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Physics of massive superstrings

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PHYSICS OF MASSIVE SUPERSTRINGS

A dissertation presented
by

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To my parents, and my wife

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ABSTRACT OF DISSERTATION

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Abstract

This thesis is based on the author's work [1–3] conducted from September 2009 to May 2012 at Northeastern University, Boston, MA, USA, and also at Max-Planck-Institut für Physik, Munich, Germany as a visiting student, under supervision of Professor Tomasz Taylor.

In this thesis, we focus on the massive superstring states. Starting from a short review on the first massive level open string states in ten dimensions, we investigate the four-dimensional physical open string states under $\mathcal{N} = 4, 2, 1$ compactifications. We find these physical states split into certain supermultiplets by working out explicitly their supersymmetry transformations. We then focus on the universal states which are common to all compactifications, their scattering amplitudes are the most appealing and important for these scattering processes could generate model-independent stringy signals which could be tested at the Large Hadron Collider if the fundamental string mass scale is as low as a few TeVs. Finally, we explore some general properties of higher level massive superstring states and give the first example of the use of the on-shell recursion relations of scattering amplitudes in superstring theory.

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Contents

1 Introduction

String theory is perhaps the most successful and also the most ambitious approach to the unified theory of particle physics, gravity, and quantum physics today. The fact that string theory easily includes the gauge interactions and also gravitational force at quantum level, assures string theory describe all the four known fundamental forces – gravitational, electromagnetic, weak and strong interactions, and also matter particles in a mathematically consistent system. This fact leads many physicists to believe that string theory is the correct fundamental description of the nature.

String theory was originally invented as a model for describing the spectrum and the S-matrix of hadrons. The famous Veneziano amplitudes [4] contains an infinite number of string excitations, where the hadronic particles essentially follow the Regge trajectories of vibrating strings,

$$j = j_0 + \alpha' M^2 \ , \tag{1.1}$$

with the spin j and α' the Regge slope parameter which indicates the fundamental string scale $M^2_{\text{string}} = \alpha'^{-1}$. As a theory of hadrons, the string scale has to be chosen of order of the relevant hadronic mass, i.e., $M_{\text{string}} = \mathcal{O}(\text{GeV})$. However, due to several difficulties within the theory, and QCD was recognized to be the correct theory of strong interactions, and string theory as a model for hadrons was left aside.

In 1974 a radical change of paradigm in string theory took place, when Scherk and Schwarz proposed string theory as a fundamental theory for quantum gravity [5]. The massless spin two closed string excitation has all properties of the graviton particle in the quantized version of general relativity. As pointed out by the authors, this observation seemingly implies that the string scale now has to be identified with the fundamental scale of gravity – the Planck scale $M_{\text{Planck}} \simeq 10^{19}$ GeV. If this is the case, the masses of all string excitations are as high as Planck scale and thus not accessible for direct production and discovery at current accelerators like the LHC.

The situation drastically changes when one compactifies the ten-dimensional superstring theory to four spacetime dimensions to make contact with the Standard Model of particles physics (for a review see [6]). Meanwhile, the uniqueness of string theory was destroyed during the compactification, since string compactifications allow for a huge number of these possible ground states, which could be as large as order of 10^{500} [7, 8], which was referred as the landscape problem. Each of these vacua in the string landscape corresponds to a different universe with different physics and different cosmological properties.

However, after compactification, the string scale M_{string} is not necessarily at the order of the Planck mass, it is a free parameter, which can be as low as a few TeV [9, 10].[1] This is particularly true in D-brane compactifications, where the Standard Model is living on a lower dimensional brane, that might be embedded into the internal, compact six-dimensional Calabi-Yau space used for compactification.

In this brane world scenario the elementary particles, such as quarks, leptons, gluons, photons, weak bosons and the Higgs particles arise as open string excitations, whose ends are attached to the world volumes of the intersecting D-branes. Specifically, gauge bosons are due to strings attached to stacks of D-branes and chiral matter due to strings stretching between intersecting D-branes. Hence, now the colored quarks and gluons are elementary open strings, from which it follows that there exist higher spin Regge excitations of the quarks, gluons, and of all Standard Model fields.

It has been shown in [13, 14] that the production cross sections of gluons and quarks at the LHC into massive string excitations can be computed in a completely universal, model independent way, allowing for universal string predictions in case the string scale is low. The corresponding tree level string cross sections are independent from in the internal geometry and hence independent from the particular location of the model in the string landscape. This observation nullifies the notorious landscape problem.

At the energy around few TeVs, stringy corrections due to new colored Regge modes will become important. Their production and subsequent decay will then lead the discovery of these universal heavy string excitations. Eventually there will be a full tower of elementary open string Regge modes. Direct detection of such vibrating string modes is possible at the LHC. For a survey of low-mass superstring phenomenology and early references, see [15, 16]. At first, one would see Regge excitations indirectly, in the excess of photons [17, 18], jets [19, 20], heavy quarks [21] and leptons due to the resonant enhancement of their production rates. In another paper [22] by the author, we discussed the possible signals of low mass string resonances in e^+e^- and $\gamma\gamma$ collisions at future lepton colliders. The effects of Regge resonances and KK gravitons are

[1] A low mass string scale below 4 TeV has been already excluded by CMS with recent data of LHC7 run [11, 12].

also discussed in [23–26].

Once the center-of-mass energies of the colliding partons is crossed the mass threshold M, one would also see free Regge states produced directly, in association with jets, photons and other particles. We will discuss direct production of the lightest universal Regge particles, i.e., the quanta of fundamental string harmonics with masses equal M. In fact, the first massive gluonic resonances can be shown to have spin 0,1, and 2, whereas the first quarks resonances have spin 1/2 and spin 3/2, respectively. These lowest excitations will be found exactly at mass $M_{\rm string}$, and they are followed by the infinite tower of higher Regge excitations. So in case $M_{\rm string} \simeq \mathcal{O}(\mathrm{TeV})$, these universal string Regge excitations should be easily found at the LHC. These universal scattering amplitudes involving one massive particle and two or three massless ones are computed. The amplitudes relevant to the direct production of string resonances at the LHC are $p_1 p_2 \to p_3 R$, where $p\ (= g, q, \bar{q})$ are partons and R is a massive string state. These universal amplitudes will be one of the main focus on this thesis.

However, other than these "universal" states, the spectrum of Regge excitations are highly model-dependent. For example, in the toroidal compactifications of a single ten-dimensional D9-brane one encounters 128 bosons and 128 fermions at the first massive level. Most of these particles are tied to $\mathcal{N}$=4 supersymmetry of toroidal compactifications. We then, take a detailed analysis to exploit some of the basic supersymmetry properties of the first massive level superstring states in four dimensions, originating from supersymmetric type II compactifications. Besides world-sheet conformal invariance, supersymmetry plays a key role for the consistency of string theory, both on the world-sheet as well as in target space. In ten spacetime dimensions, the type IIB(A) superstring exhibits extended (non-)chiral $\mathcal{N} = 2$ spacetime supersymmetry with in total 32 supersymmetry charges. It follows that all massless as well as all massive closed string states are organized in supermultiplets of the ten-dimensional $\mathcal{N} = 2$ supersymmetry algebra. This leads to a very subtle interplay between massive string excitations with different higher spins that belong to common supersymmetry multiplets. In fact, the covariant world-sheet vertex operators of the higher spin states must transform into each other when acting on them with the supersymmetry charge operators. Hence, spacetime supersymmetry must be reflected in the structure of the world-sheet BRST cohomology on each mass level of the higher spin excitations.

Going from ten to lower dimensions, parts or all of spacetime supersymmetry can be preserved during the compactification process. As it is known already for several years [27–30], there exists a deep relation between the number of spacetime supersymmetries, preserved by the compactification, and the number of world-sheet supersymmetries of the corresponding internal superconformal field theory. Specifically, for type II compactifications on six-dimensional Calabi-Yau spaces, which correspond to $\hat{c} = 6$ SCFT's with $(2, 2)$ world sheet supersymmetry, one obtains in the closed string sector four-dimensional $\mathcal{N} = 2$ effective supergravity theories with 8 preserved supercharges in the bulk. Second, type II compactification on $K3 \times T^2$ with four-dimensional $\mathcal{N} = 4$ spacetime supersymmetry (16 bulk supercharges) can be described by the direct product of two SCFT's with central charges $\hat{c} = 4$ and $\hat{c} = 2$, where the $\hat{c} = 4$ part possesses $(4, 4)$ supersymmetry on the world-sheet. Finally, compactifications on a six-dimensional torus leads to effective type II supergravity theories with maximal $\mathcal{N} = 8$ supersymmetry (32 bulk supercharges).

However, when also including D-branes and open strings, the number of spacetime supersymmetries is reduced by half compared to the closed string bulk sector, we just discussed above. First, toroidal compactifications of type II superstrings lead to Yang-Mills open string sectors with $\mathcal{N} = 4$ supersymmetry in $D = 4$. Next, the IIB $K3 \times T^2$ orientifolds with D5/D9-branes lead to $\mathcal{N} = 2$ supersymmetric Yang-Mills theories in four dimensions. These theories originate upon compactification on T^2 from $D = 6$, IIB theories on $K3$ with (1,1) spacetime supersymmetry. And finally, the effective, four-dimensional Yang-Mills theories of type IIB, Calabi-Yau orientifolds with D3/D7-branes or with D5/D9-branes (or type IIA Calabi-Yau orientifolds with intersecting D6-branes) possess just $\mathcal{N} = 1$ supersymmetry.

We are going to systematically construct the covariant vertex operators of the lowest massive open string supermultiplets for all three cases of $\mathcal{N} = 4, 2, 1$ spacetime supersymmetry on the corresponding D-branes. We will focus in particular on those massive supermultiplets and their SUSY transformations in the universal sector, which are always present in any four-dimensional orientifold models:

- For $\mathcal{N} = 4$ super Yang-Mills, there is a single massive, spin two supermultiplet with 128 bosonic as well as 128 fermionic degrees of freedom.

- Finally, for $\mathcal{N} = 2$ super Yang-Mills we are dealing with 40 + 40 massive open string states, being organized in one spin two plus two spin one massive supermultiplets.

- The supermultiplets of the universal $\mathcal{N}=1$ sector contains one spin two supermultiplet and two spin 1/2 representations with in total 12 + 12 bosonic and fermion degrees of freedom.

In this way we extend the analysis of [30] about the relation between world-sheet and spacetime supersymmetries and their closed string (massless) supermultiplet structure to the case of the massive, open string supermultiplets. At the same time we are giving here a massive version of the SUSY multiplet analysis in [31], where it was shown that SUSY Ward identities among scattering amplitudes are valid to all orders in α', and where the spinor helicity methods were applied to make efficient use of these Ward identities.

Finally, we go one further step – studying some general properties of arbitrary higher level massive superstring states, as recently, there is also growing interest in the dynamics of higher spin states in string theory [32–42]. We are particularly interested in massive particles that couple to massless gauge bosons according to "(anti)self-dual" selection rules. These particles decay into two gauge bosons with the same (say ++) helicities only and to more gluons in "mostly plus" helicity configurations. We rely on the factorization techniques [43]. They allow identifying not only the spins of Regge resonances propagating in a given channel, but also their couplings and decay rates. We construct the vertex operators for all "universal" bosons of the Neveu-Schwarz (NS) sector in the second massive level. We compute the amplitudes involving one such state and two or three gluons, focusing on the decays of the (anti)self-dual massive (complex) vector fields.

The amplitudes describing decays of heavy states into gauge bosons are also important for the superstring generalization of Britto-Cachazo-Feng-Witten (BCFW) recursion relations [44–47] to disk amplitudes with arbitrary number of external gauge bosons. Recently, it has been argued that the BCFW-deformed full-fledged string amplitudes have no singularities at the infinite value of the deformation parameter, therefore BCFW recursion relations should be valid also in string theory [48–53]. This approach to constructing the scattering amplitudes is however highly impractical because in order to increase the number of external massless particles from N to $N+1$, one needs to compute an infinite number of amplitudes involving one massive state and $N-1$ massless ones, for all mass levels. It may be useful, however, for revealing some general properties of the amplitudes. We show that at least the four-gluon amplitude can be obtained by a BCFW deformation of a factorized sum involving on-shell amplitudes of one massive Regge state and two gauge bosons.

This thesis is organized as follows. In Chapter 2 we introduce the covariant quantization and the techniques of conformal field theory (CFT) as the tool to construct the physical string states. With some review of massive superstring states in ten-dimensions, we then present the physical superstring states in four-dimensions after the compactifications. In Chapter 3 we find all the physical states are connected by the supersymmetry (SUSY) relations, and they form certain supermultiplets. To understand the structure of these supermultiplets, we use the massive helicity formalism, and investigate the interplays between individual polarization states. In Chapter 4, we focus on the universal states which common to all compactifications, compute the scattering amplitudes with the Standard Model (SM) particles, which are zero mode of quantized string. If the fundamental string scale is around a few TeVs, there is a chance of observing the stringy signals at the Large Hadron Collider (LHC). In Chapter 5, we investigate some general properties of higher level massive string states. We also give the first example of the use of BCFW recursion relations in the superstring theory. In the end, we make some final remarks in Chapter 6.

2 Physical superstring states in the first massive level

As the main focus of this thesis, we are going to discuss the physics of massive superstring states in four dimensions, especially, which belong to the first massive level. In this chapter, we will first review some basic knowledge of BRST quantization, and vertex operators of ten-dimensional physical open string states. They comprise in total 128 + 128 bosonic as well as fermionic states. We verify that these states form a massive representation of the ten-dimensional (type I) $\mathcal{N}=1$ SUSY algebra. Next, we consider the SCFT's of string vacua in four dimensions, and discuss the relation between the extended world-sheet superconformal algebras and the spacetime $\mathcal{N}=4,2,1$ SUSY algebras and the covariant vertex operators for the corresponding supercharge operators. Then we construct the physical massive open string states in NS and R sectors for the three cases of $\mathcal{N}=4$, $\mathcal{N}=2$ and $\mathcal{N}=1$ supersymmetry in four dimensions respectively.

This chapter is based on the paper [3].

2.1 The first mass level in ten dimensions

The lightest Regge excitations of open superstring theory in ten-dimensional Minkowski spacetime were firstly constructed in 1987 [54]. Let us briefly review the general method to construct heavy string excitations as well as the explicit results of [54] and then offer a covariant approach to the excited Ramond sector states.

We are going to use the covariant methods (BRST) to construct the physical states. Physical states belong to the cohomology of the BRST operator $Q_{\rm BRST}$. In the world-sheet variables of the RNS formalism, it splits into three pieces of different superghost charge:

$$Q_{\rm BRST} = Q_0 + Q_1 + Q_2 \,, \tag{2.1}$$

$$Q_0 = \oint \frac{\mathrm{d}z}{2\pi i} \left(c\,(T+T_{\beta,\gamma}) + b\,c\,\partial c \right) \,, \tag{2.2}$$

$$Q_1 = -\oint \frac{\mathrm{d}z}{2\pi i}\,\gamma\,G = -\oint \frac{\mathrm{d}z}{2\pi i}\,e^{\phi}\,\eta\,G \,, \tag{2.3}$$

$$Q_2 = -\frac{1}{4}\oint \frac{\mathrm{d}z}{2\pi i}\,b\,\gamma^2 = -\frac{1}{4}\oint \frac{\mathrm{d}z}{2\pi i}\,b\,e^{2\phi}\,\eta\,\partial\eta \,. \tag{2.4}$$

We denote the $c=15$ stress tensor and supercurrent of the matter fields[2] $i\partial X^m, \psi^n$ by T and G, respectively, whereas $T_{\beta,\gamma}$ captures the β,γ superghost system of $c=11$. The latter is partially bosonized in terms of exponentials $e^{q\phi}$ (with ϕ denoting a free chiral boson) and completed by a pair of $h=1,0$ fermions η,ξ. The Grassmann odd ghost system (b,c) is well-known from the bosonic string.

States of uniform superghost charge are BRST closed only if they are annihilated by Q_0, Q_1 and Q_2 separately. Closure under Q_0 forces vertex operators to be a Virasoro primary of unit weight, while Q_2 does not contribute in the ghost pictures considered in this paper. Hence, given a vertex operator ansatz of suitable conformal weight, only the Q_1 constraint involving the supercurrent

$$G(z) = \frac{1}{2\sqrt{2\alpha'}}\, i\partial X_m(z)\, \psi^m(z) \tag{2.6}$$

has to be evaluated separately.

2.1.1 Physical states in the NS sector

The lowest mass $m^2=-k^2=1/\alpha'$ for Regge excitations assigns conformal weight $h=-1$ to the plane wave $e^{ik\cdot X}$ which introduces spacetime momentum into vertex operators. In the NS sector of canonical superghost charge -1, it can combine with the $h=\frac{1}{2}$ field $e^{-\phi}$ and a $h=\frac{3}{2}$ combination of $i\partial X^m, \psi^n$ oscillators to form a Virasoro primary of unit conformal weight in total. (Hence, neglecting the plane wave $e^{ik\cdot X}$ contribution, the massive states at first mass level always correspond to vertex operators with conformal dimension $h=2$.)

The most general $h=1$ ansatz for the first massive NS sector states involves three[3] $h=\frac{3}{2}$ operators $i\partial X^m \psi^n$, $\psi^m\psi^n\psi^p$ and $\partial\psi^m$ along with polarization wavefunctions B_{mn}, E_{mnp}, H_m:

$$V^{(-1)}(B,E,H,k,z) = \Big(B_{mn}\, i\partial X^m\, \psi^n \;+\; E_{mnp}\, \psi^m\, \psi^n\, \psi^p \;+\; H_m\, \partial\psi^m \Big)\, e^{-\phi}\, e^{ik\cdot X}\ . \tag{2.7}$$

The BRST constraints arising from Q_1 admit two physical solutions,[4] namely a (traceless and symmetric)

[2] Our normalization conventions for the world-sheet matter fields are fixed by

$$i\partial X^m(z)\, i\partial X^n(w) \;\sim\; \frac{2\alpha'\, \eta^{mn}}{(z-w)^2} \;+\; \ldots\,, \qquad \psi^m(z)\, \psi^n(w) \;\sim\; \frac{\eta^{mn}}{z-w} \;+\; \ldots\ . \tag{2.5}$$

[3] The addition of $\xi_m \psi^m \partial\phi e^{-\phi}$ is neglected because it can be absorbed into a total derivative.

[4] Throughout this and the next chapter, we set vertex operator normalization factor $g_{\rm A} = \sqrt{2\alpha'} g_{\rm YM}$ from [13, 14] to unity.

spin two tensor B_{mn} and a three-form E_{mnp}:

$$V^{(-1)}(B,k,z) = \frac{1}{\sqrt{2\alpha'}}\, B_{mn}\, i\partial X^m\, \psi^n\, e^{-\phi}\, e^{ik\cdot X} \,, \qquad k^m\, B_{mn} \;=\; B_m{}^m \;=\; B_{[mn]} \;=\; 0 \,, \tag{2.8}$$

$$V^{(-1)}(E,k,z) = \frac{1}{6}\, E_{mnp}\, \psi^m\, \psi^n\, \psi^p\, e^{-\phi}\, e^{ik\cdot X} \,, \qquad k^m\, E_{mnp} = 0 \,. \tag{2.9}$$

Both polarizations are transverse and therefore naturally fall into representations of the stabilizer group $SO(9)$ of massive momenta. The number of degrees of freedom is $\frac{9\cdot 10}{2} - 1 = 44$ for B_{mn} and $\frac{9\cdot 8\cdot 7}{1\cdot 2\cdot 3} = 84$ for E_{mnp}, i.e. we have $44 + 84 = 128$ bosonic states in total.

Some of the solutions to the BRST constraint turn out to be Q_{BRST} exact:

$$\begin{aligned}
\big[\, Q_{\mathrm{BRST}} \,,\, e^{-2\phi}\, \Sigma_{[mn]}\, \psi^m\, \psi^n\, \partial\xi\, e^{ik\cdot X} \,\big] \;&\sim\; \Big(2\, \Sigma_{[mn]}\, i\partial X^m\, \psi^n \;+\; \Sigma_{[mn}\, k_{p]}\, \psi^m\, \psi^n\, \psi^p \Big)\, e^{-\phi}\, e^{ik\cdot X} \,, \\
\big[\, Q_{\mathrm{BRST}} \,,\, e^{-2\phi}\, \pi_m\, i\partial X^m\, \partial\xi\, e^{ik\cdot X} \,\big] \;&\sim\; \Big(\pi_m\, \partial\psi^m \;+\; \pi_m\, k_n\, i\partial X^m\, \psi^n \Big)\, e^{-\phi}\, e^{ik\cdot X} \,, \\
\big[\, Q_{\mathrm{BRST}} \,,\, \partial e^{-2\phi}\, \partial\xi\, e^{ik\cdot X} \,\big] \;&\sim\; \Big(\Big[\frac{\eta_{mn}}{2\alpha'} \;+\; 2\, k_m\, k_n \Big]\, i\partial X^m\, \psi^n \;+\; 3\, k_m\, \partial\psi^m \Big)\, e^{-\phi}\, e^{ik\cdot X} \,.
\end{aligned} \tag{2.10}$$

These spurious states parametrized by a two-form $\Sigma_{[mn]}$, a vector π_m and a scalar of $SO(9)$ (i.e. subject to $k^m \Sigma_{mn} = k^m \pi_m = 0$) decouple from physical states.

2.1.2 Excited spin fields and physical states in the R sector

In the R sector, the canonical superghost vacuum is created by the $h = \frac{3}{8}$ field $e^{-\phi/2}$. Masses $m^2 = 1/\alpha'$ allow for an $h = \frac{13}{8}$ operator to complete fermionic vertex operators for the first mass level. The matter sector of the R ground states corresponds to $h = \frac{5}{8}$ spin fields S_α transforming as left-handed spinors of the Lorentz group [55, 56]. The right-handed chirality is forbidden by GSO projection. The role of S_α to open or close branch cuts for the ψ^m is reflected in the OPE

$$\psi^m(z)\, S_\alpha(w) \;\sim\; \frac{\gamma^m_{\alpha\dot\beta}}{\sqrt{2}\, (z-w)^{1/2}}\, S^{\dot\beta}(w) \;+\; \dots \,. \tag{2.11}$$

The nontrivial three-point interactions between ψ^m and S_α render their covariant correlation functions inaccessible to the Wick theorem, one has to use techniques of [57, 58] instead to compute higher order correlators. Only by breaking $SO(1,9)$ to its $SU(5)$ subgroup, one can relate the ψ^m and S_α to a free field system of chiral bosons $H_{1,2,\ldots,5}$:

$$i\partial H_k(z)\, i\partial H_l(w) \ \sim \ \frac{\delta_{kl}}{(z-w)^2} \ + \ i\partial H_k(w)\, i\partial H_l(w) \ + \ \ldots \tag{2.12}$$

This technique is known as bosonization[5] [56]:

$$\psi^m \ \leftrightarrow \ e^{\pm iH_m}\,, \qquad S_\alpha \ \leftrightarrow \ e^{\pm iH_1/2}\, e^{\pm iH_2/2}\, e^{\pm iH_3/2}\, e^{\pm iH_4/2}\, e^{\pm iH_5/2}\,. \tag{2.13}$$

It is clear from this bosonized representation that the subleading term $\sigma(z-w)^{1/2}$ of the OPE (2.11) involves $e^{\pm 3iH_k/2}$ primary operators, in addition to the derivatives $\partial e^{\pm iH_k/2}$. The covariant description of these new excited primary fields requires an irreducible vector spinor

$$S^{\dot\beta}_m \ \leftrightarrow \ e^{\pm i3H_1/2}\, e^{\pm iH_2/2}\, e^{\pm iH_3/2}\, e^{\pm iH_4/2}\, e^{\pm iH_5/2}\,, \qquad \gamma^m_{\alpha\dot\beta}\, S^{\dot\beta}_m = 0 \tag{2.14}$$

of weight $h=\frac{13}{8}$, where the gamma tracelessness condition subtracts the descendant components $\partial S_\alpha \leftrightarrow \partial(e^{\pm iH_1/2}e^{\pm iH_2/2}e^{\pm iH_3/2}e^{\pm iH_4/2}e^{\pm iH_5/2})$. The introduction of $S^{\dot\beta}_m$ and ∂S_α is the covariant way to disentangle the primary field- and descendant components within the operator $\psi_m \psi^n S_\alpha \gamma_n^{\alpha\dot\beta}$ used in [54]. The completion of the OPE (2.11) to the subleading level reads

$$\psi^m(z)\, S_\alpha(w) \ \sim \ \frac{\gamma^m_{\alpha\dot\beta}\, S^{\dot\beta}(w)}{\sqrt{2}\,(z-w)^{1/2}} \ + \ (z-w)^{1/2}\left[S^m_\alpha(w) \ + \ \frac{2}{\sqrt{2}\,5}\, \gamma^m_{\alpha\dot\beta}\, \partial S^{\dot\beta}(w)\right] \ + \ \ldots \tag{2.15}$$

in $D=10$. A more exhaustive list of OPEs involving ψ^m, S_α and $S^{\dot\beta}_m$ (and their counterparts of opposite $SO(1,9)$ chirality) can be found in appendix B.1.

After the GSO projection, the most general vertex operator for spacetime fermions at the first mass level involves the $h=\frac{13}{8}$ operators $i\partial X^m S_\alpha$, $S^{\dot\beta}_m$ and ∂S_α and therefore two vector spinor wavefunctions $v^\alpha_m, \bar\rho^m_{\dot\beta}$

[5] We should admit that our discussion neglects Jordan-Wigner cocycle factors [56]. These are additional algebraic objects accompanying the exponentials to ensure that $e^{\pm iH_k}$ and $e^{\pm iH_l}$ associated with different bosons $k \neq l$ anticommute. We drop cocycle factors to simplify the notation, it suffices to remember that they are implicitly present and that the bosonized representation of ψ^μ still obeys Fermi statistics. The instance where they contribute a phase is commented on above (3.27).

as well as spinor wavefunction u^α:

$$V^{(-1/2)}(v,\bar\rho,u,k,z) = \Big(v_m^\alpha \, i\partial X^m \, S_\alpha \ + \ \bar\rho_{\dot\beta}^m \, S_m^{\dot\beta} \ + \ u^\alpha \, \partial S_\alpha \Big) \, e^{-\phi/2} \, e^{ik\cdot X} \ . \tag{2.16}$$

Since $\bar\rho$ is contracted with the excited spin field $S_m^{\dot\beta}$, we can regard it as γ traceless, i.e. $\bar\rho_{\dot\beta}^m \bar\gamma_m^{\dot\beta\alpha} = 0$. The independent Q_1 BRST constraints for (2.16) can be summarized as

$$0 = 2\alpha' \, v_m^\alpha \, \not{k}_{\alpha\dot\beta} \ + \ \sqrt{2} \, \bar\rho_{m,\dot\beta} \ + \ \tfrac{1}{2} \, u^\alpha \, \gamma_{m\alpha\dot\beta} \ , \tag{2.17}$$

$$0 = 2\sqrt{2} \, k_\mu \, \bar\rho_{\dot\beta}^\mu \ - \ \tfrac{3}{2} \, u^\alpha \, \not{k}_{\alpha\dot\beta} \ . \tag{2.18}$$

Disentangling the $SO(1,9)$ irreducibles of the former allows to express u^α and $\bar\rho_{\dot\beta}^m$ in terms of v_m^α,

$$\bar\rho_{\dot\beta}^m \ = \ - \sqrt{2} \, \alpha' \left(v^{m\alpha} \, \not{k}_{\alpha\dot\beta} \ + \ \frac{1}{10} \, v_p^\alpha \, (\not{k} \, \bar\gamma^p \, \gamma^m)_{\alpha\dot\beta} \right) \ , \tag{2.19}$$

$$u^\alpha \ = \ \frac{2\alpha'}{5} \, v_m^\beta \, (\not{k} \, \gamma^m)_\beta{}^\alpha \ , \tag{2.20}$$

whereas (2.18) yields an extra constraint on the only independent polarization v_m^α:

$$v_m^\alpha \, \gamma_{\alpha\dot\beta}^m = 2\alpha' \, k^m \, v_m^\alpha \, \not{k}_{\alpha\dot\beta} \ . \tag{2.21}$$

As recognized in [54], there is a physical solution $v_m^\alpha \equiv \chi_m^\alpha$ of spin 3/2

$$V^{(-1/2)}(\chi,k) = \frac{1}{\sqrt{2}\alpha'^{1/4}} \, \Big(\chi_m^\alpha \, i\partial X^m \, S_\alpha \ - \ \sqrt{2} \, \alpha' \, \chi^{m\alpha} \, \not{k}_{\alpha\dot\beta} \, S_m^{\dot\beta} \Big) \, e^{-\phi/2} \, e^{ik\cdot X} \tag{2.22}$$

$$0 = k^m \, \chi_m^\alpha = \chi_m^\alpha \, \gamma_{\alpha\dot\beta}^m$$

and one spurious state associated with the gamma trace choice $v_m^\alpha = k_m \Theta^\alpha + \frac{1}{4} \Theta^\beta (\not{k} \gamma_m)_\beta{}^\alpha$

$$\begin{aligned} \big[Q_{\mathrm{BRST}} \, , \, e^{-3\phi/2} \, \partial\xi \, \Theta^\alpha \, \not{k}_{\alpha\dot\beta} \, S^{\dot\beta} \, e^{ik\cdot X} \big] \ \sim \ & \Big(\big[k_m \, \Theta^\alpha \ + \ \tfrac{1}{4} \, \Theta^\beta \, (\not{k} \, \gamma_m)_\beta{}^\alpha \big] \, i\partial X^m \, S_\alpha \\ & - \frac{1}{\sqrt{2}} \, \big[\alpha' \, k^m \, \Theta^\alpha \, \not{k}_{\alpha\dot\beta} \ + \ \tfrac{1}{10} \, \Theta^\alpha \, \gamma_{\alpha\dot\beta}^m \big] \, S_m^{\dot\beta} \ + \ \frac{6}{5} \, \Theta^\alpha \, \partial S_\alpha \Big) \, e^{-\phi/2} \, e^{ik\cdot X} \end{aligned} \tag{2.23}$$

which allows to gauge away the u^α wavefunction.

2.1.3 Ten-dimensional SUSY transformations

The SUSY charge in open superstring theory is given by the massless gaugino vertex at zero momentum [27]:

$$\mathcal{Q}_\alpha^{(-1/2)} = \frac{1}{\alpha'^{1/4}} \oint \frac{\mathrm{d}z}{2\pi i}\, S_\alpha\, e^{-\phi/2} \ . \tag{2.24}$$

It transforms R sector states in their canonical $-1/2$ superghost picture into canonical NS vertex operators $\sigma e^{-\phi}$. The contour integral is evaluated by performing OPEs between the S_α and $e^{-\phi/2}$ fields from the supercharge at point z and the vertex operator $V^{(-1/2)}(w)$ of the fermion in question. Appendix B.1 gathers the required OPEs for the $D = 10$ case.

The inverse transformation from the NS sector to the R sector requires the $+1/2$ picture representative of the SUSY generator

$$\mathcal{Q}_\alpha^{(+1/2)} = \frac{1}{2\alpha'^{3/4}} \oint \frac{\mathrm{d}z}{2\pi i}\, i\partial X_m\, \gamma^m_{\alpha\dot\beta}\, S^{\dot\beta}\, e^{+\phi/2} \ . \tag{2.25}$$

The latter allows to write down the ghost neutral $\mathcal{N} = 1$ SUSY algebra in ten dimensions,

$$\big\{ \mathcal{Q}_\alpha^{(+1/2)} \ , \ \mathcal{Q}_\beta^{(-1/2)} \big\} = (\gamma^m\, C)_{\alpha\beta}\, P_m \, , \qquad P_m = \frac{1}{2\alpha'} \oint \frac{\mathrm{d}z}{2\pi i}\, i\partial X_m \ . \tag{2.26}$$

Let us list the SUSY variations of the physical $D = 10$ vertex operators. The NS sector states (2.8) and (2.9) have already been discussed in [54]

$$\big[\eta^\alpha\, \mathcal{Q}_\alpha^{(+1/2)} \ , \ V^{(-1)}(B,k) \big] \ = \ V^{(-1/2)} \Big(\chi^\alpha_m \ = \ \tfrac{1}{\sqrt{2}}\, B_{mn}\, (\eta \ \not{k}\, \gamma^n)^\alpha \Big) \ , \tag{2.27}$$

$$\begin{aligned}\big[\eta^\alpha\, \mathcal{Q}_\alpha^{(+1/2)} \ , \ V^{(-1)}(E,k) \big] \ = \ V^{(-1/2)} \Big(\chi^\alpha_m \ = \ &\tfrac{1}{12\,\sqrt{\alpha'}} \big[E_{mnp}\, (\eta\, \gamma^{np})^\alpha \ - \ \tfrac{1}{3}\, E_{npq}\, (\eta\, \gamma_m\, \gamma^{npq})^\alpha \\ &- \ \tfrac{\alpha'}{3}\, k_m\, E_{npq}\, (\eta \ \not{k}\, \gamma^{npq})^\alpha \big] \Big) \ .\end{aligned} \tag{2.28}$$

In addition, we use the covariant OPEs from appendix B.1 to compute the SUSY variation of the massive gravitino (2.22):

$$\left[\eta^\alpha\, \mathcal{Q}_\alpha^{(-1/2)}\,,\, V^{(-1/2)}(\chi,k)\right] \;=\; V^{(-1)}\left(B_{mn} \,=\, \tfrac{\alpha'}{\sqrt{2}}\,(\eta\, \not{k}\,\chi_{(m)}\,k_{n)} \,+\, \tfrac{1}{\sqrt{2}}\,(\eta\,\gamma_{(m}\,\chi_{n)})\right)$$
$$+\; V^{(-1)}\left(E_{mnp} \,=\, 3\alpha'^{1/2}\,(\eta\,\gamma_{[m}\,\chi_n)\,k_{p]} \,-\, \tfrac{3}{2}\,\alpha'^{1/2}\,(\eta\,\gamma_{[np}\,\not{k}\,\chi_{m]})\right)\;. \tag{2.29}$$

2.2 CFTs of supersymmetric string vacua in four dimensions

In this section we will first review some basic facts about extended supersymmetry algebras in four spacetime dimensions and about the general relation between extended spacetime supersymmetries and world-sheet supersymmetries. In part, we are following the work in references [28–30]. Our conventions for indices w.r.t. Lorentz symmetry $SO(1,3)$ and R-symmetries $SO(6)$ or $SU(2)$ are gathered in appendix A.

2.2.1 The four-dimensional spacetime supersymmetry algebra

The $\mathcal{N}$ supercharges $\mathcal{Q}_a^I$ as well as the complex conjugate operators $\bar{\mathcal{Q}}_{\bar{I}}^{\dot{a}}$ satisfy the $\mathcal{N}$-extended supersymmetry algebra $(I,\bar{I}=1\ldots,\mathcal{N})$

$$\begin{aligned}\{\mathcal{Q}_a^I\,,\,\bar{\mathcal{Q}}_{\bar{J}}^{\dot{b}}\} &= C_{\bar{J}}^I\,(\sigma^\mu\,\varepsilon)_a{}^{\dot{b}}\,P_\mu\,,\\ \{\mathcal{Q}_a^I\,,\,\mathcal{Q}_b^J\} &= \varepsilon_{ab}\,\mathcal{Z}^{IJ}\,.\end{aligned} \tag{2.30}$$

P^μ is the momentum operator and the $\mathcal{Z}^{IJ}$ are central charges, which are antisymmetric in I,J and can therefore appear in the $\mathcal{N}\geq 2$ supersymmetry algebra only.

Next let us discuss the representations of the extended supersymmetry algebras, namely how the supercharges in general act on massless and on massive states. Let us first recall the case of massless states. Here we can choose a frame where the momenta are $k^\mu=(E,0,0,E)$, the supercharges are

$$\mathcal{Q}_1^I \;\equiv\; \mathcal{Q}^I\,,\qquad \bar{\mathcal{Q}}_{\bar{I}}^{\dot{1}} \;\equiv\; \bar{\mathcal{Q}}_I\,,\qquad \text{whereas}\quad 0=\mathcal{Q}_1^2=\bar{\mathcal{Q}}_{\bar{I}}^{\dot{2}}\,. \tag{2.31}$$

In terms of $\mathcal{Q}^I$ and $\bar{\mathcal{Q}}_I$ the supersymmetry algebra takes the form

$$\begin{aligned}\left\{\mathcal{Q}^I\,,\,\bar{\mathcal{Q}}_J\right\} &= \delta^I_J\,,\\ \left\{\mathcal{Q}^I\,,\,\mathcal{Q}^J\right\} &= \left\{\bar{\mathcal{Q}}_I\,,\,\bar{\mathcal{Q}}_J\right\} \;=\; 0\,,\end{aligned}\tag{2.32}$$

where we have rescaled the supersymmetry charges by $\sqrt{E}$. The $2\mathcal{N}$ supercharges $\mathcal{Q}^I$ and $\bar{\mathcal{Q}}_I$ build an $SO(2\mathcal{N})$ Clifford algebra

$$\begin{aligned}\Gamma_{2I-1} &= \mathcal{Q}^I + \bar{\mathcal{Q}}_{\bar{I}}\,, \quad \Gamma_{2I} \;=\; i(\mathcal{Q}^I - \bar{\mathcal{Q}}_{\bar{I}})\,,\\ \{\Gamma_i, \Gamma_j\} &= 2\delta_{ij}\,, \quad i,j = 1,\ldots,2\mathcal{N}\end{aligned}\tag{2.33}$$

whose representations have dimension $2^{\mathcal{N}}$. The generators for $SO(2\mathcal{N})$ rotations are

$$\Lambda_{ij} \;=\; \frac{1}{4i}\,[\Gamma_i\,,\,\Gamma_j]\;.\tag{2.34}$$

This group contains a $SU(\mathcal{N})\times U(1)$ subgroup specified by the following generators

$$\begin{aligned}\Lambda^I_J &= \frac{1}{2}\,[\mathcal{Q}^I\,,\,\bar{\mathcal{Q}}_{\bar{J}}] \;-\; \frac{1}{2\mathcal{N}}\,\delta^I_J\,[\mathcal{Q}^K\,,\,\bar{\mathcal{Q}}_{\bar{K}}] \;\text{ for }\; SU(\mathcal{N})\;,\\ \Lambda &= \frac{1}{4}\,[\mathcal{Q}^I\,,\,\bar{\mathcal{Q}}_I] \;\text{ for }\; U(1)\;.\end{aligned}\tag{2.35}$$

For massless states, this $SU(\mathcal{N})$ commutes with the $SO(2)$ helicity group. Hence this group classifies massless states. The eigenvalue of the supercharge under the $U(1)$, which is called intrinsic helicity, is the same as under spacetime helicity. Therefore one can define a new generator Λ' through a shift by the z component j^3 of the spin, called superhelicity,

$$\Lambda' = j^3 \;-\; \Lambda\,,\tag{2.36}$$

which commutes with $\mathcal{Q}^I$.

Next let us consider massive states rotated into their rest frame $k^\mu = (m,0,0,0)$. Now also the second helicity components of the supercharge spinors become active, i.e. give rise to nonzero supersymmetry transformations on massive states. We will denote them as follows:

$$\mathcal{Q}^I_2 \;\equiv\; \tilde{\mathcal{Q}}^I\,, \quad \bar{\mathcal{Q}}^{\dot 2}_{\bar I} \;\equiv\; \bar{\tilde{\mathcal{Q}}}_{\bar I}\,. \tag{2.37}$$

The supersymmetry algebra between the $\tilde{\mathcal{Q}}$ looks like

$$\left\{\tilde{\mathcal{Q}}^I\,,\,\bar{\tilde{\mathcal{Q}}}_{\bar J}\right\} = m\, C^I_{\bar J}\,, \quad \left\{\tilde{\mathcal{Q}}^I\,,\,\tilde{\mathcal{Q}}^J\right\} = \left\{\bar{\tilde{\mathcal{Q}}}_I\,,\,\bar{\tilde{\mathcal{Q}}}_J\right\} = 0\;. \tag{2.38}$$

Now the $(\mathcal{Q}^I, \bar{Q}_{\bar I})$ and $(\tilde{\mathcal{Q}}^I, \bar{\tilde{\mathcal{Q}}}_{\bar I})$ build an $SO(4\mathcal{N})$ Clifford algebra on the states without central charges. Consequently, the dimension of massive representations is a multiple of $2^{2\mathcal{N}}$. The maximal subalgebra that commutes with the $SO(3)$ little group of the massive states is $USp(2\mathcal{N})$. Therefore massive states without central charges build representations of $USp(2\mathcal{N})$. As for the massless states one can consider an $SU(\mathcal{N}) \times U(1)$ subgroup with generators $\Lambda_{tot} = \Lambda + \tilde{\Lambda}$ where the $\tilde{\Lambda}$ are defined from the $\tilde{\mathcal{Q}}$ as in (2.35). In section 3.2 we will introduce an organization scheme for massive SUSY representations based on spinor helicity methods which keeps track of the spin quantum numbers along a reference axis of choice.

However, in the presence of central charges $\mathcal{Z}^{IJ}$, the operators $\mathcal{Q}^I$ and $\tilde{\mathcal{Q}}^I$ generate a smaller $SO(2\mathcal{N})$ Clifford algebra, whose maximal subalgebra is $SO(3) \times Sp(\mathcal{N})$. Therefore states with central charges only build representations of $Sp(\mathcal{N})$.

2.2.2 CFT realization of extended $D = 4$ SUSY

As it is well known, there is a beautiful relation between the $\mathcal{N}$-extended spacetime supersymmetry algebras and the n-extended internal superconformal algebras with corresponding Kac-Moody symmetry g. We will assume in the following that we are dealing with holomorphic spacetime supercharges that all originate from the right-moving sector of the compactified string theory, as it is always the case for heterotic string compactifications. As we will discuss, for purely holomorphic supercharges, the massive BPS states with non-vanishing central charges are of perturbative nature. However in type II compactifications, the supercharges can originate from the left-moving as well as the right-moving sector of the string theory. In this case, some of the massive BPS states with central charges are non-perturbative, as they are given in terms of wrapped type II D-branes. These non-perturbative states will not be discussed in this paper.

In SCFT, the holomorphic supercharges $\mathcal{Q}^I$ and $\bar{\mathcal{Q}}_{\bar{I}}$ can be always realized by the world-sheet fields of the uncompactified four-dimensional Minkowski spacetime together with those of the internal Kac-Moody symmetries. This fact allows for a completely model-independent realization of the spacetime supersymmetry algebra without any reference to "geometrical" details of the internal SCFT. To be more specific, compactifications to four-dimensional Minkowski spacetime which allow for a CFT description, still have $SO(1,3)$ vectors $i\partial X^\mu$ and ψ^μ in their world-sheet theory, the first four components of the ten-dimensional ancestors $i\partial X^m$ and ψ^m. Similarly, the ten-dimensional $SO(1,9)$ spin field S_α factorizes into separate $h = \frac{1}{4}$ and $h = \frac{3}{8}$ primaries S_a and Σ, the former being a Weyl spinor of $SO(1,3)$ and the latter falling into representations of the R-symmetry. In fact, both $SO(1,3)$ chiralities can occur, i.e.

$$S_\alpha \quad \equiv \quad S_a \, \Sigma^I \ \oplus \ S^{\dot{b}} \, \bar{\Sigma}_I \ . \tag{2.39}$$

The number of $(\Sigma^I, \bar{\Sigma}_I)$ species coincides with the number of spacetime supersymmetries, we will discuss the $\mathcal{N} = 4, 1, 2$ cases below. In each case, the (left- and right-handed) supercharges in their canonical ghost picture are given by

$$\mathcal{Q}_a^{(-1/2)I} = \frac{1}{\alpha'^{1/4}} \oint \frac{\mathrm{d}z}{2\pi i} \, S_a \, \Sigma^I \, e^{-\phi/2} \, , \qquad \bar{\mathcal{Q}}_{\bar{J}}^{(-1/2),\dot{b}} = \frac{1}{\alpha'^{1/4}} \oint \frac{\mathrm{d}z}{2\pi i} \, S^{\dot{b}} \, \bar{\Sigma}_{\bar{J}} \, e^{-\phi/2} \ . \tag{2.40}$$

Independent on the fate of the internal spin fields $\Sigma^I, \bar{\Sigma}_I$, the interactions of the $h = \frac{1}{4}$ spacetime spin fields $S_a, S^{\dot{b}}$ with the NS fermions is governed by

$$\psi^\mu(z) \, S_a(w) \quad \sim \quad \frac{\sigma^\mu_{a\dot{b}} \, S^{\dot{b}}(w)}{\sqrt{2} \, (z-w)^{1/2}} \ + \ (z-w)^{1/2} \left[S^\mu_a(w) \ + \ \frac{1}{\sqrt{2}} \, \sigma^\mu_{a\dot{b}} \, \partial S^{\dot{b}}(w) \right] \ + \ \dots \ . \tag{2.41}$$

In lines with the discussion of subsection 2.1.2, one can bosonize the left- and right-handed spin fields as $e^{\pm i(H_1+H_2)/2}$ and $e^{\pm i(H_1-H_2)/2}$, respectively. In order to reconcile bosonization techniques with $SO(1,3)$ symmetry, we align $e^{\pm 3iH_j/2}$ components showing up in the subleading term of the OPE (2.41) into covariant excited spin fields $S^{\dot b}_\mu, S^\mu_a$ of weight $h = \frac{5}{4}$:

$$S^\mu_a \quad \leftrightarrow \quad e^{\pm 3iH_1/2}\, e^{\pm iH_2/2}\,, \qquad S^{\dot b}_\mu \quad \leftrightarrow \quad e^{\pm 3iH_1/2}\, e^{\mp iH_2/2}\,, \qquad \bar\sigma_\mu^{\dot b a}\, S^\mu_a = \sigma^\mu_{\ a\dot b}\, S^{\dot b}_\mu = 0\ . \tag{2.42}$$

A large list of OPEs between $(\psi^\mu, S_a, S^{\dot b}, S^{\dot b}_\mu, S^\mu_a)$ including subleading singularities can be found in appendix B.2.

2.2.3 CFT operators in $\mathcal{N} = 4$ compactifications

The internal SCFT in maximally supersymmetric $\mathcal{N} = 4$ compactifications to $D = 4$ dimensions can be understood in terms of free fields $i\partial Z^m, \Psi^m$ with $m = 4, 5, \ldots, 9$ which represent the internal components of the ten-dimensional $i\partial X^{m=0,1,\ldots,9}, \psi^{m=0,1,\ldots,9}$ and transform as vectors of the internal rotation group $SO(6)$. The corresponding $h = \frac{3}{8}$ spin fields Σ^I and $\bar\Sigma_{\bar J}$, responsible for branch cuts of Ψ^m, transform as spinors of the $SO(6) \equiv SU(4)$ with left-handed (right-handed) index I ($\bar J$). They enter the dimensional reduction $SO(1,9) \to SO(1,3) \times SO(6)$ of the $D = 10$ SUSY charges

$$\mathcal{Q}^{(-1/2)I}_a = \frac{1}{\alpha'^{1/4}} \oint \frac{\mathrm{d}z}{2\pi i}\, S_a\, \Sigma^I\, e^{-\phi/2}\,, \qquad \bar{\mathcal{Q}}^{(-1/2),\dot b}_{\bar J} = \frac{1}{\alpha'^{1/4}} \oint \frac{\mathrm{d}z}{2\pi i}\, S^{\dot b}\, \bar\Sigma_{\bar J}\, e^{-\phi/2} \tag{2.43}$$

where the internal $SO(6) \equiv SU(4)$ is interpreted as the R-symmetry group. The ten-dimensional bosonization prescription can be straightforwardly applied to $\Psi^m, \Sigma^I, \bar\Sigma_{\bar J}$ (e.g. $\Sigma^I \leftrightarrow e^{\pm i(H_3+H_4+H_5)/2}$), and excited spin fields Σ^I_m and $\bar\Sigma^m_{\bar J}$ of weight $h = \frac{11}{8}$ are constructed in close analogy to their ten- and four-dimensional counterparts (2.14) and (2.42):

$$\Sigma^I_m \quad \leftrightarrow \quad e^{\pm 3iH_3/2}\, e^{\pm iH_4/2}\, e^{\pm iH_5/2}\,, \qquad \gamma^m_{\bar J I}\, \Sigma^I_m = \bar\gamma_m^{I\bar J}\, \bar\Sigma^m_{\bar J} = 0\ . \tag{2.44}$$

The internal supercurrent is built from the $m = 4, 5, \ldots, 9$ components of its ten-dimensional ancestor (2.6)

$$G_{\mathrm{int}} = \frac{1}{2\sqrt{2\alpha'}}\, i\partial Z_m\, \Psi^m \tag{2.45}$$

and gives rise to internal central charge[6] $c = 9$. OPEs among the $\Psi^k, \Sigma^I, \bar{\Sigma}_{\bar{J}}$ and $\Sigma^I_m, \bar{\Sigma}^m_{\bar{J}}$ are gathered in appendix B.3. Identities between six-dimensional gamma and charge conjugation matrices can for instance be found in the appendix of [58]. The following Fig. 1 aims to give an overview of the conformal fields in the spacetime and $\mathcal{N} = 4$ internal CFTs[7]

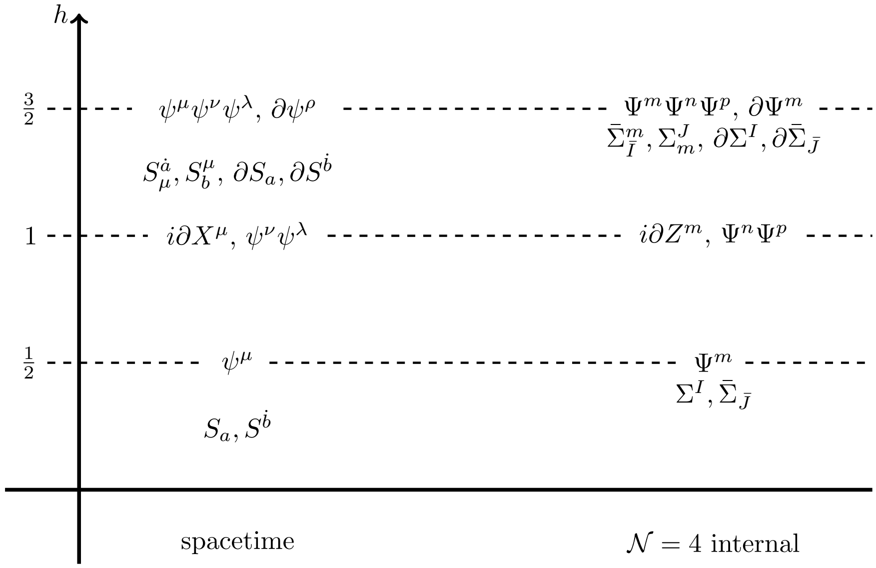

Figure 1: Conformal fields in the spacetime CFT and the internal CFT of $\mathcal{N} = 4$ supersymmetric compactifications

The higher ghost picture version of the SUSY generators (2.43) is given by

$$\mathcal{Q}^{(+1/2),I}_a = \frac{1}{2\alpha'^{3/4}} \oint \frac{\mathrm{d}z}{2\pi i} \left[i\partial X_\mu \, \sigma^\mu_{a\dot{b}} \, S^{\dot{b}} \, \Sigma^I \ + \ S_a \, i\partial Z^m \, \gamma_m^{I\bar{J}} \, \bar{\Sigma}_{\bar{J}} \right] e^{+\phi/2} \ , \tag{2.47}$$

$$\bar{\mathcal{Q}}^{(+1/2),\dot{b}}_{\bar{J}} = \frac{1}{2\alpha'^{3/4}} \oint \frac{\mathrm{d}z}{2\pi i} \left[i\partial X^\mu \, \sigma_\mu^{\dot{b}a} \, S_a \, \bar{\Sigma}_{\bar{J}} \ + \ S^{\dot{b}} \, i\partial Z_m \, \bar{\gamma}^m_{\bar{J}I} \, \Sigma^I \right] e^{+\phi/2} \ , \tag{2.48}$$

their anticommutator with the $(-1/2)$ picture analogues (2.43) yields the following ghost-neutral SUSY

[6] The underlying OPEs are

$$i\partial Z_m(z) \, i\partial Z_n(w) \ \sim \ \frac{2\alpha' \, \delta^{(6)}_{mn}}{(z-w)^2} \ + \ \dots \, , \qquad \Psi_m(z) \, \Psi_n(w) \ \sim \ \frac{\delta^{(6)}_{mn}}{z-w} \ + \ \dots \, . \tag{2.46}$$

[7] The fermionic bilinear states $\psi^\nu\psi^\lambda$ and $\Psi^n\Psi^p$ at weight $h = 1$ by themselves should be eliminated by the GSO projection, but trilinear combinations $\Psi^m\psi^\nu\psi^\lambda$ and $\psi^\mu\Psi^n\Psi^p$ which mix between spacetime components and internal fields would survive after the GSO projection. That is why we include the bilinears into the bookkeeping.

algebra with nontrivial central charges $\mathcal{Z}^{IJ}$ and $\bar{\mathcal{Z}}_{\bar{I}\bar{J}}$:

$$\{\, \mathcal{Q}_a^{(+1/2),I} \,,\; \bar{\mathcal{Q}}_{\bar{J}}^{(-1/2),\dot{b}} \,\} = C^I{}_{\bar{J}}\, (\sigma^\mu\, \varepsilon)_a{}^{\dot{b}}\, P_\mu\,, \qquad P_\mu = \frac{1}{2\alpha'} \oint \frac{\mathrm{d}z}{2\pi i}\; i\partial X_\mu \;, \tag{2.49}$$

$$\{\, \mathcal{Q}_a^{(+1/2),I} \,,\; \mathcal{Q}_b^{(-1/2),J} \,\} = \varepsilon_{ab}\, \mathcal{Z}^{IJ}\,, \qquad \mathcal{Z}^{IJ} = \frac{1}{2\alpha'} \oint \frac{\mathrm{d}z}{2\pi i}\; i\partial Z^m\, (\gamma_m\, C)^{IJ} \;, \tag{2.50}$$

$$\{\, \bar{\mathcal{Q}}_{\bar{I}}^{(+1/2),\dot{a}} \,,\; \bar{\mathcal{Q}}_{\bar{J}}^{(-1/2),\dot{b}} \,\} = \varepsilon^{\dot{a}\dot{b}}\, \bar{\mathcal{Z}}_{\bar{I}\bar{J}}\,, \qquad \bar{\mathcal{Z}}_{\bar{I}\bar{J}} = \frac{1}{2\alpha'} \oint \frac{\mathrm{d}z}{2\pi i}\; i\partial Z_m\, (\bar{\gamma}^m\, C)_{\bar{I}\bar{J}} \;. \tag{2.51}$$

The central charges arise due to poles in the operator product expansion of $\mathcal{Q}_a^{(+1/2),I}$ and $\mathcal{Q}_b^{(-1/2),J}$ caused by internal free fermions and bosons Ψ^m and ∂Z_m. The latter appear in the internal supercurrent $G_{\rm int} \sigma i\partial Z_m \Psi^m$ and generate an internal Kac-Moody algebra

$$g = SO(6) \times [U(1)]^6 \tag{2.52}$$

with dimension one currents

$$j^{mn}_{SO(6)}(z) = \Psi^m\, \Psi^n(z)\,, \qquad j^m_{U(1)^6}(z) = i\partial Z^m(z)\,. \tag{2.53}$$

The fields $Z_m(z)$ can be viewed as the coordinates of a (holomorphic) torus compactification on a six-dimensional torus T^6. Their world-sheet superpartners Ψ^m generate a $U(1)^6$ spacetime gauge symmetry, and the six spacetime gauge bosons are the six graviphotons, which arise in any compactification on a (holomorphic) six-torus. States that carry non-vanishing internal momenta p^m on the (holomorphic) six-torus always have the following field as part of their vertex operator:

$$|p^m\rangle \quad \sim \quad e^{ip^m Z_m(z)}\,. \tag{2.54}$$

Switching to the more convenient bispinor basis, the six central charge operators (in the zero ghost picture) of the $\mathcal{N}=4$ supersymmetry algebra are nothing else than the free bosons Z^m:

$$\mathcal{Z}^{IJ}(z) = \frac{1}{2\alpha'}\; (\gamma_m\, C)^{IJ}\; i\partial Z^m(z) \;. \tag{2.55}$$

It follows that the internal momentum states $|p^m\rangle$ are precisely those states that carry non-vanishing $\mathcal{N}=4$ central charges. They break the internal world-sheet $SO(6)$ symmetry to $SO(5)$. At the same time, states with non-vanishing momenta p^m build representations of the spacetime automorphism group for massive states with central charges, which is $Sp(4) \cong SO(5)$. On the other hand, states with vanishing internal momenta, $|p^m = 0\rangle$, build internal $SO(6)$ representations, respectively at the same time representations of the group $USp(8)$, which classifies massive states without central charges. The subsequent discussions only take into account the states at zero internal momentum ($p^m = 0$).

2.2.4 CFT operators in $\mathcal{N}=2$ compactifications

In superstring compactifications which preserve $\mathcal{N}=2$ spacetime SUSY, it can be shown along the lines of [29, 30] that the internal CFT splits into two decoupled sectors with central charges $c=6$ and $c=3$, respectively. Starting point are the two supercharges

$$\mathcal{Q}_a^{(-1/2),i} = \frac{1}{\alpha'^{1/4}} \oint \frac{\mathrm{d}z}{2\pi i}\, S_a\, \Sigma^i\, e^{-\phi/2}\,, \qquad \bar{\mathcal{Q}}^{(-1/2),\dot{b}i} = \frac{1}{\alpha'^{1/4}} \oint \frac{\mathrm{d}z}{2\pi i}\, S^{\dot{b}}\, \bar{\Sigma}^i\, e^{-\phi/2}\,, \tag{2.56}$$

containing two species of spin fields $\Sigma^{i=1,2}$ and $\bar{\Sigma}^{i=1,2}$. The latter turn out to factorize into decoupled primaries λ^i and $e^{\pm iH/2}$ from the $c=6$ and $c=3$ sector, respectively:

$$\Sigma^i = \lambda^i\, e^{+iH/2}\,, \qquad \bar{\Sigma}^i = \lambda^i\, e^{-iH/2}\,. \tag{2.57}$$

The $c=3$ part can be represented in terms of a single free chiral boson H subject to (2.12). Its contribution $\frac{1}{2}(i\partial H)^2$ to the $c=3$ energy momentum tensor assigns conformal weight $h(e^{\pm iH/2}) = 1/8$ (or more generally, $h(e^{iqH}) = q^2/2$). Moreover, OPEs of the partial spin fields $e^{\pm iH/2}$ introduce $h=\frac{1}{2}$ fermions $e^{\pm iH}$ and excited spin fields $e^{\pm 3iH/2}$ of weight $h=\frac{9}{8}$.

On the other hand, the λ^i fields from the $c=6$ sector have weight $h(\lambda^i) = 1/4$ and form an $SU(2)$

doublet. Their operator algebra[8] gives rise to an $SU(2)$ triplet of $h=1$ currents $\mathcal{J}^{A=1,2,3}$:

$$\lambda^i(z)\,\lambda^j(w) \quad\sim\quad \frac{\varepsilon^{ij}}{(z-w)^{1/2}} \;+\; \frac{1}{\sqrt{2}}\,(z-w)^{1/2}\,(\tau_A\,\varepsilon)^{ij}\,\mathcal{J}^A(w) \;+\; \ldots \tag{2.59}$$

The τ_A denote the standard (traceless) $SU(2)$ Pauli matrices $\left\{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}0&-i\\i&0\end{smallmatrix}\right),\ \left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)\right\}$ subject to the multiplication rule $\tau_A\tau_B = \delta_{AB} + i\varepsilon_{ABC}\tau^C$.

The currents obey the $SU(2)$ current algebra at level $k=1$, we use normalization conventions

$$\mathcal{J}^A(z)\,\mathcal{J}^B(w) \quad\sim\quad \frac{\delta^{AB}}{(z-w)^2} \;+\; \frac{i\,\sqrt{2}\,\varepsilon^{ABC}\,\mathcal{J}_C(w)}{z-w} \;+\; \ldots \tag{2.60}$$

in which their interaction with the spin fields is governed by

$$\mathcal{J}^A(z)\,\lambda^i(w) \quad\sim\quad \frac{(\tau^A)^i{}_j\,\lambda^j(w)}{\sqrt{2}\,(z-w)} \;+\; \sqrt{2}\,(\tau^A)^i{}_j\,\partial\lambda^j(w) \;+\; \ldots \tag{2.61}$$

$$\lambda^i(z)\,\mathcal{J}^A(w) \quad\sim\quad \frac{(\tau^A)^i{}_j\,\lambda^j(w)}{\sqrt{2}\,(z-w)} \;-\; \frac{1}{\sqrt{2}}\,(\tau^A)^i{}_j\,\partial\lambda^j(w) \;+\; \ldots\ . \tag{2.62}$$

Note that also the λ^i and $\mathcal{J}^A$ fit into a bosonization scheme according to

$$\mathcal{J}^{A=3} \;\equiv\; i\partial H_3\,, \qquad \mathcal{J}^{A=1} \;\pm\; i\mathcal{J}^{A=2} \;\equiv\; \sqrt{2}\,e^{\pm i\sqrt{2}H_3}\,, \qquad \lambda^{i=1,2} = e^{\pm iH_3/\sqrt{2}} \tag{2.63}$$

with H_3 being nonsingular with respect to the $c=3$ boson H. This fixes the choice of the $SU(2)$ Cartan subalgebra.

The world-sheet supercurrents associated with the two decoupled CFTs,

$$G_{\rm int} \;\equiv\; G_{c=3} \;+\; G_{c=6}\ , \tag{2.64}$$

can be split according to their charges under the $h=1$ currents. In the $c=3$ sector, we find a free field

[8] The contraction rules for the antisymmetric $\varepsilon^{ij},\varepsilon_{ij}$ tensors introduce signs in some of the OPEs:

$$\lambda_i(z)\,\lambda^j(w) \;\sim\; \frac{+\,\delta_i^j}{(z-w)^{1/2}}\ , \quad \lambda_i(z)\,\lambda_j(w) \;\sim\; \frac{-\,\varepsilon_{ij}}{(z-w)^{1/2}}\ , \quad \lambda^i(z)\,\lambda_j(w) \;\sim\; \frac{-\,\delta^i_j}{(z-w)^{1/2}}\ . \tag{2.58}$$

representation in terms of internal $h = 1$ coordinates[9] $i\partial Z^{\pm}$,

$$G_{c=3} = \frac{1}{2\sqrt{2\alpha'}} \left(i\partial Z^{+} e^{-iH} + i\partial Z^{-} e^{iH} \right) , \tag{2.66}$$

The fermions $\Psi^{\pm}(z) = e^{\pm iH(z)}$ together with the free bosons $Z^{\pm}$ generate an internal Kac-Moody algebra

$$g = SO(2) \times [U(1)]^2 \tag{2.67}$$

with dimension one currents

$$j_{SO(2)}(z) = \Psi^{+} \Psi^{-}(z) = i\partial H(z) , \qquad j^{\pm}_{U(1)}(z) = i\partial Z^{\pm}(z) . \tag{2.68}$$

As for the $\mathcal{N} = 4$ case, the fields $Z_{\pm}(z)$ can be viewed as the coordinates of a (holomorphic) torus compactification on a two-dimensional torus T^2.

Also the supercurrent of the $c = 6$ sector cannot be fully built from the bosonization prescription (2.63), it additionally requires the introduction of an $SU(2)$ doublet of $h = 5/4$ fields g_i:

$$G_{c=6} = \frac{1}{\sqrt{2}} \left(e^{iH_3/\sqrt{2}} g_1 + e^{-iH_3/\sqrt{2}} g_2 \right) = \frac{1}{\sqrt{2}} \lambda^i g_i . \tag{2.69}$$

The g_i decouple from the λ^i and $\mathcal{J}^A$, and their OPE[10]

$$g_i(z)\, g_j(w) \sim \frac{\varepsilon_{ij}}{(z-w)^{5/2}} + \frac{0}{(z-w)^{3/2}} + \dots \tag{2.71}$$

makes sure that the supercurrents satisfy the required $\mathcal{N} = 4$ superconformal algebra at $c = 6$. A summary of operators in the internal SCFTs common to $\mathcal{N} = 2$ compactifications are presented in Fig. 2.

[9] As usual, the OPEs between $i\partial Z^{\pm}$ are normalized as

$$i\partial Z^{\pm}(z)\, i\partial Z^{\mp}(w) \sim \frac{2\alpha'}{(z-w)^2} + \dots , \qquad i\partial Z^{\pm}(z)\, i\partial Z^{\pm}(w) \sim i\partial Z^{\pm}(w)\, i\partial Z^{\pm}(w) + \dots . \tag{2.65}$$

[10] ε contractions yield signs opposite to the $\lambda^i \lambda_j$ case:

$$g^i(z)\, g_j(w) \sim \frac{+\,\delta^i_j}{(z-w)^{5/2}} , \quad g^i(z)\, g^j(w) \sim \frac{-\,\varepsilon^{ij}}{(z-w)^{5/2}} , \quad g_i(z)\, g^j(w) \sim \frac{-\,\delta^j_i}{(z-w)^{5/2}} . \tag{2.70}$$

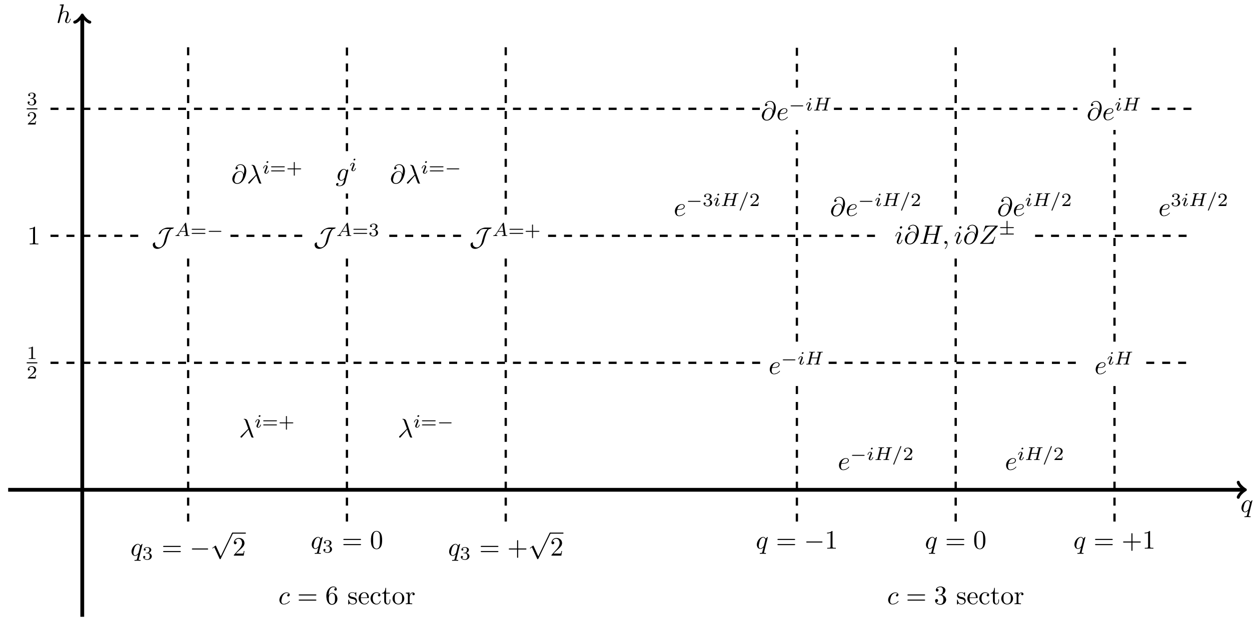

Figure 2: Universal operator content of the internal CFT associated with $\mathcal{N} = 2$ spacetime SUSY, including weight h and charges q_3, q under $i\partial H_3$ and $i\partial H$, respectively.

