Higher-Spin Gauge Fields and Duality

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Abstract. We review the construction of free gauge theories for gauge fields in arbitrary representations of the Lorentz group in D dimensions. We describe the multi-form calculus which gives the natural geometric framework for these theories. We also discuss duality transformations that give different field theory representations of the same physical degrees of freedom, and discuss the example of gravity in D dimensions and its dual realisations in detail.

Based on the lecture presented by C.M. Hull at the First Solvay Workshop on Higher-Spin Gauge Theories, held in Brussels on May 12-14, 2004

1 Introduction

Tensor fields in exotic higher-spin representations of the Lorentz group arise as massive modes in string theory, and limits in which such fields might become massless are of particular interest. In such cases, these would have to become higher-spin gauge fields with appropriate gauge invariance. Such exotic gauge fields can also arise as dual representations of more familiar gauge theories [1], [2]. The purpose here is to review the formulation of such exotic gauge theories that was developed in collaboration with Paul de Medeiros in [3], [4].

Free massless particles in D-dimensional Minkowski space are classified by representations of the little group $SO(D-2)$. A bosonic particle is associated with a tensor field $A_{ij\ldots k}$ in some irreducible tensor representation of $SO(D-2)$ and in physical gauge (i.e. in light-cone gauge) the particle is described by a field $A_{ij\ldots k}$ depending on all D coordinates of Minkowski space and satisfying a free wave equation

$$\Box A = 0 \ . \tag{1}$$

For $D = 4$, the bosonic representations of the little group $SO(2)$ are classified by an integer, the spin s, while for $D > 4$ the representation theory is more involved, although it is common to still refer to generic tensors as being of 'higher spin'.

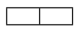
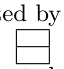
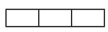
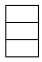
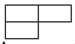
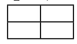
The main topic to be considered here is the construction of the Lorentz-covariant gauge theory corresponding to these free physical-gauge theories. The first step is finding the appropriate covariant tensor gauge field. For example, an n'th rank antisymmetric tensor physical-gauge field $A_{i_1\ldots i_n} = A_{[i_1\ldots i_n]}$ (where $i,j = 1,\ldots,D-2$) arises from a covariant n'th rank antisymmetric tensor gauge field $A_{\mu_1\ldots\mu_n} = A_{[\mu_1\ldots\mu_n]}$ (where $\mu,\nu = 0,1,\ldots,D-1$) with gauge symmetry $\delta A = d\lambda$, while a graviton represented by a traceless symmetric tensor $h_{ij} = h_{ji}$ with $h_i{}^i = 0$ arises from a covariant tensor gauge field $h_{\mu\nu}$ which is symmetric but not traceless, with the usual gauge transformations corresponding to linearised diffeomorphisms. The general rule is to replace an irreducible tensor representation of $SO(D-2)$, given by some tensor field $A_{ij\ldots k}$ with suitable trace-free constraints, by the corresponding tensor field $A_{\mu\nu\ldots\rho}$ with the same symmetry properties as $A_{ij\ldots k}$, but with no constraints on the traces, so that it can be viewed as a tensor representation of $GL(D,\mathbb{R})$. There are some subtleties in this step which we shall return to shortly. The covariant gauge field must transform under gauge symmetries that are sufficient to remove all negative-norm states and to allow the recovery of the physical-gauge theory on gauge fixing.

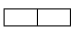
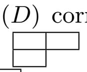
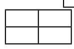
The next step is the construction of a gauge-invariant field equation and action. For antisymmetric tensors or gravitons, this is straightforward, but for generic higher spin representations the situation is more complicated. One of the simplest cases is that of totally symmetric tensor gauge fields $A_{\mu_1\ldots\mu_n} = A_{(\mu_1\ldots\mu_n)}$. For these, covariant field equations were found by Fronsdal in [5] and reformulated in a geometric language by de Wit and Freedman in [6], but these suffered from the drawback that the gauge fields were constrained, corresponding to a partial fixing of the gauge invariance. This was generalised to arbitrary representations by Siegel and Zwiebach [7], and the duality properties analysed. Covariant field equations and actions have very recently been constructed for totally symmetric tensor gauge fields by Francia and Sagnotti [8], [9] (for a review see the contribution to these proceedings [10]). These have an elegant geometrical structure, being constructed in terms of covariant field strengths, but have the surprising feature of being non-local in general. Nonetheless, on partially fixing the gauge invariance the non-locality is eliminated and the field equations of [5], [6] are recovered. It appears that this non-locality is inescapable in the covariant formulation of higher-spin gauge theories, and it would be interesting to understand whether this has any physical consequences.

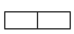
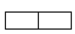
Recently, this has been generalised to general higher spin gauge fields in any tensor representation [3], [4], [11], [12]. The formulation of [3], [4] uses an elegant mathematical structure, the multiform calculus, developed in [3], [4] and in [13], [14], [15]. It is the approach of [3], [4] which will be reviewed here. The theory is formulated in terms of covariant field strengths or curvatures, and is non-local but reduces to a local theory on gauge-fixing.

In general, it turns out that a given particle theory corresponding to a particular irreducible tensor representation of $SO(D-2)$ can arise from a number of different covariant field theories, and these covariant field theories are said to give dual realisations of the same theory [1], [2]. For example, consider an n-form representation of $SO(D-2)$ with field $A_{i_1 \dots i_n}$. This is equivalent to the $\tilde{n}$-form representation, where $\tilde{n} = D - 2 - n$ and so the theory could instead be represented in terms of an $\tilde{n}$-form field $\tilde{A}_{i_1 \dots i_{\tilde{n}}} \equiv \frac{1}{n!} \epsilon_{i_1 \dots i_{\tilde{n}} j_1 \dots j_n} A^{j_1 \dots j_n}$. One can then construct a covariant gauge theory based on an n-form gauge field $A_{\mu_1 \dots \mu_n}$ or a $\tilde{n}$-form gauge field $\tilde{A}_{\mu_1 \dots \mu_{\tilde{n}}}$. These are physically equivalent classically, as they both give equivalent theories in physical gauge. The key feature here is that n-form and $\tilde{n}$-form representations are equivalent for $SO(D-2)$ but distinct for $GL(D, \mathbb{R})$. For the general case, there are a number of distinct representations of $GL(D, \mathbb{R})$ that give rise to equivalent representations of $SO(D-2)$ and so lead to dual formulations of the same physical degrees of freedom. Such dualities [1], [2] can be considered in multi-form gauge theories and in general interchange field equations and Bianchi identities and will also be briefly reviewed here.

