

CONNECTION OF ELASTIC ELECTROMAGNETIC NUCLEON FORM FACTORS AT LARGE
 Q^2 AND DEEP INELASTIC STRUCTURE FUNCTIONS NEAR THRESHOLD*

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It is suggested that if the structure function νW_2 for deep inelastic electron-proton scattering behaves near threshold as

$$\nu W_2 \sim \frac{1}{\omega} \left(1 - \frac{1}{\omega}\right)^p \text{ for } \omega \equiv \frac{2M\nu}{Q^2} \rightarrow 1$$

then the elastic electromagnetic form factor of the proton, F_1 , behaves for large momentum transfers as

$$F_1(Q^2) \sim \left(\frac{1}{Q^2}\right)^{\frac{p+1}{2}} \text{ for } Q^2 \rightarrow \infty.$$

Recent data on inelastic electron scattering show that the structure functions of the proton depend only on one variable $\omega = 2M\nu/Q^2$, i. e. on the ratio of the energy transfer to the proton, $\nu \equiv P \cdot q/M > 0$, to the invariant momentum transfer $Q^2 \equiv -q^2 > 0$ in the region of large Q^2 and $M\nu \gg M^2$, with $\omega \equiv 2M\nu/Q^2$ finite. This is true in particular for the structure function νW_2 which has been studied extensively at SLAC¹ over a broad range of energy and momentum transfers in this kinematic region referred to as the Bjorken limit. This so-called scaling behavior of the structure function νW_2 supports Bjorken's prediction.²

A natural interpretation of this scaling behavior can be found in a picture of the proton as made up out of constituents - called "partons" by Feynman - that are instantaneously free during the sudden impulse bearing a high frequency ν from the scattered electron in the Bjorken limit. The associated physical picture is that the ω dependence of νW_2 probes the longitudinal momentum distribution of the charged partons as viewed in an infinite momentum frame of the initial proton³; specifically

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$\nu W_2 \propto \frac{1}{\omega} \times \{\text{Probability that a parton scattering the electron has a fraction } \eta = \frac{1}{\omega} \text{ of the proton's momentum } \underline{P} \text{ in the } \underline{P} \rightarrow \infty \text{ coordinate frame}\}.$

In this letter we will explore what can be inferred about the elastic electromagnetic nucleon form factors, particularly for large Q^2 , from the parton model and its apparent successes with νW_2 . In particular, we will suggest a connection between the behavior of νW_2 near $\omega \sim 1$ and the rate of decrease of the elastic form factors for $Q^2 \rightarrow \infty$. Our work is based on the canonical field theoretic formalism developed earlier⁴ for deriving the parton model and the Bjorken limiting behavior from any reasonable - i. e. renormalizable in the usual sense - canonical field theory of strong interactions. A basic ingredient in this derivation of the parton model was the assumption that there exists an asymptotic region in which Q^2 can be made greater than the components of momenta transverse to the direction of \underline{P} of all particles involved - i. e. of the constituents of the proton.

To develop this approach and identify the partons we introduce the familiar unitary U matrix which undresses the Heisenberg fields and currents $U(t) \equiv \left(-i \int_{-\infty}^t d\tau H_I(\tau) \right)_+^e$ where $H_I(\tau)$ is the interaction Hamiltonian of the hadrons, so that for example $J_\mu(x) = U^{-1}(t) j_\mu(x) U(t)$ where $J_\mu(x)$ and $j_\mu(x)$ are the hadronic electromagnetic current operators in the Heisenberg and interaction pictures respectively.

Then if $|P\rangle$ denotes the one proton eigenstate with momentum P, we have

$$\begin{aligned}
 |UP\rangle &= \sqrt{Z_2} \left\{ |P\rangle + \sum_n \frac{\langle n | H_I | P \rangle}{E_P - E_n} + \sum_{m, n} \frac{m \langle m | H_I | n \rangle \langle n | H_I | P \rangle}{(E_P - E_n)(E_P - E_m)} + \dots \right\} \\
 &\equiv \sqrt{Z_2} |P\rangle + \sum_{n=1}^{\infty} \int \prod_{i=1}^{n+1} \frac{d^3 k_i}{\sqrt{P^n}} \delta^3(\sum_{i=1}^{n+1} \underline{k}_i - \underline{P}) f_P(\underline{k}_1, \underline{k}_2, \dots, \underline{k}_{n+1}) |k_1 k_2 \dots k_{n+1}\rangle
 \end{aligned} \tag{1}$$

