.4.9}$$

where the path integral is now done in a free theory, $s_k = b^{-1}\langle 2Q - \sum_i \alpha_i, \omega_k\rangle$, and we used the screening charges

$$Q_k^{\pm} = \oint \mathrm{d}z\,\mathrm{d}\bar{z}\, V_{b^{\pm 1}e_k}(z)\,. \tag{5.4.10}$$

The operator $(Q_k^+)^{s_k}$ in (5.4.9) only makes sense if all s_k are non-negative integers. Taking residues since $\Gamma(-s_k)$ has a pole at such values, we deduce

$$\begin{aligned} &\operatorname*{res}_{\substack{\langle 2Q-\sum\alpha_j,\omega_k\rangle = bs_k\\ \forall 1\leq k\leq N-1}} \langle V_{\alpha_1}\cdots V_{\alpha_l}\rangle \\ &= \frac{(-\mu)^{s_1+\cdots+s_{N-1}}}{s_1!\cdots s_{N-1}!}\langle V_{\alpha_1}\cdots V_{\alpha_l}(Q_1^+)^{s_1}\cdots(Q_{N-1}^+)^{s_{N-1}}\rangle_{\mu=0}\,. \end{aligned} \tag{5.4.11}$$

Note that the sum of momenta in the free-field correlator is $2Q$ because of how s_k are defined. The free-field correlator is then evaluated using

$$\Biggl\langle \prod_i V_{\beta_i}(z_i,\bar{z}_i)\Biggr\rangle_{\mu=0} = \delta\Bigl(\sum_i \beta_i - 2Q\Bigr)\prod_{i<j}|z_i - z_j|^{2\langle\beta_i,\beta_j\rangle}\,. \tag{5.4.12}$$

This approach, the Coulomb gas formalism, expresses correlators of vertex operators obeying the screening condition

$$\sum_j \alpha_j = 2Q - \sum_{k=1}^{N-1} b s_k e_k \tag{5.4.13}$$

for some integers $s_k \geq 0$ in terms of $\sum_k s_k$ dimensional integrals. The quantum theory is invariant under $b \to \frac{1}{b}$, so the screening charges Q_k^+ and Q_k^- play the same role. The Coulomb gas formalism then extends to the more general screening condition

$$\sum_j \alpha_j = 2Q - \sum_{k=1}^{N-1}(b s_k^+ + b^{-1}s_k^-)e_k\,, \tag{5.4.14}$$

again when all $s_k^{\pm}$ are non-negative integers.

Let us determine which two-point functions are be non-zero. From Section 5.1.4 we know that $\langle V_\alpha V_\beta\rangle \neq 0$ fixes the quantum numbers of V_β

uniquely in terms of those of V_α. The Coulomb gas formalism (with all $s_k^\pm = 0$) implies

$$\langle V_\alpha V_{2Q-\alpha}\rangle = \langle V_\alpha V_{2Q-\alpha}\rangle_{\mu=0} = 1 \neq 0\,, \tag{5.4.15}$$

hence the constraint on V_β must be that its quantum numbers are equal to those of $V_{2Q-\alpha}$. Therefore, $\langle V_\alpha V_\beta\rangle \neq 0$ implies that β is Weyl symmetric to $2Q-\alpha$ (or equivalently to α^C). Note that the screening condition (5.4.14) allows an infinite set of values of β; a tedious calculation confirms that the Coulomb gas integral vanishes unless all $s_k^\pm = 0$. We will not need to know two-point functions other than (5.4.15), but we could deduce them from the fact that our normalized vertex operators (5.4.3) are invariant under Weyl symmetry with no reflexion amplitude.

Next we evaluate structure constants $C^\beta_{-bh_1,\alpha}$ appearing in the OPE $V_{-bh_1}\times V_\alpha = \sum_\beta C^\beta_{-bh_1,\alpha}[V_\beta]$ of a degenerate operator V_{-bh_1} with another vertex operator V_α, where brackets denote contributions from descendants, and we ignored the coordinate dependence. To this end, we consider (the residue of) a three-point function $\langle V_{2Q-\beta}(\infty)V_{-bh_1}(1)V_\alpha(0)\rangle$, where the switch from β to $2Q-\beta$ is due to the non-zero two-point function (5.4.15). The screening condition (5.4.14) forces β to take discrete values labelled by integers $s_k^\pm \geq 0$. In fact, Coulomb gas integrals turn out to vanish except when β takes the following values:

$$\beta = \alpha - bh_p = \alpha_1 - bh_1 + \sum_{k=1}^{p-1} be_k\,. \tag{5.4.16}$$

For a given p we get

$$\frac{C^{\alpha-bh_p}_{-bh_1,\alpha}}{(-\pi\mu)^{p-1}} = \left\langle V_{2Q-\alpha+bh_p}(\infty)V_{-bh_1}(1)V_\alpha(0)\prod_{k=1}^{p-1}\int \mathrm{d}t_k\,\mathrm{d}\bar{t}_k\,V_{be_k}(t_k,\bar{t}_k)\right\rangle_{\mu=0} \tag{5.4.17}$$

$$= \int \mathrm{d}^{p-1}t\,\mathrm{d}^{p-1}\bar{t}\prod_{k=1}^{p-1}\left[|t_k|^{-2\langle\alpha,be_k\rangle}|t_k-t_{k-1}|^{2b^2}\right] \tag{5.4.18}$$

where we introduced $t_0 = 1$ for convenience. The integrals can then be performed starting from $t_{p-1}, \bar{t}_{p-1}$ all the way to $t_1, \bar{t}_1$ by recognizing the Euler integral for the Beta function, or rather an analoguous function $\beta(x,y) = \gamma(x)\gamma(y)/\gamma(x,y)$. The result is

$$C^{\alpha-bh_p}_{-bh_1,\alpha} = (-\pi\mu)^{p-1}\prod_{j=1}^{p-1}\beta(1+b^2, b\langle\alpha-Q,h_j-h_p\rangle) \tag{5.4.19}$$

$$= \left[-\pi\mu\gamma(1+b^2)\right]^{p-1}\prod_{j=1}^{p-1}\frac{\gamma(b\langle\alpha-Q,h_j-h_p\rangle)}{\gamma(1+b^2+b\langle\alpha-Q,h_j-h_p\rangle)}\,. \tag{5.4.20}$$

For normalized vertex operators (5.4.3) and (5.4.4), the structure constants are

$$\widehat{C}^{Q-\mathrm{i}a-bh_p}_{-bh_1,Q-\mathrm{i}a} = b^{-N\langle 2\mathrm{i}a+bh_p,bh_p\rangle} \prod_{s\neq p}^{N} \gamma(b\langle \mathrm{i}a, h_p - h_s\rangle) \,. \tag{5.4.21}$$

A similar computation of three-point functions involving one degenerate momentum $-b\omega_K$ shows that the fusion of $\widehat{V}_{-b\omega_K}$ with $\widehat{V}_\alpha$ yields conformal families with momenta $\alpha - bh$ for all weights h of the K-th antisymmetric representation of A_N. The structure constant for a weight $h = h_{p_1} + \cdots + h_{p_K}$ for $1 \leq p_1 < \cdots < p_K \leq N$ is

$$\widehat{C}^{Q-\mathrm{i}a-bh}_{-b\omega_K,Q-\mathrm{i}a} = b^{-N\langle 2\mathrm{i}a+bh,bh\rangle} \prod_{s\notin\{p\}}^{N} \prod_{t\in\{p\}} \gamma(b\langle \mathrm{i}a, h_t - h_s\rangle) \,. \tag{5.4.22}$$

In Section 5.4.3 we determine three-point functions which involve a semi-degenerate momentum $\varkappa h_1$. Setting $\varkappa = -Kb$ leads to the fusion rule for $\widehat{V}_{-Kbh_1}$ with $\widehat{V}_\alpha$. This fusion yields momenta $\alpha - bh$ for all weights h of the K-th symmetric representation, and the structure constant for a weight $h = \sum_{s=1}^N n_s h_s$ is

$$\widehat{C}^{Q-\mathrm{i}a-bh}_{-Kbh_1,Q-\mathrm{i}a} = \frac{b^{-N\langle 2\mathrm{i}a+bh,bh\rangle}}{\prod_{\nu=1}^K \gamma(-\nu b^2)} \prod_{s,t=1}^{N} \prod_{\nu=0}^{n_t-1} \gamma(b\langle \mathrm{i}a, h_t - h_s\rangle + (\nu - n_s)b^2) \,. \tag{5.4.23}$$

Setting $K = 1$ in either (5.4.22) or (5.4.23) gives (5.4.21). A natural generalization of the fusion rules above is that

$$\widehat{V}_{-b\omega-\omega'/b} \times \widehat{V}_{Q-\mathrm{i}a} = \sum_{h\in\mathcal{R}(\omega)} \sum_{h'\in\mathcal{R}(\omega')} \widehat{C}^{Q-\mathrm{i}a-bh-h'/b}_{-b\omega-\omega'/b,Q-\mathrm{i}a} \big[\widehat{V}_{Q-\mathrm{i}a-bh-h'/b}\big] \,, \tag{5.4.24}$$

where the sums run over weights of $\mathcal{R}(\omega)$ and $\mathcal{R}(\omega')$. We are not aware of a proof.

The fusion rules simplify for a semi-degenerate momentum $\alpha = \lambda h_1$. Indeed, (5.4.19) contains a factor $1/\gamma(0) = 0$ (for $j = p - 1$) whenever $p \geq 3$. Thus, the fusion rule reduces to

$$V_{-bh_1} \times V_{\lambda h_1} = [V_{\lambda h_1 - bh_1}] + [V_{\lambda h_1 - bh_2}] \,. \tag{5.4.25}$$

Note that we had to work with the vertex operators V_α since the normalization $\widehat{V}_\alpha$ is singular at $\alpha = \lambda h_1$. A similar calculation using $C^{\alpha-bh}_{-b\omega_K,\alpha}$ deduced from (5.4.22) shows that

$$V_{-b\omega_K} \times V_{\lambda h_1} = [V_{\lambda h_1 - b\omega_K}] + [V_{\lambda h_1 - b(\omega_{K+1} - h_1)}] \,. \tag{5.4.26}$$

Finally, the symmetric case (5.4.23) yields

$$V_{-Kbh_1} \times V_{\lambda h_1} = \sum_{j=0}^{K} [V_{(\lambda - Kb)h_1 + jb(h_1 - h_2)}] \,. \tag{5.4.27}$$

We discuss such fusion rules further in Section 5.5 and give a proposal for the fusion of an arbitrary degenerate operator with a semi-degenerate operator.

