A Matrix Guide trough String Theory
From D-branes and Matrices towards the Big Bang

Paul de Lange
plange@science.uva.nl

August 8, 2011

Contents

Introduction **3**

1 The web of dualities and M-Theory **6**
- 1.1 The five SUSY string theories and their content . . . 8
  - 1.1.1 Effective Actions . . . 11
  - 1.1.2 The Symmetries of IIB . . . 14
- 1.2 T-duality . . . 18
  - 1.2.1 T-duality from the states . . . 18
- 1.3 M-theory . . . 21
- 1.4 F-theory . . . 24
- 1.5 Large N field theory and the Gauge/Gravity correspondance . 26

2 Matrix theory **30**
- 2.1 M(atrix) Theory . . . 31
- 2.2 Dimensional reduction of a SYM . . . 32
- 2.3 The M2-brane . . . 34
- 2.4 5-branes in Matrix theory . . . 37
  - 2.4.1 Dirac Quantization in M(atrix) . . . 39
- 2.5 Matrix interactions . . . 41
- 2.6 Matrix String Theory . . . 44

3 Matrix Cosmology **48**
- 3.1 A Matrix Big Bang . . . 49
  - 3.1.1 A specific dilaton dependence . . . 50
- 3.2 Matrix Black Holes . . . 52

4 Layman Summary **55**

Bilbliography **58**

Introduction

From a highbrow point of view, (perturbative) string theory is a topological extension of the quantum field theory (QFT) of particle physics. Whereas QFT, in its perturbative approach, is a theory of summing over different connected and closed weighted *graphs* - networks of connected worldlines -, the perturbative picture of string theory is that of a theory of summing over *surfaces* - networks of worldsheets. This already establishes one of the successes and beauties of string theory. The graphs of QFT are not manifolds, have annoying vertices, and the set of graphs with many faces gets extremely complicated (the complexity in the perturbation rises with a factorial in the power of the weight). In string-theory however, the surfaces are two-folds, and summing over in-equivalent surfaces is way more easy: we know everything about the classification of two-folds as we have the elementary result that the only distinction between two-folds is it's genus. The perturbation is simply in this genus.

But we know from particle physics that not all is captured by the perturbative analysis: solitons, monopoles, symmetry braking, dualities, etc. have not been discovered trough graph computations but are extremely important in our understanding of the theory. With these lessons learned from QFT, string theorists have always tried to capture non-perturbative effects of the theory. In fact the biggest successes and breakthroughs in the field are non-perturbative ones: the discovery of D-branes and the proposal of M-theory and certain dualities. Indeed, most dualities and M-theory are only certainly known to hold in low-energy regimes, but the conviction of many workers in the field is that those results are in fact exact.

Anyway, at present there is not yet a full, non-perturbative picture of the entire body of string theory, and after forty years of hard work we must be a little pessmistic about ever finding such a description. For one, in the late 90's there has emerged quite some research on one proposal that might

capture a microscopic and non-perturbative description of M-theory, which is known under the name M(atrix)-theory[7]. The proposal to describe the fundamental theory in terms of a matrix quantum mechanics has proven to be successful in quite some extent, and indeed has proven to describe a lot of the objects and physics of strings[8] and M-theory. But it certainly had some flaws - some missing objects, some high-order mismatches - which led to a diminishing in its popularity (together with the expanding popularity of AdS/CFT which emerged at the same time of the M(atrix) proposal).
Recently however, it has become clear that the Matrix theory looks quite successful in handling time-like singularities, surprisingly shifting its applications from the microscopic to the cosmological. This is of course exiting, as we would really like to have some clue about what happened around time-like singularities like the big-bang, or the center of a black-hole. That matrix theory might provide a proper frame-work for addressing these extreme physical situations lies in the nature of its description of space-time. Many cosmologists and high-energy physicists in general do suspect that around these extreme time-like singularities the nature of space-time itself may need replacement and that concepts of space and time are actually emergent entities that open up only at "low energies". This vague intuition finds concreteness in matrix theory, where space-like coordinates get replaced by non-commuting matrices. A system of N fundamental objects is represented by a set of $N \times N$ matrices. Originally the coordinates of N fundamental objects were described by N parameters per dimension. This idea is replaced: in extremely high energies or at extremely low length-scales, positions are described by one matrix per dimensions, N diagonal term and the off-diagonal terms. The emergence of every-day space is in the low-energy limit, where the matrices are diagonalized and indeed to reduce to a set of N degrees of freedom.