2 Young Tableaux

Representations of $GL(D, \mathbb{R})$ can be represented by Young tableaux, with each index μ of a tensor $T_{\mu\nu \dots \rho}$ corresponding to a box in the diagram; see [16] for a full discussion. Symmetrized indices are represented by boxes arranged in a row, so that e.g. a 2nd rank symmetric tensor $h_{\mu\nu}$ is represented by  , while anti-symmetrized indices are represented by boxes arranged in a column, so that e.g. a 2nd rank anti-symmetric tensor $B_{\mu\nu}$ is represented by  . A general 3rd rank tensor $E_{\mu\nu\rho}$ can be decomposed into a totally symmetric piece $E_{(\mu\nu\rho)}$ represented by the tableau  , a totally anti-symmetric piece $E_{[\mu\nu\rho]}$ represented by the tableau  , and the remaining piece $D_{\mu\nu\rho} \equiv E_{\mu\nu\rho} - E_{(\mu\nu\rho)} - E_{[\mu\nu\rho]}$, which is said to be of mixed symmetry, is represented by the "hook" tableau:  . This satisfies $D_{[\mu\nu\rho]} = 0$ and $D_{(\mu\nu\rho)} = 0$ and is an irreducible representation of $GL(D, \mathbb{R})$. As another example, a fourth-rank tensor $R_{\mu\nu\rho\sigma}$ with the symmetries of the Riemann tensor corresponds to the diagram  .

The same diagrams can be used also to classify representations of $SO(D)$, but with the difference that now all traces must be removed to obtain an irreducible representation. For example, the diagram  now regarded as a tableau for $SO(D)$ corresponds to 2nd rank symmetric tensor $h_{\mu\nu}$ that is traceless, $\delta^{\mu\nu} h_{\mu\nu} = 0$. The hook tableau  now corresponds to a tensor $D_{\mu\nu\rho}$ that is traceless, $\delta^{\mu\rho} D_{\mu\nu\rho} = 0$. Similarly, the diagram  now corresponds to a tensor with the algebraic properties of the Weyl tensor.

Then given a field in physical gauge in a representation of $SO(D-2)$ corresponding to some Young tableau, the corresponding covariant field in the construction outlined above is in the representation of $GL(D, \mathbb{R})$ corresponding to the *same* Young tableau, now regarded as a tableau for $GL(D, \mathbb{R})$. For example, a graviton is represented in physical gauge by a transverse traceless tensor h_{ij} (with $\delta^{ij} h_{ij} = 0$) of $SO(D-2)$ corresponding to the Young tableau  , so the covariant formulation is the $GL(D, \mathbb{R})$ representation with tableau  , which is a symmetric tensor $h_{\mu\nu}$ with no constraints on its trace.

It will be convenient to label tableaux by the lengths of their columns, so that a tableau with columns of length $n_1, n_2, \dots, n_p$ will be said to be of type $[n_1, n_2, \dots, n_p]$. It is conventional to arrange these in decreasing order, $n_1 \geq n_2 \geq \dots \geq n_p$.

3 Duality

Free gauge theories typically have a number of dual formulations. For example, electromagnetism in flat D dimensional space is formulated in terms of a 2-form field strength $F = \frac{1}{2} F_{\mu\nu} \, dx^\mu \wedge \, dx^\nu$ satisfying $dF = 0$ and $d * F = 0$, where $*F$ denotes the Hodge dual $D-2$ form with components

$$*F_{\mu_1 \dots \mu_{D-2}} \equiv \frac{1}{2} F^{\nu\rho} \epsilon_{\nu\rho\mu_1 \dots \mu_{D-2}} \, . \tag{2}$$

The equation $dF = 0$ can be interpreted as a Bianchi identity and solved in terms of a 1-form potential A as $F = dA$, with $d * F = 0$ regarded as a field equation for A. Alternatively, one can view $d * F = 0$

as the Bianchi identity $d\tilde{F} = 0$ for $\tilde{F} \equiv *F$, and this implies that $\tilde{F}$ can be written in terms of a $D-3$ form potential $\tilde{A}$ with $\tilde{F} = d\tilde{A}$. Then $dF = 0$ becomes $d * \tilde{F} = 0$ which can be regarded as a field equation for $\tilde{A}$. The theory can be formulated either in terms of the one-form A or in terms of the $D-3$ form potential $\tilde{A}$, giving two dual formulations.

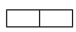
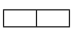
This can be understood from the point of view of the little group $SO(D-2)$. In physical gauge or light-cone gauge, the degrees of freedom are represented by a transverse vector field A_i in the $D-2$ dimensional vector representation of $SO(D-2)$, with $i = 1 \ldots D-2$. This is equivalent to the $(D-3)$-form representation of $SO(D-2)$, so the theory can equivalently be formulated in physical gauge in terms of a $(D-3)$-form

$$\tilde{A}_{j_1 \cdots j_n} = \epsilon_{j_1 \cdots j_n i} A^i \ . \tag{3}$$

where $n = D-3$. These representations of $SO(D-2)$ can be associated with Young tableaux. The vector representation of $SO(D-2)$ is described by a single-box Young tableau,  , while the $(D-3)$-form is associated with a tableau that has one column of $D-3$ boxes. For example in $D=5$, this is a one-column, two-box tableau,  .

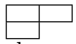
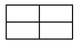
In physical gauge, changing from a 1-form field A_i to a $D-3$ form field $\tilde{A}_{j_1 \cdots j_n}$ is the local field redefinition (3) and so is a trivial rewriting of the theory. However, these lead to two different formulations of the covariant theory: the same physical degrees of freedom can be obtained either from a covariant 1-form gauge field A_μ transforming as a vector under $SO(D-1,1)$, or from a $D-3$ form gauge field $\tilde{A}_{\mu_1 \cdots \mu_n}$. The one-form field has a gauge symmetry $\delta A = d\lambda$ while the $D-3$ form field has a gauge symmetry $\delta \tilde{A} = d\tilde{\lambda}$ and these can be used to eliminate the unphysical degrees of freedom and go to physical gauge. Thus two formulations that are equivalent in physical gauge correspond to two covariant formulations that are distinct covariant realisations of the theory.

This is the key to understanding the generalisations to other gauge fields in other representations of the Lorentz group. A scalar field is a singlet of the little group, and this is equivalent to the $D-2$ form representation of $SO(D-2)$, represented by a tableau with one column consisting of $D-2$ boxes. The scalar field ϕ then has a dual covariant formulation as a $D-2$ form gauge field $\tilde{\phi}_{\mu_1 \cdots \mu_{D-2}}$.