The internal supercurrent yields the following higher ghost picture SUSY charges:

$$\mathcal{Q}_a^{(+1/2),i} = \frac{1}{\sqrt{2}\alpha'^{3/4}} \oint \frac{\mathrm{d}z}{2\pi i} \left[\frac{1}{\sqrt{2}} \, i\partial X_\mu \, \sigma^\mu_{a\dot{b}} \, S^{\dot{b}} \, \lambda^i \, e^{iH/2} \; + \; i\partial Z^+ \, S_a \, \lambda^i \, e^{-iH/2} \right.$$
$$\left. - \; 2\sqrt{\alpha'} \; g^i \, S_a \, e^{iH/2} \right] e^{\phi/2} \; , \tag{2.72}$$

$$\bar{\mathcal{Q}}^{(+1/2),\dot{b}i} = \frac{1}{\sqrt{2}\alpha'^{3/4}} \oint \frac{\mathrm{d}z}{2\pi i} \left[\frac{1}{\sqrt{2}} \, i\partial X^\mu \, \bar{\sigma}_\mu^{\dot{b}a} \, S_a \, \lambda^i \, e^{-iH/2} \; + \; i\partial Z^- \, S^{\dot{b}} \, \lambda^i \, e^{iH/2} \right.$$
$$\left. - \; 2\sqrt{\alpha'} \; g^i \, S^{\dot{b}} \, e^{-iH/2} \right] e^{\phi/2} \; . \tag{2.73}$$

The anticommutator of equal chirality generators gives rise to a complex central charge operator, which can be written in terms of the free bosons $Z^{\pm}$:

$$\left\{ \, \mathcal{Q}_a^{(+1/2),i} \, , \; \mathcal{Q}_b^{(-1/2),j} \, \right\} = \varepsilon_{ab} \, \mathcal{Z}^{ij} \, , \qquad \mathcal{Z}^{ij} = \frac{\varepsilon^{ij}}{\sqrt{2}\,\alpha'} \oint \frac{\mathrm{d}z}{2\pi i} \, i\partial Z^+ \; , \tag{2.74}$$

$$\left\{ \, \bar{\mathcal{Q}}^{(+1/2),\dot{a}i} \, , \; \bar{\mathcal{Q}}^{(-1/2),\dot{b}j} \, \right\} = \varepsilon^{\dot{a}\dot{b}} \, \bar{\mathcal{Z}}^{ij} \, , \qquad \bar{\mathcal{Z}}^{ij} = \frac{\varepsilon^{ij}}{\sqrt{2}\,\alpha'} \oint \frac{\mathrm{d}z}{2\pi i} \, i\partial Z^- \; . \tag{2.75}$$

It again follows that the internal momentum states $|p^{\pm}\rangle$ of the two-torus are precisely those states that carry non-vanishing $\mathcal{N} = 2$ central charges. They completely break the internal world-sheet $SO(2)$ symmetry. On the other hand, states with vanishing internal momenta, $|p^{\pm} = 0\rangle$, build internal $SO(2)$ representations, resp. representations of the group $USp(4)$, which classifies the $\mathcal{N} = 2$ massive states without central charges.

2.2.5 CFT operators in $\mathcal{N} = 1$ compactifications

In this subsection, we summarize universal aspects of internal $c = 9$ SCFTs describing $D = 4$ superstring compactifications which preserve $\mathcal{N} = 1$ SUSY in spacetime [28–30]. The existence of one supercharge species

$$\mathcal{Q}_a^{(-1/2)} = \frac{1}{\alpha'^{1/4}} \oint \frac{\mathrm{d}z}{2\pi i} \, S_a \, \Sigma^+ \, e^{-\phi/2} \, , \qquad \bar{\mathcal{Q}}^{(-1/2)\dot{b}} = \frac{1}{\alpha'^{1/4}} \oint \frac{\mathrm{d}z}{2\pi i} \, S^{\dot{b}} \, \Sigma^- \, e^{-\phi/2} \tag{2.76}$$

with $h = \frac{3}{8}$ spin fields $\Sigma^{\pm}$ implies that the world-sheet supersymmetry is enhanced to $\mathcal{N} = 2$. This can be traced back to the existence of a $U(1)$ Kac-Moody current $\mathcal{J}$ of $h = 1$ which emerges from the mutual OPEs of spin fields with opposite charge:

$$\Sigma^{\pm}(z) \, \Sigma^{\mp}(w) \quad \sim \quad \frac{1}{(z-w)^{3/4}} \ \pm \ \frac{\sqrt{3}}{2} \, (z-w)^{1/4} \, \mathcal{J}(w) \ + \ \ldots \tag{2.77}$$

The internal supercurrents $G^{\pm}_{\mathrm{int}}$ can be split into two components of opposite $U(1)$ charge,

$$G_{\mathrm{int}} = \frac{1}{\sqrt{2}} \left(G^+_{\mathrm{int}} \ + \ G^-_{\mathrm{int}} \right) \, , \tag{2.78}$$

subject to the superconformal $\mathcal{N} = 2$ algebra[11]

$$\mathcal{J}(z) \, \mathcal{J}(w) \quad \sim \quad \frac{1}{(z-w)^2} \ + \ \mathcal{J}(w) \, \mathcal{J}(w) \ + \ \ldots \tag{2.79}$$

$$\mathcal{J}(z) \, G^{\pm}_{\mathrm{int}}(w) \quad \sim \quad \pm \, \frac{G^{\pm}_{\mathrm{int}}(w)}{\sqrt{3} \, (z-w)} \ + \ \mathcal{J}(w) \, G^{\pm}_{\mathrm{int}}(w) \ + \ \ldots \tag{2.80}$$

$$G^{\pm}_{\mathrm{int}}(z) \, G^{\pm}_{\mathrm{int}}(w) \quad \sim \quad G^{\pm}_{\mathrm{int}}(w) \, G^{\pm}_{\mathrm{int}}(w) \ + \ \ldots \tag{2.81}$$

$$G^{\pm}_{\mathrm{int}}(z) \, G^{\mp}_{\mathrm{int}}(w) \quad \sim \quad \frac{3/2}{(z-w)^3} \ \pm \ \frac{\sqrt{3} \, \mathcal{J}(w)}{2 \, (z-w)^2} + \frac{2 \, T_{\mathrm{int}}(w) \ \pm \ \sqrt{3} \, \partial \mathcal{J}(w)}{4 \, (z-w)} \ + \ \ldots \tag{2.82}$$

[11] In contrast to [28–30], we normalize $\mathcal{J}$ such that it has canonical two-point functions $\langle \mathcal{J}(z) \mathcal{J}(w) \rangle = 1 \cdot (z-w)^{-2}$. This simplifies (subleading) OPE coefficients and normalization factors in vertex operators.

with internal $c=9$ energy momentum tensor $T_{\rm int}$. The OPE of alike spin fields gives rise to new $h=\frac{3}{2}$ Virasoro primary operators

$$\Sigma^{\pm}(z)\,\Sigma^{\pm}(w) \ \ \sim \ \ (z-w)^{3/4}\,\mathcal{O}^{\pm}(w) \ + \ \dots \tag{2.83}$$

with twice the $U(1)$ charge of the spin fields, and iterated OPEs with $\Sigma^{\pm}$ create an infinite tower of further conformal primaries with higher weights and charges.

A large sector of the internal CFT can be captured by bosonization. Let $H(z)$ denote a canonically normalized free & chiral boson, then we have the following representation for some for the aforementioned operators:

$$\mathcal{J} \ \ \equiv \ \ i\partial H\,, \qquad \Sigma^{\pm} \ \ \equiv \ \ e^{\pm i\sqrt{3}H/2}\,, \qquad \mathcal{O}^{\pm} \ \ \equiv \ \ e^{\pm i\sqrt{3}H} \ . \tag{2.84}$$

However, the internal supercurrent (or energy momentum tensor) cannot be fully bosonized. Instead, we can represent $G^{\pm}_{\rm int}$ as

$$G^{\pm}_{\rm int} = \sqrt{\frac{3}{2}}\,e^{\pm\frac{i}{\sqrt{3}}H}\,g^{\pm} \ , \tag{2.85}$$

where the $h=\frac{4}{3}$ operators $g^{\pm}$ are local with respect to H and satisfy

$$g^{\pm}(z)\,g^{\mp}(w) \ \ \sim \ \ \frac{1}{(z-w)^{8/3}} \ + \ \frac{0}{(z-w)^{5/3}} \ + \ \dots \tag{2.86}$$

$$g^{\pm}(z)\,g^{\pm}(w) \ \ \sim \ \ \frac{g^{\pm}(w)\,g^{\pm}(w)}{(z-w)^{1/3}} \ + \ \dots \ . \tag{2.87}$$

On these grounds, we can understand the OPE of the supercurrent with internal spin fields,

$$G^{\pm}_{\rm int}(z)\,\Sigma^{\mp}(w) \ \ \sim \ \ \sqrt{\frac{3}{2}}\,\frac{\tilde{\Sigma}^{\mp}(w)}{(z-w)^{1/2}} \ + \ \dots \tag{2.88}$$

$$G^{\pm}_{\rm int}(z)\,\Sigma^{\pm}(w) \ \ \sim \ \ (z-w)^{1/2}\,g^{\pm}\,e^{\pm\frac{5i}{2\sqrt{3}}}(w) \ + \ \dots \tag{2.89}$$

which introduces excited spin fields $\tilde{\Sigma}^{\pm}$ of $h=\frac{11}{8}$ in case of opposite $U(1)$ charges $G^{\pm}_{\rm int}\leftrightarrow\Sigma^{\mp}$,

$$\tilde{\Sigma}^{\pm} \ \ \equiv \ \ g^{\mp}\,e^{\pm\frac{i}{2\sqrt{3}}H} \ . \tag{2.90}$$

Fig. 3 gives an overview of the universal Virasoro primaries in the internal $c = 9$ SCFT. More detailed OPEs including subleading singularities can be found in appendix B.4.

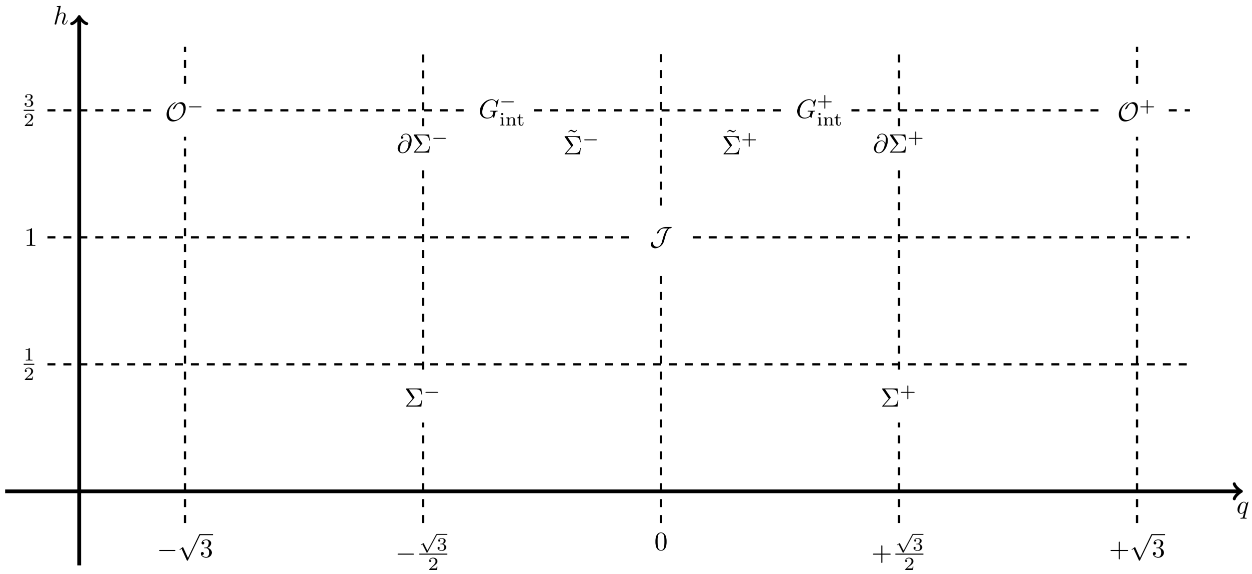

Figure 3: Conformal fields in the $\mathcal{N} = 1$ internal CFT, together with their weight h and $U(1)$ charge q.

From these OPEs, we obtain the following $+1/2$ ghost picture version for the SUSY charge

$$\mathcal{Q}_a^{(+1/2)} = \oint \frac{\mathrm{d}z}{2\pi i} \left[\frac{\sqrt{3}}{\alpha'^{1/4}} \, S_a \, \tilde{\Sigma}^+ \; + \; \frac{1}{2\alpha'^{3/4}} \, i\partial X_\mu \, \sigma^\mu_{a\dot{b}} \, S^{\dot{b}} \, \Sigma^+ \right] e^{+\phi/2} \,, \tag{2.91}$$

$$\bar{\mathcal{Q}}^{(+1/2)\dot{b}} = \oint \frac{\mathrm{d}z}{2\pi i} \left[\frac{\sqrt{3}}{\alpha'^{1/4}} \, S^{\dot{b}} \, \tilde{\Sigma}^- \; + \; \frac{1}{2\alpha'^{3/4}} \, i\partial X^\mu \, \bar{\sigma}_\mu^{\dot{b}a} \, S_a \, \Sigma^- \right] e^{+\phi/2} \,, \tag{2.92}$$

which yield the $\mathcal{N} = 1$ SUSY algebra

$$\left\{ \, \mathcal{Q}_a^{(+1/2)} \, , \, \bar{\mathcal{Q}}^{(-1/2),\dot{b}} \right\} = (\sigma^\mu \, \varepsilon)_a{}^{\dot{b}} \, P_\mu \,, \qquad P_\mu = \frac{1}{2\alpha'} \oint \frac{\mathrm{d}z}{2\pi i} \, i\partial X_\mu \,. \tag{2.93}$$

2.2.6 Summary of CFT operators

To conclude this section on the internal SCFTs associated with $D = 4$ compactifications of different supercharges, Fig. 4 summarizes the field content of the different sectors. This is a good reference to build the most general ansatz for physical vertex operators.

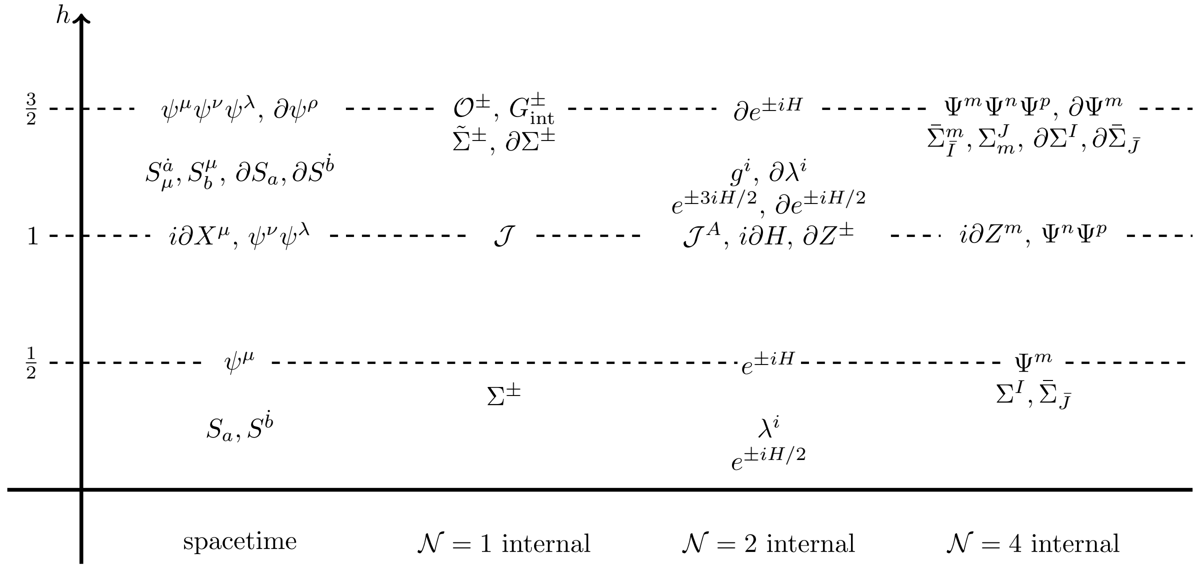

Figure 4: Conformal fields together with their weight in various decoupling CFT sectors

2.3 Physical states of $\mathcal{N} = 4, 2, 1$ compactifications

2.3.1 Physical states of $\mathcal{N} = 4$ SUSY

Having introduced the CFT setup for the construction of massive string state, let us now turn to explicit vertex operators on the first mass level. We will first of all examine the four-dimensional field content of maximally supersymmetric superstring compactifications to $D = 4$ with $\mathcal{N} = 4$ SUSY. This is the dimensional reduction of the ten-dimensional multiplet, so we will again find all the 256 states which have been discussed from the $D = 10$ viewpoint in Chapter 2.1. They form a massive $\mathcal{N} = 4$ multiplet in four dimensions for which we will work out the spin and R-symmetry content as well as the SUSY transformations.

NS sector

With the internal CFT operators from Fig. 1 at hand, the following $h = 3/2$ combinations must be considered in the most general NS vertex operator at first mass level:

$$\begin{aligned} V^{(-1)} =\Big(& \alpha_{\mu\nu}\, i\partial X^\mu\, \psi^\nu \ + \ e_{\mu\nu\lambda}\, \psi^\mu\, \psi^\nu\, \psi^\lambda \ + \ h_\mu\, \partial\psi^\mu \ + \ \beta^m_\mu\, i\partial X^\mu\, \Psi_m \\ & + \ \gamma^m_\mu\, \psi^\mu\, i\partial Z_m \ + \ d^{mn}_\mu\, \psi^\mu\, \Psi_m\, \Psi_n \ + \ Y^m\, \partial\Psi_m \ + \ \omega^m_{\mu\nu}\, \psi^\mu\, \psi^\nu\, \Psi_m \\ & + \ \zeta^{mn}\, i\partial Z_m\, \Psi_n \ + \ \Omega^{mnp}\, \Psi_m\, \Psi_n\, \Psi_p \Big)\, e^{-\phi}\, e^{ik\cdot X} \ . \end{aligned} \tag{2.94}$$

Requiring vanishing Q_1 variation for (2.94) implies the following on-shell constraints for the ten wavefunctions above:

$$
\begin{aligned}
0 &= \alpha_\mu{}^\mu \ + \ k^\mu h_\mu \ + \ \zeta_m{}^m & \qquad 0 &= 2\alpha' Y^m \ + \ k^\mu \gamma_\mu^m \\
0 &= \alpha_{[\mu\nu]} \ + \ 3\, k^\lambda e_{\lambda\mu\nu} & \qquad 0 &= \beta_\mu^m \ - \ \gamma_\mu^m \ + \ 2\, k^\lambda \omega_{\lambda\mu}^m \\
0 &= 2\alpha' \alpha_{\mu\nu} k^\nu \ + \ h_\mu & \qquad 0 &= k^\mu d_\mu^{mn} \ + \ \zeta^{[mn]}
\end{aligned}
\tag{2.95}
$$

This leaves the following 128 physical solutions

- one transverse and traceless spin two tensor

$$
V_\alpha^{(-1)} = \frac{1}{\sqrt{2\alpha'}}\, \alpha_{\mu\nu}\, i\partial X^\mu\, \psi^\nu\, e^{-\phi}\, e^{ik\cdot X}\,, \qquad k^\mu\, \alpha_{\mu\nu} = \alpha_{[\mu\nu]} = \alpha_\mu{}^\mu = 0 \tag{2.96}
$$

- 27 transverse vectors (in the vector and two-form representations of the R-symmetry $SO(6)$)

$$
V_d^{(-1)} = \frac{1}{2}\, d_\mu^{mn}\, \psi^\mu\, \Psi_m\, \Psi_n\, e^{-\phi}\, e^{ik\cdot X}\,, \qquad k^\mu\, d_\mu^{mn} = 0 \tag{2.97}
$$

$$
\begin{aligned}
V_{\beta^\pm}^{(-1)} = \frac{1}{2\sqrt{2\alpha'}}\, \beta_\mu^{\pm,m} \Big(& i\partial X^\mu\, \Psi_m \ + \ i\partial Z_m\, \psi^\mu \\
& \pm \ i\alpha'\, \varepsilon^{\mu\nu\lambda\rho}\, k_\nu\, \psi_\lambda\, \psi_\rho\, \Psi_m \Big)\, e^{-\phi}\, e^{ik\cdot X}\,, \qquad k^\mu\, \beta_\mu^{\pm,m} = 0
\end{aligned}
\tag{2.98}
$$

- 42 scalar degrees of freedom (scalars, spin two and and three-form with respect to $SO(6)$)

$$
\begin{aligned}
V_{\Phi^\pm}^{(-1)} = \frac{1}{2\sqrt{2\alpha'}}\, \Phi^\pm \Big[& (\eta_{\mu\nu} \ + \ 2\alpha'\, k_\mu\, k_\nu)\, i\partial X^\mu\, \psi^\nu \ + \ 2\alpha'\, k_\mu\, \partial\psi^\mu \\
& \pm \ \frac{i\alpha'}{3}\, \varepsilon_{\mu\nu\lambda\rho}\, \psi^\mu\, \psi^\nu\, \psi^\lambda\, k^\rho \Big]\, e^{-\phi}\, e^{ik\cdot X}
\end{aligned}
\tag{2.99}
$$

$$
V_\zeta^{(-1)} = \frac{1}{\sqrt{2\alpha'}}\, \zeta^{mn}\, i\partial Z_m\, \Psi_n\, e^{-\phi}\, e^{ik\cdot X}\,, \qquad \zeta^{[mn]} = \zeta^m{}_m = 0 \tag{2.100}
$$

$$
V_\Omega^{(-1)} = \Omega^{mnp}\, \Psi_m\, \Psi_n\, \Psi_p\, e^{-\phi}\, e^{ik\cdot X}\ . \tag{2.101}
$$

The 46 spurious NS sector states from ten dimensions are aligned into six representations of $SO(1,3)\times SO(6)$. They can be constructively obtained as BRST variations of ghost charge -2 objects, see (2.10):

$$
V_{\pi(\mathrm{sp})}^{(-1)} \ \sim \ \Big[(\pi_\mu\, k_\nu \ + \ k_\mu\, \pi_\nu)\, i\partial X^\mu\, \psi^\nu \ + \ 2\, \pi_\mu\, \partial\psi^\mu \Big]\, e^{-\phi}\, e^{ik\cdot X}\,, \qquad k^\mu\, \pi_\mu = 0\ , \tag{2.102}
$$

$$V^{(-1)}_{\Sigma(\mathrm{sp})} \sim \Big[2\,\Sigma_{[\mu\nu]}\, i\partial X^\mu\, \psi^\nu \;+\; 2\alpha'\,\Sigma_{[\mu\nu}\, k_{\lambda]}\, \psi^\mu\, \psi^\nu\, \psi^\lambda \Big]\, e^{-\phi}\, e^{ik\cdot X}\,, \qquad k^\mu\,\Sigma_{\mu\nu} = 0\,, \tag{2.103}$$

$$V^{(-1)}_{\Lambda_1(\mathrm{sp})} \sim \Lambda_1 \Big[(\eta_{\mu\nu} \;+\; 4\alpha'\, k_\mu\, k_\nu)\, i\partial X^\mu\, \psi^\nu \;+\; 6\alpha'\, k_\mu\, \partial\psi^\mu \;+\; i\partial Z_m\, \Psi^m \Big]\, e^{-\phi}\, e^{ik\cdot X}\,, \tag{2.104}$$

$$V^{(-1)}_{\Lambda_2(\mathrm{sp})} \sim \Lambda_2^m \Big(k_\mu \big[i\partial X^\mu\, \Psi_m \;+\; i\partial Z_m\, \psi^\mu \big] \;+\; 2\,\partial\Psi_m \Big)\, e^{-\phi}\, e^{ik\cdot X}\,, \tag{2.105}$$

$$V^{(-1)}_{\Lambda_3(\mathrm{sp})} \sim \Lambda_3^{[mn]} \big[i\partial Z_m\, \Psi_n \;+\; \alpha'\, k_\mu\, \psi^\mu\, \Psi_m\, \Psi_n \big]\, e^{-\phi}\, e^{ik\cdot X}\,, \tag{2.106}$$

$$V^{(-1)}_{\Lambda_4(\mathrm{sp})} \sim \Lambda^m_{4\,\mu} \Big(i\partial X^\mu\, \Psi_m \;-\; i\partial Z_m\, \psi^\mu \;-\; 2\alpha'\, k_\nu\, \psi^\mu\, \psi^\nu\, \Psi_m \Big)\, e^{-\phi}\, e^{ik\cdot X}\,, \qquad k^\mu\, \Lambda^m_{4\,\mu} = 0\,. \tag{2.107}$$

Each spurious state corresponds to a gauge freedom. The first one (2.102) admits to gauge away the longitudinal component of the rank two tensor $\alpha_{\mu\nu}$ whereas the second one (2.103) identifies the antisymmetric part $\alpha_{[\mu\nu]}$ together with the longitudinal three-form $e_{\mu\nu\lambda}\sigma k_{[\mu}\Sigma_{\nu\lambda]}$ as unphysical. Similarly, (2.105), (2.106) and (2.107) eliminate the longitudinal components of $(\beta^m_\mu + \gamma^m_\mu), d^{mn}_\mu$ and $\omega^m_{\mu\nu}$ as well as the antisymmetric parts $\beta^m_\mu - \gamma^m_\mu$ and $\zeta_{[mn]}$. The trace of $\alpha_{\mu\nu}$ can be gauged away using (2.104).

Once the three- and two-forms $e_{\mu\nu\lambda}$ and $\omega^k_{\mu\nu}$ are reduced to there transverse part, contraction with $\varepsilon^{\mu\nu\lambda\rho}k_\rho$ dualizes them to a scalar and a vector, respectively. As we will see below, supersymmetry suggests to include these dualized states into the complex combinations (2.98) and (2.99).

R sector

In the R sector, the SCFT operators of appropriate weight give rise to a vertex operator ansatz with six wavefunctions:

$$\begin{aligned} V^{(-\frac{1}{2})} = \Big(& v^a_{\mu,I}\, i\partial X^\mu\, S_a\, \Sigma^I \;+\; \bar\rho^\mu_{\dot b,I}\, S^{\dot b}_\mu\, \Sigma^I \;+\; u^a_I\, \partial S_a\, \Sigma^I \\ & +\; y^a_I\, S_a\, \partial\Sigma^I \;+\; \bar r^{\bar J}_{m,\dot b}\, i\partial Z^m\, S^{\dot b}\, \bar\Sigma_{\bar J} \;+\; s^{a,\bar J}_m\, S_a\, \bar\Sigma^m_{\bar J} \Big)\, e^{-\phi/2}\, e^{ik\cdot X}\,. \end{aligned} \tag{2.108}$$

The same set of states also exists with opposite chiralities with respect to both $SO(1,3)$ and $SO(6)$ (e.g. $v^a_{\mu,I} S_a \Sigma^I \leftrightarrow \bar v^{\bar J}_{\mu,\dot b} S^{\dot b} \bar\Sigma_{\bar J}$). However, the BRST constraints for the polarizations in (2.108) decouple from those of the other chirality sector which we did not display, so the discussion will be limited to the six wavefunctions shown in (2.108) for the moment. The full list of physical and spurious states follows from doubling the solutions of the on-shell constraints. Imposing invariance under Q_1 yields the following three independent constraints:

$$
\begin{aligned}
0 &= 2\alpha' \, v_I^{\mu,a} \; \not{k}_{a\dot b} \; + \; \sqrt{2}\, \bar\rho^{\mu}_{\dot b, I} \; + \; \tfrac{1}{2}\, u_I^a \, \sigma^{\mu}_{a\dot b} \; , \\
0 &= 2\alpha' \, \bar r^{\bar J}_{m,\dot b} \; \not{k}^{\dot b a} \; + \; \sqrt{2}\, s_m^{a,\bar J} \; - \; \tfrac{1}{2}\, y_I^a \, \gamma_m^{I\bar J} \; , \\
0 &= k_\mu \, \bar\rho^{\mu}_{\dot b, I} \; + \; \tfrac{1}{2\sqrt{2}}\, \bar r^{\bar J}_{m,\dot b} \, \bar\gamma^m_{\bar J I} \; .
\end{aligned}
\tag{2.109}
$$

The first two equations can be further disentangled into a trace and a traceless part with respect to the σ^μ and γ_m matrices. Since excited spin fields are σ and γ traceless, the associated wavefunctions satisfy $\bar\rho^{\mu}_{\dot b,I}\, \bar\sigma^{\dot b a}_{\mu} = s_m^{a,\bar J} \bar\gamma^m_{\bar J I} = 0$ by construction. Hence, the aforementioned projections simplify the BRST constraints to

$$
\begin{aligned}
u_I^a &= \alpha' \, v^b_{\mu,I} \, (\not{k} \, \bar\sigma^\mu)_b{}^a \\
\bar\rho^{\mu}_{\dot b, I} &= - \sqrt{2}\, \alpha' \left(v_I^{\mu,a} \; \not{k}_{a\dot b} \; + \; \tfrac{1}{4}\, v^a_{\lambda, I} \, (\not{k} \, \bar\sigma^\lambda \, \sigma^\mu)_{a\dot b} \right) \\
y_I^a &= - \, \frac{2\alpha'}{3} \, \bar r^{\bar J}_{m,\dot b} \, \bar\gamma^m_{\bar J I} \; \not{k}^{\dot b a} \\
s_m^{a,\bar J} &= - \sqrt{2}\, \alpha' \left(\bar r^{\bar J}_{m,\dot b} \; \not{k}^{\dot b a} \; + \; \tfrac{1}{6}\, \bar r^{\bar I}_{n,\dot b} \, (\bar\gamma^n \, \gamma_m)_{\bar I}{}^{\bar J} \; \not{k}^{\dot b a} \right) \\
\bar r^{\bar J}_{m,\dot b} \, \bar\gamma^m_{\bar J I} &= 2\alpha' \, k_\mu \, v_I^{\mu,a} \; \not{k}_{a\dot b} \; - \; v^a_{\mu,I} \, \sigma^\mu_{a\dot b}
\end{aligned}
\tag{2.110}
$$

where $\bar\rho, u, y$ and s are expressed in terms of v and $\bar r$. It turns out that both spin 3/2 and spin 1/2 components of the vector spinors v_I as well as the γ traceless components of $\bar r$ give rise to an independent physical solution. The former is the $D = 4$ analogue of the ten-dimensional spin 3/2 state (2.22). But additionally, we find spin 1/2 Dirac fermions $(a^b, \bar r^{\bar I}_{m,a})$ – both in the fundamental spinor- and in the spin 3/2 representations of the R-symmetry $SO(6)$. To summarize the physical states built from (2.108) and its opposite chirality counterpart:

- eight transverse and σ traceless spin 3/2 vector spinors

$$
V_\chi^{(-\frac{1}{2})} = \frac{1}{\sqrt{2}\alpha'^{1/4}} \; \chi^a_{\mu,I} \left(i\partial X^\mu \, S_a \; - \; \sqrt{2}\, \alpha' \; \not{k}_{a\dot b} \, S^{\mu \dot b} \right) \Sigma^I \, e^{-\phi/2} \, e^{ik\cdot X} \; , \tag{2.111}
$$

$$
V_{\bar\chi}^{(-\frac{1}{2})} = \frac{1}{\sqrt{2}\alpha'^{1/4}} \; \bar\chi^{\bar I}_{\mu,\dot a} \left(i\partial X^\mu \, S^{\dot a} \; - \; \sqrt{2}\, \alpha' \; \not{k}^{\dot a b} \, S^\mu_b \right) \bar\Sigma_{\bar I} \, e^{-\phi/2} \, e^{ik\cdot X} \; , \tag{2.112}
$$

$$
0 = k^\mu \, \chi^a_{\mu,I} = \chi^a_{\mu,I} \, \sigma^\mu_{a\dot b} = k_\mu \, \bar\chi^{\mu,\bar J}_{\dot b} = \bar\chi^{\mu,\bar J}_{\dot b} \, \bar\sigma^{\dot b a}_\mu \tag{2.113}
$$

- 48 spin 1/2 fermions (eight in the fundamental and 40 in spin 3/2 representations of $SO(6)$)

$$V_a^{(-\frac{1}{2})} = \frac{\alpha'^{1/4}}{2} \, a_I^b \left((\sigma_\mu \, \not{k})_b{}^a \, S_a \, i\partial X^\mu \; - \; 4 \, \partial S_b \right) \Sigma^I \, e^{-\phi/2} \, e^{ik\cdot X} \; , \tag{2.114}$$

$$V_{\bar{a}}^{(-\frac{1}{2})} = \frac{\alpha'^{1/4}}{2} \, \bar{a}_{\dot{b}}^{\bar{I}} \left((\bar{\sigma}_\mu \, \not{k})^{\dot{b}}{}_{\dot{a}} \, S^{\dot{a}} \, i\partial X^\mu \; - \; 4 \, \partial S^{\dot{b}} \right) \bar{\Sigma}_{\bar{I}} \, e^{-\phi/2} \, e^{ik\cdot X} \; , \tag{2.115}$$

$$V_r^{(-\frac{1}{2})} = \frac{1}{\sqrt{2}\alpha'^{1/4}} \, r_{m,I}^a \left(i\partial Z^m \, \Sigma^I \, S_a \; - \; \sqrt{2} \, \alpha' \; \not{k}_{a\dot{b}} \, S^{\dot{b}} \, \Sigma^{m,I} \right) e^{-\phi/2} \, e^{ik\cdot X} \; , \tag{2.116}$$

$$V_{\bar{r}}^{(-\frac{1}{2})} = \frac{1}{\sqrt{2}\alpha'^{1/4}} \, \bar{r}_{m,\dot{a}}^{\bar{I}} \left(i\partial Z^m \, \bar{\Sigma}_{\bar{I}} \, S^{\dot{a}} \; - \; \sqrt{2} \, \alpha' \; \not{k}^{\dot{a}b} \, S_b \, \bar{\Sigma}_{\bar{I}}^m \right) e^{-\phi/2} \, e^{ik\cdot X} \; . \tag{2.117}$$

The following spurious solutions have been subtracted to remove internal derivatives $\partial\Sigma^I$ from the vertex operators:

$$\begin{aligned} V_{\Theta(\mathrm{sp})}^{(-\frac{1}{2})} \;\; &\sim \;\; \Theta_I^a \Big[(\not{k}_{a\dot{b}} \, \bar{\sigma}_\mu^{\dot{b}b} \; + \; 4k_\mu \, \delta_a^b) \, i\partial X^\mu \, S_b \, \Sigma^I \; - \; 2\sqrt{2} \left(\alpha' \, k^\mu \, \not{k}_{a\dot{b}} \; + \; \tfrac{1}{4} \, \sigma^\mu_{a\dot{b}} \right) S_\mu^{\dot{b}} \, \Sigma^I \\ &\quad + \; 6 \, \partial S_a \, \Sigma^I \; + \; 4 \, S_a \, \partial \Sigma^I \; + \; \not{k}_{a\dot{b}} \, \gamma_m^{I\bar{J}} \, i\partial Z^m \, S^{\dot{b}} \, \bar{\Sigma}_{\bar{I}} \Big] \, e^{-\phi/2} \, e^{ik\cdot X}, \end{aligned} \tag{2.118}$$

$$\begin{aligned} V_{\bar{\Theta}(\mathrm{sp})}^{(-\frac{1}{2})} \;\; &\sim \;\; \bar{\Theta}_{\dot{b}}^{\bar{I}} \Big[(\not{k}^{\dot{b}a} \, \sigma_{a\dot{a}}^\mu \; + \; 4k^\mu \, \delta_{\dot{a}}^{\dot{b}}) \, i\partial X_\mu \, S^{\dot{a}} \, \bar{\Sigma}_{\bar{I}} \; - \; 2\sqrt{2} \left(\alpha' \, k_\mu \, \not{k}^{\dot{b}a} \; + \; \tfrac{1}{4} \, \bar{\sigma}_\mu^{\dot{b}a} \right) S_a^\mu \, \bar{\Sigma}_{\bar{I}} \\ &\quad + \; 6 \, \partial S^{\dot{b}} \, \bar{\Sigma}_{\bar{I}} \; + \; 4 \, S^{\dot{b}} \, \partial \bar{\Sigma}_{\bar{I}} \; + \; \not{k}^{\dot{b}a} \, \gamma^m_{\bar{I}J} \, i\partial Z_m \, S_a \, \Sigma^J \Big] \, e^{-\phi/2} \, e^{ik\cdot X}. \end{aligned} \tag{2.119}$$

They are the dimensional reduction of the ten-dimensional spurious state (2.23).

2.3.2 Physical states of $\mathcal{N} = 2$ SUSY

In this section, we will show that the first mass level in compactifications with $\mathcal{N} = 2$ spacetime SUSY is populated by 80 universal states which are aligned into one 24+24 state multiplet of highest spin two and two 8+8 state multiplets of maximum spin one.

NS sector

According to the CFT operator content shown in Fig. 2, we make the following general ansatz for an NS state at the first mass level:[12]

$$
\begin{aligned}
V^{(-1)} = \Big(& \alpha_{\mu\nu}\, i\partial X^\mu\, \psi^\nu \;+\; e_{\mu\nu\lambda}\, \psi^\mu\, \psi^\nu\, \psi^\lambda \;+\; h_\mu\, \partial\psi^\mu \;+\; Y_+\, \partial e^{iH} \;+\; Y_-\, \partial e^{-iH} \\
& + \beta^+_\mu\, i\partial X^\mu\, e^{iH} \;+\; \beta^-_\mu\, i\partial X^\mu\, e^{-iH} \;+\; \gamma^+_\mu\, \psi^\mu\, i\partial Z^+ \;+\; \gamma^-_\mu\, \psi^\mu\, i\partial Z^- \\
& + \xi_\mu\, \psi^\mu\, i\partial H \;+\; d^A_\mu\, \mathcal{J}_A\, \psi^\mu \;+\; \Omega^A_+\, \mathcal{J}_A\, e^{iH} \;+\; \Omega^A_-\, \mathcal{J}_A\, e^{-iH} \\
& + \zeta_{++}\, i\partial Z^+\, e^{iH} \;+\; \zeta_{--}\, i\partial Z^-\, e^{-iH} \;+\; \zeta_{-+}\, i\partial Z^-\, e^{iH} \;+\; \zeta_{+-}\, i\partial Z^+\, e^{-iH} \\
& + \omega^+_{\mu\nu}\, \psi^\mu\, \psi^\nu\, e^{iH} \;+\; \omega^-_{\mu\nu}\, \psi^\mu\, \psi^\nu\, e^{-iH} \;+\; c_i{}^j\, \lambda^i\, g_j \Big)\, e^{-\phi}\, e^{ik\cdot X}\ .
\end{aligned}
\tag{2.120}
$$

Requiring BRST invariance under Q_1 yields the following on-shell conditions:

$$
\begin{aligned}
& 0 = \alpha_\mu{}^\mu \;+\; k^\mu\, h_\mu \;+\; \zeta_{+-} \;+\; \zeta_{-+} \;-\; \alpha'^{-1/2}\, c_i{}^i \\
& 0 = 2\alpha'\, \alpha_{\mu\nu}\, k^\nu \;+\; h_\mu && 0 = k^\mu\, d^A_\mu \;+\; \tfrac{1}{\sqrt{2\alpha'}}\, (\tau^A)^i{}_j\, c_i{}^j \\
& 0 = \alpha_{[\mu\nu]} \;+\; 3\, e_{\mu\nu\lambda}\, k^\lambda && 0 = Y_\pm \;+\; 2\alpha'\, \gamma^\pm_\mu\, k^\mu \\
& 0 = \beta^\pm_\mu \;-\; \gamma^\pm_\mu \;+\; 2\, k^\nu\, \omega^\pm_{\nu\mu} && 0 = k^\mu\, \xi_\mu \;+\; \zeta_{-+} \;-\; \zeta_{+-}
\end{aligned}
\tag{2.121}
$$

These BRST constraints admit 40 physical solutions:

- one transverse and traceless spin two tensor

$$
V^{(-1)}_\alpha = \frac{1}{\sqrt{2\alpha'}}\, \alpha_{\mu\nu}\, i\partial X^\mu\, \psi^\nu\, e^{-\phi}\, e^{ik\cdot X}\ , \qquad k^\mu\, \alpha_{\mu\nu} = \alpha_{[\mu\nu]} = \alpha_\mu{}^\mu = 0 \tag{2.122}
$$

- eight transverse vectors three of which form an R-symmetry triplet (note the sign difference in the pseudovector parts of $\beta^\pm$ and $\omega^\pm$)

$$
V^{(-1)}_\xi = \xi_\mu\, \psi^\mu\, i\partial H\, e^{-\phi}\, e^{ik\cdot X}\ , \qquad k^\mu\, \xi_\mu = 0 \tag{2.123}
$$

$$
V^{(-1)}_d = d^A_\mu\, \psi^\mu\, \mathcal{J}_A\, e^{-\phi}\, e^{ik\cdot X}\ , \qquad k^\mu\, d^A_\mu = 0 \tag{2.124}
$$

[12] Recall that we have non-Abelian R-symmetry $SU(2)$ in this setting, and $i, j = 1, 2$ denote its spinor indices whereas $A = 1, 2, 3$ are adjoint indices.

$$V^{(-1)}_{\beta^\pm} = \frac{1}{2\sqrt{2\alpha'}}\,\beta^\pm_\mu\,\big(\,i\partial X^\mu\,e^{\pm iH}\ +\ i\partial Z^\pm\,\psi^\mu$$

$$\pm\ i\alpha'\,\varepsilon^{\mu\nu\lambda\rho}\,k_\nu\,\psi_\lambda\,\psi_\rho\,e^{\pm iH}\,\big)\,e^{-\phi}\,e^{ik\cdot X}\,, \qquad k^\mu\,\beta^\pm_\mu = 0 \tag{2.125}$$

$$V^{(-1)}_{\omega^\pm} = \frac{1}{2\sqrt{2\alpha'}}\,\omega^\pm_\mu\,\big(\,i\partial X^\mu\,e^{\pm iH}\ +\ i\partial Z^\pm\,\psi^\mu$$

$$\mp\ i\alpha'\,\varepsilon^{\mu\nu\lambda\rho}\,k_\nu\,\psi_\lambda\,\psi_\rho\,e^{\pm iH}\,\big)\,e^{-\phi}\,e^{ik\cdot X}\,, \qquad k^\mu\,\omega^\pm_\mu = 0 \tag{2.126}$$

- eleven real scalar degrees of freedom

$$V^{(-1)}_{\Phi^\pm} = \frac{\Phi^\pm}{2\sqrt{2\alpha'}}\,\Big[\,(\eta_{\mu\nu}\ +\ 2\alpha'\,k_\mu\,k_\nu)\,i\partial X^\mu\,\psi^\nu\ +\ 2\alpha'\,k_\mu\,\partial\psi^\mu$$

$$\pm\ \frac{i\alpha'}{3}\,\varepsilon_{\mu\nu\lambda\rho}\,\psi^\mu\,\psi^\nu\,\psi^\lambda\,k^\rho\,\Big]\,e^{-\phi}\,e^{ik\cdot X}\ , \tag{2.127}$$

$$V^{(-1)}_{\phi} = \frac{1}{\sqrt{6\alpha'}}\,\phi\,\Big[\,i\partial Z^+\,e^{-iH}\ +\ i\partial Z^-\,e^{iH}\ +\ \sqrt{\alpha'}\,G^i{}_i\,\Big]\,e^{-\phi}\,e^{ik\cdot X}\ , \tag{2.128}$$

$$V^{(-1)}_{\Omega^\pm} = \Omega^\pm_A\,e^{\pm iH}\,\mathcal{J}^A\,e^{-\phi}\,e^{ik\cdot X}\ , \tag{2.129}$$

$$V^{(-1)}_{\zeta^\pm} = \frac{1}{\sqrt{2\alpha'}}\,\zeta^\pm\,i\partial Z^\pm\,e^{\pm iH}\,e^{-\phi}\,e^{ik\cdot X}\ . \tag{2.130}$$

In addition, we have numerous spurious states:

$$V^{(-1)}_{\pi(\mathrm{sp})} \ \sim\ \Big[\,(\pi_\mu\,k_\nu\ +\ k_\mu\,\pi_\nu)\,i\partial X^\mu\,\psi^\nu\ +\ 2\,\pi_\mu\,\partial\psi^\mu\,\Big]\,e^{-\phi}\,e^{ik\cdot X}\,, \qquad k^\mu\,\pi_\mu = 0\ , \tag{2.131}$$

$$V^{(-1)}_{\Sigma(\mathrm{sp})} \ \sim\ \Big[\,2\,\Sigma_{[\mu\nu]}\,i\partial X^\mu\,\psi^\nu\ +\ 2\alpha'\,\Sigma_{[\mu\nu}\,k_{\lambda]}\,\psi^\mu\,\psi^\nu\,\psi^\lambda\,\Big]\,e^{-\phi}\,e^{ik\cdot X}\,, \qquad k^\mu\,\Sigma_{\mu\nu} = 0\ , \tag{2.132}$$

$$V^{(-1)}_{\Lambda_0(\mathrm{sp})} \ \sim\ \Lambda^A_0\,\Big[\,k_\mu\,\psi^\mu\,\mathcal{J}_A\ +\ \tfrac{1}{\sqrt{2\alpha'}}\,(\tau_A)^j{}_i\,G^i{}_j\,\Big]\,e^{-\phi}\,e^{ik\cdot X}\ , \tag{2.133}$$

$$V^{(-1)}_{\Lambda_1(\mathrm{sp})} \ \sim\ \Lambda_1\,\Big[\,(\eta_{\mu\nu}\ +\ 4\alpha'\,k_\mu\,k_\nu)\,i\partial X^\mu\,\psi^\nu\ +\ 6\alpha'\,k_\mu\,\partial\psi^\mu$$

$$+\ i\partial Z^+\,e^{-iH}\ +\ i\partial Z^-\,e^{iH}\ -\ 2\sqrt{\alpha'}\,G^i{}_i\,\Big]\,e^{-\phi}\,e^{ik\cdot X}\ , \tag{2.134}$$

$$V^{(-1)}_{\Lambda^\pm_2(\mathrm{sp})} \ \sim\ \Lambda^\pm_2\,\Big[\,k_\mu\,(i\partial Z^\pm\,\psi^\mu\ +\ i\partial X^\mu\,e^{\pm iH})\ +\ 2\,\partial e^{\pm iH}\,\Big]\,e^{-\phi}\,e^{ik\cdot X}\ , \tag{2.135}$$

$$V^{(-1)}_{\Lambda_3(\mathrm{sp})} \ \sim\ \Lambda_3\,\Big[\,2\alpha'\,k_\mu\,\psi^\mu\,i\partial H\ +\ i\partial Z^-\,e^{iH}\ -\ i\partial Z^+\,e^{-iH}\,\Big]\,e^{-\phi}\,e^{ik\cdot X}\ , \tag{2.136}$$

$$V^{(-1)}_{\Lambda^\pm_4(\mathrm{sp})} \ \sim\ \Lambda^\pm_{4\,\mu}\,\Big[\,2\alpha'\,k_\nu\,\psi^\nu\,\psi^\mu\,e^{\pm iH}\ +\ i\partial X^\mu\,e^{\pm iH}\ -\ i\partial Z^\pm\,\psi^\mu\,\Big]\,e^{-\phi}\,e^{ik\cdot X}\,, \qquad k^\mu\,\Lambda^\pm_{4\,\mu} = 0\ . \tag{2.137}$$

They allow to eliminate the longitudinal components of six vectors and of the two-forms $\omega^\pm_{\mu\nu}$. The latter therefore dualize to transverse pseudovectors entering the $\beta^\pm_\mu$ and $\omega^\pm_\mu$ states. By combining with the Λ_1 spurious state, one can transform the ϕ solution into a form without internal $c = 6$ supercurrents:

$$V^{(-1)}_\phi = \frac{1}{\sqrt{6\alpha'}}\, \phi \Big[(\eta_{\mu\nu} \ + \ 4\alpha'\, k_\mu\, k_\nu)\, i\partial X^\mu\, \psi^\nu \ + \ 6\alpha'\, k_\mu\, \partial\psi^\mu \\ + \ 3\, (i\partial Z^+\, e^{-iH} \ + \ i\partial Z^-\, e^{iH}) \Big]\, e^{-\phi}\, e^{ik\cdot X}. \tag{2.138}$$

R sector

In the R sector of the first mass level in $\mathcal{N} = 2$ scenarios, the vertex operator ansatz in one chirality sector includes nine SCFT operators:

$$V^{(-\frac{1}{2})} = \Big\{ v^a_{\mu i}\, i\partial X^\mu\, S_a\, \lambda^i\, e^{iH/2} \ + \ \bar\rho^\mu_{\dot b i}\, S^{\dot b}_\mu\, \lambda^i\, e^{iH/2} \ + \ u^a_i\, \partial S_a\, \lambda^i\, e^{iH/2} \ + \ \bar r_{+\dot b i}\, i\partial Z^+\, S^{\dot b}\, \lambda^i\, e^{-iH/2} \\ + \ \bar r_{-\dot b i}\, i\partial Z^-\, S^{\dot b}\, \lambda^i\, e^{-iH/2} \ + \ \omega^a_i\, S_a\, \lambda^i\, \partial e^{iH/2} \ + \ y^a_i\, S_a\, \partial\lambda^i\, e^{iH/2} \\ + \ \bar\ell_{\dot b i}\, S^{\dot b}\, g^i\, e^{iH/2} \ + \ \psi^a_i\, S_a\, \lambda^i\, e^{-3iH/2} \Big\}\, e^{-\phi/2}\, e^{ik\cdot X} \ . \tag{2.139}$$

The system of BRST constraints can be reduced to the following independent set:

$$\begin{array}{l} 0 = 2\, k_\mu\, \bar\rho^\mu_{\dot b i} \ + \ \bar r_{+\dot b i} \ - \ \sqrt{\frac{1}{\alpha'}}\, \bar\ell_{\dot b i} \\ 0 = 2\alpha'\, v^{\mu,a}_i\, \not k_{a\dot b} \ + \ \sqrt{2}\, \bar\rho^\mu_{\dot b i} \ + \ \frac{1}{2}\, u^a_i\, \sigma^\mu_{a\dot b} \end{array} \quad , \quad \begin{array}{l} 0 = \omega^a_i \ + \ 2\sqrt{2}\, \alpha'\, \bar r_{+\dot b i}\, \not k^{\dot b a} \\ 0 = \psi^a_i \ + \ \sqrt{2}\, \alpha'\, \bar r_{-\dot b i}\, \not k^{\dot b a} \\ 0 = y^a_i \ - \ \sqrt{2\alpha'}\, \bar\ell_{\dot b i}\, \not k^{\dot b a} \end{array} \ . \tag{2.140}$$

Adding a sector of opposite chirality and internal charge gives rise to 40 physical solutions. All of them transform in the fundamental representation of the $SU(2)$ R-symmetry:

- four transverse and σ traceless spin 3/2 vector spinors

$$V^{(-\frac{1}{2})}_\chi = \frac{1}{\sqrt{2}\alpha'^{1/4}}\, \chi^a_{\mu,i}\, \Big(i\partial X^\mu\, S_a \ - \ \sqrt{2}\, \alpha'\, \not k_{a\dot b}\, S^{\mu\dot b} \Big)\, \lambda^i\, e^{iH/2}\, e^{-\phi/2}\, e^{ik\cdot X} \ , \tag{2.141}$$

$$V^{(-\frac{1}{2})}_{\bar\chi} = \frac{1}{\sqrt{2}\alpha'^{1/4}}\, \bar\chi^\mu_{\dot b,i}\, \Big(i\partial X_\mu\, S^{\dot b} \ - \ \sqrt{2}\, \alpha'\, \not k^{\dot b a}\, S_{\mu a} \Big)\, \lambda^i\, e^{-iH/2}\, e^{-\phi/2}\, e^{ik\cdot X} \ , \tag{2.142}$$

$$0 = k^\mu \, \chi^a_{\mu i} = \chi^a_{\mu i} \, \sigma^\mu_{a\dot b} = k_\mu \, \bar\chi^\mu_{\dot b i} = \bar\chi^\mu_{\dot b i} \, \bar\sigma^{\dot b a}_\mu \tag{2.143}$$

- six spin 1/2 fermions:

$$V_a^{(-\frac{1}{2})} = \frac{\alpha'^{1/4}}{2} \, a^b_i \left((\sigma_\mu \, \not{k})_b{}^a \, S_a \, i\partial X^\mu \; - \; 4 \, \partial S_b \right) \lambda^i \, e^{iH/2} \, e^{-\phi/2} \, e^{ik\cdot X} \;\; , \tag{2.144}$$

$$V_{\bar a}^{(-\frac{1}{2})} = \frac{\alpha'^{1/4}}{2} \, \bar a_{\dot a, i} \left((\bar\sigma_\mu \, \not{k})^{\dot a}{}_{\dot b} \, S^{\dot b} \, i\partial X^\mu \; - \; 4 \, \partial S^{\dot a} \right) \lambda^i \, e^{-iH/2} \, e^{-\phi/2} \, e^{ik\cdot X} \;\; , \tag{2.145}$$

$$V_r^{(-\frac{1}{2})} = \frac{1}{\sqrt{2}\alpha'^{1/4}} \, r^a_i \left(i\partial Z^+ \, S_a \, e^{iH/2} \; - \; \sqrt{2} \, \alpha' \, \not{k}_{a\dot b} \, S^{\dot b} \, e^{3iH/2} \right) \lambda^i \, e^{-\phi/2} \, e^{ik\cdot X} \;\; , \tag{2.146}$$

$$V_{\bar r}^{(-\frac{1}{2})} = \frac{1}{\sqrt{2}\alpha'^{1/4}} \, \bar r_{\dot b, i} \left(i\partial Z^- \, S^{\dot b} \, e^{-iH/2} \; - \; \sqrt{2} \, \alpha' \, \not{k}^{\dot b a} \, S_a \, e^{-3iH/2} \right) \lambda^i \, e^{-\phi/2} \, e^{ik\cdot X} \;\; , \tag{2.147}$$

$$\begin{aligned} V_s^{(-\frac{1}{2})} = \frac{1}{\sqrt{3}\alpha'^{1/4}} \, s^a_i \Big(& i\partial Z^- \, S_a \, \lambda^i \, e^{iH/2} \; + \; \sqrt{\alpha'} \, S_a \, g^i \, e^{-iH/2} \\ & + \; \sqrt{2} \, \alpha' \, \not{k}_{a\dot b} \left(S^{\dot b} \, \partial\lambda^i \, e^{-iH/2} \; - \; 2 \, S^{\dot b} \, \lambda^i \, \partial e^{-iH/2} \right) \Big) \, e^{-\phi/2} \, e^{ik\cdot X} \;\; , \end{aligned} \tag{2.148}$$

$$\begin{aligned} V_{\bar s}^{(-\frac{1}{2})} = \frac{1}{\sqrt{3}\alpha'^{1/4}} \, \bar s_{\dot b, i} \Big(& i\partial Z^+ \, S^{\dot b} \, \lambda^i \, e^{-iH/2} \; + \; \sqrt{\alpha'} \, S^{\dot b} \, g^i \, e^{iH/2} \\ & + \; \sqrt{2} \, \alpha' \, \not{k}^{\dot b a} \left(S_a \, \partial\lambda^i \, e^{iH/2} \; - \; 2 \, S_a \, \lambda^i \, \partial e^{iH/2} \right) \Big) \, e^{-\phi/2} \, e^{ik\cdot X} \;\; . \end{aligned} \tag{2.149}$$

Again, there is a spurious fermion which can be used to remove some internal SCFT fields from the vertex operators:

$$\begin{aligned} V_{\Theta(\mathrm{sp})}^{(-\frac{1}{2})} \;\; \sim \;\; \Theta^a_i \Big[& (\not{k}_{a\dot b} \, \bar\sigma^{\dot b b}_\mu \; + \; 4 k_\mu \, \delta^b_a) \, i\partial X^\mu \, S_b \, \lambda^i \, e^{iH/2} \; - \; 2\sqrt{2} \left(\alpha' \, k^\mu \, \not{k}_{a\dot b} \; + \; \tfrac{1}{4} \, \sigma^\mu_{a\dot b} \right) S^{\dot b}_\mu \, \lambda^i \, e^{iH/2} \\ & + \; 6 \, \partial S_a \, \lambda^i \, e^{iH/2} \; + \; 4 \, S_a \, \partial\lambda^i \, e^{iH/2} \; + \; 4 \, S_a \, \lambda^i \, \partial e^{iH/2} \\ & + \; 2\sqrt{2\alpha'} \, \not{k}_{a\dot b} \, S^{\dot b} \, g^i \, e^{iH/2} \; - \; \sqrt{2} \, \not{k}_{a\dot b} \, i\partial Z^+ \, S^{\dot b} \, \lambda^i \, e^{-iH/2} \Big] \, e^{-\phi/2} \, e^{ik\cdot X}, \end{aligned} \tag{2.150}$$

$$\begin{aligned} V_{\bar\Theta(\mathrm{sp})}^{(-\frac{1}{2})} \;\; \sim \;\; \bar\Theta_{\dot b, i} \Big[& (\not{k}^{\dot b a} \, \sigma^\mu_{a\dot a} \; + \; 4 k^\mu \, \delta^{\dot b}_{\dot a}) \, i\partial X_\mu \, S^{\dot a} \, \lambda^i \, e^{-iH/2} \; - \; 2\sqrt{2} \left(\alpha' \, k_\mu \, \not{k}^{\dot b a} \; + \; \tfrac{1}{4} \, \bar\sigma^{\dot b a}_\mu \right) S^\mu_a \, \lambda^i \, e^{-iH/2} \\ & + \; 6 \, \partial S^{\dot b} \, \lambda^i \, e^{-iH/2} \; + \; 4 \, S^{\dot b} \, \partial\lambda^i \, e^{-iH/2} \; + \; 4 \, S^{\dot b} \, \lambda^i \, \partial e^{-iH/2} \\ & + \; 2\sqrt{2\alpha'} \, \not{k}^{\dot b a} \, S_a \, g^i \, e^{-iH/2} \; - \; \sqrt{2} \, \not{k}^{\dot b a} \, i\partial Z^- \, S_a \, \lambda^i e^{iH/2} \Big] \, e^{-\phi/2} \, e^{ik\cdot X}. \end{aligned} \tag{2.151}$$

2.3.3 Physical states of $\mathcal{N}=1$ SUSY

This section is devoted to the universal SUSY multiplets common to all $D=4$ superstring compactifications which preserve $\mathcal{N}=1$ spacetime SUSY. Only the $\mathcal{N}=1$ SUSY can provide us chiral fermion spectrum which is phenomenologically most important.

NS sector

By assembling $h=3/2$ combinations of the conformal fields of Fig. 3, one arrives at the following general form of an NS state at mass $m^2 = 1/\alpha'$:

$$\begin{aligned} V^{(-1)} =& \Big(\alpha_{\mu\nu} \, i\partial X^\mu \, \psi^\nu \ + \ e_{\mu\nu\lambda} \, \psi^\mu \, \psi^\nu \, \psi^\lambda \ + \ h_\mu \, \partial\psi^\mu \ + \ \xi_\mu \, \psi^\mu \, \mathcal{J} \\ & + \ \Omega_+ \, \mathcal{O}^+ \ + \ \Omega_- \, \mathcal{O}^- \ + \ c_+ \, G^+_{\rm int} \ + \ c_- \, G^-_{\rm int} \Big) \, e^{-\phi} \, e^{ik\cdot X} \ . \end{aligned} \tag{2.152}$$

This is BRST invariant if the polarization tensors satisfy

$$\begin{aligned} 0 &= \alpha_\mu{}^\mu \ + \ k^\mu \, h_\mu \ + \ \tfrac{3}{2\sqrt{\alpha'}} \, (c_+ + c_-) \qquad & 0 &= \alpha_{[\mu\nu]} \ + \ 3 \, e_{\mu\nu\lambda} \, k^\lambda \\ 0 &= k^\mu \, \xi_\mu \ + \ \tfrac{\sqrt{3}}{2\sqrt{\alpha'}} \, (c_- - c_+) \qquad & 0 &= 2\alpha' \, \alpha_{\mu\nu} \, k^\nu \ + \ h_\mu \end{aligned} \tag{2.153}$$

Twelve physical states solve this system of equations:

- one transverse and traceless spin two tensor

$$V^{(-1)}_\alpha = \frac{1}{\sqrt{2\alpha'}} \, \alpha_{\mu\nu} \, i\partial X^\mu \, \psi^\nu \, e^{-\phi} \, e^{ik\cdot X} \, , \qquad k^\mu \, \alpha_{\mu\nu} = \alpha_{[\mu\nu]} = \alpha_\mu{}^\mu = 0 \tag{2.154}$$

- one transverse vector

$$V^{(-1)}_d = d_\mu \, \psi^\mu \, \mathcal{J} \, e^{-\phi} \, e^{ik\cdot X} \, , \qquad k^\mu \, d_\mu = 0 \tag{2.155}$$

- two complex scalars

$$V_{\Phi^\pm}^{(-1)} = \frac{\Phi^\pm}{2\sqrt{2\alpha'}} \Big[(\eta_{\mu\nu} \; + \; 2\alpha' \, k_\mu \, k_\nu) \, i\partial X^\mu \, \psi^\nu \; + \; 2\alpha' \, k_\mu \, \partial\psi^\mu \\ \pm \; \frac{i\alpha'}{3} \, \varepsilon_{\mu\nu\lambda\rho} \, \psi^\mu \, \psi^\nu \, \psi^\lambda \, k^\rho \Big] \, e^{-\phi} \, e^{ik\cdot X} \; , \tag{2.156}$$

$$V_{\Omega^\pm}^{(-1)} = \Omega^\pm \, \mathcal{O}^\pm \, e^{-\phi} \, e^{ik\cdot X} \; . \tag{2.157}$$

In addition, we have spurious solutions to the BRST constraints:

$$V_{\pi(\text{sp})}^{(-1)} \;\; \sim \;\; \Big[(\pi_\mu \, k_\nu \; + \; k_\mu \, \pi_\nu) \, i\partial X^\mu \, \psi^\nu \; + \; 2 \, \pi_\mu \, \partial\psi^\mu \Big] \, e^{-\phi} \, e^{ik\cdot X} \, , \qquad k^\mu \, \pi_\mu = 0 \; , \tag{2.158}$$

$$V_{\Sigma(\text{sp})}^{(-1)} \;\; \sim \;\; \Big[2 \, \Sigma_{[\mu\nu]} \, i\partial X^\mu \, \psi^\nu \; + \; 2\alpha' \, \Sigma_{[\mu\nu} \, k_{\lambda]} \, \psi^\mu \, \psi^\nu \, \psi^\lambda \Big] \, e^{-\phi} \, e^{ik\cdot X} \, , \qquad k^\mu \, \Sigma_{\mu\nu} = 0 \; , \tag{2.159}$$

$$V_{\Lambda_0(\text{sp})}^{(-1)} \;\; \sim \;\; \Lambda_0 \Big[(G_{\text{int}}^+ \; - \; G_{\text{int}}^-) \; - \; \sqrt{3\alpha'} \, k_\mu \, \psi^\mu \, \mathcal{J} \Big] \, e^{-\phi} \, e^{ik\cdot X} \; , \tag{2.160}$$

$$V_{\Lambda_1(\text{sp})}^{(-1)} \;\; \sim \;\; \Lambda_1 \Big[(\eta_{\mu\nu} \; + \; 4\alpha' \, k_\mu \, k_\nu) \, i\partial X^\mu \, \psi^\nu \; + \; 6\alpha' \, k_\mu \, \partial\psi^\mu \; , \\ + \; 2\sqrt{\alpha'} \, (G_{\text{int}}^+ \; + \; G_{\text{int}}^-) \Big] \, e^{-\phi} \, e^{ik\cdot X} \; . \tag{2.161}$$

The last two spurious states allow to gauge away both the $c^\pm$ scalars and the longitudinal component of the massive vector $\xi_\mu \sigma k_\mu$.