5.4.3 Three-point function

In the normalizations (5.4.3) and (5.4.4), equation (1.39) of [FL07] with all momenta conjugated is

$$\begin{aligned}
&\widehat{C}(Q-\mathrm{i}a_1, Q-\mathrm{i}a_2, \varkappa h_1) \\
&\quad = \frac{\hat{\mu}^{-\langle \mathrm{i}a_1+\mathrm{i}a_2-\varkappa h_1,\rho\rangle} C(Q-\mathrm{i}a_1^C, Q-\mathrm{i}a_2^C, \varkappa h_1^C)}{(\Upsilon(b))^{N-1}\Upsilon(\varkappa)\prod_{s<t}^N \Upsilon(\langle \mathrm{i}a_1, h_s-h_t\rangle)\Upsilon(\langle \mathrm{i}a_2, h_s-h_t\rangle)} \\
&\quad = \frac{1}{\prod_{s,t=1}^N \Upsilon\big(\frac{\varkappa}{N}+\langle \mathrm{i}a_1,h_s\rangle+\langle \mathrm{i}a_2,h_t\rangle\big)} \,.
\end{aligned} \tag{5.4.28}$$

In this section we derive this three-point function using results from Section 5.2.1.

There, we studied a four-point function involving a degenerate operator V_{-bh_1} and obtained

$$\langle V_{\alpha_\infty}(\infty) V_{(\varkappa+b)h_1}(1) V_{-bh_1}(x,\bar{x}) V_{\alpha_0}(0)\rangle = \sum_{p=1}^N C_p^{(\mathrm{s})}\, \mathcal{F}_p^{(\mathrm{s})}(x)\, \mathcal{F}_p^{(\mathrm{s})}(\bar{x}) \tag{5.4.29}$$

with explicit expressions (5.2.17) for the N s-channel conformal blocks $\mathcal{F}_p^{(\mathrm{s})}(x)$. We also found the braiding matrix (5.2.19). Expanding the holomorphic and antiholomorphic s-channel conformal blocks in the u-channel yields

$$\sum_p C_p^{(\mathrm{s})}\, \mathcal{F}_p^{(\mathrm{s})}(x)\, \mathcal{F}_p^{(\mathrm{s})}(\bar{x}) = \sum_{p,s,t} C_p^{(\mathrm{s})} \mathbf{B}_{ps}\mathbf{B}_{pt}\, \mathcal{F}_s^{(\mathrm{u})}(x)\, \mathcal{F}_t^{(\mathrm{u})}(\bar{x}) \,. \tag{5.4.30}$$

The four-point function must be single valued: this forbids terms with $s \neq t$. Thus, the N coefficients $C_p^{(\mathrm{s})}$ must be such that $\sum_p C_p^{(\mathrm{s})}\mathbf{B}_{ps}\mathbf{B}_{pt}$ vanishes for $s \neq t$. This highly overdetermined system of equations for $C_p^{(\mathrm{s})}$ fixes these coefficients up to a constant. The unique single-valued combination is

$$\begin{aligned}
&\langle V_{\alpha_\infty}(\infty) V_{(\varkappa+b)h_1}(1) V_{-bh_1}(x,\bar{x}) V_{\alpha_0}(0)\rangle \\
&\quad \sim \sum_{p=1}^N \frac{\prod_{s\neq p}^N \gamma(b\langle Q-\alpha_0, h_p-h_s\rangle)}{\prod_{s=1}^N \gamma(b\varkappa/N + b\langle Q-\alpha_0,h_p\rangle + b\langle Q-\alpha_\infty, h_s\rangle)}\, \mathcal{F}_p^{(\mathrm{s})}(x)\, \mathcal{F}_p^{(\mathrm{s})}(\bar{x})
\end{aligned} \tag{5.4.31}$$

up to a multiplicative constant which may depend on the momenta, but not on p nor on positions.

On the other hand, the coefficients $C_p^{(\mathrm{s})}$ are products of three-point functions

$$C_p^{(\mathrm{s})} = C(\alpha_\infty, (\varkappa+b)h_1, \alpha_0-bh_p) C^{\alpha_0-bh_p}_{-bh_1,\alpha_0} \tag{5.4.32}$$

and $C^{\alpha_0-bh_p}_{-bh_1,\alpha_0}$ are known. We deduce a relation between $C(\alpha_\infty,(\varkappa+b)h_1,\alpha_0-bh_p)$ for different p. This relates three-point functions with the generic momentum α_0 shifted by $b(h_p-h_r)$. Thanks to the $(b,\hat{\mu})\to(\frac{1}{b},\hat{\mu})$ symmetry there is another relation shifting a generic momentum by $\frac{1}{b}(h_p-h_r)$, and by symmetry these shifts can be applied to α_∞ as well. For generic real b^2 (in other words, generic $c>1$) the system of shift relations fixes the three-point function up to a multiplicative constant which depends on the semi-degenerate momentum.

The dependence on $\varkappa$ is fixed by comparing the s-channel and t-channel coefficients. More precisely, the t-channel coefficient for an internal momentum of $(\varkappa+b)h_1-bh_1$ (as opposed to $(\varkappa+b)h_1-bh_2$ which has multiplicity) is

$$C_1^{(\mathrm{t})}=C(\alpha_\infty,\varkappa h_1,\alpha_0)C^{\varkappa h_1}_{-bh_1,(\varkappa+b)h_1} \tag{5.4.33}$$

and can be expressed in terms of (any of the) $C_p^{(\mathrm{s})}$. Since structure constants with a momentum $-bh_1$ are known, we deduce a relation between $C(\alpha_\infty,\varkappa h_1,\alpha_0)$ and $C(\alpha_\infty,(\varkappa+b)h_1,\alpha_0-bh_p)$. A similar relation holds with $(b,\hat{\mu})\to(\frac{1}{b},\hat{\mu})$. Assuming again that b^2 is real and generic, these shift relations fix the $\varkappa$ dependence up to an overall constant factor.

We conclude that (5.4.28) is correct up to a constant factor, provided that it leads to a single-valued four-point function. Denoting $\alpha_0=Q-\mathrm{i}a_0$ and $\alpha_\infty=Q-\mathrm{i}a_\infty$, compute

$$\begin{aligned}
&\widehat{C}^{\alpha_0-bh_p}_{-bh_1,\alpha_0}\widehat{C}(\alpha_\infty,\alpha_0-bh_p,(\varkappa+b)h_1)\ /\ \widehat{C}(\alpha_\infty,\alpha_0,\varkappa h_1) \\
&\quad=\widehat{C}^{\alpha_0-bh_p}_{-bh_1,\alpha_0}\prod_{s,t=1}^{N}\left[\frac{\Upsilon(\frac{\varkappa}{N}+\langle\mathrm{i}a_0,h_s\rangle+\langle\mathrm{i}a_\infty,h_t\rangle)}{\Upsilon(\frac{\varkappa}{N}+\langle\mathrm{i}a_0,h_s\rangle+\langle\mathrm{i}a_\infty,h_t\rangle+b\delta_{ps})}\right] \\
&\quad=b^{2b\varkappa-N(1+b^2)+b^2}\frac{\prod_{s\neq p}^{N}\gamma(b\langle\mathrm{i}a_0,h_p-h_s\rangle)}{\prod_{t=1}^{N}\gamma(\frac{b\varkappa}{N}+b\langle\mathrm{i}a_0,h_p\rangle+b\langle\mathrm{i}a_\infty,h_t\rangle)}\,.
\end{aligned} \tag{5.4.34}$$

Up to a constant $\widehat{C}(\alpha_\infty,\alpha_0,\varkappa h_1)b^{2b\varkappa-N(1+b^2)+b^2}$ the product of three-point functions reproduces coefficients in (5.4.31).

A last comment is that the three-point function (5.4.28) is invariant under Weyl transformations of each $\widehat{V}_{Q-\mathrm{i}a_i}$, as they simply permute the $\langle\mathrm{i}a_i,h_s\rangle$. Hence, the normalized $\widehat{V}_{Q-\mathrm{i}a}$ are Weyl invariant, as claimed earlier. The three-point function is also invariant under conjugation of all momenta, followed by the Weyl transformation (5.1.28) which maps $(\varkappa h_1)^C\to\varkappa^D h_1=(N(b+1/b)-\varkappa)h_1$: indeed, $\langle\mathrm{i}a_i,h_s\rangle\to-\langle\mathrm{i}a_i,h_s\rangle$ and $\varkappa/N\to(b+1/b)-\varkappa/N$ under this transformation, and $\Upsilon(b+1/b-x)=\Upsilon(x)$.

5.4.4 Conformal blocks

This section collects expressions for correlators and conformal blocks with two generic momenta $\alpha_\infty=Q-\mathrm{i}a_\infty$ and $\alpha_0=Q-\mathrm{i}a_0$ at ∞ and 0, one semi-degenerate momentum $\hat{m}=\lambda h_1$ at 1, and one or more degenerate momenta

$-b\Omega_j$ inserted at the positions $(x_j,\bar{x}_j)$ for $1\leq j\leq n$. The expressions are direct translations of the gauge theory vortex partition functions through the correspondence described in Chapter 3. We only consider conformal blocks in the s-channel (the region $1>|x_n|>\cdots>|x_1|>0$), which are series in powers of x_n, x_{n-1}/x_n, $\ldots$, x_1/x_2.