where Σ' denotes a sum over all states $|m\rangle$ other than $|P\rangle$; Z_2 is the standard wave function renormalization constant of the proton state as required to insure $\langle P' | P \rangle = \langle UP' | UP \rangle = \delta^3(\underline{P}' - \underline{P})$. The second form expresses the expansion in terms of a sum over numbers of constituents n (the "physical" pions, nucleons, and anti-nucleons in a conventional

pion-nucleon field theory; indices for other quantum numbers are suppressed). These are the partons. The probabilities for different numbers, charges, momenta, etc. are specified by the matrix elements in (1). In particular we have seen that we must set $Z_2 = 0$ so that the elastic form factor vanishes as $Q^2 \rightarrow \infty$; hence⁵ the single physical proton state is absent from $UP\rangle$.

For computing the inelastic and elastic structure functions we choose as a convenient infinite momentum frame for the proton

$$P^\mu = \left(P + \frac{M^2}{2P}, 0, 0, P \right); \quad q^\mu = \left(\frac{M\nu}{P}, q_\perp, 0 \right) \quad (2)$$

with

$$|q_\perp|^2 = Q^2 + 0(1/P^2)$$

In this frame the longitudinal and transverse momenta of the constituents in the states

$|k_1 \dots k_{n+1}\rangle$ in (1) are defined by

$$k_i = \eta_i P + k_{i\perp}; \quad k_{i\perp} \cdot P = 0 \quad (3)$$

The momentum conserving delta function fixes

$$\eta_{n+1} = 1 - \sum_{i=1}^n \eta_i; \quad k_{n+1\perp} = - \sum_{i=1}^n k_{i\perp} \quad (4)$$

As established in the analysis above Eq. (78) in Paper II, the structure functions W_1 and νW_2 in the Bjorken limit can be written as a sum of contributions from each term f_P in (1) of the form

$$(\nu W_2)_n^a = \lambda_a^2 \frac{1}{\omega} \int \prod_{i=1}^n d^2 k_i d\eta_i \theta \left(1 - \sum_i \eta_i \right) \delta \left(\eta_a - \frac{1}{\omega} \right) |f_P(\dots \eta_i \dots k_{i\perp})|^2 \quad (5)$$

where all longitudinal momenta are along the P direction - i. e. $0 < \eta_i < 1$; λ_a is the charge on the a^{th} constituent (viz. π^\pm, P, \bar{P}) in the particular state f_P , and the spin average over constituents' states is assumed in writing (5). In particular we note that the behavior of f_P when $\eta_a = \frac{1}{\omega} \rightarrow 1$ and all other η_i are within $(1 - \frac{1}{\omega})$ of zero determines the threshold behavior of $(\nu W_2)_n^a$ near $\omega = 1$. Recall that in approaching the threshold we must still satisfy the inequality $|Q^2(\omega-1)| \gg M^2$ in order to stay in the Bjorken limiting region as required⁴.

For the elastic form factor of the proton we write

$$\langle P' | J_\mu | P \rangle = \langle UP' | j_\mu | UP \rangle$$

In order to compute the two scalar form factors F_1 and F_2 or G_E and G_M as customarily defined we need only work with the two good current components $\mu = 0$ or 3 . Then according to the discussion in Paper II (see especially Eqs. (10) and following) to leading order in $P \rightarrow \infty$ all constituent particles in f_P will be moving along the direction \underline{P} (and along \underline{P}' in $f_{P'}$) and the operator j_0 or j_3 will simply scatter one of the charged constituents changing the magnitude but not the sign of its momentum projection along \underline{P} or \underline{P}' . Furthermore we can separate the two form factors according to their spin dependence. In terms of the Pauli two-component spinors χ' and χ and in the $P \rightarrow \infty$ frame (2)

$$\langle UP' | j^\mu | UP \rangle = \frac{1}{(2\pi)^3} \chi'^* \left[F_1(q^2) - \frac{\sigma_{3a} \underline{q}_a}{2M} F_2(q^2) \right] \chi \quad \mu = 0 \text{ or } 3$$

Taking the spin average as in (5) for νW_2 we obtain

$$F_1(q^2) = (2\pi)^3 \langle UP' | j^\mu | UP \rangle \quad \mu = 0 \text{ or } 3 \quad (6)$$