For spin 2, the graviton in D dimensions is a field $h_{\mu\nu}$ which is a symmetric second rank (with trace) represented by the Young tableau  for $GL(D,\mathbb{R})$. The reduction to physical gauge gives a transverse, symmetric, traceless tensor of $SO(D-2)$ h_{ij}, corresponding to the $SO(D-2)$ tableau  . (Recall that for $GL(D,\mathbb{R})$, each box in the tableau represents an index in the D-dimensional representation and has a trace in general, while for $SO(D-2)$ each box in the tableau represents an index in the $(D-2)$-dimensional representation and traces are removed using the $SO(D-2)$ metric, so that the symmetric tensor h_{ij} satisfies $\delta^{ij} h_{ij} = 0$.) The physical gauge graviton h_{ij} can be dualized on one or both of its indices giving respectively

$$D_{i_1 \cdots i_n k} = \epsilon_{i_1 \cdots i_n l} \, h^l{}_k \ , \tag{4}$$

$$C_{i_1 \cdots i_n j_1 \cdots j_n} = \epsilon_{i_1 \cdots i_n l} \, \epsilon_{j_1 \cdots j_n k} \, h^{lk} \ . \tag{5}$$

These give equivalent representations of the little group $SO(D-2)$, with appropriate trace conditions. The tracelessness condition $\delta^{ij} h_{ij} = 0$ implies $D_{[i_1 \cdots i_n k]} = 0$, while the symmetry $h_{[ij]} = 0$ implies the tracelessness $\delta^{i_n k} D_{i_1 \cdots i_n k} = 0$. Then D is represented by the $[n,1]$ hook diagram with one column of length $n = D-3$ and one of length one, so that in dimension $D=5$, D_{ijk} corresponds to the "hook" tableau for $SO(D-2)$:  . The field $C_{i_1 \cdots i_n j_1 \cdots j_n}$ corresponds to the tableau for $GL(D-2,\mathbb{R})$ of type $[n,n]$ with two columns each of $n = D-3$ boxes, so that for $D=5$ C_{ijkl} corresponds to the "window", the two-times-two tableau:  . However, it turns out that $C_{i_1 \cdots i_n j_1 \cdots j_n}$ is not in the $[n,n]$ representation for $SO(D-2)$. In general, the $[m,m]$ representation of $GL(D-2,\mathbb{R})$ would decompose into the representations $[m,m] \oplus [m-1,m-1] \oplus [m-2,m-2] \oplus ....$ of $SO(D-2)$, corresponding to multiple traces. For $m = n = D-3$, it turns out that all the trace-free parts vanish identically, so that only the $[1,1]$ and singlet representations of $SO(D-2)$ survive resulting from $n-1$ and n traces respectively, so that

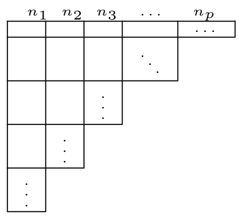
$$C_{i_1 \cdots i_n}{}^{j_1 \cdots j_n} = \delta_{[i_1}{}^{[j_1} \cdots \delta_{i_{n-1}}{}^{j_{n-1}} C_{i_n]}{}^{j_n]} + \delta_{[i_1}{}^{[j_1} \cdots \delta_{i_{n-1}}{}^{j_{n-1}} \delta_{i_n]}{}^{j_n]} C$$

for some C_{ij}, C with traceless C_{ij}. The definition (5) and the tracelessness of h_{ij} then imply that taking n traces of $C_{i_1\cdots i_n j_1\cdots j_n}$ gives zero, so that $C = 0$ and C_{ij} is traceless and in the representation $[1, 1]$, and in fact C_{ij} is proportional to h_{ij}.

For arbitrary spin in dimension D the general form for a gauge field in light-cone gauge will be

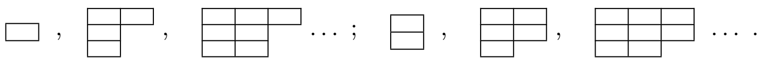
$$D_{[i_1\cdots i_{n_1}][j_1\cdots j_{n_2}]\cdots} ,$$

corresponding to an arbitrary representation of the little group $SO(D-2)$, described by a Young tableau with an arbitrary number of columns of lenghts $n_1, n_2 \cdots$:



Dual descriptions of such fields can be obtained by dualising any column, i.e. by replacing one of length m with one of length $D - 2 - m$ (and re-ordering the sequence of columns, if necessary), or by simultaneously dualising a number of columns [2]. Then any of the equivalent physical gauge fields can be covariantised to a gauge field associated with the same tableau, but now viewed as defining a representation of $GL(D, \mathbb{R})$. The set of Young tableaux for these dual representations of the same theory define distinct representations of $GL(D, \mathbb{R})$, but all reduce to equivalent representations of the little group $SO(D - 2)$.

In fact, there are yet further dual representations. For $SO(D - 2)$, a column of length $D - 2$ is a singlet, and given any tableau for $SO(D - 2)$, one can obtain yet more dual formulations by adding any number of columns of length $D - 2$, then reinterpreting as a tableau for $GL(D, \mathbb{R})$ [7]. Thus for a vector field in $D = 5$, there are dual representations with gauge fields in the representations of $GL(D, \mathbb{R})$ corresponding to the following tableaux:



4 Bi-forms

Before turning to general gauge fields in general representations, we consider the simplest new case, that of gauge fields in representations corresponding to Young tableaux with two columns. It is useful to consider first bi-forms, which are reducible representations in general, arising from the tensor product of two forms, and then at a later stage project onto the irreducible representation corresponding to a Young tableau with two columns. In this section we review the calculus for bi-forms of [3] and generalise to multi-forms and general tableaux in section 7.

A *bi-form* of type (p, q) is an element T of $X^{p,q}$, where $X^{p,q} \equiv \Lambda^p \otimes \Lambda^q$ is the $GL(D, \mathbb{R})$ - reducible tensor product of the space Λ^p of p-forms with the space Λ^q of q-forms on $\mathbb{R}^D$. In components:

$$T = \frac{1}{p!\,q!} T_{\mu_1\cdots\mu_p\nu_1\cdots\nu_q} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \otimes dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_q} \ . \tag{6}$$

and is specified by a tensor $T_{\mu_1\cdots\mu_p\nu_1\cdots\nu_q}$ which is antisymmetric on each of the two sets of p and q indices $T_{\mu_1\cdots\mu_p\nu_1\cdots\nu_q} = T_{[\mu_1\cdots\mu_p][\nu_1\cdots\nu_q]}$, and no other symmetries are assumed. One can define a number of operations on bi-forms: here we only describe the ones needed for the forthcoming discussion, referring to [3] for a more complete development.