First comes the case of a single degenerate momentum $-b\omega_K$ labelled by the K-th antisymmetric representation of A_{N-1}. The correlator splits in the s-channel as a sum over weights $h_{\{p\}}=h_{p_1}+\cdots+h_{p_K}$ of $\mathcal{R}(\omega_K)$:

$$\begin{aligned}&\langle \widehat{V}_{\alpha_\infty}(\infty)\widehat{V}_{\hat{m}}(1)\widehat{V}_{-b\omega_K}(x,\bar{x})\widehat{V}_{\alpha_0}(0)\rangle \\ &=\sum_{\{p\}}\widehat{C}^{\alpha_0-bh_{\{p\}}}_{-b\omega_K,\alpha_0}\widehat{C}(\alpha_\infty,\alpha_0-bh_{\{p\}},\lambda h_1)\left|\mathcal{F}^{(\mathrm{s})}_{Q-\mathrm{i}a_0-bh_{\{p\}}}\begin{bmatrix}\lambda h_1 & -b\omega_K\\ Q-\mathrm{i}a_\infty & Q-\mathrm{i}a_0\end{bmatrix}\right|^2.\end{aligned} \tag{5.4.35}$$

Factoring out $\widehat{C}(\alpha_\infty,\alpha_0,(\lambda-Kb)h_1)$, the product of three-point functions is given by

$$\begin{aligned}&\widehat{C}^{\alpha_0-bh_{\{p\}}}_{-b\omega_K,\alpha_0}\widehat{C}(\alpha_\infty,\alpha_0-bh_{\{p\}},\lambda h_1)\ /\ \widehat{C}(\alpha_\infty,\alpha_0,(\lambda-Kb)h_1) \\ &=\widehat{C}^{\alpha_0-bh_{\{p\}}}_{-b\omega_K,\alpha_0}\prod_{s,t=1}^{N}\left[\frac{\Upsilon\left(\frac{\lambda-Kb}{N}+\langle\mathrm{i}a_0,h_t\rangle+\langle\mathrm{i}a_\infty,h_s\rangle\right)}{\Upsilon\left(\frac{\lambda-Kb}{N}+\langle\mathrm{i}a_0,h_t\rangle+\langle\mathrm{i}a_\infty,h_s\rangle+b\delta_{t\in\{p\}}\right)}\right] \\ &=b^{2Kb\lambda-KN(1+b^2)-K^2b^2}\prod_{t\in\{p\}}\left[\frac{\prod_{s\notin\{p\}}^{N}\gamma(-b\langle\mathrm{i}a_0,h_s-h_t\rangle)}{\prod_{s=1}^{N}\gamma\left(\frac{b\lambda-Kb^2}{N}+b\langle\mathrm{i}a_0,h_t\rangle+b\langle\mathrm{i}a_\infty,h_s\rangle\right)}\right].\end{aligned} \tag{5.4.36}$$

The four-point conformal blocks are

$$\begin{aligned}&\mathcal{F}^{(\mathrm{s})}_{Q-\mathrm{i}a_0-bh_{\{p\}}}\begin{bmatrix}\lambda h_1 & -b\omega_K\\ Q-\mathrm{i}a_\infty & Q-\mathrm{i}a_0\end{bmatrix}(x) \\ &=x^{-b\langle\mathrm{i}a_0,h_{\{p\}}\rangle+\frac{K(N-K)}{2}(b^2+1)}(1-x)^{K(b^2+1-b\lambda/N)} \\ &\quad\sum_{k\geq0}\prod_{j=1}^{K}\frac{(x^{k_j}/k_j!)\prod_{s=1}^{N}(1-b(\lambda-Kb)/N-b\langle\mathrm{i}a_0,h_{p_j}\rangle-b\langle\mathrm{i}a_\infty,h_s\rangle)_{k_j}}{\prod_{i\neq j}^{K}(b\langle\mathrm{i}a_0,h_{p_i}-h_{p_j}\rangle-k_i)_{k_j}\prod_{s\notin\{p\}}^{N}(1+b\langle\mathrm{i}a_0,h_s-h_{p_j}\rangle)_{k_j}}\end{aligned} \tag{5.4.37}$$

as shown in Section 5.2.2 by comparing the CFT braiding matrix and the braiding matrix for this series expression. As we will explain in Section 3.4, the correlator is equal to the sphere partition function (3.4.1) of a surface defect carrying $\mathcal{N}=(2,2)$ SQCD, up to simple factors. Indeed, the three-point factors (5.4.36) match with one-loop determinants of SQCD, and conformal blocks (5.4.37) reproduce vortex partition functions.

Our next four-point function involves the degenerate field $\widehat{V}_{-Kbh_1}$, labelled by the K-th symmetric representation $\mathcal{R}(Kh_1)$ of A_{N-1}. We give evidence in Section 3.5 that up to simple factors the correlator is equal to the sphere partition function (3.5.1) of a surface defect carrying $\mathcal{N}=(2,2)$ SQCD with additional adjoint matter. This is confirmed in Section 3.5 by checking that the leading terms in the s-channel, t-channel, and u-channel expansions of the correlator are reproduced by explicit expressions of the sphere partition function. The correlator splits in the s-channel as a sum over the weights $h_{[n]} = \sum_{s=1}^{N} n_s$ of $\mathcal{R}(Kh_1)$, with $n_s \geq 0$ summing to K:

$$\begin{aligned}&\langle \widehat{V}_{\alpha_\infty}(\infty)\widehat{V}_{\hat{m}}(1)\widehat{V}_{-Kbh_1}(x,\bar{x})\widehat{V}_{\alpha_0}(0)\rangle \\ &= \sum_{[n]} \widehat{C}^{\alpha_0-bh_{[n]}}_{-Kbh_1,\alpha_0}\widehat{C}(\alpha_\infty,\alpha_0-bh_{[n]},\lambda h_1)\left|\mathcal{F}^{(\mathrm{s})}_{Q-\mathrm{i}a_0-bh_{[n]}}\begin{bmatrix}\lambda h_1 & -Kbh_1\\ Q-\mathrm{i}a_\infty & Q-\mathrm{i}a_0\end{bmatrix}\right|^2 .\end{aligned} \tag{5.4.38}$$

The product of three-point functions

$$\begin{aligned}&\widehat{C}^{\alpha_0-bh_{[n]}}_{-Kbh_1,\alpha_0}\widehat{C}(\alpha_\infty,\alpha_0-bh_{[n]},\lambda h_1)\ /\ \widehat{C}(\alpha_\infty,\alpha_0,(\lambda-Kb)h_1) \\ &= \frac{b^{2Kb\lambda-KN(1+b^2)-K^2b^2}}{\prod_{\nu=1}^{K}\gamma(-\nu b^2)}\prod_{s,t=1}^{N}\prod_{\nu=0}^{n_t-1}\frac{\gamma(b\langle\mathrm{i}a_0,h_t-h_s\rangle+(\nu-n_s)b^2)}{\gamma(\frac{b\lambda-Kb^2}{N}+b\langle\mathrm{i}a_0,h_t\rangle+b\langle\mathrm{i}a_\infty,h_s\rangle+\nu b^2)}\end{aligned} \tag{5.4.39}$$

matches with one-loop determinants of SQCD plus adjoint matter, up to a factor. Trusting the matching with gauge theory leads to the following proposal for conformal blocks:

$$\begin{aligned}&\mathcal{F}^{(\mathrm{s})}_{Q-\mathrm{i}a_0-bh_{[n]}}\begin{bmatrix}\lambda h_1 & -Kbh_1\\ Q-\mathrm{i}a_\infty & Q-\mathrm{i}a_0\end{bmatrix}(x) \\ &= x^{\Delta(Q-\mathrm{i}a_0-bh_{[n]})-\Delta(Q-\mathrm{i}a_0)-\Delta(-Kbh_1)}(1-x)^{\Delta(\lambda h_1-Kbh_2)-\Delta(\lambda h_1)-\Delta(-Kbh_1)} \\ &\quad\sum_{k\geq 0}\prod_{(s,\mu)}^{[n]}\Bigg[x^{k_{s\mu}}\prod_{t=1}^{N}\frac{(1-b(\lambda-Kb)/N-b\langle\mathrm{i}a_0,h_s\rangle-b\langle\mathrm{i}a_\infty,h_t\rangle-\mu b^2)_{k_{s\mu}}}{(1+b\langle\mathrm{i}a_0,h_t-h_s\rangle+(n_t-\mu)b^2)_{k_{s\mu}}} \\ &\qquad\cdot\frac{\prod_{t=1}^{N}(1+b\langle\mathrm{i}a_0,h_t-h_s\rangle+(n_t-\mu)b^2+k_{s\mu}-k_{t(n_t-1)})_{k_{t(n_t-1)}}}{\prod_{(t,\nu)\in I}(1+b\langle\mathrm{i}a_0,h_t-h_s\rangle+(\nu-\mu)b^2+k_{s\mu}-k_{t\nu})_{k_{t\nu}-k_{t(\nu-1)}}}\Bigg]\end{aligned} \tag{5.4.40}$$

where $\prod_{(s,\mu)}^{[n]} = \prod_{s=1}^{N}\prod_{\mu=0}^{n_s-1}$. In Section 5.3.1 we compute the braiding matrix for these proposed conformal blocks. A special case of shift relations found in Section 5.3.2 expresses the actual CFT braiding matrix for some K to that for $K \to (K-1)$. The braiding matrix of Section 5.3.1 obeys this recurrence relation, thus is the correct CFT braiding matrix. We conclude that the conformal block (5.4.40) is correct.

We now come to the case of $(n+3)$-point correlators with two generic, one semi-degenerate, and n degenerate momenta $-b\Omega_j = -b\Omega(K_j,\epsilon_j)$, where $\Omega(K,-1) = \omega_K$ and $\Omega(K,+1) = Kh_1$. We only consider the s-channel $1 > |x_n| > \ldots > |x_1|$, and we only write down the conformal block, as the product of three-point functions is straightforward to compute. In the s-channel, the internal momentum running between the punctures at x_j and x_{j+1} (here $x_{n+1} = 1$) has the form $\alpha_0 - bh_{[n^j]} = \alpha_0 - b\sum_{t=1}^{N} n_t^j h_t$, for some integers $n_t^j \geq 0$. These integers must be such that $h_{[n^j]} - h_{[n^{j-1}]}$ is a weight of

$\mathcal{R}(\Omega_j)$ for each $1 \le j \le n$ (here $n_t^0 = 0$). Explicitly, $\sum_{t=1}^N (n_t^j - n_t^{j-1}) = K_j$, and $n_t^j - n_t^{j-1}$ is in $\mathbb{Z}_{\ge 0}$ if $\epsilon_j = +1$ and in $\{0, 1\}$ if $\epsilon_j = -1$.