Introducing the expansion (1) gives then

$$F_1(q^2) = \sum_{n=1}^{\infty} \int \prod_{i=1}^n d^2 k_{i\perp} d\eta_i \theta(1 - \sum_{i=1}^n \eta_i) \sum_a \lambda_a \times \int_{P'}^* (\eta_1 \dots \eta_n; \underline{k}_{1\perp} - \eta_1 \underline{q}, \dots, \underline{k}_{a\perp} + (1 - \eta_a) \underline{q}, \dots, \underline{k}_{n\perp} - \eta_n \underline{q}) f_P(\eta_1 \dots \eta_n; \underline{k}_{1\perp}, \dots, \underline{k}_{n\perp}) \quad (7)$$

Each η_i in the initial is the same as in the final wave function because no longitudinal momentum is introduced by \underline{q} according to (2) and the rotation from the direction \underline{P} to $\underline{P}' = \underline{P} + \underline{q}$ alters the longitudinal projection of η_a only by corrective terms $\sim 1/P^2$ which we consistently neglect. This displacement of the transverse projections by $-\eta_i \underline{q}$ for each i is just an expression of this very rotation: momentum $\underline{k}_{i\perp}$ transverse to \underline{P} is identical to order $1/P$ with $\underline{k}_{i\perp} - \eta_i \underline{q}$ as reckoned relative to \underline{P}' . Only the constituent a has its momentum altered by \underline{q} as a result of the scattering by the current.

To determine the asymptotic behavior of $F_1(q^2)$ we must consider the various possible ranges of η_i that contribute to the overlap integral (7).

- (i) If $(1 - \eta_a)$ does not take an extreme value within $1/q$ of $1 - \eta_a = 0$, i. e. $1 - \eta_a$

lies in the region $0 < c < 1 - \eta_a < 1$, then $(1 - \eta_a) \underline{q}$ increases with q as $q \rightarrow \infty$. The energy denominator associated with the scattered final state is given by

$$\frac{1}{2P} \left\{ \frac{[\underline{k}_{a\perp} + (1 - \eta_a) \underline{q}]^2 + m_a^2}{\eta_a} + \sum_{i \neq a} \frac{[\underline{k}_{i\perp} - \eta_i \underline{q}]^2 + m_i^2}{\eta_i} \right\}. \quad (8)$$

Thus there is at least one energy denominator of order Q^2/P since a heavy state of (mass)² of order Q^2 is formed from an interaction creating a large transverse momentum squared proportional to Q^2 . In addition, due to the momentum mismatch between f_P and $f_{P'}$, at least one vertex matrix element in (1) will be suppressed by a transverse momentum cutoff $g(Q^2)$. In this case therefore we have⁶

$$F_1(q^2) \lesssim \frac{1}{Q^2} g(Q^2) ; g(Q^2) \rightarrow 0 \text{ as } Q^2 \rightarrow \infty. \quad (9)$$

i. e. $F_1(q^2)$ decreases more rapidly than $1/Q^2$. To say more than this we require detailed models of the cutoff. However, any association of the fall-off of $g(Q^2)$ with the observed transverse momentum distribution from high energy collision data⁷ will generally predict a too rapid decrease of $F_1(q^2)$ in (9). Furthermore, a variety of

specific calculations in this region of parameters leads to a q independent ratio of $F_2(q^2)/F_1(q^2)$ and thus to a ratio $G_M(q^2)/G_E(q^2) \equiv \frac{F_1 + \kappa F_2}{F_1 + \frac{q}{4M^2} \kappa F_2} \propto \frac{1}{Q^2}$ in defiance

of the "desired" scaling law for the elastic form factors. A few examples of these calculations are illustrated in Figure 1. All these indications suggest to us that the contribution of primary importance does not come from this region.

(ii) Suppose then that the more important region is $0 < 1 - \eta_a \lesssim m/q$ where m is some characteristic mass, so that $\underline{k}_{a\perp} + (1 - \eta_a) \underline{q}$ in (7) remains bounded as q increases. According to (4), all the other $\underline{k}_{i\perp} - \eta_i \underline{q}$ with $i \neq a$ are also bounded. For all i we write

$$\begin{aligned} \underline{k}_{i\perp} - \eta_i \underline{q} &\rightarrow \underline{k}'_{i\perp} = \underline{k}_{i\perp} + \bar{\underline{k}}_{i\perp} \\ \underline{k}_{a\perp} + (1 - \eta_a) \underline{q} &\rightarrow \underline{k}'_{a\perp} = \underline{k}_{a\perp} + \bar{\underline{k}}_{a\perp} \end{aligned}$$