This thesis reviews the ideas of matrix string theory and its cosmological implications. The first chapter introduces important back ground concepts of superstring theory itself, such as the D-brane, M-theory and some of the dualities of string theory. It does not treat the bosonic string in any way, but still this document tries to be self-contained to some extent. The five supersymmetric strings are simply introduces without a lot derivation, and I should refer to the bilbliography for excellent text-books where both the bosonic string and the background of superstring theory are explained. I also refrained from conformal analysis.

The second chapter introduces Matrix theory and discusses some of the objects it reproduces. Also a scattering computation is verified. The final chapter deals with the cosmological implications of the Matrix proposal.

Chapter 1

The web of dualities and M-Theory

At present, since the second revolution in string theory, the string is not seen anymore as the unique and defining object in string theory. In fact, we now know that string theory consists of more and equally fundamental objects. I will quite hands on set up the content of the theory that will be of interest in this document. Based on superconformal grounds, we will embed the two-dimensional world-sheet of the superstring into a D=10 dimensional target space.
There are five different physically consistent string theories, all with different objects and fields. These five are called:

TYPE-I, TYPE-IIA, TYPE-IIB, HETEROTIC $SO(32)$, HETEROTIC $E_8 \times E_8$

Without giving complete derivations or historical motivations, I will briefly describe what defines the differences between the theories.

Type-IIA and Type-IIB theories are in a way the most *straightforward* string set-ups. They constructed by identifying the modes of the expansion that solves the bulk equation of motion for the fermionic part of the action. Identifying right and left movers, and noting that these are independent, the theory is split in a left and right sector. Each mover can admit one of two physically admissable boundary conditions, called Ramond (R) and Neveu-Schwarz (NS), introducing four pairs or sectors. This theory is not yet consistent as the spectrum contains a tachyon. By consistently projecting out suitable subsets of the full spectrum, physical consistency is obtained. Such sub-selection (called GSO-projection) can be done in two ways, one yielding a theory with four sectors, such that the left and right-moving sectors have equal chirality (Type-IIB), and one such that the left-moving Ramond sector has negative chirality while the right Ramond sector has positive chirality (Type-IIA).

From the equi-chiral Type-IIB, which is clearly an $\mathcal{N} = 2$ supersymmetric (susy) theory, one can build the so-called Type-I theory by projecting out even further specific subsets of the spectrum. From its construction is is clear that the group $\mathbf{Z}_2$ acts on Type-IIB in an invariant way, since left and right-movers have the same chirality. That is, the theory is left invariant under $\Pi : IIB \to IIB$, $\Pi(\sigma) = -\sigma$. We can project out half of the Type-IIB theory by constructing $P = \frac{1}{2}(1 + \Pi)$ and only keeping states with positive eigenvalue under P. This (naive) construction yields an $\mathcal{N} = 1$ superstring theory. The construction is naive in the sense that it yields an anomalous theory, an objects have to be added in order to make it stable. We will discuss the content below.

Perhaps the most counterintuitive string-theoretic constructions are the Heterotic strings. The way they are build might seem a little strange, but they have always been perceived as the most promising in making contact with the standard model trough a Grand Unification program.
Recall that also in the bosonic string we have left and right movers, and that this theory lives - critically - in 26 dimensions. Now let's take only the left movers, and perceive the fact that the target-space is 26-dimensional merely as the statement that we have 26 left moving degrees of freedom. Let's take at the same time the right-movers of the superstring in ten dimensions and look at them too as ten right-moving degrees of freedom.
Now the construction of the Heterotic string, is to combine these two observations into one string, namely by taking a string with as right-movers superstring degrees of freedom and as left-movers the bosonic ones. The model consists of an X^μ boson, a Majorana-Weyl fermion ψ^μ and a Majorana-Weyl fermion λ^A, where $\mu = 0, \ldots, 9$ and $A = 1, \ldots, 32$ where the number of 32 follows from anomaly cancelations. As the heterotic string is not of our main interest, we will not delve deeply into it at all. The λ^A then may carry either an $SO(32)$ or an $E_8 \times E_8$ gauge symmetry. This choice defines the two different heterotic string types: Heterotic-SO(32) or Heterotic-$E_x \times E_8$

I will quite boldly and top-down state their field-content and the different objects that are present in the theories, not following the text-book approach starting with the derivations and motivations - starting with the action and variations or setting up the different movers etcetera. I will, along the way, take some theories apart and discuss in more depth it's features to motivate why they are a member of this arena of five.