Two exterior derivatives, acting on the two sets of indices, are defined as

$$d : X^{p,q} \to X^{p+1,q} \ , \qquad \textit{left derivative}$$

$$\tilde{d}: X^{p,q} \to X^{p,q+1} \ ,$$ *right derivative* whose action on the elements of $X^{p,q}$ is

$$\begin{aligned} d\,T &= \frac{1}{p!\,q!}\partial_{[\mu}T_{\mu_1\cdots\mu_p]\nu_1\cdots\nu_q}dx^{\mu}\wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}\otimes dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_q} \ , \\ \tilde{d}\,T &= \frac{1}{p!\,q!}\partial_{[\nu}T_{|\mu_1\cdots\mu_p|\nu_1\cdots\nu_q]}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}\otimes dx^{\nu}\wedge dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_q} \ , \end{aligned} \tag{7}$$

where the two sets of antisymmetric indices are separated by vertical bars. One can verify that

$$d^2 = 0 = \tilde{d}^2, \qquad\qquad d\,\tilde{d} = \tilde{d}\,d \ . \tag{8}$$

With the help of these two exterior derivatives, one can also define the *total derivative*

$$\mathcal{D} \equiv d + \tilde{d} \ , \qquad \text{such that} \qquad \mathcal{D}^3 = 0 \ , \tag{9}$$

where the nilpotency of $\mathcal{D}$ is a straightforward consequence of the nilpotency of d and $\tilde{d}$. Such nilpotent differential operators were considered by [13], [14], [15]. In a similar fashion, restricting to reducible representations of $SO(D-1,1)$, one can introduce two distinct Hodge-duals:

$$* : X^{p,q} \to X^{D-p,q} \ , \qquad \textit{left dual}$$
$$\tilde{*} : X^{p,q} \to X^{p,D-q} \ , \qquad \textit{right dual}$$

defined as

$$\begin{aligned} *\,T &= \frac{1}{p!\,(D-p)!\,q!}T_{\mu_1\cdots\mu_p\nu_1\cdots\nu_q}\epsilon^{\mu_1\cdots\mu_p}{}_{\mu_{p+1}\cdots\mu_D}dx^{\mu_{p+1}}\wedge\cdots\wedge dx^{\mu_D}\otimes dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_q} \ , \\ \tilde{*}\,T &= \frac{1}{p!\,q!\,(D-q)!}T_{\mu_1\cdots\mu_p\nu_1\cdots\nu_q}\epsilon^{\nu_1\cdots\nu_q}{}_{\nu_{q+1}\cdots\nu_D}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}\otimes dx^{\nu_{q+1}}\wedge\cdots\wedge dx^{\nu_D} \ . \end{aligned} \tag{10}$$

These definitions imply that

$$*^2 = (-1)^{1+p(D-p)}, \qquad \tilde{*}^2 = (-1)^{1+q(D-q)} \ , \qquad *\tilde{*} = \tilde{*}* \ , \tag{11}$$

as can be verified recalling the contraction identity for the Ricci-tensor in D dimensions:

$$\epsilon^{\alpha_1\ldots\alpha_k\alpha_{k+1}\ldots\alpha_D}\epsilon_{\alpha_1\ldots\alpha_k\beta_{k+1}\ldots\beta_D} \;=\; -(D-k)!\cdot(\delta^{[\alpha_{k+1}}_{\beta_{k+1}}\ldots\delta^{\alpha_D]}_{\beta_D}) \ , \tag{12}$$

where we are using the "mostly plus" flat background metric.

There are three operations on bi-forms that enter the Bianchi identities and the equations of motion, and into the projections onto irreducible representations: a trace, a dual trace, and a transposition.

A trace operator acts on a pair of indices belonging to different sets, so that

$$\tau : X^{p,q} \to X^{p-1,q-1} \ ,$$

and is defined by

$$\tau\,T \equiv \frac{1}{(p-1)!\,(q-1)!}\eta^{\mu_1\nu_1}T_{\mu_1\cdots\mu_p\nu_1\cdots\nu_q}dx^{\mu_2}\wedge\cdots\wedge dx^{\mu_p}\otimes dx^{\nu_2}\wedge\cdots\wedge dx^{\nu_q} \ . \tag{13}$$

Combining the τ operator with the Hodge duals, one can also define two distinct dual traces:

$$\begin{aligned} \sigma &\equiv (-1)^{1+D(p+1)} * \tau * : X^{p,q} \to X^{p+1,q-1} \ , \\ \tilde{\sigma} &\equiv (-1)^{1+D(q+1)}\tilde{*}\tau\tilde{*} : X^{p,q} \to X^{p-1,q+1} \ , \end{aligned} \tag{14}$$

that antisymmetrize one index in a set with respect to the whole other set:

$$\begin{aligned} \sigma\,T &= \frac{(-1)^{p+1}}{p!\,(q-1)!}T_{[\mu_1\cdots\mu_p\nu_1]\cdots\nu_q}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}\otimes dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_q} \ , \\ \tilde{\sigma}\,T &= \frac{(-1)^{q+1}}{(p-1)!\,q!}T_{\mu_1\cdots[\mu_p\nu_1\cdots\nu_q]}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}\otimes dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_q} \ . \end{aligned} \tag{15}$$

Again, the proof of (15) relays on the identity (12). For example, for a $(2,3)$ form in $D=5$ one has (omitting combinatorial factors):

$$
\begin{aligned}
* \, T &\sim T_{p_1p_2,q_1q_2q_3}\epsilon^{p_1p_2}{}_{p_3p_4p_5} \ , \\
\tau * T &\sim T_{p_1p_2,q_1q_2q_3}\epsilon^{p_1p_2q_1}{}_{p_4p_5} \ , \\
*\tau * T &\sim T_{p_1p_2,q_1q_2q_3}\epsilon^{p_1p_2q_1}{}_{p_4p_5}\epsilon^{p_4p_5}{}_{\alpha_1\alpha_1\alpha_3} \sim T_{p_1p_2,q_1q_2q_3}\delta^{[p_1p_2q_1]}_{\alpha_1\alpha_2\alpha_3} \ .
\end{aligned} \tag{16}
$$

Finally, the transposition operator simply interchanges the two sets of indices:

$$t : X^{p,q} \to X^{q,p} \ ,$$

so that

$$(t\,T)_{\nu_1\cdots\nu_q\mu_1\cdots\mu_p} = T_{\mu_1\cdots\mu_p\nu_1\cdots\nu_q}$$

and

$$t\,T \equiv \frac{1}{p!q!}T_{\nu_1\cdots\nu_q\mu_1\cdots\mu_p}dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_q}\otimes dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p} \ . \tag{17}$$