In Section 3.6 we find a quiver gauge theory whose vacua are labelled by the same data, and perform various checks that its partition function is equal to the Toda correlator we are now considering. Up to simple factors, the conformal blocks are thus equal to the vortex partition functions, themselves a sum of residues in the Coulomb branch representation of the partition function. Let us introduce the sets $I_j = \{(s,\mu) \mid 0 \le \nu < n_s^j, 1 \le s \le N\}$ (I_0 is empty), the notation $\mathrm{i}a_{s,\mu} = \langle \mathrm{i}a_0, h_s\rangle + \mu b$, and the parameters $q_{n+1} = b^2/2$ and $q_j = \epsilon_j(b^2/2 + 1/4) - 1/4$. We find

$$\mathcal{F}^{(\mathrm{s})}\left[\begin{array}{ccccccccc} & \lambda h_1 & & -b\Omega_n & & -b\Omega_2 & & -b\Omega_1 & \\ \alpha_\infty & \downarrow & \alpha_0 - bh_{[n^n]} & \downarrow & \cdots & \downarrow & \alpha_0 - bh_{[n^1]} & \downarrow & \alpha_0 \end{array}\right](x)$$

$$= \prod_{j=1}^n \left[x_j^{\Delta(\alpha_0 - bh_{[n^j]}) - \Delta(\alpha_0 - bh_{[n^{j-1}]}) - \Delta(-b\Omega_j)} [1 - x_j]^{(1 + b^2 - \frac{b\lambda}{N})K_j} \right] \prod_{i<j}^n \left[1 - \frac{x_i}{x_j} \right]^{(1 + 2q_j + b^2 \frac{K_j}{N})K_i}$$
$$\cdot \sum_{\{k_{j,s,\mu} \ge 0\}} \left\{ \prod_{j=1}^n \prod_{(s,\mu)\in I_j} \left[\left[\frac{x_j}{x_{j+1}} \right]^{k_{j,s,\mu}} \prod_{(t,\nu)\in I_j} \frac{(1 + b\mathrm{i}a_{s,\mu} - b\mathrm{i}a_{t,\nu})_{k_{j,t,\nu} - k_{j,s,\mu}}}{(1 + q_j + q_{j+1} + b\mathrm{i}a_{s,\mu} - b\mathrm{i}a_{t,\nu})_{k_{j,t,\nu} - k_{j,s,\mu}}} \right.\right.$$
$$\left. \cdot \prod_{(t,\nu)\in I_{j-1}} \frac{(1 + 2q_j + b\mathrm{i}a_{s,\mu} - b\mathrm{i}a_{t,\nu})_{k_{j-1,t,\nu} - k_{j,s,\mu}}}{(1 + b\mathrm{i}a_{s,\mu} - b\mathrm{i}a_{t,\nu})_{k_{j-1,t,\nu} - k_{j,s,\mu}}} \right]$$
$$\left. \cdot \prod_{s=1}^N \prod_{(t,\nu)\in I_n} \frac{(1 - b(\lambda - b\sum_{j=1}^n K_j)/N - b\langle \mathrm{i}a_\infty, h_s\rangle - b\mathrm{i}a_{t,\nu})_{k_{n,t,\nu}}}{(1 + b\langle \mathrm{i}a_0, h_s\rangle - b\mathrm{i}a_{t,\nu})_{k_{n,t,\nu}}} \right\}. \tag{5.4.41}$$

As discussed in Section 3.6, when all $\epsilon_j = -1$, placing all degenerate punctures at the same position $x_j = x$ yields the conformal block for one particular fusion of the degenerate momenta, which turns out to be

$$-b\Omega = -b\sum_{j=1}^n \Omega_j = -b\sum_{j=1}^n \omega_{K_j}\,. \tag{5.4.42}$$

This provides an explicit expression for the four-point conformal block of two generic and one semi-degenerate momentum, and one degenerate momentum labelled by an arbitrary representation of A_{N-1}. Fusing degenerate punctures in several sets gives conformal blocks with several arbitrary degenerate momenta $-b\Omega$, but these quickly become unwieldy.

5.5 Fusion rules

From the fusion rule of an arbitrary degenerate vertex operator with another vertex operator (for which we have no proof), we deduce the fusion (5.5.12) of two semi-degenerate operators, and the fusion (5.5.24) of a semi-degenerate operator with an arbitrary fully-degenerate operator. We propose that operators resulting from the latter fusion appear with multiplicity (5.5.25) in the fusion of two generic operators. This section is Appendix A.4 of [GLF14]. Since we decided to rederive in this thesis many Toda CFT results, references made to Appendix B of [GLF10] could be replaced by references to earlier sections of this chapter, but the original text was kept.

Null vectors among W_N descendants of a fully degenerate vertex operator $V_{-b\omega-\omega'/b}$ constrain its three-point function with arbitrary vertex operators V_α and V_β. Namely, the three-point function vanishes unless $\alpha+\beta = 2Q + bh + h'/b$ for some weights h of $\mathcal{R}(\omega)$ and h' of $\mathcal{R}(\omega')$. This results in the fusion rule

$$V_\alpha \times V_{-b\omega-\omega'/b} = \sum_{h\in\mathcal{R}(\omega)} \sum_{h'\in\mathcal{R}(\omega')} V_{\alpha-bh-h'/b}\,, \tag{5.5.1}$$

with outgoing momenta $\alpha - bh - h'/b = 2Q - \beta$: the degenerate operator shifts the incoming momentum by $-bh - h'/b$. Each operator $V_{\alpha-bh-h'/b}$ appears in (5.5.1) with a multiplicity equal to the product of the multiplicity of h in $\mathcal{R}(\omega)$ and that of h' in $\mathcal{R}(\omega')$. Henceforth, we take $\omega' = 0$, thus $h' = 0$.

Later in this appendix, we find that the fusion (5.5.24) of a semi-degenerate operator $V_{\varkappa h_1}$ with $V_{-b\omega}$ only allows some of the shifts $-bh$ of (5.5.1). Let us first describe the case $\omega = h_1$ based on [GLF10, Appendix B]: the fusion of $-\varkappa h_N$ and $-bh_1$ yields the momenta $-(\varkappa+b)h_N$ and $-\varkappa h_N - bh_1$. After the Weyl rotation $(1\,2\cdots N)$, we get

$$V_{\varkappa h_1} \times V_{-bh_1} = V_{\varkappa h_1 - bh_1} + V_{\varkappa h_1 - bh_2}\,. \tag{5.5.2}$$

The s-channel expansion of $\langle V_{\alpha_\infty}(\infty) V_{\varkappa h_1}(1) V_{-bh_1}(x,\bar{x}) V_{\alpha_0}(0)\rangle$ is the sum of N products of holomorphic and antiholomorphic conformal blocks. The t-channel expansion only features two momenta (5.5.2), and takes the form

$$\sum_{p=1}^{2} |1-z|^{2[\Delta(\varkappa h_1 - bh_p) - \Delta(\varkappa h_1) - \Delta(-bh_1)]}(\cdots) \tag{5.5.3}$$

where $(\cdots)$ are two series in powers of $(1-z)$ and $(1-\bar{z})$. The first series $(p=1)$ factorizes as the product of a holomorphic and an anti-holomorphic conformal blocks, multiplied by $C^{\varkappa h_1 - bh_1}_{-bh_1,\varkappa h_1} C(\alpha_0, \alpha_\infty, (\varkappa - b)h_1)$. The second $(p=2)$ does not, but can be written non-canonically as a sum of $N-1$ products of the same form. This multiplicity implies that the fusion V_{α_0} and V_{α_∞} includes $N-1$ copies of the representations of the W_N algebra generated by $V_{\varkappa h_1 - bh_2}$, while it only includes one copy of the representation generated by any semi-degenerate operator $V_{\varkappa h_1}$. We generalize the statement to all momenta of the form $\varkappa h_1 - b\omega$ in (5.5.25).

5.5.1 Two semi-degenerates

To reach more complicated degenerate operators, we first find which momenta result from the fusion of two semi-degenerate momenta $-\varkappa h_N$ and λh_1. In principle, one could write null vectors descending from $V_{-\varkappa h_N}$ and $V_{\lambda h_1}$ for a given N and, through those, constrain the momenta which arise in the OPE. Such constraints are polynomial in the momenta, and any constraint shown for generic $(b, \varkappa, \lambda)$ must hold for all $(b, \varkappa, \lambda)$ by continuity: in other words, fusion rules for more specific momenta can only become more restrictive. We are thus free to assume that $(b, \varkappa, \lambda)$ is generic.

Since null vectors are very difficult to write down for general N, we use a different route: the braiding matrix relating the s-channel ($x \to 0$) and u-channel ($x \to \infty$) conformal blocks of $\langle V_{\lambda h_1}(\infty) V_{-\varkappa h_N}(1) V_{-bh_1}(x, \bar{x}) V_{\alpha_0}(0)\rangle$ should only lead to u-channel conformal blocks with internal momenta $\lambda h_1 - bh_1$ and $\lambda h_1 - bh_2$, and all other components must vanish. Specifically, we take $\alpha_2 = 2Q - \lambda h_1$, $\hat{m} = -\varkappa h_N$, $\mu = -bh_1$ and $\alpha_1 = 2Q - \alpha + bh_l$ in equation (B.12) of [GLF10]:

$$\begin{aligned}
&\mathcal{F}^{(\mathrm{s})}_{2Q-\alpha}\begin{bmatrix} -\varkappa h_N & -bh_1 \\ \lambda h_1 & 2Q-\alpha+bh_l \end{bmatrix}(x) \\
&= \sum_{k=1}^{N} e^{i\pi\epsilon(\phi_{kl}-b\varkappa/N)} \prod_{j\neq l} \frac{\Gamma(1+b^2+b\langle \alpha-Q, h_j-h_l\rangle)}{\Gamma(1+b^2-\phi_{kj})} \\
&\quad \cdot \prod_{j\neq k} \frac{\Gamma(b\langle \lambda h_1 - Q, h_j-h_k\rangle)}{\Gamma(\phi_{jl})} \mathcal{F}^{(\mathrm{u})}_{\lambda h_1 - bh_k}\begin{bmatrix} -\varkappa h_N & -bh_1 \\ \lambda h_1 & 2Q-\alpha+bh_l \end{bmatrix}(x)
\end{aligned} \tag{5.5.4}$$

where

$$\phi_{st} = b\langle -\varkappa h_N, h_1\rangle + b\langle \lambda h_1 - Q, h_s\rangle - b\langle \alpha - Q, h_t\rangle \,. \tag{5.5.5}$$

The coefficient must vanish for all $k \notin \{1, 2\}$ and all l, hence one of the denominator Gamma functions must have a non-positive integer argument:

$$\forall k \in [\![3, N]\!]\ \forall l \in [\![1, N]\!]\ -\phi_{jl} \in \mathbb{Z}_{\geq 0} \text{ or } \phi_{kj} - 1 - b^2 = \phi_{(k-1)j} \in \mathbb{Z}_{\geq 0} \,. \tag{5.5.6}$$

If for each $1 \leq s \leq N$ one had $\phi_{p_s s} = n_s$ for some integers $1 \leq p_s \leq N$ and n_s, then summing over s would yield

$$0 = \sum_{s=1}^{N} b\langle \alpha - Q, h_s\rangle = \sum_{s=1}^{N} \left(\frac{b\varkappa}{N} + b\langle \lambda h_1 - Q, h_{p_s}\rangle - n_s\right) = b\varkappa + k_1 b\lambda + k_2 b^2 + k_3 \tag{5.5.7}$$

for some integers k_i: this cannot happen for generic $(b, \varkappa, \lambda)$. Thus there exists $1 \leq u \leq N$ such that none of the ϕ_{pu} are integers. The condition (5.5.6) for $l = u$ then implies that for each $3 \leq k \leq N$, $\phi_{(k-1)t_k} \in \mathbb{Z}_{\geq 0}$ for some $1 \leq t_k \leq N$. No two t_k can be equal, because $\phi_{(k-1)t} - \phi_{(l-1)t} = (k-l)(b^2+1)$

is non-integer for $k \neq l$. We can thus permute the components of $\alpha - Q$ through a Weyl transformation so that $t_k = k-1$:

$$b\langle \alpha - Q, h_{k-1}\rangle = b\langle -\varkappa h_N, h_1\rangle + b\langle \lambda h_1 - Q, h_{k-1}\rangle - n_k \tag{5.5.8}$$

for all $3 \leq k \leq N$, where $n_k \geq 0$ are some integers. We deduce that

$$\alpha = (\lambda+\nu)h_1 - (\varkappa+\nu)h_N + \frac{1}{b}\sum_{k=3}^{N} n_k(h_1 - h_{k-1})\,. \tag{5.5.9}$$