1.1 The five SUSY string theories and their content

The fact that there are five different theories finds its origin in the evaluation of the fermionic Lagrangian, and the notion that upon demanding the variation of the action to vanish, some conditions emerge on the boundary of the world-sheet. Different theories emerge by making different GSO-projections of physical states, to protect the vacuum from being tachyonic, or they are built by concisely choosing proper degrees of freedom for different sections of the space of solutions for the bulk. I will state the content of the theories and some of the fundamental objects we encounter therein. As mentioned, the motivation and some brief derivations will become clear as we go along.

Type-IIA

Fields:

For our purposes this realm is of particular interest. The field-content is divided into four sectors, denoted by $(left, right)$, referring to left- and right-moving modes (levels of solutions of the bulk-field equations), such that each left and right segment can be either in the Ramond (R) or in the Neveu-Schwarz (NS) (nicknames for overall relative sign of chirality between the left and right sector). By field-content we mean the body of zero-mode, massless fields.

(R,R): a one-form $C^{(1)}$ in the **8** and a three-form $C^{(3)}$ in the **56**.
(NS,NS): a scalar $\Phi^{(0)}$ in the **1**, an anti-symmetric two-form $B^{(2)}$ in the **28** and a traceless symmetric two-form $G^{(2)}$ in the **35**.

(NS,R): a chiral spin-$\frac{1}{2}$ spinor in **8** dimensions and a spin-$\frac{3}{2}$ field of **56** dimensions.
(R,NS): idem as the (NS,R) but with opposite chiralities.
Thus this is a theory with two gravitinos, and this can only be understood from a supersymmetric gravitational theory with two supercharges. Hence we are dealing with a $\mathcal{N} = 2$ supersymmetry.

Objects:

The superstring, or fundamental string (F-string). The free objects are closed but can get broken into open strings in the presence of D-branes.Inspired by the coupling of the point particle to the electromagnetic field, we can write down the coupling $\int_{\mathcal{M}_F} B^{(2)}$.
Focussing on the (R,R) sector, an F-string couples to a D-brane. In the Type-IIA, the F-string couples electrically to the 0-brane and the 2-brane due to the $C^{(1)}$, $C^{(3)}$ respectively. It couples magnetically to a 4- and 6-brane due to the $\star C^{(3)}, \star C^{(1)}$ respectively. Here $\star$ denotes the Hodge-star, and in ten dimensions, $\star C^{(n)}$ is a $(8-n)-$form. There is room for defining an 8-brane. This would require a 9-form $C^{(9)}$ with fieldstrengt $F^{(10)} = dC^{(9)}$. Such a ten-form is static in the full ten dimensions, but nevertheless can get interesting when embedding the IIA theory in a higher-dimensional manifold.
For reasons explained when delving deeper into IIB-theory and *dualities*, we also have an object dual to the F-string that couples to the (NS,NS) dual $B^{(2)}$ form $\star B^{(2)}$: the $NS5-$brane.
When writing down the solutions of effective Type-IIA actions, we also encounter gravitational wave-like objects (GW) and their dual objects, Kaluza-Klein monopoles (KK).

Type-IIB

Fields:

The field-content of the (NS,NS) sector is the same as in the type II-A. Also, the (NS,R) and (R,NS) sector are the same with the difference that they have the same chiralty. The big difference is seen in the (R,R) sector:
(R,R): a scalar $C^{(0)}$ in the **1**, an anti-symmetric $C^{(2)}$ two-from in the **28** and

a self-dual anti-symmetric $C^{(4)} = \star C^{(4)}$ in the $\mathbf{35}$. It is interesting that in this theory, some (R,R) and (NS,NS) fields have the same dimensions. By times it will prove handy to arrange them in doublets, a construction we will not yet pursue at this point.
Here too we are dealing with an $\mathcal{N} = 2$ supergravity (sugra), which is different from the IIA sugra due to the opposite relative helicity structure.