R sector

For $D = 4$ fermions at mass $m^2 = 1/\alpha'$, the most general vertex operators built from $\mathcal{N} = 1$ internal SCFT fields reads

$$V^{(-\frac{1}{2})} = \Big(v_\mu^a \, i\partial X^\mu \, S_a \, \Sigma^+ \; + \; \bar{\rho}_{\dot{b}}^\mu \, S_\mu^{\dot{b}} \, \Sigma^+ \; + \; u^a \, \partial S_a \, \Sigma^+ \\ + \; y^a \, S_a \, \partial\Sigma^+ \; + \; \bar{\omega}_{\dot{b}} \, S^{\dot{b}} \, \tilde{\Sigma}^+ \Big) \, e^{-\phi/2} \, e^{ik\cdot X} \; , \tag{2.162}$$

see Fig. 3. Invariance under Q_1 yields three independent BRST constraints:

$$
\begin{aligned}
0 &= 2\alpha' \, v^{\mu,a} \, \not{k}_{a\dot{b}} \; + \; \sqrt{2}\, \bar{\rho}^{\mu}_{\dot{b}} \; + \; \tfrac{1}{2}\, u^a \, \sigma^{\mu}_{a\dot{b}} \, , \\
0 &= k_\mu \, \bar{\rho}^{\mu}_{\dot{b}} \; + \; \tfrac{1}{2} \sqrt{\tfrac{3}{2\alpha'}} \, \bar{\omega}_{\dot{b}} \, , \\
0 &= y^a \; + \; 2 \sqrt{\tfrac{\alpha'}{3}} \, \bar{\omega}_{\dot{b}} \; \not{k}^{\dot{b}a} \, .
\end{aligned}
\tag{2.163}
$$

They allow one to express any wavefunction in terms of v^a_μ

$$
\begin{aligned}
u^a &= \alpha' \, v^b_\mu \, (\not{k} \, \bar{\sigma}^\mu)_b{}^a \\
\bar{\rho}_{\mu\dot{b}} &= -\sqrt{2}\, \alpha' \left(v^a_\mu \; \not{k}_{a\dot{b}} \; + \; \tfrac{1}{4}\, v^a_\lambda \, (\not{k} \, \bar{\sigma}^\lambda \, \sigma_\mu)_{a\dot{b}} \right) \\
\bar{\omega}_{\dot{b}} &= \sqrt{\tfrac{\alpha'}{3}} \left(2\alpha' \, k^\mu \, v^a_\mu \; \not{k}_{a\dot{b}} \; - \; v^a_\mu \, \sigma^\mu_{a\dot{b}} \right) \\
y^a &= \frac{2\alpha'}{3} \left(v^b_\mu \, (\sigma^\mu \, \not{k})_b{}^a \; - \; 2\, k^\mu \, v^a_\mu \right) .
\end{aligned}
\tag{2.164}
$$

The same set of states exists with opposite $SO(1,3)$ chirality and internal $U(1)$ charge. Including them, we have four physical solutions to (2.164) and four solutions to the conjugate system of equations:

- two transverse and σ traceless spin 3/2 vector spinors

$$
V^{(-\frac{1}{2})}_{\chi} = \frac{1}{\sqrt{2}\alpha'^{1/4}} \, \chi^a_\mu \left(i\partial X^\mu \, S_a \; - \; \sqrt{2}\, \alpha' \; \not{k}_{a\dot{b}} \, S^{\mu\dot{b}} \right) \Sigma^+ \, e^{-\phi/2} \, e^{ik\cdot X} \, , \tag{2.165}
$$

$$
V^{(-\frac{1}{2})}_{\bar{\chi}} = \frac{1}{\sqrt{2}\alpha'^{1/4}} \, \bar{\chi}^\mu_{\dot{a}} \left(i\partial X_\mu \, S^{\dot{a}} \; - \; \sqrt{2}\, \alpha' \; \not{k}^{\dot{a}b} \, S_{\mu b} \right) \Sigma^- \, e^{-\phi/2} \, e^{ik\cdot X} \, , \tag{2.166}
$$

$$
0 = k^\mu \, \chi^a_\mu = \chi^a_\mu \, \sigma^\mu_{a\dot{b}} = k_\mu \, \bar{\chi}^\mu_{\dot{a}} = \bar{\chi}^\mu_{\dot{a}} \, \bar{\sigma}^{\dot{a}b}_\mu \tag{2.167}
$$

- two spin 1/2 fermions

$$
V^{(-\frac{1}{2})}_{a} = \frac{\alpha'^{1/4}}{2} \, a^b \left((\sigma_\mu \; \not{k})_b{}^a \, S_a \, i\partial X^\mu \; - \; 4\, \partial S_b \right) \Sigma^+ \, e^{-\phi/2} \, e^{ik\cdot X} \, , \tag{2.168}
$$

$$
V^{(-\frac{1}{2})}_{\bar{a}} = \frac{\alpha'^{1/4}}{2} \, \bar{a}_{\dot{b}} \left((\bar{\sigma}_\mu \; \not{k})^{\dot{b}}{}_{\dot{a}} \, S^{\dot{a}} \, i\partial X^\mu \; - \; 4\, \partial S^{\dot{b}} \right) \Sigma^- \, e^{-\phi/2} \, e^{ik\cdot X} \, . \tag{2.169}
$$

Spurious solutions can gauge away the internal excitations with wavefunctions y^a and $\bar{\omega}_{\dot{b}}$:

$$
\begin{aligned}
V^{(-\frac{1}{2})}_{\Theta(\mathrm{sp})} \;&\sim\; \Theta^a \Big[(\not{k}_{a\dot{a}}\, \bar{\sigma}^{\dot{a}b}_{\mu} \;+\; 4k_{\mu}\, \delta^b_a)\, i\partial X^{\mu}\, S_b\, \Sigma^+ \;-\; 2\sqrt{2}\, \big(\alpha'\, k^{\mu}\, \not{k}_{a\dot{b}} \;+\; \tfrac{1}{4}\, \sigma^{\mu}_{\ a\dot{b}}\big)\, S^{\dot{b}}_{\mu}\, \Sigma^+ \\
&\qquad +\; 6\, \partial S_a\, \Sigma^+ \;+\; 4\, S_a\, \partial\Sigma^+ \;-\; 2\sqrt{3\alpha'}\, \not{k}_{a\dot{b}}\, S^{\dot{b}}\, \tilde{\Sigma}^+ \Big]\, e^{-\phi/2}\, e^{ik\cdot X}, \qquad (2.170) \\
V^{(-\frac{1}{2})}_{\bar{\Theta}(\mathrm{sp})} \;&\sim\; \bar{\Theta}_{\dot{b}} \Big[(\not{k}^{\dot{b}a}\, \sigma^{\mu}_{a\dot{a}} \;+\; 4k^{\mu}\, \delta^{\dot{b}}_{\dot{a}})\, i\partial X_{\mu}\, S^{\dot{a}}\, \Sigma^- \;-\; 2\sqrt{2}\, \big(\alpha'\, k_{\mu}\, \not{k}^{\dot{b}a} \;+\; \tfrac{1}{4}\, \bar{\sigma}^{\dot{b}a}_{\mu}\big)\, S^{\mu}_{a}\, \Sigma^- \\
&\qquad +\; 6\, \partial S^{\dot{b}}\, \Sigma^- \;+\; 4\, S^{\dot{b}}\, \partial\Sigma^- \;-\; 2\sqrt{3\alpha'}\, \not{k}^{\dot{b}a}\, S_a\, \tilde{\Sigma}^- \Big]\, e^{-\phi/2}\, e^{ik\cdot X}. \qquad (2.171)
\end{aligned}
$$

3 Massive supermultiplets in the first massive level

In this chapter, we show explicitly the physical states we constructed in the previous chapter form certain supermultiplets for cases $\mathcal{N}=1$, $\mathcal{N}=2$ and $\mathcal{N}=4$ respectively. First of all, we study their supersymmetry transformations; and then, we study in more detail the helicity structure of the various on-shell supermultiplets. Differently than in the previous chapter, we will start from massive states under $\mathcal{N}=1$ compactification since this has the simplest field content. This chapter is based on the paper [3].

3.1 Supersymmetry relations of massive supermultiplets

With all the higher order OPEs (c.f., Appendix B) and physical spectrum in hands, we are able to compute the SUSY transformations by acting with the supercharge operators on the physical states and evaluating the corresponding contour integral.

3.1.1 $\mathcal{N}=1$ supermultiplets

$\mathcal{N}=1$ multiplets are the simplest compare to $\mathcal{N}=2,4$ multiplets because of the Abelian R-symmetry group $U(1)$. The supercharge operators do not carry any R-symmetry indices, only an Abelian charge of $\pm\sqrt{3}/2$.

For our convenience, we choose these SUSY parameters to have mass dimension $[M^{-\frac{1}{2}}]$. As we will verify case by case, action of the supercharges $\mathcal{Q}_a$ and $\bar{\mathcal{Q}}^{\dot{b}}$ given by Eqs. (2.76), (2.91) and (2.92) takes bosonic (fermionic) vertex operators exactly into fermionic (bosonic) vertex operators, including their couplings. The polarization wavefunction of the $\mathcal{Q}$ image state is expressed in terms of $\eta^a, \bar{\eta}_{\dot{a}}$ and the pre-image wavefunction.[13] Once we perform the SUSY variations, besides physical fields in the spectrum, we will also get certain spurious states. We will drop out all these spurious states in our final results for simplicity.

After performing SUSY variation on all the bosonic and fermionic states in $\mathcal{N}=1$ SUSY, we find that these states split into three separate massive supermultiplets – a spin two multiplet $\{\alpha, \chi, \bar{\chi}, d\}$, two spin-$\frac{1}{2}$ multiplets $\{\Phi^+, \bar{a}, \Omega^-\}$ and $\{\Omega^+, a, \Phi^-\}$, see Fig. 5 below. We will show our results of the SUSY transformations in order.

[13] In our settings, all the wavefunctions of bosonic fields have mass dimension 0, and all the wavefunction of fermionic fields have mass dimension $\frac{1}{2}$, see Appendix C for their explicit construction in terms of (massive) spinor helicity variables.

$$
\begin{array}{ccccc}
\Omega^{+} & \longleftrightarrow & a^{b} & \longleftrightarrow & \Phi^{-} \\[2em]
\chi^{a}_{\mu} & \longleftrightarrow & \alpha_{\mu\nu} \oplus d_{\mu} & \longleftrightarrow & \bar{\chi}^{\mu}_{\dot{b}} \\[2em]
\Phi^{+} & \longleftrightarrow & \bar{a}_{\dot{b}} & \longleftrightarrow & \Omega^{-}
\end{array}
$$

Figure 5: The three disconnected $\mathcal{N}=1$ SUSY multiplets at the first mass level: As before, $\mathcal{Q}_a$ ($\bar{\mathcal{Q}}^{\dot{b}}$) action takes states along a left (right) arrow.

SUSY variation of the spin two supermultiplet

The spin two multiplet includes a spin two boson $\alpha_{\mu\nu}$, a vector d_μ, and two spin-$\frac{3}{2}$ fermions χ^a_μ, $\bar{\chi}_{\mu,\dot{a}}$ with opposite chirality. The SUSY transformation of the bosonic states are:

$$
\big[\eta^a \mathcal{Q}_a^{(+\frac{1}{2})}, V_\alpha^{(-1)}\big] = V_\chi^{(-\frac{1}{2})}\Big(\chi^b_\mu = \frac{1}{\sqrt{2}}\eta^a \alpha_{\mu\nu}(\not{k}\bar{\sigma}^\nu)_a{}^b\Big), \tag{3.1}
$$

$$
\big[\bar{\eta}_{\dot{a}} \bar{\mathcal{Q}}^{(+\frac{1}{2}),\dot{a}}, V_\alpha^{(-1)}\big] = V_{\bar{\chi}}^{(-\frac{1}{2})}\Big(\bar{\chi}_{\mu,\dot{b}} = \frac{1}{\sqrt{2}}\bar{\eta}_{\dot{a}} \alpha_{\mu\nu}(\not{k}\sigma^\nu)^{\dot{a}}{}_{\dot{b}}\Big), \tag{3.2}
$$

$$
\big[\eta^a \mathcal{Q}_a^{(+\frac{1}{2})}, V_d^{(-1)}\big] = V_\chi^{(-\frac{1}{2})}\Big(\chi^b_\mu = \frac{-1}{2\sqrt{3\alpha'}}\eta^a \big[3d_\mu \delta_a{}^b + (\not{d}\bar{\sigma}_\mu + \alpha' k_\mu \not{d}\not{k})_a{}^b\big]\Big), \tag{3.3}
$$

$$
\big[\bar{\eta}_{\dot{a}} \bar{\mathcal{Q}}^{(+\frac{1}{2}),\dot{a}}, V_d^{(-1)}\big] = V_{\bar{\chi}}^{(-\frac{1}{2})}\Big(\bar{\chi}_{\mu,\dot{b}} = \frac{1}{2\sqrt{3\alpha'}}\bar{\eta}_{\dot{a}} \big[3d_\mu \delta^{\dot{a}}{}_{\dot{b}} + (\not{d}\sigma_\mu + \alpha' k_\mu \not{d}\not{k})^{\dot{a}}{}_{\dot{b}}\big]\Big). \tag{3.4}
$$

The SUSY transformation of the fermionic states are:

$$
\big[\eta^a \mathcal{Q}_a^{(-\frac{1}{2})}, V_\chi^{(-\frac{1}{2})}\big] = 0, \tag{3.5}
$$

$$
\begin{aligned}
\big[\bar{\eta}_{\dot{a}} \bar{\mathcal{Q}}^{(-\frac{1}{2}),\dot{a}}, V_\chi^{(-\frac{1}{2})}\big] &= V_\alpha^{(-1)}\Big(\alpha_{\mu\nu} = \frac{1}{\sqrt{2}}\bar{\eta}_{\dot{a}}\big(\bar{\sigma}^{\dot{a}a}_{(\mu}\chi_{\nu)a} + \alpha' \not{k}^{\dot{a}a} k_{(\mu}\chi_{\nu),a}\big)\Big) \\
&\quad + V_d^{(-1)}\Big(d_\mu = \frac{\sqrt{3\alpha'}}{2}\bar{\eta}_{\dot{a}} \not{k}^{\dot{a}a}\chi_{\mu,a}\Big),
\end{aligned} \tag{3.6}
$$

$$
\begin{aligned}
\big[\eta^a \mathcal{Q}_a^{(-\frac{1}{2})}, V_{\bar{\chi}}^{(-\frac{1}{2})}\big] &= V_\alpha^{(-1)}\Big(\alpha_{\mu\nu} = \frac{1}{\sqrt{2}}\eta^a\big(\sigma_{(\mu|a\dot{a}|}\bar{\chi}^{\dot{a}}_{\nu)} + \alpha' \not{k}_{a\dot{a}} k_{(\mu}\bar{\chi}^{\dot{a}}_{\nu)}\big)\Big) \\
&\quad + V_d^{(-1)}\Big(d_\mu = -\frac{\sqrt{3\alpha'}}{2}\eta^a \not{k}_{a\dot{a}}\bar{\chi}^{\dot{a}}_\mu\Big),
\end{aligned} \tag{3.7}
$$

$$
\big[\bar{\eta}_{\dot{a}} \bar{\mathcal{Q}}^{(-\frac{1}{2}),\dot{a}}, V_{\bar{\chi}}^{(-\frac{1}{2})}\big] = 0. \tag{3.8}
$$

Note that the signs of the SUSY transformations between spin-$\frac{3}{2}$ and spin one are sensitive to the chirality, see the relative signs between (3.3) and (3.4) as well as (3.6) and (3.7). This is necessary for consistent closure of the SUSY algebra and can be neatly represented by a chirality matrix γ^5 when passing to Dirac spinor notation.

SUSY variation of the spin 1/2 supermultiplets

The first spin-$\frac{1}{2}$ multiplet $\{\Phi^+, \bar{a}, \Omega^-\}$ includes a right-handed spin-$\frac{1}{2}$ fermion $\bar{a}$ and two scalars Φ^+, Ω^-. It is governed by the following SUSY transformations:

$$\left[\eta^b \mathcal{Q}_b^{(+\frac{1}{2})}, V_{\Phi^+}^{(-1)}\right] = 0, \tag{3.9}$$

$$\left[\bar{\eta}_{\dot{b}} \bar{\mathcal{Q}}^{(+\frac{1}{2}),\dot{b}}, V_{\Phi^+}^{(-1)}\right] = V_{\bar{a}}^{(-\frac{1}{2})}\Big(\bar{a}_{\dot{b}} = -\alpha'^{-\frac{1}{2}}\Phi^+\bar{\eta}_{\dot{b}}\Big), \tag{3.10}$$

$$\left[\eta^b \mathcal{Q}_b^{(+\frac{1}{2})}, V_{\Omega^-}^{(-1)}\right] = V_{\bar{a}}^{(-\frac{1}{2})}\Big(\bar{a}_{\dot{b}} = \Omega^- \eta^b \not{k}_{b\dot{b}}\Big), \tag{3.11}$$

$$\left[\bar{\eta}_{\dot{b}} \bar{\mathcal{Q}}^{(+\frac{1}{2}),\dot{b}}, V_{\Omega^-}^{(-1)}\right] = 0, \tag{3.12}$$

and

$$\left[\eta^b \mathcal{Q}_b^{(-\frac{1}{2})}, V_{\bar{a}}^{(-\frac{1}{2})}\right] = V_{\Phi^+}^{(-1)}\Big(\Phi^+ = \sqrt{\alpha'}\eta^b \not{k}_{b\dot{b}}\bar{a}^{\dot{b}}\Big), \tag{3.13}$$

$$\left[\bar{\eta}_{\dot{b}} \bar{\mathcal{Q}}^{(-\frac{1}{2}),\dot{b}}, V_{\bar{a}}^{(-\frac{1}{2})}\right] = V_{\Omega^-}^{(-1)}\Big(\Omega^- = \bar{\eta}_{\dot{b}}\bar{a}^{\dot{b}}\Big). \tag{3.14}$$

For $\{\Omega^+, a, \Phi^-\}$ multiplet of opposite R-symmetry charges and fermion chirality, we obtain

$$\left[\eta^b \mathcal{Q}_b^{(+\frac{1}{2})}, V_{\Phi^-}^{(-1)}\right] = V_{a}^{(-\frac{1}{2})}\Big(a^b = -\alpha'^{-\frac{1}{2}}\Phi^-\eta^b\Big), \tag{3.15}$$

$$\left[\bar{\eta}_{\dot{b}} \bar{\mathcal{Q}}^{(+\frac{1}{2}),\dot{b}}, V_{\Phi^-}^{(-1)}\right] = 0, \tag{3.16}$$

$$\left[\eta^b \mathcal{Q}_b^{(+\frac{1}{2})}, V_{\Omega^+}^{(-1)}\right] = 0, \tag{3.17}$$

$$\left[\bar{\eta}_{\dot{b}} \bar{\mathcal{Q}}^{(+\frac{1}{2}),\dot{b}}, V_{\Omega^+}^{(-1)}\right] = V_{a}^{(-\frac{1}{2})}\Big(a^b = \Omega^+\bar{\eta}_{\dot{b}}\not{k}^{\dot{b}b}\Big), \tag{3.18}$$

and

$$\left[\eta^b \mathcal{Q}_b^{(-\frac{1}{2})}, V_{a}^{(-\frac{1}{2})}\right] = V_{\Omega^+}^{(-1)}\Big(\Omega^+ = \eta^b a_b\Big), \tag{3.19}$$

$$\big[\bar{\eta}_{\dot{b}}\bar{\mathcal{Q}}^{(-\frac{1}{2}),\dot{b}}, V_a^{(-\frac{1}{2})}\big] = V_{\Phi^-}^{(-1)}\Big(\Phi^- = \sqrt{\alpha'}\bar{\eta}_{\dot{b}}\not{k}^{\dot{b}b}a_b\Big). \tag{3.20}$$

We will explore the helicity structure of these results in section 3.2.

3.1.2 $\mathcal{N}=2$ supermultiplets

The charges of $\mathcal{N}=2$ SUSY are spinors of the internal $SU(2)$ R-symmetry and therefore carry an extra index i. In this sector, universal states at the first mass level split into three separate massive supermultiplets – a spin two multiplet $\{\alpha,\chi,\bar{\chi},d,\xi,\beta^{\pm},s,\bar{s},\phi\}$ as well as two spin one multiplets $\{\omega^-,\bar{a},\bar{r},\Phi^+,\zeta^-,\Omega_A^-\}$ and $\{\omega^+,a,r,\Phi^-,\zeta^+,\Omega_A^+\}$, see Fig. 6 below for their structure.

$$\begin{array}{ccccccccccc}
 & & \zeta^+ & \longleftrightarrow & r_i^b & \longleftrightarrow & \omega_\mu^+ \oplus \Omega_A^+ & \longleftrightarrow & a_i^b & \longleftrightarrow & \Phi^- & & \\
\beta_\mu^+ & \longleftrightarrow & & & \chi_{\mu,i}^a \oplus \bar{s}_{\dot{b},i} & \longleftrightarrow & \begin{array}{c}\alpha_{\mu\nu} \oplus d_\mu^A \\ \oplus\, \xi_\mu \oplus \phi\end{array} & \longleftrightarrow & \bar{\chi}^\mu_{\dot{b},i} \oplus s_i^a & \longleftrightarrow & & & \beta_\mu^- \\
 & & \Phi^+ & \longleftrightarrow & \bar{a}_{\dot{b},i} & \longleftrightarrow & \omega_\mu^- \oplus \Omega_A^- & \longleftrightarrow & \bar{r}_{\dot{b},i} & \longleftrightarrow & \zeta^- & &
\end{array}$$

Figure 6: Three disconnected $\mathcal{N}=2$ SUSY multiplets

SUSY variation of the spin two supermultiplet

The spin two multiplet includes a spin two boson $\alpha_{\mu\nu}$, six vectors ξ_μ, $d_\mu^{A=1,2,3}$, $\beta_\mu^{\pm}$, one scalar ϕ, two spin-$\frac{3}{2}$ fermions χ_μ^a, $\bar{\chi}_{\mu,\dot{a}}$ and two spin-$\frac{1}{2}$ fermions s^a, $\bar{s}_{\dot{a}}$. Their SUSY transformations are:

$$\big[\eta_i^a\mathcal{Q}_a^{(+\frac{1}{2}),i}, V_\alpha^{(-1)}\big] = V_\chi^{(-\frac{1}{2})}\Big(\chi_{\mu,i}^b = \frac{1}{\sqrt{2}}\eta_i^a\alpha_{\mu\nu}(\not{k}\bar{\sigma}^\nu)_a{}^b\Big), \tag{3.21}$$

$$\big[\bar{\eta}_{\dot{a},i}\bar{\mathcal{Q}}^{(+\frac{1}{2}),\dot{a},i}, V_\alpha^{(-1)}\big] = V_{\bar{\chi}}^{(-\frac{1}{2})}\Big(\bar{\chi}_{\mu,\dot{a},i} = \frac{1}{\sqrt{2}}\bar{\eta}_{\dot{a},i}\alpha_{\mu\nu}(\not{k}\sigma^\nu)^{\dot{a}}{}_{\dot{b}}\Big). \tag{3.22}$$

For the four spin one fields, we have the following results – the SUSY variations of ξ_μ field read,

$$\begin{aligned}\big[\eta_i^a\mathcal{Q}_a^{(+\frac{1}{2}),i}, V_\xi^{(-1)}\big] = V_\chi^{(-\frac{1}{2})}&\Big(\chi_{\mu,i}^b = -\frac{1}{6\sqrt{\alpha'}}\eta_i^a\big[3\xi_\mu\delta_a{}^b + (\not{\xi}\bar{\sigma}_\mu + \alpha'\not{\xi}\not{k}k_\mu)_a{}^b\big]\Big)\\
&+ V_{\bar{s}}^{(-\frac{1}{2})}\Big(\bar{s}_{\dot{a},i} = -\frac{1}{\sqrt{3\alpha'}}\eta_i^a\not{\xi}_{a\dot{a}}\Big),\end{aligned} \tag{3.23}$$

$$\left[\bar{\eta}_{\dot{a},i}\bar{\mathcal{Q}}^{(+\frac{1}{2}),\dot{a},i}, V_{\xi}^{(-1)}\right] = V_{\bar{\chi}}^{(-\frac{1}{2})}\Big(\bar{\chi}_{\mu,\dot{b},i} = \frac{1}{6\sqrt{\alpha'}}\bar{\eta}_{\dot{a},i}\big[3\xi_\mu\delta^{\dot{a}}{}_{\dot{b}} + (\not{\xi}\bar{\sigma}_\mu + \alpha'\not{\xi}\not{k}k_\mu)^{\dot{a}}{}_{\dot{b}}\big]\Big)$$

$$+ V_s^{(-\frac{1}{2})}\Big(s_i^a = \frac{1}{\sqrt{3\alpha'}}\bar{\eta}_{\dot{a},i}\not{\xi}^{\dot{a}a}\Big), \tag{3.24}$$

the $SU(2)$ triplet d_μ^A transforms to,

$$\left[\eta_i^a\mathcal{Q}_a^{(+\frac{1}{2}),i}, V_d^{(-1)}\right] = V_{\chi}^{(-\frac{1}{2})}\Big(\chi_{\mu,j}^b = -\frac{1}{3\sqrt{2\alpha'}}\eta_i^a\big[3d_\mu^A\delta_a{}^b + (\not{d}^A\bar{\sigma}_\mu + \alpha' k_\mu\not{d}^A\not{k})_a{}^b\big](\tau_A)^i{}_j\Big)$$

$$+ V_{\bar{s}}^{(-\frac{1}{2})}\Big(\bar{s}_{\dot{a},j} = \frac{1}{\sqrt{6\alpha'}}\eta_i^a\not{d}^A_{a\dot{a}}(\tau_A)^i{}_j\Big), \tag{3.25}$$

$$\left[\bar{\eta}_{\dot{a},i}\bar{\mathcal{Q}}^{(+\frac{1}{2}),\dot{a},i}, V_d^{(-1)}\right] = V_{\bar{\chi}}^{(-\frac{1}{2})}\Big(\bar{\chi}_{\mu,\dot{b},i} = -\frac{1}{3\sqrt{2\alpha'}}\bar{\eta}_{\dot{a},i}\big[3d_\mu^A\delta^{\dot{a}}{}_{\dot{b}} + (\not{d}^A\bar{\sigma}_\mu + \alpha' k_\mu\not{d}^A\not{k})^{\dot{a}}{}_{\dot{b}}\big](\tau_A)^i{}_j\Big)$$

$$+ V_s^{(-\frac{1}{2})}\Big(s_j^a = \frac{1}{\sqrt{6\alpha'}}\bar{\eta}_{\dot{a},i}\not{d}^{A,\dot{a}a}(\tau_A)^i{}_j\Big), \tag{3.26}$$

and the complex vectors $\beta_\mu^\pm$ are varied to,[14]

$$\left[\eta_i^a\mathcal{Q}_a^{(+\frac{1}{2}),i}, V_{\beta^+}^{(-1)}\right] = 0, \tag{3.27}$$

$$\left[\bar{\eta}_{\dot{b},i}\bar{\mathcal{Q}}^{(+\frac{1}{2}),\dot{b},i}, V_{\beta^+}^{(-1)}\right] = V_{\chi}^{(-\frac{1}{2})}\Big(\chi_{\mu,i}^b = \frac{1}{3}\bar{\eta}_{\dot{b},i}\big[3\beta_\mu^+\not{k}^{\dot{b}b} - (k_\mu\not{\beta}^+ + \not{\beta}^+\not{k}\bar{\sigma}_\mu)^{\dot{b}b}\big]\Big)$$

$$+ V_{\bar{s}}^{(-\frac{1}{2})}\Big(\bar{s}_{\dot{c},i} = \frac{1}{\sqrt{3}}\bar{\eta}_{\dot{b},i}(\not{\beta}^+\not{k})^{\dot{b}}{}_{\dot{c}}\Big), \tag{3.28}$$

$$\left[\eta_i^b\mathcal{Q}_b^{(+\frac{1}{2}),i}, V_{\beta^-}^{(-1)}\right] = V_{\bar{\chi}}^{(-\frac{1}{2})}\Big(\bar{\chi}_{\mu,\dot{b},i} = \frac{1}{3}\eta_i^b\big[3\beta_\mu^-\not{k}_{b\dot{b}} - (k_\mu\not{\beta}^- + \not{\beta}^-\not{k}\sigma_\mu)_{b\dot{b}}\big]\Big)$$

$$+ V_s^{(-\frac{1}{2})}\Big(s_i^c = \frac{1}{\sqrt{3}}\eta_i^b(\not{\beta}^-\not{k})_b{}^c\Big), \tag{3.29}$$

$$\left[\bar{\eta}_{\dot{a},i}\bar{\mathcal{Q}}^{(+\frac{1}{2}),\dot{a},i}, V_{\beta^-}^{(-1)}\right] = 0\ . \tag{3.30}$$

The SUSY action on the unique scalar field ϕ is given by

$$\left[\eta_i^a\mathcal{Q}_a^{(+\frac{1}{2}),i}, V_\phi^{(-1)}\right] = V_{\bar{s}}^{(-\frac{1}{2})}\Big(\bar{s}_{\dot{a},i} = \frac{1}{\sqrt{2}}\phi\eta_i^a\not{k}_{a\dot{a}}\Big), \tag{3.31}$$

$$\left[\bar{\eta}_{\dot{a},i}\bar{\mathcal{Q}}^{(+\frac{1}{2}),\dot{a},i}, V_\phi^{(-1)}\right] = V_s^{(-\frac{1}{2})}\Big(s_i^a = \frac{1}{\sqrt{2}}\phi\bar{\eta}_{\dot{a},i}\not{k}^{\dot{a}a}\Big). \tag{3.32}$$

[14] Cocycles would introduce additional minus signs in the computations (and several analogous ones at later points). However, we are able to eliminate these extra minus signs in a consistent way.

Now we turn to analyze the fermionic states. For χ and $\bar{\chi}$ at spin-$\frac{3}{2}$, we have SUSY relations,

$$\big[\eta^a_i \mathcal{Q}^{(-\frac{1}{2}),i}_a, V^{(-\frac{1}{2})}_\chi\big] = V^{(-1)}_{\beta^+}\Big(\beta^+_\mu = \eta^a_i \chi_{\mu,a,i}\Big), \tag{3.33}$$

$$\begin{aligned}\big[\bar\eta_{\dot a,i} \bar{\mathcal{Q}}^{(-\frac{1}{2}),\dot a,i}, V^{(-\frac{1}{2})}_\chi\big] &= V^{(-1)}_{\alpha}\Big(\alpha_{\mu\nu} = \frac{1}{\sqrt{2}}\bar\eta_{\dot a,i}\big(\bar\sigma^{\dot a a}_{(\mu}\chi^i_{\nu),a} + \alpha' \not{k}^{\dot a a}\chi^i_{(\mu|,a|}k_{\nu)}\big)\Big) \\ &\quad + V^{(-1)}_{\xi}\Big(\xi_\mu = -\frac{\sqrt{\alpha'}}{2}\bar\eta_{\dot a,i}\not{k}^{\dot a a}\chi^i_{\mu,a}\Big) \\ &\quad + V^{(-1)}_{d}\Big(d^A_\mu = \sqrt{\frac{\alpha'}{2}}\bar\eta_{\dot a,i}\not{k}^{\dot a a}\chi_{\mu,a,j}(\tau^A\varepsilon)^{ij}\Big),\end{aligned} \tag{3.34}$$

and

$$\begin{aligned}\big[\eta^a_i \mathcal{Q}^{(-\frac{1}{2}),i}_a, V^{(-\frac{1}{2})}_{\bar\chi}\big] &= V^{(-1)}_{\alpha}\Big(\alpha_{\mu\nu} = \frac{1}{\sqrt{2}}\eta^a_i\big(\sigma_{(\mu|a\dot a|}\bar\chi^{\dot a,i}_{\nu)} + \alpha'\not{k}_{a\dot a}\bar\chi^{\dot a,i}_{(\mu}k_{\nu)}\big)\Big) \\ &\quad + V^{(-1)}_{\xi}\Big(\xi_\mu = \frac{\sqrt{\alpha'}}{2}\eta^a_i\not{k}_{a\dot a}\bar\chi^{\dot a,i}_\mu\Big) \\ &\quad + V^{(-1)}_{d}\Big(d^A_\mu = \sqrt{\frac{\alpha'}{2}}\eta^a_i\not{k}_{a\dot a}\bar\chi^{\dot a}_{\mu,j}(\tau^A\varepsilon)^{ij}\Big),\end{aligned} \tag{3.35}$$

$$\big[\bar\eta_{\dot a,i} \bar{\mathcal{Q}}^{(-\frac{1}{2}),\dot a,i}, V^{(-\frac{1}{2})}_{\bar\chi}\big] = V^{(-1)}_{\beta^-}\Big(\beta^-_\mu = \bar\eta_{\dot a,i}\bar\chi^{\dot a,i}_\mu\Big). \tag{3.36}$$

The spin-$\frac{1}{2}$ states s and $\bar{s}$, on the other hand, transform to

$$\begin{aligned}\big[\eta^a_i \mathcal{Q}^{(-\frac{1}{2}),i}_a, V^{(-\frac{1}{2})}_{s}\big] &= V^{(-1)}_{\xi}\Big(\xi_\mu = \sqrt{\frac{\alpha'}{3}}\eta^a_i\big[k_\mu\delta_a{}^b + (\sigma_\mu\not{k})_a{}^b\big]s^i_b\Big) \\ &\quad + V^{(-1)}_{d}\Big(d^A_\mu = \sqrt{\frac{\alpha'}{6}}\eta^a_i\big[k_\mu\delta_a{}^b + (\sigma_\mu\not{k})_a{}^b\big]s_{b,j}(\tau^A\varepsilon)^{ij}\Big) \\ &\quad + V^{(-1)}_{\phi}\Big(\phi = \frac{1}{\sqrt{2}}\eta^a_i s^i_a\Big),\end{aligned} \tag{3.37}$$

$$\big[\bar\eta_{\dot a,i} \bar{\mathcal{Q}}^{(-\frac{1}{2}),\dot a,i}, V^{(-\frac{1}{2})}_{s}\big] = V^{(-1)}_{\beta^-}\Big(\beta^-_\mu = -\frac{1}{\sqrt{3}}\bar\eta_{\dot a,i}\big(\bar\sigma^{\dot a a}_\mu + \alpha' k_\mu\not{k}^{\dot a a}\big)s^i_a\Big), \tag{3.38}$$

and

$$\big[\eta^a_i \mathcal{Q}^{(-\frac{1}{2}),i}_a, V^{(-\frac{1}{2})}_{\bar s}\big] = V^{(-1)}_{\beta^+}\Big(\beta^+_\mu = -\frac{1}{\sqrt{3}}\eta^a_i\big(\sigma_{\mu a\dot a} + \alpha' k_\mu\not{k}_{a\dot a}\big)\bar{s}^{\dot a,i}\Big), \tag{3.39}$$

$$\begin{aligned}\big[\bar\eta_{\dot a,i} \bar{\mathcal{Q}}^{(-\frac{1}{2}),\dot a,i}, V^{(-\frac{1}{2})}_{\bar s}\big] &= V^{(-1)}_{\xi}\Big(\xi_\mu = \sqrt{\frac{\alpha'}{3}}\bar\eta_{\dot a,i}\big[k_\mu\delta^{\dot a}{}_{\dot b} + (\bar\sigma_\mu\not{k})^{\dot a}{}_{\dot b}\big]\bar{s}^{\dot b,i}\Big) \\ &\quad + V^{(-1)}_{d}\Big(d^A_\mu = \sqrt{\frac{\alpha'}{6}}\bar\eta_{\dot a,i}\big[k_\mu\delta^{\dot a}{}_{\dot b} + (\bar\sigma_\mu\not{k})^{\dot a}{}_{\dot b}\big]\bar{s}^{\dot b,i}(\tau^A\varepsilon)^{ij}\Big)\end{aligned}$$

$$+ V_{\phi}^{(-1)}\Big(\phi = \frac{1}{\sqrt{2}}\bar{\eta}_{\dot{a},i}\bar{r}_{+}^{\dot{a},i}\Big). \tag{3.40}$$

SUSY variation of the spin one supermultiplets

The first spin one multiplet $\{\omega^-, \bar{a}, \bar{r}, \Phi^+, \zeta^-, \Omega_A^-\}$ contains one vector ω_μ^-, two right-handed fermions $\bar{a}_{\dot{b}}$ and $\bar{r}_{\dot{b}}$ of spin 1/2 each, and three scalars Φ^+, ζ^- and Ω_A^-. The SUSY relations for the spin one ω_μ^- read,

$$\big[\eta_i^b \mathcal{Q}_b^{(+\frac{1}{2}),i}, V_{\omega^-}^{(-1)}\big] = V_{\bar{a}}^{(-1)}\Big(\bar{a}_{\dot{b},i} = -\frac{1}{\sqrt{2\alpha'}}\eta_i^b \not{\omega}^-_{b\dot{b}}\Big), \tag{3.41}$$

$$\big[\bar{\eta}_{\dot{a},i}\bar{\mathcal{Q}}^{(+\frac{1}{2}),\dot{a},i}, V_{\omega^-}^{(-1)}\big] = V_{\bar{r}}^{(+\frac{1}{2})}\Big(\bar{r}_{\dot{b},i} = -\frac{1}{\sqrt{2}}\bar{\eta}_{\dot{a},i}(\not{\omega}^- \not{k})^{\dot{a}}{}_{\dot{b}}\Big). \tag{3.42}$$

For the fermions $\bar{a}_{\dot{b}}$ and $\bar{r}_{\dot{b}}$, we have,

$$\big[\eta_i^b \mathcal{Q}_b^{(-\frac{1}{2}),i}, V_{\bar{a}}^{(-\frac{1}{2})}\big] = V_{\Phi^+}^{(-1)}\Big(\Phi^+ = \sqrt{\alpha'}\eta_i^b \not{k}_{b\dot{b}}\bar{a}^{\dot{b},i}\Big), \tag{3.43}$$

$$\begin{aligned}\big[\bar{\eta}_{\dot{b},i}\bar{\mathcal{Q}}^{(-\frac{1}{2}),\dot{b},i}, V_{\bar{a}}^{(-\frac{1}{2})}\big] &= V_{\omega^-}^{(-1)}\Big(\omega_\mu^- = \sqrt{\frac{\alpha'}{2}}\bar{\eta}_{\dot{b},i}\big[k_\mu \delta^{\dot{b}}{}_{\dot{c}} + (\not{k}\sigma_\mu)^{\dot{b}}{}_{\dot{c}}\big]\bar{a}^{\dot{c},i}\Big) \\ &\quad + V_{\Omega^-}^{(-1)}\Big(\Omega_A^- = -\frac{1}{\sqrt{2}}\bar{\eta}_{\dot{b},i}(\tau_A\varepsilon)^{ij}\bar{a}_j^{\dot{b}}\Big),\end{aligned} \tag{3.44}$$

and

$$\begin{aligned}\big[\eta_i^a \mathcal{Q}_a^{(-\frac{1}{2}),i}, V_{\bar{r}}^{(-\frac{1}{2})}\big] &= V_{\omega^-}^{(-1)}\Big(\omega_\mu^- = \frac{1}{\sqrt{2}}\eta_i^a\big(\sigma_{\mu a\dot{a}} + \alpha' k_\mu \not{k}_{a\dot{a}}\big)\bar{r}^{\dot{a},i}\Big) \\ &\quad + V_{\Omega^-}^{(-1)}\Big(\Omega_A^- = \sqrt{\frac{\alpha'}{2}}\eta_i^a \not{k}_{a\dot{a}}\bar{r}_j^{\dot{a}}(\tau_A\varepsilon)^{ij}\Big),\end{aligned} \tag{3.45}$$

$$\big[\bar{\eta}_{\dot{a},i}\bar{\mathcal{Q}}^{(-\frac{1}{2}),\dot{a},i}, V_{\bar{r}}^{(-\frac{1}{2})}\big] = V_{\zeta^-}^{(-1)}\Big(\zeta^- = \bar{\eta}_{\dot{a},i}\bar{r}^{\dot{a},i}\Big). \tag{3.46}$$

The results for the scalar fields are:

$$\big[\eta_i^b \mathcal{Q}_b^{(+\frac{1}{2}),i}, V_{\Phi^+}^{(-1)}\big] = 0, \tag{3.47}$$

$$\big[\bar{\eta}_{\dot{b},i}\bar{\mathcal{Q}}^{(+\frac{1}{2}),\dot{b},i}, V_{\Phi^+}^{(-1)}\big] = V_{\bar{a}}^{(-\frac{1}{2})}\Big(\bar{a}_{\dot{b},i} = -\alpha'^{-\frac{1}{2}}\Phi^+\bar{\eta}_{\dot{b},i}\Big), \tag{3.48}$$

and

$$\big[\eta^a_i\mathcal{Q}^{(+\frac{1}{2}),i}_a, V^{(-1)}_{\zeta^-}\big] = V^{(-\frac{1}{2})}_{\bar{r}}\Big(\bar{r}_{\dot{a},i} = \zeta^-\eta^a_i \not{k}_{a\dot{a}}\Big), \tag{3.49}$$

$$\big[\bar{\eta}_{\dot{a},i}\bar{\mathcal{Q}}^{(+\frac{1}{2}),\dot{a},i}, V^{(-1)}_{\zeta^-}\big] = 0, \tag{3.50}$$

and

$$\big[\eta^b_i\mathcal{Q}^{(+\frac{1}{2}),i}_b, V^{(-1)}_{\Omega^-}\big] = V^{(-\frac{1}{2})}_{\bar{a}}\Big(\bar{a}_{\dot{b},j} = -\frac{1}{\sqrt{2}}\eta^b_i \not{k}_{b\dot{b}}\Omega^-_A(\tau^A)^i{}_j\Big), \tag{3.51}$$

$$\big[\bar{\eta}_{\dot{a},i}\bar{\mathcal{Q}}^{(+\frac{1}{2}),\dot{a},i}, V^{(-1)}_{\Omega^-}\big] = V^{(-\frac{1}{2})}_{\bar{r}}\Big(\bar{r}_{\dot{a},j} = \frac{1}{\sqrt{2\alpha'}}\bar{\eta}_{\dot{a},i}\Omega^-_A(\tau^A)^i{}_j\Big). \tag{3.52}$$

The second spin one multiplet $\{\omega^+, a, r, \Phi^-, \zeta^+, \Omega^+_A\}$ is just the complex conjugate of the former, so let us simply list the analogous SUSY transformations:

$$\big[\eta^a_i\mathcal{Q}^{(+\frac{1}{2}),i}_a, V^{(-1)}_{\omega^+}\big] = V^{(+\frac{1}{2})}_{r}\Big(r^b_i = -\frac{1}{\sqrt{2}}\eta^a_i(\not{\omega}^+\not{k})_a{}^b\Big), \tag{3.53}$$

$$\big[\bar{\eta}_{\dot{b},i}\bar{\mathcal{Q}}^{(+\frac{1}{2}),\dot{b},i}, V^{(-1)}_{\omega^+}\big] = V^{(-1)}_{a}\Big(a^b_i = -\frac{1}{\sqrt{2\alpha'}}\bar{\eta}_{\dot{b},i}\not{\omega}^{+,\dot{b}b}\Big), \tag{3.54}$$

$$\begin{aligned}\big[\eta^b_i\mathcal{Q}^{(-\frac{1}{2}),i}_b, V^{(-\frac{1}{2})}_{a}\big] &= V^{(-1)}_{\omega^+}\Big(\omega^+_\mu = \sqrt{\frac{\alpha'}{2}}\eta^b_i\big[k_\mu\delta_b{}^c + (\not{k}\bar{\sigma}_\mu)_b{}^c\big]a^i_c\Big) \\ &\quad + V^{(-1)}_{\Omega^+}\Big(\Omega^+_A = -\frac{1}{\sqrt{2}}\eta^b_i(\tau_A\varepsilon)^{ij}a_{b,j}\Big),\end{aligned} \tag{3.55}$$

$$\big[\bar{\eta}_{\dot{b},i}\bar{\mathcal{Q}}^{(-\frac{1}{2}),\dot{b},i}, V^{(-\frac{1}{2})}_{a}\big] = V^{(-1)}_{\Phi^-}\Big(\Phi^- = \sqrt{\alpha'}\bar{\eta}_{\dot{b},i}\not{k}^{\dot{b}b}a^i_b\Big), \tag{3.56}$$

$$\big[\eta^a_i\mathcal{Q}^{(-\frac{1}{2}),i}_a, V^{(-\frac{1}{2})}_{r}\big] = V^{(-1)}_{\zeta^+}\Big(\zeta^+ = \eta^a_i r^i_a\Big), \tag{3.57}$$

$$\begin{aligned}\big[\bar{\eta}_{\dot{a},i}\bar{\mathcal{Q}}^{(-\frac{1}{2}),\dot{a},i}, V^{(-\frac{1}{2})}_{r}\big] &= V^{(-1)}_{\omega^+}\Big(\omega^+_\mu = \frac{1}{\sqrt{2}}\bar{\eta}_{\dot{a},i}\big(\bar{\sigma}^{\dot{a}a}_\mu + \alpha' k_\mu\not{k}^{\dot{a}a}\big)r^i_a\Big) \\ &\quad + V^{(-1)}_{\Omega^+}\Big(\Omega^+_A = \sqrt{\frac{\alpha'}{2}}\bar{\eta}_{\dot{a},i}\not{k}^{\dot{a}a}r_{a,j}(\tau_A\varepsilon)^{ij}\Big),\end{aligned} \tag{3.58}$$

$$\big[\eta^b_i\mathcal{Q}^{(+\frac{1}{2}),i}_b, V^{(-1)}_{\Phi^-}\big] = V^{(-\frac{1}{2})}_{a}\Big(a^b_i = -\alpha'^{-\frac{1}{2}}\Phi^-\eta^b_i\Big), \tag{3.59}$$

$$\big[\bar{\eta}_{\dot{b},i}\bar{\mathcal{Q}}^{(+\frac{1}{2}),\dot{b},i}, V^{(-1)}_{\Phi^-}\big] = 0, \tag{3.60}$$

$$\big[\eta^a_i\mathcal{Q}^{(+\frac{1}{2}),i}_a, V^{(-1)}_{\zeta^+}\big] = 0, \tag{3.61}$$

$$\big[\bar{\eta}_{\dot{a},i}\bar{\mathcal{Q}}^{(+\frac{1}{2}),\dot{a},i}, V^{(-1)}_{\zeta^+}\big] = V^{(-\frac{1}{2})}_{r}\Big(r^a_i = \zeta^+\bar{\eta}_{\dot{a},i}\not{k}^{\dot{a}a}\Big), \tag{3.62}$$

$$\left[\eta_i^a \mathcal{Q}_a^{(+\frac{1}{2}),i}, V_{\Omega^+}^{(-1)}\right] = V_r^{(-\frac{1}{2})}\Big(r_j^a = \frac{1}{\sqrt{2\alpha'}}\eta_i^a \Omega_A^+ (\tau^A)^i{}_j\Big), \tag{3.63}$$

$$\left[\bar\eta_{\dot b,i} \bar{\mathcal{Q}}^{(+\frac{1}{2}),\dot b,i}, V_{\Omega^+}^{(-1)}\right] = V_a^{(-\frac{1}{2})}\Big(a_j^b = -\frac{1}{\sqrt{2}}\bar\eta_{\dot b,i} \not{k}^{\dot b b}\Omega_A^+ (\tau^A)^i{}_j\Big). \tag{3.64}$$

3.1.3 $\mathcal{N} = 4$ supermultiplet

In $\mathcal{N} = 4$ SUSY, the SUSY parameters $\eta_I^a, \bar\eta_{\dot a}^{\bar I}$ are chiral spinors of both the $SO(1,3)$ Lorentz group and the internal $SO(6)$ R-symmetry group. All the physical states form one big supermultiplet of $\mathcal{N} = 4$. The structure of the explicit SUSY variations listed in this section is summarized in Fig. 7 below. This diagram will be refined in section 3.2 to take helicity quantum numbers into account.

$$\Phi^+ \quad \leftrightarrow \quad \bar a_{\dot b}^{\bar J} \quad \leftrightarrow \quad \begin{matrix} \beta_\mu^{-,m} \\ \Omega_-^{mnp} \end{matrix} \quad \leftrightarrow \quad \begin{matrix} \chi_{\mu,I}^a \\ \bar r_{\dot b,m}^{\bar J} \end{matrix} \quad \leftrightarrow \quad \begin{matrix} \alpha_{\mu\nu} \\ d_\mu^{mn} \\ \zeta^{mn} \end{matrix} \quad \leftrightarrow \quad \begin{matrix} \bar\chi_{\dot b}^{\mu,\bar J} \\ r_I^{a,m} \end{matrix} \quad \leftrightarrow \quad \begin{matrix} \beta_\mu^{+,m} \\ \Omega_+^{mnp} \end{matrix} \quad \leftrightarrow \quad a_I^b \quad \leftrightarrow \quad \Phi^-$$

Figure 7: $\mathcal{N} = 4$ SUSY multiplet: action of the left-handed SUSY charge $\mathcal{Q}_a^I$ transforms a state into (a combination of) its left neighbors, whereas $\bar{\mathcal{Q}}_{\bar J}^{\dot b}$ action maps states into right neighbors.

The pattern of SUSY variations depicted in Fig. 7 justifies the complex combinations (2.98) of vectors and (2.102) of scalars: The complex conjugates appear on widely separated positions of the multiplet (i.e. the β^+ and β^- are separated by four $\mathcal{Q}$ actions whereas $\Phi^+ \leftrightarrow \Phi^-$ requires eight supercharge applications). Also, the internal scalar Ω^{mnp} splits into self-dual and anti-self-dual components $\Omega_\pm^{mnp}$ which sit at different points of the multiplet.

There are group theoretic selection rules for the possible outcome of a physical state's SUSY variations, based on the $SO(1,3)\times SO(6)$ symmetry. Firstly, according to its eigenvalue under diagonal Lorentz currents, $\mathcal{Q}$ can only change the spin by $\pm\frac{1}{2}$. Secondly, transformations have to compatible with the $SO(6)$ quantum numbers involved. Representation of the $SO(6) \equiv SU(4)$ R-symmetry group are referred to using their Dynkin Labels $[k,p,q]$.[15] The SUSY variation of a state $\in [k,p,q]$ aligns into the tensor product with $[0,1,0] \ni \mathcal{Q}^I$ or $[0,0,1] \ni \bar{\mathcal{Q}}_{\bar J}$ of the SUSY charge. Table 1 gives an overview of the R-symmetry representations involved (see the following subsection for the $\Omega^\pm$ splitting).

[15] Our conventions for the Dynkin labels $[k,p,q]$ are such that $[1,0,0]$ labels the vector representation, and $[0,1,0]$ and $[0,0,1]$ are left- and right-handed spinor. A generic representation with labels $[k,p,q]$ has dimension

$$D_{[k,p,q]} = \frac{1}{12}(k+p+q+3)(k+p+2)(k+q+2)(k+1)(p+1)(q+1)\ , \tag{3.65}$$

and tensor products act as follows on Dynkin labels:

$$[k,p,q]\otimes[0,1,0] = [k,p,q-1]\oplus[k,p+1,q]\oplus[k+1,p-1,q]\oplus[k-1,p,q+1], \tag{3.66}$$

$$[k,p,q]\otimes[0,0,1] = [k,p,q+1]\oplus[k,p-1,q]\oplus[k+1,p,q-1]\oplus[k-1,p+1,q], \tag{3.67}$$

$$\begin{aligned}[k,p,q]\otimes[1,0,0] = {}& [k,p+1,q-1]\oplus[k,p-1,q+1]\oplus[k+1,p,q] \\ & \oplus[k+1,p-1,q-1]\oplus[k-1,p,q]\oplus[k-1,p+1,q+1].\end{aligned} \tag{3.68}$$

Spin	Wavefunctions	$SO(6)$ rep.	Spin	Wavefunctions	$SO(6)$ rep.
2	$\alpha_{\mu\nu}$	[0,0,0]	3/2	$\chi^a_{\mu,I}$	[0,1,0]
1	$\beta^{\pm,m}_\mu$	[1,0,0]	3/2	$\bar{\chi}^{\mu,J}_{\dot{b}}$	[0,0,1]
1	$d^{[mn]}_\mu$	[0,1,1]	1/2	$r^a_{m,I}$	[1,1,0]
0	$\zeta^{(mn)}$	[2,0,0]	1/2	$\bar{r}^{m,J}_{\dot{b}}$	[1,0,1]
0	Ω^+_{mnl}	[0,2,0]	1/2	a^b_I	[0,1,0]
0	Ω^-_{mnl}	[0,0,2]	1/2	$\bar{a}^J_{\dot{b}}$	[0,0,1]
0	$\Phi^\pm$	[0,0,0]			

Table 1: R-symmetry content of the massive $\mathcal{N}=4$ multiplet in $SO(6)$ Dynkin label notation

SUSY transformation of bosonic states

In this subsubsection, we will analyze supercharge acting on the bosonic states. The spin two field $\alpha_{\mu\nu}$ transforms into left- and right-handed spin-$\frac{3}{2}$ fermions $\mathcal{Q}\alpha \to \chi$ and $\bar{\mathcal{Q}}\alpha \to \bar{\chi}$ in lines with $[0,0,0]\otimes[0,1,0] \to [0,1,0]$ for the R-symmetry scalar $\alpha_{\mu\nu}$. The SUSY variations of this field are parallel to (2.27) in ten dimensions:

$$\big[\eta^a_I \mathcal{Q}^{(+\frac{1}{2}),I}_a, V^{(-1)}_\alpha\big] = V^{(-\frac{1}{2})}_\chi \Big(\chi^b_{\mu,I} = \frac{1}{\sqrt{2}}\eta^a_I \alpha_{\mu\nu}(\not{k}\bar{\sigma}^\nu)_a{}^b\Big), \tag{3.69}$$

$$\big[\bar{\eta}^{\bar{I}}_{\dot{a}} \bar{\mathcal{Q}}^{(+\frac{1}{2}),\dot{a}}_{\bar{I}}, V^{(-1)}_\alpha\big] = V^{(-\frac{1}{2})}_{\bar{\chi}} \Big(\bar{\chi}^{\bar{I}}_{\mu,\dot{a}} = \frac{1}{\sqrt{2}}\bar{\eta}^{\bar{I}}_{\dot{a}} \alpha_{\mu\nu}(\not{k}\sigma^\nu)^{\dot{a}}{}_{\dot{b}}\Big). \tag{3.70}$$

The spin one fields fall into vector and two-form representations $[1,0,0]$ and $[0,1,1]$ of the R-symmetry, so their SUSY image belongs to $[0,1,0]\otimes[1,0,0] \to [1,1,0]\oplus[0,0,1]$ and $[0,1,0]\otimes[0,1,1] \to [0,1,0]\oplus[0,2,1]\oplus[1,0,1]$, respectively (note that $[0,2,1]$ does not occur in our multiplet). This implies that $\beta^{\pm,m}_\mu$ can transform into an internal left-handed fermion $r^a_{m,I} \in [1,1,0]$, and right-handed spin-$\frac{3}{2}$ fermions $\bar{\chi}^{\bar{I}}_{\mu\dot{a}}$ or a spin-$\frac{1}{2}$ fermions $\bar{a}^{\bar{I}}_{\dot{b}}$, in short: $\mathcal{Q}\beta^\pm \to \bar{\chi}+\bar{a}+r$. For the $SO(6)$ two-form $d^{[mn]}$, we will get the opposite chirality configuration, $\mathcal{Q}d \to \chi + a + \bar{r}$. The explicit results for the left-handed $\mathcal{Q}^I_a$ are given as follows,[16]

$$\begin{aligned}\big[\eta^b_I \mathcal{Q}^{(+\frac{1}{2}),I}_b, V^{(-1)}_{\beta^+}\big] &= V^{(-\frac{1}{2})}_{\bar{\chi}} \Big(\bar{\chi}^{\bar{I}}_{\mu,\dot{b}} = \frac{1}{3\sqrt{2}}\eta^b_I\big[3\beta^{+,m}_\mu \not{k}_{b\dot{b}} - k_\mu \not{\beta}^{+,m}_{b\dot{b}} - (\not{\beta}^{+,m}\not{k}\sigma_\mu)_{b\dot{b}}\big]\gamma^{I\bar{I}}_m\Big) \\ &\quad + V^{(-\frac{1}{2})}_r \Big(r^c_{n,J} = -\frac{1}{6\sqrt{2}}\eta^b_I(\not{\beta}^{+,m}\not{k})_b{}^c\big[6\delta^{(6)}_{mn}\delta^I_J + (\gamma_m\bar{\gamma}_n)^I{}_J\big]\Big),\end{aligned} \tag{3.71}$$

$$\big[\eta^b_I \mathcal{Q}^{(+\frac{1}{2}),I}_b, V^{(-1)}_{\beta^-}\big] = V^{(-\frac{1}{2})}_{\bar{a}} \Big(\bar{a}^{\bar{I}}_{\dot{b}} = -\frac{1}{2\sqrt{\alpha'}}\eta^b_I \not{\beta}^{-,m}_{b\dot{b}}\gamma^{I\bar{I}}_m\Big), \tag{3.72}$$

[16]There is a subtlety in these computations (and also for some later ones) related to the fact that gamma matrices associated with spacetime and internal dimensions are anticommuting.

$$\big[\eta^b_I \mathcal{Q}^{(+\frac{1}{2}),I}_b, V^{(-1)}_d\big] = V^{(-\frac{1}{2})}_{\chi}\Big(\chi^c_{\mu,J} = \frac{1}{6\sqrt{\alpha'}}\eta^b_I\big[3d^{mn}_\mu \delta_b{}^c + (\not{d}^{mn}\bar{\sigma}_\mu + \alpha' k_\mu \not{d}^{mn}\not{k})_b{}^c\big](\gamma_m\bar{\gamma}_n)^I{}_J\Big)$$
$$+ V^{(-\frac{1}{2})}_{\bar{r}}\Big(\bar{r}^{\bar{I}}_{l,\dot{b}} = \frac{1}{6\sqrt{\alpha'}}\eta^b_I \not{d}^{mn}_{b\dot{b}}\gamma^{I\bar{I}}_n\big[6\delta^{(6)}_{ml}\delta_{\bar{I}}{}^{\bar{J}} + (\bar{\gamma}_m\gamma_l)_{\bar{I}}{}^{\bar{J}}\big]\Big), \tag{3.73}$$

whereas the action of right-handed $\bar{\mathcal{Q}}^{\dot{b}}_{\bar{J}}$ yields

$$\big[\bar{\eta}^{\bar{I}}_{\dot{b}} \bar{\mathcal{Q}}^{(+\frac{1}{2}),\dot{b}}_{\bar{I}}, V^{(-1)}_{\beta^+}\big] = V^{(-\frac{1}{2})}_a\Big(a^b_I = -\frac{1}{2\sqrt{\alpha'}}\bar{\eta}^{\bar{I}}_{\dot{b}}\not{\beta}^{+,m,\dot{b}b}\bar{\gamma}_{m,\bar{I}I}\Big), \tag{3.74}$$

$$\big[\bar{\eta}^{\bar{I}}_{\dot{b}} \bar{\mathcal{Q}}^{(+\frac{1}{2}),\dot{b}}_{\bar{I}}, V^{(-1)}_{\beta^-}\big] = V^{(-\frac{1}{2})}_{\chi}\Big(\chi^b_{\mu,I} = \frac{1}{3\sqrt{2}}\bar{\eta}^{\bar{I}}_{\dot{b}}\big[3\beta^{-,m}_\mu \not{k}^{\dot{b}b} - k_\mu \not{\beta}^{-,m,\dot{b}b} - (\not{\beta}^{-,m}\not{k}\bar{\sigma}_\mu)^{\dot{b}b}\big]\bar{\gamma}_{m,\bar{I}I}\Big)$$
$$+ V^{(-\frac{1}{2})}_{\bar{r}}\Big(\bar{r}^{\bar{J}}_{n,\dot{c}} = -\frac{1}{6\sqrt{2}}\bar{\eta}^{\bar{I}}_{\dot{b}}(\not{\beta}^{-,m}\not{k})^{\dot{b}}{}_{\dot{c}}\big[6\delta^{(6)}_{mn}\delta_{\bar{I}}{}^{\bar{J}} + (\bar{\gamma}_m\gamma_n)_{\bar{I}}{}^{\bar{J}}\big]\Big), \tag{3.75}$$

$$\big[\bar{\eta}^{\bar{I}}_{\dot{b}} \bar{\mathcal{Q}}^{(+\frac{1}{2}),\dot{b}}_{\bar{I}}, V^{(-1)}_d\big] = V^{(-\frac{1}{2})}_{\bar{\chi}}\Big(\bar{\chi}^{\bar{J}}_{\mu,\dot{c}} = \frac{1}{6\sqrt{\alpha'}}\bar{\eta}^{\bar{I}}_{\dot{b}}\big[3d^{mn}_\mu \delta^{\dot{b}}{}_{\dot{c}} + (\not{d}^{mn}\sigma_\mu + \alpha' k_\mu \not{d}\not{k})^{\dot{b}}{}_{\dot{c}}\big](\bar{\gamma}_m\gamma_n)_{\bar{I}}{}^{\bar{J}}\Big)$$
$$+ V^{(-\frac{1}{2})}_r\Big(r^b_{J,p} = \frac{1}{6\sqrt{\alpha'}}\bar{\eta}^{\bar{I}}_{\dot{b}}\not{d}^{mn,\dot{b}b}\bar{\gamma}_{n,\bar{I}I}\big[6\delta^{(6)}_{mp}\delta^I{}_J + (\gamma_m\bar{\gamma}_p)^I{}_J\big]\Big). \tag{3.76}$$

Then we are left with the $SO(1,3)$ scalar fields $\Phi^\pm$, $\zeta^{(mn)}$ and Ω_{mnl}. The internal states Ω_{mnl} represent both self-dual and anti-self-dual three-forms of $SO(6)$. We will denote their irreducible components as $\Omega^+_{mnl} \in [0,2,0]$ and $\Omega^-_{mnl} \in [0,0,2]$, for the self-dual and anti-self-dual part, respectively. Their defining irreducibility constraint is

$$\Omega^-_{mnl}(\gamma^{mnl})_{I\bar{I}} = \Omega^+_{mnl}(\bar{\gamma}^{mnl})^{\bar{I}I} = 0 \ . \tag{3.77}$$

The $SO(6)$ selection rules constrain $\mathcal{Q}^I\zeta^{(mn)} \in [0,1,0]\otimes[2,0,0] \to [2,1,0]\oplus[1,0,1]$ as well as $\mathcal{Q}^I\Omega^+_{mnl} \in [0,1,0]\otimes[0,2,0] \to [0,3,0]\oplus[1,1,0]$ and $\mathcal{Q}^I\Omega^-_{mnl} \in [0,1,0]\otimes[0,0,2] \to [0,1,2]\oplus[0,0,1]$. Thus, we expect the internal spin-$\frac{1}{2}$ fermion $\bar{r}$ or r by performing the SUSY transformation $\mathcal{Q}\zeta \to \bar{r}$, and $\bar{\mathcal{Q}}\zeta \to r$. Three-forms, on the other hand, are mapped to either r or $\bar{a}$, depending on the self-duality property $\mathcal{Q}\Omega^+ \to r$ or $\mathcal{Q}\Omega^- \to \bar{a}$. The supercharges acting on $\Phi^\pm$ and $\zeta^{(mn)}$ yield

$$\big[\eta^b_I \mathcal{Q}^{(+\frac{1}{2}),I}_b, V^{(-1)}_{\Phi^+}\big] = 0, \qquad \big[\bar{\eta}^{\bar{I}}_{\dot{b}} \bar{\mathcal{Q}}^{(+\frac{1}{2}),\dot{b}}_{\bar{I}}, V^{(-1)}_{\Phi^-}\big] = 0, \tag{3.78}$$

$$\big[\bar{\eta}^{\bar{I}}_{\dot{b}} \bar{\mathcal{Q}}^{(+\frac{1}{2}),\dot{b}}_{\bar{I}}, V^{(-1)}_{\Phi^+}\big] = V^{(-\frac{1}{2})}_{\bar{a}}\Big(\bar{a}^{\bar{I}}_{\dot{b}} = -\alpha'^{-\frac{1}{2}}\Phi^+\bar{\eta}^{\bar{I}}_{\dot{b}}\Big), \tag{3.79}$$

$$\big[\eta^b_I \mathcal{Q}^{(+\frac{1}{2}),I}_b, V^{(-1)}_{\Phi^-}\big] = V^{(-\frac{1}{2})}_a\Big(a^b_I = -\alpha'^{-\frac{1}{2}}\Phi^-\eta^b_I\Big), \tag{3.80}$$

and

$$\left[\eta_I^b \mathcal{Q}_b^{(+\frac{1}{2}),I}, V_\zeta^{(-1)}\right] = V_{\bar{r}}^{(-\frac{1}{2})}\left(\bar{r}_{\dot{b}}^{m,\bar{I}} = \frac{1}{\sqrt{2}}\eta_I^b \zeta^{(mn)} \not{k}_{b\dot{b}} \gamma_n^{I\bar{I}}\right), \tag{3.81}$$

$$\left[\bar{\eta}_{\dot{b}}^{\bar{I}} \bar{\mathcal{Q}}_{\bar{I}}^{(+\frac{1}{2}),\dot{b}}, V_\zeta^{(-1)}\right] = V_r^{(-\frac{1}{2})}\left(r_I^{m,b} = \frac{1}{\sqrt{2}}\bar{\eta}_{\dot{b}}^{\bar{I}} \zeta^{(mn)} \not{k}^{\dot{b}b} \bar{\gamma}_{n,\bar{I}I}\right). \tag{3.82}$$

On the three-forms $\Omega^{\pm}_{mnl}$, we obtain

$$\left[\eta_I^b \mathcal{Q}_b^{(+\frac{1}{2}),I}, V_{\Omega^+}^{(-1)}\right] = V_r^{(-\frac{1}{2})}\left(r_{k,J}^b = -\frac{1}{4\sqrt{\alpha'}}\eta_I^b \Omega^+_{mnl}(\gamma_k \bar{\gamma}^{mnl})^I{}_J\right), \tag{3.83}$$

$$\left[\bar{\eta}_{\dot{b}}^{\bar{I}} \bar{\mathcal{Q}}_{\bar{I}}^{(+\frac{1}{2}),\dot{b}}, V_{\Omega^+}^{(-1)}\right] = V_a^{(-\frac{1}{2})}\left(a_I^b = \frac{1}{2\sqrt{2}}\bar{\eta}_{\dot{b}}^{\bar{I}} \Omega^+_{mnl} \not{k}^{\dot{b}b} (\bar{\gamma}^{mnl})_{\bar{I}I}\right), \tag{3.84}$$

and

$$\left[\eta_I^b \mathcal{Q}_b^{(+\frac{1}{2}),I}, V_{\Omega^-}^{(-1)}\right] = V_{\bar{a}}^{(-\frac{1}{2})}\left(\bar{a}_{\dot{b}}^{\bar{I}} = \frac{1}{2\sqrt{2}}\eta_I^b \Omega^-_{mnl} \not{k}_{b\dot{b}} (\gamma^{mnl})^{I\bar{I}}\right), \tag{3.85}$$

$$\left[\bar{\eta}_{\dot{b}}^{\bar{I}} \bar{\mathcal{Q}}_{\bar{I}}^{(+\frac{1}{2}),\dot{b}}, V_{\Omega^-}^{(-1)}\right] = V_{\bar{r}}^{(-\frac{1}{2})}\left(\bar{r}_{k,\dot{b}}^{\bar{J}} = -\frac{1}{4\sqrt{\alpha'}}\bar{\eta}_{\dot{b}}^{\bar{I}} \Omega^-_{mnl}(\bar{\gamma}_k \gamma^{mnl})_{\bar{I}}{}^{\bar{J}}\right). \tag{3.86}$$

SUSY transformation of fermionic states

In this subsubsection, we investigate the (anti-)supercharge acting on the fermionic states. Following the strategy outlined before, we first derive a selection rule from group theory and then perform SUSY variations to get the expression of the bosonic wavefunctions explicitly. All the transformations are symmetric under simultaneous exchange of chiralities on the supercharges and the states (where $\Phi^+, \beta^+, \Omega^+ \leftrightarrow \Phi^-, \beta^-, \Omega^-$). We will only comment on one out of two inequivalent cases in the text but also give the formulae for the images under chirality reversal.

Since both the spin-$\frac{3}{2}$ fermions $(\chi, \bar{\chi})$ and the spin-$\frac{1}{2}$ states $(a, \bar{a})$ fall into (anti-)fundamental R-symmetry representations, the $SO(6)$ content of their SUSY variation is $[0,1,0] \otimes [0,1,0] \rightarrow [0,2,0] \oplus [1,0,0]$ and $[0,0,1] \otimes [0,1,0] \rightarrow [0,0,0] \oplus [0,1,1]$. The (anti-)supercharge acting on $\chi^a_{\mu,I}(\bar{\chi}^{\bar{I}}_{\mu,\dot{a}})$ will give us vectors $\beta^{\pm,m}_\mu$. In the cases $\mathcal{Q}^I \bar{\chi}^{\bar{I}}_{\mu,\dot{a}}$ and $\bar{\mathcal{Q}}_{\bar{J}} \chi^a_{\mu,I}$ of opposite chirality, the spin two field $\alpha_{\mu\nu}$ and the vector $d^{[mn]}_\mu$ can emerge.