The same considerations applied to the braiding of $-\frac{1}{b}h_1$ and $-\varkappa h_N$ yield the constraint above with $\frac{1}{b}$ replaced by b. We can thus restrict to momenta (5.5.9) which also have, up to a Weyl transformation, the $b \to \frac{1}{b}$ form. All in all, the fusion of two semi-degenerate operators can only allow a one-parameter set of momenta, and some isolated momenta

$$\begin{aligned} V_{-\varkappa h_N} \times V_{\lambda h_1} = &\int \mathrm{d}\nu\, V_{(\lambda+\nu)h_1 - (\varkappa+\nu)h_N} \\ &+ \sum_{n\in\mathbb{Z}}\sum_{n'\in\mathbb{Z}}\sum_{k=1}^{N} V_{(\lambda-\varkappa)h_1 + (n/b)(h_1-h_k) + [n'b - (N-k)/b](h_1-h_N)}\,. \end{aligned} \tag{5.5.10}$$

In the case $N=3$, we wrote down explicitly null vectors descending from $V_{-\varkappa h_N}$ and $V_{\lambda h_1}$ (higher W_N algebras are not tractable), and found that the isolated momenta are in fact not allowed. We propose that this holds for general N. After performing some Weyl reflexions of momenta on the left and right-hand side and redefining ν, we deduce the fusion rules

$$V_{-\varkappa h_N} \times V_{\lambda h_1} = \int \mathrm{d}\nu\, V_{-\varkappa h_N + \lambda h_1 + \nu(h_1 - h_N)} \tag{5.5.11}$$

$$V_{\varkappa h_1} \times V_{\lambda h_1} = \int \mathrm{d}\nu\, V_{\varkappa h_1 + \lambda h_1 + \nu(h_1 - h_2)} \tag{5.5.12}$$

$$V_{-\varkappa h_N} \times V_{-\lambda h_N} = \int \mathrm{d}\nu\, V_{-\varkappa h_N - \lambda h_N + \nu(h_{N-1} - h_N)}\,. \tag{5.5.13}$$

For completeness, we find the corresponding structure constant as the main residue of $C(\alpha_1, \alpha_2, \varkappa h_1)$ at $\alpha_1 = \lambda h_1$ and $\alpha_2 = 2Q - (\varkappa+\lambda+\nu)h_1 + \nu h_2$, after removing our normalization from (5.4.28), and recognize a Liouville CFT three-point function:

$$\begin{aligned} C^{(\varkappa+\lambda+\nu)h_1 - \nu h_2}_{\varkappa h_1, \lambda h_1} &= \frac{\hat{\mu}^{\nu}\Upsilon(b)^{N-1}\Upsilon(\varkappa)\Upsilon(\lambda)\Upsilon(\varkappa+\lambda+2\nu-b-1/b)}{\Upsilon(-\nu)\Upsilon(\varkappa+\nu)\Upsilon(\lambda+\nu)\Upsilon(\varkappa+\lambda+\nu-b-1/b)} \\ &= \Upsilon(b)^{N-2}\, C_{\text{Liouville}}\left(\frac{\varkappa}{2}, \frac{\lambda}{2}, b + \frac{1}{b} - \frac{\varkappa+\lambda}{2} - \nu\right). \end{aligned} \tag{5.5.14}$$

The equality is true by construction for $N=2$, as a Liouville momentum of $\varkappa/2$ corresponds in the Toda CFT language to a momentum of $(\varkappa/2)(h_1 - h_2) = \varkappa h_1$. More generally, the equality may hint to a deeper relation between Toda CFTs for different values of N.

5.5.2 Semi-degenerate and degenerate operators

We are now ready to tackle the fusion of other degenerate vertex operators $V_{-b\omega}$ with semi-degenerate operators $V_{\varkappa h_1}$.

For $\omega = Kh_1$ the fusion is a special case of (5.5.12) with $\lambda = -Kb$, hence only allows the momenta $(\varkappa - Kb)h_1 + \nu(h_1 - h_2)$. Given the fusion rule (5.5.1) of a degenerate operator, $(K-\nu/b)h_1 + (\nu/b)h_2$ must be a weight of $\mathcal{R}(Kh_1)$ hence $\nu = nb$ with $0 \leq n \leq K$, and

$$V_{\varkappa h_1} \times V_{-Kbh_1} = \sum_{n=0}^{K} V_{(\varkappa-(K-n)b)h_1 - nbh_2} \tag{5.5.15}$$

with no multiplicity since the weight $(K-n)h_1 + nh_2$ of $\mathcal{R}(Kh_1)$ has no multiplicity. Through the Weyl rotation $(N \cdots 2\,1)$, an equivalent statement is that the fusion of $-\varkappa h_N$ and $-Kbh_1$ yields the momenta $-nbh_1 - (\varkappa + (K-n)b)h_N$.

The correlator $\langle V_{\alpha_\infty}(\infty) V_{(\varkappa'+lb)h_1}(1) V_{-lbh_1}(x,\bar{x}) V_{\alpha_0}(0)\rangle$ has $\dim(\mathcal{R}(lh_1))$ s-channel conformal blocks, and must have the same number of t-channel conformal blocks. The fusion (5.5.15) allows the t-channel internal momenta $\varkappa' h_1 + nb(h_1 - h_2)$ for $0 \leq n \leq l$, with no multiplicity, hence any multiplicity is due to the fusion of V_{α_0} and V_{α_∞}. The number of t-channel conformal blocks is thus

$$\sum_{n=0}^{l} N_{\alpha_0,\,\alpha_\infty}^{\varkappa' h_1 + nb(h_1-h_2)} = \dim \mathcal{R}(lh_1) = \binom{N+l-1}{l} \tag{5.5.16}$$

where V_β appears $N^{\beta}_{\alpha_0,\alpha_\infty}$ times in the fusion of V_{α_0} and V_{α_∞}. Solving, we find

$$\begin{aligned} N_{\alpha_0,\,\alpha_\infty}^{\varkappa' h_1 + nb(h_1-h_2)} &= \dim \mathcal{R}(nh_1) - \dim \mathcal{R}((n-1)h_1) \\ &= \binom{N+n-1}{n} - \binom{N+n-2}{n-1} = \binom{N+n-2}{n} . \end{aligned} \tag{5.5.17}$$

None of these multiplicities vanish, so all $K+1$ momenta of (5.5.15) do appear in the fusion.

Restricting the fusion rule (5.5.15) to $\varkappa h_1 = -Jbh_1$ with $J \geq K$, we retrieve the decomposition into irreducible representations of the tensor product of two symmetric representations, given by the Littlewood–Richardson rule:

$$\mathcal{R}(Jh_1) \otimes \mathcal{R}(Kh_1) = \bigoplus_{n=0}^{K} \mathcal{R}\big((J+K-n)h_1 + nh_2\big) . \tag{5.5.18}$$

One could go in the other direction: the decomposition (5.5.18) for $J \geq K$ implies that the fusion of V_{-Jbh_1} with V_{-Kbh_1} yields the momenta $-(J +$

$K-n)bh_1 - nbh_2$. This set of $K+1$ momenta only involves $-Jbh_1$ as an overall constant part, hence the natural generalization from V_{-Jbh_1} to $V_{\varkappa h_1}$ is (5.5.15). We will apply a similar reasoning[3] to guess the fusion of other degenerate operators with a semi-degenerate operator.

The tensor product of an antisymmetric and a symmetric representations of A_{N-1} is the sum of two irreducible representations,

$$\mathcal{R}(Jh_1)\otimes\mathcal{R}(\omega_K) = \mathcal{R}(Jh_1+\omega_K)\oplus\mathcal{R}((J-1)h_1+\omega_{K+1})\,. \qquad (5.5.19)$$

This naturally generalizes to the fusion rule

$$V_{\varkappa h_1}\times V_{-b\omega_K} = V_{\varkappa h_1 - b\omega_K} + V_{\varkappa h_1 - b(\omega_{K+1}-h_1)}\,. \qquad (5.5.20)$$

For completeness, a Weyl reflexion yields the fusion of $-\varkappa h_N$ and $-b\omega_K$, which features the momenta $-(\varkappa+b)h_N - b\omega_{K-1}$ and $-\varkappa h_N - b\omega_K$.

We show in Section 3.4 together with Section 5.2.2 that the Toda CFT correlator of two generic operators with $V_{\varkappa h_1}$ and $V_{-b\omega_K}$ is equal to the partition function of a surface operator. At the end of Section 3.4.1 we expand the partition function in a limit which corresponds to the fusion of $V_{\varkappa h_1}$ and $V_{-b\omega_K}$. The exponents found there prove the fusion rule (5.5.20). Once more, the number of t-channel and s-channel conformal blocks must be equal:

$$N^{\varkappa h_1 - b\omega_K}_{\alpha_0,\,\alpha_\infty} + N^{(\varkappa+b)h_1 - b\omega_{K+1}}_{\alpha_0,\,\alpha_\infty} = \binom{N}{K}\,. \qquad (5.5.21)$$

We deduce for each $n\geq 0$ that for all $\varkappa$,

$$N^{\varkappa h_1 - b\omega_{n+1}}_{\alpha_0,\,\alpha_\infty} = \binom{N-1}{n}\,. \qquad (5.5.22)$$

This is consistent with multiplicities of the two powers of $|1-x|^2$ found at the end of Section 3.4.1, and matches with (5.5.17) for $n=1$ and $n=0$.