Objects:

The different objects are of course due to the (R,R) sector. The F-string can now couple to a (-1)-brane, which is an object solely localized in time, hence interpreted as an instantonic mode. The coupling is electric with the $C^{(0)}$ scalar. We also encounter a 1-brane and a 3-brane, with electric couplings with the $C^{(2)}$ and $C^{(4)}$ respectively. The 3-brane is a bit peculiar because of the self-duality of the form it is coupled to. It plays an important role in for example Ads/CFT duality. Analogously to the II-A case, we also have a 5- and 7-brane with magnetic coupling to the $\star C^{(2)}$ and $\star C^{(0)}$ respectively. These two objects are also interesting in their own. We will come across the 7-brane in the topic of F-theory and will discuss 5-branes when investigating the full spectrum of M(atrix)-theory. Here too we can define a 9-brane, even tough it's field-strenght is not defined in ten dimensions. Yet again, if for example we would like to embed II-B theory into say an 11-dimensional theory, the 9-brane would couple to a (static) eleven-form.
The IIB also has an $NS5-$brane. Also in the II-B spectrum we encounter gravitational waves, and their KK monopoles

Type-I

Fields:

Recall that the Type-I is constructed from Type-IIB by projecting out all the content that has negative eigenvalue of the projection operator $P = \frac{1}{2}(1+\Pi)$. First of all, P maps the IIB (NS,R) sector to the (R,NS) sector and vice-versa, so they alone will not survive, but if we build a sector (NS,R)$\oplus$(R,NS), defined by taking the linear combinations of the field-content, this and only this will end up in Type-I. This reveals the $\mathcal{N} = 1$ susy structure. Now from the (NS,NS) sepctrum, we see that P acts odd on the two-from $B^{(2)}$ and is projected out, while in the (R,R) sector we have to dismiss $C^{(0)}$ and

$C^{(4)}$. In summing up, we keep the **1**,**8**,**28**,**35**, but all with multiplicity of one.

Objects:

Projecting out all these fields is accompanied by a great loss of objets. What remains is an $F-string$, but in the Type-I it is not orientable, due to the $\mathbf{Z}_2$-projection, and it's open. Moreover, looking at the remaining Ramond-fields, we see that only the D1-brane and the D5-brane survived the projection. Also the NS5-brane has disappeared.

HETEROTIC $SO(32)$ AND $E_8 \times E_8$

We now have fermions only on the right-moving sector, so we have a single NS or R branch. Here we already see how heterotic strings will be $\mathcal{N} = 1$ susys. We distiguish between the zero modes in the de fact physical ten dimensions, and the ones on the sixteen extra's, which are embedded on a self-dual compactified 16-torus.
In the physical dimensions we have again in the NS sector a two-form $A^{(2)}$ splitting up in a scalar (**1**), an antisymmetric part $B^{(2)}$ (**28**) and a traceless symmetric part $G^{(2)}$ (**35**). In the Ramond we have vector fermion consisting of a one chiral fermion array in the **8** and the spin-$\frac{3}{2}$ gravitino in the **56**.
Furthermore, on compactified dimensions we must distuish between two representations, where the difference between the two heterotic theories is defined. As it does not concern us in this document and the spectrum is a bit less straightforward I will only state here that the Ramond sectors of both heterotic theories are equal and have left-moving fermions in the $\mathbf{16} \times \mathbf{8}$ and $\mathbf{480} \times \mathbf{8}$, and that the difference is in the NS sector, where we have one eight-component vector in the $\frac{1}{2} \cdot 32 \cdot 31$ dimensional representation of either $SO(32)$ or $E_8 \times E_8$.