Indeed,

$$\big[\eta_I^a\mathcal{Q}_a^{(-\frac{1}{2}),I},V_\chi^{(-\frac{1}{2})}\big]=V_{\beta^-}^{(-1)}\Big(\beta_\mu^{-,m}=\frac{1}{\sqrt{2}}\eta_I^a\chi_{\mu,a,J}(\gamma^m C)^{IJ}\Big),\tag{3.87}$$

$$\begin{aligned}\big[\bar\eta_{\dot a}^{\bar I}\bar{\mathcal{Q}}_{\bar I}^{(-\frac{1}{2}),\dot a},V_\chi^{(-\frac{1}{2})}\big]&=V_\alpha^{(-1)}\Big(\alpha_{\mu\nu}=\frac{1}{\sqrt{2}}\bar\eta_{\dot a}^{\bar I}\big(\bar\sigma_{(\mu}^{\dot a a}\chi_{\nu),a,I}+\alpha'\not{k}^{\dot a a}k_{(\mu}\chi_{\nu),a,I}\big)C_{\bar I}^{I}\Big)\\&\quad+V_d^{(-1)}\Big(d_\mu^{[mn]}=-\frac{\sqrt{\alpha'}}{4}\bar\eta_{\dot a}^{\bar I}\not{k}^{\dot a a}\chi_{\mu,a,I}(\bar\gamma^{mn}C)_{\bar I}{}^{I}\Big),\end{aligned}\tag{3.88}$$

and[17]

$$\begin{aligned}\big[\eta_I^a\mathcal{Q}_a^{(-\frac{1}{2}),I},V_{\bar\chi}^{(-\frac{1}{2})}\big]&=V_\alpha^{(-1)}\Big(\alpha_{\mu\nu}=\frac{1}{\sqrt{2}}\eta_I^a\big(\sigma_{(\mu|a\dot a|}\bar\chi_{\nu)}^{\dot a,\bar I}+\alpha'\not{k}_{a\dot a}k_{(\mu}\bar\chi_{\nu)}^{\dot a,\bar I}\big)C_{\bar I}^{I}\Big)\\&\quad+V_d^{(-1)}\Big(d_\mu^{[mn]}=-\frac{\sqrt{\alpha'}}{4}\eta_I^a\not{k}_{a\dot a}\bar\chi_\mu^{\dot a,\bar I}(\gamma^{mn}C)^{I}{}_{\bar I}\Big),\end{aligned}\tag{3.89}$$

$$\big[\bar\eta_{\dot a}^{\bar I}\bar{\mathcal{Q}}_{\bar I}^{(-\frac{1}{2}),\dot a},V_{\bar\chi}^{(-\frac{1}{2})}\big]=V_{\beta^+}^{(-1)}\Big(\beta_\mu^{+,m}=\frac{1}{\sqrt{2}}\bar\eta_{\dot a}^{\bar I}\bar\chi_\mu^{\dot a,\bar J}(\bar\gamma^m C)_{\bar I\bar J}\Big).\tag{3.90}$$

The supercharge action on a_I^b and $\bar a_{\dot b}^{\bar I}$ follows the same selection rules with respect to $SO(6)$ but different ones with respect to spacetime spin. The corresponding SUSY transformations read

$$\begin{aligned}\big[\eta_I^b\mathcal{Q}_b^{(-\frac{1}{2}),I},V_a^{(-\frac{1}{2})}\big]&=V_{\beta^+}^{(-1)}\Big(\beta_\mu^{+,m}=\frac{\sqrt{\alpha'}}{2}\eta_I^b\big[k_\mu\delta_b{}^c+(\not{k}\bar\sigma_\mu)_b{}^c\big]a_{c,J}(\gamma^m C)^{IJ}\Big)\\&\quad+V_{\Omega^+}^{(-1)}\Big(\Omega_{mnl}^+=\frac{1}{12\sqrt{2}}\eta_I^b a_{b,J}(\gamma_{mnl}C)^{IJ}\Big),\end{aligned}\tag{3.91}$$

$$\big[\bar\eta_{\dot b}^{\bar I}\bar{\mathcal{Q}}_{\bar I}^{(-\frac{1}{2}),\dot b},V_a^{(-\frac{1}{2})}\big]=V_{\Phi^-}^{(-1)}\Big(\Phi^-=\sqrt{\alpha'}\bar\eta_{\dot b}^{\bar I}\not{k}^{\dot b b}a_{b,I}C_{\bar I}^{I}\Big),\tag{3.92}$$

and

$$\big[\eta_I^b\mathcal{Q}_b^{(-\frac{1}{2}),I},V_{\bar a}^{(-\frac{1}{2})}\big]=V_{\Phi^+}^{(-1)}\Big(\Phi^+=\sqrt{\alpha'}\eta_I^b\not{k}_{b\dot b}\bar a^{\dot b,\bar I}C_{\bar I}^{I}\Big),\tag{3.93}$$

$$\begin{aligned}\big[\bar\eta_{\dot b}^{\bar I}\bar{\mathcal{Q}}_{\bar I}^{(-\frac{1}{2}),\dot b},V_{\bar a}^{(-\frac{1}{2})}\big]&=V_{\beta^-}^{(-1)}\Big(\beta_\mu^{-,m}=\frac{\sqrt{\alpha'}}{2}\bar\eta_{\dot b}^{\bar I}\big[k_\mu\delta^{\dot b}{}_{\dot c}+(\not{k}\sigma_\mu)^{\dot b}{}_{\dot c}\big]\bar a^{\dot c,\bar J}(\bar\gamma^m C)_{\bar I\bar J}\Big)\\&\quad+V_{\Omega^-}^{(-1)}\Big(\Omega_{mnl}^-=\frac{1}{12\sqrt{2}}\bar\eta_{\dot b}^{\bar I}\bar a^{\dot b,\bar J}(\bar\gamma_{mnl}C)_{\bar I\bar J}\Big).\end{aligned}\tag{3.94}$$

Notice we do not get a vector $d_\mu^{[mn]}$ in the SUSY transformations, although it is allowed by $SO(1,3)\times SO(6)$.

[17]The notation $M_{\mu_1\mu_2\cdots(\mu_i\cdots\mu_{j-1}|\mu_j\cdots\mu_k|\mu_{k+1}\cdots\mu_l)\cdots\mu_n}$ indicates we symmetrize over the indices $\mu_i,\cdots,\mu_{j-1},\mu_{k+1},\cdots,\mu_l$, but not over the indices $\mu_j,\ldots\mu_k$ enclosed between the bars.

Now we are left with the internal spin-$\frac{1}{2}$ fermions r and $\bar{r}$. Group theory admits SUSY variations in $[0,1,0]\otimes[1,1,0] \to [1,2,0]\oplus[2,0,0]\oplus[0,1,1]$ and $[0,0,1]\otimes[1,1,0] \to [1,1,1]\oplus[1,0,0]\oplus[0,2,0]$ corresponding to vectors $d_\mu^{[mn]}$ and internal scalars $\zeta^{(mn)}$ in the former case and $\bar{\mathcal{Q}}r \to \beta^{\pm}+\Omega^{+}$ in the latter. The left-handed supercharge yields

$$\begin{aligned}
\big[\eta_I^a \mathcal{Q}_a^{(-\frac{1}{2}),I}, V_r^{(-\frac{1}{2})}\big] &= V_d^{(-1)}\Big(d_\mu^{[mn]} = \frac{\sqrt{\alpha'}}{2}\eta_I^a\big[k_\mu \delta_a{}^b + (\sigma_\mu \not{k})_a{}^b\big] r_{b,J}^{[m}(\gamma^{n]}C)^{IJ}\Big) \\
&\quad + V_\zeta^{(-1)}\Big(\zeta^{(mn)} = \frac{1}{\sqrt{2}}\eta_I^a r_{a,J}^{(m}(\gamma^{n)}C)^{IJ}\Big), \qquad (3.95) \\
\big[\bar{\eta}_{\dot{a}}^{\bar{I}} \bar{\mathcal{Q}}_{\bar{I}}^{(-\frac{1}{2}),\dot{a}}, V_r^{(-\frac{1}{2})}\big] &= V_{\beta^+}^{(-1)}\Big(\beta_\mu^{+,m} = \frac{1}{\sqrt{2}}\bar{\eta}_{\dot{a}}^{\bar{I}}\big(\bar{\sigma}_\mu^{\dot{a}a} + \alpha' k_\mu \not{k}^{\dot{a}a}\big) r_{a,I}^m C_{\bar{I}}^I\Big) \\
&\quad + V_{\Omega_+}^{(-1)}\Big(\Omega_+^{mnl} = -\frac{\sqrt{\alpha'}}{4}\bar{\eta}_{\dot{a}}^{\bar{I}}\not{k}^{\dot{a}a} r_{a,I}^{[m}(\bar{\gamma}^{nl]}C)_{\bar{I}}{}^I\Big), \qquad (3.96)
\end{aligned}$$

and the right-handed counterpart reads

$$\begin{aligned}
\big[\eta_I^a \mathcal{Q}_a^{(-\frac{1}{2}),I}, V_{\bar{r}}^{(-\frac{1}{2})}\big] &= V_{\beta^-}^{(-1)}\Big(\beta_\mu^{-,m} = \frac{1}{\sqrt{2}}\eta_I^a\big(\sigma_{\mu a\dot{a}} + \alpha' k_\mu \not{k}_{a\dot{a}}\big)\bar{r}_{\bar{I}}^{m,\dot{a}} C_I^{\bar{I}}\Big) \\
&\quad + V_{\Omega_-}^{(-1)}\Big(\Omega_-^{mnl} = -\frac{\sqrt{\alpha'}}{4}\eta_I^a \not{k}_{a\dot{a}}\bar{r}^{[m|,\dot{a},\bar{I}|}(\gamma^{nl]}C)^I{}_{\bar{I}}\Big), \qquad (3.97) \\
\big[\bar{\eta}_{\dot{a}}^{\bar{I}} \bar{\mathcal{Q}}_{\bar{I}}^{(-\frac{1}{2}),\dot{a}}, V_{\bar{r}}^{(-\frac{1}{2})}\big] &= V_d^{(-1)}\Big(d_\mu^{[mn]} = \frac{\sqrt{\alpha'}}{2}\bar{\eta}_{\dot{a}}^{\bar{I}}\big[k_\mu \delta^{\dot{a}}{}_{\dot{b}} + (\bar{\sigma}_\mu \not{k})^{\dot{a}}{}_{\dot{b}}\big]\bar{r}^{[m|,\dot{b},\bar{J}|}(\bar{\gamma}^{n]}C)_{\bar{I}\bar{J}}\Big) \\
&\quad + V_\zeta^{(-1)}\Big(\zeta^{(mn)} = \frac{1}{\sqrt{2}}\bar{\eta}_{\dot{a}}^{\bar{I}}\bar{r}^{(m|,\dot{a},\bar{J}|}(\bar{\gamma}^{n)}C)_{\bar{I}\bar{J}}\Big). \qquad (3.98)
\end{aligned}$$

This completes the list of SUSY transformations within the $\mathcal{N}=4$ multiplet. We will revisit these results from the spinor helicity viewpoint in section 3.2.

3.2 Helicity structure of massive on-shell multiplets

In this section, we apply the massive version of the spinor helicity formalism [59–61] to obtain a refined understanding of the structure of the previously constructed SUSY multiplets. A brief summary of the spinor techniques is collected in Appendix C, including the explicit form of massive wavefunctions associated with different spin components. The spin quantization axis is chosen covariantly by decomposing the timelike momentum k into two arbitrary light-like reference momenta p and q:

$$k^\mu = p^\mu + q^\mu, \qquad k^2 = -m^2 = 2pq, \qquad p^2 = q^2 = 0. \tag{3.99}$$

As was explained in detail in [62], the supercharges can be expanded in the basis of the momentum spinors $p_a, p^{*\dot{a}}$ and $q_a, q^{*\dot{a}}$ defined by $p_\mu \sigma^\mu_{a\dot{a}} = -p_a p^*_{\dot{a}}$ and $q_\mu \sigma^\mu_{a\dot{a}} = -q_a q^*_{\dot{a}}$:

$$\mathcal{Q}_a = \frac{[q\mathcal{Q}]}{[qp]} p_a + \frac{[p\mathcal{Q}]}{[pq]} q_a = \mathcal{Q}_+ p_a + \mathcal{Q}_- q_a, \tag{3.100}$$

$$\bar{\mathcal{Q}}^{\dot{a}} = \frac{\langle p\bar{\mathcal{Q}}\rangle}{\langle pq\rangle} q^{*\dot{a}} + \frac{\langle q\bar{\mathcal{Q}}\rangle}{\langle qp\rangle} p^{*\dot{a}} = \bar{\mathcal{Q}}_+ q^{*\dot{a}} + \bar{\mathcal{Q}}_- p^{*\dot{a}}. \tag{3.101}$$

This defines the supercharge components $\mathcal{Q}_\pm$ and $\bar{\mathcal{Q}}_\pm$ to be

$$\mathcal{Q}_+ \equiv \frac{[q\mathcal{Q}]}{[qp]}, \qquad \mathcal{Q}_- \equiv \frac{[p\mathcal{Q}]}{[pq]}, \tag{3.102}$$

$$\bar{\mathcal{Q}}_+ \equiv \frac{\langle p\bar{\mathcal{Q}}\rangle}{\langle pq\rangle}, \qquad \bar{\mathcal{Q}}_- \equiv \frac{\langle q\bar{\mathcal{Q}}\rangle}{\langle qp\rangle}. \tag{3.103}$$

The $\mathcal{Q}_+$ and $\bar{\mathcal{Q}}_+$ raise the spin quantum j_z number along the quantization axis by $1/2$, while $\mathcal{Q}_-$ and $\bar{\mathcal{Q}}_-$ lower it by $1/2$. The corresponding Lorentz generator which is diagonalized with eigenvalues j_z reads

$$J_z = \frac{1}{m^2}\, \varepsilon^{\mu\nu\lambda\rho}\, P_\mu\, q_\nu\, M_{\lambda\rho}\ , \tag{3.104}$$

where P_μ denotes the translation operator and $M_{\lambda\rho}$ an $SO(1,3)$ rotation.

A convenient way of organizing representations of the super Poincaré group is to pick a highest weight state which is annihilated by half the supercharges – either the left-handed $\mathcal{Q}_a$ or the right-handed $\bar{\mathcal{Q}}^{\dot{b}}$. States with this property are referred to as (anti-)Clifford vacua, and we shall use the vacuum eliminated by the left-handed $\mathcal{Q}_a$ by convention. The rest of the supermultiplet is then constructed by applying the nontrivially acting $\bar{\mathcal{Q}}_+$ and $\bar{\mathcal{Q}}_-$, see the figures in this section. In our notation, each diamond shaped diagram represents one supermultiplet. The dashed lines connecting bosonic and fermionic states indicate $\mathcal{Q}_\pm$ and

$\bar{\mathcal{Q}}_\pm$ applications, and we assign the following directions:

$$\nearrow \equiv \bar{\mathcal{Q}}_+, \qquad \searrow \equiv \bar{\mathcal{Q}}_- \qquad \text{and} \qquad \nwarrow \equiv \mathcal{Q}_+, \qquad \swarrow \equiv \mathcal{Q}_-. \tag{3.105}$$

The Clifford vacuum state being annihilated by the left-handed $\mathcal{Q}_\pm$ is located on the far left of the diamond, and we can construct the full supermultiplet by repeated action of $\bar{\mathcal{Q}}_\pm$.[18] In this section, we will show how $\bar{\mathcal{Q}}_\pm$ transform all the states in the multiplet from the left side of the diamond all the way to the right. The SUSY algebras $\{\mathcal{Q}_\pm, \bar{\mathcal{Q}}_\mp\} = 1$ and $\{\mathcal{Q}_\pm, \bar{\mathcal{Q}}_\pm\} = 0$ [19] imply that $\mathcal{Q}_\pm$ undoes $\bar{\mathcal{Q}}_\pm$ applications and transforms states from right to left in the diamond.

This section starts with the $\mathcal{N} = 1$ situation to illustrate the methods, and the additional features of extended SUSY are explained in the later subsections on $\mathcal{N} = 2, 4$ supermultiplets. To make everything simple and clear, instead of using our old notation of vertex operators in the previous sections, we will use the "ket" notation to express the states inside the diamonds. For example, the spin two boson with $j_z = +2$ is expressed by

$$|\alpha, +2\rangle \;\equiv\; V_\alpha^{(-1)}\Big(\, \alpha^{\mu\nu} = \frac{1}{2m^2}\,\bar{\sigma}^{\mu\dot{a}a}\bar{\sigma}^{\nu\dot{b}b} p^*_{\dot{a}} q_a p^*_{\dot{b}} q_b \Big)\,, \tag{3.106}$$

and a combined state $\{\alpha, d\}$ with $j_z = +1$ is expressed by $|\alpha \oplus d, +1\rangle$. The commutators of $\mathcal{Q}_a$ and $\bar{\mathcal{Q}}^{\dot{b}}$ with vertex operators are replaced by SUSY transformations acting directly on the states.

3.2.1 $\mathcal{N} = 1$ supermultiplets

According to the strategy outlined above, it suffices to evaluate the anti-supercharge components $\bar{\mathcal{Q}}_\pm$ on the helicity states in the $\mathcal{N} = 1$ supermultiplets. The decomposition $\bar{\mathcal{Q}}^{\dot{a}} = \bar{\mathcal{Q}}_+ q^{\dot{a}*} + \bar{\mathcal{Q}}_- p^{\dot{a}*}$ corresponds to the

[18] Alternatively, we can also construct this supermultiplet starting from the anti-Clifford vacuum state on the right side of this diamond, which is eliminated by the anti-supercharge $\bar{\mathcal{Q}}_\pm$, and the remaining states follow by acting $\mathcal{Q}_\pm$ on it.

[19] To show this, we simply plug the supercharge decompositions (3.100) and (3.101) into the $\mathcal{N} = 1$ SUSY algebra (2.93), and obtain,

$$\begin{aligned}\{\mathcal{Q}_a^{(+\frac{1}{2})}, \bar{\mathcal{Q}}^{(-\frac{1}{2}),\dot{a}}\} &= p_a p^{*\dot{a}}\{\mathcal{Q}_+, \bar{\mathcal{Q}}_-\} + p_a q^{*\dot{a}}\{\mathcal{Q}_+, \bar{\mathcal{Q}}_+\} + q_a q^{*\dot{a}}\{\mathcal{Q}_-, \bar{\mathcal{Q}}_+\} + q_a p^{*\dot{a}}\{\mathcal{Q}_-, \bar{\mathcal{Q}}_-\} \\ &= (\sigma^\mu\varepsilon)_a{}^{\dot{a}} P_\mu \sigma(\sigma^\mu\varepsilon)_a{}^{\dot{a}} k_\mu = p_a p^{*\dot{a}} + q_a q^{*\dot{a}}.\end{aligned}$$

Thus we arrive at,

$$\begin{pmatrix} \{\mathcal{Q}_+, \bar{\mathcal{Q}}_-\} & \{\mathcal{Q}_+, \bar{\mathcal{Q}}_+\} \\ \{\mathcal{Q}_-, \bar{\mathcal{Q}}_-\} & \{\mathcal{Q}_-, \bar{\mathcal{Q}}_+\} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

mass dimension $[M^{-\frac{1}{2}}]$ choices for $\bar{\eta}$:

$$\bar{\mathcal{Q}}_{\pm} = \bar{\eta}^{\pm}_{\dot{a}}\, \mathcal{Q}^{\dot{a}} \quad \longleftrightarrow \quad \bar{\eta}^{+}_{\dot{a}} = \frac{p^{*}_{\dot{a}}}{\langle pq \rangle}, \ \ \bar{\eta}^{-}_{\dot{a}} = \frac{q^{*}_{\dot{a}}}{\langle qp \rangle}. \tag{3.107}$$

Spin one half supermultiplets

We firstly consider the $\{\Phi^+, \bar{a}, \Omega^-\}$ multiplet of highest spin 1/2 whose scalar Clifford vacuum $|\Phi^+\rangle$ is eliminated by the supercharge $\mathcal{Q}^a$. By repeated actions of the anti-supercharge $\bar{\mathcal{Q}}_{\pm}$ on Φ^+, we can construct the remainder of the multiplet, see Fig. 8.

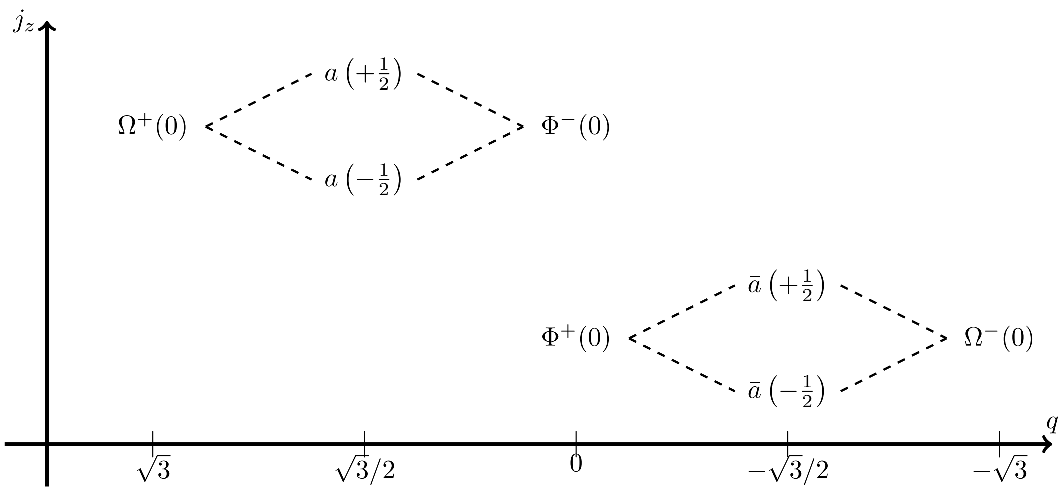

Figure 8: $\mathcal{N}=1$ SUSY multiplets with scalar Clifford vacuum: In $\mathcal{N}=1$ scenarios, the $U(1)$ charge q with respect to the internal current $\mathcal{J}$ is plotted along the horizontal axis. The SUSY charges have eigenvalue $\pm\sqrt{3}/2$ under $\mathcal{J}$ and therefore change q by a fixed offset.

The spin-$\frac{1}{2}$ multiplet is the minimal massive representation of the $\mathcal{N}=1$ SUSY algebra, since it only contains four states. Very straightly, we obtain, up to a phase,

$$\bar{\mathcal{Q}}_{\pm}\ |\Phi^+, 0\rangle = |\bar{a}, \pm\tfrac{1}{2}\rangle, \tag{3.108}$$

and

$$\bar{\mathcal{Q}}_{\pm}\ |\bar{a}, \mp\tfrac{1}{2}\rangle = |\Omega^-, 0\rangle, \qquad \bar{\mathcal{Q}}_{\pm}\ |\bar{a}, \pm\tfrac{1}{2}\rangle = 0. \tag{3.109}$$

The anti-Clifford vacuum $|\Omega^-\rangle$ is then annihilated by $\bar{\mathcal{Q}}_\pm$ action,

$$\bar{\mathcal{Q}}_\pm \ |\Omega^-, 0\rangle = 0. \tag{3.110}$$

Secondly, we consider the mirror multiplet $\{\Omega^+, a, \Phi^-\}$ which is also summarized in Fig. 8. Starting from the Clifford vacuum $|\Omega^+\rangle$, c.f. (3.17), we obtain,

$$\bar{\mathcal{Q}}_\pm \ |\Omega^+, 0\rangle = |a, \pm\tfrac{1}{2}\rangle, \tag{3.111}$$

and

$$\bar{\mathcal{Q}}_\pm \ |a, \mp\tfrac{1}{2}\rangle = |\Phi^-, 0\rangle, \qquad \bar{\mathcal{Q}}_\pm \ |a, \pm\tfrac{1}{2}\rangle = 0. \tag{3.112}$$

Spin two supermultiplet

In addition to the two minimal spin 1/2 multiplets, there is a larger multiplet $\{\alpha, \chi, \bar{\chi}, d\}$ with spins up to $j_z = 2$ in each $\mathcal{N} = 1$ scenario. All the left-handed spin 3/2 states $|\chi, j_z\rangle$ with $-3/2 \leq j_z \leq +3/2$ are annihilated by $\mathcal{Q}_a$, c.f. (3.5). Hence, the Clifford vacuum transforms in a nontrivial $SO(1,3)$ representation. Starting from the four states $|\chi, j_z\rangle$, we build the full spin two multiplet by $\bar{\mathcal{Q}}_\pm$ application, see Fig. 9. The spin-$\frac{3}{2}$ states with wavefunction $\bar{\chi}^\mu_{\dot{a}}$ of opposite chirality are obtained by $|\bar{\chi}, j_z\rangle = \bar{\mathcal{Q}}_+\bar{\mathcal{Q}}_-|\chi, j_z\rangle$, so they form the anti-Clifford vacua.

The helicity SUSY transformations are such that normalized states are either mapped to equally normalized states or annihilated. This becomes particularly interesting at the intersection points $\bar{\mathcal{Q}}_- \, |\chi, j_z\rangle \leftrightarrow \bar{\mathcal{Q}}_+ \, |\chi, j_z - 1\rangle$ within the diamond where combination states of type $|\alpha \oplus d\rangle$ arise. From the $j_z = \pm\frac{3}{2}$ components, we obtain

$$\bar{\mathcal{Q}}_\pm \, |\chi, \pm\tfrac{3}{2}\rangle = |\alpha, \pm 2\rangle, \tag{3.113}$$

$$\bar{\mathcal{Q}}_\mp \, |\chi, \pm\tfrac{3}{2}\rangle = \frac{1}{2}|\alpha, \pm 1\rangle \pm \frac{\sqrt{3}}{2}|d, \pm 1\rangle \equiv |\alpha \pm d, \pm 1\rangle, \tag{3.114}$$

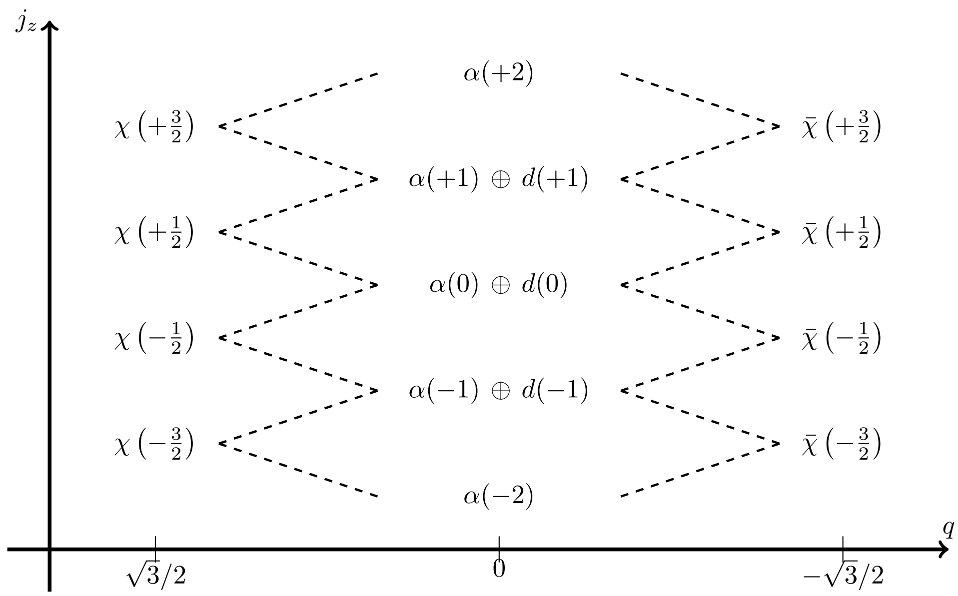

Figure 9: $\mathcal{N}=1$ SUSY multiplets with spin 3/2 Clifford vacuum

whereas $\bar{\mathcal{Q}}_\pm$ action on $j_z = \pm\frac{1}{2}$ components yields

$$\bar{\mathcal{Q}}_\pm\,|\chi,\pm\tfrac{1}{2}\rangle = \frac{\sqrt{3}}{2}|\alpha,\pm 1\rangle \mp \frac{1}{2}|d,\pm 1\rangle \equiv |\alpha \mp d, \pm 1\rangle, \tag{3.115}$$

$$\bar{\mathcal{Q}}_\mp\,|\chi,\pm\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2}}|\alpha,0\rangle \pm \frac{1}{\sqrt{2}}|d,0\rangle \equiv |\alpha \pm d, 0\rangle. \tag{3.116}$$

We use canonical normalization conventions for vertex operators as well as helicity wavefunctions: Let $|\psi, j_z\rangle$ denote some physical state with polarization tensor ψ and spin component j_z along the quantization axis. Then, $|\psi,+j_z\rangle$ has unit scalar product with $|\psi,-j_z\rangle$ and is orthogonal to all states whose wavefunction belongs to a different $SO(3)$ representation. We can see from above results that all the states on the right-hand sides of (3.113) to (3.116) have unit norm. Furthermore, we find that the combined states $|\alpha \pm d, \pm 1\rangle$ obtained from $\bar{\mathcal{Q}}_\mp\,|\chi,\pm\frac{3}{2}\rangle$ are orthogonal to $|\alpha \mp d, \pm 1\rangle$ from distinct Clifford vacuum components $\bar{\mathcal{Q}}_\pm\,|\chi,\pm\frac{1}{2}\rangle$, as expected.

To complete the other half of the diamond, we have,

$$\bar{\mathcal{Q}}_\pm\,|\alpha,+2\rangle = 0, \qquad \bar{\mathcal{Q}}_\mp\,|\alpha,+2\rangle = |\bar{\chi},\pm\tfrac{3}{2}\rangle, \tag{3.117}$$

and

$$\bar{\mathcal{Q}}_{\pm}\,|\alpha \pm d, \pm 1\rangle = |\bar{\chi}, \pm\tfrac{3}{2}\rangle, \qquad \bar{\mathcal{Q}}_{\mp}\,|\alpha \pm d, \pm 1\rangle = 0, \tag{3.118}$$

$$\bar{\mathcal{Q}}_{\mp}\,|\alpha \mp d, \pm 1\rangle = |\bar{\chi}, \pm\tfrac{1}{2}\rangle, \qquad \bar{\mathcal{Q}}_{\pm}\,|\alpha \mp d, \pm 1\rangle = 0, \tag{3.119}$$

and

$$\bar{\mathcal{Q}}_{\pm}\,|\alpha \pm d, 0\rangle = |\bar{\chi}, \pm\tfrac{1}{2}\rangle, \qquad \bar{\mathcal{Q}}_{\pm}\,|\alpha \mp d, 0\rangle = 0. \tag{3.120}$$

The diamond is symmetric about the $j_z = 0$ line. In other words, once we obtained all the transformations for the states in its upper half, the lower half can be filled up by interchanging momentum spinors $p \leftrightarrow q$. This holds by the construction of the massive helicity wavefunctions in Appendix C, see also [1] and [2].

3.2.2 $\mathcal{N} = 2$ supermultiplets

The new feature of extended $\mathcal{N} = 2$ SUSY is the non-Abelian $SU(2)$ R-symmetry group. The supercharges are spinors with respect to this $SU(2)$ and therefore carry fundamental indices i. That is why we have to introduce a bookkeeping Grassmann variable η_i which decouples from the spacetime spinor index structure. In other words, this η_i is a spinor of the R-symmetry but a scalar with respect to the spacetime $SO(1,3)$. We define supercharge components $\bar{\mathcal{Q}}_{\pm}(\eta)$ which are associated with the choices $\bar{\eta}^{+}_{\dot{a},i} = \eta_i p^*_{\dot{a}}/\langle pq\rangle$ and $\bar{\eta}^{-}_{\dot{a},i} = \eta_i q^*_{\dot{a}}/\langle qp\rangle$:

$$\bar{\mathcal{Q}}_{+}(\eta) = \eta_i\,\frac{p^*_{\dot{a}}}{\langle pq\rangle}\,\bar{\mathcal{Q}}^{\dot{a},i}, \tag{3.121}$$

$$\bar{\mathcal{Q}}_{-}(\eta) = \eta_i\,\frac{q^*_{\dot{a}}}{\langle qp\rangle}\,\bar{\mathcal{Q}}^{\dot{a},i}. \tag{3.122}$$

In the construction of $\mathcal{N} = 2$ supermultiplets from their Clifford vacua, we obtain states in nontrivial representations of the $SU(2)$ R-symmetry.[20] Their $SU(2)$ tensor structures will be displayed inside the ket vectors, right after the J_z eigenvalue, separated by a semicolon.[21]

[20] In fact, it is a peculiar feature of the first mass level that its Clifford vacua are R-symmetry scalars.

[21] In the literature, on-shell supersymmetry is usually described by the notion of supercharge eigenstates – Grassmann coherent states, firstly in [63], and recently in [62] and also [64]. Our presentation of SUSY transformations including internal

Spin one supermultiplets

Again, we start our presentation with the smaller multiplets of lower spin. The universal sector due to $\mathcal{N}=2$ SUSY encompasses two spin one multiplets with scalar Clifford vacua, see Figs. 10 and 11 below.

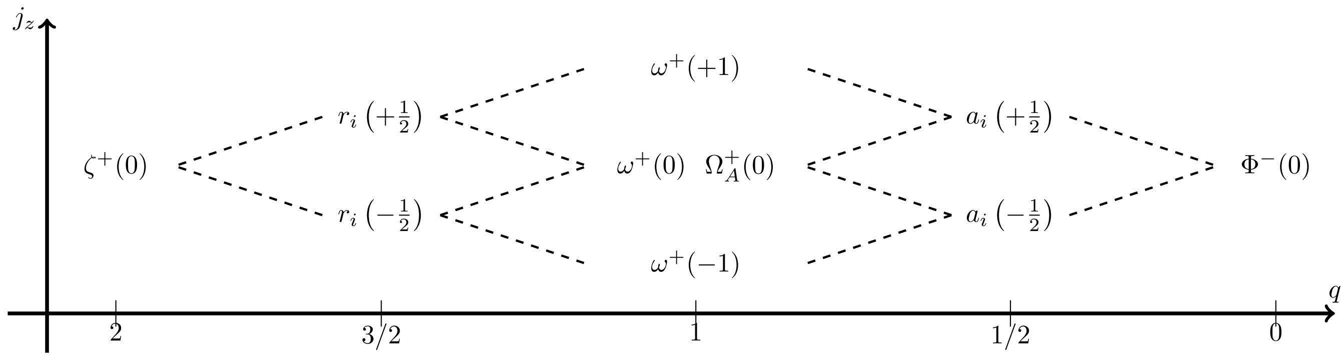

Figure 10: $\mathcal{N}=2$ SUSY multiplet with scalar Clifford vacuum: In $\mathcal{N}=2$ scenarios, the $U(1)$ charge q with respect to the internal toroidal directions is plotted along the horizontal axis. Since the world-sheet fields $i\partial Z^{\pm}$ and e^{iqH} have charge ± 1 and q, respectively, the SUSY generators built from $e^{\pm iH/2}$ and $i\partial Z^{\pm}e^{\mp iH/2}$ change q by the fixed offset $\pm 1/2$.

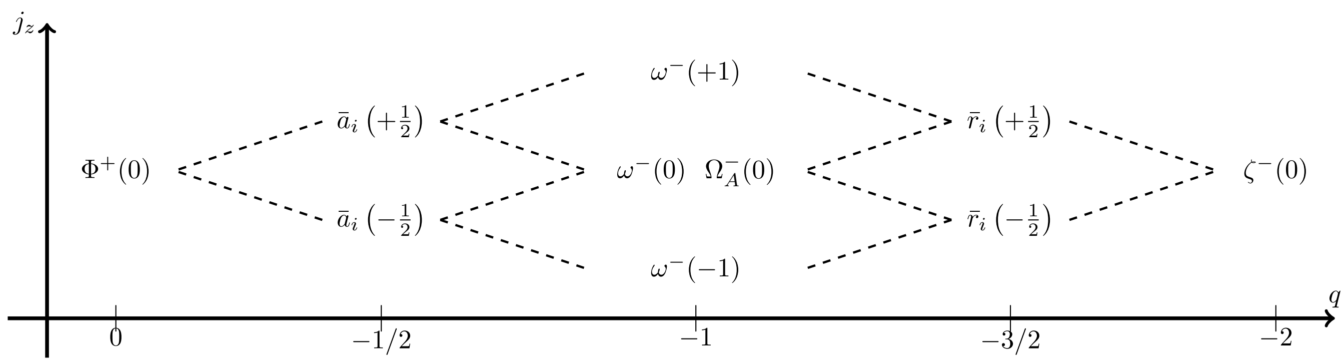

Figure 11: conjugate $\mathcal{N}=2$ SUSY multiplet with scalar Clifford vacuum

The first multiplet $\{\omega^+, a, r, \Phi^-, \zeta^+, \Omega_A^+\}$ is constructed from a scalar Clifford vacuum ζ^+, c.f. (3.61). Omitting all the vanishing results, we obtain

$$\bar{\mathcal{Q}}_{\pm}(\eta_i)\,|\Phi^+, 0; 1\rangle = |\bar{a}, \pm\tfrac{1}{2}; \eta_i\rangle, \tag{3.123}$$

wavefunctions (carrying the R-symmetry quantum numbers) are an equivalent way of expressing their information content.

and

$$\bar{\mathcal{Q}}_{\pm}(\epsilon_j)\,|\bar{a},\pm\tfrac{1}{2},\eta_i\rangle = |\omega^-,\pm 1;(\epsilon\eta)\rangle, \tag{3.124}$$

$$\begin{aligned}\bar{\mathcal{Q}}_{\mp}(\epsilon_j)\,|\bar{a},\pm\tfrac{1}{2},\eta_i\rangle &= \frac{1}{\sqrt{2}}|\omega^-,0;(\epsilon\eta)\rangle \pm \frac{1}{\sqrt{2}}|\Omega^-,0;\epsilon_j(\tau_A\varepsilon)^{ji}\eta_i\rangle \\ &\equiv |\omega^- \pm \Omega^-_A,0\rangle,\end{aligned} \tag{3.125}$$

where $(\epsilon\eta) = \epsilon_j\varepsilon^{ji}\eta_i$. The ω^- and Ω^- states in the center of the diamond transform to

$$\bar{\mathcal{Q}}_{\mp}(\eta_i)\,|\omega^-,\pm 1;(\epsilon\eta)\rangle = \bar{\mathcal{Q}}_{\pm}(\eta_i)\,|\omega^- \pm \Omega^-_A,0\rangle = |\bar{r},\pm\tfrac{1}{2};(\epsilon\eta)\eta_i\rangle, \tag{3.126}$$

$$\bar{\mathcal{Q}}_{\mp}(\eta_i)\,|\omega^- \pm \Omega^-_A,0\rangle = 0, \tag{3.127}$$

and

$$\bar{\mathcal{Q}}_{\mp}(\epsilon_j)\,|\bar{r},\pm\tfrac{1}{2};(\epsilon\eta)\eta_i\rangle = |\zeta^-,0;(\epsilon\eta)^2\rangle. \tag{3.128}$$

Similar results are obtained for the mirror spin one multiplet $\{\omega^-,\bar{a},\bar{r},\Phi^+,\zeta^-,\Omega^-_A\}$, which is constructed from the scalar Clifford vacuum. The helicity SUSY transformations are

$$\bar{\mathcal{Q}}_{\pm}(\eta_i)\,|\zeta^+,0;1\rangle = |r,\pm\tfrac{1}{2};\eta_i\rangle, \tag{3.129}$$

and

$$\bar{\mathcal{Q}}_{\pm}(\epsilon_j)\,|r,\pm\tfrac{1}{2};\eta_i\rangle = |\omega^+,\pm 1;(\epsilon\eta)\rangle, \tag{3.130}$$

$$\begin{aligned}\bar{\mathcal{Q}}_{\mp}(\epsilon_j)\,|r,\pm\tfrac{1}{2};\eta_i\rangle &= \frac{1}{\sqrt{2}}|\omega^+,0;(\epsilon\eta)\rangle \pm \frac{1}{\sqrt{2}}|\Omega^+,0;\epsilon_j(\tau_A\varepsilon)^{ji}\eta_i\rangle \\ &\equiv |\omega^+ \pm \Omega^+,0\rangle,\end{aligned} \tag{3.131}$$

and

$$\bar{\mathcal{Q}}_{\mp}(\eta_i)\,|\omega^+,\pm 1;(\epsilon\eta)\rangle = \bar{\mathcal{Q}}_{\pm}(\eta_i)\,|\omega^+ \pm \Omega^+,0\rangle = |a,\pm\tfrac{1}{2};(\epsilon\eta)\eta_i\rangle, \tag{3.132}$$

$$\bar{\mathcal{Q}}_{\mp}(\eta_i)\,|\omega^+ \pm \Omega^+, 0\rangle = 0, \tag{3.133}$$

and

$$\bar{\mathcal{Q}}_{\mp}(\epsilon_j)\,|a, \pm\tfrac{1}{2}; (\epsilon\eta)\eta_i\rangle = |\Phi^-, 0; (\epsilon\eta)^2\rangle. \tag{3.134}$$

Spin two supermultiplet

The highest spin state of the first mass level populate a spin two multiplet $\{\alpha, \chi, \bar{\chi}, d, \xi, \beta^{\pm}, s, \bar{s}, \phi\}$ (see Fig. 12), which is built from a vector Clifford vacuum β^+_μ state, c.f. (3.27).

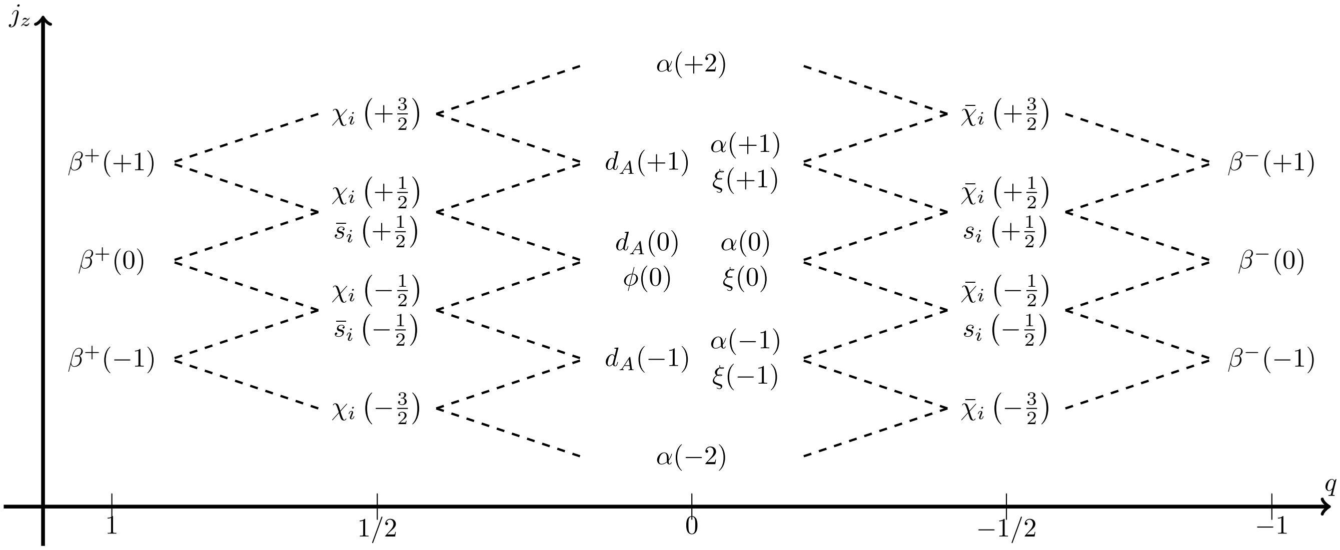

Figure 12: $\mathcal{N} = 2$ SUSY multiplet with vector Clifford vacuum

The supermultiplet structure is more complicated here due to intersection points in the diamond like $\bar{\mathcal{Q}}_-(\eta_i)\,|\beta^+, +1; 1\rangle \leftrightarrow \bar{\mathcal{Q}}_+(\eta_i)\,|\beta^+, 0; 1\rangle$. Since $j_z \mapsto -j_z$ reflection can be implemented by $p \leftrightarrow q$ exchange, we will only show the transformations for the upper half of the diamond. Omitting all the trivial relations, we obtain

$$\bar{\mathcal{Q}}_+(\eta_i)\,|\beta^+, +1; 1\rangle = |\chi, +\tfrac{3}{2}; \eta_i\rangle, \tag{3.135}$$

$$\bar{\mathcal{Q}}_-(\eta_i)\,|\beta^+, +1; 1\rangle = \frac{1}{\sqrt{3}}|\chi, +\tfrac{1}{2}; \eta_i\rangle + \frac{\sqrt{2}}{\sqrt{3}}|\bar{s}, +\tfrac{1}{2}; \eta_i\rangle \equiv |\chi \oplus \bar{s}, +\tfrac{1}{2}\rangle_1, \tag{3.136}$$

$$\bar{\mathcal{Q}}_+(\eta_i)\,|\beta^+, 0; 1\rangle = \frac{\sqrt{2}}{\sqrt{3}}|\chi, +\tfrac{1}{2}; \eta_i\rangle - \frac{1}{\sqrt{3}}|\bar{s}, +\tfrac{1}{2}; \eta_i\rangle \equiv |\chi \oplus \bar{s}, +\tfrac{1}{2}\rangle_2, \tag{3.137}$$

where $|\chi \oplus \bar{s}, +\frac{1}{2}\rangle_1$ is orthogonal to $|\chi \oplus \bar{s}, +\frac{1}{2}\rangle_2$. For the helicity SUSY transformation of the second column of Fig. 12, we have

$$\bar{\mathcal{Q}}_+(\epsilon_j)\,|\chi, +\tfrac{3}{2}; \eta_i\rangle = |\alpha, +2; (\epsilon\eta)\rangle, \tag{3.138}$$

$$\begin{aligned}\bar{\mathcal{Q}}_-(\epsilon_j)\,|\chi, +\tfrac{3}{2}; \eta_i\rangle &= -\frac{1}{2}|\alpha, +1; (\epsilon\eta)\rangle + \frac{1}{2}|\xi, +1; (\epsilon\eta)\rangle - \frac{1}{\sqrt{2}}|d, +1; (\epsilon\eta)\rangle \\ &\equiv |\alpha \oplus \xi \oplus d, +1\rangle_1,\end{aligned} \tag{3.139}$$

$$\begin{aligned}\bar{\mathcal{Q}}_+(\epsilon_j)\,|\chi \oplus \bar{s}, +\tfrac{1}{2}\rangle_1 &= -\frac{1}{2}|\alpha, +1; (\epsilon\eta)\rangle + \frac{1}{2}|\xi, +1; (\epsilon\eta)\rangle + \frac{1}{\sqrt{2}}|d, +1; (\epsilon\eta)\rangle \\ &\equiv |\alpha \oplus \xi \oplus d, +1\rangle_2,\end{aligned} \tag{3.140}$$

$$\begin{aligned}\bar{\mathcal{Q}}_+(\epsilon_j)\,|\chi \oplus \bar{s}, +\tfrac{1}{2}\rangle_2 &= -\frac{1}{\sqrt{2}}|\alpha, +1; (\epsilon\eta)\rangle - \frac{1}{\sqrt{2}}|\xi, +1; (\epsilon\eta)\rangle \\ &\equiv |\alpha \oplus \xi \oplus d, +1\rangle_3.\end{aligned} \tag{3.141}$$

One can easily check that the three states $|\alpha \oplus \xi \oplus d, +1\rangle_{1,2,3}$ are orthonormal. Moreover,

$$\begin{aligned}\bar{\mathcal{Q}}_-(\epsilon_j)\,|\chi \oplus \bar{s}, +\tfrac{1}{2}\rangle_1 &= -\frac{1}{\sqrt{6}}|\alpha, 0; (\epsilon\eta)\rangle + \frac{1}{\sqrt{2}}|\xi, 0; (\epsilon\eta)\rangle + \frac{1}{\sqrt{3}}|\phi, 0; (\epsilon\eta)\rangle \\ &\equiv |\alpha \oplus \xi \oplus d \oplus \phi\rangle_1,\end{aligned} \tag{3.142}$$

$$\begin{aligned}\bar{\mathcal{Q}}_-(\epsilon_j)\,|\chi \oplus \bar{s}, +\tfrac{1}{2}\rangle_2 &= -\frac{1}{\sqrt{3}}|\alpha, 0; (\epsilon\eta)\rangle - \frac{1}{\sqrt{2}}|d, 0; (\epsilon\eta)\rangle - \frac{1}{\sqrt{6}}|\phi, 0; (\epsilon\eta)\rangle \\ &\equiv |\alpha \oplus \xi \oplus d \oplus \phi\rangle_2.\end{aligned} \tag{3.143}$$

By interchanging $p \leftrightarrow q$ we get the states

$$|\alpha \oplus \xi \oplus d \oplus \phi\rangle'_1 = |\alpha \oplus \xi \oplus d \oplus \phi\rangle_1(p \leftrightarrow q) = -\frac{1}{\sqrt{6}}|\alpha, 0; (\epsilon\eta)\rangle - \frac{1}{\sqrt{2}}|\xi, 0; (\epsilon\eta)\rangle + \frac{1}{\sqrt{3}}|\phi, 0; (\epsilon\eta)\rangle, \tag{3.144}$$

$$|\alpha \oplus \xi \oplus d \oplus \phi\rangle'_2 = |\alpha \oplus \xi \oplus d \oplus \phi\rangle_2(p \leftrightarrow q) = -\frac{1}{\sqrt{3}}|\alpha, 0; (\epsilon\eta)\rangle + \frac{1}{\sqrt{2}}|d, 0; (\epsilon\eta)\rangle - \frac{1}{\sqrt{6}}|\phi, 0; (\epsilon\eta)\rangle, \tag{3.145}$$

which are the results obtained from $\bar{\mathcal{Q}}_+(\eta_j)\,|\chi \oplus \bar{s}, -\frac{1}{2}\rangle$. Clearly, $|\alpha \oplus \xi \oplus d \oplus \phi\rangle_{1(2)}$ is orthogonal to $|\alpha \oplus \xi \oplus$

$d \oplus \phi\rangle'_{1(2)}$. The helicity SUSY transformations of this column are

$$\bar{\mathcal{Q}}_-(\eta_i)\,|\alpha,+2;(\epsilon\eta)\rangle = \bar{\mathcal{Q}}_+(\eta_i)\,|\alpha\oplus\xi\oplus d,+1\rangle_1 = |\bar{\chi},+\tfrac{3}{2};(\epsilon\eta)\eta_i\rangle, \tag{3.146}$$

$$\bar{\mathcal{Q}}_+(\eta_i)\,|\alpha\oplus\xi\oplus d,+1\rangle_2 = \bar{\mathcal{Q}}_+(\eta_i)\,|\alpha\oplus\xi\oplus d,+1\rangle_3 = 0, \tag{3.147}$$

and

$$\bar{\mathcal{Q}}_-(\eta_i)\,|\alpha\oplus\xi\oplus d,+1\rangle_1 = 0, \tag{3.148}$$

$$\bar{\mathcal{Q}}_-(\eta_i)\,|\alpha\oplus\xi\oplus d,+1\rangle_2 = \frac{1}{\sqrt{3}}|\bar{\chi},+\tfrac{1}{2};(\epsilon\eta)\eta_i\rangle - \frac{\sqrt{2}}{\sqrt{3}}|s,+\tfrac{1}{2};(\epsilon\eta)\eta_i\rangle \equiv |\bar{\chi}\oplus s,+\tfrac{1}{2}\rangle_1, \tag{3.149}$$

$$\bar{\mathcal{Q}}_-(\eta_i)\,|\alpha\oplus\xi\oplus d,+1\rangle_3 = \frac{\sqrt{2}}{\sqrt{3}}|\bar{\chi},+\tfrac{1}{2};(\epsilon\eta)\eta_i\rangle + \frac{1}{\sqrt{3}}|s,+\tfrac{1}{2};(\epsilon\eta)\eta_i\rangle \equiv |\bar{\chi}\oplus s,+\tfrac{1}{2}\rangle_2. \tag{3.150}$$

States in the center of the diamond transform as

$$\bar{\mathcal{Q}}_+(\eta_i)\,|\alpha\oplus\xi\oplus d\oplus\phi\rangle'_1 = \bar{\mathcal{Q}}_+(\eta_i)\,|\alpha\oplus\xi\oplus d\oplus\phi\rangle'_2 = 0, \tag{3.151}$$

$$\bar{\mathcal{Q}}_+(\eta_i)\,|\alpha\oplus\xi\oplus d\oplus\phi\rangle_1 = |\bar{\chi}\oplus s,+\tfrac{1}{2}\rangle_1, \tag{3.152}$$

$$\bar{\mathcal{Q}}_+(\eta_i)\,|\alpha\oplus\xi\oplus d\oplus\phi\rangle_2 = |\bar{\chi}\oplus s,+\tfrac{1}{2}\rangle_2, \tag{3.153}$$

where $|\bar{\chi}\oplus s,+\frac{1}{2}\rangle_1$ and $|\bar{\chi}\oplus s,+\frac{1}{2}\rangle_2$ are orthogonal to each other. Now we are left with the transformations to the anti-Clifford vacuum states $|\beta^-\rangle$ in last column of the diamond:

$$\bar{\mathcal{Q}}_-(\epsilon_j)\,|\bar{\chi},+\tfrac{3}{2};(\epsilon\eta)\eta_i\rangle = \bar{\mathcal{Q}}_+(\epsilon_j)\,|\bar{\chi}\oplus s,+\tfrac{1}{2}\rangle_1 = |\beta^-,+1;(\epsilon\eta)^2\rangle, \tag{3.154}$$

$$\bar{\mathcal{Q}}_+(\epsilon_j)\,|\bar{\chi}\oplus s,+\tfrac{1}{2}\rangle_2 = \bar{\mathcal{Q}}_-(\epsilon_j)\,|\bar{\chi}\oplus s,+\tfrac{1}{2}\rangle_1 = 0, \tag{3.155}$$

$$\bar{\mathcal{Q}}_-(\epsilon_j)\,|\bar{\chi}\oplus s,+\tfrac{1}{2}\rangle_2 = |\beta^-,0;(\epsilon\eta)^2\rangle. \tag{3.156}$$

This completes the helicity SUSY transformations for the upper half of the diamond representing the spin two supermultiplet of $\mathcal{N}=2$.

3.2.3 $\mathcal{N} = 4$ supermultiplet

In $\mathcal{N} = 4$ SUSY, the supercharges carry internal $SO(6) \equiv SU(4)$ spinor indices I or $\bar{I}$. Similar to $\mathcal{N} = 2$ case, we introduce the internal spinors η_I and $\bar{\eta}^{\bar{I}}$. Then the components of the (right-handed) anti-supercharge can be written as

$$\bar{\mathcal{Q}}_+ = \bar{\eta}^{\bar{I}} \frac{p^*_{\dot{a}}}{\langle pq\rangle} \bar{\mathcal{Q}}^{\dot{a}}_{\bar{I}}, \qquad \bar{\mathcal{Q}}_- = \bar{\eta}^{\bar{I}} \frac{q^*_{\dot{a}}}{\langle qp\rangle} \bar{\mathcal{Q}}^{\dot{a}}_{\bar{I}}. \tag{3.157}$$

We only have one big spin two supermultiplet in $\mathcal{N} = 4$, see Fig. 13. Starting from the Clifford vacuum Φ^+, c.f. (3.78), the remainder of the multiplet is filled by $\bar{\mathcal{Q}}_\pm$ application. Following the symmetry argument of the last subsections, we will only show the helicity SUSY transformation of the states in the upper half $j_z \geq 0$ of the diamond. And again, the internal wavefunctions of the physical states are displayed right behind the semicolon in the ket.

We start from Clifford vacuum state $|\Phi^+, 0; 1\rangle$ located at the far left of the diamond. The helicity SUSY transformations read

$$\bar{\mathcal{Q}}_+(\bar{\eta}^{\bar{I}})\,|\Phi^+, 0; 1\rangle = |\bar{a}, +\tfrac{1}{2}; \bar{\eta}^{\bar{I}}\rangle, \tag{3.158}$$

and

$$\bar{\mathcal{Q}}_+(\bar{\epsilon}^{\bar{J}})\,|\bar{a}, +\tfrac{1}{2}; \bar{\eta}^{\bar{I}}\rangle = |\beta^-, +1; \tfrac{1}{\sqrt{2}}\bar{\epsilon}^{\bar{J}}(\bar{\gamma}_m C)_{\bar{J}\bar{I}}\bar{\eta}^{\bar{I}}\rangle, \tag{3.159}$$

$$\begin{aligned}\bar{\mathcal{Q}}_-(\bar{\epsilon}^{\bar{J}})\,|\bar{a}, +\tfrac{1}{2}; \bar{\eta}^{\bar{I}}\rangle &= -\frac{1}{\sqrt{2}}|\beta^-, 0; \tfrac{1}{\sqrt{2}}\bar{\epsilon}^{\bar{J}}(\bar{\gamma}_m C)_{\bar{J}\bar{I}}\bar{\eta}^{\bar{I}}\rangle + \frac{1}{\sqrt{2}}|\Omega^-, 0; \tfrac{1}{12}\bar{\epsilon}^{\bar{J}}(\bar{\gamma}_{mnl} C)_{\bar{J}\bar{I}}\bar{\eta}^{\bar{I}}\rangle \\ &\equiv |\beta^- \oplus \Omega^-, 0\rangle,\end{aligned} \tag{3.160}$$

and

$$\bar{\mathcal{Q}}_+(\bar{\xi}^{\bar{K}})\,|\beta^-, +1; \tfrac{1}{\sqrt{2}}\bar{\epsilon}^{\bar{J}}(\bar{\gamma}_m C)_{\bar{J}\bar{I}}\bar{\eta}^{\bar{I}}\rangle = |\chi, +\tfrac{3}{2}; \varepsilon_{\bar{I}\bar{J}\bar{K}\bar{L}}\bar{\eta}^{\bar{I}}\bar{\epsilon}^{\bar{J}}\bar{\xi}^{\bar{K}} C^{\bar{L}}_{L}\rangle, \tag{3.161}$$

$$\begin{aligned}\bar{\mathcal{Q}}_-(\bar{\xi}^{\bar{K}})\,|\beta^-, +1; \tfrac{1}{\sqrt{2}}\bar{\epsilon}^{\bar{J}}(\bar{\gamma}_m C)_{\bar{J}\bar{I}}\bar{\eta}^{\bar{I}}\rangle &= -\frac{1}{\sqrt{3}}|\chi, +\tfrac{1}{2}; \varepsilon_{\bar{I}\bar{J}\bar{K}\bar{L}}\bar{\eta}^{\bar{I}}\bar{\epsilon}^{\bar{J}}\bar{\xi}^{\bar{K}} C^{\bar{L}}_{L}\rangle + \frac{\sqrt{2}}{\sqrt{3}}|\bar{r}, +\tfrac{1}{2}; \bar{r}_\beta\rangle \\ &\equiv |\chi \oplus \bar{r}, +\tfrac{1}{2}\rangle_1,\end{aligned} \tag{3.162}$$

$$\bar{\mathcal{Q}}_+(\bar{\xi}^{\bar{K}})\,|\beta^- \oplus \Omega^-, 0\rangle = \frac{1}{\sqrt{3}}|\chi, +\tfrac{1}{2}; \varepsilon_{\bar{I}\bar{J}\bar{K}\bar{L}}\bar{\eta}^{\bar{I}}\bar{\epsilon}^{\bar{J}}\bar{\xi}^{\bar{K}}C^{\bar{L}}_{L}\rangle + \frac{1}{\sqrt{6}}|\bar{r}, +\tfrac{1}{2}; \bar{r}_\beta\rangle + \frac{1}{\sqrt{2}}|\bar{r}, +\tfrac{1}{2}; \bar{r}_\Omega\rangle$$

$$\equiv |\chi \oplus \bar{r}, +\tfrac{1}{2}\rangle_2, \tag{3.163}$$

where

$$\bar{r}_\beta = \frac{\sqrt{3}}{2}\bar{\epsilon}^{\bar{J}}(\bar{\gamma}^m C)_{\bar{J}\bar{I}}\bar{\eta}^{\bar{I}}\bar{\xi}^{\bar{K}}\big(\delta^{(6)}_{mn}\delta_{\bar{K}}{}^{\bar{L}} + \frac{1}{6}(\bar{\gamma}_m\gamma_n)_{\bar{K}}{}^{\bar{L}}\big), \tag{3.164}$$

$$\bar{r}_\Omega = \frac{1}{48}\bar{\epsilon}^{\bar{J}}(\bar{\gamma}_{mnl}C)_{\bar{J}\bar{I}}\bar{\eta}^{\bar{I}}\bar{\xi}^{\bar{K}}(\bar{\gamma}_k\gamma^{mnl})_{\bar{K}}{}^{\bar{L}}. \tag{3.165}$$

Note that $\bar{r}_\beta$ and $\bar{r}_\Omega$ represent different and mutually orthogonal internal wavefunctions of $\bar{r}$.

The left-handed spin 3/2 states in the third column of the $\mathcal{N}=4$ diamond transform to

$$\bar{\mathcal{Q}}_+(\bar{\theta}^{\bar{M}})\,|\chi, +\tfrac{3}{2}; \varepsilon_{\bar{I}\bar{J}\bar{K}\bar{L}}\bar{\eta}^{\bar{I}}\bar{\epsilon}^{\bar{J}}\bar{\xi}^{\bar{K}}C^{\bar{L}}_{L}\rangle = |\alpha, +2; \varepsilon(\bar{\eta}\bar{\epsilon}\bar{\xi}\bar{\theta})\rangle, \tag{3.166}$$

$$\bar{\mathcal{Q}}_-(\bar{\theta}^{\bar{M}})\,|\chi, +\tfrac{1}{2}; \varepsilon_{\bar{I}\bar{J}\bar{K}\bar{L}}\bar{\eta}^{\bar{I}}\bar{\epsilon}^{\bar{J}}\bar{\xi}^{\bar{K}}C^{\bar{L}}_{L}\rangle = -\frac{1}{2}|\alpha, +1; \varepsilon(\bar{\eta}\bar{\epsilon}\bar{\xi}\bar{\theta})\rangle + \frac{\sqrt{3}}{2}|d, +1; d_\chi\rangle \equiv |\alpha \oplus d, +1\rangle_1, \tag{3.167}$$

$$\bar{\mathcal{Q}}_+(\bar{\theta}^{\bar{M}})\,|\chi \oplus \bar{r}, +\tfrac{1}{2}\rangle_1 = -\frac{1}{2}|\alpha, +1; \varepsilon(\bar{\eta}\bar{\epsilon}\bar{\xi}\bar{\theta})\rangle - \frac{1}{2\sqrt{3}}|d, +1; d_\chi\rangle + \frac{\sqrt{2}}{\sqrt{3}}|d, +1; d_{\bar{r}_\beta}\rangle$$

$$\equiv |\alpha \oplus d, +1\rangle_2, \tag{3.168}$$

$$\bar{\mathcal{Q}}_+(\bar{\theta}^{\bar{M}})\,|\chi \oplus \bar{r}, +\tfrac{1}{2}\rangle_2 = \frac{1}{2}|\alpha, +1; \varepsilon(\bar{\eta}\bar{\epsilon}\bar{\xi}\bar{\theta})\rangle + \frac{1}{2\sqrt{3}}|d, +1; d_\chi\rangle + \frac{1}{\sqrt{6}}|d, +1; d_{\bar{r}_\beta}\rangle$$

$$+ \frac{1}{\sqrt{2}}|d, +1; d_{\bar{r}_\Omega}\rangle \equiv |\alpha \oplus d, +1\rangle_3, \tag{3.169}$$

$$\bar{\mathcal{Q}}_-(\bar{\theta}^{\bar{M}})\,|\chi \oplus \bar{r}, +\tfrac{1}{2}\rangle_1 = \frac{1}{\sqrt{6}}|\alpha, 0; \varepsilon(\bar{\eta}\bar{\epsilon}\bar{\xi}\bar{\theta})\rangle - \frac{1}{\sqrt{6}}|d, 0; d_\chi\rangle + \frac{1}{\sqrt{3}}|d, 0; d_{\bar{r}_\beta}\rangle + \frac{1}{\sqrt{3}}|\zeta, 0; \zeta_{\bar{r}_\beta}\rangle$$

$$\equiv |\alpha \oplus d \oplus \zeta, 0\rangle_1, \tag{3.170}$$

$$\bar{\mathcal{Q}}_-(\bar{\theta}^{\bar{M}})\,|\chi \oplus \bar{r}, +\tfrac{1}{2}\rangle_2 = -\frac{1}{\sqrt{6}}|\alpha, 0; \varepsilon(\bar{\eta}\bar{\epsilon}\bar{\xi}\bar{\theta})\rangle + \frac{1}{\sqrt{6}}|d, 0; d_\chi\rangle + \frac{1}{2\sqrt{3}}|d, 0; d_{\bar{r}_\beta}\rangle$$

$$+ \frac{1}{2}|d, 0; d_{\bar{r}_\Omega}\rangle + \frac{1}{2\sqrt{3}}|\zeta, 0; \zeta_{\bar{r}_\beta}\rangle + \frac{1}{2}|\zeta, 0; \zeta_{\bar{r}_\Omega}\rangle$$

$$\equiv |\alpha \oplus d \oplus \zeta, 0\rangle_2, \tag{3.171}$$

where we have used the following abbreviations:

$$\varepsilon(\bar{\eta}\bar{\epsilon}\bar{\xi}\bar{\theta}) = \varepsilon_{\bar{I}\bar{J}\bar{K}\bar{L}}\bar{\eta}^{\bar{I}}\bar{\epsilon}^{\bar{J}}\bar{\xi}^{\bar{K}}\bar{\theta}^{\bar{L}}, \tag{3.172}$$

$$d_{\chi} = \frac{1}{2\sqrt{3}}\bar{\theta}^{\bar{M}}(\bar{\gamma}^{[m}\gamma^{n]})_{\bar{M}}{}^{\bar{L}}\varepsilon_{\bar{I}\bar{J}\bar{K}\bar{L}}\bar{\eta}^{\bar{I}}\bar{\epsilon}^{\bar{J}}\bar{\xi}^{\bar{K}}, \tag{3.173}$$

$$d_{\bar{r}_\beta} = \frac{1}{\sqrt{2}}\bar{\theta}^{\bar{M}}\bar{r}_{\beta}^{[m|,\bar{L}|}(\bar{\gamma}^{n]}C)_{\bar{M}\bar{L}}, \qquad d_{\bar{r}_\Omega} = \frac{1}{\sqrt{2}}\bar{\theta}^{\bar{M}}\bar{r}_{\Omega}^{[m|,\bar{L}|}(\bar{\gamma}^{n]}C)_{\bar{M}\bar{L}}, \tag{3.174}$$

$$\zeta_{\bar{r}_\beta} = \frac{1}{\sqrt{2}}\bar{\theta}^{\bar{M}}\bar{r}_{\beta}^{(m|,\bar{L}|}(\bar{\gamma}^{n)}C)_{\bar{M}\bar{L}}, \qquad \zeta_{\bar{r}_\Omega} = \frac{1}{\sqrt{2}}\bar{\theta}^{\bar{M}}\bar{r}_{\Omega}^{(m|,\bar{L}|}(\bar{\gamma}^{n)}C)_{\bar{M}\bar{L}}. \tag{3.175}$$

Similarly, d_χ, $d_{\bar{r}_\beta}$, $d_{\bar{r}_\Omega}$ and $\zeta_{\bar{r}_\beta}$, $\zeta_{\bar{r}_\Omega}$ are two pairs of orthogonal states with respect to the internal R-symmetry. Thus, the explicit computation confirms that different states located at the same point inside the diamond (with the same j_z) are orthogonal to each other.