Introducing this notation into (7) we see that $F_1(q^2)$ becomes a series of overlap integrals in each of which the transverse momenta are displaced by a bounded, finite amount $\bar{\underline{k}}_{i\perp}$

In this case the longitudinal (normalized) momenta η_i are confined in a similar manner as discussed below (5): All but one η_i are within $1/q$ of zero whereas in (5) they are within $\left(1 - \frac{1}{\omega}\right)$ of zero in the inelastic threshold region; and for η_a , $1 - \eta_a \lesssim 1/q$ here and $\sim \left(1 - \frac{1}{\omega}\right)$ in (5). Since the transverse momentum overlap integrals will be generally finite and q independent, with numerical upper bounds according to a simple application of the Schwartz inequality⁸, we look to the η integrals for the functional dependence on q . Here we see that (5) and (7) differ only by the appearance of the $\delta(\eta_a - 1/\omega)$ in (5) which removes one of the $d\eta$ integrals and thereby avoids one additional factor $\sim \left(1 - \frac{1}{\omega}\right)$. Thus we conclude that the leading contribution of this region in (7) can be written

$$F_1(q^2) \sim (1/q)^{p+1} \sim \left(\frac{1}{Q^2}\right)^{\frac{p+1}{2}} \text{ as } q \rightarrow \infty \quad (10)$$

if the leading term contributing to the inelastic scattering in (5) varies as

$$\nu W_2 \sim \frac{1}{\omega} \left(1 - \frac{1}{\omega}\right)^p \text{ as } \frac{1}{\omega} \rightarrow 1. \quad (11)$$

The diagrams in Figure 1 are examples dominated by this region of parameters. These examples also lead to a decreasing ratio for $F_2(q^2)/F_1(q^2)$ as q increases. Therefore it remains a possibility that the so-called scaling law for the elastic form factors is valid if indeed this is the dominant region of contribution⁹.

(iii) Finally we must consider the region in between (i) and (ii) - i. e. the region $m/q \leq \eta < c < 1$. Generally we expect that this region can be ignored by choosing a sufficiently small value of c if region (i) dominates, and by a proper choice of m if region (ii) dominates. Beyond this, we have not been able to derive any general statements. To proceed further we resort to "empirical mathematics" - i. e. specific calculations of types of diagrams in Figure 1 and others. All these show that this region never dominates and can always be incorporated in the manner described above. In fact, the overlap integral (7) decreases as one increases the range of the η integration beyond the limit of region (ii) and toward region (i). This results essentially from the growing energy denominators (8) since the "masses" increase with η .

On the basis of the above discussion we infer - i. e. we conjecture - that the

connection described by (10) and (11) is generally valid. Their physical connection is that near threshold νW_2 measures the probability that all but a fraction $\sim \left(1 - \frac{1}{\omega}\right)$ of the proton's momentum is concentrated on one charged parton in the $\underline{P} \rightarrow \infty$ frame as indicated in (5). Similarly the dominant contribution to $F_1(q^2)$ for asymptotically large q measures the probability that all but a fraction $\sim 1/q$ of the proton's momentum is concentrated on one charged parton. In this case the other partons emitted before the scattering by the virtual photon can rejoin with the scattered one without introducing a large transverse momentum mismatch $\sim q$ at the vertex, as occurred in (9). The probability that $UP >$ dissociates into only the physical proton $P >$ - i. e. into one parton - has been set to zero by choosing $Z_2 = 0$ as required⁵ in order to insure that $F(q^2)$ vanishes as $q \rightarrow \infty$. This has often been discussed¹⁰ in the literature as the bootstrap or composite particle condition. In our present application it is interesting to note that the two requirements that both the nucleon and the pion wave function renormalization constants vanish so that their electromagnetic form factors will do likewise as $q \rightarrow \infty$ present two constraints on the two parameters in the calculation, the pion nucleon coupling constant $g^2/4\pi$, which nominally ≈ 15 , and the cutoff momentum $k_{\perp \max}$, which is characteristically ≈ 400 MeV as observed in high energy secondary particle production events. Although lowest order perturbation calculations are notoriously dangerous frameworks on which to base speculations it is intriguing to note that to order $g^2/4\pi$ the conditions¹¹ $Z_2 = Z_3 = 0$ fix the values $g^2/4\pi = 17$ and $k_{\perp \max}^2 = 0.2 \text{ GeV}^2$.