1.1.1 Effective Actions

Having established the field-content of the theories it will be intstructive for averything that follows to think about the actions that are associated with these fields. We should keep in mind that these actions only refer to the massless level of the spectrum and thereby make sense only in the low-energy regime. But since string theory has a mass-gap of order $\frac{1}{l_s^2}$, where l_s dentotes the string lenght, it is exact in the window $[0, 1/l_s^2]$. One note about the string length: the basis paramter of relativistic string theory is the string tension - the mass per unit length if you like - which is related to the Regge-slope. Originally, string theory was formulated to model Regge-trajectories of mesonic interactions, where the Regge-slope measures the ratio of spin and mass of meson collisions. Anyway, the tension, T, of the string theory is related to the string length l_s and the Regge-slope α' trough $T = \frac{1}{2\pi\alpha'} = \frac{1}{2\pi l_s^2}$. Also string theory has it's string couling constant g_s which measures the strength of string self-interactions in string Feynman diagrams. As string theory models gravitons as massless exitations of closed strings, Newton's constant G_N is a paramter set by g_s.

Starting with the Type-II theories, we note that the actions for the (NS,NS) sectors are the same. It is in fact straightforward to write it down:

$$S_{NS}^{II} = \frac{1}{4\pi\alpha'} \int d^2\sigma\sqrt{h}\Big(G^{(2)} + B^{(2)}\Big)\partial X\partial X + \alpha' R\Phi^{(0)} =$$
$$\frac{1}{4\pi\alpha'} \int d^2\sigma\sqrt{h}(h^{ab}G_{\mu\nu} + i\epsilon^{ab}B_{\mu\nu})\partial_a X^\mu \partial_b X^\nu + \alpha' R\Phi^{(0)} \tag{1.0}$$

In this action, h denotes the world-sheet metric and R the world-sheet Ricci-scalar. Note that the dilaton contribution is of an order in α' higher then the two-form fields. Furthermore it is comforting that variations in the fields X proportional to Killing vectors leave the action invariant. This is the action for the worldsheet of the $F-string$, where it's coupling with $B^{(2)}$ is manifest. It might be interpreted as a direct generalization of the Polyakov action, and in fact reduces to this if $B^{(2)} = 0$ and the dilaton vanishes everywhere.

To set up an action involving the (R,R) fields, we should keep in mind that these fields couple to higher dimensional objects, anything up top the 8-brane. Hence we would like an action that described the field-content in the full ten-dimensional space-time. We can describe integrate low-dimensional forms on target-space by means of the pull-back map. In case of the (NS,NS) anti-symmetric two-form we might for example take $\phi^* B_{ab} = B_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$, as we already implicitly did with the generalized Polyakov.

The (R,R) sector has, for the full Type-II theory, form-fields

$$\begin{aligned} \mathcal{C} &= C \bigcup \star C \\ C = \{C^{(0)}, C^{(1)}, \ldots, C^{(4)}\},\ \star C &= \{\star C^{(0)}, \star C^{(1)}, \ldots, \star C^{(4)}\} = \{C^{(8)}, \ldots, C^{(4)}\} \end{aligned}$$

When explicitly writing down the action, pulling everything back to target-space, for the field-strengths $G^{(n+1)} = dC^{(n)}$ a symmetric factor $\frac{1}{2(n+1)!}$ is convenient. Now the (R,R) sector action should at least have:

$$S_{IIA} = \int d^{10}x\, \Sigma_{n=1}^{2} \frac{1}{2(2n-1)!} dC^{(2n-1)} \wedge dC^{(2n-1)} + \int d^{10}x \frac{1}{4}(G^{(2)})^2 + \frac{1}{48}(G^{(4)})^2 \qquad (1.\text{-}2)$$

Like in ordinary field theories, we might supplement this with a Chern-Simons term, something proportional to for example

$$\int B^{(2)} dC^3 dC^3$$

Indeed such terms will only contribute on a (non-trivial) boundary. Here I make a distinction between $G^{(4)}$ and $dC^{(3)}$ because in the full picture, $G^{(4)} = dC^{(3)} + dB^{(2)} \wedge C^{(1)}$, which is a result of dimensional reduction from eleven to ten dimensions, a construction which will be discussed later on.
Now the target-space field-equations for the (NS,NS) sector deserves some attention. As above, we wil denote the field-strength of $B^{(2)}$ as $H^{(3)} = dB^{(2)}$. Furthermore we demand that the dilaton obeys a covariant Klein-Gordan wave equation, hence introducing an action proportianol to $S_\Phi \propto \int d^{10}x(\nabla\Phi)^2$. Furthermore we take the same symmetry-convention for the field-strecht as in the (R,R) sector, thus the $B-$field generates a term $S_B \propto \int d^{10}x \frac{1}{12}(H^{(3)})^2$. And the graviton is, just as in General Relativity, implemented in the field-theory by means of the Ricci-scalar. We recall that the string couling constant is related to the dilaton v.e.v. throug: $g_s = e^\Phi$. We nog can write down the full bosonic type-IIA Supergravitational action[1]