Now we are left with the helicity SUSY transformations for the right half of the diamond. After some manipulations, we obtain

$$\bar{\mathcal{Q}}_-(\bar{\eta}^{\bar{I}})\,|\alpha,+2;\varepsilon(\bar{\eta}\bar{\epsilon}\bar{\xi}\bar{\theta})\rangle = \bar{\mathcal{Q}}_+(\bar{\eta}^{\bar{I}})\,|\alpha\oplus d,+1\rangle_1 = |\bar{\chi},+\tfrac{3}{2};\varepsilon(\bar{\eta}\bar{\epsilon}\bar{\xi}\bar{\theta})\bar{\eta}^{\bar{I}}\rangle, \tag{3.176}$$

$$\begin{aligned}\bar{\mathcal{Q}}_-(\bar{\eta}^{\bar{I}})\,|\alpha\oplus d,+1\rangle_2 &= \bar{\mathcal{Q}}_+(\bar{\eta}^{\bar{I}})\,|\alpha\oplus d\oplus\zeta,0\rangle_1\\ &= \frac{1}{\sqrt{3}}|\bar{\chi},+\tfrac{1}{2};\varepsilon(\bar{\eta}\bar{\epsilon}\bar{\xi}\bar{\theta})\bar{\eta}^{\bar{I}}\rangle - \frac{\sqrt{2}}{\sqrt{3}}|r,+\tfrac{1}{2};\varepsilon(\bar{\eta}\bar{\epsilon}\bar{\xi}\bar{\theta})\bar{\eta}^{\bar{I}}\bar{\gamma}^{m}_{\bar{I}I}\big(\delta^{(6)}_{mn}\delta^{I}{}_{J} + (\gamma_m\bar{\gamma}_n)^{I}{}_{J}\big)\rangle\\ &\equiv |\bar{\chi}\oplus r,+\tfrac{1}{2}\rangle_1,\end{aligned} \tag{3.177}$$

$$\begin{aligned}\bar{\mathcal{Q}}_-(\bar{\eta}^{\bar{I}})\,|\alpha\oplus d,+1\rangle_3 &= \bar{\mathcal{Q}}_+(\bar{\eta}^{\bar{I}})\,|\alpha\oplus d\oplus\zeta,0\rangle_2\\ &= \frac{1}{\sqrt{3}}|\bar{\chi},+\tfrac{1}{2};\varepsilon(\bar{\eta}\bar{\epsilon}\bar{\xi}\bar{\theta})\bar{\eta}^{\bar{I}}\rangle + \frac{1}{\sqrt{6}}|r,+\tfrac{1}{2};\varepsilon(\bar{\eta}\bar{\epsilon}\bar{\xi}\bar{\theta})\bar{\eta}^{\bar{I}}\bar{\gamma}^{m}_{\bar{I}I}\big(\delta^{(6)}_{mn}\delta^{I}{}_{J} + (\gamma_m\bar{\gamma}_n)^{I}{}_{J}\big)\rangle\\ &\quad+\frac{1}{\sqrt{2}}|r,+\tfrac{1}{2};\varepsilon(\bar{\eta}\bar{\epsilon}\bar{\xi}\bar{\theta})\bar{\eta}^{\bar{I}}\bar{\gamma}^{mnl}_{\bar{I}I}(\gamma_k\bar{\gamma}_{mnl})^{I}{}_{J}\rangle\\ &\equiv |\bar{\chi}\oplus r,+\tfrac{1}{2}\rangle_2,\end{aligned} \tag{3.178}$$

and

$$\bar{\mathcal{Q}}_-(\bar{\epsilon}^{\bar{J}})\,|\bar{\chi},+\tfrac{3}{2};\varepsilon(\bar{\eta}\bar{\epsilon}\bar{\xi}\bar{\theta})\bar{\eta}^{\bar{I}}\rangle = \bar{\mathcal{Q}}_+(\bar{\epsilon}^{\bar{J}})\,|\bar{\chi}\oplus r,+\tfrac{1}{2}\rangle_1 = |\beta^+,+1;\tfrac{1}{\sqrt{2}}\varepsilon(\bar{\eta}\bar{\epsilon}\bar{\xi}\bar{\theta})\bar{\epsilon}^{\bar{J}}(\bar{\gamma}_m C)_{\bar{J}\bar{I}}\bar{\eta}^{\bar{I}}\rangle, \tag{3.179}$$

$$\bar{\mathcal{Q}}_-(\bar{\epsilon}^{\bar{J}})\,|\bar{\chi}\oplus r,+\tfrac{1}{2}\rangle_2 = -\frac{1}{\sqrt{2}}|\beta^+,0;\tfrac{1}{\sqrt{2}}\varepsilon(\bar{\eta}\bar{\epsilon}\bar{\xi}\bar{\theta})\bar{\epsilon}^{\bar{J}}(\bar{\gamma}_m C)_{\bar{J}\bar{I}}\bar{\eta}^{\bar{I}}\rangle$$

$$+\frac{1}{\sqrt{2}}|\Omega^+, 0; \tfrac{1}{12}\varepsilon(\bar{\eta}\bar{\epsilon}\bar{\xi}\bar{\theta})\bar{\epsilon}^{\bar{J}}(C\gamma_{mnl})_{\bar{J}\bar{I}}\bar{\eta}^{\bar{I}}\rangle \equiv |\beta^+ \oplus \Omega^+, 0\rangle, \tag{3.180}$$

and

$$\bar{\mathcal{Q}}_-(\bar{\xi}^{\bar{K}})\,|\beta^+, +1; \tfrac{1}{\sqrt{2}}\varepsilon(\bar{\eta}\bar{\epsilon}\bar{\xi}\bar{\theta})\bar{\epsilon}^{\bar{J}}(\bar{\gamma}_m C)_{\bar{J}\bar{I}}\bar{\eta}^{\bar{I}}\rangle = \bar{\mathcal{Q}}_+(\bar{\xi}^{\bar{K}})\,|\beta^+ \oplus \Omega^+, 0\rangle = |a, +\tfrac{1}{2}; \varepsilon(\bar{\eta}\bar{\epsilon}\bar{\xi}\bar{\theta})\varepsilon_{\bar{I}\bar{J}\bar{K}\bar{L}}\bar{\eta}^{\bar{I}}\bar{\epsilon}^{\bar{J}}\bar{\xi}^{\bar{K}}C_L^{\bar{L}}\rangle, \tag{3.181}$$

and finally we have

$$\bar{\mathcal{Q}}_-(\bar{\theta}^{\bar{L}})\,|a, +\tfrac{1}{2}; \varepsilon(\bar{\eta}\bar{\epsilon}\bar{\xi}\bar{\theta})\varepsilon_{\bar{I}\bar{J}\bar{K}\bar{L}}\bar{\eta}^{\bar{I}}\bar{\epsilon}^{\bar{J}}\bar{\xi}^{\bar{K}}C_L^{\bar{L}}\rangle = |\Phi^+, 0; [\varepsilon(\bar{\eta}\bar{\epsilon}\bar{\xi}\bar{\theta})]^2\rangle. \tag{3.182}$$

This completes the chain of transformations that take the Clifford vacuum $|\Phi^+\rangle$ into its anti-Clifford counterpart $|\Phi^-\rangle$.

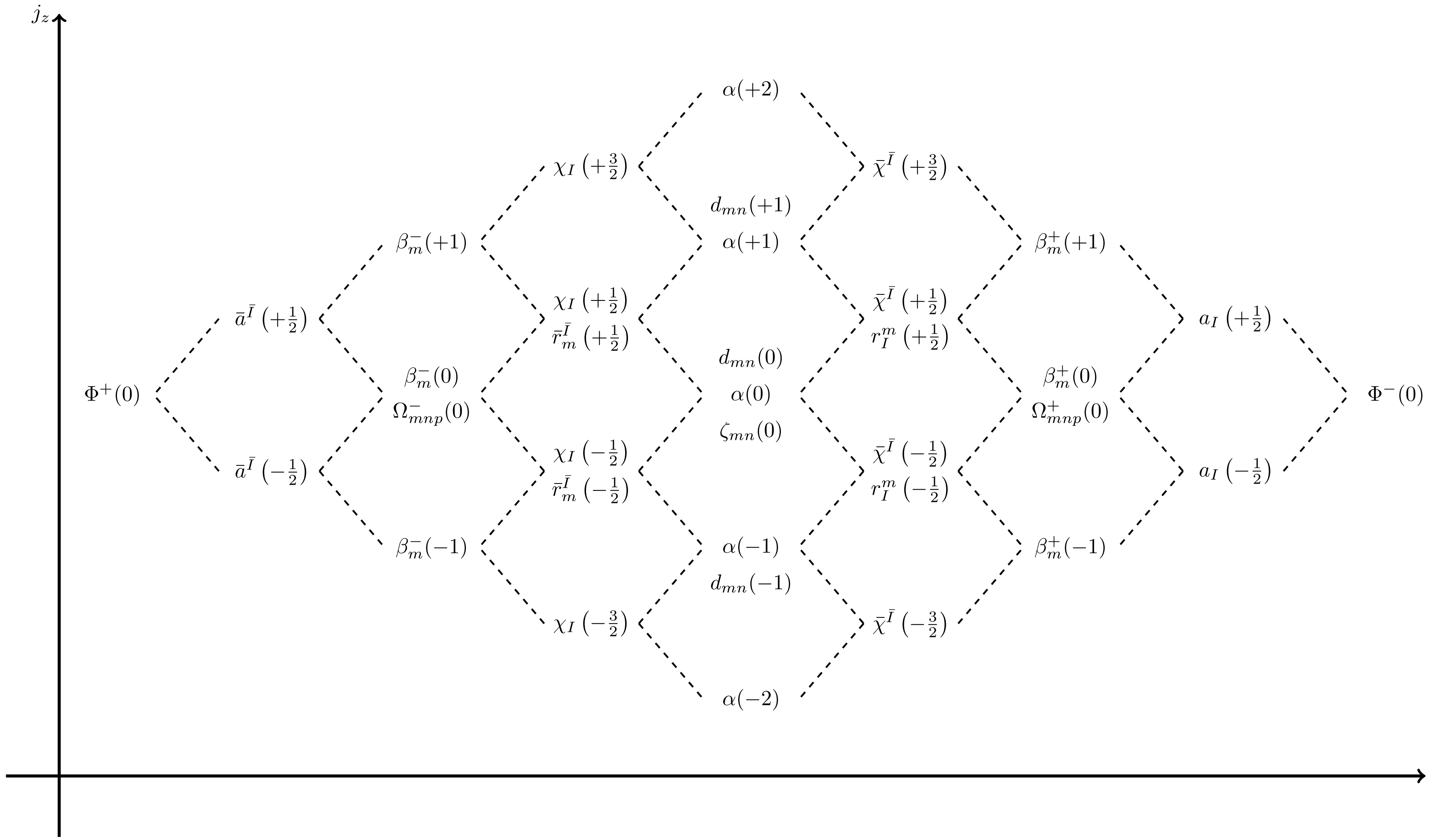

Figure 13: $\mathcal{N} = 4$ SUSY multiplet with scalar Clifford vacuum

4 Direct production of lightest massive superstrings

In this chapter, we discuss the direct production of the lightest Regge excitations. We will first review some basic knowledge of intersecting branes realization of the SM and how to compute the string scattering processes. Then we discuss some factorization properties the of string amplitudes. In Chapter 2, we already found the full physical fields contents for the first massive level. Here we focus on the universal excitations of gauge bosons originating from a stack of N D-branes extending into higher dimensions, which include one spin two particle, one vector and two complex scalars, with only the spin two and a single scalar coupled to massless gauge bosons directly at the disk level. These Regge excitations of SM fields are independent from the details of the internal geometry of the compactifications of the ten-dimensional superstring theory. On the other hand, quarks exist in the excited spin 3/2 and 1/2 states. The amplitudes with only two fermions are also universal. Next, we compute all amplitudes involving one of the universal Regge excitations and up to three massless partons. These amplitudes acquire a very simple form in the helicity basis, which also reveals certain selection rules similar (and related) to the vanishing of "all-plus" amplitudes at the zero mass level [31]. Finally, we square the appropriately crossed amplitudes for $p_1p_2 \to p_3R$, average over initial helicities and colors and sum over the colors and spin directions of the outgoing particles. In order to facilitate phenomenological applications of the partonic cross sections, we tabulate squared amplitudes according to the production processes: gluon fusion, gluon-quark absorption, quark-antiquark annihilation and quark-quark scattering.

This chapter is based on the paper [1].

4.1 Intersecting D-branes realization of the SM

In this section we review briefly how the SM is realized by intersecting D-branes.

Large extra dimensions can appear in string theory when the string mass scale M_{string} is very low, at the order of TeV [9,10]. There's a relation between the Planck mass M_{Planck}, the string mass scale M_{string} and the sizes of the compactified internal directions R_j. For type II superstring theory, we have [13],

$$g_{Dp}^2 M_{Planck} = 2^{\frac{5}{2}} \pi M_{string}^{7-p} \Big(\prod_{i=1}^{d_\parallel} R_i^{\parallel} \Big)^{-\frac{1}{2}} \Big(\prod_{j=1}^{d_\perp} R_j^{\perp} \Big)^{\frac{1}{2}} \tag{4.1}$$

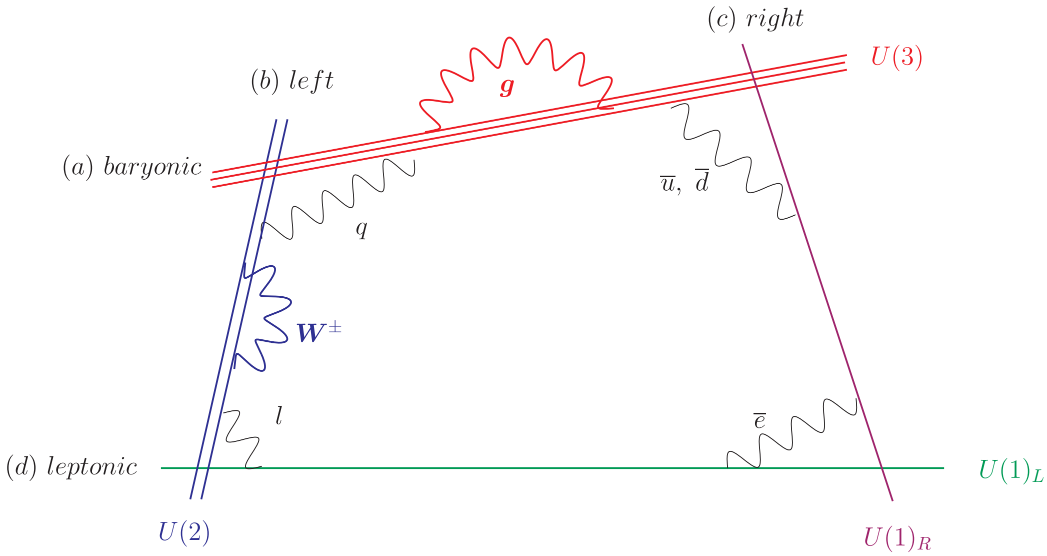

Figure 14: Intersection pattern of four stacks of D6-branes giving rise to the MSSM

Thus, by enlarging some of the transverse compactifications radii $R_j^{\perp}$, the string scale has to become lower in order to achieve the correct Planck mass.

One of the most common ways to realize the SM is by considering four stacks of D-branes. As it is shown in Fig. 14, the SM particles can be locally realized as massless open string excitations that live on a local four stacks of intersecting D-branes. The corresponding SM gauge group is given by

$$U(3)_a \times U(2)_b \times U(1)_c \times U(1)_d. \tag{4.2}$$

Gauge bosons are originated from open strings attached to the same stack of D-branes; chiral fermions are due to open string stretching between different stacks of D-branes.

The SM gauge bosons which are in the adjoint representations of the gauge group are originated from open strings attached to the same stack of D-branes. The SM matter (chiral) fields such as quark and leptons are due to open string stretching between different stacks of D-branes. They transform under bifundamental representations of the four gauge group factors, and they can also be in the antisymmetric representation **3** of $SU(3)_a$, in case the color stack of D-branes is intersected by its orientifold image.

In order to combine D-branes with the SM particle content and large extra dimensions, the local setup of intersecting D-branes (like the example above) which give rise to the spectrum of the SM have to be embedded into a global large volume CY-manifold to get a consistent compactification. We will not go through details of a fully realistic string compactification, since our focusing point – the universal tree-level open string scattering amplitudes only requires the local information. However, to use the CFT techniques to compute string amplitudes, we need to assume that the SM D-branes are wrapped around flat, toroidal like cycles. A fully consistent global orientifold model with all their tadpole and stability conditions satisfied is beyond the scope of this work, for these consistency conditions depend on the details of the compactification such as background fluxes. Yet even for models-dependent four-fermion couplings are argued in [13] that they only depend only on the local structure of the brane intersections, but not on the global Calabi Yau geometry.

4.2 Tree-level superstring amplitudes

4.2.1 Computation of tree-level superstring amplitudes

n-point string amplitudes are obtained by calculating the n-point correlation functions of associate vertex operators on the boundary of the disk, which read,

$$\mathscr{A}^{(n)} = \sum V_{\mathrm{CKG}}^{-1} \int (\prod_{i=1}^{n} \mathrm{d}z_i) \langle V(z_1) V(z_2) V(z_3) V(z_4) \cdots V(z_n) \rangle \tag{4.3}$$

where the sum runs over all the cyclic ordering of the n vertices on the boundary of the disk. In three-point amplitude, there are two different orderings; in four-point amplitude, there are six. In order to cancel the total background ghost charge -2 of the disk D_2, we should choose the vertex operators in the correlator in appropriate ghost "pictures" which makes the total ghost number to be -2. In addition, the factor V_{CKG} is defined to be the volume of the *conformal Killing group* of the disk after choosing the conformal gauge, which would be canceled by fixing three vertices and introducing respective c-ghost fields into the vertex operators. Because of *PSL(2,* $\mathbb{R}$*)* invariance on the disk, we can fix three vertex operators on the boundary of the disk. Mathematically, $V_{\rm CKG}^{-1}$ can be written as

$$V_{\rm CKG}^{-1}=c(z_i)c(z_j)c(z_k)\delta(z_i-\omega_i)\delta(z_j-\omega_j)\delta(z_k-\omega_k)\,, \tag{4.4}$$

where $\omega_i,\omega_j,\omega_k$ are totally arbitrary, however it is the most convenient to choose these three position to be 0, 1 and ∞. Then we integrate over other $n-3$ points and get the amplitude [27,65] which could be written in the form

$$\mathscr{A}=\sum\langle c_1V(z_1)c_2V(z_2)c_3V(z_3)\int(\prod_{i=4}^{n}\mathrm{d}z_i)V(z_4)\cdots V(z_n)\rangle\,. \tag{4.5}$$

Following the conventions in [65], here we list several the most important correlation functions of the world-sheet fields:

$$\langle c_1c_2c_3\rangle=|z_{12}z_{13}z_{23}|\,, \tag{4.6}$$

$$\langle e^{-\phi}(z_1)e^{-\phi}(z_2)\rangle=z_{12}^{-1}\,, \tag{4.7}$$

$$\langle\psi^\mu(z_1)\psi^\nu(z_2)\rangle=\eta^{\mu\nu}z_{12}^{-1}\,, \tag{4.8}$$

$$\langle X^\mu(z_1)X^\nu(z_2)\rangle_{D_2}=-2\alpha'\eta^{\mu\nu}\ln|z_{12}|\,, \tag{4.9}$$

$$\langle\prod_{i=1}^{n}e^{ik_iX(z_i)}\rangle_{D_2}=iC_{D_2}(2\pi)^d\delta^d(\Sigma k_i)\prod_{\substack{i,j=1\\1<j}}^{n}|z_{ij}|^{2\alpha' k_i\cdot k_j}\,, \tag{4.10}$$

where $z_{ij}=z_i-z_j$ and $C_{D_2}=1/(g^2\alpha'^2)$ are normalized in [13]. The last two correlators are for the X fields on the D_2 boundary, and are obtained by "doubling trick". We also need to use,

$$\begin{aligned}\langle\prod_{i=1}^{n}e^{ik_iX(z_i)}\prod_{j=1}^{p}\partial X^{\mu_j}(z_j)\rangle_{D_2}&=iC_{D_2}(2\pi)^d\delta^d(\Sigma k_i)\prod_{\substack{i,j=1\\1<j}}^{n}|z_{ij}|^{2\alpha' k_i\cdot k_j}\\&\times\prod_{j=1}^{p}\big[\sum_{\substack{i=1\\i\neq j}}^{n}(-2i\alpha'\frac{k_i^{\mu_j}}{z_{ji}})+q^{\mu_j}(z_j)\big]\,,\end{aligned} \tag{4.11}$$

where q's term are contracted by $-2\alpha'\eta^{\mu_j\mu_i}(z_{ji})^{-2}$.

4.2.2 Universality of four-dimensional tree-level amplitudes

Although we have adapted our whole setup to full-fledged ten-dimensional superstring theory with spacetime filling D9-branes, it turns out that all the results presented in the previous sections except for the four-fermion amplitude can be taken over to lower dimensional Dp-brane world volumes and compactification geometries.

As we already saw in Chapter 2, dimensional reduction of the spacetime gluon vertex operators simply replace $SO(1,9)$ and ten-dimensional world-sheet fields by four-dimensional fields. Correlation functions involving exclusively the world-sheet fields X and ψ do not depend on their dimensions. Moreover, we do not have any contractions which would get the dimension of the spacetime D (e.g., identity such as δ^μ_μ will give rise to quantities dependent on the spacetime dimensions D). Thus the n-gluon amplitudes do not depend on the spacetime dimensions.

Under dimension reduction, the ten-dimensional spin fields S_α can be factorize as $S_a \otimes s^i_{\text{int}}$ or $S^{\dot b} \otimes \bar{s}^j_{\text{int}}$ into four-dimensional spinors $S_a, S^{\dot b}$ and internal components $s^i_{\text{int}}, \bar{s}^j_{\text{int}}$. We already saw that for the four-dimensional gaugino the internal components $(s^i_{\text{int}}, \bar{s}^j_{\text{int}}) \equiv (\Sigma, \bar\Sigma)$ in Chapter 2. However, for the chiral matter that are located at D-brane intersections, $s^i_{\text{int}}, \bar{s}^j_{\text{int}}$ are identified with the boundary changing operators Ξ and $\bar\Xi$ will be defined in Eq. (4.67), with OPE

$$\langle \Xi^{a\cap b}(z_1)\, \bar\Xi^{a\cap b}(z_2)\rangle = \frac{1}{(z_1 - z_2)^{3/4}} \ . \tag{4.12}$$

As we can see, two-point function in both spin fields sector and internal sector are completely determined by their conformal weights, which shows, the four-dimensional amplitudes involving only two fermion fields are also universal to all compactifications.

4.3 Parton amplitudes and factorization on massive poles

Regge excitations may appear in resonance channels of SM processes or may be directly produced as external states. While the first effect has been extensively studied in [13–15] the latter effect will be discussed in this work. A first look at the couplings of massless SM particles to massive Regge states is made by considering the factorization of higher–point amplitudes involving massless external states. In what follows we shall discuss this factorization on general grounds. We consider scattering amplitudes involving massless SM model open string fields Φ_i as external particles. These amplitudes are described[22] by the exchange of the light (massless) SM fields and the tower of infinite many higher Regge excitations.

Due to the extended nature of strings the string amplitudes are generically non–trivial functions of α' in addition to the usual dependence on the kinematic invariants and degrees of freedom of the external states. In the effective field theory description this α'–dependence gives rise to a series of infinite many resonance channels due to Regge excitations and new contact interactions involving massless SM fields and massive Regge states. As a consequence of unitarity an N–point tree–level string amplitude[23] $\mathcal{M}(\Phi_1,\ldots,\Phi_N)$ can be written as an infinite sum over exchanges of (massive) intermediate string states $|J,n\rangle$ coupling to N_1 and N_2 external massless string states, with $N_1+N_2=N$. For each level this pole expansion gives rise to (new) N_1+1– and N_2+1–point couplings between the massive string states $|J,n\rangle$ and the N_1 and N_2 external massless string states, respectively.

In the following we illustrate this at the four–gluon amplitude, i.e., $\Phi_i=g_i$ and $N_1=N_2=2$. The latter gives rise to an infinite series of three–point couplings involving two massless gluons and massive string state $|J,n\rangle$. The general expression for the color ordered four–gluon amplitude is

$$\begin{aligned}\mathcal{M}(g_1,g_2,g_3,g_4) &= 2\ g_{YM}^2\ K_4(\varepsilon_1,k_1;\varepsilon_2,k_2;\varepsilon_3,k_3;\varepsilon_4,k_4)\\ &\times \left\{ T^{a_1a_2a_3a_4}\ \frac{B(\alpha's,\alpha'u)}{\alpha't}+T^{a_2a_3a_1a_4}\ \frac{B(\alpha't,\alpha'u)}{\alpha's}+T^{a_3a_1a_2a_4}\ \frac{B(\alpha's,\alpha't)}{\alpha'u}\right\}\ ,\end{aligned} \tag{4.14}$$

[22]There may be additional resonance channels due to the exchange of KK and winding states, as it is the case for amplitudes involving at least four quarks or leptons.

[23]Disk amplitudes $\mathcal{M}(\Phi_1,\ldots,\Phi_N)$ involving N open string states Φ_i as external states decompose into a sum over all possible orderings ρ of the corresponding vertex operators V_{Φ_i} along the boundary of the disk

$$\mathcal{M}(\Phi_1,\ldots,\Phi_N)=\sum_{\rho\in S_N}\mathcal{M}_\rho(\Phi_1,\ldots,\Phi_N)\sum_{\rho\in S_N}\mathrm{Tr}(T^{a_{1_\rho}}\ldots T^{a_{N_\rho}})\ A(1_\rho,\ldots,N_\rho)\ , \tag{4.13}$$

with $i_\rho=\rho(i)$ and the partial ordered amplitudes $A(1_\rho,\ldots,N_\rho)$. Furthermore, T^{a_i} is the Chan–Paton factor accounting for the gauge degrees of freedom of the two ends of the ith open string.

with the kinematic factor [66, 67]

$$\begin{aligned}K_4(\varepsilon_1,k_1;\varepsilon_2,k_2;\varepsilon_3,k_3;\varepsilon_4,k_4)&=\alpha' t\alpha' u\ (\varepsilon_1\varepsilon_2)\ (\varepsilon_3\varepsilon_4)+\alpha' s\alpha' t\ (\varepsilon_1\varepsilon_4)\ (\varepsilon_2\varepsilon_3)+\alpha' s\alpha' u\ (\varepsilon_1\varepsilon_3)\ (\varepsilon_2\varepsilon_4)\\ &+\alpha' s\ [\ (\varepsilon_1\varepsilon_3)(\varepsilon_2k_3)(\varepsilon_4k_1)+(\varepsilon_1\varepsilon_4)(\varepsilon_2k_4)(\varepsilon_3k_1)+(\varepsilon_2\varepsilon_3)(\varepsilon_1k_3)(\varepsilon_4k_2)+(\varepsilon_2\varepsilon_4)(\varepsilon_1k_4)(\varepsilon_3k_2)\ ]\\ &+\alpha' t\ [\ (\varepsilon_1\varepsilon_2)(\varepsilon_3k_2)(\varepsilon_4k_1)+(\varepsilon_1\varepsilon_4)(\varepsilon_2k_1)(\varepsilon_3k_4)+(\varepsilon_2\varepsilon_3)(\varepsilon_1k_2)(\varepsilon_4k_3)+(\varepsilon_3\varepsilon_4)(\varepsilon_1k_4)(\varepsilon_2k_3)\ ]\\ &+\alpha' u\ [\ (\varepsilon_1\varepsilon_2)(\varepsilon_3k_1)(\varepsilon_4k_2)+(\varepsilon_1\varepsilon_3)(\varepsilon_2k_1)(\varepsilon_4k_3)+(\varepsilon_2\varepsilon_4)(\varepsilon_1k_2)(\varepsilon_3k_4)+(\varepsilon_3\varepsilon_4)(\varepsilon_1k_3)(\varepsilon_2k_4)\ ]\,.\end{aligned}\tag{4.15}$$

and the color factor:

$$T^{a_1a_2a_3a_4}=\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})+\mathrm{Tr}(T^{a_4}T^{a_3}T^{a_2}T^{a_1})\ .\tag{4.16}$$

Above, ε_i are the polarization vectors and k_i the external momenta of the four gluons. Furthermore, we have the kinematic invariants $\hat s=2\alpha' k_1k_2$, $\hat t=2\alpha' k_1k_3$ and $\hat u=2\alpha' k_1k_4$.

In what follows we shall concentrate on the partial amplitude $\mathcal{M}_{(1234)}$. According to the definition (4.13) we have: $\mathcal{M}_{(1234)}=\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\ A(1,2,3,4)$. With (c.f. Ref. [13])

$$\frac{B(\hat s,\hat u)}{\hat t}=\frac{1}{\hat t\hat u}\ \frac{\Gamma(\hat s)\ \Gamma(1+\hat u)}{\Gamma(\hat s+\hat u)}=\sum_{n=0}^{\infty}\ \frac{\gamma(n)}{\hat s+n}\ ,\tag{4.17}$$

and

$$\gamma(n)=\frac{1}{n!}\ \frac{\Gamma(\alpha' u+n)}{\Gamma(\alpha' u+1)}=\frac{1}{n!}\ \frac{1}{\alpha' u}\ \prod_{j=1}^{n}(\alpha' u-1+j)\tag{4.18}$$

the amplitude (4.14) can be written as an infinite sum over s-channel poles at the masses of the Regge excitations:

$$\mathcal{M}_{(1234)}(g_1,g_2,g_3,g_4)=2\ g_{YM}^2\ \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\ K_4(\varepsilon_1,k_1;\varepsilon_2,k_2;\varepsilon_3,k_3;\varepsilon_4,k_4)\ \sum_{n=0}^{\infty}\ \frac{\gamma(n)}{\alpha' s+n}\ .\tag{4.19}$$

In (4.19) to each residue at $\alpha' s=-n$ a class of three–point couplings of two massless and one massive Regge state $|J,n\rangle$ of a specific spin J is associated, c.f. Fig. 15.

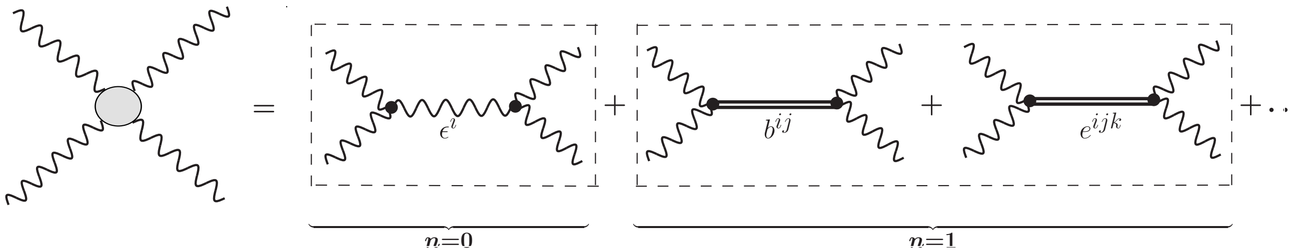

Figure 15: Factorization of four-gluon amplitude into pairs of two three-point couplings.

From (4.19) the three–point couplings are determined by the product $2\ g_{YM}^2\ \gamma(n)\ K_4$. To cast the residue of (4.19) into suitable form non–trivial factorization properties of the kinematic factor (4.15) have to hold.

We shall now evaluate for the amplitude (4.19) the contribution to the residue of the pole in $\alpha' s$ at $\alpha' s = -n$, with $n = 0, 1, \ldots$. At the level $n = 0$ only a massless gluon with polarization ε^i and spin $J = 1$ is exchanged. Hence, we obtain the following residue at $\alpha' s = 0$

$$\begin{aligned}
\mathrm{Res}_{\substack{\alpha' s=0 \\ \alpha' u=-\alpha' t}}\ \mathcal{M}_{(1234)}(g_1, g_2, g_3, g_4) &= 2\ g_{YM}^2\ \mathrm{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4}) \\
&\times\ \gamma(0)\ K_4(\varepsilon_1, k_1; \varepsilon_2, k_2; \varepsilon_3, k_3; \varepsilon_4, k_4)|_{\substack{\alpha' s=0 \\ \alpha' u=-\alpha' t}} \\
&= \sum_{\varepsilon(k)} K_{3,0}(\varepsilon_1, k_1; \varepsilon_2, k_2; \varepsilon, k)\ K_{3,0}(\varepsilon_3, k_3; \varepsilon_4, k_4; \varepsilon, -k)\ ,
\end{aligned} \tag{4.20}$$

with the YM three–vertex:

$$\begin{aligned}
K_{3,0}(\varepsilon_1, k_1; \varepsilon_2, k_2; \varepsilon_3, k_3) &= g_{YM}\ \mathrm{Tr}(T^{a_1}[T^{a_2}, T^{a_3}]) \\
&\times\ \{\ (\varepsilon_1\varepsilon_2)\ (\varepsilon_3 k_1) + (\varepsilon_1\varepsilon_3)\ (\varepsilon_2 k_3) + (\varepsilon_2\varepsilon_3)\ (\varepsilon_1 k_2)\ \}\ .
\end{aligned} \tag{4.21}$$

Furthermore we have applied the completeness relations

$$\sum_a \mathrm{Tr}(T^{a_1} T^{a_2} T^a)\ \mathrm{Tr}(T^a T^{a_3} T^{a_4}) = \frac{1}{2}\ \mathrm{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4})\ , \tag{4.22}$$

width the sum over the Chan–Paton wavefunction of the intermediate state.

At the $n = 1$ level exchanges of a spin $J = 2$ state b^{ij} and a $J = 0$ state e^{ijk} occur [43]. For the amplitude

(4.19) we obtain the following residue at $\alpha' s = -1$

$$
\begin{aligned}
\mathrm{Res}_{\substack{\alpha' s=-1 \\ \alpha' u = 1-\alpha' t}} \mathcal{M}_{(1234)}(g_1,g_2,g_3,g_4) &= 2\, g_{YM}^2\ \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) \\
&\times\ \gamma(1)\ K_4(\varepsilon_1,k_1;\varepsilon_2,k_2;\varepsilon_3,k_3;\varepsilon_4,k_4)\big|_{\substack{\alpha' s=-1 \\ \alpha' u = 1-\alpha' t}} \\
&= \sum_{e(k)} K_{3,1}(\varepsilon_1,k_1;\varepsilon_2,k_2;e,k)\ K_{3,1}(\varepsilon_3,k_3;\varepsilon_4,k_4;e,-k) \\
&+ \sum_{b(k)} K_{3,2}(\varepsilon_1,k_1;\varepsilon_2,k_2;b,k)\ K_{3,2}(\varepsilon_3,k_3;\varepsilon_4,k_4;b,-k)
\end{aligned}
\tag{4.23}
$$

with the two three–point vertices

$$
\begin{aligned}
K_{3,1}(\varepsilon_1,k_1;\varepsilon_2,k_2;e,k) &= 6\ g_{YM}\ \{\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}) + \mathrm{Tr}(T^{a_2}T^{a_1}T^{a_3})\}\ e^{ijk}\ \varepsilon_{1i}\ \varepsilon_{2j}\ k_{1k}\ , \\
K_{3,2}(\varepsilon_1,k_1;\varepsilon_2,k_2;b,k) &= g_{YM}\ \{\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}) + \mathrm{Tr}(T^{a_2}T^{a_1}T^{a_3})\} \\
&\times\ b^{ij}\ \{\ (k_1k_2)\ \varepsilon_{1i}\varepsilon_{2j} - k_{1i}\varepsilon_{2j}\ (\varepsilon_1 k_2) - k_{2i}\varepsilon_{1j}\ (\varepsilon_2 k_1) + k_{1i}k_{2j}\ (\varepsilon_1\varepsilon_2)\ \}\ ,
\end{aligned}
\tag{4.24}
$$

involving two massless gluons and one massive string state e^{ijk} and b^{ij}, respectively. In (4.23) the second equality follows by applying results from [68–70].

4.4 Two- and three-particle decay amplitudes

With all the physical vertex operators at hand, we are ready to compute the amplitudes describing two- and three-particle decays of the bosons – $\alpha(J=2)$, $d(J=1)$, $\Phi_{\pm}(J=0)$, $\Omega(J=0)$ with vertex operators written as we obtain one complex scalar $\Phi^a \equiv \Phi^{a+}$ ($\overline{\Phi}^a \equiv \Phi^{a-}$) and one $J=2$ particle B^a, with the vertices given by

$$
V^{(-1)}_{\alpha^a} = \frac{g_A}{\sqrt{2\alpha'}}\, T^a\, e^{-\phi}\, \alpha_{\mu\nu}\, i\partial X^\mu \psi^\nu\, e^{ikX}, \tag{4.25}
$$

$$
V^{(-1)}_{d^a} = g_A\ T^a\ e^{-\phi}\ \xi_\mu\ \psi^\mu\ \mathcal{J}\ e^{ikX}\ , \tag{4.26}
$$

$$
\begin{aligned}
V^{(-1)}_{\Phi^{a\pm}} = \frac{g_A}{2\sqrt{2\alpha'}}\, T^a\, e^{-\phi} \Big[& (g_{\mu\nu}\ +\ 2\alpha' k_\mu k_\nu)\, i\partial X^\mu\, \psi^\nu\ +\ 2\alpha' k_\mu \partial\psi^\mu \\
& \pm\ \frac{i}{6}\, 2\alpha'\, \varepsilon_{\mu\nu\rho\lambda}\, k^\lambda\, \psi^\mu\, \psi^\nu\, \psi^\rho \Big]\, e^{ikX}\ ,
\end{aligned}
\tag{4.27}
$$

$$
V^{(-1)}_{\Omega^{a\pm}} = g_A\ T^a\ e^{-\phi}\ \mathcal{O}^{\pm}\ e^{ikX}\ , \tag{4.28}
$$

with the on-shell conditions $k^\mu \alpha_{\mu\nu} = \eta^{\mu\nu}\alpha_{\mu\nu} = k^\mu d_\mu = 0$. The vertex operators of fermion $\chi(J=3/2)$ read

$$V^{(-\frac{1}{2})}_{\chi^\alpha_\beta} = \frac{g_A}{\sqrt{2}\alpha'^{1/4}} \big(T^\alpha_\beta\big)^{\beta_1}_{\alpha_1}\, \chi^a_\mu \Big(i\partial X^\mu\, S_a \;-\; \sqrt{2}\,\alpha'\; \not{k}_{a\dot{b}}\, S^{\mu\dot{b}} \Big)\, \Xi^{a\cap b}\, e^{-\phi/2}\, e^{ik\cdot X} \;, \tag{4.29}$$

$$V^{(-\frac{1}{2})}_{\bar{\chi}^\beta_\alpha} = \frac{g_A}{\sqrt{2}\alpha'^{1/4}} \big(T^\beta_\alpha\big)^{\alpha_1}_{\beta_1}\, \bar{\chi}^\mu_{\dot{a}} \Big(i\partial X_\mu\, S^{\dot{a}} \;-\; \sqrt{2}\,\alpha'\; \not{k}^{\dot{a}b}\, S_{\mu b} \Big)\, \bar{\Xi}^{a\cap b}\, e^{-\phi/2}\, e^{ik\cdot X} \;, \tag{4.30}$$

with the on-shell condition $k^\mu\, \chi^a_\mu = \chi^a_\mu\, \sigma^\mu_{a\dot{b}} = k_\mu\, \bar{\chi}^\mu_{\dot{a}} = \bar{\chi}^\mu_{\dot{a}}\, \bar{\sigma}^{\dot{a}b}_\mu$. And for $a(J=1/2)$, we have

$$V^{(-\frac{1}{2})}_{a^\alpha_\beta} = \frac{\alpha'^{1/4} g_A}{2} \big(T^\alpha_\beta\big)^{\beta_1}_{\alpha_1}\; a^b \Big((\sigma_\mu\; \not{k})_b{}^a\, S_a\, i\partial X^\mu \;-\; 4\,\partial S_b \Big)\, \Xi^{a\cap b}\, e^{-\phi/2}\, e^{ik\cdot X} \;, \tag{4.31}$$

$$V^{(-\frac{1}{2})}_{\bar{a}^\beta_\alpha} = \frac{\alpha'^{1/4} g_A}{2} \big(T^\beta_\alpha\big)^{\alpha_1}_{\beta_1}\; \bar{a}_{\dot{b}} \Big((\bar{\sigma}_\mu\; \not{k})^{\dot{b}}{}_{\dot{a}}\, S^{\dot{a}}\, i\partial X^\mu \;-\; 4\,\partial S^{\dot{b}} \Big)\, \bar{\Xi}^{a\cap b}\, e^{-\phi/2}\, e^{ik\cdot X} \;. \tag{4.32}$$

All the above vertex operators that describe the physical open string states are normalized by using the factorization techniques introduced in the previous subsection. The coupling $g_A = (2\alpha')^{1/2} g$ where g is the gauge coupling.

Our notation for Mandelstam variables are

$$s = (k_1 + k_2)^2 \;, \qquad t = (k_1 + k_3)^2 \;, \qquad u = (k_1 + k_4)^2 \tag{4.33}$$

with all momenta incoming and on-shell:

$$\sum_{i=1}^{4} k_i = 0 \;, \qquad k_1^2 = k_2^2 = k_3^2 = 0 \;, \qquad k_4^2 = -m^2 = -\tfrac{1}{\alpha'} \;, \tag{4.34}$$

which implies the following relation for the dimensionless variables:

$$\alpha' s + \alpha' t + \alpha' u = -1 \;. \tag{4.35}$$

The spinor products will be abbreviated as

$$\langle k_i | k_j \rangle = \langle ij \rangle \;, \qquad [k_i | k_j] = [ij] \;. \tag{4.36}$$

Finally, we recall the string formfactor

$$V_t = V(s,t,u) = \frac{\Gamma(1+\alpha' s)\Gamma(1+\alpha' u)}{\Gamma(1+\alpha' s+\alpha' u)} \ , \qquad V_s = V_t(t \leftrightarrow s) \ , \quad V_u = V_t(t \leftrightarrow u) \ . \tag{4.37}$$

Note that once the kinematic constraint (4.35) is implemented,

$$V_t = \frac{\Gamma(1+\alpha' s)\Gamma(1+\alpha' u)}{\Gamma(\alpha' t)} \ , \tag{4.38}$$

4.4.1 Massive spin two boson $\alpha(J=2)$

We begin with the B-decays into gluons. The two-gluon channel is described by the amplitude

$$\mathscr{A}[\alpha;\varepsilon_1,\varepsilon_2] = (2\, d^{a_1a_2a_3})\ (4g\sqrt{2\alpha'})\ \alpha_{\mu\nu}\, [\, (\varepsilon_2 k_1)k_2^\mu\varepsilon_1^\nu + (\varepsilon_1 k_2)k_1^\mu\varepsilon_2^\nu - (k_1k_2)\varepsilon_2^\mu\varepsilon_1^\nu - (\varepsilon_1\varepsilon_2)k_1^\mu k_2^\nu\,] \ . \tag{4.39}$$

In the prefactor, we singled out the color factor,

$$2\, d^{a_1a_2a_3} = \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}) + \mathrm{Tr}(T^{a_2}T^{a_1}T^{a_3}) \ , \tag{4.40}$$

which appears after adding the contributions of the two orderings of the vertex operators inserted at the disk boundary. It is convenient to rewrite the amplitude (4.39) as

$$\mathscr{A}[\alpha;\varepsilon_1,\varepsilon_2] = 4g\ (2\, d^{a_1a_2a_3})\ (2\alpha')^{3/2}\ \mathcal{A}\,[\alpha;\varepsilon_1,\varepsilon_2] \tag{4.41}$$

Substitute the helicity wave functions of α which can be find in Appendix C. We find

$$\mathcal{A}\,[\alpha;\pm,\pm] = 0 \ , \tag{4.42}$$

thus non-vanishing amplitudes must necessarily involve two gluons with opposite polarizations. They read:

$$\mathcal{A}\,[\alpha(-2);+,-] \quad = \quad -\tfrac{1}{4}\langle p2\rangle^2[q1]^2 \ ,$$

$$
\begin{aligned}
\mathcal{A}\,[\alpha(-1);+,-] &= -\tfrac{1}{2}\langle p2\rangle^2[q1][p1]\ , \\
\mathcal{A}\,[\alpha(0);+,-] &= -\tfrac{\sqrt{6}}{4}\langle p2\rangle^2[p1]^2\ , \\
\mathcal{A}\,[\alpha(+1);+,-] &= +\tfrac{1}{2}\langle p2\rangle\langle q2\rangle[p1]^2\ , \\
\mathcal{A}\,[\alpha(+2);+,-] &= -\tfrac{1}{4}\langle q2\rangle^2[p1]^2\ .
\end{aligned}
\tag{4.43}
$$

As a first check of the above result, we can examine the probability for the decay of unpolarized B into a specific $(+,-)$ helicity configuration, by computing the sum

$$
\sum_{j=-2}^{+2}|\mathcal{A}[\alpha(j);+,-]|^2 = \frac{8}{\alpha'}\,g^2(2\,d^{a_1a_2a_3})^2\ , \tag{4.44}
$$

which does indeed turn out to be independent of the choice of reference vectors p and q. Now we can check if the result is consistent with string factorization. From Ref. [43] we know that only the spin 2 resonance appears in the s-channel of the four-gluon amplitude $\mathcal{M}[g_1^+,g_2^-,g_3^-,g_4^+]$, where it yields the following residue at $s=M^2=1/\alpha'$

$$
\mathrm{Res}_{s=1/\alpha'}\ \mathcal{M}[g_1^+,g_2^-,g_3^-,g_4^+] = 4\,g^2\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\,\alpha'\langle 23\rangle^2[14]^2+\ldots\ , \tag{4.45}
$$

where we picked up just one partial amplitude contribution. In order to compare our B-decay amplitude with the residue, we compute

$$
\sum_{j=-2}^{+2}\mathcal{A}[\alpha(j);+,-]^*\ \mathcal{A}[\alpha(j);+,-]\big|_{(1\to3,2\to4)}\ , \tag{4.46}
$$

with the color factor associated to the first ordering in Eq. (4.40), c.f. Eq. (4.22). The simplest way to perform the sum (4.46) is to set $p=k_1$ and $q=k_2$ because then only $J_z=-2$ contributes. Indeed, after combining the spin and color sums we recover Eq. (4.45), thus confirming the correct normalization of the vertex operator (4.25). Eqs. (4.41) and (4.43) can be also checked by comparing directly with Eq. (25) of Ref. [43].

Three-gluon α–decays are described by the following amplitude

$$\begin{aligned}
\mathscr{A}[\alpha;\varepsilon_1,\varepsilon_2,\varepsilon_3] \;&=\; 4\,g^2\sqrt{2\alpha'}\,\Big(V_t\,t^{a_1a_2a_3a_4} \;+\; V_s\,t^{a_2a_3a_1a_4} \;+\; V_u\,t^{a_3a_1a_2a_4}\Big)\\
\times\Bigg\{&\frac{1}{s}\Big[(\varepsilon_2\,\varepsilon_3)\,(\varepsilon_1\,k_2)\,(k_3^\mu\,\alpha_{\mu\nu}\,k_3^\nu) \;-\; (\varepsilon_1\,\varepsilon_3)\,(\varepsilon_2\,k_1)\,(k_3^\mu\,\alpha_{\mu\nu}\,k_3^\nu) \;+\; (\varepsilon_1\,\varepsilon_2)\,(\varepsilon_3\,k_2)\,(k_1^\mu\,\alpha_{\mu\nu}\,k_3^\nu)\\
&-(\varepsilon_1\,\varepsilon_2)\,(\varepsilon_3\,k_1)\,(k_2^\mu\,\alpha_{\mu\nu}\,k_3^\nu) \;+\; (\varepsilon_1\,k_3)\,(\varepsilon_2\,k_1)\,(k_3^\mu\,\alpha_{\mu\nu}\,\varepsilon_3^\nu) \;-\; (\varepsilon_2\,k_3)\,(\varepsilon_1\,k_2)\,(k_3^\mu\,\alpha_{\mu\nu}\,\varepsilon_3^\nu)\\
&+(\varepsilon_2\,k_1)\,(\varepsilon_3\,k_4)\,(k_3^\mu\,\alpha_{\mu\nu}\,\varepsilon_1^\nu) \;-\; (\varepsilon_1\,k_2)\,(\varepsilon_3\,k_4)\,(k_3^\mu\,\alpha_{\mu\nu}\,\varepsilon_2^\nu)\\
&+\frac{1}{2\alpha'}\,(\varepsilon_1\,k_2)\,(\varepsilon_2^\mu\,\alpha_{\mu\nu}\,\varepsilon_3^\nu) \;-\; \frac{1}{2\alpha'}\,(\varepsilon_2\,k_1)\,(\varepsilon_1^\mu\,\alpha_{\mu\nu}\,\varepsilon_3^\nu)\\
&-\frac{t}{2}\,(\varepsilon_1\,\varepsilon_2)\,(k_2^\mu\,\alpha_{\mu\nu}\,\varepsilon_3^\nu) \;+\; \frac{u}{2}\,(\varepsilon_1\,\varepsilon_2)\,(k_1^\mu\,\alpha_{\mu\nu}\,\varepsilon_3^\nu)\Big]\\
+\;&\frac{1}{u}\Big[(\varepsilon_1\,\varepsilon_3)\,(\varepsilon_2\,k_3)\,(k_1^\mu\,\alpha_{\mu\nu}\,k_1^\nu) \;-\; (\varepsilon_1\,\varepsilon_2)\,(\varepsilon_3\,k_2)\,(k_1^\mu\,\alpha_{\mu\nu}\,k_1^\nu) \;+\; (\varepsilon_2\,\varepsilon_3)\,(\varepsilon_1\,k_3)\,(k_1^\mu\,\alpha_{\mu\nu}\,k_2^\nu)\\
&-(\varepsilon_2\,\varepsilon_3)\,(\varepsilon_1\,k_2)\,(k_1^\mu\,\alpha_{\mu\nu}\,k_3^\nu) \;+\; (\varepsilon_2\,k_1)\,(\varepsilon_3\,k_2)\,(k_1^\mu\,\alpha_{\mu\nu}\,\varepsilon_1^\nu) \;-\; (\varepsilon_3\,k_1)\,(\varepsilon_2\,k_3)\,(k_1^\mu\,\alpha_{\mu\nu}\,\varepsilon_1^\nu)\\
&+(\varepsilon_3\,k_2)\,(\varepsilon_1\,k_4)\,(k_1^\mu\,\alpha_{\mu\nu}\,\varepsilon_2^\nu) \;-\; (\varepsilon_2\,k_3)\,(\varepsilon_1\,k_4)\,(k_1^\mu\,\alpha_{\mu\nu}\,\varepsilon_3^\nu)\\
&+\frac{1}{2\alpha'}\,(\varepsilon_2\,k_3)\,(\varepsilon_1^\mu\,\alpha_{\mu\nu}\,\varepsilon_3^\nu) \;-\; \frac{1}{2\alpha'}\,(\varepsilon_3\,k_2)\,(\varepsilon_1^\mu\,\alpha_{\mu\nu}\,\varepsilon_2^\nu)\\
&-\frac{s}{2}\,(\varepsilon_2\,\varepsilon_3)\,(k_3^\mu\,\alpha_{\mu\nu}\,\varepsilon_1^\nu) \;+\; \frac{t}{2}\,(\varepsilon_2\,\varepsilon_3)\,(k_2^\mu\,\alpha_{\mu\nu}\,\varepsilon_1^\nu)\Big]\\
+\;&\frac{1}{t}\Big[(\varepsilon_1\,\varepsilon_2)\,(\varepsilon_3\,k_1)\,(k_2^\mu\,\alpha_{\mu\nu}\,k_2^\nu) \;-\; (\varepsilon_2\,\varepsilon_3)\,(\varepsilon_1\,k_3)\,(k_2^\mu\,\alpha_{\mu\nu}\,k_2^\nu) \;+\; (\varepsilon_1\,\varepsilon_3)\,(\varepsilon_2\,k_1)\,(k_2^\mu\,\alpha_{\mu\nu}\,k_3^\nu)\\
&-(\varepsilon_1\,\varepsilon_3)\,(\varepsilon_2\,k_3)\,(k_1^\mu\,\alpha_{\mu\nu}\,k_2^\nu) \;+\; (\varepsilon_3\,k_2)\,(\varepsilon_1\,k_3)\,(k_2^\mu\,\alpha_{\mu\nu}\,\varepsilon_2^\nu) \;-\; (\varepsilon_1\,k_2)\,(\varepsilon_3\,k_1)\,(k_2^\mu\,\alpha_{\mu\nu}\,\varepsilon_2^\nu)\\
&+(\varepsilon_1\,k_3)\,(\varepsilon_2\,k_4)\,(k_2^\mu\,\alpha_{\mu\nu}\,\varepsilon_3^\nu) \;-\; (\varepsilon_3\,k_1)\,(\varepsilon_2\,k_4)\,(k_2^\mu\,\alpha_{\mu\nu}\,\varepsilon_1^\nu)\\
&+\frac{1}{2\alpha'}\,(\varepsilon_3\,k_1)\,(\varepsilon_1^\mu\,\alpha_{\mu\nu}\,\varepsilon_2^\nu) \;-\; \frac{1}{2\alpha'}\,(\varepsilon_1\,k_3)\,(\varepsilon_2^\mu\,\alpha_{\mu\nu}\,\varepsilon_3^\nu)\\
&-\frac{u}{2}\,(\varepsilon_1\,\varepsilon_3)\,(k_1^\mu\,\alpha_{\mu\nu}\,\varepsilon_2^\nu) \;+\; \frac{s}{2}\,(\varepsilon_1\,\varepsilon_3)\,(k_3^\mu\,\alpha_{\mu\nu}\,\varepsilon_2^\nu)\Big]\\
-\;&\frac{1}{2}\Big[(\varepsilon_1\,\varepsilon_2)\,\varepsilon_3^\mu\,\alpha_{\mu\nu}\,(k_2^\nu-k_1^\nu) \;+\; (\varepsilon_2\,\varepsilon_3)\,\varepsilon_1^\mu\,\alpha_{\mu\nu}\,(k_3^\nu-k_2^\nu) \;+\; (\varepsilon_1\,\varepsilon_3)\,\varepsilon_2^\mu\,\alpha_{\mu\nu}\,(k_1^\nu-k_3^\nu)\Big]\Bigg\}\,,
\end{aligned}\tag{4.47}$$

with the color factor

$$\begin{aligned}
t^{a_1a_2a_3a_4} \;&=\; \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) - \mathrm{Tr}(T^{a_4}T^{a_3}T^{a_2}T^{a_1})\\
&=\; i\,(d^{a_1a_4n}f^{a_2a_3n} - d^{a_2a_3n}f^{a_1a_4n})\,,
\end{aligned}\tag{4.48}$$

i.e., $t^{a_1a_2a_3a_4} = \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) - \mathrm{Tr}(T^{a_1}T^{a_4}T^{a_3}T^{a_2})$, $t^{a_2a_3a_1a_4} = \mathrm{Tr}(T^{a_1}T^{a_4}T^{a_2}T^{a_3}) - \mathrm{Tr}(T^{a_1}T^{a_3}T^{a_2}T^{a_4})$ and $t^{a_3a_1a_2a_4} = \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_4}T^{a_3}) - \mathrm{Tr}(T^{a_1}T^{a_3}T^{a_4}T^{a_2})$. Note that the massless (4.14) and massive (4.47) amplitudes have different group structures, c.f., Eq.(4.48) and (4.16), respectively. This is explained below.

Generally, under world–sheet parity an N–point open superstring amplitude $\mathscr{A}(1,\ldots,N)$ (recall the definition (4.13)) behaves as

$$\mathscr{A}(1,\ldots,N) = \left(\prod_{i=1}^{N}(-1)^{\alpha' m_i^2+\varepsilon}\right)\ \mathscr{A}(N,\ldots,1)\ , \tag{4.49}$$

with m_i^2 the masses of the external open string states. Furthermore, for $SO(N)$ representations we have $\varepsilon = 1$ and $\varepsilon = 0$ for $USp(N)$ representations [71]. Further relations between subamplitudes are obtained by analyzing their monodromy behavior w.r.t. to contour integrals in the complex plane [72]. As a consequence for amplitudes involving only massless external string states ($m_i^2 = 0$) the full set of relations allows to reduce the number of independent subamplitudes to $(N-3)!$ [72,73]. On the other hand, the set of relations for the massless case does not hold in the case if $m_i^2 \neq 0$ and new monodromy relations have to be derived.

For the case at hand, i.e., $m_i = 0,\ i = 1,2,3$ and $m_4^2 = \alpha'^{-1}$, the partial amplitudes are odd under the parity transformation. Hence from (4.49) we deduce:

$$\begin{aligned}
\mathscr{A}(1,2,3,4) &= -\mathscr{A}(1,4,3,2)\ ,\\
\mathscr{A}(1,2,4,3) &= -\mathscr{A}(1,3,4,2)\ ,\\
\mathscr{A}(1,3,2,4) &= -\mathscr{A}(1,4,2,3)\ .
\end{aligned} \tag{4.50}$$

This fact is manifest in the full amplitude (4.47) due to the color factor. After applying the contour arguments of [72] the following monodromy relation can be established for the case at hand:

$$\mathscr{A}(1,2,3,4) - e^{i\pi\alpha' s}\ \mathscr{A}(1,2,4,3) - e^{-i\pi\alpha' u}\ \mathscr{A}(1,4,2,3) = 0\ , \tag{4.51}$$

Together with (4.50) this relation allows to express all six partial amplitudes in terms of one, say $\mathscr{A}(1,2,3,4)$:

$$\mathscr{A}(1,4,3,2) = -\mathscr{A}(1,2,3,4)\,,$$

$$\mathscr{A}(1,2,4,3) = -\mathscr{A}(1,3,4,2) = \frac{\sin(\pi\alpha' u)}{\sin(\pi\alpha' t)}\mathscr{A}(1,2,3,4)\,,$$
$$\mathscr{A}(1,3,2,4) = -\mathscr{A}(1,4,2,3) = -\frac{\sin(\pi\alpha' s)}{\sin(\pi\alpha' t)}\mathscr{A}(1,2,3,4)\,. \tag{4.52}$$

Note, that (4.51) differs from the monodromy relation for the massless case, c.f. Eq. (4.8) of [72]. As a consequence also the solution (4.52) is different than in the massless case, c.f., Eq. (4.10) of [72]. It is easy to see that the relations (4.52) are indeed satisfied by the result (4.47).

In order to represent the amplitude (4.47) in the helicity basis, we rewrite it as

$$\mathscr{A}[\alpha;\varepsilon_1,\varepsilon_2,\varepsilon_3] \;=\; 8\,g^2\big(\,V_t\,t^{a_1a_2a_3a_4}+V_s\,t^{a_2a_3a_1a_4}+V_u\,t^{a_3a_1a_2a_4}\,\big)\,\sqrt{2\alpha'}\;\mathcal{A}\,[\alpha;\varepsilon_1,\varepsilon_2,\varepsilon_3]\;. \tag{4.53}$$

We find

$$\mathcal{A}\,[\alpha;\pm,\pm,\pm] = 0\;, \tag{4.54}$$

therefore non-vanishing amplitudes always involve one gluon of a given helicity and two of the opposite one. They have a very simple form:

$$\begin{aligned}
\mathcal{A}\,[\alpha(-2);+,+,-] \;&=\; \frac{1}{2\sqrt{2}}\,\frac{\langle p3\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\;,\\
\mathcal{A}\,[\alpha(-1);+,+,-] \;&=\; \frac{1}{\sqrt{2}}\;\frac{\langle p3\rangle^3\langle 3q\rangle}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\;,\\
\mathcal{A}\,[\alpha(0);+,+,-] \;&=\; \frac{\sqrt{3}}{2}\;\frac{\langle p3\rangle^2\langle 3q\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\;,\\
\mathcal{A}\,[\alpha(+1);+,+,-] \;&=\; \frac{1}{\sqrt{2}}\;\frac{\langle q3\rangle^3\langle 3p\rangle}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\;,\\
\mathcal{A}\,[\alpha(+2);+,+,-] \;&=\; \frac{1}{2\sqrt{2}}\,\frac{\langle q3\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\;.
\end{aligned} \tag{4.55}$$

Similarly to the two-gluon case, we can consider the case of unpolarized B decaying into a specific helicity configuration of the three gluons. By using Eqs. (4.53) and (4.55), we obtain

$$\sum_{\alpha=-2}^{+2}|\mathscr{A}(\alpha;+,+,-)|^2 = 16\,g^4\,\frac{(1-\alpha' s)^4}{\alpha'^3\,s\,t\,u}\,|V_t\,t^{a_1a_2a_3a_4}+V_s\,t^{a_2a_3a_1a_4}+V_u\,t^{a_3a_1a_2a_4}|^2\;. \tag{4.56}$$

Next, we turn to α-decays into fermions. The quark-antiquark channel is described by the amplitude

$$\mathscr{A}[\alpha; u_1, \bar{u}_2] = (T^a)^{\alpha_1}_{\alpha_2}(g\sqrt{2\alpha'})\, k_1^\mu \alpha_{\mu\nu}\, u_1^\lambda \sigma^\nu_{\lambda\dot{\rho}} \bar{u}_2^{\dot{\rho}} \ , \tag{4.57}$$

which we rewrite as:

$$\mathscr{A}[\alpha; u_1, \bar{u}_2] = g\, (T^a)^{\alpha_1}_{\alpha_2}\ (2\alpha')^{3/2} \mathcal{A}\left[\alpha; u_1, \bar{u}_2\right] \ . \tag{4.58}$$

For the specific $(+\frac{1}{2}, -\frac{1}{2})$ helicity configuration of the antiquark-quark pair, we obtain:

$$\begin{aligned}
\mathcal{A}\left[\alpha(-2); +\tfrac{1}{2}, -\tfrac{1}{2}\right] &= \tfrac{1}{2}\ \langle p1\rangle\langle p2\rangle[q1]^2 \ , \\
\mathcal{A}\left[\alpha(-1); +\tfrac{1}{2}, -\tfrac{1}{2}\right] &= \tfrac{1}{4}\ \langle p2\rangle[q1]\left(\langle q1\rangle[1q] - 3\langle p1\rangle[1p]\right) \ , \\
\mathcal{A}\left[\alpha(0); +\tfrac{1}{2}, -\tfrac{1}{2}\right] &= \tfrac{\sqrt{6}}{4}\ \langle 2p\rangle[p1]\left(\langle p1\rangle[1p] - \langle q1\rangle[1q]\right) \ , \\
\mathcal{A}\left[\alpha(+1); +\tfrac{1}{2}, -\tfrac{1}{2}\right] &= \tfrac{1}{4}\ \langle q2\rangle[p1]\left(\langle p1\rangle[1p] - 3\langle q1\rangle[1q]\right) \ , \\
\mathcal{A}\left[\alpha(+2); +\tfrac{1}{2}, -\tfrac{1}{2}\right] &= \tfrac{1}{2}\ \langle q1\rangle\langle q2\rangle[p1]^2 \ .
\end{aligned} \tag{4.59}$$

Adding up the moduli squares of the amplitudes, we obtain

$$\sum_{j=-2}^{+2} |\mathscr{A}[\alpha(j); +\tfrac{1}{2}, -\tfrac{1}{2}]|^2 = \frac{1}{2\alpha'}\, g^2\, [(T^a)^{\alpha_1}_{\alpha_2}]^2 \ , \tag{4.60}$$

which does not depend on the choice of the reference vectors. As a further check, we can compare our result with the residue of the two-gluon – quark-antiquark amplitude

$$\mathrm{Res}_{s=1/\alpha'}\ \mathcal{M}[q_1^-, \bar{q}_2^+, g_3^-, g_4^+] = 2\, g^2 \{T^{a_3} T^{a_4}\}^{\alpha_1}_{\alpha_2}\ \alpha' tu\, \frac{\langle 13\rangle^2}{\langle 14\rangle\langle 24\rangle} \ , \tag{4.61}$$

which is known to receive contributions from the spin two resonance only [43]. Indeed, the residue is correctly reproduced by

$$\sum_{j=-2}^{+2} \mathscr{A}[\alpha(j); +\tfrac{1}{2}, -\tfrac{1}{2}]^*\, \mathscr{A}[\alpha(j); -, +]\Big|_{(1\to 3, 2\to 4)} \ . \tag{4.62}$$

The amplitude with one gluon in addition to the quark-antiquark pair in the final state can be written as:

$$\mathscr{A}[\alpha;u_1,\bar{u}_2,\varepsilon_3]=2\,g^2\,\left[V_t(T^{a_3}T^{a_4})^{\alpha_1}_{\alpha_2}+V_u(T^{a_4}T^{a_3})^{\alpha_1}_{\alpha_2}\right]\,\sqrt{2\alpha'}\,\mathcal{A}\left[\alpha;u_1,\bar{u}_2,\varepsilon_3\right]\ , \tag{4.63}$$

with:

$$\begin{aligned}
\mathcal{A}\left[\alpha(-2);+\tfrac{1}{2},-\tfrac{1}{2},+\right] &= \frac{1}{\sqrt{2}}\,\frac{\langle p1\rangle\langle p2\rangle^3}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\ ,\\
\mathcal{A}\left[\alpha(-1);+\tfrac{1}{2},-\tfrac{1}{2},+\right] &= \frac{1}{2\sqrt{2}}\,\frac{\langle p2\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\big(\langle q1\rangle\langle p2\rangle+3\langle p1\rangle\langle q2\rangle\big)\ ,\\
\mathcal{A}\left[\alpha(0);+\tfrac{1}{2},-\tfrac{1}{2},+\right] &= \frac{\sqrt{3}}{2}\,\frac{\langle p2\rangle\langle q2\rangle}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\,\big(\langle q1\rangle\langle p2\rangle+\langle p1\rangle\langle q2\rangle\big)\ ,\\
\mathcal{A}\left[\alpha(+1);+\tfrac{1}{2},-\tfrac{1}{2},+\right] &= \frac{1}{2\sqrt{2}}\,\frac{\langle q2\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\big(3\langle q1\rangle\langle p2\rangle+\langle p1\rangle\langle q2\rangle\big)\ ,\\
\mathcal{A}\left[\alpha(+2);+\tfrac{1}{2},-\tfrac{1}{2},+\right] &= \frac{1}{\sqrt{2}}\,\frac{\langle q1\rangle\langle q2\rangle^3}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\ .
\end{aligned} \tag{4.64}$$

For the gluon with opposite helicity we have:

$$\mathcal{A}\left[\alpha;+\tfrac{1}{2},-\tfrac{1}{2},-\right]=\mathcal{A}^*\left[\alpha;+\tfrac{1}{2},-\tfrac{1}{2},+\right]\Big|_{(p\leftrightarrow q),(1\leftrightarrow 2)}\ . \tag{4.65}$$

The sum of the squared moduli of the corresponding amplitudes reads

$$\begin{aligned}
\sum_{j=-2}^{+2}\left|\mathscr{A}[\alpha(j);+\tfrac{1}{2},-\tfrac{1}{2},+]\right|^2 &= g^4\big|V_t(T^{a_3}T^{a_4})^{\alpha_1}_{\alpha_2}+V_u(T^{a_4}T^{a_3})^{\alpha_1}_{\alpha_2}\big|^2\\
&\quad\times\frac{(1-\alpha' t)^2}{\alpha'^2 s\,t\,u}\,\left(s+4\alpha' tu\right)\ ,
\end{aligned} \tag{4.66}$$

and a similar expression with $(k_1\leftrightarrow k_2, a_3\leftrightarrow a_4)$ for the gluon with opposite helicity.

4.4.2 Massive spin one boson $d(J=1)$

The spin one vector resonance has a different character than spin two, because it is tied to space–time SUSY. The internal part of the corresponding vertex operator (4.26) contains the current $\mathcal{J}$, which plays an important role in the world–sheet SCFT describing superstrings propagating on CYMs. As we have described in the third section the most natural way of thinking about this particle is as a two–gluino bound state. Indeed, with Σ the internal Ramond field associated to the gluino, the current $\mathcal{J}$ appears as a subleading term in the OPE (2.77). It is clear that the current $\mathcal{J}$ and the existence of the resonance $d(J=1)$ is a universal property of all $\mathcal{N}=1$ SUSY compactifications. At the disk level, this particle does not couple to purely gluonic processes. Its main decay channel is into two gluinos and its mass will be affected by the SUSY breaking mechanism. The reason why we include it in our discussion is that it also couples to the quark sector, therefore it can be a priori directly produced at the LHC.