How well this connection in (10) and (11) can be tested experimentally is not certain at present. The elastic form factors, assuming that they have already reached their asymptotic behavior by $q^2 \sim 25 \text{ GeV}^2$, come close¹² to $p + 1 = 4$ in (10). However should the data lie just on the verge of becoming asymptotic it is also possible that¹³ $p + 1 = 6$. The curvature of νW_2 near $\omega = 1$, extrapolated from points with $|Q^2(\omega-1)| \gg 1$, is just beginning to be determined.¹⁴ On the basis of our earlier analysis we suggested¹⁵ that interactions with the part of electromagnetic current due to boson currents should dominate over that part due to charged fermions near the threshold region. If this is true we would expect p to be an even power.

At this time the problem of determining p in (11) by comparison with experiment is the following. Since $Q^2 \lesssim 10 \text{ GeV}^2$ is a restriction on existing data¹ and $Q^2(\omega - 1) \gg 1$ is a requirement for our theoretical model we must consider a range of values $1.2 \lesssim \omega \lesssim 1.5$. Thus our resulting numerical fit is greatly affected depending on whether we write $\nu W_2 \sim (\omega - 1)^p$ which is its limiting threshold form, or $\nu W_2 \sim \frac{1}{\omega} \left(1 - \frac{1}{\omega}\right)^p$ which is the natural form emerging from (5). Clearly we can make no quantitative statement when the difference between these forms controls the fit. As written Eq. (11) is consistent with present data if we fix $p = 3$ from Eq. (10). Experiments at higher Q^2 and smaller $(\omega - 1)$ values, both for the deep inelastic scattering and annihilation processes, will be required before the two forms (10) and (11) become strong mutual constraints on the theory.

According to our model an odd integral value for p , such as $p = 3$, is necessary if the nucleon current (or generally a spin 1/2 current) contribution is dominant. If this is the case it also follows that the ratio of longitudinal to transverse cross sections is small - i. e. $\frac{MW_1}{\nu W_2} \rightarrow \frac{\omega}{2}$, or in the notation of Ref. 1, $R \rightarrow 0$. The present data are¹ consistent with $R \lesssim 0.2$ near threshold indicating that this and not even integral p is the case.

This region near threshold is of considerable interest not only for testing the connection given by (10) and (11). The field theoretical formalism on which the present discussion is based shows that this is also the region in which the constituents are far off their mass shells - i. e. they are very virtual. It is here then that one is indeed probing very small space-time intervals by the study of deep inelastic scattering.

FOOTNOTES AND REFERENCES

1. E. Bloom et al, Phys. Rev. Letters, 23, 930 (1969). M. Breidenbach et al, *ibid*, 23, 935, (1969). R. Taylor, Invited talk at Daresbury Conference, SLAC-PUB-677 (1969).
2. J. D. Bjorken, Phys., 179, 1547 (1969).
3. R. P. Feynman, unpublished. J. D. Bjorken, invited paper at the American Physical Society New York Meeting, Feb. 3, 1969. SLAC-PUB-571 (1969).
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4. S. D. Drell, D. J. Levy, T. M. Yan, Phys. Rev. Letters. 22, 744 (1969); and SLAC-PUB-606, 645 (1969) (to be published). The last two papers will be referred to as Paper I, and II, respectively. The motivation for constructing this field theory framework was to provide the machinery for accomplishing crossing to the annihilation channel for study of the Bjorken limiting behavior in the reaction $e^- + e^+ \rightarrow p + \text{anything}$. The same formalism has been used to compute inelastic neutrino cross sections and derive correlations in the final states when two particles are detected. For more details of these applications see S. D. Drell, D. J. Levy, T. M. Yan, SLAC-PUB-685 (1969), T. M. Yan, S. D. Drell, SLAC-PUB-692 (1969), and S. D. Drell, T. M. Yan, to be published. In conformity with the standard notation in Ref. 1 we henceforth designate by ω the variable called w in these references.
5. See discussion in the last section of Paper I cited in Ref. 4.
6. We assume that the effective cutoffs for the transverse momentum integration permit the limit $Q^2 \rightarrow \infty$ for the energy denominator (8) to be taken inside the integrand. The resulting integration for any η_i with $i \neq a$ can diverge no more strongly than logarithmically near the end point $\eta_i \approx 0$, since otherwise the original integral will be infinite in violation of the physical requirement that the form factors are finite. The same conclusion can also be arrived at by counting powers of η appearing in the vertices and energy denominators from specific field theoretic models such as the pseudoscalar or scalar coupling for spinless meson, spin $-\frac{1}{2}$ nucleon systems. Due to the possible logarithmic divergences similar to the one just mentioned, our conclusion about the asymptotic behavior of the form factors is valid only up to logarithmic factors in Q^2 .
7. J. L. Day et al, Phys. Rev. Letters, 23, 1055 (1969).
D. B. Smith et al, Phys. Rev. Letters, 23, 1064 (1969).
8. Application of the Schwartz inequality gives