The bi-forms are a reducible representation of $GL(D,\mathbb{R})$. It is useful to introduce the Young symmetrizer $\mathcal{Y}_{[p,q]}$ which projects a bi-form T of type (p,q) onto the part $\hat{T} = \mathcal{Y}_{[p,q]}\,T$ lying in the irreducible representation corresponding to a tableau of type $[p,q]$, with two columns of length p and q, respectively (we use round brackets for reducible (p,q) bi-forms and square ones for irreducible representations). The projected part $\hat{T}$ satisfies the additional constraints (for $p \geq q$):

$$
\begin{aligned}
&\sigma\,\hat{T} = 0 \ , \\
&t\,\hat{T} = \hat{T}, \qquad \text{if} \quad p = q \ .
\end{aligned} \tag{18}
$$

5 D-Dimensional Linearised Gravity

It is straightforward to formulate gauge field theories of bi-forms; a gauge field A in the space $X^{p,q}$ can be thought of as a linear combination of terms arising from the tensor product of a p-form gauge field and a q-form gauge field. It transforms under the gauge transformation

$$\delta\,A = d\,\alpha^{\,p-1,q} + \tilde{d}\,\alpha^{\,p,q-1}, \tag{19}$$

with gauge parameters that are themselves bi-forms in $X^{\,p-1,q}$, $X^{\,p,q-1}$. Clearly,

$$F = d\,\tilde{d}\,A \tag{20}$$

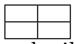
is a gauge invariant field-strength for A. This is a convenient starting point for describing gauge fields in irreducible representations. We now show how to project the bi-form gauge theory using Young projections to obtain irreducible gauge theories, starting with one of the simplest examples, that of linearised gravity in D dimensions.

The graviton field is a rank-two tensor in an irreducible representation of $GL(D,\mathbb{R})$ described by a Young tableau of type $[1,1]$, i.e. a two-column, one-row Young tableau, $\square\!\square$. The starting point in our case is thus a bi-form $h \in X^{1,1}$, corresponding to a 2nd rank tensor $h_{\mu\nu}$, and we would like to project on the $GL(D,\mathbb{R})-$irreducible tensor of type $[1,1]$: $\widehat{h} = \mathcal{Y}_{[1,1]}\,h$ using the Young projector $\mathcal{Y}_{[1,1]}$. Then the constraints (18) become

$$
\begin{aligned}
&\sigma\,\widehat{h} = 0 \ , \\
&t\,\widehat{h} = \widehat{h} \ .
\end{aligned} \tag{21}
$$

In this case, the two conditions are equivalent and simply imply that $\widehat{h}$ is symmetric, $\widehat{h}_{\mu\nu} = h_{(\mu\nu)}$. The gauge transformation for the graviton is the Young-projection of $(19)^1$, which gives $\delta\,\widehat{h}_{\mu\nu} = \partial_{(\mu}\lambda_{\nu)}$ where $\lambda_\mu = \alpha^{1,0}_\mu + \alpha^{0,1}_\mu$. The invariant field strength is given by [2]

$$R = \tilde{d}\,d\,\widehat{h}\ . \tag{22}$$

This is the $[2,2]$ Young tableau  describing the linearized Riemann tensor. The nilpotency of the exterior derivatives and the irreducibility imply that the Bianchi identities

$$d\,R = 0\ , \qquad\qquad \tilde{d}\,R = 0\ , \tag{23}$$

$$\sigma\,R = 0\ , \tag{24}$$

are satisfied, while acting with the τ operator gives the Einstein equation in $D \geq 4$ [3] :

$$\tau\,R = 0\ . \tag{25}$$

or in components, $R_{\mu\nu} = 0$.

We now return to the issue of duality. In Section 3 we described the triality of linearised gravity in D dimensions, for which there are three different fields that can be used for describing the degrees of freedom of the graviton. The discussion can be expressed succinctly in terms of bi-forms. In light-cone gauge the fields are tensors in irreducible representations of $SO(D-2)$, and so are trace-less. The graviton arises from projecting a $[1,1]$ form h onto a symmetric tensor $\widehat{h}_{ij}$ that is traceless $\widehat{h}^i{}_i = 0$. Now one can easily dualise the field $\widehat{h}$ in one or both indices, by applying (10), where the $*$-operator is now the $SO(D-2)$-covariant dual. The dual light-cone fields are

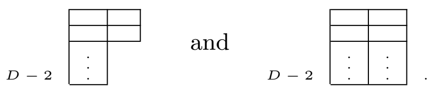
$$D \quad = *\,\widehat{h}\ , \tag{26}$$
$$C \quad = *\,\tilde{*}\,\widehat{h}\ , \tag{27}$$

and all have the same number of independent components.

In the covariant theory one dualises the field strengths rather than the gauge fields and this is easily analysed using the bi-form formalism developed. Indeed, starting from the $[2,2]$ field strength R one can define the Hodge duals

$$S \quad \equiv *\,R, \tag{28}$$
$$G \quad \equiv *\,\tilde{*}\,R, \tag{29}$$

which are respectively of type $[D-2,2]$ and $[D-2,D-2]$, associated with the tableaux:



The other possible dual, $\tilde{S} \equiv \tilde{*}\,R$ is not independent, since $\tilde{S} = t\,S$ (this would not be the case for the generalisation to a general (p,q)-form with $p \neq q$). In components:

$$S_{\mu_1\cdots\mu_{D-2}\nu_1\nu_2} \quad = \quad \frac{1}{2} R^{\alpha\beta}{}_{\nu_1\nu_2}\,\epsilon_{\alpha\beta\mu_1\cdots\mu_{D-2}}\ , \tag{30}$$

$$G_{\mu_1\cdots\mu_{D-2}\nu_1\cdots\nu_{D-2}} \quad = \quad \frac{1}{4} R^{\alpha\beta\gamma\delta}\,\epsilon_{\alpha\beta\mu_1\cdots\mu_{D-2}}\,\epsilon_{\gamma\delta\nu_1\cdots\nu_{D-2}}\ . \tag{31}$$

[1]Note that acting on a tensor in an irreducible representation with d or $\tilde{d}$ gives a *reducible* form in general, so that a Young projection is necessary in order to obtain irreducible tensors.