In the intersecting D–brane models, the internal part of the quark vertex operators contains the boundary-changing operators [13]

$$\Xi^{a\cap b}(z)=\prod_{j=1}^{3}e^{i\,(\frac{1}{2}-\theta^j)\,H_j(z)}\,\sigma_{\theta^j}(z)\,,\qquad \bar{\Xi}^{a\cap b}(z)=\prod_{j=1}^{3}e^{-i\,(\frac{1}{2}-\theta^j)\,H_j(z)}\,\sigma_{-\theta^j}(z)\ ,\tag{4.67}$$

where σ_θ is the bosonic twist operator associated to the intersection angle θ. The angles θ^i are associated to the three complex planes subject to the $\mathcal{N}=1$ SUSY constraint:

$$\sum_{k=1}^{3}\theta^k=0\ .\tag{4.68}$$

Note, that in the limit $\theta\to 0$ bosonic twist fields σ_θ become the identity operator and we have:

$$\lim_{\theta^j\to 0}\Xi^{a\cap b}=\prod_{j=1}^{3}e^{\frac{i}{2}\,H_j}=\Sigma\,,\qquad \lim_{\theta^j\to 0}\bar{\Xi}^{a\cap b}=\prod_{j=1}^{3}e^{-\frac{i}{2}\,H_j}=\bar{\Sigma}\ .\tag{4.69}$$

Therefore, up to Chan-Paton and normalization factors in this limit the quark vertex operators turn into the gaugino vertex operators [13]. With the explicit free field representation of the $U(1)$ current $\mathcal{J}$

$$\mathcal{J}=i\partial H=i\ \sum_{j=1}^{3}\partial H_j\ ,\tag{4.70}$$

the three-point function relevant to the W coupling to a quark-antiquark pair reads:

$$\langle\mathcal{J}(z_1)\,\Xi^{a\cap b}(z_2)\,\bar{\Xi}^{a\cap b}(z_3)\rangle\ =\ \sum_{j=1}^{3}\big(\tfrac{1}{2}\ -\ \theta^j\big)\,\frac{z_{23}^{1/4}}{z_{12}\,z_{13}}=\frac{\sqrt{3}\,z_{23}^{1/4}}{2\,z_{12}\,z_{13}}$$

$$= \quad \langle \mathcal{J}(z_1)\Sigma(z_2)\,\bar{\Sigma}(z_3)\rangle\ . \tag{4.71}$$

The corresponding amplitude is:

$$\mathscr{A}[\mathrm{d},u_1,\bar{u}_2] \;=\; \frac{\sqrt{3}}{4}\,g(T^a)^{\alpha_1}_{\alpha_2}\,\xi_\mu u_1^\lambda \sigma^\mu_{\lambda\dot\rho}\bar{u}_2^{\dot\rho} \;\equiv\; \sqrt{\frac{3\alpha'}{2}}\,g(T^a)^{\alpha_1}_{\alpha_2}\,\mathcal{A}\left[d,u_1,\bar{u}_2\right]\ . \tag{4.72}$$

For the specific $(+\frac{1}{2},-\frac{1}{2})$ helicity configuration of the antiquark-quark pair, we obtain

$$\begin{aligned} \mathcal{A}\left[d(-1);+\tfrac{1}{2},-\tfrac{1}{2}\right] &= \langle p2\rangle[q1]\ ,\\ \mathcal{A}\left[d(0);+\tfrac{1}{2},-\tfrac{1}{2}\right] &= \sqrt{2}\langle p2\rangle[p1]\ ,\\ \mathcal{A}\left[d(+1);+\tfrac{1}{2},-\tfrac{1}{2}\right] &= \langle q2\rangle[p1]\ . \end{aligned} \tag{4.73}$$

From Eq. (4.71) it follows that the W-coupling to two gauginos can be obtained from Eq. (4.72) by the replacement $(T^a)^{\alpha_1}_{\alpha_2} \to 4d^{a_1a_2a}$. The normalization of the above couplings can be checked by comparing with Eq. (39) of [43].

The amplitude with one gluon in addition to the quark-antiquark pair in the final state can be written as:

$$\mathscr{A}[d;u_1,\bar{u}_2,\varepsilon_3] = \sqrt{3}\,g^2\,\left[V_t(T^{a_3}T^{a_4})^{\alpha_1}_{\alpha_2} + V_u(T^{a_4}T^{a_3})^{\alpha_1}_{\alpha_2}\right]\,\mathcal{A}\left[\xi;u_1,\bar{u}_2,\varepsilon_3\right]\ . \tag{4.74}$$

with

$$\begin{aligned} \mathcal{A}\left[d(-1);+\tfrac{1}{2},-\tfrac{1}{2},+\right] &= \frac{\langle p2\rangle^2}{\langle 13\rangle\langle 23\rangle}\ ,\\ \mathcal{A}\left[d(0);+\tfrac{1}{2},-\tfrac{1}{2},+\right] &= \sqrt{2}\,\frac{\langle p2\rangle\langle q2\rangle}{\langle 13\rangle\langle 23\rangle}\ ,\\ \mathcal{A}\left[d(+1);+\tfrac{1}{2},-\tfrac{1}{2},+\right] &= -\frac{\langle q2\rangle^2}{\langle 13\rangle\langle 23\rangle}\ . \end{aligned} \tag{4.75}$$

For the gluon with opposite helicity we have:

$$\mathcal{A}\left[d;+\tfrac{1}{2},-\tfrac{1}{2},-\right] = \mathcal{A}^*\left[d;+\tfrac{1}{2},-\tfrac{1}{2},+\right]\Big|_{(p\leftrightarrow q),(1\leftrightarrow 2)}\ . \tag{4.76}$$

The sum of the squared moduli of the corresponding amplitudes reads

$$\sum_{j=-1}^{+1} \left|\mathscr{A}[d(j);+\tfrac{1}{2},-\tfrac{1}{2},+]\right|^2 = 3\,g^4 \left|V_t(T^{a_3}T^{a_4})^{\alpha_1}_{\alpha_2} + V_u(T^{a_4}T^{a_3})^{\alpha_1}_{\alpha_2}\right|^2 \frac{(1-\alpha' t)^2}{\alpha'^2\, t\, u} \tag{4.77}$$

and a similar expression with $(k_1 \leftrightarrow k_2, a_3 \leftrightarrow a_4)$ for the gluon with opposite helicity.

4.4.3 The universal scalar $\Phi(J=0)$

It has been originally pointed out in Ref. [43] that the lowest scalar resonance propagating in two-particle channels of multi-gluon amplitudes must couple to the product of "self-dual" gauge field strengths, with the coupling to two gluons that is non-vanishing only if they carry the same helicities, say $(+,+)$. Such couplings arise naturally from $\mathcal{N}=1$ supersymmetric F-terms $\int d^2\theta\,\Phi W^\alpha W_\alpha$ where W^α is the gauge field strength superfield. The scalar and pseudoscalar components of complex $\Phi \equiv \Phi_+$ ($\Phi_- = \bar{\Phi}$) are combined with the relative weight that enforces this selection rule.

The two-gluon decay of Φ with momentum k is described by the amplitude

$$\begin{aligned} \mathscr{A}[\Phi_\pm,\varepsilon_1,\varepsilon_2] \;=\; & 4g\;(2d^{a_1a_2a})\;\sqrt{2\alpha'}\;\Big\{\;(g_{\mu\nu}+2\alpha' k_\mu k_\nu)\big[(\varepsilon_2 k_1)k_2^\mu\varepsilon_1^\nu + (\varepsilon_1 k_2)k_1^\mu\varepsilon_2^\nu \\ & -(k_1k_2)\varepsilon_2^\mu\varepsilon_1^\nu - (\varepsilon_1\varepsilon_2)k_1^\mu k_2^\nu\big] \pm \varepsilon_{\mu\nu\rho\lambda}k^\lambda\varepsilon_1^\mu\varepsilon_2^\nu k_2^\rho\;\Big\}\;. \end{aligned} \tag{4.78}$$

In the helicity basis,

$$\mathscr{A}[\Phi_+,-,-] = \mathscr{A}[\Phi_+,-,+] = \mathscr{A}[\Phi_+,+,-] = 0\;, \tag{4.79}$$

and

$$\mathscr{A}[\Phi_+,+,+] \;=\; 2g(2d^{a_1a_2a})\sqrt{2\alpha'}\,[12]^2\;. \tag{4.80}$$

The conjugate scalar Φ_- couples to $(-,-)$ configuration only, with the complex conjugate coupling. Our results correctly reproduce Eq. (25) of [43].

The three-gluon decay amplitudes obey similar selection rules:

$$\mathscr{A}[\Phi_+,-,-,-] = \mathscr{A}[\Phi_+,-,-,+] = \mathscr{A}[\Phi_+,-,+,-] = \mathscr{A}[\Phi_+,+,-,-] = 0\;, \tag{4.81}$$

while the non-vanishing ones are the "all plus" amplitude

$$\mathscr{A}[\Phi_+,+,+,+] \;=\; 4\,g^2\big(\,V_t\,t^{a_1a_2a_3a_4}+V_s\,t^{a_2a_3a_1a_4}+V_u\,t^{a_3a_1a_2a_4}\,\big)\frac{(\alpha')^{-3/2}}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} \tag{4.82}$$

and three "mostly plus" amplitudes that can be obtained from

$$\mathscr{A}[\Phi_+,+,+,-] \;=\; 4\,g^2\big(\,V_t\,t^{a_1a_2a_3a_4}+V_s\,t^{a_2a_3a_1a_4}+V_u\,t^{a_3a_1a_2a_4}\,\big)\sqrt{\alpha'}\,\frac{[12]^4}{[12][23][31]}\ , \tag{4.83}$$

by cyclically permuting $(1,2,3)$.

The Φ resonance couples to the quark-antiquark pair and one gluon only if the gluon is in appropriate polarization state: $+$ for Φ^+ and $-$ for Φ^-. The amplitude reads

$$\mathscr{A}[\Phi_+,+\tfrac{1}{2},-\tfrac{1}{2},+]=2g^2\big[V_t(T^{a_3}T^{a_4})_{\alpha_2}^{\alpha_1}+V_u(T^{a_4}T^{a_3})_{\alpha_2}^{\alpha_1}\big]\,\sqrt{\alpha'}\,\frac{[13]^2}{[12]}\ . \tag{4.84}$$

4.4.4 The Calabi-Yau scalar $\Omega(J=0)$

The universal Ω scalar in (4.28) is associated to world–sheet operator $\mathcal{O}^{\pm}$ appearing in the $\mathcal{N}=1$ OPEs (2.83). Hence the field $\Omega(J=0)$ does not couple to purely gluonic processes at the disk level, similarly to $d(J=1)$. It can couple though to two fermions of the same helicity. The coupling to two quarks (color triplets) is not allowed because Ω is a color octet, but the coupling to two gluinos is non-vanishing and can be used to determine the normalization factor of the respective vertex operator. The LHC production rate of this particle is suppressed at least by $\mathcal{O}(\alpha_s^2)$ compared to other resonances, therefore we do not discuss it here any further.

4.4.5 Massive spin 3/2 quark $\chi(J=3/2)$

Massive quarks are color triplets [in general, in the fundamental representation of $U(N)$]. Their main decay channels are into a quark and a gluon. For the spin 3/2 resonance

$\chi(J=3/2)$, the respective amplitude reads

$$\begin{aligned}\mathscr{A}[\chi;\varepsilon_1,u_2] &= (T^{a_1})_\alpha^{\alpha_2}\, g\,\sqrt{2\alpha'}(k_{1\mu}\varepsilon_{1\nu}-k_{1\nu}\varepsilon_{1\mu})\,u_2^\lambda\sigma^\mu_{\lambda\dot\rho}\bar\chi^{\nu\dot\rho}\\ &\equiv (T^{a_1})_\alpha^{\alpha_2}\sqrt{2}g\,\alpha'\mathcal{A}^\star(R;\varepsilon_1,u_2)\ . \end{aligned}\tag{4.85}$$

In the helicity basis,

$$\mathscr{A}[\chi;+,+\tfrac{1}{2}]=0 \tag{4.86}$$

and:

$$\mathcal{A}\left[\chi(-\tfrac{3}{2});-,+\tfrac{1}{2}\right]=\langle p1\rangle^2[2q]\ , \tag{4.87}$$

$$\mathcal{A}\left[\chi(-\tfrac{1}{2});-,+\tfrac{1}{2}\right]=\sqrt{3}\langle p1\rangle\langle q1\rangle[q2]\ , \tag{4.88}$$

$$\mathcal{A}\left[\chi(+\tfrac{1}{2});-,+\tfrac{1}{2}\right]=\sqrt{3}\langle p1\rangle\langle q1\rangle[p2]\ , \tag{4.89}$$

$$\mathcal{A}\left[\chi(+\tfrac{3}{2});-,+\tfrac{1}{2}\right]=\langle q1\rangle^2[2p]\ . \tag{4.90}$$

The above result agrees with Eq. (47) of Ref. [43]. Adding up the moduli squares of the amplitudes, we obtain:

$$\sum_{j=-3/2}^{+3/2}|\mathscr{A}(\chi(j);-,+\tfrac{1}{2})|^2=g^2\,|(T^{a_1})_\alpha^{\alpha_2}|^2\,\frac{2}{\alpha'}\ . \tag{4.91}$$

The amplitude with one quark and two gluons in the final state reads:

$$\begin{aligned}\mathscr{A}[\chi;\varepsilon_1,\varepsilon_2,u_3] \ &=\ 2\,g^2\sqrt{2\alpha'}\,[\,V_t\,(T^{a_1}T^{a_2})_{\alpha_4}^{\alpha_3}\ -\ V_u\,(T^{a_2}T^{a_1})_{\alpha_4}^{\alpha_3}\,]\\ &\times\Big\{\frac{1}{s}\Big[(\varepsilon_2\,k_1)\,k_2^\mu\,(\bar\chi_\mu\,\not\varepsilon_1u_3)\ -\ (\varepsilon_1\,k_2)\,k_1^\mu\,(\bar\chi_\mu\,\not\varepsilon_2u_3)\ +\ (\varepsilon_2\,k_1)\,k_1^\mu\,(\bar\chi_\mu\,\not\varepsilon_1u_3)\\ &\quad-\ (\varepsilon_1\,k_2)\,k_2^\mu\,(\bar\chi_\mu\,\not\varepsilon_2u_3)+\ (\varepsilon_1\,\varepsilon_2)\,k_1^\mu\,(\bar\chi_\mu\,\not k_2u_3)\ -\ (\varepsilon_1\,\varepsilon_2)\,k_2^\mu\,(\bar\chi_\mu\,\not k_1u_3)\\ &\quad-\ (\varepsilon_1\,k_2)\,\varepsilon_2^\mu\,(\bar\chi_\mu\,\not k_4u_3)\ +\ (\varepsilon_2\,k_1)\,\varepsilon_1^\mu\,(\bar\chi_\mu\,\not k_4u_3)\Big]\\ &\quad+\ \frac{1}{t}\Big[(\varepsilon_1\,k_3)\,k_2^\mu\,(\bar\chi_\mu\,\not\varepsilon_2u_3)\ -\ (\varepsilon_1\,k_3)\,\varepsilon_2^\mu\,(\bar\chi_\mu\,\not k_2u_3)\\ &\qquad-\ \frac{1}{2}\,\varepsilon_2^\mu\,(\bar\chi_\mu\,\not k_2\,\not\varepsilon_1\,\not k_1u_3)\ -\ \frac{1}{2}\,k_2^\mu\,(\bar\chi_\mu\,\not\varepsilon_2\,\not k_1\,\not\varepsilon_1u_3)\Big]\end{aligned}$$

$$
\begin{aligned}
&+\ \frac{1}{u}\Big[\,(\varepsilon_2\,k_3)\,\varepsilon_1^{\mu}\,(\bar\chi_\mu\,\not{k}_1 u_3)\ -\ (\varepsilon_2\,k_3)\,k_1^{\mu}\,(\bar\chi_\mu\,\not{\varepsilon}_1 u_3)\\
&\qquad +\ \frac{1}{2}\,k_1^{\mu}\,(\bar\chi_\mu\,\not{\varepsilon}_1\,\not{k}_2\,\not{\varepsilon}_2 u_3)\ +\ \frac{1}{2}\,\varepsilon_1^{\mu}\,(\bar\chi_\mu\,\not{k}_1\,\not{\varepsilon}_2\,\not{k}_2 u_3)\Big]\\
&-\ \frac{1}{2}\,\varepsilon_1^{\mu}\,(\bar\chi_\mu\,\not{\varepsilon}_2 u_3)\ +\ \frac{1}{2}\,\varepsilon_2^{\mu}\,(\bar\chi_\mu\,\not{\varepsilon}_1 u_3)\Big\}\ .
\end{aligned}
\tag{4.92}
$$

It is convenient to rewrite this amplitude as:

$$
\mathscr{A}[\chi;\varepsilon_1,\varepsilon_2,u_3]\ =\ 2\,g^2\,[\,V_t\,(T^{a_1}T^{a_2})_{\alpha_4}^{\alpha_3}\ -\ V_u\,(T^{a_2}T^{a_1})_{\alpha_4}^{\alpha_3}\,]\,\mathcal{A}[\chi;\varepsilon_1,\varepsilon_2,u_3]\ . \tag{4.93}
$$

We find the selection rule:

$$
\mathcal{A}[\chi;+,+,+\tfrac{1}{2}]=0\ . \tag{4.94}
$$

For two gluons in the $(-,-)$ helicity configuration, the amplitude reads:

$$
\begin{aligned}
\mathcal{A}\left[\chi(-\tfrac{3}{2});-,-,+\tfrac{1}{2}\right] &= \frac{[3q]^3}{[12][23][31]}\ ,\\
\mathcal{A}\left[\chi(-\tfrac{1}{2});-,-,+\tfrac{1}{2}\right] &= \sqrt{3}\frac{[3q]^2[p3]}{[12][23][31]}\ ,\\
\mathcal{A}\left[\chi(+\tfrac{1}{2});-,-,+\tfrac{1}{2}\right] &= \sqrt{3}\frac{[3p]^2[q3]}{[12][23][31]}\ ,\\
\mathcal{A}\left[\chi(+\tfrac{3}{2});-,-,+\tfrac{1}{2}\right] &= \frac{[3p]^3}{[12][23][31]}\ .
\end{aligned}
\tag{4.95}
$$

When the gluons carry opposite helicities, then:

$$
\begin{aligned}
\mathcal{A}\left[\chi(-\tfrac{3}{2});+,-,+\tfrac{1}{2}\right] &= \sqrt{\alpha'}\frac{\langle p2\rangle^3}{\langle 12\rangle\langle 13\rangle}\ ,\\
\mathcal{A}\left[\chi(-\tfrac{1}{2});+,-,+\tfrac{1}{2}\right] &= \sqrt{3\alpha'}\,\frac{\langle p2\rangle^2\langle q2\rangle}{\langle 12\rangle\langle 13\rangle}\ ,\\
\mathcal{A}\left[\chi(+\tfrac{1}{2});+,-,+\tfrac{1}{2}\right] &= -\sqrt{3\alpha'}\frac{\langle q2\rangle^2\langle p2\rangle}{\langle 12\rangle\langle 13\rangle}\ ,\\
\mathcal{A}\left[\chi(+\tfrac{3}{2});+,-,+\tfrac{1}{2}\right] &= -\sqrt{\alpha'}\frac{\langle q2\rangle^3}{\langle 12\rangle\langle 13\rangle}\ .
\end{aligned}
\tag{4.96}
$$

and a similar expression with $(1\leftrightarrow 2)$ for gluons with flipped helicities.

The sums of the squared moduli of the amplitudes read:

$$\begin{aligned}
\sum_{j=-3/2}^{+3/2} |\mathscr{A}[\chi(j);-,-,+\tfrac{1}{2}]|^2 &= 4\,g^4\,|\,V_t\,(T^{a_1}T^{a_2})^{\alpha_3}_{\alpha_4} - V_u\,(T^{a_2}T^{a_1})^{\alpha_3}_{\alpha_4}\,|^2 \frac{(1-\alpha' s)^3}{\alpha'^3 stu}\ , \\
\sum_{j=-3/2}^{+3/2} |\mathscr{A}[\chi(j);+,-,+\tfrac{1}{2}]|^2 &= 4\,g^4\,|\,V_t\,(T^{a_1}T^{a_2})^{\alpha_3}_{\alpha_4} - V_u\,(T^{a_2}T^{a_1})^{\alpha_3}_{\alpha_4}\,|^2 \frac{(1-\alpha' t)^3}{\alpha'^2 st}\ , \\
\sum_{j=-3/2}^{+3/2} |\mathscr{A}[\chi(j);-,+,+\tfrac{1}{2}]|^2 &= 4\,g^4\,|\,V_t\,(T^{a_1}T^{a_2})^{\alpha_3}_{\alpha_4} - V_u\,(T^{a_2}T^{a_1})^{\alpha_3}_{\alpha_4}\,|^2 \frac{(1-\alpha' u)^3}{\alpha'^2 su}\ .
\end{aligned} \tag{4.97}$$

One important comment is here in order. Since in our conventions all particles are incoming, the helicities of the final quark and gluons must be reversed in the physical amplitudes describing decays of the excited quarks. Thus if the χ fermion considered above decays into a number of gluons and only one quark, the quark must be a left-handed $SU(2)$ doublet associated to the intersection of the QCD and electro-weak branes [the $SU(2)$ index is just a spectator]. In order to produce a right-handed quark one would have to start from another χ excitation, an $SU(2)$ singlet associated to a different intersection of the QCD brane. Thus $\chi(J = 3/2)$ and $a(J = 1/2)$ are the massive excitations of chiral fermions. In superstring theory, there is no conventional "doubling" of massive quarks because chiral fermions generate their own Regge trajectories.

Massive quark excitations can also decay into more fermions. The minimal case involves one quark and a fermion-antifermion pair in the final state. The structure of the corresponding amplitudes is similar to four-fermion processes discussed in Ref. [13]. Although lepton pairs can be produced in this way, we focus on the case of two quarks and one antiquark, as the most relevant to the direct production of $\chi(J = 3/2)$ and $a(J = 1/2)$ in quark-quark scattering and quark-antiquark annihilation at the LHC. Even in this case, two qualitatively different computations need to be performed depending whether the processes involve quarks form the intersection of the QCD brane with a single brane (thus either four $SU(2)$ doublets or four $SU(2)$ singlets) or from two intersections (amplitudes with both $SU(2)$ doublets and singlets). In order to keep track of all gauge indices, it is convenient to display them explicitly in the amplitudes. The lower α indices will label $SU(3)$ triplets (stack a), the upper β indices will label electroweak $SU(2)$ doublets (stack b) and upper γ (stack c) indices electroweak singlets. Thus, for instance, $\mathscr{A}[\chi^{\beta}_{\alpha}; u^{\alpha_1}_{\beta_1}, \bar{u}^{\beta_2}_{\alpha_2}, u^{\alpha_3}_{\beta_3}]$ will denote the amplitude with the (incoming) $Q^\star$ Regge excitation of a left-handed quark, $\bar{q}_{1R}, q_{2L}$ and $\bar{q}_{3R}$. On the other hand, $\mathscr{A}[\chi^{\beta}_{\alpha}; \bar{u}^{\alpha_1}_{\gamma_1}, u^{\gamma_2}_{\alpha_2}, u^{\alpha_3}_{\beta_3}]$ will denote the amplitude with the same Regge excitation, $\bar{q}_{1L}, q_{2R}$ and $\bar{q}_{3R}$.

We begin with the case of two stacks, say a and b, intersecting at angles $\theta_j = \theta_{bj} - \theta_{aj},\ j = 1, 2, 3$. By following the lines of [13], we obtain:

$$
\begin{aligned}
&\mathscr{A}[\chi^{\beta}_{\alpha}; u^{\alpha_1}_{\beta_1}, \bar{u}^{\beta_2}_{\alpha_2}, u^{\alpha_3}_{\beta_3}] = (2\alpha')^{3/2} e^{\phi_{10}} \int_0^1 \mathrm{d}x\ I(x, \theta^j) \\
&\times\ \Big\{ \delta^{\alpha_1}_{\alpha} \delta^{\alpha_3}_{\alpha_2} \delta^{\beta_2}_{\beta_1} \delta^{\beta}_{\beta_3}\ Z^{ba}_{\rm inst}(x) - \delta^{\alpha_3}_{\alpha} \delta^{\alpha_1}_{\alpha_2} \delta^{\beta_2}_{\beta_3} \delta^{\beta}_{\beta_1}\ Z^{ba}_{\rm inst}(1-x) \Big\} \\
&\times\ \Big\{ x^{-\alpha' s}\, (1-x)^{-\alpha' u - 1} \big[\, (u_1 u_3)\, (\bar{u}_2 \bar{\chi}^{\mu})\, k^1_{\mu}\ +\ \tfrac{1}{4}\, (u_1\, \not{k}_4\, \bar{\chi}^{\mu})\, (u_3 \sigma_{\mu} \bar{u}_2) \\
&\qquad +\ \tfrac{1}{4}\, (u_3\, \not{k}_4\, \bar{\chi}^{\mu})\, (u_1 \sigma_{\mu} \bar{u}_2) \,\big] \\
&\quad +\ x^{-\alpha' s - 1}\, (1-x)^{-\alpha' u} \big[\, -\ (u_1 u_3)\, (\bar{u}_2 \bar{\chi}^{\mu})\, k^3_{\mu}\ +\ \tfrac{1}{4}\, (u_1\, \not{k}_4\, \bar{\chi}^{\mu})\, (u_3 \sigma_{\mu} \bar{u}_2) \\
&\qquad +\ \tfrac{1}{4}\, (u_3\, \not{k}_4\, \bar{\chi}^{\mu})\, (u_1 \sigma_{\mu} \bar{u}_2) \,\big] \Big\}\ .
\end{aligned}
\tag{4.98}
$$

Here, $Z^{ba}_{\rm inst}$ is the instanton partition function [13]. The function $I(x, \theta^j)$, written explicitly in [13], is the correlation function of four boundary-changing operators and it is symmetric under $x \to 1-x$. It is convenient to define:

$$
\begin{aligned}
Q_{su} &= \alpha' e^{\phi_{10}} \int_0^1 \mathrm{d}x\ Z^{ba}_{\rm inst}(x)\ I(x, \theta^j)\ x^{-\alpha' s}\, (1-x)^{-\alpha' u - 1}\ . \\
\widetilde{Q}_{su} &= \alpha' e^{\phi_{10}} \int_0^1 \mathrm{d}x\ Z^{ba}_{\rm inst}(x)\ I(x, \theta^j)\ x^{-\alpha' s - 1}\, (1-x)^{-\alpha' u}\ .
\end{aligned}
\tag{4.99}
$$

Note that the amplitude (4.98) exhibits kinematical singularities due to the propagation of massless gauge bosons in the respective channels:

$$
Q_{su} \stackrel{u \to 0}{\longrightarrow}\ -\frac{g_b^2}{u}\ , \qquad \widetilde{Q}_{su} \stackrel{s \to 0}{\longrightarrow}\ -\frac{g_a^2}{s}\ . \tag{4.100}
$$

where $g_a = g$ and g_b are the QCD [more precisely $U(3)$] and electro-weak coupling constants, respectively.

In order to obtain the helicity amplitudes, it is convenient to rewrite Eq. (4.98) as

$$\mathscr{A}[\chi_\alpha^\beta; u_{\beta_1}^{\alpha_1}, \bar{u}_{\alpha_2}^{\beta_2}, u_{\beta_3}^{\alpha_3}] \;=\; \delta_\alpha^{\alpha_1}\delta_{\alpha_2}^{\alpha_3}\delta_{\beta_1}^{\beta_2}\delta_{\beta_3}^{\beta}\,\alpha'\,\mathcal{A}[\chi(j);+\tfrac{1}{2},-\tfrac{1}{2},+\tfrac{1}{2}] \;-\; (1\leftrightarrow 3)\;, \tag{4.101}$$

where:

$$\begin{aligned}
\mathcal{A}[\chi(-\tfrac{3}{2});+\tfrac{1}{2},-\tfrac{1}{2},+\tfrac{1}{2}] &= Q_{su}\,\langle p2\rangle^2[q1][23] \;+\; \widetilde{Q}_{su}\,\langle p2\rangle^2[3q][12]\;,\\
\mathcal{A}[\chi(-\tfrac{1}{2});+\tfrac{1}{2},-\tfrac{1}{2},+\tfrac{1}{2}] &= \frac{1}{\sqrt{3}}Q_{su}\,\langle p2\rangle[23]\Big\{2\langle q2\rangle[q1]-\langle p2\rangle[p1]\Big\}\\
&\qquad\qquad +\;\frac{1}{\sqrt{3}}\widetilde{Q}_{su}\,\langle p2\rangle[21]\Big\{2\langle q2\rangle[q3]-\langle p2\rangle[p3]\Big\}\;,\\
\mathcal{A}[\chi(+\tfrac{1}{2});+\tfrac{1}{2},-\tfrac{1}{2},+\tfrac{1}{2}] &= \frac{1}{\sqrt{3}}Q_{su}\,\langle q2\rangle[23]\Big\{2\langle p2\rangle[p1]-\langle q2\rangle[q1]\Big\}\\
&\qquad\qquad +\;\frac{1}{\sqrt{3}}\widetilde{Q}_{su}\,\langle q2\rangle[21]\Big\{2\langle p2\rangle[p3]-\langle q2\rangle[q3]\Big\}\;,\\
\mathcal{A}[\chi(+\tfrac{3}{2});+\tfrac{1}{2},-\tfrac{1}{2},+\tfrac{1}{2}] &= Q_{su}\,\langle q2\rangle^2[p1][23] \;+\; \widetilde{Q}_{su}\,\langle q2\rangle^2[3p][12]\;.
\end{aligned} \tag{4.102}$$

Finally, we consider the case of three stacks, say a, b and c, intersecting at angles $\theta_j = \theta_{bj} - \theta_{aj},\ \nu_j = \theta_{cj} - \theta_{aj},\ j = 1,2,3$. Then the four-point correlation function of boundary-changing operators depends on the additional set of angles: $I = I(x,\theta^j,\nu^j)$ [13], however the rest of the computation is very similar to the two-stack case. Let us define R_{su} and $\widetilde{R}_{su}$ as the integrals (4.99) with $I(x,\theta^j)$ replaced by $I(x,\theta^j,\nu^j)$ in the integrand, i.e., $Q_{su}\to R_{su},\ \widetilde{Q}_{su}\to\widetilde{R}_{su}$ upon $I(x,\theta^j)\to I(x,\theta^j,\nu^j)$. Then the relevant amplitude can be written as

$$\mathscr{A}[\chi_\alpha^\beta; \bar{u}_{\gamma_1}^{\alpha_1}, u_{\alpha_2}^{\gamma_2}, u_{\beta_3}^{\alpha_3}] \;=\; \delta_\alpha^{\alpha_1}\delta_{\alpha_2}^{\alpha_3}\delta_{\gamma_1}^{\gamma_2}\delta_{\beta_3}^{\beta}\,\alpha'\,\mathcal{A}[\chi(j);-\tfrac{1}{2},+\tfrac{1}{2},+\tfrac{1}{2}]\;, \tag{4.103}$$

where:

$$\begin{aligned}
\mathcal{A}[\chi(-\tfrac{3}{2});-\tfrac{1}{2},+\tfrac{1}{2},+\tfrac{1}{2}] &= R_{st}\,\langle p1\rangle^2[q2][13] \;+\; \widetilde{R}_{st}\,\langle p1\rangle^2[3q][21]\;,\\
\mathcal{A}[\chi(-\tfrac{1}{2});-\tfrac{1}{2},+\tfrac{1}{2},+\tfrac{1}{2}] &= \frac{1}{\sqrt{3}}R_{st}\,\langle p1\rangle[13]\Big\{2\langle q1\rangle[q2]-\langle p1\rangle[p2]\Big\}\\
&\qquad\qquad +\;\frac{1}{\sqrt{3}}\widetilde{R}_{st}\,\langle p1\rangle[12]\Big\{2\langle q1\rangle[q3]-\langle p1\rangle[p3]\Big\}\;,
\end{aligned}$$

$$\mathcal{A}[\chi(+\tfrac{1}{2}); -\tfrac{1}{2}, +\tfrac{1}{2}, +\tfrac{1}{2}] = \frac{1}{\sqrt{3}} R_{st}\, \langle q1\rangle[13]\Big\{ 2\langle p1\rangle[p2] - \langle q1\rangle[q2] \Big\} \qquad (4.104)$$

$$+ \;\; \frac{1}{\sqrt{3}} \widetilde{R}_{st}\, \langle q1\rangle[12]\Big\{ 2\langle p1\rangle[p3] - \langle q1\rangle[q3] \Big\} \ ,$$

$$\mathcal{A}[\chi(+\tfrac{3}{2}); -\tfrac{1}{2}, +\tfrac{1}{2}, +\tfrac{1}{2}] = R_{st}\, \langle q1\rangle^2[p2][13] \; + \; \widetilde{R}_{st}\, \langle q1\rangle^2[3p][21] \ .$$

4.4.6 Massive spin 1/2 quark $a(J = 1/2)$

The amplitude describing the decay of $a(J = 1/2)$ into one quark and a gluon is given by

$$\begin{aligned} \mathscr{A}[a; \varepsilon_1, u_2] \;&=\; (T^{a_1})_{\alpha}^{\alpha_2}\, g\, \alpha'(k_{1\mu}\varepsilon_{1\nu} - k_{1\nu}\varepsilon_{1\mu})k_3^{\nu}\, u_2^{\lambda}\sigma^{\mu}_{\lambda\dot{\rho}}\bar{\chi}^{\dot{\rho}} \\ &\equiv\; (T^{a_1})_{\alpha}^{\alpha_2}\sqrt{2}g\, \alpha' \mathcal{A}(a; \varepsilon_1, u_2) \ . \end{aligned} \qquad (4.105)$$

The selection rule

$$\mathcal{A}[a; -, +\tfrac{1}{2}] = 0 \qquad (4.106)$$

is complementary to Eq. (4.86) of its higher spin partner $\chi(J = 3/2)$. The non-vanishing amplitudes are:

$$\mathcal{A}[a(-\tfrac{1}{2}); +, +\tfrac{1}{2}] = \langle p2\rangle[12]^2 \ , \qquad (4.107)$$

$$\mathcal{A}[a(+\tfrac{1}{2}); +, +\tfrac{1}{2}] = \langle q2\rangle[12]^2 \ . \qquad (4.108)$$

The amplitude with one quark and two gluons in the final state is given by a lengthy expression similar to Eq. (4.92), however, as usual, it simplifies in the helicity basis. It is convenient to write it as:

$$\mathscr{A}[a; \varepsilon_1, \varepsilon_2, u_3] \;=\; g^2\, (\alpha')^{-1}[\, V_t\, (T^{a_1}T^{a_2})_{\alpha_4}^{\alpha_3} \;-\; V_u\, (T^{a_2}T^{a_1})_{\alpha_4}^{\alpha_3}\,]\, \mathcal{A}[a; \varepsilon_1, \varepsilon_2, u_3] \ . \qquad (4.109)$$

In this case, the selection rule complementary to (4.94) is

$$\mathcal{A}[a; -, -, +\tfrac{1}{2}] = 0 \ . \qquad (4.110)$$

For two gluons in the $(+,+)$ helicity configuration, the amplitude reads:

$$
\begin{aligned}
\mathcal{A}[a(-\tfrac{1}{2});+,+,+\tfrac{1}{2}] &= \frac{\langle p3\rangle}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\ , \\
\mathcal{A}[a(+\tfrac{1}{2});+,+,+\tfrac{1}{2}] &= \frac{\langle q3\rangle}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\ .
\end{aligned}
\tag{4.111}
$$

When the gluons carry opposite helicities, then

$$
\begin{aligned}
\mathcal{A}[a(-\tfrac{1}{2});+,-,+\tfrac{1}{2}] &= \ \alpha'^{3/2}\ \frac{[q1][13]^2}{[12][23]}\ , \\
\mathcal{A}[a(+\tfrac{1}{2});+,-,+\tfrac{1}{2}] &= -\alpha'^{3/2}\frac{[p1][13]^2}{[12][23]}\ ,
\end{aligned}
\tag{4.112}
$$

and a similar expression with $(1 \leftrightarrow 2)$ for gluons with flipped helicities.

The sums of the squared moduli of the amplitudes read:

$$
\begin{aligned}
\sum_{j=-1/2}^{+1/2} \left|\mathscr{A}[a(j);+,+,+\tfrac{1}{2}]\right|^2 &= g^4\,|\,V_t\,(T^{a_1}T^{a_2})^{\alpha_3}_{\alpha_4} - V_u\,(T^{a_2}T^{a_1})^{\alpha_3}_{\alpha_4}\,|^2\,\frac{1-\alpha' s}{\alpha'^3 stu}\ , \\
\sum_{j=-1/2}^{+1/2} \left|\mathscr{A}[a(j);+,-,+\tfrac{1}{2}]\right|^2 &= g^4\,|\,V_t\,(T^{a_1}T^{a_2})^{\alpha_3}_{\alpha_4} - V_u\,(T^{a_2}T^{a_1})^{\alpha_3}_{\alpha_4}\,|^2\,\frac{t^2(1-\alpha' u)}{su}\ , \\
\sum_{j=-1/2}^{+1/2} \left|\mathscr{A}[a(j);-,+,+\tfrac{1}{2}]\right|^2 &= g^4\,|\,V_t\,(T^{a_1}T^{a_2})^{\alpha_3}_{\alpha_4} - V_u\,(T^{a_2}T^{a_1})^{\alpha_3}_{\alpha_4}\,|^2\,\frac{u^2(1-\alpha' t)}{st}\ .
\end{aligned}
\tag{4.113}
$$

The amplitudes describing Q-decays into two quarks and one antiquark are described by formulas similar to (4.101), (4.102) in the two-stack case and (4.103), (4.104) in the three-stack case. All what one has to do in order to obtain the corresponding amplitudes is to replace Eqs. (4.102) and (4.104) by

$$
\begin{aligned}
\mathcal{A}[a(-\tfrac{1}{2});+\tfrac{1}{2},-\tfrac{1}{2},+\tfrac{1}{2}] &= Q_{su}\,\langle p1\rangle\langle 23\rangle[13]^2\ +\ \widetilde{Q}_{su}\,\langle p3\rangle\langle 21\rangle[13]^2\ , \\
\mathcal{A}[a(+\tfrac{1}{2});+\tfrac{1}{2},-\tfrac{1}{2},+\tfrac{1}{2}] &= Q_{su}\,\langle q1\rangle\langle 23\rangle[13]^2\ +\ \widetilde{Q}_{su}\,\langle q3\rangle\langle 21\rangle[13]^2\ ,
\end{aligned}
\tag{4.114}
$$

and

$$
\mathcal{A}[a(-\tfrac{1}{2});-\tfrac{1}{2},+\tfrac{1}{2},+\tfrac{1}{2}] = R_{st}\,\langle p2\rangle\langle 13\rangle[23]^2\ +\ \widetilde{R}_{su}\,\langle p3\rangle\langle 12\rangle[23]^2\ ,
$$

$$\mathcal{A}[a(+\tfrac{1}{2});-\tfrac{1}{2},+\tfrac{1}{2},+\tfrac{1}{2}] = R_{st}\,\langle q2\rangle\langle 13\rangle[23]^2 \;+\; \widetilde{R}_{su}\,\langle q3\rangle\langle 12\rangle[23]^2\ , \tag{4.115}$$

respectively.

4.5 Cross sections for the direct production

After discussing the amplitudes (and their squared moduli) involving one lowest Regge excitation (R, mass $M = 1/\alpha'$) and three massless partons ($p = g, q, \bar{q}$), we collect the results for the subprocesses $p_1(k_1)p_2(k_2) \to p_3(k_3)R(k_4)$ relevant to the production of Regge resonances at the LHC. For the applications to jet-associated Regge production, we square the moduli of the amplitudes, average over helicities and colors of the incident partons and sum over spin directions (helicity of p_3 and J_z of R) and colors of the outgoing particles. In all these processes, quark flavor is a spectator.

The kinematic Mandelstam variables s, t and u have been defined in Eq. (4.33) in such a way that after reverting to the conventional $(+---)$ metric signature, and crossing to the physical (outgoing) momenta, $k_3 \to -k_3,\ k_4 \to -k_4$, they become

$$s = (k_1 + k_2)^2\ , \qquad t = (k_1 - k_3)^2\ , \qquad u = (k_1 - k_4)^2\ , \tag{4.116}$$

satisfying the constraint

$$s + t + u = M^2 \tag{4.117}$$

due to the momentum conservation $k_1 + k_2 = k_3 + k_4$ and on-shell conditions $k_1^2 = k_2^2 = k_3^2 = 0$, $k_4^2 = M^2$. Their physical domain is

$$s\ > M^2\ , \qquad t\ <\ 0\ , \qquad u\ <\ 0\ . \tag{4.118}$$

There are some subtleties encountered when analyzing the flow of gauge charges in the scattering amplitudes, related to the presence of massless and massive intermediate states expected either to acquire masses due to quantum effects or to be eliminated by electro-weak symmetry breaking. For example, quark-quark elastic scattering processes involve exchanges of massless abelian ("color singlet") gauge bosons associated to the $U(1)$ "baryon number" subgroup of $U(N)$ [13]. However, it is well-known that the $U(1)$ anomaly generates their masses at the one loop level, and certainly affects whole Regge trajectory. Other processes, like multi-gluon scattering can be also affected by mass shifts on such a deformed Regge trajectory. In the processes involving external Regge excitations, this problem becomes even more pronounced because massless color singlets contribute to all processes with one or more external quark-antiquark pairs. As an example, consider the $\alpha(J=2)$ decay into one gluon and one quark-antiquark pair, described by Eqs. (4.63) and (4.64). Let us focus on the prefactor

$$V_t(T^{a_3}T^{a_4})^{\alpha_1}_{\alpha_2} + V_u(T^{a_4}T^{a_3})^{\alpha_1}_{\alpha_2} = \big[2d^{a_3a_4a_n}(V_t+V_u) + \frac{i}{2}f^{a_3a_4a_n}(V_t-V_u)\big](T^{a_n})^{\alpha_1}_{\alpha_2} \tag{4.119}$$

which multiplies the function$\sim \langle 12\rangle^{-1}\langle 23\rangle^{-1}\langle 31\rangle^{-1}$. Now consider the limit $\langle 12\rangle \to 0$ ($s\to 0$, allowed in the decay channel). Since $V_t = V_u = 1$ in this limit, the amplitude exhibits a massless pole $\langle 12\rangle^{-1}$, with the residue $\sim d^{a_3a_4a_n}(T^{a_n})^{\alpha_1}_{\alpha_2}$. The pole is due to intermediate gauge bosons, produced in the B decay together with one free gluon [see Eq. (4.41)], and subsequently decaying into the quark-antiquark pair. Note that the $U(1)$ generator ($T^{a_n} = Q_A I\!I_N,\ Q_A = 1/\sqrt{2N}$) is among these gauge bosons and there is no obvious way to remove it from the disk amplitude. A formal $N\to\infty$ limit would help in that respect by suppressing such singlet contributions. When collecting the squared amplitudes describing direct production of Regge resonances, we set the number of colors to $N=3$, but we display the abelian coupling $Q_A = 1/\sqrt{6}$ explicitly. We always assume that the external partons are either color octet gluons or color triplet quarks (or antitriplet antiquarks), however we allow the possibility of color singlet Regge excitations α_0 and Φ_0 labeled by an additional subscript 0.

The following formulas, valid for general N, are useful for summing over the non-abelian color indices:

$$\sum_{a_1,a_2,a_3} d^{a_1a_2a_3}d^{a_1a_2a_3} = \frac{(N^2-1)(N^2-4)}{16N}\ , \qquad \big(d^{a_1a_20} = \frac{Q_A}{2}\delta^{a_1a_2}\big)$$

$$\sum_{a_1,a_2} f^{i_1a_1a_2}f^{i_2a_1a_2} = N\ \delta^{i_1i_2}\ , \qquad \big(f^{a_1a_20}=0\big)$$

$$\sum_{a_1,a_2,a_3,a_4} t^{a_1a_2a_3a_4}t^{a_1a_2a_3a_4} = -\frac{(N^2-1)(N^2-4)}{8}$$

$$\sum_{a_1,a_2,a_3,a_4} t^{a_1a_2a_3a_4} t^{a_2a_3a_1a_4} = 0 \tag{4.120}$$

In the Tables below, we collect the squared amplitudes for all disk-level production mechanisms of Regge resonances, listed in order of the initial two-particle channels: gg followed by gq and $q\bar{q}$. The quark-quark channel can be obtained from $q\bar{q}$ by trivial crossing. Except for the case of four-fermion processes, we factored out the QCD coupling factor g^4.

Table 1: *Gluon fusion*

subprocess	$\lvert\mathcal{M}\rvert^2/g^4$
$gg \to g\alpha$	$\frac{5}{8}\big(\,V_s^2 + V_t^2 + V_u^2\,\big)\,\frac{(s-M^2)^4+(t-M^2)^4+(u-M^2)^4}{M^2 stu}$
$gg \to g\alpha_0$	$\frac{3}{4}Q_A^2\big(\,V_s + V_t + V_u\,\big)^2\,\frac{(s-M^2)^4+(t-M^2)^4+(u-M^2)^4}{M^2 stu}$
$gg \to g\Phi$	$\frac{5}{8}\big(\,V_s^2 + V_t^2 + V_u^2\,\big)\,\frac{s^4+t^4+u^4+M^8}{M^2 stu}$
$gg \to g\Phi_0$	$\frac{3}{4}Q_A^2\big(\,V_s + V_t + V_u\,\big)^2\,\frac{s^4+t^4+u^4+M^8}{M^2 stu}$
$gg \to \bar{q}a$	$\frac{1}{4}\Big[\frac{3}{32}(V_t + V_u)^2 + \big(\frac{5}{96} + \frac{Q_A^2}{8}\big)(V_t - V_u)^2\Big]\frac{(s-M^2)M^6+(t-M^2)u^3+(u-M^2)t^3}{M^2 stu}$
$gg \to \bar{q}\chi$	$\Big[\frac{3}{32}(V_t + V_u)^2 + \big(\frac{5}{96} + \frac{Q_A^2}{8}\big)(V_t - V_u)^2\Big]\frac{(s-M^2)^3M^2+(t-M^2)^3u+(u-M^2)^3t}{M^2 stu}$

Table 2: *Quark-gluon absorption*

subprocess	$\vert\mathcal{M}\vert^2/g^4$
$qg \to q\alpha$	$-\frac{1}{16}\left[(V_s - V_u)^2 + (\frac{5}{9} + \frac{4Q_A^2}{3})(V_s + V_u)^2\right] \frac{[(s-M^2)^2+(u-M^2)^2](tM^2+4su)}{M^2stu}$
$gq \to q\alpha_0$	$-\frac{Q_A^2}{12}(V_s + V_u)^2 \frac{[(s-M^2)^2+(u-M^2)^2](tM^2+4su)}{M^2stu}$
$qg \to q\Phi$	$-\frac{1}{4}\left[(V_s - V_u)^2 + (\frac{5}{9} + \frac{4Q_A^2}{3})(V_s + V_u)^2\right] \frac{s^2+u^2}{M^2t}$
$qg \to q\Phi_0$	$-\frac{Q_A^2}{3}(V_s + V_u)^2 \frac{s^2+u^2}{M^2t}$
$qg \to qd$	$-\frac{3}{16}\left[(V_s - V_u)^2 + (\frac{5}{9} + \frac{4Q_A^2}{3})(V_s + V_u)^2\right] \frac{(s-M^2)^2+(u-M^2)^2}{su}$
$qg \to qd_0$	$-\frac{Q_A^2}{4}(V_s + V_u)^2 \frac{(s-M^2)^2+(u-M^2)^2}{su}$
$gq \to ga$	$-\frac{1}{16}\left[(V_s + V_u)^2 + (\frac{5}{9} + \frac{4Q_A^2}{3})(V_s - V_u)^2\right] \frac{(t-M^2)^3M^2+(u-M^2)^3s+(s-M^2)^3u}{M^2stu}$
$gq \to g\chi$	$-\frac{1}{4}\left[(V_s + V_u)^2 + (\frac{5}{9} + \frac{4Q_A^2}{3})(V_s - V_u)^2\right] \frac{(t-M^2)M^6+(u-M^2)s^3+(s-M^2)u^3}{M^2stu}$

Table 3: *Quark-antiquark annihilation*

subprocess	$\vert\mathcal{M}\vert^2/g^4$
$\bar{q}q \to gX$ $X = \alpha, d, \Phi$	$\vert\mathcal{M}(\bar{q}q \to gX)\vert^2 = -\frac{8}{3}\vert\mathcal{M}(qg \to qX)\vert^2(s \to u,\ u \to t,\ t \to s)$
$\bar{q}q \to \bar{q}a$	$\Big\{ -\frac{t^2}{4}(su\vert Q_{su} + \widetilde{Q}_{su}\vert^2 + ut\vert Q_{su}\vert^2 + st\vert\widetilde{Q}_{su}\vert^2) + \frac{t^2}{12}\left[su(Q_{su} + \widetilde{Q}_{su})(Q_{us} + \widetilde{Q}_{us})^* + utQ_{su}\widetilde{Q}^*_{us} + st\widetilde{Q}_{su}Q^*_{us}\right] - \frac{u^2}{4}(st\vert R_{st} + \widetilde{R}_{st}\vert^2 + ut\vert R_{st}\vert^2 + su\vert\widetilde{R}_{st}\vert^2 \Big\} + \Big\{ s \leftrightarrow u \Big\}$
$\bar{q}q \to \bar{q}'a'$	$-\frac{t^2}{4}(su\vert Q_{su} + \widetilde{Q}_{su}\vert^2 + ut\vert Q_{su}\vert^2 + st\vert\widetilde{Q}_{su}\vert^2) + \Big(Q \to R\ ;\ u \leftrightarrow t \Big)$
$\bar{q}q \to \bar{q}\chi$	$\Big\{ -\frac{(M^2-t)^2}{4}(su\vert Q_{su} + \widetilde{Q}_{su}\vert^2 + ut\vert Q_{su}\vert^2 + st\vert\widetilde{Q}_{su}\vert^2) + \frac{t(M^2-t)}{6}\vert uQ_{su} - s\widetilde{Q}_{su}\vert^2 + \frac{(M^2-t)^2}{12}\left[su(Q_{su} + \widetilde{Q}_{su})(Q_{us} + \widetilde{Q}_{us})^* + utQ_{su}\widetilde{Q}^*_{us} + st\widetilde{Q}_{su}Q^*_{us}\right] + \frac{t(M^2-t)}{18}(uQ_{su} - s\widetilde{Q}_{su})(u\widetilde{Q}_{us} - sQ_{us})^* - \frac{(M^2-u)^2}{4}(st\vert R_{st} + \widetilde{R}_{st}\vert^2 + ut\vert R_{st}\vert^2 + su\vert\widetilde{R}_{st}\vert^2) + \frac{u(M^2-u)}{6}\vert tR_{st} - s\widetilde{R}_{st}\vert^2 \Big\} + \Big\{ s \leftrightarrow u \Big\}$
$\bar{q}q \to \bar{q}'\chi'$	$-\frac{(M^2-t)^2}{4}(su\vert Q_{su} + \widetilde{Q}_{su}\vert^2 + ut\vert Q_{su}\vert^2 + st\vert\widetilde{Q}_{su}\vert^2) + \frac{t(M^2-t)}{6}\vert uQ_{su} - s\widetilde{Q}_{su}\vert^2 + \Big(Q \to R\ ;\ u \leftrightarrow t \Big)$

5 Physics of higher massive level superstrings

In the previous chapter, we presented a detailed discussion of the "universal" part of the first massive level, common to all D-brane embeddings of the standard model. In this chapter, we extend it to the second massive level, and discuss some general properties of higher levels. We are particularly interested in massive particles that couple to massless gauge bosons according to "(anti)self-dual" selection rules. These particles decay into two gauge bosons with the same (say $++$) helicities only and to more gluons in "mostly plus" helicity configurations. We rely on the factorization techniques. They allow identifying not only the spins of Regge resonances propagating in a given channel, but also their couplings and decay rates.

To start, we perform the spin decomposition of the well-known four-gluon maximally helicity violating (MHV) amplitude in the s-channels of $(--)$ and $(-+)$ gluons. We examine decay rates of heavy states into two gluons, for masses much larger than M, *i.e.* in the large n limit. We find that for any particle with spin $j \leq n+1$, the maximum partial decay width into two gluons is n-independent – it never exceeds M. Particles with $j \sim \sqrt{n} = M_n/M$ have largest widths. We also find that for $j \sim n$, the decay rate into two gluons is exponentially suppressed. We then study the second massive level in detail. We construct the vertex operators for all "universal" bosons of the NS sector. We compute the amplitudes involving one such state and two or three gluons, focusing on the decays of the (anti)self-dual massive (complex) vector fields. As were argued that the BCFW-deformed full-fledged string amplitudes have no singularities at the infinite value of the deformation parameter, therefore BCFW recursion relations should be valid also in string theory [48–53]. In the last section of this chapter, we show that at least the four-gluon amplitude can be obtained by a BCFW deformation of a factorized sum involving on-shell amplitudes of one massive Regge state and two gauge bosons.

This chapter is base on the paper [2].

5.1 Properties of higher massive level superstring

In this section, we study the general properties of higher massive level (universal) superstring states by factorization of the four-gluon amplitudes. The amplitudes describing the scattering of massless superstring states (zero modes) encode many important properties of massive excitations. The spin content of intermediate massive particles, their decay rates *etc.* can be extracted by factorizing massless amplitudes on their Regge poles [43]. We are primarily interested in the properties of particles that couple to gauge bosons, *i.e.* of those that can be detected at particle accelerators if the fundamental string mass scale happens to be sufficiently low. As we will see below, even the simplest, four-gluon amplitudes contain some interesting information.

We will be using the helicity basis to describe gluon polarizations. For four gluons, all non-vanishing amplitudes can be obtained from a single, maximally helicity violating (MHV) configuration. Our starting point is the well-known MHV amplitude [74, 75]

$$\begin{aligned}\mathcal{M}(g_1^-, g_2^-, g_3^+, g_4^+) = 4g^2\langle 12\rangle^4 \Big[&\frac{V_t}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4} + T^{a_2}T^{a_1}T^{a_4}T^{a_3}) \\ &+ \frac{V_u}{\langle 13\rangle\langle 34\rangle\langle 42\rangle\langle 21\rangle}\mathrm{Tr}(T^{a_2}T^{a_1}T^{a_3}T^{a_4} + T^{a_1}T^{a_2}T^{a_4}T^{a_3}) \\ &+ \frac{V_s}{\langle 14\rangle\langle 42\rangle\langle 23\rangle\langle 31\rangle}\mathrm{Tr}(T^{a_1}T^{a_3}T^{a_2}T^{a_4} + T^{a_3}T^{a_1}T^{a_4}T^{a_2})\Big],\end{aligned} \tag{5.1}$$

where the Veneziano “formfactor” function reads

$$V_t = V(s,t,u) = \frac{\Gamma(1 - s/M^2)\Gamma(1 - u/M^2)}{\Gamma(1 + t/M^2)}. \tag{5.2}$$

Here, $M^2 = 1/\alpha'$ is the fundamental string mass scale. s, t, u are the Mandelstam variables defined in Eq. (4.33).

The momenta and helicities are specified for *incoming* particles, therefore they need appropriate crossing to the relevant physical domains. In particular, $u < 0$ and $t < 0$ describing a $g_1 g_2 \to g_3 g_4$ scattering process with $s > 0$ can be expressed in terms of the scattering angle in the center of mass frame:

$$u = -\frac{s}{2}(1 + \cos\theta), \qquad t = -\frac{s}{2}(1 - \cos\theta), \tag{5.3}$$

so that $\theta = 0$ describes forward scattering. Finally, $a_1, \ldots, a_4$ are the gluon color indices. For future reference, it is convenient to absorb the gauge coupling g into the color factors and define the combinations:

$$S^{a_1a_2}_{a_3a_4} = 4g^2\mathrm{Tr}(\{T^{a_1}T^{a_2}\}\{T^{a_3}T^{a_4}\}) \ , \qquad A^{a_1a_2}_{a_3a_4} = 4g^2\mathrm{Tr}([T^{a_1}T^{a_2}][T^{a_3}T^{a_4}]) \ . \tag{5.4}$$

which are symmetric and antisymmetric, respectively, in the color indices of initial (and final) gluons.

Using the expansion in terms of s-channel resonances

$$B(-s/M^2, -u/M^2) = -\sum_{n=0}^{\infty} \frac{M^{2-2n}}{n!} \frac{1}{s - nM^2} \left[\prod_{J=1}^{n}(u + M^2 J)\right], \tag{5.5}$$

we obtain, near the nth level pole ($s \to nM^2$),

$$V_t(n) = V(s,t,u) \approx \frac{1}{s - nM^2} \times \frac{M^{2-2n}}{(n-1)!} \prod_{J=0}^{n-1}(u + M^2 J). \tag{5.6}$$

The spin content of Regge resonances can be disentangled by analyzing the angular distributions of scattered gluons, that is by decomposing the residue of each Regge pole in the basis of Wigner d-matrix elements $d^{(j)}_{m',m}(\theta)$.[24] In this context, $d^{(j)}_{m',m}(\theta)$ describe the angular distribution (in the center of mass frame) of the final gluons with the helicity difference $m = \lambda_3 - \lambda_4$, produced in a decay of spin j resonance; $m' = \lambda_2 - \lambda_1$ is the helicity difference of incident gluons [43]. Thus $m, \ m' = 0, \pm 2$.

We begin with the amplitude $\mathcal{M}(g_1^-, g_2^+, g_3^+, g_4^-)$ which can be obtained from (5.1) by interchanging $2 \leftrightarrow 4$. Near the lowest mass pole, associated to the "fundamental" $n = 1$ string mode,

$$\mathcal{M}(g_1^-, g_2^+, g_3^+, g_4^-) \xrightarrow{n=1} S^{a_1a_2}_{a_3a_4} \frac{M^2}{s - M^2} d^{(2)}_{2,2} \tag{5.7}$$

which reflects the obvious fact [1] that in order to create two gluons with opposite helicities ($+-$) one needs a resonance with $j \geq 2$, which is the highest spin at this level. At the next $n = 2$ level,

$$\mathcal{M}(g_1^-, g_2^+, g_3^+, g_4^-) \xrightarrow{n=2} -A^{a_1a_2}_{a_3a_4} \frac{M^2}{s - 2M^2} (\frac{2}{3})(d^{(3)}_{2.2} + 2d^{(2)}_{2,2}), \tag{5.8}$$

[24]Appendix D contains a brief introduction to Wigner d-matrices.

in agreement with [21]. Near the $n = 3$ string resonance, we find,

$$\mathcal{M}(g_1^-, g_2^+, g_3^+, g_4^-) \xrightarrow{n=3} S^{a_1a_2}_{a_3a_4} \frac{M^2}{s-3M^2} \frac{3}{56}(9d^{(4)}_{2,2} + 21d^{(3)}_{2,2} + 26d^{(2)}_{2,2}) \tag{5.9}$$

In general, at the nth massive level, states with all spins from 2 up to $n+1$ appear in the s-channel, decaying into two opposite helicity gluons. The residues of Regge poles factorize as

$$\mathrm{Res}_{s=nM^2}\mathcal{M}(g_1^-, g_2^+, g_3^+, g_4^-) = \sum_{j=2}^{n+1}\sum_a F^{aj}_{+-;a_1a_2}(F^{aj}_{+-;a_3a_4})^* d^{(j)}_{2,2}(\theta) \tag{5.10}$$

where F are the matrix elements for the decay of a spin j resonance, in the $m_j = 2$ eigenstate (in the the center of mass frame), into two gluons moving along the $\pm z$ axis, with helicities ± 1, respectively [43]. In the above expression, the sum over intermediate color indices appears after rewriting the color factors as

$$S^{a_1a_2}_{a_3a_4} = \sum_a (4\sqrt{2}gd^{a_1a_2a})(4\sqrt{2}gd^{a_3a_4a}) \tag{5.11}$$

$$-A^{a_1a_2}_{a_3a_4} = \sum_a (\sqrt{2}gf^{a_1a_2a})(\sqrt{2}gf^{a_3a_4a}) \tag{5.12}$$

where f are the gauge group structure constants while d are the symmetrized traces:

$$d^{a_1a_2a_3} = \mathrm{STr}(T^{a_1}T^{a_2}T^{a_3}) \ . \tag{5.13}$$

The matrix elements involve totally symmetric group factors at odd levels and antisymmetric ones at even levels. This can be understood as a consequence of world-sheet parity [1, 43]. Note that the numerical factors multiplying d-functions in Eqs.(5.7)-(5.9) and at higher n are positive, as required by unitarity, c.f., Eq.(5.10).

The amplitude $\mathcal{M}(g_1^-, g_2^+, g_3^-, g_4^+)$ can be obtained from (5.1) by interchanging $2 \leftrightarrow 3$, however there is no need to repeat calculations because it can be also obtained from $\mathcal{M}(g_1^-, g_2^+, g_3^+, g_4^-)$ by interchanging the color indices $a_3 \leftrightarrow a_4$ combined with the reflection $\theta \to \pi - \theta$, for which

$$\begin{cases} d^{(j)}_{2,2}(-\cos\theta) = (-1) d^{(j)}_{2,-2}(\cos\theta) & j \text{ odd} \\ d^{(j)}_{2,2}(-\cos\theta) = d^{(j)}_{2,-2}(\cos\theta) & j \text{ even} \end{cases}$$

As a result, $d^{(j)}_{2,2} \to d^{(j)}_{2,-2}$ and the coefficients acquire alternating $(-1)^{n+j+1}$ signs, for instance

$$\mathcal{M}(g_1^-, g_2^+, g_3^-, g_4^+) \xrightarrow{n=3} S^{a_1 a_2}_{a_3 a_4} \frac{M^2}{s - 3M^2} \frac{3}{56} (9 d^{(4)}_{2,-2} - 21 d^{(3)}_{2,-2} + 26 d^{(2)}_{2,-2}). \tag{5.14}$$

Next, we turn to the amplitude $\mathcal{M}(g_1^-, g_2^-, g_3^+, g_4^+)$. This case is very interesting because the resonances appearing in the s-channel couple only to (anti)self-dual gauge field configurations, *i.e.* to gluons in $(++)$ or $(--)$ helicity configurations. In the previous work [1], we discussed the first massive level and identified a complex scalar Φ (2 degrees of freedom $\Phi \equiv \Phi_+$ and $\bar{\Phi} \equiv \Phi_-$) which couples to gluons according to the selection rules

$$\mathscr{A}[\Phi_+, -, -] = \mathscr{A}[\Phi_+, +, -] = \mathscr{A}[\Phi_-, +, +] = \mathscr{A}[\Phi_-, +, -] = 0. \tag{5.15}$$

This scalar is the sole resonance contributing to

$$\mathcal{M}(g_1^-, g_2^-, g_3^+, g_4^+) \xrightarrow{n=1} S^{a_1 a_2}_{a_3 a_4} \frac{M^2}{s - M^2} d^{(0)}_{0,0} \, . \tag{5.16}$$

At higher levels, there are more such particles, with higher spins:

$$\mathcal{M}(g_1^-, g_2^-, g_3^+, g_4^+) \xrightarrow{n=2} -A^{a_1 a_2}_{a_3 a_4} \frac{2M^2}{s - 2M^2} (d^{(1)}_{0,0}), \tag{5.17}$$

$$\mathcal{M}(g_1^-, g_2^-, g_3^+, g_4^+) \xrightarrow{n=3} S^{a_1 a_2}_{a_3 a_4} \frac{3M^2}{s - 3M^2} (\frac{3}{4} d^{(2)}_{0,0} + \frac{1}{4} d^{(0)}_{0,0}), \tag{5.18}$$

$$\mathcal{M}(g_1^-, g_2^-, g_3^+, g_4^+) \xrightarrow{n=4} -A^{a_1 a_2}_{a_3 a_4} \frac{4M^2}{s - 4M^2} (\frac{8}{15} d^{(3)}_{0,0} + \frac{7}{15} d^{(1)}_{0,0}), \tag{5.19}$$

$$\mathcal{M}(g_1^-, g_2^-, g_3^+, g_4^+) \xrightarrow{n=5} S^{a_1 a_2}_{a_3 a_4} \frac{5M^2}{s - 5M^2} (\frac{125}{336} d^{(4)}_{0,0} + \frac{125}{252} d^{(2)}_{0,0} + \frac{19}{144} d^{(0)}_{0,0}). \tag{5.20}$$

In order to proceed to higher n, we first note that, in this case,

$$d^{(l)}_{0,0}(\theta) = P_l(\cos\theta), \tag{5.21}$$

see Appendix A, therefore the resonance coefficients can be obtained by decomposing the angular dependence in the basis of Legendre polynomials:

$$\mathcal{M}(g_1^-, g_2^-, g_3^+, g_4^+) \xrightarrow{\text{odd } n} S^{a_1 a_2}_{a_3 a_4} \frac{M^2}{s - nM^2} \sum_{k=0,2\cdots}^{n-1} c_k^{(n)} P_k(\cos\theta) \ , \tag{5.22}$$

$$\mathcal{M}(g_1^-, g_2^-, g_3^+, g_4^+) \xrightarrow{\text{even } n} -A^{a_1 a_2}_{a_3 a_4} \frac{M^2}{s - nM^2} \sum_{k=1,3\cdots}^{n-1} c_k^{(n)} P_k(\cos\theta) \ . \tag{5.23}$$

The above expansions involve even Legendre polynomials only for odd n and odd ones for even n, reflecting the $g_3 \leftrightarrow g_4$ ($a_3 \leftrightarrow a_4$, $\theta \to \pi - \theta$) symmetry of the amplitude. A straightforward, but tedious computation, outlined in Appendix B of [2], yields the following coefficients:

$$c_k^{(n)} = \frac{n}{(n-1)!} \sum_{j=0}^{\frac{n-1-k}{2}} \sum_{i=0}^{(n-1-k-2j)} \frac{(-1)^{n-1-k-2j}}{2^{2j+i-1}} \frac{(2k+1)(k+j+1)!(k+2j+i)!}{i!j!(2k+2j+2)!} \times (n)^{k+2j}(n-2)^i s(n-1, k+2j+i), \tag{5.24}$$

where $s(n, k)$ is the Stirling number of the first kind, defined through the expansion of the Pochhammer symbol:

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = x(x+1)...(x+n-1) = \sum_{k=0}^{n} (-1)^{n-k} s(n,k) x^k. \tag{5.25}$$

We want to see how the decay rates of Regge resonances at a given mass level n depend on their spin j and in general, on the n, j dependence of their partial widths into two gluons, in the large n limit. It has been often suggested that string perturbation theory breaks down at energies much higher than the fundamental string mass, with the onset of non-perturbative effects marked by large widths of Regge particles, covering up the mass gap between subsequent resonances. To that end, we examine the k-dependence of the coefficients $c_k^{(n)}$, see Eq.(5.24), in the large n limit. Since we could not find a compact expression for Stirling numbers, we had to resort to numerical methods. On Fig. 16, we plot $c_k^{(n)}$ as a function of k for two typical values,

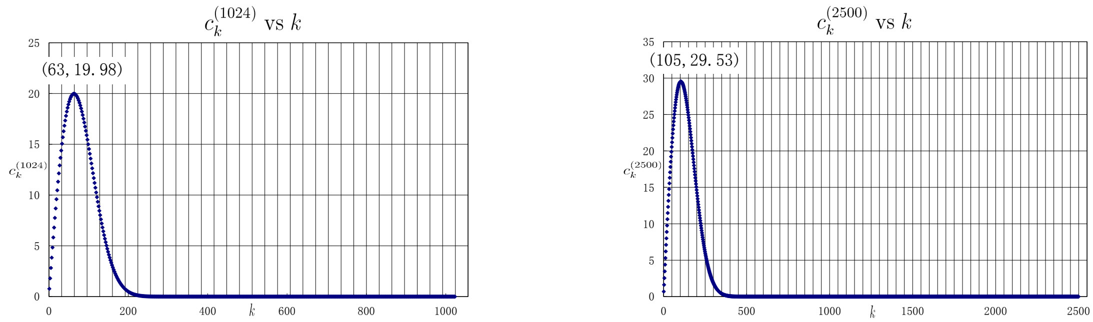

Figure 16: The coefficients of $P_k(\cos\theta)$ at the 1024th and 2500th massive levels. On the x-axis, we mark multiples of $\sqrt{n}$ to display the peaks at $k \approx 2\sqrt{n}$

$n = 32^2$ and $n = 50^2$. For small k, roughly $k \sim \sqrt{n}$, one finds a sharp peak at $k \approx 2\sqrt{n}$, with $c^{(n)}_{2\sqrt{n}} \approx \sqrt{n}/2$. For large $k \sim n$, the coefficients are exponentially suppressed. For example,

$$c^{(n)}_{n-1} = \frac{n^n(n-1)!}{(2n-2)!} \xrightarrow{\text{large } n} \left(\frac{2n}{\sqrt{e}}\right)^{-2n}, \tag{5.26}$$

see Appendix B. Since $\sum_{j \text{ odd}}^{n-1} c_j^{(n)} = \sum_{j \text{ even}}^{n-1} c_j^{(n)} = n$, we conclude that the sums are saturated by spins ranging from 0 to $j \sim \sqrt{n}$, with the maximum $c_{\max} \sim \sqrt{n}$.