$$\lambda_a \int \prod_{i=1}^n d^2 k_{i\perp} \prod_{i \neq a}^n d\eta_i \theta(1 - \sum_{i=1}^n \eta_i) f_{P'}^* f_P \leq |\lambda_a| \left[\int \prod_{i=1}^n d^2 k_{i\perp} \prod_{i \neq a}^n d\eta_i \theta(1 - \sum_{i=1}^n \eta_i) |f_P|^2 \right]^{1/2}$$

$$\left[\int \prod_{i=1}^n d^2 k'_{i\perp} \prod_{i \neq a}^n d\eta_i \theta(1 - \sum_{i=1}^n \eta_i) |f_{P'}|^2 \right]^{1/2} = \frac{1}{|\lambda_a|} \left[\omega (\nu W_2)^a \right] \omega = \frac{1}{\eta_a}$$

9. We should point out that examples in Figure 1 neither give the observed q dependence of F_1 nor necessarily the correct ratio of F_2/F_1 which should be proportional to $1/q^2$ if the scaling law is to hold. Nor do the corresponding diagrams for νW_2 predict the correct threshold behavior; see the discussion in Paper I. We only use the general qualitative features of these field theoretic models in order to correlate νW_2 and F_1 .
10. M. Gell-Mann and F. Zachariasen, Phys. Rev. 123, 1065 (1961). See also the discussion and references cited by S. Weinberg in Brandeis Summer School Lectures, 1964.
11. See Eqs. (18) plus (19) and (20) of Paper II. A somewhat different use of these conditions was discussed by A. Salam in Proceedings of 1962 International Conference on High Energy Physics at CERN, p. 686.
12. See W. K. H. Panofsky, International Conference on High Energy Physics, Vienna, 1968, p. 27.
13. A. Zichichi, International Conference on High Energy Physics, Vienna, 1968, p. 87.
14. This is a less specific statement than given in Paper I below Eq. (30) and is made in the light of all the data now available. We thank Drs. E. Bloom and R. Taylor of SLAC for discussions of the data in its present state.
15. See the discussion below Eq. (30) in Paper I.

FIGURE CAPTION

Fig. 1 - Typical graphs contributing to elastic form factors as computed for $Q^2 \rightarrow \infty$ with γ_5 coupling of pions (dashed lines) to nucleons (solid lines)

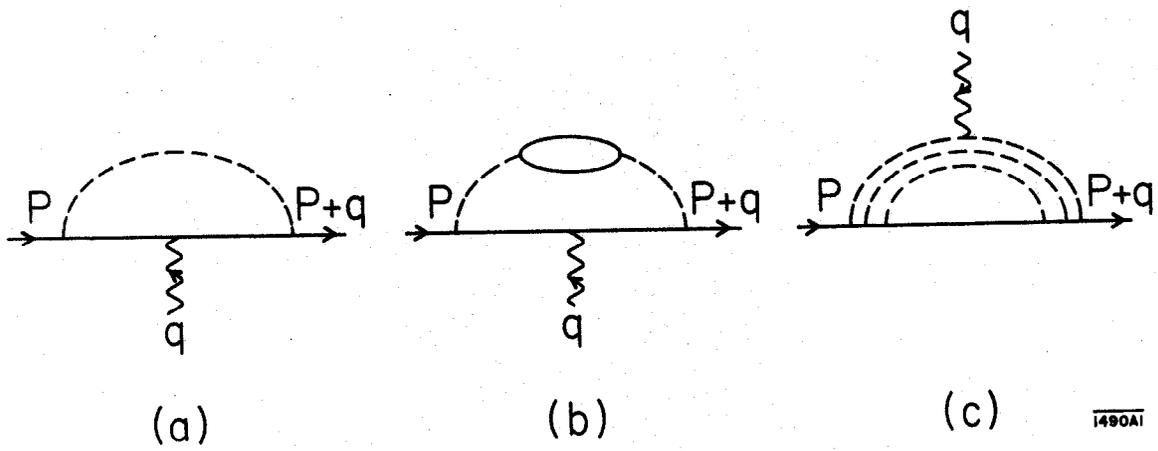


Fig. 1