Consider now an arbitrary highest weight $\omega = \sum_{j=1}^{N-1} n_j\omega_j$ of A_{N-1}. For each j from $N-1$ to 1, its Young diagram has n_j columns with j boxes. Through the Littlewood–Richardson rule, we find a decomposition valid for $J\geq\sum_{j=1}^{N-1} n_j$,

$$\mathcal{R}(Jh_1)\otimes\mathcal{R}(\omega) = \bigoplus_{k_{N-1}=0}^{n_{N-1}}\cdots\bigoplus_{k_1=0}^{n_1}\mathcal{R}\Big(Jh_1+\omega+\sum_{j=1}^{N-1}\Big[k_j(h_{j+1}-h_1)\Big]\Big) \qquad (5.5.23)$$

[3] In principle, one could go further, and guess the fusion rule (5.5.12) for two semi-degenerate operators by replacing $-Kbh_1\to\lambda h_1$ and allowing shifts by continuous multiples of h_2-h_1. It could be interesting to obtain a continuous analogue of the Littlewood–Richardson rule along those lines.

into $\prod_{j=1}^{N-1}(n_j+1)$ irreducible representations. We thus propose the fusion rule

$$V_{\varkappa h_1} \times V_{-b\sum_{j=1}^{N-1} n_j\omega_j} = \sum_{k_1=0}^{n_1} \cdots \sum_{k_{N-1}=0}^{n_{N-1}} V_{\varkappa h_1 - b\sum_{j=1}^{N-1}[n_j\omega_j + k_j(h_{j+1}-h_1)]} . \tag{5.5.24}$$

As a natural generalization of (5.5.17) and (5.5.22), we propose that vertex operators with a momentum $\varkappa h_1 - b\sum_{j=1}^{N-1} l_j\omega_j$ appear with multiplicity

$$N_{\alpha_0,\alpha_\infty}^{\varkappa h_1 - b\sum_{j=1}^{N-1} l_j\omega_j} = \dim \mathcal{R}_{A_{N-2}}\left(\sum_{j=1}^{N-1} l_j\omega_{j-1}\right), \tag{5.5.25}$$

where the $j=1$ term can be absorbed in a shift of $\varkappa$, and the right-hand side is the dimension of the representation of A_{N-2} whose Young diagram is obtained from that of $\mathcal{R}\big(\sum_{j=1}^{N-1} l_j\omega_j\big)$ by removing the first row: $h_1 \to 0$ and $h_i \to h_{i-1}$. Besides reproducing the correct multiplicities for the symmetric and antisymmetric case, the proposal (5.5.25) correctly leads to equally many s-channel and t-channel conformal blocks in the four-point function $\langle V_{\alpha_\infty}(\infty)V_{\varkappa h_1}(1)V_{-b\omega}(x,\bar{x})V_{\alpha_0}(0)\rangle$ since

$$\dim \mathcal{R}\left(\sum_{j=1}^{N-1} n_j\omega_j\right) = \sum_{k_1=0}^{n_1} \cdots \sum_{k_{N-1}=0}^{n_{N-1}} \dim \mathcal{R}_{A_{N-2}}\left(\sum_{j=1}^{N-1}\Big[n_j\omega_{j-1} + k_j h_j\Big]\right). \tag{5.5.26}$$

The equality holds because the representations on the right-hand side are the decomposition of $\mathcal{R}(\omega)$ into irreducible representations of the subalgebra A_{N-2} of A_{N-1}.

5.6 Irregular punctures

This section is Appendix A.6 of [GLF14].

We study irregular punctures obtained as collision limits of vertex operators in the Toda CFT. Such collisions were studied for Virasoro primaries in [GT12], and extended to other algebras in [KMST13; GLP13]. We give evidence that the limit

$$\mathbb{V}_{c_0;c_1,\bar{c}_1;\cdots;c_K,\bar{c}_K}(w,\bar{w}) = \lim_{(w_I,\bar{w}_I)\to(w,\bar{w})} \prod_{I<J} |w_J - w_I|^{2\langle\alpha_J,\alpha_I\rangle} \prod_{I=0}^{K} V_{\alpha_I}(w_I,\bar{w}_I) \tag{5.6.1}$$

exists, provided that the momenta α_I of vertex operators, and their position $(w_J,\bar{w}_J)$, vary in such a way that

$$C_j = \sum_{I=0}^{K}(w_I - w)^j\alpha_I \to c_j \qquad \bar{C}_j = \sum_{I=0}^{K}(\bar{w}_I - \bar{w})^j\alpha_I \to \bar{c}_j \tag{5.6.2}$$

for all $j \geq 0$. Not every choice of c_j and $\bar{c}_j$ can appear (for a given rank K). Firstly, $\bar{c}_0 = c_0$. Secondly, $\bar{c}_j = c_j = 0$ for all $j > K$. Indeed, any C_j with $j > K$ is a linear combination $C_j = \sum_{k=0}^{K} P_{j,k}(\{w_I - w\})C_k$ whose coefficients $P_{j,k}$ are homogeneous polynomial of degree $j - k \geq 1$ in the variables $w_I - w$, and such polynomials vanish as $w_I \to w$. The limits of C_j and $\bar{C}_j$ are thus described by the $2K + 1$ momenta $(c_0; c_1, \bar{c}_1; \cdots; c_K, \bar{c}_K)$, as indicated by the notation in (5.6.1).

There is (at least) one other condition on the c_j and $\bar{c}_j$: for each $0 \leq m \leq K$ the vectors $\{c_n, \bar{c}_n \mid m \leq n \leq K\}$ must span a space of dimension at most $K - m + 1$, for instance c_K and $\bar{c}_K$ must be collinear. This third restriction relies on

$$\sum_{j=0}^{n}\Bigg(C_j \sum_{\substack{S\in[\![0,n-1]\!]\\ \#S=n-j}} \prod_{I\in S}(w - w_I)\Bigg) = \sum_{J=n}^{K}\prod_{I=0}^{n-1}(w_J - w_I)^n \alpha_J\,, \tag{5.6.3}$$

whose left-hand side goes to c_n in our limit, and on its analogue for $\bar{c}_n$. Since rank is lower semicontinuous, the rank of the space spanned by $\{c_n, \bar{c}_n \mid m \leq n \leq K\}$ is at most that of the space spanned by (5.6.3) and by their antiholomorphic counterparts (for $m \leq n \leq K$). This second space lies within the span of $\{\alpha_J \mid m \leq J \leq K\}$, which has rank at most $K - m + 1$.

5.6.1 OPE with the stress-energy tensor

Our first piece of evidence is to write the OPE of the stress-energy tensor with

$$\mathbf{V}_{\{\alpha_I\}}(\{w_I, \bar{w}_I\}) = \prod_{I<J} |w_J - w_I|^{2\langle\alpha_J,\alpha_I\rangle} \prod_{I=0}^{K} V_{\alpha_I}(w_I, \bar{w}_I) \tag{5.6.4}$$

in the limit which defines $\mathbb{V}_{c_0;\cdots;c_K,\bar{c}_K}$. The operators V_{α_I} are primary, hence

$$T(z)\mathbf{V}_{\{\alpha_I\}}(\{w_I, \bar{w}_I\}) \sim \prod_{I<J}|w_J - w_I|^{2\langle\alpha_J,\alpha_I\rangle} \sum_{I=0}^{K}\left(\frac{\Delta(\alpha_I)}{(z-w_I)^2} + \frac{1}{z-w_I}\partial_{w_I}\right)\prod_{I=0}^{K} V_{\alpha_I}(w_I,\bar{w}_I) \tag{5.6.5}$$

$$= \sum_{I=0}^{K}\left(\frac{\Delta(\alpha_I)}{(z-w_I)^2} + \frac{1}{z-w_I}\left(\partial_{w_I} + \sum_{J\neq I}\frac{\langle\alpha_I,\alpha_J\rangle}{w_J - w_I}\right)\right)\mathbf{V}_{\{\alpha_I\}}(\{w_I,\bar{w}_I\}) \tag{5.6.6}$$

$$= \left(\langle Q, \partial_z\partial_z\varphi_{\text{sing}}\rangle - \frac{1}{2}\langle\partial_z\varphi_{\text{sing}}, \partial_z\varphi_{\text{sing}}\rangle + \sum_{I=0}^{K}\frac{\partial_{w_I}}{z-w_I}\right)\mathbf{V}_{\{\alpha_I\}}(\{w_I,\bar{w}_I\}) \tag{5.6.7}$$

where in the last line we use $\Delta(\alpha_I) = \langle Q, \alpha_I\rangle - \frac{1}{2}\langle \alpha_I, \alpha_I\rangle$ to express all but the ∂_{w_I} piece in terms of

$$\partial_z \varphi_{\text{sing}} = \sum_{I=0}^{K} \frac{-\alpha_I}{z - w_I} = \sum_{n\geq 0} \frac{-\sum_{I=0}^{K}(w_I - w)^n \alpha_I}{(z-w)^{n+1}} \to \sum_{n=0}^{K} \frac{-c_n}{(z-w)^{n+1}} \,. \tag{5.6.8}$$

In the domain where all $|w_I - w| < |z - w|$, we can expand the derivative term as

$$\sum_{I=0}^{K} (z - w_I)^{-1} \partial_{w_I} = \sum_{n\geq -1} (z-w)^{-n-2} \sum_{I=0}^{K} (w_I - w)^{n+1} \partial_{w_I} \,. \tag{5.6.9}$$

The term with $n = -1$ is $\sum_{I=0}^{K} \partial_{w_I}$, which translates all vertex operators, hence its limit is ∂_w. The other terms do not have such a simple geometrical interpretation. Instead, let us write their action on C_m for $0 \leq m \leq K$:

$$\sum_{I=0}^{K} (w_I - w)^{n+1} \partial_{w_I} C_m = \sum_{I=0}^{K} (w_I - w)^{n+1} \partial_{w_I} \sum_{J=0}^{K} (w_J - w)^m \alpha_J \tag{5.6.10}$$

$$= \sum_{I=0}^{K} m (w_I - w)^{n+m} \alpha_I = m C_{n+m} \,. \tag{5.6.11}$$

The limit of $\sum_{I=0}^{K} (w_I - w)^{n+1} \partial_{w_I}$ must thus be a differential operator which maps $c_m \to m c_{n+m}$ for all $0 \leq m \leq K - n$ and $c_m \to 0$ for $K - n < m \leq K$. This is naturally realized by

$$\sum_{I=0}^{K} (w_I - w)^{n+1} \partial_{w_I} \to \sum_{j=1}^{K-n} j \langle c_{n+j}, \partial_{c_j} \rangle \,. \tag{5.6.12}$$