The partial width of mass $M_n = \sqrt{n}M$, spin j resonance $R_{n,j}$ into a pair of gluons is given by [43]

$$\Gamma(R_{n,j} \to gg) = g^2 \delta \, \frac{c_j^{(n)} M^2}{32(2j+1)\pi M_n} \tag{5.27}$$

where $\delta \sim 1$ is the gauge group factor. In the denominator, the number $2j+1$ comes from averaging over spin components, and provides additional suppression for large j, however we will not take it into account because it is purely statistical. Thus the width size is determined by the ratio $c_j^{(n)} M^2/M_n$. From our discussion of the coefficients, it follows that the largest possible widths are n-independent, $(2j+1)\Gamma(n \to \infty, j \sim \sqrt{n}) \sim M$, the same as for low-lying Regge resonances. We conclude that the leading order (disk) approximation gives a perfectly sensible result for the decays of higher level Regge resonances. Note that the exponential suppression of direct decays of very high spin ($j \sim n$) particles into two massless gluons is akin to the Sudakov formfactor. These particles will cascade into lower mass, lower spin states, decaying at the end into a large number of gluons.

5.2 The Second Massive Level: Physical States, Vertices and Amplitudes

The second massive level has been previously discussed in Ref. [37], in the context of ten-dimensional heterotic superstrings. Here, we focus on four-dimensional open string excitations, especially on those that can be created by the fusion of gauge bosons associated to strings ending on D-branes. Such particles appear in the NS sector and are universal to the whole landscape of models because their vertices do not contain internal parts associated to compact dimensions. We will be using the *Old Covariant Quantization (OCQ)* method for identifying the physical states, which is equivalent to the covariant quantization (BRST) as we did in Chapter 2. However it's more straightforward in analyzing the physical open string states in the NS sector. As a warm-up, we start from the first massive level universal fields, which have been already worked out in Chapter 2, using it as a check of the method. Then we study in detail of the second massive level universal open string states. We compute the amplitudes involving one such state and two or three gluons, focusing on the decays of the (anti)self-dual massive (complex) vector fields.

5.2.1 The first massive level

In the NS sector, the four-dimensional string states are created by $SO(3,1)$ Lorentz-covariant creation operators acting on the vacuum. At the first massive level, their numbers must add up to $-3/2$, therefore the states can be written as

$$|n=1\rangle = \Big(\chi_{1\mu}\psi^{\mu}_{-\frac{3}{2}} + \chi_{2\mu\nu}\alpha^{\mu}_{-1}\psi^{\nu}_{-\frac{1}{2}} + \chi_{3\mu\nu\rho}\psi^{\mu}_{-\frac{1}{2}}\psi^{\nu}_{-\frac{1}{2}}\psi^{\rho}_{-\frac{1}{2}}\Big)|0;k\rangle, \tag{5.28}$$

where $|0;k\rangle$ is the open string vacuum state in the NS sector. Here, the Greek letters denote $D=4$ spacetime indices. Note that $\chi_{3\mu\nu\rho}$ is totally antisymmetric due to anticommuting ψ operators. The physical state conditions are:

$$(L_0 - \frac{1}{2})|n=1\rangle = 0, \qquad L_1|n=1\rangle = 0, \qquad G_{\frac{3}{2}}|n=1\rangle = G_{\frac{1}{2}}|n=1\rangle = 0, \tag{5.29}$$

where the superconformal Virasoro generators read,

$$L_m = \frac{1}{2}\sum_n :\alpha^\lambda_{m-n}\alpha_{n\lambda}: + \frac{1}{4}\sum_r (2r-m) :\psi^\lambda_{m-r}\psi_{r\lambda}: + a\delta_{m,0}, \tag{5.30}$$

$$G_r = \sum_n \alpha^\lambda_n \psi_{(r-n)\lambda}, \tag{5.31}$$

and $a = 0$ in the NS sector. The first condition in (5.29) gives the mass shell condition $k^2 = 1/\alpha' = M^2$ for the first massive level, as expected. By using the commutation relations of the bosonic and fermionic operators: $[\alpha^\mu_m, \alpha^\nu_n] = m\eta^{\mu\nu}\delta_{m,-n}$, $\{\psi^\mu_r, \psi^\nu_s\} = \eta^{\mu\nu}\delta_{r,-s}$, the three remaining conditions of Eqs.(5.29) yield:

$$\sqrt{2\alpha'}\chi_{1\mu}k^\mu + \chi_{2\mu\nu}\eta^{\mu\nu} = 0, \tag{5.32}$$

$$\chi_{1\mu} + \sqrt{2\alpha'}\chi_{2\mu\nu}k^\nu = 0, \tag{5.33}$$

$$\chi_{2\mu\nu} - \chi_{2\nu\mu} + 6\sqrt{2\alpha'}\chi_{3\mu\nu\rho}k^\rho = 0. \tag{5.34}$$

In order to simplify the above constraints, it is convenient to decompose

$$\chi_{2\mu\nu} = S_{2(\mu\nu)} + A_{2[\mu\nu]}, \tag{5.35}$$

where $S_{2(\mu\nu)}$ and $A_{2[\mu\nu]}$ are the symmetric and antisymmetric parts of $\chi_{2\mu\nu}$ respectively. Then the symmetric and antisymmetric parts decouple in (5.32)-(5.34). The symmetric one is subject to

$$\begin{cases} \sqrt{2\alpha'}\chi_{1\mu}k^\mu + S_{2(\mu\nu)}\eta^{\mu\nu} = 0 \\ \chi_{1\mu} + \sqrt{2\alpha'}S_{2(\mu\nu)}k^\nu = 0 \end{cases}, \tag{5.36}$$

which is fairly easy to resolve. We obtain the following solutions:

1. $S_{2(\mu\nu)} = \alpha_{\mu\nu}$ and $\chi_{1\mu} = 0$, where $\alpha_{\mu\nu}$ is a spin-2 field satisfying $\alpha_{\mu\nu}k^\nu = \alpha_{\mu\nu}\eta^{\mu\nu} = 0$.

2. $S_{2(\mu\nu)} = \sqrt{2\alpha'}(k_\mu\xi_\nu + \xi_\mu k_\nu)$ and $\chi_{1\mu} = 2\xi_\mu$, where ξ_μ represents a spin-1 field satisfying $\xi_\mu k^\mu = 0$.

3. $S_{2(\mu\nu)} = \eta_{\mu\nu} + 2\alpha' k_\mu k_\nu$ and $\chi_{1\mu} = \sqrt{2\alpha'}k_\mu$.

In this way, we obtain a spin-2 field, a vector field and a scalar field. At this point, let us count the physical degrees of freedom to make sure we are not losing any states. We started from one symmetric Lorentz 2-tensor $S_{2(\mu\nu)}$ which has 10 d.o.f. and one Lorentz vector $\chi_{1\mu}$ which has 4 d.o.f.. On the other hand, Eqs.(5.36) gave us $1 + 4 = 5$ constraints. Thus we are left with $14 - 5 = 9$ d.o.f., which are exactly the degrees of freedom of a spin-2 field (5 d.o.f.), a vector field (3 d.o.f.) and a scalar (1 d.o.f.).

The antisymmetric part of $\chi_{2\mu\nu}$ is also easy to handle. Eqs.(5.32)-(5.34) boil down to

$$A_{2[\mu\nu]} + 3\sqrt{2\alpha'}\chi_{3\mu\nu\rho}k^{\rho} = 0. \tag{5.37}$$

The solutions are:

1. $\chi_{3\mu\nu\rho} = i\varepsilon_{\mu\nu\rho\sigma}k^{\sigma}$, $A_{2[\mu\nu]} = 0$, and $\varepsilon_{\mu\nu\rho\sigma}$ is the Levi-Civita symbol.

2. $\chi_{3\mu\nu\rho} = \varepsilon_{\mu\nu\rho\sigma}\xi'^{\sigma}$ and $A_{2[\mu\nu]} = -3\varepsilon_{\mu\nu\rho\sigma}k^{\rho}\xi'^{\sigma}$. ξ'_{μ} is another spin-1 field satisfying $\xi'_{\mu}k^{\mu} = 0$.

In this way, we obtain a pseudo-vector (3 d.o.f.) and a pseudo-scalar (1 d.o.f.). To recapitulate, we started from a 3-form $\chi_{3\mu\nu\rho}$ (4 d.o.f.) and an antisymmetric 2-tensor $A_{2[\mu\nu]}$ (6 d.o.f.). Eq.(5.37) gave us 6 constraints. Thus we are left with $10 - 6 = 4$ d.o.f., which are exactly what we get.

In order to construct the vertex operators, we use the state-operator correspondence and replace the bosonic and fermionic creation operators with world-sheet bosons and fermions as follows:

$$\alpha^{\mu}_{-m} \to i\sqrt{\frac{1}{2\alpha'}}\frac{1}{(m-1)!}\partial^{m}X^{\mu}, \tag{5.38}$$

$$\psi^{\mu}_{-r} \to \frac{1}{(r-\frac{1}{2})!}\partial^{r-\frac{1}{2}}\psi^{\mu}. \tag{5.39}$$

Therefore, we have the following vertices, universal to all $D = 4$ compactifications, which satisfy the physical state conditions. They are: a spin-2 field,

$$V_{\alpha} = \alpha_{\mu\nu}\sqrt{\frac{1}{2\alpha'}}i\partial X^{\mu}\psi^{\nu}e^{-\phi}e^{ikX}, \tag{5.40}$$

with $\alpha_{\mu\nu}k^\nu = \alpha_{\mu\nu}\eta^{\mu\nu} = 0$; one spin-1 field and one pseudo spin-1 field,

$$V_\xi = (\xi_\mu k_\nu + k_\mu \xi_\nu) i\partial X^\mu \psi^\nu e^{-\phi} e^{ikX} + 2\xi_\mu \partial\psi^\mu e^{-\phi} e^{ikX}, \tag{5.41}$$

$$V_{\xi'} = \varepsilon_{\mu\nu\rho\sigma} \xi'^\sigma \psi^\mu \psi^\nu \psi^\rho e^{-\phi} e^{ikX} - 3\varepsilon_{\mu\nu\rho\sigma} k^\rho \xi'^\sigma i\partial X^\mu \psi^\nu e^{-\phi} e^{ikX}, \tag{5.42}$$

with $\xi_\mu k^\mu = \xi'_\mu k^\mu = 0$; plus one scalar and one pseudo-scalar,

$$V_{ps} = \sqrt{2\alpha'} i\varepsilon_{\mu\nu\rho\sigma} k^\sigma \psi^\mu \psi^\nu \psi^\rho e^{-\phi} e^{ikX}, \tag{5.43}$$

$$V_s = \Big[(\eta_{\mu\nu} + 2\alpha' k_\mu k_\nu)\sqrt{\frac{1}{2\alpha'}} i\partial X^\mu \psi^\nu + \sqrt{2\alpha'} k_\mu \partial\psi^\mu\Big] e^{-\phi} e^{ikX}. \tag{5.44}$$

It is well known that not all fields satisfying the physical state conditions like (5.29) appear in the spectrum. Actually, both spin-1 vertices (5.41) and (5.42) represent such null, spurious states, decoupled from the rest of the spectrum.[25] This can be demonstrated by computing their two-point correlation functions and showing that they do not contain poles appropriate to physical propagators. It is also easy to show that they do not couple to two gauge bosons in any helicity configuration: the three-point amplitude involving two gauge bosons and one such massive state is zero.

To summarize, at the first massive level of NS sector, we have a total of 7 universal degrees of freedom. They are a spin-2 field $\alpha_{\mu\nu}$, plus a scalar and a pseudoscalar. As explained in [1], it is natural to combine Eqs.(5.43) and (5.44) into one vertex of a "self-dual" complex scalar,

$$V_{\Phi_\pm} = \Big[(\eta_{\mu\nu} + 2\alpha' k_\mu k_\nu)\sqrt{\frac{1}{2\alpha'}} i\partial X^\mu \psi^\nu + \sqrt{2\alpha'} k_\mu \partial\psi^\mu \pm \frac{i}{6}\sqrt{2\alpha'} \varepsilon_{\mu\nu\rho\sigma} k^\sigma \psi^\mu \psi^\nu \psi^\rho\Big] e^{-\phi} e^{ikX} \ , \tag{5.45}$$

which satisfies the selection rules written in Eq.(5.15). We will find similar complex vector resonances at the second level.

[25] A spurious state is defined to be a state that is orthogonal to all the physical states, and a null state is defined to be a spurious state that satisfies the physical state conditions [65].

5.2.2 The second massive Level

At the second level, the number of creation operators add up to $-5/2$:

$$|n=2\rangle = \Big(\zeta_{1\mu}\psi^{\mu}_{-\frac{5}{2}} + \zeta_{2\mu\nu}\alpha^{\mu}_{-1}\psi^{\nu}_{-\frac{3}{2}} + \zeta'_{2\mu\nu}\alpha^{\mu}_{-2}\psi^{\nu}_{-\frac{1}{2}} + \zeta_{3\mu\nu\rho}\alpha^{\mu}_{-1}\alpha^{\nu}_{-1}\psi^{\rho}_{-\frac{1}{2}} + \zeta'_{3\mu\nu\rho}\psi^{\mu}_{-\frac{1}{2}}\psi^{\nu}_{-\frac{1}{2}}\psi^{\rho}_{-\frac{3}{2}}$$
$$+ \zeta_{4\mu\nu\rho\sigma}\alpha^{\mu}_{-1}\psi^{\nu}_{-\frac{1}{2}}\psi^{\rho}_{-\frac{1}{2}}\psi^{\sigma}_{-\frac{1}{2}} + \zeta_{5\mu\nu\rho\sigma\gamma}\psi^{\mu}_{-\frac{1}{2}}\psi^{\nu}_{-\frac{1}{2}}\psi^{\rho}_{-\frac{1}{2}}\psi^{\sigma}_{-\frac{1}{2}}\psi^{\gamma}_{-\frac{1}{2}}\Big)|0;k\rangle. \quad (5.46)$$

The physical state conditions are:

1. $(L_0 - \frac{1}{2})|n=2\rangle = 0$,
2. $L_2|n=2\rangle = L_1|n=2\rangle = 0$,
3. $G_{\frac{5}{2}}|n=2\rangle = G_{\frac{3}{2}}|n=2\rangle = G_{\frac{1}{2}}|n=2\rangle = 0$,

with the superconformal Virasoro generators written in (5.30) and (5.31). Here again, the first condition amounts to $k^2 = 2/\alpha' = 2M^2$. To solve the remaining constraints, it is convenient to decompose the tensors, especially those of higher rank, into representations that are symmetric or antisymmetric in groups of Lorentz indices. This is most succinctly done by using Young tableaux. Our analysis parallels to the discussion of the heterotic case (in ten dimensions) presented in [37]. The tensors $\zeta_{2\mu\nu}$ and $\zeta'_{2\mu\nu}$ can be decomposed into symmetric and antisymmetric parts:

$$\zeta_{2\mu\nu} = S_{2(\mu\nu)} + A_{2[\mu\nu]}, \qquad \zeta'_{2\mu\nu} = S'_{2(\mu\nu)} + A'_{2[\mu\nu]}. \quad (5.47)$$

The rank 3 tensors $\zeta_{3\mu\nu\rho}$ and $\zeta'_{3\mu\nu\rho}$ can be decomposed as

$$\zeta_{3\mu\nu\rho} \to S_{3(\mu\nu\rho)} + B_{3(\mu[\nu)\rho]} + D_{3[\mu(\nu]\rho)} + A_{3[\mu\nu\rho]}, \quad (5.48)$$

$$\zeta'_{3\mu\nu\rho} \to S'_{3(\mu\nu\rho)} + B'_{3(\mu[\nu)\rho]} + D'_{3[\mu(\nu]\rho)} + A'_{3[\mu\nu\rho]}, \quad (5.49)$$

corresponding to

$$\boxed{\mu} \otimes \boxed{\nu} \otimes \boxed{\rho} = \begin{array}{|c|c|c|}\hline \mu & \nu & \rho \\ \hline\end{array} \oplus \begin{array}{|c|c|}\hline \mu & \nu \\ \hline \rho \\ \cline{1-1}\end{array} \oplus \begin{array}{|c|c|}\hline \mu & \rho \\ \hline \nu \\ \cline{1-1}\end{array} \oplus \begin{array}{|c|}\hline \mu \\ \hline \nu \\ \hline \rho \\ \hline\end{array}\,, \quad (5.50)$$

or by dimensions,

$$\mathbf{4} \otimes \mathbf{4} \otimes \mathbf{4} = \mathbf{20} \oplus \mathbf{20} \oplus \mathbf{20} \oplus \mathbf{4}\ . \tag{5.51}$$

Due to the (anti)commutation properties of the creation operators in (5.46), we can set $D_{3[\mu(\nu]\rho)} = A_{3[\mu\nu\rho]} = S'_{3(\mu\nu\rho)} = B'_{3(\mu[\nu)\rho]} = 0$. We are left with

$$\zeta_{3\mu\nu\rho} = S_{3(\mu\nu\rho)} + B_{3(\mu[\nu)\rho]}, \qquad \zeta'_{3\mu\nu\rho} = D'_{3[\mu(\nu]\rho)} + A'_{3[\mu\nu\rho]}. \tag{5.52}$$

Similarly, the rank 4 tensor $\zeta_{4\mu\nu\rho\sigma}$ can be decomposed as

$$\begin{aligned} &\begin{ytableau} \mu \end{ytableau} \otimes \begin{ytableau} \nu \end{ytableau} \otimes \begin{ytableau} \rho \end{ytableau} \otimes \begin{ytableau} \sigma \end{ytableau} = \begin{ytableau} \mu & \nu & \rho & \sigma \end{ytableau} \oplus \begin{ytableau} \mu & \nu & \rho \\ \sigma \end{ytableau} \oplus \begin{ytableau} \mu & \nu & \sigma \\ \rho \end{ytableau} \oplus \begin{ytableau} \mu & \rho & \sigma \\ \nu \end{ytableau} \oplus \begin{ytableau} \mu & \nu \\ \rho & \sigma \end{ytableau} \oplus \begin{ytableau} \mu & \rho \\ \nu & \sigma \end{ytableau} \\ &\qquad \oplus \begin{ytableau} \mu & \nu \\ \rho \\ \sigma \end{ytableau} \oplus \begin{ytableau} \mu & \rho \\ \nu \\ \sigma \end{ytableau} \oplus \begin{ytableau} \mu & \sigma \\ \nu \\ \rho \end{ytableau} \oplus \begin{ytableau} \mu \\ \nu \\ \rho \\ \sigma \end{ytableau}, \end{aligned} \tag{5.53}$$

or by dimensions,

$$\mathbf{4} \otimes \mathbf{4} \otimes \mathbf{4} \otimes \mathbf{4} = \mathbf{35} \oplus \mathbf{45} \times 3 \oplus \mathbf{20} \times 2 \oplus \mathbf{15} \times 3 \oplus \mathbf{1}\ . \tag{5.54}$$

Here again, we can ignore all but the last four Young diagrams. Actually, due to the anticommutation of ψ operators in the respective term of (5.46), the three 3-row diagrams would lead to the same state, therefore we are allowed to pick just one of of them, say the one symmetric in μ and ν. Thus the 4-tensor is decomposed as

$$\zeta_{4\mu\nu\rho\sigma} \to B_{4(\mu[\nu)\rho\sigma]} + A_{4[\mu\nu\rho\sigma]}. \tag{5.55}$$

Finally, the term involving completely antisymmetric $\zeta_{5\mu\nu\rho\sigma\gamma}$ must necessarily involve one internal index, therefore we do not discuss it any further.

The second physical state condition, $L_2|n=2\rangle = L_1|n=2\rangle = 0$, yields,

$$2\zeta_{1\mu} + \sqrt{2\alpha'}(S_{2(\mu\nu)} - A_{2[\mu\nu]})k^\nu = 0, \tag{5.56}$$

$$A'_{3[\mu\nu\rho]} + \sqrt{2\alpha'}(B_{4(\sigma[\mu)\nu\rho]} + A_{4[\sigma\mu\nu\rho]})k^\sigma = 0, \tag{5.57}$$

$$(S_{2(\mu\nu)} + A_{2[\mu\nu]}) + 2(S'_{2(\mu\nu)} + A'_{2[\mu\nu]}) + \sqrt{2\alpha'}(2S_{3(\rho\mu\nu)} + B_{3(\rho[\mu)\nu]} + B_{3(\mu[\rho)\nu]})k^\rho = 0, \tag{5.58}$$

$$\frac{3}{2}\zeta_{1\mu} + 2\sqrt{2\alpha'}(S'_{2(\mu\nu)} - A'_{2[\mu\nu]})k^\nu + (S_{3(\nu\rho\mu)} + B_{3(\nu[\rho)\mu]})\eta^{\nu\rho} + \frac{\eta^{\nu\rho}}{2}(D'_{3[\nu(\mu]\rho)} - D'_{3[\mu(\nu]\rho)}) = 0. \tag{5.59}$$

We are left with the third set of conditions. From $G_{\frac{5}{2}}|n=2\rangle = 0$, we obtain,

$$\sqrt{2\alpha'}\zeta_{1\mu}k^\mu + (S_{2(\mu\nu)} + 2S'_{2(\mu\nu)})\eta^{\mu\nu} = 0. \tag{5.60}$$

From $G_{\frac{3}{2}}|n=2\rangle = 0$, we obtain,

$$\zeta_{1\mu} + \sqrt{2\alpha'}(S_{2(\mu\nu)} + A_{2[\mu\nu]})k^\nu + 2S_{3(\mu\nu\rho)}\eta^{\nu\rho} + (B_{4(\mu[\nu)\rho]} + B_{4(\nu[\mu)\rho]})\eta^{\nu\rho} = 0, \tag{5.61}$$

$$(B_{4(\rho[\sigma)\mu\nu]} - B_{4(\rho[\mu)\sigma\nu]} + B_{4(\rho[\mu)\nu\sigma]})\eta^{\rho\sigma} + 2A'_{2[\mu\nu]} + \sqrt{2\alpha'}(D'_{3[\mu(\nu]\rho)} + A'_{3[\mu\nu\rho]})k^\rho = 0. \tag{5.62}$$

Finally, $G_{\frac{1}{2}}|n=2\rangle = 0$ yields,

$$A_{4[\mu\nu\rho\sigma]} = 0, \tag{5.63}$$

$$\zeta_{1\mu} + \sqrt{2\alpha'}(S'_{2(\mu\nu)} + A'_{2[\mu\nu]})k^\nu = 0, \tag{5.64}$$

$$S_{2(\mu\nu)} + \sqrt{2\alpha'}(S_{3(\mu\nu\rho)} + B_{3(\mu[\nu)\rho]})k^\rho = 0, \tag{5.65}$$

$$A_{2[\mu\nu]} + 2A'_{2[\mu\nu]} + \sqrt{2\alpha'}(D'_{3[\rho(\mu]\nu)} - D'_{3[\mu(\rho]\nu)} + 2A'_{3[\mu\nu\rho]})k^\rho = 0, \tag{5.66}$$

$$3\sqrt{2\alpha'}B_{4(\mu[\nu)\rho\sigma]}k^\sigma + B_{3(\mu[\nu)\rho]} + \frac{1}{2}B_{3(\nu[\mu)\rho]} - \frac{1}{2}B_{3(\rho[\mu)\nu]} + D'_{3[\nu(\rho]\mu)} + A'_{3[\mu\nu\rho]} = 0. \tag{5.67}$$

First, we take care of simplest conditions. We get $A_{4[\mu\nu\rho\sigma]} = 0$ directly from Eq.(5.63). Similarly, Eq.(5.67) requires $A'_{3[\mu\nu\rho]} = 0$. Thus, Eq.(5.57) now reads

$$B_{4(\sigma[\mu)\nu\rho]}k^\sigma = 0. \tag{5.68}$$

Furthermore, by examining all equations involving $B_{4(\mu[\nu)\rho\sigma]}$, we find the consistency condition $B_{4(\mu[\nu)\rho\sigma]}k^\sigma = 0$ which, together with Eq.(5.68), impose transversality of $B_{4(\mu[\nu)\rho\sigma]}$ with respect to all indices:

$$B_{4\mu_1\mu_2\mu_3\mu_4}k^{\mu_i} = 0, \qquad (i = 1,2,3,4). \tag{5.69}$$

Notice that now, Eq.(5.67) becomes

$$B_{3(\mu[\nu)\rho]} + \frac{1}{2}B_{3(\nu[\mu)\rho]} - \frac{1}{2}B_{3(\rho[\mu)\nu]} + D'_{3[\nu(\rho]\mu)} = 0 \tag{5.70}$$

Next, Eq.(5.59) splits into

$$S_{2(\mu\nu)} + 2S'_{2(\mu\nu)} + \sqrt{2\alpha'}2S_{3(\rho\mu\nu)}k^\rho = 0, \tag{5.71}$$

$$A_{2[\mu\nu]} + 2A'_{2[\mu\nu]} + \sqrt{2\alpha'}(B_{3(\rho[\mu)\nu]} + B_{3(\mu[\rho)\nu]})k^\rho = 0. \tag{5.72}$$

Note also that Eq.(5.66) becomes

$$A_{2[\mu\nu]} + 2A'_{2[\mu\nu]} + \sqrt{2\alpha'}(D'_{3[\rho(\mu]\nu)} - D'_{3[\mu(\rho]\nu)})k^\rho = 0. \tag{5.73}$$

After multiplying both sides by k^μ, we obtain

$$(A_{2[\mu\nu]} + 2A'_{2[\mu\nu]})k^\mu = 0. \tag{5.74}$$

On the other hand, Eq.(5.59)$-\frac{1}{2}\times$Eq.(5.61)$-\frac{\eta^{\mu\nu}}{2}\times$Eq.(5.67) gives us

$$\zeta_{1\mu} + 2\sqrt{2\alpha'}(S'_{2(\mu\nu)} - A'_{2[\mu\nu]})k^\nu - \frac{1}{2}\sqrt{2\alpha'}(S_{2(\mu\nu)} + A_{2[\mu\nu]})k^\nu = 0. \tag{5.75}$$

After inserting this into Eq.(5.56) and Eq.(5.64) we find

$$-4A'_{2[\mu\nu]}k^\nu - A_{2[\mu\nu]}k^\nu = 0. \tag{5.76}$$

In this way, we obtain

$$A_{2[\mu\nu]}k^{\mu} = A'_{2[\mu\nu]}k^{\mu} = 0. \tag{5.77}$$

Taking into account all equations allows decoupling $\zeta_{1\mu}$, $S_{2(\mu\nu)}$, $S'_{2(\mu\nu)}$ and $S_{3(\mu\nu\rho)}$ from other fields. After removing all the dependent relations, we obtain the following set:

$$\begin{cases} 2\zeta_{1\mu} + \sqrt{2\alpha'}S_{2(\mu\nu)}k^{\nu} = 0 \\ S_{2(\mu\nu)} + \sqrt{2\alpha'}S_{3(\mu\nu\rho)}k^{\rho} = 0 \\ 2S'_{2(\mu\nu)} + \sqrt{2\alpha'}S_{3(\mu\nu\rho)}k^{\rho} = 0 \\ \sqrt{2\alpha'}S'_{2(\mu\nu)}k^{\nu} + 2S_{3(\mu\nu\rho)}\eta^{\nu\rho} = 0 \end{cases}. \tag{5.78}$$

The solutions are enumerated below:

1. $S_{3(\mu\nu\rho)} = \sigma_{\mu\nu\rho}$ and $S_{2(\mu\nu)} = S'_{2(\mu\nu)} = \zeta_{1\mu} = 0$. $\sigma_{\mu\nu\rho}$ is a spin-3 field which satisfies

$$\sigma_{\mu\nu\rho}k^{\rho} = \sigma_{\mu\nu\rho}\eta^{\mu\nu} = 0, \tag{5.79}$$

and its vertex operator reads

$$V_{\sigma} = \frac{1}{2\alpha'}\sigma_{\mu\nu\rho}i\partial X^{\mu}i\partial X^{\nu}\psi^{\rho}e^{-\phi}e^{ikX}. \tag{5.80}$$

2. $S_{3(\mu\nu\rho)} = \sqrt{2\alpha'}(\pi_{\mu\nu}k_{\rho} + \pi_{\mu\rho}k_{\nu} + \pi_{\nu\rho}k_{\mu})$, $S_{2(\mu\nu)} = 4\pi_{\mu\nu}$, $S'_{2(\mu\nu)} = 2\pi_{\mu\nu}$, and $\zeta_{1\mu} = 0$, where $\pi_{\mu\nu}$ is a spin-2 field satisfying $\pi_{\mu\nu}k^{\nu} = \pi_{\mu\nu}\eta^{\mu\nu} = 0$. The corresponding spin-2 vertex operator is

$$\begin{aligned} V_{\pi} = \Big(&\sqrt{\frac{1}{2\alpha'}}(\pi_{\mu\nu}k_{\rho} + \pi_{\mu\rho}k_{\nu} + \pi_{\nu\rho}k_{\mu})i\partial X^{\mu}i\partial X^{\nu}\psi^{\rho} \\ &+ 4\pi_{\mu\nu}\sqrt{\frac{1}{2\alpha'}}i\partial X^{\mu}\partial\psi^{\nu} + 2\pi_{\mu\nu}\sqrt{\frac{1}{2\alpha'}}i\partial^2 X^{\mu}\psi^{\nu}\Big)e^{-\phi}e^{ikX}. \end{aligned} \tag{5.81}$$

3. $S_{3(\mu\nu\rho)} = \tilde{\zeta}_{3\mu\nu\rho}$, $S_{2(\mu\nu)} = \tilde{\zeta}_{2\mu\nu}$, $S'_{2(\mu\nu)} = \tilde{\zeta}'_{2\mu\nu}$ and $\zeta_{1\mu} = \tilde{\zeta}_{1\mu}$, where

$$\tilde{\zeta}_{3\mu\nu\rho} = \eta_{\mu\nu}\xi_{\rho} + \eta_{\mu\rho}\xi_{\nu} + \eta_{\nu\rho}\xi_{\mu} + c(2\alpha')(k_{\mu}k_{\nu}\xi_{\rho} + k_{\mu}\xi_{\nu}k_{\rho} + \xi_{\mu}k_{\nu}k_{\rho}), \tag{5.82}$$

$$\tilde{\zeta}_{2\mu\nu} = (4c-1)\sqrt{2\alpha'}(k_\mu \xi_\nu + \xi_\mu k_\nu), \tag{5.83}$$

$$\tilde{\zeta}'_{2\mu\nu} = \frac{1}{2}(4c-1)\sqrt{2\alpha'}(k_\mu \xi_\nu + \xi_\mu k_\nu), \tag{5.84}$$

$$\tilde{\zeta}_{1\mu} = 2(4c-1)\xi_\mu, \tag{5.85}$$

with $c = 7/8$. ξ_μ is a vector field satisfying $\xi_\mu k^\mu = 0$. The corresponding vector vertex operator reads

$$\begin{aligned} V_\xi^{(1)} = \Big(&\tilde{\zeta}_{3\mu\nu\rho}\frac{1}{2\alpha'}i\partial X^\mu i\partial X^\nu \psi^\rho + \tilde{\zeta}_{2\mu\nu}\sqrt{\frac{1}{2\alpha'}}i\partial X^\mu \partial\psi^\nu \\ &+ \tilde{\zeta}'_{2\mu\nu}\sqrt{\frac{1}{2\alpha'}}i\partial^2 X^\mu \psi^\nu + \tilde{\zeta}_{1\mu}\frac{1}{2}\partial^2\psi^\mu\Big)e^{-\phi}e^{ikX}. \end{aligned} \tag{5.86}$$

4. $S_{3\mu\nu\rho} = \hat{\zeta}_{3\mu\nu\rho}$, $S_{2\mu\nu} = \hat{\zeta}_{2\mu\nu}$, $S'_{2\mu\nu} = \hat{\zeta}'_{2\mu\nu}$ and $\zeta_{1\mu} = \hat{\zeta}_{1\mu}$, where

$$\hat{\zeta}_{3\mu\nu\rho} = \big[\sqrt{2\alpha'}(\eta_{\mu\nu}k_\rho + \eta_{\mu\rho}k_\nu + \eta_{\nu\rho}k_\mu) + d(2\alpha')^{\frac{3}{2}}k_\mu k_\nu k_\rho\big]\varphi, \tag{5.87}$$

$$\hat{\zeta}_{2\mu\nu} = \big[4\eta_{\mu\nu} - 2\alpha'(2-4d)k_\mu k_\nu\big]\varphi, \tag{5.88}$$

$$\hat{\zeta}'_{2\mu\nu} = \big[2\eta_{\mu\nu} - \alpha'(2-4d)k_\mu k_\nu\big]\varphi, \tag{5.89}$$

$$\hat{\zeta}_{1\mu} = \big[-2(3-4d)\sqrt{2\alpha'}k_\mu\big]\varphi, \tag{5.90}$$

with $d = 9/8$. φ is a scalar field, and its vertex operator is

$$\begin{aligned} V_\varphi = \Big(&\hat{\zeta}_{3\mu\nu\rho}\frac{1}{2\alpha'}i\partial X^\mu i\partial X^\nu \psi^\rho + \hat{\zeta}_{2\mu\nu}\sqrt{\frac{1}{2\alpha'}}i\partial X^\mu \partial\psi^\nu \\ &+ \hat{\zeta}'_{2\mu\nu}\sqrt{\frac{1}{2\alpha'}}i\partial^2 X^\mu \psi^\nu + \hat{\zeta}_{1\mu}\frac{1}{2}\partial^2\psi^\mu\Big)e^{-\phi}e^{ikX}. \end{aligned} \tag{5.91}$$

Let us check if we identified all independents degrees of freedom. We started from a totally symmetric 3-tensor S_3 (20 d.o.f.), two symmetric 2-tensors S_2 and S'_2 (20 d.o.f.) and one vector ξ_1 (4 d.o.f.), a total of 44 d.o.f. The set (5.78) contains $4+10+10+4 = 28$ constraints. Thus we are left with 16 independent d.o.f., which are one spin-3 field σ (7 d.o.f.), one spin-2 field π (5 d.o.f.), one vector ξ (3 d.o.f.) and one scalar φ (1 d.o.f.). Next, we examine the vertex operators to check if any of the above degrees of freedom happens to represent a null state. Indeed, we find that the spin-2 field π and the scalar φ are null states.

They do not couple to two massless gluons in any helicity configurations, just like the two vectors found at the first massive level, c.f., Eqs.(5.41) and (5.42).

Now we turn to remaining fields. With our previous analysis, after eliminating all the dependent relations, we arrive to the following set:

$$\begin{cases} (B_{4(\rho[\sigma)\mu\nu]} - B_{4(\rho[\mu)\sigma\nu]} + B_{4(\rho[\mu)\nu\sigma]})\eta^{\rho\sigma} + 2A'_{2[\mu\nu]} + \sqrt{2\alpha'}D'_{3[\mu(\nu]\rho)}k^{\rho} = 0 \\ B_{4(\mu[\nu)\rho\sigma]}k^{\mu} = B_{4(\mu[\nu)\rho\sigma]}k^{\sigma} = 0 \\ A_{2[\mu\nu]} + 2A'_{2[\mu\nu]} + \sqrt{2\alpha'}(B_{3(\rho[\mu)\nu]} + B_{3(\mu[\rho)\nu]})k^{\rho} = 0 \\ A_{2[\mu\nu]} + 2A'_{2[\mu\nu]} + \sqrt{2\alpha'}(D'_{3[\rho(\mu]\nu)} - D'_{3[\mu(\rho]\nu)})k^{\rho} = 0 \\ B_{3(\mu[\nu)\rho]} + \frac{1}{2}B_{3(\nu[\mu)\rho]} - \frac{1}{2}B_{3(\rho[\mu)\nu]} + D'_{3[\nu(\rho]\mu)} = 0 \\ B_{3(\mu[\nu)\rho]}k^{\rho} = 0 \end{cases} . \tag{5.92}$$

The solutions are:

1. $B_{3\mu\nu\rho} = \eta^{\perp}_{\mu\nu}\xi_{\rho} - \frac{1}{4}\xi_{\mu}\eta^{\perp}_{\nu\rho} - \frac{1}{4}\xi_{\nu}\eta^{\perp}_{\mu\rho}$, $D'_{3\mu\nu\rho} = \frac{1}{2}\xi_{\mu}\eta^{\perp}_{\nu\rho} - 2\xi_{\nu}\eta^{\perp}_{\mu\rho}$, $B_4 = A_2 = A'_2 = 0$, where ξ_{μ} is a spin-1 wavefunction satisfying $\xi_{\mu}k^{\mu} = 0$.

2. $B_{3\mu\nu\rho} = -\frac{1}{2}D'_{3\mu\nu\rho} = k^{\sigma}\varepsilon_{\sigma\mu\rho\gamma}\pi'^{\gamma\lambda}\eta_{\lambda\nu} + k^{\sigma}\varepsilon_{\sigma\nu\rho\gamma}\pi'^{\gamma\lambda}\eta_{\lambda\mu}$, $B_4 = A_2 = A'_2 = 0$, where $\pi'_{\mu\nu}$ is another spin-2 field satisfying $\pi'_{\mu\nu}k^{\nu} = \pi'_{\mu\nu}\eta^{\mu\nu} = 0$.

3. $B_{4\mu\nu\rho\sigma} = x\upsilon_{(\mu}E_{\nu)\rho\sigma}$, [26] $B_{3\mu\nu\rho} = y\sqrt{2\alpha'}k_{\mu}\upsilon^{\tau}E_{\tau\nu\rho}$, $D'_{3\mu\nu\rho} = -y\sqrt{2\alpha'}(\upsilon^{\tau}E_{\tau\mu\nu}k_{\rho} + \frac{1}{2}\upsilon^{\tau}E_{\tau\rho\nu}k_{\mu} - \frac{1}{2}\upsilon^{\tau}E_{\tau\rho\mu}k_{\nu})$, $A'_{2\mu\nu} = -(x+2y)\upsilon^{\tau}E_{\tau\mu\nu}$, $A_{2\mu\nu} = (2x+8y)\upsilon^{\tau}E_{\tau\mu\nu}$, where the vector υ is transverse, $\upsilon_{\mu}k^{\mu} = 0$, and the 3-form $E_{\mu\nu\rho} = \frac{i}{6}\sqrt{2\alpha'}\varepsilon_{\mu\nu\rho\sigma}k^{\sigma}$. Although only one massive vector field υ is involved in our solution, we still have two parameters x, y available, thus we get two pseudo-vectors, $\upsilon_{\mu}(x_1, y_1)$ and $\upsilon_{\mu}(x_2, y_2)$. There is a natural choice for the coefficients x, y, dictated by the complexification of vector fields, to be made after discussing the helicity-dependence of their couplings to gauge bosons.

[26] The choice of the wave function of $B_{4\mu\nu\rho\sigma}$ is not unique. Indeed we find another solution with $B_{4\mu\nu\rho\sigma} = 2\eta^{\perp}_{\mu\nu}\upsilon^{\tau}E_{\tau\rho\sigma}$, where $\eta^{\perp}_{\mu\nu} \equiv \eta_{\mu\nu} - k_{\mu}k_{\nu}/k^2$ and $E_{\mu\nu\rho}$ is the same 3-form. However, this solution does not give us an extra physical field. When we compute the scattering amplitudes involving the physical fields subject to these two solutions, we find the results are exactly the same. That is to say, these two different solutions represent the same physical vector state. We will stick to the first solution in our later discussions.

Let us count the number of degrees of freedom again. We started from a hook 4-index tensor B_4 with 15 d.o.f., two hook 3-tensors, B_3 and D'_3, with $20 \times 2 = 40$ d.o.f., and two antisymmetric 2-tensors A_2, A'_2 with $6 \times 2 = 12$ d.o.f. The set (5.92) contains $6 + 12 + 6 + 6 + 20 + 6 = 56$ constraints[27,28]. We are left with 11 d.o.f.: two vector fields ξ_μ and υ_μ (3 d.o.f. each) and one spin-2 field π' (5 d.o.f.).

The vertex operators of these physical fields are:

$$V^{(2)}_\xi = \Big[\xi_\mu \eta^\perp_{\nu\rho} \frac{1}{2\alpha'} (\psi^\mu i\partial X^\nu i\partial X^\rho - \frac{1}{2} i\partial X^\mu i\partial X^\nu \psi^\rho) + \frac{5}{2} \xi_\mu \eta^\perp_{\nu\rho} \psi^\mu \psi^\nu \partial\psi^\rho\Big] e^{-\phi} e^{ikX}, \tag{5.93}$$

$$V_{\pi'} = (k^\sigma \varepsilon_{\sigma\mu\rho\gamma} \pi'^{\gamma\lambda} \eta_{\lambda\nu} + k^\sigma \varepsilon_{\sigma\nu\rho\gamma} \pi'^{\gamma\lambda} \eta_{\lambda\mu}) \Big[(\frac{1}{2\alpha'}) i\partial X^\mu i\partial X^\nu \psi^\rho - 2\partial\psi^\mu \psi^\nu \psi^\rho\Big] e^{-\phi} e^{ikX}, \tag{5.94}$$

and

$$\begin{aligned} V_{\upsilon(x,y)} = {} & \frac{x}{\sqrt{2\alpha'}} \Big[\big(\upsilon^\tau E_{\tau\mu\nu} i\partial^2 X^\mu \psi^\nu - 2\upsilon^\tau E_{\tau\mu\nu} i\partial X^\mu \partial\psi^\nu\big) + \upsilon_{(\mu} E_{\nu)\rho\sigma} i\partial X^\mu \psi^\nu \psi^\rho \psi^\sigma\Big] e^{-\phi} e^{ikX} \\ & + \frac{y}{\sqrt{2\alpha'}} \Big[k_\mu \upsilon^\tau E_{\tau\nu\rho} i\partial X^\mu i\partial X^\nu \psi^\rho - 2(2\alpha') \upsilon^\tau E_{\tau\mu(\nu} k_{\rho)} \psi^\mu \psi^\nu \partial\psi^\rho \\ & \qquad + \big(8\upsilon^\tau E_{\tau\mu\nu} i\partial X^\mu \partial\psi^\nu - 2\upsilon^\tau E_{\tau\mu\nu} i\partial^2 X^\mu \psi^\nu\big)\Big] e^{-\phi} e^{ikX}. \end{aligned} \tag{5.95}$$

To summarize, we identified one spin-3 field, two spin-2 fields, four vector fields (two vectors and two pseudo-vectors), and one real scalar satisfying the physical state conditions. The scalar φ and the spin-2 field ϕ are null states; all other fields are physical. Thus the number of universal physical degrees of freedom at the second level of NS sector is 24. The spin-3 field and the spin-2 field π couple to two massless gluons with opposite helicities – these are the particles responsible for the Regge pole in Eq.(5.8). The four spin-1 fields will pair up to form two complex vectors that can decay into two gluons with the same helicities only, c.f., the pole in Eq.(5.17).

[27] Relation $B_{4(\mu[\nu)\rho\sigma]} k^\mu = B_{4(\mu[\nu)\rho\sigma]} k^\sigma = 0$ give total $8 + 4 = 12$ constraints. First of all $B_{4(\mu[\nu)\rho\sigma]} k^\mu = 0$ kills the second box in the first row, so we are left with a Young diagram $\yng(1,1,1)$, which corresponds to $\frac{4\times3\times2}{3\times2} = 4$ constraints. In addition, $B_{4(\mu[\nu)\rho\sigma]} k^\sigma = 0$ kills the box in the third row, so we are left with $\yng(2,1)$, which gives us $\frac{3\times4\times2}{3} = 8$ more constraints. The subtlety here is when one of the antisymmetric indices is eliminated, once we calculate the dimensions of the Young diagram, the number we put in the first box is 3 instead of 4.

[28] Similarly to the previous case, $B_{3(\mu[\nu)\rho]} k^\rho = 0$ represents 6 constraints. $B_{3(\mu[\nu)\rho]} k^\rho = 0$ corresponds to $\yng(2)$ so it gives us $\frac{3\times4}{2} = 6$ constraints. Note that again the number we put in the first box is 3 instead of 4 because one of the antisymmetric indices is eliminated.

Both even- and odd-parity particles couple to (++) and (−−) gluon helicity configurations. The relative normalization of their couplings is dictated by supersymmetry which forbids non-vanishing "all-plus" and "all-minus" scattering amplitudes [31]. Thus similarly to the scalar Φ at the first level, the vectors and pseudo-vectors of the second level must combine to form complex vector fields that couple to gluons with the selection rules similar to Eq.(5.15). To that end, we introduce two complex vector fields, $\Xi^{\pm}_{1,2}$, with the vertices

$$\begin{aligned}
V_{\Xi_1^\pm} &= V_\xi^{(1)} \pm V_{\upsilon(x_1,y_1)}(\xi) \\
&= CT^a \Big\{ \big[\big(\frac{3}{2\alpha'} \xi_{(\mu}\eta_{\nu\rho)} + \frac{21}{8} \xi_{(\mu}k_\nu k_{\rho)} \big) i\partial X^\mu i\partial X^\nu \psi^\rho \\
&\qquad + 5\xi_{(\mu}k_{\nu)} i\partial X^\mu \partial\psi^\nu + \frac{5}{2}\xi_{(\mu}k_{\nu)} i\partial^2 X^\mu \psi^\nu + \frac{5}{2}\xi_\mu \partial^2\psi^\mu \big] \\
&\quad \pm \big\{ \frac{x_1}{\sqrt{2\alpha'}} \big[\big(\xi^\tau E_{\tau\mu\nu} i\partial^2 X^\mu \psi^\nu - 2\xi^\tau E_{\tau\mu\nu} i\partial X^\mu \partial\psi^\nu \big) + \xi_{(\mu}E_{\nu)\rho\sigma} i\partial X^\mu \psi^\nu \psi^\rho \psi^\sigma \big] \\
&\quad + \frac{y_1}{\sqrt{2\alpha'}} \big[k_\mu \xi^\tau E_{\tau\nu\rho} i\partial X^\mu i\partial X^\nu \psi^\rho - 2(2\alpha')\xi^\tau E_{\tau\mu(\nu}k_{\rho)} \psi^\mu \psi^\nu \partial\psi^\rho \\
&\qquad + \big(8\xi^\tau E_{\tau\mu\nu} i\partial X^\mu \partial\psi^\nu - 2\xi^\tau E_{\tau\mu\nu} i\partial^2 X^\mu \psi^\nu \big) \big] \big\} \Big\} e^{-\phi} e^{ikX}.
\end{aligned} \tag{5.96}$$

$$\begin{aligned}
V_{\Xi_2^\pm} &= V_\xi^{(2)} \pm V_{\upsilon(x_2,y_2)}(\xi) \\
&= CT^a \Big\{ \big[\xi_\mu \eta^\perp_{\nu\rho} \frac{1}{2\alpha'} (\psi^\mu i\partial X^\nu i\partial X^\rho - \frac{1}{2} i\partial X^\mu i\partial X^\nu \psi^\rho) + \frac{5}{2}\xi_\mu \eta^\perp_{\nu\rho} \psi^\mu \psi^\nu \partial\psi^\rho \big] \\
&\quad \pm \big\{ \frac{x_2}{\sqrt{2\alpha'}} \big[\big(\xi^\tau E_{\tau\mu\nu} i\partial^2 X^\mu \psi^\nu - 2\xi^\tau E_{\tau\mu\nu} i\partial X^\mu \partial\psi^\nu \big) + \xi_{(\mu}E_{\nu)\rho\sigma} i\partial X^\mu \psi^\nu \psi^\rho \psi^\sigma \big] \\
&\quad + \frac{y_2}{\sqrt{2\alpha'}} \big[k_\mu \xi^\tau E_{\tau\nu\rho} i\partial X^\mu i\partial X^\nu \psi^\rho - 2(2\alpha')\xi^\tau E_{\tau\mu(\nu}k_{\rho)} \psi^\mu \psi^\nu \partial\psi^\rho \\
&\qquad + \big(8\xi^\tau E_{\tau\mu\nu} i\partial X^\mu \partial\psi^\nu - 2\xi^\tau E_{\tau\mu\nu} i\partial^2 X^\mu \psi^\nu \big) \big] \big\} \Big\} e^{-\phi} e^{ikX}.
\end{aligned} \tag{5.97}$$

The coefficients (x_1, y_1) and (x_2, y_2) will be fixed by requiring that $\Xi^{+}_{1,2}$ couple to two gluons in (++) configurations and to three gluons in mostly plus configurations only (at least two gluons carrying positive helicities). The overall normalization factors C will be fixed by the usual factorization arguments.

5.2.3 Complex vector couplings to two gluons

Three-point amplitudes with one massive vector (with the momentum and color indices labeled by 1) and two massless gluons are very simple because the positions of three vertices can be fixed by using $PSL(2,R)$ invariance of the disk world-sheet and there are no integrals involved in the computations. The three-point amplitude of the pseudo-vector (vertex $V_{\upsilon(x,y)}$) and two gluons reads

$$\begin{aligned}\mathscr{A}^{(3)}(\upsilon_{(x,y)}(\xi),\epsilon_2,\epsilon_3) \;=\;& C_{D_2}C\sqrt{2\alpha'}g^2 f^{a_1a_2a_3}(2\alpha')^{\frac{3}{2}}\Big\{4x\varepsilon_{\mu\nu\rho}\epsilon_2^\mu\epsilon_3^\nu k_3^\rho(\xi\cdot k_3)\\ &-(x+2y)\xi^\rho\varepsilon_{\rho\mu\nu}\big[\epsilon_2^\mu k_3^\nu(\epsilon_3\cdot k_2)+\epsilon_3^\mu k_3^\nu(\epsilon_2\cdot k_3)+\frac{1}{\alpha'}\epsilon_2^\mu\epsilon_3^\nu\big]\Big\}.\end{aligned} \tag{5.98}$$

where $C_{D_2}=g^{-2}\alpha'^{-2}$ is the universal disk factor [65]. In the helicity basis, this corresponds to

$$\mathscr{A}^{(3)}(\xi,+,+)=(\frac{x}{6}-\frac{y}{3})C_{D_2}C\sqrt{2\alpha'}g^2 f^{a_1a_2a_3}(2\alpha')^2[23]^2(\xi\cdot k_2), \tag{5.99}$$

$$\mathscr{A}^{(3)}(\xi,+,-)=0, \tag{5.100}$$

$$\mathscr{A}^{(3)}(\xi,-,-)=-(\frac{x}{6}-\frac{y}{3})C_{D_2}C\sqrt{2\alpha'}g^2 f^{a_1a_2a_3}(2\alpha')^2\langle 23\rangle^2(\xi\cdot k_2). \tag{5.101}$$

The three-point amplitude of the vector $\xi_{(1)}$ (vertex $V_\xi^{(1)}$) and two gluons is

$$\begin{aligned}\mathscr{A}^{(3)}(\xi_{(1)},\epsilon_2,\epsilon_3)=&\,C_{D_2}C\sqrt{2\alpha'}g^2 f^{a_1a_2a_3}(2\alpha')^2\Big\{\big(3\xi_{(\mu}\eta_{\nu\rho)}+\frac{21}{4}\alpha'\xi_{(\mu}k_\nu k_{\rho)}\big)\big[\frac{1}{2\alpha'}\epsilon_2^\mu\epsilon_3^\nu k_3^\rho\\ &-\frac{1}{2\alpha'}\epsilon_2^\mu\epsilon_3^\nu k_2^\rho-\epsilon_2^\mu k_2^\nu k_3^\rho(\epsilon_3\cdot k_2)-\epsilon_3^\mu k_2^\nu k_2^\rho(\epsilon_2\cdot k_3)+k_2^\mu k_2^\nu k_3^\rho(\epsilon_2\cdot\epsilon_3)\big]\\ &+\frac{5}{2}\xi_{(\mu}k_{\nu)}\big[\epsilon_2^\mu k_3^\nu(\epsilon_3\cdot k_2)+\epsilon_3^\mu k_2^\nu(\epsilon_2\cdot k_3)-k_2^\mu k_3^\nu(\epsilon_2\cdot\epsilon_3)\big]\Big\}.\end{aligned} \tag{5.102}$$

The corresponding helicity amplitudes are

$$\mathscr{A}^{(3)}(\xi_{(1)},+,+)=\frac{5}{8}C_{D_2}C\sqrt{2\alpha'}g^2 f^{a_1a_2a_3}(2\alpha')^2[23]^2(\xi\cdot k_2), \tag{5.103}$$

$$\mathscr{A}^{(3)}(\xi_{(1)},+,-)=0, \tag{5.104}$$

$$\mathscr{A}^{(3)}(\xi_{(1)},-,-)=\frac{5}{8}C_{D_2}C\sqrt{2\alpha'}g^2 f^{a_1a_2a_3}(2\alpha')^2\langle 23\rangle^2(\xi\cdot k_2). \tag{5.105}$$

The three-point amplitude of the vector $\xi_{(2)}$ (vertex $V_\xi^{(2)}$) and two gluons is

$$\begin{aligned}\mathscr{A}^{(3)}(\xi_{(2)},\epsilon_2,\epsilon_3) = C_{D_2}C\sqrt{2\alpha'}g^2 f^{a_1a_2a_3}(2\alpha')^2\Big\{\big[(\xi\cdot\epsilon_2)(\eta^\perp_{\mu\nu}k_3^\mu k_3^\nu)(\epsilon_3\cdot k_2)+(\xi\cdot k_3)(\eta^\perp_{\mu\nu}k_3^\mu k_3^\nu)(\epsilon_2\cdot\epsilon_3)\\ -(\xi\cdot\epsilon_3)(\eta^\perp_{\mu\nu}k_3^\mu k_3^\nu)(\epsilon_2\cdot k_3)+\frac{1}{\alpha'}(\xi\cdot\epsilon_2)(\eta^\perp_{\mu\nu}\epsilon_3^\mu k_3^\nu)\big]-\frac{1}{2}\big[(\xi\cdot k_3)(\eta^\perp_{\mu\nu}\epsilon_2^\mu k_3^\nu)(\epsilon_3\cdot k_2)\\ +(\xi\cdot k_3)(\eta^\perp_{\mu\nu}k_3^\mu k_3^\nu)(\epsilon_2\cdot\epsilon_3)-(\xi\cdot k_3)(\eta^\perp_{\mu\nu}\epsilon_3^\mu k_3^\nu)(\epsilon_2\cdot k_3)+\frac{1}{2\alpha'}(\xi\cdot k_3)(\eta^\perp_{\mu\nu}\epsilon_2^\mu\epsilon_3^\nu)\\ +\frac{1}{2\alpha'}(\xi\cdot\epsilon_3)(\eta^\perp_{\mu\nu}\epsilon_2^\mu k_3^\nu)\big]+\frac{5}{2}\frac{1}{2\alpha'}\big[(\xi\cdot\epsilon_3)(\eta^\perp_{\mu\nu}\epsilon_2^\mu k_3^\nu)-(\xi\cdot k_3)(\eta^\perp_{\mu\nu}\epsilon_2^\mu\epsilon_3^\nu)\big]\Big\}.\end{aligned}\tag{5.106}$$

The corresponding helicity amplitudes are

$$\mathscr{A}^{(3)}(\xi_{(2)},+,+) = -\frac{5}{8}C_{D_2}C\sqrt{2\alpha'}g^2 f^{a_1a_2a_3}(2\alpha')^2[23]^2(\xi\cdot k_2),\tag{5.107}$$

$$\mathscr{A}^{(3)}(\xi_{(2)},+,-) = 0,\tag{5.108}$$

$$\mathscr{A}^{(3)}(\xi_{(2)},-,-) = -\frac{5}{8}C_{D_2}C\sqrt{2\alpha'}g^2 f^{a_1a_2a_3}(2\alpha')^2\langle 23\rangle^2(\xi\cdot k_2).\tag{5.109}$$

In the basis of complex vectors $\Xi^\pm_{1,2}(x,y)$, the above amplitudes correspond to

$$\mathscr{A}^{(3)}(\Xi_1^\pm(\xi),+,+) = \big[\frac{5}{8}\pm(\frac{x_1}{6}-\frac{y_1}{3})\big]C_{D_2}C\sqrt{2\alpha'}g^2 f^{a_1a_2a_3}(2\alpha')^2[23]^2(\xi\cdot k_2),\tag{5.110}$$

$$\mathscr{A}^{(3)}(\Xi_1^\pm(\xi),+,-) = 0,\tag{5.111}$$

$$\mathscr{A}^{(3)}(\Xi_1^\pm(\xi),-,-) = \big[\frac{5}{8}\mp(\frac{x_1}{6}-\frac{y_1}{3})\big]C_{D_2}C\sqrt{2\alpha'}g^2 f^{a_1a_2a_3}(2\alpha')^2\langle 23\rangle^2(\xi\cdot k_2).\tag{5.112}$$

and

$$\mathscr{A}^{(3)}(\Xi_2^\pm(\xi),+,+) = \big[-\frac{5}{8}\pm(\frac{x_2}{6}-\frac{y_2}{3})\big]C_{D_2}C\sqrt{2\alpha'}g^2 f^{a_1a_2a_3}(2\alpha')^2[23]^2(\xi\cdot k_2),\tag{5.113}$$

$$\mathscr{A}^{(3)}(\Xi_2^\pm(\xi),+,-) = 0,\tag{5.114}$$

$$\mathscr{A}^{(3)}(\Xi_2^\pm(\xi),-,-) = \big[-\frac{5}{8}\mp(\frac{x_2}{6}-\frac{y_2}{3})\big]C_{D_2}C\sqrt{2\alpha'}g^2 f^{a_1a_2a_3}(2\alpha')^2\langle 23\rangle^2(\xi\cdot k_2).\tag{5.115}$$

By requiring that $\Xi^+_{1,2}$ couple to $(+,+)$ only (and respectively, $\Xi^-_{1,2}$ to $(-,-)$ only), we obtain the following constraints:

$$\frac{5}{8}-(\frac{x_1}{6}-\frac{y_1}{3})=0, \tag{5.116}$$

$$-\frac{5}{8}-(\frac{x_2}{6}-\frac{y_2}{3})=0. \tag{5.117}$$

5.2.4 Complex vector couplings to three gluons

We consider four-point amplitudes involving one massive vector and three gluons.[29] The kinematic variables are defined in Eq.(4.33). Now, k_1 is the momentum of the massive particle, $k_1^2=2M^2$, and the Mandelstam variables satisfy

$$s+t+u=2M^2=\frac{2}{\alpha'}. \tag{5.118}$$

All other quantum numbers associated to the massive vector will be also labeled by 1.

We begin with the amplitudes involving three all-plus and all-minus gluons,$(+++)$ and $(---)$, respectively. For all-plus configurations, they contain the common factors

$$\begin{aligned}\mathscr{F}(j_z=+1,+,+,+)=C_{D_2}C\sqrt{2\alpha'}g^3\mathscr{T}^{a_1a_2a_3a_4}_{n=2}V_t(2\alpha')^3\frac{\langle qp\rangle}{m}\Big\{&\frac{(1-\alpha'u)}{\alpha'^2}\frac{[2q][q3]}{\langle 34\rangle\langle 42\rangle}\\&+\frac{(1-\alpha's)}{\alpha'^2}\frac{[3q][q4]}{\langle 23\rangle\langle 42\rangle}+\frac{(1-\alpha't)}{\alpha'^2}\frac{[4q][q2]}{\langle 23\rangle\langle 34\rangle}\Big\},\end{aligned} \tag{5.119}$$

$$\begin{aligned}\mathscr{F}(j_z=0,+,+,+)=C_{D_2}C\sqrt{2\alpha'}g^3\mathscr{T}^{a_1a_2a_3a_4}_{n=2}V_t(2\alpha')^3\frac{\langle qp\rangle}{\sqrt{2}m}\Big\{&\frac{(1-\alpha'u)}{\alpha'^2}\frac{\big([2q][p3]+[3q][p2]\big)}{\langle 34\rangle\langle 42\rangle}\\&+\frac{(1-\alpha's)}{\alpha'^2}\frac{\big([3q][p4]+[4q][p3]\big)}{\langle 23\rangle\langle 42\rangle}+\frac{(1-\alpha't)}{\alpha'^2}\frac{\big([4q][p2]+[2q][p4]\big)}{\langle 23\rangle\langle 34\rangle}\Big\},\end{aligned} \tag{5.120}$$

$$\begin{aligned}\mathscr{F}(j_z=-1,+,+,+)=C_{D_2}C\sqrt{2\alpha'}g^3\mathscr{T}^{a_1a_2a_3a_4}_{n=2}V_t(2\alpha')^3\frac{\langle qp\rangle}{m}\Big\{&\frac{(1-\alpha'u)}{\alpha'^2}\frac{[2p][p3]}{\langle 34\rangle\langle 42\rangle}\\&+\frac{(1-\alpha's)}{\alpha'^2}\frac{[3p][p4]}{\langle 23\rangle\langle 42\rangle}+\frac{(1-\alpha't)}{\alpha'^2}\frac{[4p][p2]}{\langle 23\rangle\langle 34\rangle}\Big\}.\end{aligned} \tag{5.121}$$

Here, $\mathscr{T}^{a_1a_2a_3a_4}_{n=2}$ is a universal factor that combines Chan-Paton factors with the kinematic variables in the

[29] The original four-point string amplitudes are very tedious, so we will only present the helicity amplitudes in this paper, which look much simpler.

following way

$$
\begin{aligned}
\mathscr{T}_{n=2}^{a_1a_2a_3a_4} &= \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4} + T^{a_4}T^{a_3}T^{a_2}T^{a_1}) \\
&+ \frac{1}{V_t}\frac{V_s}{\alpha' s - 1}\mathrm{Tr}(T^{a_2}T^{a_3}T^{a_1}T^{a_4} + T^{a_4}T^{a_1}T^{a_3}T^{a_2}) \\
&+ \frac{1}{V_t}\frac{V_u}{\alpha' u - 1}\mathrm{Tr}(T^{a_3}T^{a_1}T^{a_2}T^{a_4} + T^{a_4}T^{a_2}T^{a_1}T^{a_3}).
\end{aligned} \tag{5.122}
$$

Furthermore, p and q are the light-like reference vectors used to define the quantization axis for the polarization vector ξ. We obtain

$$
\mathscr{A}^{(4)}(\Xi_1^{\pm}(j_z),+,+,+) = \big[\frac{5}{8} \pm (\frac{x_1}{6} - \frac{y_1}{3})\big]\mathscr{F}(j_z,+,+,+), \tag{5.123}
$$

$$
\mathscr{A}^{(4)}(\Xi_2^{\pm}(j_z),+,+,+) = \big[-\frac{5}{8} \pm (\frac{x_2}{6} - \frac{y_2}{3})\big]\mathscr{F}(j_z,+,+,+) \tag{5.124}
$$

For all-minus configurations, the analogous expressions read

$$
\begin{aligned}
\mathscr{F}(j_z = +1,-,-,-) &= C_{D_2}C\sqrt{2\alpha'}g^3\mathscr{T}_{n=2}^{a_1a_2a_3a_4}V_t(2\alpha')^3\frac{[pq]}{m} \times \Big\{\frac{(1-\alpha' u)}{\alpha'^2}\frac{\langle 2p\rangle\langle p3\rangle}{[34][42]} \\
&+ \frac{(1-\alpha' s)}{\alpha'^2}\frac{\langle 3p\rangle\langle p4\rangle}{[23][42]} + \frac{(1-\alpha' t)}{\alpha'^2}\frac{\langle 4p\rangle\langle p2\rangle}{[23][34]}\Big\},
\end{aligned} \tag{5.125}
$$

$$
\begin{aligned}
\mathscr{F}(j_z = 0,-,-,-) &= C_{D_2}C\sqrt{2\alpha'}g^3\mathscr{T}_{n=2}^{a_1a_2a_3a_4}V_t(2\alpha')^3\frac{[qp]}{\sqrt{2}m} \times \Big\{\frac{(1-\alpha' u)}{\alpha'^2}\frac{\big(\langle 3q\rangle\langle p2\rangle + \langle 2q\rangle\langle p3\rangle\big)}{[34][42]} \\
&+ \frac{(1-\alpha' s)}{\alpha'^2}\frac{\big(\langle 4q\rangle\langle p3\rangle + \langle 3q\rangle\langle p4\rangle\big)}{[23][42]} + \frac{(1-\alpha' t)}{\alpha'^2}\frac{\big(\langle 4q\rangle\langle p2\rangle + \langle 2q\rangle\langle p4\rangle\big)}{[23][34]}\Big\},
\end{aligned} \tag{5.126}
$$

$$
\begin{aligned}
\mathscr{F}(j_z = -1,-,-,-) &= C_{D_2}C\sqrt{2\alpha'}g^3\mathscr{T}_{n=2}^{a_1a_2a_3a_4}V_t(2\alpha')^3\frac{[pq]}{m} \times \Big\{\frac{(1-\alpha' u)}{\alpha'^2}\frac{\langle 2q\rangle\langle q3\rangle}{[34][42]} \\
&+ \frac{(1-\alpha' s)}{\alpha'^2}\frac{\langle 3q\rangle\langle q4\rangle}{[23][42]} + \frac{(1-\alpha' t)}{\alpha'^2}\frac{\langle 4q\rangle\langle q2\rangle}{[23][34]}\Big\}.
\end{aligned} \tag{5.127}
$$

and

$$\mathscr{A}^{(4)}(\Xi_1^{\pm}(j_z),-,-,-)=\big[\frac{5}{8}\mp(\frac{x_1}{6}-\frac{y_1}{3})\big]\mathscr{F}(j_z,-,-,-), \tag{5.128}$$

$$\mathscr{A}^{(4)}(\Xi_2^{\pm}(j_z),-,-,-)=\big[-\frac{5}{8}\mp(\frac{x_2}{6}-\frac{y_2}{3})\big]\mathscr{F}(j_z,-,-,-). \tag{5.129}$$

Note that the constraints (5.116,5.117) on the parameters x, y automatically ensure the decoupling of Ξ^+ from all-minus configurations and of Ξ^- from all-plus ones.