[2]The operator $d\,\tilde{d}$, unlike d and $\tilde{d}$ separately, sends irreps to irreps, so that $\tilde{d}\,d\,\widehat{h} = \mathcal{Y}_{[1,1]}\,\tilde{d}\,d\,\widehat{h}$ and the Young projection is automatically implemented.

[3]In $D = 3$ the field equation $R_{\mu\nu} = 0$ implies $R_{\mu\nu\rho\sigma} = 0$ which only has trivial solutions; a non trivial equation is instead $\tau^2\,R = 0$, with $\tau^2\,R$ the Ricci-scalar [2], [1]. This can be generalized to (p,q)-forms, as we shall see later.

From these definitions, and the algebraic and dynamical constraints satisfied by the linearised Riemann tensor R, one can deduce a set of relations that must be obeyed by the bi-forms S and G. We will give examples of certain relations between Bianchi identities and equations of motion, referring to [3], [2], [1] for a more complete discussion. The definitions $S \equiv *R$ and $G \equiv * \tilde{*} R$, and the relations (11) imply that $*S = (-1)^D R$, and $\tilde{*} * G = R$. Then, using the definitions given in Section 4, it follows that

$$\sigma R = 0 \quad \Rightarrow \quad \sigma * S = 0 \quad \Rightarrow \quad * \sigma * S = 0 \quad \Rightarrow \quad \tau S = 0 \, ; \tag{32}$$

$$\sigma R = 0 \;\; \Rightarrow \;\; \sigma \tilde{*} * G = 0 \;\; \Rightarrow \;\; * \sigma \tilde{*} * G = 0 \;\; \Rightarrow \;\; \tau \tilde{*} G = 0 \;\; \Rightarrow \;\; \tilde{*} \tau \tilde{*} G = 0 \;\; \Rightarrow \;\; \tilde{\sigma} G = 0 \, . \tag{33}$$

That is to say, the Bianchi identity $\sigma R = 0$ for R implies the equation of motion $\tau S = 0$ for S and the Bianchi identity $\tilde{\sigma} G = 0$ for G. The equation of motion $\tau R = 0$ for R in $D > 3$ implies that

$$\tau R = 0 \quad \Rightarrow \quad * \tau * S = 0 \quad \Rightarrow \quad \sigma S = 0 \, , \tag{34}$$

and

$$\tau R = 0 \quad \Rightarrow \quad \tau \tilde{*} * G = 0 \quad \Rightarrow \quad \tau^{D-3} G = 0 \, . \tag{35}$$

giving the Bianchi identity $\sigma S = 0$ for S and the field equation $\tau^{D-3} G = 0$ for G [4].

Other consequences for S and G can be deduced starting from properties of R and making use of identities involving the various bi-form operators (see [3]). In particular the Bianchi identities $dS = \tilde{d} S = 0$ and $dG = \tilde{d} G = 0$ imply that S and G can be expressed as field-strengths of gauge potentials $\widehat{D}$ and $\widehat{C}$ respectively, which are in irreducible representations of type $[D-3, 1]$ and $[D-3, D-3]$

$$S = d \tilde{d} \widehat{D}, \qquad G = d \tilde{d} \widehat{C} \, , \tag{36}$$

whose linearized equations of motion are $\tau S = 0$ and $\tau^{D-3} G = 0$. Although these relations can be derived for gravity straightforwardly, as in [2], the bi-form formalism simplifies the discussion and generalises to general multi-form representations in a way that elucidates the geometric structure and allows simple derivations and calculations.

6 General Bi-Form Gauge Theories

The discussion of gravity extends straightforwardly to arbitrary (p, q)-forms, where without loss of generality we assume $p \geq q$. First, one can restrict from a (p, q)-form T to $\widehat{T} = \mathcal{Y}_{[p,q]} T$ which is in $[p, q]$ irrep of $GL(D, \mathbb{R})$ satisfying the constraints (18). Then one can define a field strength $F \equiv \tilde{d} d \widehat{T}$ of type $[p+1, q+1]$ that is invariant under the gauge transformations given by the projection of (19):

$$\delta \widehat{T} = \mathcal{Y}_{[p,q]} \, (d \, \alpha^{p-1,q} + \tilde{d} \, \alpha^{p,q-1}), \tag{37}$$

and satisfies the Bianchi identities

$$d F = \tilde{d} F = 0 \, , \qquad \sigma F = 0 \, , \tag{38}$$

together with $t F = F$ if $p = q$. We now turn to the generalisation of the "Einstein equation" $\tau R = 0$. The natural guess is

$$\tau F = 0 \, , \tag{39}$$

However, we have seen that for gravity in $D = 3$, the Einstein equation $\tau R = 0$ is too strong and only has trivial solutions, but that the weaker condition $\tau^2 R = 0$ (requiring that the Ricci scalar is zero) gives a non-trivial theory. For the dual field strength G, the field equation was $\tau^{D-3} G = 0$ in D dimensions. Then it is to be expected that the "Einstein equation" $\tau R = 0$ will be generalised to $[p, q]$ forms by taking $\tau F = 0$ for large enough space-time dimension D, but for low D a number of traces

[4] Note that the equation $\tau^n G = 0$ only has trivial solutions for $n < D-3$, so that this is the simplest non-trivial field equation [2].

of the field strength may be needed to give an equation of motion $\tau^n F = 0$ for some n. It was shown in [3] that the natural equation of motion

$$\tau F = 0 \qquad \text{for} \qquad D \geq p + q + 2 \; , \tag{40}$$

is non trivial for $D \geq p + q + 2$, but for $D < p + q + 2$ that $\tau^n F = 0$ is a non-trivial field equation for $n = p + q + 3 - D$, and so we will take

$$\tau^{p+q+3-D} F = 0 \qquad \text{for} \qquad D < p + q + 2 \; . \tag{41}$$

A $[p, q]$-Young tableau can be dualized on one of the two columns, or on both, so that three duals of the field strength can be defined:

$$S \equiv * F \in X^{D-p-1,q+1} \; , \qquad \tilde{S} \equiv \tilde{*} F \in X^{p+1,D-q-1} \; , \qquad G \equiv \tilde{*} * F \in X^{D-p-1,D-q-1} \; , \tag{42}$$

and the algebraic and differential identities and equations of motion for F give analogous properties for S, $\tilde{S}$ and G. In particular, the equations of motion are [5]:

$$\tau S = 0 \; , \qquad \tau^{1+p-q} \tilde{S} = 0 \; , \qquad \tau^{D-p-q-2+n} G = 0 \; . \tag{43}$$

For example, gravity in $D = 3$ with $p = q = 1$ has dual formulations in terms of a $[1, 1]$ field strength G for a $[0, 0]$ form or scalar field C with field equation $\tau G = 0$ giving the usual scalar field equation $\partial_\mu \partial^\mu C = 0$, or to a $[2, 1]$ field strength S for a $[1, 0]$ or vector gauge field D_μ, with the usual Maxwell equation $\tau S = 0$. Then this $D = 3$ gravity theory is dual to a scalar field and to a vector field, and all describe one physical degree of freedom.