All in all, the OPE of $T(z)$ with $\mathbb{V} = \mathbb{V}_{c_0;\cdots;c_K,\bar{c}_K}(w, \bar{w})$ is

$$\begin{aligned} T(z)\mathbb{V} \sim \Bigg(& \left\langle Q, \partial_z \sum_{n=0}^{K} \frac{-c_n}{(z-w)^{n+1}} \right\rangle - \frac{1}{2} \left\langle \sum_{j=0}^{K} \frac{-c_j}{(z-w)^{j+1}}, \sum_{l=0}^{K} \frac{-c_l}{(z-w)^{l+1}} \right\rangle \\ & + \frac{\partial_w}{z-w} + \sum_{n\geq 0} \frac{1}{(z-w)^{n+2}} \sum_{j=1}^{K-n} j \langle c_{n+j}, \partial_{c_j} \rangle \Bigg) \mathbb{V} \\ = \Bigg(& \frac{1}{z-w} \partial_w + \sum_{n=0}^{2K} \frac{(n+1)\langle Q, c_n\rangle - \frac{1}{2} \sum_{j=0}^{n} \langle c_j, c_{n-j}\rangle + \sum_{j=1}^{K-n} j \langle c_{n+j}, \partial_{c_j}\rangle}{(z-w)^{n+2}} \Bigg) \mathbb{V} \end{aligned} \tag{5.6.13}$$

where we recall that $c_n = 0$ for $n > K$. The presence of singularities up to $(z - w)^{-2K-2}$ in this OPE implies that the Virasoro generators L_n act non-trivially on the state $|c\rangle = \mathbb{V}_{c_0;\cdots;c_K,\bar{c}_K}(0)|0\rangle$ for $n \leq 2K$. More precisely,

$$L_n|c\rangle = \Big((n+1)\langle Q, c_n\rangle - \frac{1}{2}\sum_{j=0}^{n}\langle c_j, c_{n-j}\rangle + \sum_{j=1}^{K-n} j\langle c_{n+j}, \partial_{c_j}\rangle\Big)|c\rangle \quad (5.6.14)$$

for $0 \leq n \leq 2K$, while L_{-1} translates w, and $L_n|c\rangle = 0$ for $n > 2K$. This is the natural generalization of equation (2.7) of [GT12].

In the rank 1 case ($c_n = 0$ for $n > 1$), we can exponentiate explicitly the action of the Virasoro generators L_n to find how large conformal transformations act. From above, we know that $L_n|c\rangle = 0$ for $n > 2$, that L_{-1} acts like ∂_w, and that

$$L_2|c\rangle = -\frac{1}{2}\langle c_1, c_1\rangle|c\rangle \quad (5.6.15)$$

$$L_1|c\rangle = \langle 2Q - c_0, c_1\rangle|c\rangle \quad (5.6.16)$$

$$L_0|c\rangle = \left(\Delta(c_0) + \langle c_1, \partial_{c_1}\rangle\right)|c\rangle \quad (5.6.17)$$

where as usual $\Delta(c_0) = \langle Q, c_0\rangle - \langle c_0, c_0\rangle/2$. Omitting the parameters $\bar{z}$ and $\bar{c}_n$ which play no role for holomorphic transformations, we claim that

$$\mathbb{V}_{c_0,c_1}(z) = (\partial_z w)^{\Delta(c_0)} \exp\Big(\langle 2Q - c_0, c_1\rangle\frac{\partial_z^2 w}{\partial_z w} - \frac{1}{2}\langle c_1, c_1\rangle\Big[\frac{\partial_z^3 w}{\partial_z w} - \frac{3}{2}\frac{(\partial_z^2 w)^2}{(\partial_z w)^2}\Big]\Big)\mathbb{V}_{c_0,(\partial_z w)c_1}(w) \quad (5.6.18)$$

under a conformal map $z \to w(z)$. Indeed, this transformation is transitive and has the correct infinitesimal behaviour: for $\partial_z w = 1 + \epsilon$,

$$\mathbb{V}_{c_0,c_1}(z) = \Big(1 + \epsilon(\Delta(c_0) + \langle c_1, \partial_{c_1}\rangle) + \langle 2Q - c_0, c_1\rangle\partial_z\epsilon - \frac{1}{2}\langle c_1, c_1\rangle\partial_z^2\epsilon + O(\epsilon^2)\Big)\mathbb{V}_{c_0,c_1}(w)\,. \quad (5.6.19)$$

5.6.2 Free field realization

Our derivation of (5.6.13) only relies on the OPE of $T(z)$ with vertex operators V_α. This OPE has a free field realization as the OPE of $T_Q^{\text{free}} = \langle Q, \partial\partial\varphi\rangle - \frac{1}{2}{:}\langle\partial\varphi, \partial\varphi\rangle{:}$ with $V_\alpha^{\text{free}} = {:}e^{\langle\alpha,\varphi\rangle}{:}$. We rederive (5.6.13) more directly by first building the collision limit $\mathbb{V}^{\text{free}}$ of vertex operators V_α^{free}, then computing its OPE with T_Q^{free}. We then go further and consider the OPE of higher spin currents of the W_N algebra with $\mathbb{V}^{\text{free}}$.

First, ${:}e^{\langle\alpha,\varphi(z,\bar{z})\rangle}{:}{:}e^{\langle\beta,\varphi(w,\bar{w})\rangle}{:} = |z-w|^{-2\langle\alpha,\beta\rangle}{:}e^{\langle\alpha,\varphi(z,\bar{z})\rangle+\langle\beta,\varphi(w,\bar{w})\rangle}{:}$ implies by induction

$$\prod_{I<J}|w_I - w_J|^{2\langle\alpha_I,\alpha_J\rangle}\prod_{I=0}^{K}{:}e^{\langle\alpha_I,\varphi(w_I,\bar{w}_I)\rangle}{:} = {:}e^{\sum_{I=0}^{K}\langle\alpha_I,\varphi(w_I,\bar{w}_I)\rangle}{:}\,. \quad (5.6.20)$$

Expand $\varphi(w_I,\bar w_I) = \varphi(w,\bar w) + \sum_{n\geq 1}\frac{1}{n!}\big((w_I-w)^n\partial^n\varphi(w) + (\bar w_I-\bar w)^n\bar\partial^n\varphi(\bar w)\big)$ thanks to $\partial\bar\partial\varphi = 0$ and use the limit $\sum_{I=0}^K (w_I-w)^n\langle\alpha_I,\partial^n\varphi\rangle \to \langle c_n,\partial^n\varphi\rangle$ and its antiholomorphic counterpart to obtain the free field collision limit

$$\mathbb{V}^{\text{free}}_{c_0;\cdots}(w,\bar w) = :\exp\biggl[\langle c_0,\varphi(w,\bar w)\rangle + \sum_{n=1}^{K}\frac{\langle c_n,\partial^n\varphi(w)\rangle + \langle\bar c_n,\bar\partial^n\varphi(\bar w)\rangle}{n!}\biggr]:. \tag{5.6.21}$$

The stress-energy tensor $T_Q^{\text{free}}(z)$ and higher spin currents are polynomials in $\partial\varphi(z)$ and its derivatives. We thus evaluate

$$\partial\varphi(z)\mathbb{V}^{\text{free}}_{c_0;\cdots}(w,\bar w) = :\biggl(\partial\varphi(z) + \sum_{n\geq 0}\frac{c_n}{n!}\partial_w^n\frac{-1}{z-w}\biggr)\mathbb{V}^{\text{free}}_{c_0;\cdots}(w,\bar w): \tag{5.6.22}$$

$$= :\biggl(\sum_{n\geq 1}(z-w)^{n-1}n\partial_{c_n} - \sum_{n\geq 0}\frac{c_n}{(z-w)^{n+1}}\biggr)\mathbb{V}^{\text{free}}_{c_0;\cdots}(w,\bar w): \tag{5.6.23}$$

where the first equality relies on $\partial\overline{\varphi(z)\varphi}(w,\bar w) = -1/(z-w)$, and the second on the Taylor expansion of $\partial\varphi(z)$ and on $\partial^n\varphi\mathbb{V}^{\text{free}}_{c_0;\cdots} = n!\partial_{c_n}\mathbb{V}^{\text{free}}_{c_0;\cdots}$. The OPE of $\mathbb{V}^{\text{free}}_{c_0;\cdots}(w,\bar w)$ with any polynomial in derivatives of $\partial\varphi(z)$ is thus obtained by replacing all

$$\begin{aligned}\partial_z^{l+1}\varphi(z) &\to \partial_z^l\biggl(\sum_{n\geq 1}(z-w)^{n-1}n\partial_{c_n} - \sum_{n\geq 0}\frac{c_n}{(z-w)^{n+1}}\biggr)\\ &= -\partial_z^l\sum_{n\in\mathbb{Z}}\frac{\theta_n c_n + \theta_{-n}n\partial_{c_{-n}}}{(z-w)^{n+1}}\end{aligned} \tag{5.6.24}$$

where $\theta_n = 1$ if $n\geq 0$ and 0 if $n<0$, then dropping terms that are regular as $z\to w$.

In particular,

$$T_Q^{\text{free}}(z)\mathbb{V}^{\text{free}}(w,\bar w) = \Bigl(\langle Q,\partial\partial\varphi\rangle - \frac{1}{2}{:}\langle\partial\varphi,\partial\varphi\rangle{:}\Bigr)\mathbb{V}^{\text{free}}_{c_0;\cdots;c_K,\bar c_K} \tag{5.6.25}$$

$$\sim\Biggl(\biggl\langle Q,\sum_{n\geq 0}\frac{(n+1)c_n}{(z-w)^{n+2}}\biggr\rangle - \frac{1}{2}\biggl\langle\sum_{i\geq 0}\frac{c_i}{(z-w)^{i+1}},\sum_{j\geq 0}\frac{c_j}{(z-w)^{j+1}}\biggr\rangle + \biggl\langle\sum_{i\geq 0}\frac{c_i}{(z-w)^{i+1}},\sum_{j\geq 1}(z-w)^{j-1}j\partial_{c_j}\biggr\rangle\Biggr)\mathbb{V}^{\text{free}} \tag{5.6.26}$$

$$= \Biggl(\sum_{n=0}^{2K}\frac{(n+1)\langle Q,c_n\rangle - \frac{1}{2}\sum_{i=0}^n\langle c_i,c_{n-i}\rangle}{(z-w)^{n+2}} + \sum_{n=-1}^{K-1}\frac{\sum_{j=1}^{K-n}\langle c_{j+n},j\partial_{c_j}\rangle}{(z-w)^{n+2}}\Biggr)\mathbb{V}^{\text{free}}. \tag{5.6.27}$$

Upper bounds could be omitted since $c_m = 0$ for $m\geq K$. Note the presence of $\partial_{c_{K+1}}$ in the last term for $n=-1$ and $j=K+1$. This derivative is inconvenient as it involves irregular punctures with a rank higher than $\mathbb{V}^{\text{free}}$. It turns out that the terms with $n = -1$ combine nicely into

$$\sum_{j=1}^{K+1} \frac{\langle c_{j-1}, \partial^j \varphi(w)\rangle}{(j-1)!} \mathbb{V}^{\text{free}}_{c_0;\cdots;c_K,\bar{c}_K}(w,\bar{w}) = \partial_w \mathbb{V}^{\text{free}}_{c_0;\cdots;c_K,\bar{c}_K}(w,\bar{w}) \,. \tag{5.6.28}$$

As expected, the free field OPE reproduces the OPE (5.6.13).