Next, we turn to mostly plus configuration $(++-)$. For each $j_z = 0, \pm 1$, there are two kinematic structures common to these amplitudes:

$$\mathscr{K}_1(j_z=+1,+,+,-)=C_{D_2}C\sqrt{2\alpha'}g^3\mathscr{T}^{a_1a_2a_3a_4}_{n=2}V_t(2\alpha')^3\frac{\langle pq\rangle}{2m}\frac{[23]^2}{[24][34]}\frac{1-\alpha' u}{\alpha'}[2q][q3], \tag{5.130}$$

$$\mathscr{K}_2(j_z=+1,+,+,-)=C_{D_2}C\sqrt{2\alpha'}g^3\mathscr{T}^{a_1a_2a_3a_4}_{n=2}V_t(2\alpha')^3\frac{[pq]}{2m}(-)\langle p4\rangle^2[23]^2, \tag{5.131}$$

and

$$\begin{aligned}\mathscr{K}_1(j_z=0,+,+,-)=C_{D_2}C\sqrt{2\alpha'}g^3\mathscr{T}^{a_1a_2a_3a_4}_{n=2}V_t(2\alpha')^3\frac{1}{2\sqrt{2}}\frac{\langle pq\rangle}{m}\\ \times\frac{[23]^2}{[24][34]}\frac{1-\alpha' u}{\alpha'}\big([2p][q3]+[3p][q2]\big),\end{aligned} \tag{5.132}$$

$$\mathscr{K}_2(j_z=0,+,+,-)=C_{D_2}C\sqrt{2\alpha'}g^3\mathscr{T}^{a_1a_2a_3a_4}_{n=2}V_t(2\alpha')^3\frac{1}{2\sqrt{2}}\frac{[pq]}{m}2\langle p4\rangle\langle q4\rangle[23]^2, \tag{5.133}$$

and lastly

$$\mathscr{K}_1(j_z=-1,+,+,-)=C_{D_2}C\sqrt{2\alpha'}g^3\mathscr{T}^{a_1a_2a_3a_4}_{n=2}V_t(2\alpha')^3\frac{\langle pq\rangle}{2m}\frac{(-)[23]^2}{[24][34]}\frac{1-\alpha' u}{\alpha'}[2p][p3], \tag{5.134}$$

$$\mathscr{K}_2(j_z=-1,+,+,-)=C_{D_2}C\sqrt{2\alpha'}g^3\mathscr{T}^{a_1a_2a_3a_4}_{n=2}V_t(2\alpha')^3\frac{[pq]}{2m}\langle q4\rangle^2[23]^2. \tag{5.135}$$

We obtain

$$\mathscr{A}^{(4)}(\Xi_1^{\pm}(j_z),+,+,-)=\big[\frac{5}{8}\pm(\frac{x_1}{6}-\frac{y_1}{3})\big]\mathscr{K}_1(j_z)+\big[\frac{23}{16}\pm(\frac{7x_1}{12}+\frac{5y_1}{6})\big]\mathscr{K}_2(j_z), \tag{5.136}$$

$$\mathscr{A}^{(4)}(\Xi_2^{\pm}(j_z),+,+,-) = \big[-\frac{5}{8} \pm (\frac{x_2}{6} - \frac{y_2}{3})\big]\mathscr{K}_1(j_z) + \big[\frac{13}{16} \pm (\frac{7x_2}{12} + \frac{5y_2}{6})\big]\mathscr{K}_2(j_z). \tag{5.137}$$

For the opposite, $(--+)$ helicity configurations,

$$\mathscr{A}^{(4)}(\Xi_1^{\pm}(j_z),-,-.+) = \big[\frac{5}{8} \mp (\frac{x_1}{6} - \frac{y_1}{3})\big]\mathscr{K}_1(-j_z)^* + \big[\frac{23}{16} \mp (\frac{7x_1}{12} + \frac{5y_1}{6})\big]\mathscr{K}_2(-j_z)^*, \tag{5.138}$$

$$\mathscr{A}^{(4)}(\Xi_2^{\pm}(j_z),-,-,+) = \big[-\frac{5}{8} \mp (\frac{x_2}{6} - \frac{y_2}{3})\big]\mathscr{K}_1(-j_z)^* + \big[\frac{13}{16} \mp (\frac{7x_2}{12} + \frac{5y_2}{6})\big]\mathscr{K}_2(-j_z)^*. \tag{5.139}$$

Note that the conditions (5.116) and (5.117) imply vanishing $\mathscr{K}_1$ parts of the $(\Xi^+,-,-,+)$ and $(\Xi^-,+,+,-)$ amplitudes. By requiring that their $\mathscr{K}_2$ parts also vanish, we obtain

$$\frac{23}{16} - (\frac{7}{12}x_1 + \frac{5}{6}y_1) = 0, \tag{5.140}$$

$$\frac{13}{16} - (\frac{7}{12}x_2 + \frac{5}{6}y_2) = 0, \tag{5.141}$$

which, combined with Eqs.(5.116) and (5.117) fixes the relative weights of vectors and pseudo-vectors to

$$\begin{cases} x_1 = 3 \\ y_1 = -3/8 \end{cases}, \quad \begin{cases} x_2 = -3/4 \\ y_2 = 3/2 \end{cases}. \tag{5.142}$$

To summarize, at the second massive level we identified two complex vectors, $\Xi_{1,2}$, with the vertex operators written in Eqs.(5.96) and (5.97) and the parameters x and y given in Eq.(5.142), which satisfy the following selection rules:

$$\mathscr{A}\left[\Xi^+,-,-\right] = \mathscr{A}\left[\Xi^{\pm},+,-\right] = \mathscr{A}\left[\Xi^-,+,+\right] = 0, \tag{5.143}$$

for three-point amplitudes and

$$\mathscr{A}\left[\Xi^+,-,-,-\right] = \mathscr{A}\left[\Xi^-,+,+,+\right] = 0, \tag{5.144}$$

$$\mathscr{A}\left[\Xi^+,+,-,-\right] = \mathscr{A}\left[\Xi^+,-,+,-\right] = \mathscr{A}\left[\Xi^+,-,-,+\right] = 0, \tag{5.145}$$

$$\mathscr{A}\left[\Xi^-, -, +, +\right] = \mathscr{A}\left[\Xi^-, +, -, +\right] = \mathscr{A}\left[\Xi^-, +, +, -\right] = 0, \tag{5.146}$$

for four-point amplitudes.The overall vertex normalization can be fixed by the usual factorization argument. It is

$$C = \frac{2}{5}\sqrt{\alpha'}g. \tag{5.147}$$

5.3 Factorization and BCFW reconstruction of the four-gluon amplitude

In this section, we consider the s-channel residue expansion of the partial four-gluon MHV amplitude, $M(p\,-, q\,-, k_1\,+, k_2\,+)$, with the external momenta p, q, k_1, k_2 and the respective Chan-Paton factor $4g^2\mathrm{Tr}(T^{a_p}T^{a_q}T^{a_1}T^{a_2})$, with the coupling constant g included. We want to compare the residues with the factorized sum[30]

$$F(p\,-, q\,-, k_1\,+, k_2\,+) \equiv \sum_{m_j,\, j<n} (p\,-, q\,-|m_j, j, n)\frac{1}{s-n}(k_2\,-, k_1\,-|m_j, j, n)^* \;, \tag{5.148}$$

where $s = 2p\cdot q$ (we also define $u = 2q\cdot k_1$ and $t = 2q\cdot k_2$), and $(p\,+, q\,+|m_j, j, n)$ are the three-point on-shell amplitudes involving two gluons and one string state at mass level n, with the spin quantum numbers j, m_j. The purpose of this exercise is to compare the three-point amplitudes with those evaluated in the previous Sections and to show explicitly how the four-gluon amplitude can be reconstructed by a BCFW deformation of the factorized sum:

$$F(p\,-, q\,-, k_1\,+, k_2\,+) \overset{BCFW}{\longrightarrow} M(p\,-, q\,-, k_1\,+, k_2\,+)\;. \tag{5.149}$$

The four-gluon amplitude is given by

$$\begin{aligned} M(p\,-, q\,-, k_1\,+, k_2\,+) &= \frac{\langle pq\rangle^4}{\langle pq\rangle\langle q1\rangle\langle 12\rangle\langle 2p\rangle}\frac{\Gamma(1-s)\Gamma(1-u)}{\Gamma(1-s-u)} \\ &= \langle pq\rangle^2[12]^2\,\frac{1}{s}\,B(1-s,-u) \end{aligned} \tag{5.150}$$

[30]In this Section, we set the mass scale $M = 1$.

By using the well-known expansion of the Beta function:

$$M(p-,q-,k_1+,k_2+) = \langle pq\rangle^2[12]^2\,\frac{1}{s}\sum_{n=1}^{\infty}\frac{1}{n-s}\,\frac{(u+1)_{n-1}}{(n-1)!}\ , \tag{5.151}$$

we find the residue associated to the mass level n is

$$\mathrm{Res}_{s=n}M(p-,q-,k_1+,k_2+) = -\langle pq\rangle^2[12]^2\,\frac{(u+1)_{n-1}}{n!}\ . \tag{5.152}$$

Note that the Pochhammer symbol contracts Lorentz indices across the s-channel (recall $u = 2q\cdot k_1$). The flow of Lorentz indices is due to the propagation of higher spin states in the s-channel. The first non-trivial contraction occurs at level $n = 2$, where it is due to massive vector particles discussed in the previous Section. At a given mass level n, not all spins j propagate: only the even ones for odd n and the odd ones for even n, up to $j = n-1$. For instance, at the next $n = 3$ level, both $j = 0$ and $j = 2$ contribute. We want to compare Eq.(5.152) with the residues of the factorized sum (5.148).

In the factorized sum (5.148), two pairs of gluons, (p,q) and (k_1,k_2) are coupled through intermediate Regge particles propagating in the s-channel. The Lorentz indices are transferred by the wave functions of intermediate particles, depending on a fixed spin quantization axis defined by the choice of reference vectors. The most convenient spin quantization axis is the direction of motion of the (p,q) pair in its center of mass frame, which is imposed by choosing p and q as the reference vectors for the massive wave functions, see Appendix C. In this case, the angular momentum conservation dictates that only $m_j = 0$ states propagate in the factorized sum. Let us illustrate this point on the example of a massive vector particle.

In the previous Section, Eq.(5.112), we found that, up to a numerical factor,

$$(p-,q-|m,j=1,n=2) = \langle pq\rangle^2(\xi_m\cdot q)\ . \tag{5.153}$$

Indeed, with the choice of (p,q) as the reference vectors for the polarization vectors ξ_m, one finds

$$\xi_{-1}\cdot q\ = \xi_{+1}\cdot q\ = 0\quad ,\qquad \xi_0\cdot q\ = \frac{1}{\sqrt{2}}(p-q)q = \frac{\sqrt{s}}{2} = \frac{1}{\sqrt{2}}\ , \tag{5.154}$$

where $\sqrt{2} = \sqrt{M_n}$ appear from the wave function normalization factors. On the other hand, with the same choice of the reference vectors,

$$
\begin{aligned}
(k_2 -, k_1 -|m=0, j=1, n=2) &= \langle 12\rangle^2 \frac{1}{\sqrt{2}}(p-q)k_1 \\
&= -\langle 12\rangle^2 \frac{1}{\sqrt{2}}(u+s/2) = -\langle 12\rangle^2 \frac{1}{\sqrt{2}}(u+1) \;.
\end{aligned}
\tag{5.155}
$$

In this way, we obtain

$$
(p-, q-|0,1,2)(k_2 -, k_1 -|0,1,2)^* = -\langle pq\rangle^2[12]^2 \frac{(u+1)}{2} \;, \tag{5.156}
$$

in agreement with the residuum (5.152) for $n=2$.

It is clear that for the above choice of reference vectors, the residues at $s=n$ of the factorized sum have the form $\langle pq\rangle^2[12]^2$ times a function of

$$
a \equiv (p-q)k_2 = -(p-q)k_1 = u + \frac{s}{2} = u + \frac{n}{2} \;, \tag{5.157}
$$

where the last step follows from the on-shell condition for the massive particle. We can obtain the factorized sum by simply setting $u = a - \frac{n}{2}$ in Eq.(5.151)

$$
\begin{aligned}
F(p-, q-, k_1+, k_2+) &= \langle pq\rangle^2[12]^2 \sum_{n=1}^{\infty} \frac{\mathrm{Res}_{(s=n\,,\ u=a-\frac{n}{2})} M(p-, q-, k_1+, k_2+)}{s-n} \\
&= \langle pq\rangle^2[12]^2 \sum_{n=1}^{\infty} \frac{1}{n-s} \frac{(a-\frac{n}{2}+1)_{n-1}}{n!} \;.
\end{aligned}
\tag{5.158}
$$

We checked the above result also at the $n=4$ level, by combining the on-shell amplitudes involving spin $j=0$ and $j=2$ Regge states, according to Eq.(5.148). It is convenient to introduce the generating function

$$
g_F(x) = \sum_{n=1}^{\infty} \frac{(a-\frac{n}{2}+1)_{n-1}}{n!} x^{n-1} \;, \tag{5.159}
$$

so that

$$F(p\,-,q\,-,k_1\,+,k_2\,+) = \langle pq\rangle^2[12]^2\int_0^1 x^{-s}\,g_F(x)\ . \tag{5.160}$$

It is easy to see that the generating function (5.159) satisfies

$$\frac{d}{dx}\big[xg_F(x)\big] = (1+\tfrac{x^2}{4})^{-1/2}\,e^{2a\mathrm{ArcSinh}(\frac{x}{2})}\ . \tag{5.161}$$

We want to stress again the the factorized sum is evaluated by using on-shell amplitudes involving one massive state and two gluons. We will show how to reconstruct the four gluon amplitude of Eq.(5.151), which involves intermediate particles propagating off-shell, by applying a BCFW deformation to F, Eq.(5.158).

It has been argued recently that the BCFW recursion relations, originally formulated for pure Yang-Mills theory [44–47], hold also in string theory [48–53]. The arguments rely crucially on proving the absence of an essential singularity at $z\to\infty$ (z is the deformation parameter) of the full-fledged string amplitudes. The proof is straightforward for four-gluon amplitudes but becomes increasingly complex for more gluons. Note that in the string case, there is an infinite number of intermediate states propagating in any channel, as seen explicitly in the factorized sum (5.148). This should be contrasted with the Yang-Mills case [76], where there are no massless on-shell states propagating in the s-channel of the deformed $(-\,-\,++)$ amplitude.

In order to force the s-channel resonances on-shell, we apply the BCFW deformation

$$p\to\hat{p}=p-zv\ ,\qquad k_2\to\hat{k_2}=k_2+zv\ , \tag{5.162}$$

where the light-like vector

$$v^\mu = \langle p|\sigma^\mu|2] \tag{5.163}$$

and z is the deformation parameter. Since $\hat{s}=s-2z\,vq$, the resonance poles appear at

$$z=\frac{s-n}{2vq}\ . \tag{5.164}$$

Under this deformation

$$a = (p-q)k_2 \ \rightarrow\ \hat{a} = a - z\, vq = a - \frac{s}{2} + \frac{n}{2} = u + \frac{n}{2} \ . \qquad (5.165)$$

Upon $a \to \hat{a}$, the generating function (5.159) transforms into

$$g_F(x) \overset{BCFW}{\longrightarrow} g_M(x) = \sum_{n=1}^{\infty} \frac{(u+1)_{n-1}}{n!}\, x^{n-1} \ , \qquad (5.166)$$

which satisfies

$$\frac{d}{dx}\big[x g_M(x)\big] = (1-x)^{-u-1} \ . \qquad (5.167)$$

In this way, we obtain

$$\begin{aligned} M(p-, q-, k_1+, k_2+) &= \langle pq\rangle^2 [12]^2 \int_0^1 x^{-s}\, g_M(x) \\ &= \langle pq\rangle^2 [12]^2\, \frac{1}{s}\, B(1-s, u) \ , \end{aligned} \qquad (5.168)$$

where we used Eq.(5.167) to integrate by parts. As usual with world-sheet duality, it is rewarding to see how the massless gluon pole appears in the u-channel after summing over the s-channel exchanges of massive string states.

Apart from providing the first explicit example of a BCFW construction in string theory, the above example seems of little or no practical importance. After all, what more can we learn by dissecting the Veneziano-Virasoro-Shapiro amplitude? It would be interesting, however, to construct all multi-gluon string disk amplitudes by using the BCFW recursion relations. Unfortunately, it is not so easy: starting from five gluons, a standard BCFW deformation, like in Eq.(5.162), yields on-shell poles in two channels, and the step leading from the factorized sums to the actual amplitude becomes quite cumbersome. Even in bosonic string theory, it is not clear how to combine much simpler factorized sums with five external tachyons to the well known five-tachyon amplitude.

6 Epilogue

My Ph.D. researches were mainly focused on both formal and phenomenological aspects of superstring scattering amplitudes. Specifically, we investigated the supersymmetry relations and coupling properties of the higher spin massive superstring states. Focusing on the first massive level universal superstring states and under the assumption of a low mass string scale as low as a few TeVs, we studied their scattering amplitudes in four dimensions and and obtain its possible collider signals explicitly.

Elementary particles are quantized vibrations of fundamental strings, and the SM particles are zero mode massless open strings. In the intersecting branes settings, gauge bosons are due to strings attached to same stacks of D-branes and chiral matters are due to strings stretching between different stacks of intersecting D-branes. The main motivation of my works is originated from the idea of low mass strings – we could test the string theory if the fundamental string scale is as low as a few TeVs. It has been shown that the production cross sections of gluons and quarks at the LHC can be computed in a model independent way, allowing for universal string predictions in case the fundamental string scale is low. The corresponding tree-level string cross sections are independent of the internal geometry and hold for all compactifications, in particular, no matter how many supercharges are preserved in the compactification. In this way, the landscape problem is circumvented. This property allows testing the low mass string scenario at the LHC.

Once the center of mass energies of the colliding partons exceed the fundamental string scale, the string excitations can be produced directly. In my first paper [1], we discussed the direct production of lightest Regge particles. There are certain higher spin Regge excitations at the first massive level, which are also completely universal. Computing production cross sections is important to identify these massive states at the LHC, since the angular distribution of their decay products is directly related to these higher spin universal Regge states. We identified these states by using BRST constraints and computed full-fledged superstring disk amplitudes of one massive Regge state with two or three gluons / quarks. Then in my second paper [22], which however is not the focus of this thesis, we discussed the possible signals of low mass string resonances in e^+e^- and $\gamma\gamma$ collisions at future lepton colliders. Then we go further in my third paper [2] to explore the properties of higher level superstring resonances. Starting from the four-gluon amplitude, we investigated the resonant structures of higher level massive superstring states. For Regge states with masses far above the fundamental string scale, we discussed the spin-dependence of their decay rates into massless gauge bosons. To illustrate the use of BCFW recursion relations in superstring theory, we showed that the four-gluon amplitude can be obtained by a BCFW deformation of a factorized sum involving on-shell amplitudes of one massive Regge state and two gauge bosons. Finally in my fourth paper [3], we expanded our discussion in [1] and investigated the SUSY relations for the physical superstring states in the first massive level. We obtained explicitly the supermultiplets of four-dimensional superstring states under $\mathcal{N} = 4, 2, 1$ compactifications.

Along the line of my previous works, there are various topics to be explored in the future: the practical use of the superstring version on-shell recursion relations; explore the general connections of massive superstring amplitudes with field theory amplitudes; investigate the massive loop amplitudes; carry out the superstring scattering processes in a fully consistent compactification; etc.

The recent discovery of the "Higgs-like" particle at the LHC [79, 80] indicates another great success of human pure thinking. However, even this new particle is confirmed to be the Higgs particle and totally complete the standard model, there are still many unsolved problems, such as dark matter, dark energy, hierarchy and naturalness problems, etc. Right now we are at the best moment to explore the unknowns of the physics beyond the standard model. I am very proud that I am one of these people, and we are so confident that we will succeed again!

Appendix

A Notation and convention

Various types of indices appear in this article, so it is essential to keep the notation as clear and unambiguous as possible. Here is a list of occurring index classes together with the preferably used alphabets and letters:

- In ten dimensions, vector indices of $SO(1,9)$ are taken from the middle of the Latin alphabet $m, n, p, \ldots$. The corresponding Weyl spinor indices are Greek letters from the beginning of the alphabet, $\alpha, \beta, \gamma, \ldots$ for left-handed spinors, and their dotted version $\dot{\alpha}, \dot{\beta}, \dot{\gamma}, \ldots$ for the right-handed counterparts.

- Vectors in four-dimensional Minkowski spacetime have indices from the middle of the Greek alphabet $\mu, \nu, \lambda, \rho, \ldots$. Spinor indices of $SO(1,3)$ are lower case Latin letters $a, b, c, \ldots$ for left-handed Weyl spinors and upper case $\dot{a}, \dot{b}, \dot{c}$ for right-handed Weyl spinors.

- The R-symmetry group of $\mathcal{N}=4$ spacetime SUSY is $SO(6) \equiv SU(4)$. We will use $m, n, p \ldots$ as vector indices and I, J, K ($\bar{I}, \bar{J}, \bar{K}$) as left-handed (right-handed) spinor indices. Confusions with the $D=10$ vector indices are excluded by the context.

- In case of $\mathcal{N}=2$ spacetime SUSY, we denote the fundamental indices of the $SU(2)$ R-symmetry by i, j, k and the corresponding adjoint indices by A, B, C.

- Chan-Paton generators carrying the color degrees of freedom of the vertex operator are suppressed throughout this work since they are the same for all members of the SUSY multiplet.

- Also, the coupling $g_{\mathrm{A}} = \sqrt{2\alpha'}\, g_{\mathrm{YM}}$ of vertex operators is suppressed, i.e. set to unity.

All these symmetry groups involve their metrics $\eta^{mn}, \eta^{\mu\nu}, \delta^{(6)}_{mn}$ as well as gamma matrices and charge conjugation matrices as Clebsch-Gordan coefficients:

- $\gamma^m_{\alpha\dot{\beta}}, \bar{\gamma}_m^{\dot{\alpha}\beta}$ and $C_\alpha{}^{\dot{\beta}}, C^{\dot{\alpha}}{}_\beta$ in $D=10$

- $\sigma^\mu_{a\dot{b}}, \bar{\sigma}_\mu^{\dot{a}b}$ and $\varepsilon_{ab}, \varepsilon^{\dot{a}\dot{b}}$ in $D=4$

- $\gamma_m^{I\bar{J}}, \bar{\gamma}^m_{\bar{I}J}$ and $C^I{}_{\bar{J}}, C_{\bar{I}}{}^J$ for the internal $SO(6)$ of $\mathcal{N}=4$ SUSY

- standard Pauli matrices ${\tau_A}^i{}_j$ and ε^{ij} for the $SU(2)$ R-symmetry of $\mathcal{N}=2$ SUSY

Our conventions for the slash notation is

$$\begin{array}{lll} \not{k}_{\alpha\dot\beta} = k_m\,\gamma^m_{\alpha\dot\beta}\,, & \not{k}^{\dot\beta\alpha} = k^m\,\bar\gamma_m^{\dot\beta\alpha} & \text{in } \ D=10 \\ \not{k}_{a\dot b} = k_\mu\,\sigma^\mu_{a\dot b}\,, & \not{k}^{\dot b a} = k^\mu\,\bar\sigma_\mu^{\dot b a} & \text{in } \ D=4 \end{array} . \tag{A.1}$$

The totally antisymmetric ε tensors are normalized to having nonzero ± 1, e.g. $\varepsilon^{\mu\nu\lambda\rho}$ for $D=4$ vectors and ε_{ABC} for the adjoint representation of $SU(2)$.

The signature of the Dirac algebras is negative in lines with the Wess & Bagger conventions:

$$\gamma^m_{\alpha\dot\beta}\,\bar\gamma^{n\dot\beta\gamma} \;+\; \gamma^n_{\alpha\dot\beta}\,\bar\gamma^{m\dot\beta\gamma} = -\,2\,\eta^{mn}\,\delta^\gamma_\alpha \tag{A.2}$$

$$\sigma^\mu_{a\dot b}\,\bar\sigma^{\nu\dot b c} \;+\; \sigma^\nu_{a\dot b}\,\bar\sigma^{\mu\dot b c} = -\,2\,\eta^{\mu\nu}\,\delta^c_a \tag{A.3}$$

$$\gamma_m^{I\bar J}\,\bar\gamma_{n\bar J K} \;+\; \gamma_n^{I\bar J}\,\bar\gamma_{m\bar J K} = -\,2\,\delta^{(6)}_{mn}\,\delta^I_K\ . \tag{A.4}$$

On the other hand, the $SU(2)$ Pauli matrices obey the multiplication rule

$$(\tau_A)^i{}_j\,(\tau_B)^j{}_k = \delta_{AB}\,\delta^i_k \;+\; i\varepsilon_{ABC}\,(\tau^C)^i{}_k \tag{A.5}$$

Useful material on spinors in various spacetime dimensions can be found in [65], the present conventions closely follow [57, 58].

B Operator product expansions

This appendix gathers the operator product expansions needed to evaluate the BRST constraints and SUSY variations. Before taking a closer look at the interacting SCFTs, let us display the free field OPEs for the sake of completeness, namely

$$i\partial X^\mu(z)\,e^{ik\cdot X}(w) \;\;\sim\;\; \left[\,\frac{2\alpha'\,k^\mu}{z-w} \;+\; i\partial X^\mu(w) \;+\; (z-w)\,i\partial^2 X^\mu(w) \;+\; \dots\right] e^{ik\cdot X}(w) \tag{B.1}$$

$$i\partial X^\mu(z)\, i\partial X^\nu(w)\, e^{ik\cdot X}(w) \quad \sim \quad \left[\frac{2\alpha'\,\eta^{\mu\nu}}{(z-w)^2} \;+\; \frac{2\alpha'\, k^\mu\, i\partial X^\nu(w)}{z-w} \;+\; i\partial X^\mu\, i\partial X^\nu(w) \;+\; \dots \right] e^{ik\cdot X}(w) \tag{B.2}$$

as well as

$$\psi^\mu(z)\,\psi^\nu(w) \quad \sim \quad \frac{\eta^{\mu\nu}}{z-w} \;+\; \psi^\mu\,\psi^\nu(w) \;+\; (z-w)\,\partial\psi^\mu\,\psi^\nu(w) \;+\; \dots \tag{B.3}$$

They are valid in any number of compactification dimensions. Another universal feature is the superghost CFT, governed by

$$\begin{aligned} e^{q_1\phi(z)}\, e^{q_2\phi(w)} =& (z-w)^{-q_1 q_2} \Big[e^{(q_1+q_2)\,\phi(w)} \;+\; q_1\,(z-w)\,\partial\phi\, e^{(q_1+q_2)\,\phi(w)} \\ &+\; \tfrac{1}{2}\,(z-w)^2 \left[q_1\,\partial^2\phi \;+\; q_1^2\,(\partial\phi)^2 \right] e^{(q_1+q_2)\,\phi(w)} \;+\; \dots \Big]\,. \end{aligned} \tag{B.4}$$

The following subsections consider the interacting RNS CFT of the ψ fermion and its spin fields S as well as its excited versions. The OPEs were pioneered in [56] and can be checked by means of correlation functions gathered in [57, 58].

B.1 Spacetime CFT in $D = 10$

Evaluating the BRST conditions on the most general fermion vertex operator at the first mass level requires OPEs

$$\psi^m(z)\, S_\alpha(w) \quad \sim \quad \frac{\gamma^m_{\alpha\dot\beta}\, S^{\dot\beta}(w)}{\sqrt{2}\,(z-w)^{1/2}} \;+\; (z-w)^{1/2} \left[S^m_\alpha(w) \;+\; \frac{2\,\gamma^m_{\alpha\dot\beta}}{\sqrt{2}\,5}\,\partial S^{\dot\beta}(w) \right] \;+\; \dots \tag{B.5}$$

$$\psi^m(z)\,\partial S_\alpha(w) \quad \sim \quad \frac{\gamma^m_{\alpha\dot\beta}\, S^{\dot\beta}(w)}{2\sqrt{2}\,(z-w)^{3/2}} \;-\; \frac{S^m_\alpha(w)}{2\,(z-w)^{1/2}} \;+\; \frac{4\,\gamma^m_{\alpha\dot\beta}\,\partial S^{\dot\beta}(w)}{5\,\sqrt{2}\,(z-w)^{1/2}} \;+\; \dots \tag{B.6}$$

$$\psi_m(z)\, S^{\dot\beta}_n(w) \quad \sim \quad \frac{\eta_{mn}\, S^{\dot\beta}(w)}{(z-w)^{3/2}} \;+\; \frac{\bar\gamma_m^{\dot\beta\alpha}\, S_{n\alpha}(w)}{\sqrt{2}\,(z-w)^{1/2}} \;-\; \frac{2\,\eta_{mn}\,\partial S^{\dot\beta}(w)}{5\,(z-w)^{1/2}} \;+\; \dots \tag{B.7}$$

in $D = 10$. The corresponding SUSY variations are computed by means of

$$S^{\dot\beta}(z)\,\psi_m(w) \quad \sim \quad \frac{\bar\gamma_m^{\dot\beta\alpha}\, S_\alpha(w)}{\sqrt{2}\,(z-w)^{1/2}} \;+\; (z-w)^{1/2} \left[S^{\dot\beta}_m(w) \;+\; \frac{3\,\bar\gamma_m^{\dot\beta\alpha}\,\partial S_\alpha(w)}{5\,\sqrt{2}} \right] \;+\; \dots \tag{B.8}$$

$$S^{\dot\beta}(z)\,\psi_m\,\psi_n\,\psi_p(w) \;\sim\; \frac{-1}{2\sqrt{2}\,(z-w)^{3/2}}\,\bar\gamma_{mnp}^{\dot\beta\alpha}\,S_\alpha(w) \;-\; \frac{3}{2\,(z-w)^{1/2}}\,(\bar\gamma_{[mn})^{\dot\beta}{}_{\dot\alpha}\,S^{\dot\alpha}_{p]}(w)$$
$$+\; \frac{1}{10\sqrt{2}\,(z-w)^{1/2}}\,\bar\gamma_{mnp}^{\dot\beta\alpha}\,\partial S_\alpha(w) \;+\;\dots \tag{B.9}$$
$$S^{\dot\beta}(z)\,\partial\psi_m(w) \;\sim\; \frac{\bar\gamma_m^{\dot\beta\alpha}\,S_\alpha(w)}{2\sqrt{2}\,(z-w)^{3/2}} \;-\; \frac{S^{\dot\beta}_m(w)}{2\,(z-w)^{1/2}} \;+\; \frac{7\,\bar\gamma_m^{\dot\beta\alpha}\,\partial S_\alpha(w)}{10\sqrt{2}\,(z-w)^{1/2}} \;+\;\dots \tag{B.10}$$

for the NS sector and

$$S_\alpha(z)\,S_\beta(w) \;\sim\; \frac{(\gamma^m C)_{\alpha\beta}\,\psi_m(w)}{\sqrt{2}\,(z-w)^{3/4}} \;+\; (z-w)^{1/4}\,\frac{(\gamma^m C)_{\alpha\beta}\,\partial\psi_m(w)}{2\sqrt{2}}$$
$$-\;(z-w)^{1/4}\,\frac{(\gamma^{mnp}C)_{\alpha\beta}\,\psi_m\,\psi_n\,\psi_p(w)}{12\sqrt{2}} \;+\;\dots \tag{B.11}$$
$$S_\alpha(z)\,S^{\dot\beta}_m(w) \;\sim\; \frac{C_\alpha{}^{\dot\beta}\,\psi_m(w)}{(z-w)^{7/4}} \;-\; \frac{C_\alpha{}^{\dot\beta}\,\partial\psi_m(w)}{2\,(z-w)^{3/4}} \;-\; \frac{(\gamma^{np}\,C)_\alpha{}^{\dot\beta}\,\psi_m\,\psi_n\,\psi_p(w)}{4\,(z-w)^{3/4}} \;+\;\dots \tag{B.12}$$
$$S_\alpha(z)\,\partial S_\beta(w) \;\sim\; \frac{3\,(\gamma^m C)_{\alpha\beta}\,\psi_m(w)}{4\sqrt{2}\,(z-w)^{7/4}} \;+\; \frac{7\,(\gamma^m C)_{\alpha\beta}\,\partial\psi_m(w)}{8\sqrt{2}\,(z-w)^{3/4}}$$
$$+\; \frac{(\gamma^{mnp}C)_{\alpha\beta}\,\psi_m\,\psi_n\,\psi_p(w)}{48\sqrt{2}\,(z-w)^{3/4}} \;+\;\dots \tag{B.13}$$

for the R sector.

B.2 Spacetime CFT in $D = 4$

In $D = 4$ spacetime dimensions, $h = \frac{1}{4}$ spin fields $S_a, S^{\dot b}$ of both chiralities are present. The OPEs between spinors and vectors or p-forms treat both chiralities on equal footing, e.g.

$$\psi^\mu(z)\,S_a(w) \;\sim\; \frac{\sigma^\mu_{a\dot b}\,S^{\dot b}(w)}{\sqrt{2}\,(z-w)^{1/2}} \;+\; (z-w)^{1/2}\left[S^\mu_a(w) \;+\; \frac{\sigma^\mu_{a\dot b}}{\sqrt{2}}\,\partial S^{\dot b}(w)\right] \;+\;\dots \tag{B.14}$$
$$\psi_\mu(z)\,S^{\dot b}(w) \;\sim\; \frac{\bar\sigma_\mu^{\dot b a}\,S_a(w)}{\sqrt{2}\,(z-w)^{1/2}} \;+\; (z-w)^{1/2}\left[S^{\dot b}_\mu(w) \;+\; \frac{\bar\sigma_\mu^{\dot b a}}{\sqrt{2}}\,\partial S_a(w)\right] \;+\;\dots \tag{B.15}$$

that is why we only display one chiral half of further OPEs:

$$\psi^\mu(z)\,\partial S_a(w) \;\sim\; \frac{\sigma^\mu_{a\dot b}\,S^{\dot b}(w)}{2\sqrt{2}\,(z-w)^{3/2}} \;-\; \frac{S^\mu_a(w)}{2\,(z-w)^{1/2}} \;+\; \frac{\sigma^\mu_{a\dot b}\,\partial S^{\dot b}(w)}{2\sqrt{2}\,(z-w)^{1/2}} \;+\;\dots \tag{B.16}$$
$$\psi_\mu(z)\,S^{\dot b}_\nu(w) \;\sim\; \frac{\eta_{\mu\nu}\,S^{\dot b}(w)}{(z-w)^{3/2}} \;+\; \frac{\bar\sigma_\mu^{\dot b a}\,S_{\nu a}(w)}{\sqrt{2}\,(z-w)^{1/2}} \;-\; \frac{\eta_{\mu\nu}\,\partial S^{\dot b}(w)}{(z-w)^{1/2}} \;+\;\dots \tag{B.17}$$

Four-dimensional SUSY variations of NS operators require

$$S^{\dot{b}}(z)\,\psi_\mu(w) \;\sim\; \frac{\bar\sigma_\mu^{\dot{b}a}\,S_a(w)}{\sqrt{2}\,(z-w)^{1/2}} \;+\; (z-w)^{1/2}\,S_\mu^{\dot{b}}(w) \;+\; \dots \tag{B.18}$$

$$S^{\dot{b}}(z)\,\psi_\mu\,\psi_\nu(w) \;\sim\; \frac{-\,(\bar\sigma_{\mu\nu})^{\dot{b}}{}_{\dot{a}}\,S^{\dot{a}}(w)}{2\,(z-w)} \;+\; \sqrt{2}\,\bar\sigma^{[\nu|\dot{b}a}\,S_a^{|\mu]}(w) \;+\; \frac{1}{2}\,(\bar\sigma_{\mu\nu})^{\dot{b}}{}_{\dot{a}}\,\partial S^{\dot{a}}(w) \;+\; \dots \tag{B.19}$$

$$\begin{aligned} S^{\dot{b}}(z)\,\psi_\mu\,\psi_\nu\,\psi_\lambda(w) \;\sim\;& \frac{-1}{2\sqrt{2}\,(z-w)^{3/2}}\,\bar\sigma_{\mu\nu\lambda}^{\dot{b}a}\,S_a(w) \;-\; \frac{3}{2\,(z-w)^{1/2}}\,(\bar\sigma_{[\mu\nu})^{\dot{b}}{}_{\dot{a}}\,S_{\lambda]}^{\dot{a}}(w) \\ &+\; \frac{1}{\sqrt{2}\,(z-w)^{1/2}}\,\bar\sigma_{\mu\nu\lambda}^{\dot{b}a}\,\partial S_a(w) \;+\; \dots \end{aligned} \tag{B.20}$$

$$S^{\dot{b}}(z)\,\partial\psi_\mu(w) \;\sim\; \frac{\bar\sigma_\mu^{\dot{b}a}\,S_a(w)}{2\sqrt{2}\,(z-w)^{3/2}} \;-\; \frac{S_\mu^{\dot{b}}(w)}{2\,(z-w)^{1/2}} \;+\; \frac{\bar\sigma_\mu^{\dot{b}a}\,\partial S_a(w)}{\sqrt{2}\,(z-w)^{1/2}} \;+\; \dots\,. \tag{B.21}$$

With two R sector states involved, the OPEs are sensitive to their relative chirality:

$$S_a(z)\,S_b(w) \;\sim\; \frac{\varepsilon_{ab}}{(z-w)^{1/2}} \;-\; \frac{1}{4}\,(z-w)^{1/2}\,(\sigma^{\mu\nu}\varepsilon)_{ab}\,\psi_\mu\,\psi_\nu(w) \;+\; \dots \tag{B.22}$$

$$\begin{aligned} S_a(z)\,S^{\dot{b}}(w) \;\sim\;& \frac{(\sigma^\mu\varepsilon)_a{}^{\dot{b}}\,\psi_\mu(w)}{\sqrt{2}} \;+\; (z-w)\,\frac{(\sigma^\mu\varepsilon)_a{}^{\dot{b}}\,\partial\psi_\mu(w)}{2\sqrt{2}} \\ &-\; (z-w)\,\frac{(\sigma^{\mu\nu\lambda}\varepsilon)_a{}^{\dot{b}}\,\psi_\mu\,\psi_\nu\,\psi_\lambda(w)}{12\sqrt{2}} \;+\; \dots \end{aligned} \tag{B.23}$$

$$S_a(z)\,S_\mu^{\dot{b}}(w) \;\sim\; \frac{(\sigma^\nu\varepsilon)_a{}^{\dot{b}}\,\psi_\mu\,\psi_\nu(w)}{\sqrt{2}\,(z-w)^{1/2}} \;+\; \dots \tag{B.24}$$

$$S_a(z)\,S_b^\mu(w) \;\sim\; \frac{\varepsilon_{ab}\,\psi^\mu(w)}{(z-w)} \;-\; \frac{\varepsilon_{ab}\,\partial\psi^\mu(w)}{2} \;-\; \frac{(\sigma_{\nu\lambda}\varepsilon)_{ab}\,\psi^\mu\,\psi^\nu\,\psi^\lambda(w)}{4} \;+\; \dots \tag{B.25}$$

$$S_a(z)\,\partial S_b(w) \;\sim\; \frac{\varepsilon_{ab}}{2\,(z-w)^{3/2}} \;+\; \frac{(\sigma^{\mu\nu}\varepsilon)_{ab}\,\psi_\mu\,\psi_\nu(w)}{8\,(z-w)^{1/2}} \;+\; \dots \tag{B.26}$$

$$S_a(z)\,\partial S^{\dot{b}}(w) \;\sim\; \frac{(\sigma^\mu\varepsilon)_a{}^{\dot{b}}\,\partial\psi_\mu(w)}{2\sqrt{2}} \;+\; \frac{(\sigma^{\mu\nu\lambda}\varepsilon)_a{}^{\dot{b}}\,\psi_\mu\,\psi_\nu\,\psi_\lambda(w)}{12\sqrt{2}} \;+\; \dots \tag{B.27}$$

B.3 Internal CFT for $\mathcal{N}=4$ SUSY

The internal components of the ten-dimensional NS fermion are denoted by Ψ_m with vector index m for the $SO(6)$ R-symmetry. Accordingly, the associated $h=\frac{3}{8}$ spin fields $\Sigma^I, \bar\Sigma_{\bar J}$ have $SO(6)$ spinor indices $I,\bar J = 1,2,3,4$. Their mutual OPEs can be covariantly expressed in terms of $SO(6)$ gamma matrices:

$$\Psi_m(z)\,\Sigma^I(w) \;\sim\; \frac{\gamma_m^{I\bar J}\,\bar\Sigma_{\bar J}(w)}{\sqrt{2}\,(z-w)^{1/2}} \;+\; (z-w)^{1/2}\left[\Sigma_m^I(w) \;+\; \frac{2\,\gamma_m^{I\bar J}}{3\sqrt{2}}\,\partial\bar\Sigma_{\bar J}(w)\right] \;+\; \dots \tag{B.28}$$

$$\Psi^m(z)\,\bar\Sigma^n_{\bar J}(w) \;\sim\; \frac{\delta^{mn}_{(6)}\,\bar\Sigma_{\bar J}(w)}{(z-w)^{3/2}} \;+\; \frac{\bar\gamma^m_{\bar J I}\,\Sigma^{n,I}(w)}{\sqrt{2}\,(z-w)^{1/2}} \;-\; \frac{2\,\delta^{mn}_{(6)}\,\partial\bar\Sigma_{\bar J}(w)}{3\,(z-w)^{1/2}} \;+\;\ldots \tag{B.29}$$

$$\Psi_m(z)\,\partial\Sigma^I(w) \;\sim\; \frac{\gamma_m^{I\bar J}\,\bar\Sigma_{\bar J}(w)}{2\sqrt{2}\,(z-w)^{3/2}} \;-\; \frac{\Sigma^I_m(w)}{2\,(z-w)^{1/2}} \;+\; \frac{2\,\gamma_m^{I\bar J}\,\partial\bar\Sigma_{\bar J}(w)}{3\sqrt{2}\,(z-w)^{1/2}} \;+\;\ldots \tag{B.30}$$

We need the following OPEs for computing SUSY transformations of bosons:

$$\bar\Sigma_{\bar J}(z)\,\Psi^m(w) \;\sim\; \frac{\bar\gamma^m_{\bar J I}\,\Sigma^I(w)}{\sqrt{2}\,(z-w)^{1/2}} \;+\; (z-w)^{1/2}\left[\bar\Sigma^m_{\bar J}(w) \;+\; \frac{\bar\gamma^m_{\bar J I}}{3\sqrt{2}}\,\partial\Sigma^I(w)\right] \;+\;\ldots \tag{B.31}$$

$$\bar\Sigma_{\bar J}(z)\,\Psi^m\,\Psi^n(w) \;\sim\; \frac{-\,(\bar\gamma^{mn})_{\bar J}{}^{\bar I}\,\bar\Sigma_{\bar I}(w)}{2\,(z-w)} \;+\; \sqrt{2}\,\bar\gamma^{[n|\bar J I}\,\Sigma^{m]}_I(w) \;+\; \frac{1}{6}\,(\bar\gamma^{mn})_{\bar J}{}^{\bar I}\,\partial\bar\Sigma_{\bar I}(w) \;+\;\ldots \tag{B.32}$$

$$\begin{aligned}\bar\Sigma_{\bar J}(z)\,\Psi^m\,\Psi^n\,\Psi^p(w) \;\sim\;& \frac{-1}{2\sqrt{2}\,(z-w)^{3/2}}\,\bar\gamma^{mnp}_{\bar J I}\,\Sigma^I(w) \;-\; \frac{3}{2\,(z-w)^{1/2}}\,(\bar\gamma^{[mn})_{\bar J}{}^{\bar I}\,\bar\Sigma^{p]}_{\bar I}(w)\\ &+\; \frac{1}{2\sqrt{2}\,(z-w)^{1/2}}\,\bar\gamma^{mnp}_{\bar J I}\,\partial\Sigma^I(w) \;+\;\ldots\end{aligned} \tag{B.33}$$

$$\bar\Sigma_{\bar J}(z)\,\partial\Psi^m(w) \;\sim\; \frac{\bar\gamma^m_{\bar J I}\,\Sigma^I(w)}{2\sqrt{2}\,(z-w)^{3/2}} \;-\; \frac{\bar\Sigma^m_{\bar I}(w)}{2\,(z-w)^{1/2}} \;+\; \frac{5\,\bar\gamma^k_{\bar J I}\,\partial\Sigma^I(w)}{6\sqrt{2}\,(z-w)^{1/2}} \;+\;\ldots\,. \tag{B.34}$$

Again, OPEs between R sector states depend on the relative chirality:

$$\Sigma^I(z)\,\bar\Sigma_{\bar J}(w) \;\sim\; \frac{C^I{}_{\bar J}}{(z-w)^{3/4}} \;-\; \frac{1}{4}\,(z-w)^{1/4}\,(\gamma_{mn}\,C)^I{}_{\bar J}\,\Psi^m\,\Psi^n(w) \;+\;\ldots \tag{B.35}$$

$$\begin{aligned}\Sigma^I(z)\,\Sigma^J(w) \;\sim\;& \frac{(\gamma_m\,C)^{IJ}\,\Psi^m(w)}{\sqrt{2}\,(z-w)^{1/4}} \;+\; (z-w)^{3/4}\,\frac{(\gamma_m\,C)^{IJ}\,\partial\Psi^m(w)}{2\sqrt{2}}\\ &-\; (z-w)^{3/4}\,\frac{(\gamma_{mnp}\,C)^{IJ}\,\Psi^m\,\Psi^n\,\Psi^p(w)}{12\sqrt{2}} \;+\;\ldots\end{aligned} \tag{B.36}$$

$$\Sigma^I(z)\,\Sigma^J_m(w) \;\sim\; \frac{(\gamma^n\,C)^{IJ}\,\Psi_m\,\Psi_n(w)}{\sqrt{2}\,(z-w)^{3/4}} \;+\;\ldots \tag{B.37}$$

$$\Sigma^I(z)\,\bar\Sigma^m_{\bar J}(w) \;\sim\; \frac{C^I{}_{\bar J}\,\Psi^m(w)}{(z-w)^{5/4}} \;-\; \frac{C^I{}_{\bar J}\,\partial\Psi^m(w)}{2\,(z-w)^{1/4}} \;-\; \frac{(\gamma_{np}\,C)^I{}_{\bar J}\,\Psi^m\,\Psi^n\,\Psi^p(w)}{4\,(z-w)^{1/4}} \;+\;\ldots \tag{B.38}$$

$$\Sigma^I(z)\,\partial\bar\Sigma_{\bar J}(w) \;\sim\; \frac{3\,C^I{}_{\bar J}}{4\,(z-w)^{7/4}} \;+\; \frac{(\gamma_{mn}\,C)^I{}_{\bar J}\,\Psi^m\,\Psi^n(w)}{16\,(z-w)^{3/4}} \;+\;\ldots \tag{B.39}$$

$$\begin{aligned}\Sigma^I(z)\,\partial\Sigma^J(w) \;\sim\;& \frac{(\gamma_m\,C)^{IJ}\,\Psi^m(w)}{4\sqrt{2}\,(z-w)^{5/4}} \;+\; \frac{5\,(\gamma_m\,C)^{IJ}\,\partial\Psi^m(w)}{8\sqrt{2}\,(z-w)^{1/4}}\\ &+\; \frac{(\gamma_{mnp}\,C)^{IJ}\,\Psi^m\,\Psi^n\,\Psi^p(w)}{16\sqrt{2}\,(z-w)^{1/4}} \;+\;\ldots\end{aligned} \tag{B.40}$$

B.4 Internal CFT for $\mathcal{N} = 1$ SUSY

Most of the OPEs relevant for the internal $c = 9$ SCFT described in subsection 2.2.5 can be derived from the CFT of a free boson:

$$i\partial H(z)\, e^{iqH}(w) \;\sim\; \left[\frac{q}{z-w} \;+\; i\partial H(w) \;+\; \dots\right] e^{iqH}(w) \tag{B.41}$$

$$e^{iqH}(z)\, i\partial H(w) \;\sim\; \left[\frac{q}{z-w} \;+\; (q^2 \,-\, 1)\, i\partial H(w) \;+\; \dots\right] e^{iqH}(w) \tag{B.42}$$

$$e^{iq_1 H}(z)\, e^{iq_2 H}(w) \;\sim\; (z-w)^{q_1 q_2} \left[1 \;+\; q_1\,(z-w)\, i\partial H \;+\; \dots\right] e^{i(q_1+q_2)H}(w) \tag{B.43}$$

This allows to reproduce (2.77) and (2.83) from the bosonized representations (2.84) of the operators $\mathcal{J}$, $\Sigma^\pm$ and $\mathcal{O}^\pm$. Moreover, we have

$$\Sigma^\pm(z)\, \mathcal{J}(w) \;\sim\; \frac{\pm\sqrt{3}\,\Sigma^\pm(w)}{2\,(z-w)} \;\mp\; \frac{\partial\Sigma^\pm(w)}{2\sqrt{3}} \;+\; \dots \tag{B.44}$$

$$\Sigma^\pm(z)\, \mathcal{O}^\mp(w) \;\sim\; (z-w)^{-3/2}\, \Sigma^\mp(w) \;-\; (z-w)^{-1/2}\, \partial\Sigma^\mp(w) \;+\; \dots \tag{B.45}$$

The excited spin fields $\tilde{\Sigma}^\pm = g^\mp e^{\pm iH/\sqrt{12}}$ are canonically normalized

$$\tilde{\Sigma}^\pm(z)\, \tilde{\Sigma}^\mp(w) \;\sim\; \frac{1}{(z-w)^{11/4}} \;\pm\; \frac{i\partial H(w)}{2\sqrt{3}\,(z-w)^{7/4}} \;+\; \dots \tag{B.46}$$

$$\tilde{\Sigma}^\pm(z)\, \tilde{\Sigma}^\pm(w) \;\sim\; \frac{g^\mp\, g^\mp\, e^{\pm\frac{iH}{\sqrt{3}}}(w)}{(z-w)^{1/4}} \;+\; \dots \tag{B.47}$$

such that the mutual singularities between standard and excited spin fields are given by

$$\tilde{\Sigma}^\pm(z)\, \Sigma^\pm(w) \;\sim\; (z-w)^{1/4}\, g^\mp\, e^{\pm\frac{2i}{\sqrt{3}}H}(w) \;+\; \dots \tag{B.48}$$

$$\tilde{\Sigma}^\pm(z)\, \Sigma^\mp(w) \;\sim\; \sqrt{\frac{2}{3}}\, \frac{G^\mp_{\text{int}}(w)}{(z-w)^{1/4}} \;+\; \dots \tag{B.49}$$

Moreover, in presence of the internal supercurrents $G^\pm_{\text{int}} = \sqrt{\frac{3}{2}} e^{\pm iH/\sqrt{3}} g^\pm$,

$$G^\pm_{\text{int}}(z)\, \tilde{\Sigma}^\pm(w) \;\sim\; \sqrt{\frac{3}{2}}\, \frac{\Sigma^\pm(w)}{(z-w)^{5/2}} \;+\; \sqrt{\frac{2}{3}}\, \frac{\partial\Sigma^\pm(w)}{(z-w)^{3/2}} \;+\; \dots \tag{B.50}$$

$$G^{\pm}_{\rm int}(z)\,\tilde\Sigma^{\mp}(w) \;\sim\; \sqrt{\frac{3}{2}}\,\frac{g^{\pm}\,g^{\pm}\,e^{\pm\frac{i}{2\sqrt{3}}H}(w)}{(z-w)^{1/2}} \;+\;\ldots \tag{B.51}$$

$$\tilde\Sigma^{\pm}(z)\,\mathcal{J}(w) \;\sim\; \pm\,\frac{\tilde\Sigma^{\pm}(w)}{2\sqrt{3}\,(z-w)} \;+\;\ldots \tag{B.52}$$

C Spinor helicity methods for massive wavefunctions

Before we proceed to introduce the massive version of the spinor helicity formalism, we will make a short review for the helicity formalism of massless spinors. For massless spin-$\frac{1}{2}$ spinors, we use the following notations,

$$|i\rangle = |k_i\rangle = u_+(k_i) = v_-(k_i) = \begin{pmatrix} 0 \\ k_i^{*\dot a}\end{pmatrix}, \tag{C.1}$$

$$|i] = |k_i] = u_-(k_i) = v_+(k_i) = \begin{pmatrix} k_{i,a} \\ 0\end{pmatrix}, \tag{C.2}$$

$$[i| = [k_i| = \bar u_+(k_i) = \bar v_-(k_i) = (k_i^a, 0)\,, \tag{C.3}$$

$$\langle i| = \langle k_i| = \bar u_-(k_i) = \bar v_+(k_i) = \left(0, k^*_{i,\dot a}\right). \tag{C.4}$$

Here the momentums with spinor indices denote two component commutative spinors. They are defined by

$$P^{\dot a a} = p_\mu \bar\sigma^{\mu\dot a a} = -p^{*\dot a}p^a, \tag{C.5}$$

$$P_{a\dot a} = p_\mu \sigma^\mu_{a\dot a} = -p_a p^*_{\dot a}, \tag{C.6}$$

where $p^{*\dot a} = (p^a)^*$ and $p^*_{\dot a} = (p_a)^*$. Spinor indices could be raised (lowered) by ε^{ab} (ε_{ab}) or a, b with dots,

$$p^a = \varepsilon^{ab}p_b, \qquad p^{*\dot a} = \varepsilon^{\dot a\dot b}p^*_{\dot b}. \tag{C.7}$$

Then we can define the notations for the spinor products,

$$\langle pq\rangle = \langle p|q\rangle = \bar u_-(p)u_+(q) = p^*_{\dot a}q^{*\dot a}, \tag{C.8}$$

$$[pq] = [p|q] = \bar u_+(p)u_-(q) = p^a q_a, \tag{C.9}$$

so that simply we have

$$[pq] = -[qp], \qquad \langle pq\rangle = -\langle qp\rangle, \tag{C.10}$$

$$\langle pq\rangle^* = -[pq], \qquad \langle pp\rangle = [pp] = 0, \tag{C.11}$$

and

$$\langle pq\rangle[qp] = -2(p\cdot q). \tag{C.12}$$

C.1 Massive spin one boson

A spin J particle contains $2J+1$ spin degrees of freedom associated to the eigenstates of J_z. The choice of the quantization axis z can be handled in an elegant way by decomposing the momentum k into two arbitrary light-like reference momenta p and q:

$$k^\mu = p^\mu + q^\mu, \qquad k^2 = -m^2 = 2pq, \qquad p^2 = q^2 = 0. \tag{C.13}$$

Then the spin quantization axis is chosen to be the direction of q in the rest frame. The $2J+1$ spin wavefunctions depend of p and q, however this dependence drops out in the amplitudes summed over all spin directions and in "unpolarized" cross sections.

The massive spin one wavefunctions ξ_μ (transverse, i.e., $\xi_\mu k^\mu = 0$) are given by the following polarization vectors [59, 60], up to a phase factor,

$$\xi^\mu_+(k) = \frac{1}{\sqrt{2}m} p^*_{\dot a}\bar\sigma^{\mu\dot a a} q_a, \tag{C.14}$$

$$\xi^\mu_0(k) = \frac{1}{2m}\bar\sigma^{\mu\dot a a}(p^*_{\dot a}p_a - q^*_{\dot a}q_a), \tag{C.15}$$

$$\xi^\mu_-(k) = -\frac{1}{\sqrt{2}m} q^*_{\dot a}\bar\sigma^{\mu\dot a a} p_a. \tag{C.16}$$

C.2 Massive spin two boson

The massive spin two boson $\alpha^{\mu\nu}$ satisfies the following conditions,

$$\alpha^{\mu\nu}(k,\lambda) = \alpha^{\nu\mu}(k,\lambda), \tag{C.17}$$

$$k_\mu \alpha^{\mu\nu}(k,\lambda) = 0, \tag{C.18}$$

$$g_{\mu\nu}\alpha^{\mu\nu}(k,\lambda) = 0, \tag{C.19}$$

where λ expresses the helicity of $\alpha^{\mu\nu}$. We do the same decomposition of the momentum, and the wavefunction of a spin two boson can be written as [59],

$$\alpha^{\mu\nu}(k,+2) = \frac{1}{2m^2}\bar{\sigma}^{\mu\dot{a}a}\bar{\sigma}^{\nu\dot{b}b} p^*_{\dot{a}} q_a p^*_{\dot{b}} q_b \ , \tag{C.20}$$

$$\alpha^{\mu\nu}(k,+1) = \frac{1}{4m^2}\bar{\sigma}^{\mu\dot{a}a}\bar{\sigma}^{\nu\dot{b}b} \left[(p^*_{\dot{a}} p_a - q^*_{\dot{a}} q_a) p^*_{\dot{b}} q_b + p^*_{\dot{a}} q_a (p^*_{\dot{b}} p_b - q^*_{\dot{b}} q_b) \right], \tag{C.21}$$

$$\alpha^{\mu\nu}(k,\ 0\ ) = \frac{1}{2\sqrt{6}m^2}\bar{\sigma}^{\mu\dot{a}a}\bar{\sigma}^{\nu\dot{b}b} \left[(p^*_{\dot{a}} p_a - q^*_{\dot{a}} q_a)(p^*_{\dot{b}} p_b - q^*_{\dot{b}} q_b) - p^*_{\dot{a}} q_a q^*_{\dot{b}} p_b - q^*_{\dot{a}} p_a p^*_{\dot{b}} q_b \right], \tag{C.22}$$

$$\alpha^{\mu\nu}(k,-1) = \frac{1}{4m^2}\bar{\sigma}^{\mu\dot{a}a}\bar{\sigma}^{\nu\dot{b}b} \left[(q^*_{\dot{a}} q_a - p^*_{\dot{a}} p_a) q^*_{\dot{b}} p_b + q^*_{\dot{a}} p_a (q^*_{\dot{a}} q_a - p^*_{\dot{b}} p_b) \right], \tag{C.23}$$

$$\alpha^{\mu\nu}(k,-2) = \frac{1}{2m^2}\bar{\sigma}^{\mu\dot{a}a}\bar{\sigma}^{\nu\dot{b}b} q^*_{\dot{a}} p_a q^*_{\dot{b}} p_b \ . \tag{C.24}$$

C.3 Massive spin 1/2 fermions

Massive spin-$\frac{1}{2}$ fermions satisfy the Dirac equation,

$$(\not{k} + m)u(k) = 0, \tag{C.25}$$

$$(\not{k} - m)v(k) = 0, \tag{C.26}$$

where $u(k)$ and $v(k)$ are positive and negative energy solutions with momentum k^μ, which correspond to fermion and anti-fermion wavefunctions respectively. Since we do not deal with the wavefunctions of the negative energy solutions, we will only present $u(k)$ wavefunction here. $u(k)$ satisfies the spin-sum relations, orthogonal condition and the normalization condition,

$$\sum_{spin} u_{\pm}(k)\bar{u}_{\pm}(k) = -\not{k} + m, \tag{C.27}$$

$$\bar{u}_{\pm}(k)u_{\mp}(k) = 0, \tag{C.28}$$

$$\bar{u}_{\pm}(k)u_{\pm}(k) = 2m, \tag{C.29}$$

Writing the four component spinor $u(k)$ as

$$u = \begin{pmatrix} \chi_a \\ \bar{\eta}^{\dot{a}} \end{pmatrix} \tag{C.30}$$

and plugging it into the Dirac equation, we get

$$k_\mu \begin{pmatrix} 0 & \sigma^\mu_{a\dot{a}} \\ \bar{\sigma}^{\mu\dot{a}a} & 0 \end{pmatrix} \begin{pmatrix} \chi_a \\ \bar{\eta}^{\dot{a}} \end{pmatrix} = -m \begin{pmatrix} \chi_a \\ \bar{\eta}^{\dot{a}} \end{pmatrix}. \tag{C.31}$$

The Dirac equation is decomposed to,

$$k_\mu \bar{\sigma}^{\mu\dot{a}a}\chi_a = -m\bar{\eta}^{\dot{a}}, \tag{C.32}$$

$$k_\mu \sigma^\mu_{a\dot{a}}\bar{\eta}^{\dot{a}} = -m\chi_a. \tag{C.33}$$

Making the same decomposition of the momentum $k^\mu = p^\mu + q^\mu$, we can obtain the wavefunction of the massive spin-$\frac{1}{2}$ fermion [60],

$$u_+(k) = \begin{pmatrix} \frac{\langle qp \rangle}{m} q_a \\ p^{*\dot{a}} \end{pmatrix}, \tag{C.34}$$

$$u_-(k) = \begin{pmatrix} p_a \\ \frac{[qp]}{m} q^{*\dot{a}} \end{pmatrix}. \tag{C.35}$$

C.4 Massive spin $3/2$ fermions

A massive spin-$\frac{3}{2}$ fermion are described by a Rarita-Schwinger spinor-vector $\Psi^{A,\mu}$ which satisfies equations,

$$(i\not{\partial} - m)^A{}_B \Psi^{B,\mu} = 0, \tag{C.36}$$

$$(\gamma_\mu)^A{}_B \Psi^{B,\mu} = 0, \tag{C.37}$$

$$\partial_\mu \Psi^{B,\mu} = 0, \tag{C.38}$$

where A and B are spinor indices. Again we only consider the positive energy solution U, it satisfies,

$$(\not{k} + m)^A{}_B U(k)^{B,\mu} = 0, \tag{C.39}$$

$$\bar{U}_{A,\mu}(k,\lambda) U^{A,\mu}(k,\lambda^{'}) = 2m\delta_{\lambda\lambda'}. \tag{C.40}$$

The wavefunction of U can be written as [60],

$$U^{A,\mu}(+\frac{3}{2}) = \frac{1}{\sqrt{2}m}\begin{pmatrix} \frac{\langle qp\rangle}{m} q_a \\ p^{*\dot{a}} \end{pmatrix} (p^*_{\dot{b}} \bar{\sigma}^{\mu\dot{b}b} q_b) \ , \tag{C.41}$$

$$U^{A,\mu}(+\frac{1}{2}) = \frac{\bar{\sigma}^{\mu\dot{b}b}}{\sqrt{6}m}\left[\begin{pmatrix} \frac{\langle qp\rangle}{m} q_a \\ p^{*\dot{a}} \end{pmatrix}(p^*_{\dot{b}} p_b - q^*_{\dot{b}} q_b) + \begin{pmatrix} \frac{\langle qp\rangle}{m} p_a \\ -q^{*\dot{a}} \end{pmatrix}(p^*_{\dot{b}} q_b)\right], \tag{C.42}$$

$$U^{A,\mu}(-\frac{1}{2}) = \frac{\bar{\sigma}^{\mu\dot{b}b}}{\sqrt{6}m}\left[\begin{pmatrix} p_a \\ \frac{[qp]}{m} q^{*\dot{a}} \end{pmatrix}(p^*_{\dot{b}} p_b - q^*_{\dot{b}} q_b) + \begin{pmatrix} -q_a \\ \frac{[qp]}{m} p^{*\dot{a}} \end{pmatrix}(q^*_{\dot{b}} p_b)\right], \tag{C.43}$$

$$U^{A,\mu}(-\frac{3}{2}) = \frac{1}{\sqrt{2}m}\begin{pmatrix} p_a \\ \frac{[qp]}{m} q^{*\dot{a}} \end{pmatrix} (q^*_{\dot{b}} \bar{\sigma}^{\mu\dot{b}b} p_b) \ . \tag{C.44}$$

D Wigner d-matrix

The Wigner D-matrix (a.k.a. Wigner rotation matrix), introduced in 1927 by Eugene Wigner, is a dimension $2j+1$ square matrix, which is in an irreducible representation of groups $SU(2)$ and $SO(3)$. The matrix is defined to be:

$$D^{(j)}_{m',m}(\alpha,\beta,\gamma) = \langle jm'|R(\alpha,\beta,\gamma)|jm\rangle = e^{-im'\alpha} d^{(j)}_{m',m}(\beta) e^{-im\gamma}, \tag{D.1}$$

where α, β, γ are Euler angles, and $d^j_{m',m}(\beta)$, known as Wigner reduced (or small) $d-$matrix, is given by a general formula [77, 78]:

$$d^{(j)}_{m',m}(\beta) = \sqrt{\frac{(j+m')!(j-m')!}{(j+m)!(j-m)!}} \sum_s (-)^{j-m'-s} \\ \times \begin{pmatrix} j+m \\ j-m'-s \end{pmatrix} \begin{pmatrix} j-m \\ s \end{pmatrix} \left(\cos\frac{\beta}{2}\right)^{m'+m+2s} \left(\sin\frac{\beta}{2}\right)^{2j-m'-m-2s}. \tag{D.2}$$

The sum over s is over such values that the factorials are non negative. Two important relations follow from the above expression:

$$d^{(l)}_{0,0}(\theta) = P_l(\cos\theta), \tag{D.3}$$

where $P_l(cos\theta)$ is the Legendre polynomial, and

$$d^{(j)}_{m',m}(\theta) = (-1)^{j-m} d^{(j)}_{m',-m}(\theta+\pi). \tag{D.4}$$

For $j \leq 4$, the following Wigner d-matrices appear in the factorized four-gluon amplitudes:

- $d^{(j)}_{0,0}(\theta)$

$$d^{(0)}_{0,0}(\theta) = P_0(\cos\theta) = 1, \tag{D.5}$$

$$d^{(1)}_{0,0}(\theta) = P_1(\cos\theta) = \cos\theta, \tag{D.6}$$

$$d^{(2)}_{0,0}(\theta) = P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1), \tag{D.7}$$

$$d^{(3)}_{0,0}(\theta) = P_3(\cos\theta) = \frac{1}{2}(5\cos^3\theta - 3\cos\theta), \tag{D.8}$$

$$d^{(4)}_{0,0}(\theta) = P_4(\cos\theta) = \frac{1}{8}(35\cos^4\theta - 30\cos^2\theta + 3). \tag{D.9}$$

- $d^{(j)}_{2,\pm 2}(\theta)$

$$d^{(2)}_{2,\pm 2}(\theta) = \left(\frac{1 \pm \cos\theta}{2}\right)^2, \tag{D.10}$$

$$d^{(3)}_{2,\pm 2}(\theta) = \frac{1}{4}(3\cos^3\theta \pm 4\cos^2\theta - \cos\theta \mp 2), \tag{D.11}$$

$$d^{(4)}_{2,\pm 2}(\theta) = \frac{1}{4}(7\cos^4\theta \pm 7\cos^3\theta - 6\cos^2\theta \mp 5\cos\theta + 1). \tag{D.12}$$

- $d^{(j)}_{2,\pm 1}(\theta)$

$$d^{(2)}_{2,\pm 1}(\theta) = \frac{1}{2}\sin\theta(1 \pm \cos\theta), \tag{D.13}$$

$$d^{(3)}_{2,\pm 1}(\theta) = \frac{\sqrt{10}}{8}\sin\theta(\pm 3\cos^2\theta + 2\cos\theta \mp 1), \tag{D.14}$$

$$d^{(4)}_{2,\pm 1}(\theta) = \frac{\sqrt{2}}{8}\sin\theta(\pm 14\cos^3\theta + 7\cos^2\theta \mp 8\cos\theta - 1). \tag{D.15}$$

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