7 Multi-Forms

The previous discussion generalizes to the case of fields in arbitrary massless representations of $SO(D-1, 1)$, including higher-spin gauge fields described by mixed-symmetry Young tableaux. As for the case of forms and bi-forms, the starting point is the definition of a larger environment, the space of *multi-forms*, in which a series of useful operations are easily defined. Then, by suitable Young projections, one can discuss the cases of irreducible gauge fields and their duality properties. In the following we shall confine ourselves to describing the main steps of the construction; further details are given in [3], [4].

A multi-form of order N is characterised by a set of N integers $(p_1, p_2, ..., p_N)$ and is a tensor of rank $\sum p_i$ whose components

$$T_{\mu^1_1 \cdots \mu^1_{p_1} \cdots \mu^N_1 \cdots \mu^N_{p_N}} = T_{[\mu^1_1 \cdots \mu^1_{p_1}] \cdots [\mu^N_1 \cdots \mu^N_{p_N}]} \; , \tag{44}$$

are totally antisymmetrized within each of N groups of p_i indices, with no other symmetry *a priori* between indices belonging to different sets. It is an element of $X^{p_1 \cdots p_N} \equiv \Lambda^{p_1} \otimes \cdots \otimes \Lambda^{p_N}$, the $GL(D, \mathbb{R})$-reducible N-fold tensor product space of p_i-forms on $\mathbb{R}^D$. The operations and the properties introduced in Section 4 generalize easily to multi-forms. For an extensive treatment see again [3]; here we restrict our attention to the operations previously discussed.

One can define an exterior derivative acting on the $i-th$ set of indices,

$$d^{(i)} : X^{p_1 \cdots p_i \cdots p_N} \to X^{p_1 \cdots p_i + 1 \cdots p_N} \; , \tag{45}$$

generalizing the properties of d and $\tilde{d}$; summing over the $d^{(i)}$'s one can then define the *total derivative*

$$\mathcal{D} \equiv \sum_{i=1}^{N} d^{(i)} \; , \tag{46}$$

such that

$$\mathcal{D}^{N+1} = 0 \; . \tag{47}$$

[5] The last equation follows from the result $\tau^n T = 0 \Rightarrow (\tau^{D-p-q+n} * \tilde{*}) T = 0$, valid for a general [p,q]-form T, applied to the case of the [p+1,q+1]-form F. Here n is the exponent such that $\tau^n F = 0$ is non trivial. So, if $D \geq p + q + 2$ then $n = 1$, and the e.o.m. for the dual tensor G reduces to $\tau^{D-p-q-1} G = 0$ [3] .

Similarly, for representations of $SO(D-1,1)$ or $SO(D)$ one can define N Hodge-duals:

$$*^{(i)} : X^{p_1,\ldots,p_i,\ldots,p_N} \to X^{p_1,\ldots,D-p_i,\ldots,p_N} \ , \tag{48}$$

each acting in the usual fashion on the i-th form, and so commuting with any $*^{(j\neq i)}$.

The operators τ, σ, $\tilde{\sigma}$ and t generalize to a set of operators, each acting on a specific pair of indices; the *trace* operators

$$\tau^{(ij)} : X^{p_1,\ldots,p_i,\ldots,p_j,\ldots,p_N} \to X^{p_1,\ldots,p_{i-1},\ldots,p_{j-1},\ldots,p_N} \ , \tag{49}$$

are defined as traces over the i-th and the j-th set; the *dual-traces* are

$$\begin{aligned} \sigma^{(ij)} &\equiv (-1)^{1+D(p_i+1)} *^{(i)} \tau^{(ij)} *^{(i)} : X^{p_1,\ldots,p_i,\ldots,p_j,\ldots,p_N} \to X^{p_1,\ldots,p_{i+1},\ldots,p_{j-1},\ldots p_N} \ , \\ \tilde{\sigma}^{(ij)} &\equiv (-1)^{1+D(p_j+1)} *^{(j)} \tau^{(ij)} *^{(j)} : X^{p_1,\ldots,p_i,\ldots,p_j,\ldots,p_N} \to X^{p_1,\ldots,p_{i-1},\ldots,p_{j+1},\ldots p_N} \ , \end{aligned} \tag{50}$$

while the *transpositions* $t^{(ij)}$ generalize the action of the t operator to exchanges between the subspaces Λ^{p_i} and Λ^{p_j} in $X^{p_1,\ldots,p_i,\ldots,p_j,\ldots,p_N}$:

$$t^{(ij)} : X^{p_1,\ldots,p_i,\ldots,p_j,\ldots,p_N} \to X^{p_1,\ldots,p_j,\ldots,p_i,\ldots,p_N} . \tag{51}$$

The Young symmetrizer $\mathcal{Y}_{[p_1,\ldots,p_N]}$ projects a multi-form of type $(p_1,\ldots,p_N)$ onto the irreducible representation associated with a Young tableau of type $[p_1,\ldots,p_N]$.

8 Multi-Form Gauge Theories

With the machinery of the last section, one can naturally extend the construction of gauge theories for general tensor gauge fields. The starting point is a multi-form gauge field of type $(p_1,\ldots,p_N)$ with gauge transformation

$$\delta T = \sum_{i=1}^{N} d^{(i)} \alpha_{(i)}^{p_1,\ldots,p_{i-1},\ldots,p_N} \ . \tag{52}$$

The restriction to irreducible representations of $GL(D,\mathbb{R})$ can be implemented using the Young symmetrizer $\mathcal{Y}_{[p_1,\ldots,p_N]}$ projecting onto the representation characterised by a Young tableau with N columns of length $p_1, p_2, \ldots, p_N$ (these are conventionally arranged in order of decreasing length, but this is not essential here). Then this projects a multi-form T onto

$$\widehat{T} = \mathcal{Y}_{[p_1,\ldots,p_N]} \, T \tag{53}$$

which satisfies the constraints

$$\begin{aligned} \sigma^{ij} \, \widehat{T} &= 0 \qquad \text{if} \quad p_i \geq p_j \\ t^{ij} \, \widehat{T} &= \widehat{T} \qquad \text{if} \quad p_i = p_j \end{aligned} \tag{54}$$