We are ready to consider higher spin currents. A basis of those currents is obtained via the Miura transformation

$$\prod_{s=N}^{1} \left(q\partial_z + \langle h_s, \partial_z \varphi(z)\rangle\right) = \sum_{p=0}^{N} W^p(z) \left(q\partial_z\right)^{N-p} \tag{5.6.29}$$

where $q = b + \frac{1}{b}$. In particular, $W^0(z) = 1$, $W^1(z) = 0$, and $W^2(z) = T^{\text{free}}_Q(z)$. The prescription (5.6.24) then yields the OPE of $W^p(z)$ with the irregular $\mathbb{V}^{\text{free}}_{c_0;\cdots}(w,\bar{w})$, but expressions quickly become very unwieldy. However, we can get valuable information by applying the prescription (5.6.24) directly to the Miura transformation (5.6.29):

$$\begin{aligned} &\sum_{p=0}^{N} W^p(z) \mathbb{V}^{\text{free}}_{c_0;\cdots}(w,\bar{w}) \left(q\partial_z\right)^{N-p} \\ &= \prod_{s=N}^{1} \left(q\partial_z + \sum_{n\in\mathbb{Z}} \frac{\langle h_s, -\theta_n c_n - \theta_{-n} n \partial_{c_{-n}}\rangle}{(z-w)^{n+1}}\right) \mathbb{V}^{\text{free}}_{c_0;\cdots}(w,\bar{w}) \end{aligned} \tag{5.6.30}$$

where ∂_{c_j} only acts on $\mathbb{V}^{\text{free}}$ and not on intervening c_j, and where $\theta_n = 1$ if $n \geq 0$ and 0 otherwise. The sums over n actually truncate to $n \leq K$ for rank K punctures, thus only a finite number of negative powers of $(z - w)$ appear in the OPE.

Let us find out the most singular terms of the OPE of a given $W^p(z)$ with $\mathbb{V}^{\text{free}}$. Thanks to the mode expansion $W^p(z) = \sum_{n\in\mathbb{Z}} W^p_n(w)(z-w)^{-n-p}$, the $(z-w)^{-n-p}$ term in the OPE encodes the action of W^p_n on the rank K puncture $|c\rangle = \mathbb{V}^{\text{free}}_{c_0;\cdots;c_K,\bar{c}_K}(w,\bar{w})|0\rangle$. Terms where $0 \leq m < p$ of the $q\partial_z$ act on some $\langle h_s, \ldots\rangle$ are at most of order $O\big((z-w)^{-(K+1)(p-m)-m}\big)$. Those involving ∂_{c_j} derivatives are of order $O\big((z-w)^{-(K+1)(p-m-1)-m}\big)$ or more regular. Thus, $W^p_n|c\rangle = 0$ for $n > pK$,

$$\begin{aligned} W^p_n|c\rangle = (-1)^p \sum_{1\leq s_1<\cdots<s_p\leq N} \Bigg[&\sum_{k_1+\cdots+k_p=pK-n} \prod_{i=1}^{p} \langle h_{s_i}, c_{K-k_i}\rangle \\ &+ \delta_{n,(p-1)K}(K+1)q \sum_{j=1}^{p} \Big((j-1) \prod_{i\neq j}^{p} \langle h_{s_i}, c_K\rangle \Big) \Bigg] |c\rangle \end{aligned} \tag{5.6.31}$$

for $(p-1)K \leq n \leq pK$, and lower components of $W^p(z)$ act with ∂_{c_j} derivatives. This is consistent with the action (5.6.14) of the Virasoro algebra for $p=2$.

For $n < (p-1)K$, the action of W^p_n on $|c\rangle$ involves derivatives ∂_{c_j} for each $1 \leq j \leq (p-1)K - n$. In particular, if $n < (p-2)K$, derivatives with $j > K$ appear: the set of rank K irregular punctures is not stable under those components W^p_n. One exception is that $L_{-1} = W^2_{-1}$ involves derivatives up to $\partial_{c_{K+1}}$ but turns out to be identical to an infinitesimal translation. The set of all (finite, integer) rank irregular punctures is stable under all W^p_n.

Before closing this section, we go back to the Toda CFT and compute various two-point functions of vertex operators with rank $K=1$ irregular punctures as a test that the collision limit is finite.

5.6.3 Two-point functions

The only irregular punctures we use in this thesis (Section 3.4.3) are the collision of a semi-degenerate and a generic vertex operators. We compute here the two-point function of the resulting rank 1 puncture with any generic vertex operator (5.6.36) in a useful normalization (5.6.33).

The collision limits of interest are a special case of the general collision limit (5.6.1) which defines rank K irregular punctures. Using notations close to the main text,

$$\mathbb{V}_{c_0;-(x/b)h_1,(\bar{x}/b)h_1}(0) = \lim_{\Lambda\to\infty}\left[\left|\frac{x}{\Lambda}\right|^{2\langle\varkappa h_1, c_0-\varkappa h_1\rangle} V_{\varkappa h_1}\left(\frac{x}{-\mathrm{i}\Lambda}, \frac{\bar{x}}{\mathrm{i}\Lambda}\right) V_{c_0-\varkappa h_1}(0)\right]_{\varkappa=\mathrm{i}\Lambda/b+O(1)} \tag{5.6.32}$$

where $\Lambda \in \mathbb{R}$ is the gauge theory cutoff scale, c_0, b, x and $\bar{x}$ are various physical parameters, and only the leading behaviour of $\varkappa$ in Λ affects the limit. We also introduce the normalization

$$\begin{aligned}\widehat{\mathbb{V}}_{c_0;-(x/b)h_1,(\bar{x}/b)h_1}(0) &= \frac{\hat{\mu}^{\langle c_0-Q,\rho\rangle}\mathbb{V}_{c_0;-(x/b)h_1,(\bar{x}/b)h_1}(0)}{\Upsilon(b)^{N-1}\prod_{2\leq s<t\leq N}\Upsilon(\langle Q-c_0, h_s-h_t\rangle)} \qquad (5.6.33)\\ &= \lim_{\Lambda\to\infty}\left[\frac{\Upsilon(\varkappa+\langle Q-c_0,h_1\rangle)^N}{|\Lambda/b|^{2\Delta(c_0)-\langle Q,Q\rangle}}\left|\frac{x}{\Lambda}\right|^{2\langle\varkappa h_1,c_0-\varkappa h_1\rangle}\widehat{V}_{\varkappa h_1}\left(\frac{x}{-\mathrm{i}\Lambda},\frac{\bar{x}}{\mathrm{i}\Lambda}\right)\widehat{V}_{c_0-\varkappa h_1}(0)\right]_{\varkappa=\mathrm{i}\Lambda/b+O(1)}\end{aligned}$$

where the second line is obtained by combining the factors (5.4.3) and (5.4.4) which relate $\widehat{V}$ and V with those relating $\widehat{\mathbb{V}}$ and $\mathbb{V}$. The only non-trivial step is that the asymptotics (5.4.7) of the Upsilon function simplify $\prod_{t=1}^N \Upsilon(\varkappa + \langle Q-c_0, h_1-h_t\rangle)$ to $\Upsilon(\varkappa+\langle Q-c_0,h_1\rangle)^N|\Lambda/b|^{\langle Q,Q\rangle-2\Delta(c_0)}$.

Let us compute the two-point function of the irregular puncture (5.6.33) with a generic vertex operator $\widehat{V}_{\alpha_0}$. Throughout the calculation, $\varkappa = \mathrm{i}\Lambda/b + O(1)$. Scale covariance and the explicit form (5.4.28) of the three-point function give

$$\frac{\Upsilon\big(\varkappa+\langle Q-c_0,h_1\rangle\big)^N}{|\Lambda/b|^{2\Delta(c_0)-\langle Q,Q\rangle}}\left|\frac{x}{\Lambda}\right|^{2\langle\varkappa h_1,c_0-\varkappa h_1\rangle}\left\langle\widehat{V}_{\alpha_0}(\infty)\widehat{V}_{\varkappa h_1}\Big(\frac{x}{-\mathrm{i}\Lambda},\frac{\bar{x}}{\mathrm{i}\Lambda}\Big)\widehat{V}_{c_0-\varkappa h_1}(0)\right\rangle$$
$$=\frac{|x/\Lambda|^{2\langle\varkappa h_1,c_0-\varkappa h_1\rangle-2\Delta(\varkappa h_1)-2\Delta(c_0-\varkappa h_1)+2\Delta(\alpha_0)}\Upsilon(\varkappa+\langle Q-c_0,h_1\rangle)^N}{|\Lambda/b|^{2\Delta(c_0)-\langle Q,Q\rangle}\prod_{s,t=1}^N\Upsilon\big(\frac{\varkappa}{N}+\langle Q-c_0+\varkappa h_1,h_s\rangle+\langle Q-\alpha_0,h_t\rangle\big)} \tag{5.6.34}$$

$$\sim\frac{|x/\Lambda|^{2\Delta(\alpha_0)-2\Delta(c_0)}|\Lambda/b|^{2\Delta(\alpha_0)-\langle Q,Q\rangle}}{|\Lambda/b|^{2\Delta(c_0)-\langle Q,Q\rangle}\prod_{s=2}^N\prod_{t=1}^N\Upsilon\big(\langle Q-c_0,h_s\rangle+\langle Q-\alpha_0,h_t\rangle\big)}\,. \tag{5.6.35}$$

All powers of Λ cancel, and we deduce that

$$\left\langle\widehat{V}_{\alpha_0}(\infty)\widehat{\mathbb{V}}_{c_0;-(x/b)h_1,(\bar{x}/b)h_1}(0)\right\rangle=\frac{|x/b|^{2\Delta(\alpha_0)-2\Delta(c_0)}}{\prod_{s=2}^N\prod_{t=1}^N\Upsilon\big(\langle Q-c_0,h_s\rangle+\langle Q-\alpha_0,h_t\rangle\big)}\,. \tag{5.6.36}$$

Note that the dependence on $|x/b|$ is as expected from the transformation (5.6.18) of rank 1 irregular punctures under a scaling. Both the OPE with W_N currents, and the two-point function we have just computed, are finite, and independent of details such as the precise value of $\varkappa$ in the limit (5.6.33). This gives credence to our claim that collision limits $\mathbb{V}_{c_0;\cdots;c_K,\bar{c}_K}$ are finite and only depend on the c_j and $\bar{c}_j$.

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