The field strength is a multi-form in the irreducible representation of type $[p_1+1,\ldots,p_N+1]$ defined as[6]

$$F \equiv \prod_{i=1}^{N} d^{(i)} \, \widehat{T} = \frac{1}{N} \mathcal{D}^N \, \widehat{T} \ , \tag{55}$$

and is invariant under the Young projection of the gauge transformation for T

$$\delta \widehat{T} = \mathcal{Y}_{[p_1,\ldots,p_N]} \sum_{i=1}^{N} d^{(i)} \alpha_{(i)}^{p_1,\ldots,p_{i-1},\ldots,p_N} \ . \tag{56}$$

[6] More generally, one can define a set of connections $\Gamma_{S_k} \equiv (\prod_{i\notin S_k} d^{(i)}) \, \widehat{T}$, corresponding to each subset $S_k = \{i_1,\ldots,i_k\} \subseteq \{1,\ldots,N\}$. These are gauge-dependent w.r.t. transformations involving parameters α^j , $j \in S_k$, while are invariant under transformations with parameters α^i , $i \notin S_k$. For a given k there are in general $\frac{N!}{(N-k)!k!}$ inequivalent possible Γ_{S_k}; in particular, the totally gauge-invariant field strenght F can be regarded both as the top of this hierarchy of connections (the one with $k=0$), or as a direct function of the connection Γ_{S_k}, being $F = (\prod_{i\in S_k} d^{(i)}) \, \Gamma_{S_k}$ [4] .

By construction, the field strength satisfies the generalized Bianchi identities

$$d^{(i)}\,F = 0\ , \tag{57}$$

$$\sigma^{(ij)}\,F = 0\ . \tag{58}$$

The simplest covariant local field equations are those proposed in [3] and these in general involve more than two derivatives. For N even, a suitable field equation is

$$\sum_{\substack{(i_1,\ldots,i_N)\\(permutations)}} \tau^{(i_1 i_2)}\ldots\tau^{(i_{N-1} i_N)}\,F = 0\ , \tag{59}$$

where the sum is over all permutations of the elements of the set $\{1,\ldots N\}$. For N odd, one first needs to define $\partial\,F$, which is the derivative $\partial_\mu\,F$ of F regarded as a rank $N+1$ multi-form of type $[p_1,\ldots,p_N,1]$. Then the equation of motion is:

$$\sum_{\substack{(i_1,\ldots,i_N)\\(permutations)}} \tau^{(i_1 i_2)}\ldots\tau^{(i_{N-2} i_{N-1})}\tau^{(i_N N+1)}\,\partial\,F = 0\ . \tag{60}$$

Here the sum is over *the same* set of permutations of the elements of the set $\{1,\ldots N\}$ as in the even case, so that the extra index is left out. These are the field equations for large enough space-time dimension D; as for the case of bi-forms, for low dimensions one needs to act with further traces.

These field equations involving multiple traces of a higher-derivative tensor are necessarily of higher order in derivatives if $N > 2$. This is unavoidable if the field equation for a higher-spin field is to be written in terms of invariant curvatures. In physical gauge, these field equations become $\Box^a A = 0$ where A is the gauge potential in physical gauge, $\Box$ is the D-dimensional d'Alembertian operator and $a = N/2$ if N is even and $a = (N+1)/2$ if N is odd. The full covariant field equation is of order $2a$ in derivatives. In order to get a second order equation, following [8, 9], one can act on these covariant field equations with $\Box^{1-a}$ to obtain equations that reduce to the second order equation $\Box A = 0$ in physical gauge. In the even case the equation (59) is of order N in derivatives, and so it is possible to write a second-order field equation dividing by $\Box^{\frac{N}{2}-1}$. Similarly, for N odd, it is necessary to divide by $\Box^{\frac{N+1}{2}-1}$. In this way, one can write second-order, non-local field equations [4]:

$$\begin{aligned}
\mathcal{G}_{\text{even}} &\equiv \sum_{\substack{(i_1,\ldots,i_N)\\(permutations)}} \tau^{(i_1 i_2)}\ldots\tau^{(i_{N-1} i_N)}\,\frac{1}{\Box^{\frac{N}{2}-1}}\,F = 0\ ,\\
\mathcal{G}_{\text{odd}} &\equiv \sum_{\substack{(i_1,\ldots,i_N)\\(permutations)}} \tau^{(i_1 i_2)}\ldots\tau^{(i_N N+1)}\,\frac{1}{\Box^{\frac{N+1}{2}-1}}\,\partial\,F = 0\ .
\end{aligned} \tag{61}$$

These then are the covariant field equations for general representations for high enough D (for low D, the appropriate field equations require further traces [4], as we saw earlier for the case of bi-forms.) These are non-local, but after fixing a suitable gauge, they become local. On fully fixing the gauge symmetry to go to light-cone gauge, the field equations reduce to the free equation $\Box A = 0$, while partially fixing the gauge gives a Fronsdal-like local covariant field equation with constraints on the traces of the gauge field and parameters of the surviving gauge symmetries. It would be interesting to understand if the non-locality of the full geometric field equation has any physical consequences, or is purely a gauge artifact. As in the Fronsdal case, only physical polarizations are propagating [9, 4, 12].

It is worth noting that these equations are not unique. As was observed in [8], and analysed in detail for the case $s = 3$ in the totally symmetric representation, one can write other second-order equations, with higher degree of non locality, by combining the least singular non-local equation with its traces and divergences. The systematics of this phenomenon was described in [4], where it was shown in the general case how to generate other field equations starting from (61). The idea is to define a new tensor $F^{(m)} \equiv \partial^m F$, by taking m partial derivatives of the field strenght F, take a suitable number of traces of the order $N+m$ tensor $\partial^m F$, and divide for the right power of the D'Alembertian operator. One can then take linear combinations of these equations with the original equations (61).

Given a field strength F of type $[p_1+1, p_2+1, \ldots, p_N+1]$, one can choose any set of columns of the Young tableau and dualise on them to obtain a dual field strength. The field equations and Bianchi identities for F then give the field equations and Bianchi identities for the dual field strength, and the new Bianchi identities imply that the dual field strength can be solved for in terms of a dual potential. There are then many dual descriptions of the same free higher-spin gauge theory.

Acknowledgements

D.F. would like to thank the Organizers for giving him the opportunity to participate in the Workshop and to collect and order these lecture notes. The work of D.F. was supported in part by I.N.F.N., by the MIUR-COFIN contract 2003-023852, by the INTAS contract 03-51-6346 and by the NATO grant PST.CLG.978785. We would like to thank Paul de Medeiros for comments on the manuscript.

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