$$S_{IIA} = \frac{g_s^2}{16\pi G_N} \int d^{10}x \sqrt{-G} \big[e^{-2\Phi}(R + 4(\nabla\Phi)^2 - \frac{1}{12}(H^{(3)})^2) - \frac{1}{4}(G^{(2)})^2 - \frac{1}{48}(G^{(4)})^2 \big] \qquad (1.\text{-}2)$$

What is most interesting and non trivial is that in the massless regime Type-IIA *is* an ($\mathcal{N} = 2$) supergravity (sugra) in ten dimensions. In fact, from the study of sugra's in ten dimensions we know that there are five distinct of such sugra's. One might be tempted to try and match every sugra with a superstring theory partner but this can not be done: one of the five sugras

[1]We adapt numerical and sign conventions of Bergshoeff et. al. [13]

can not be the action of a consistent string theory, and one action serves in a way as the effective action of both the Type-I string and the Heterotic SO(32) theory. This is a strong hint that, at least in the low energy regime, these two theories can be mapped into each other.
Likewise we van construct the supergravity action for the type-IIB case, with the subtlety that the four-form and its associated field-strength are self-dual. We will at this point exploit the fact that some of the dimensions of the fields of the NS and R sector are the same and place them en doublets. Let us thus define:

$$B_D = \begin{pmatrix} B^{(2)} \\ C^{(2)} \end{pmatrix}, \quad H_D = dB_D$$

And we introduce the important modular parameter

$$\tau = C^{(0)} + ie^{-\Phi}$$

For the scalar content of the type-IIB, we construct the matrix

$$\mathcal{M} = \frac{1}{\Im t} \begin{pmatrix} |\tau|^2 & -\Re\tau \\ -\Re\tau & 1 \end{pmatrix} \tag{1.-2}$$

This matrix is constructed such that the term $\frac{1}{4}Tr[\partial_\mu \mathcal{M} \partial^\mu \mathcal{M}^{-1}]$ gives the correct contribution to the scalar part of the action, both NS and R. Now the total two-form contribution can be written down in terms of the doublet defined above and this scalar matrix. The total action becomes:

$$S_{IIB} = \int d^{10}x\sqrt{-G}\big(R - \frac{1}{12}H_D^T \mathcal{M} H_D + \frac{1}{4}Tr[\partial_\mu \mathcal{M} \partial^\mu \mathcal{M}^{-1}]\big) \\ - \frac{1}{480}(G^{(5)})^2 \tag{1.-2}$$

Note the extra factor of one-half in front of the five-form which is due to its self-duality. Again, the five-form $G^{(5)}$ should be supplemented with an extra Ramond-NS mixed five-form, $G^{(5)} \to dC^{(4)} + B^{(2)} \wedge H_D$, which also finds it origin in a dimensional reduction. Again we did not write down the Chern-Simons term.

1.1.2 The Symmetries of IIB

Let's look closer to the sugra action of Type-IIB and its symmetries. In the form as written above, we see how the action S_{IIB} is invariant under de group op Möbius transformations,

$$SL(2,\mathbf{R}) = \{\Lambda|\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, ab-cd = 1, a, b, c, d \in \mathbf{R}\}$$

. $SL(2,\mathbf{R})$ acts on the theory as:

$$\Lambda \circ \mathcal{M} = \Lambda \mathcal{M} \Lambda^T, \quad \Lambda \circ H_D = \Lambda^{-T} H_D \tag{1.-2}$$

Thus we established full $SL(2,\mathbf{R})$-invariance of the low-energy Type-IIB theory. One specific element of these Möbius transformations generates a strong-weak, so-called S-duality, namely $\Lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ This element acts on τ as $\tau \mapsto -1/\tau$ and and on the fields as

$$H_D = \begin{pmatrix} B^{(2)} \\ C^{(2)} \end{pmatrix} \overset{S}{\mapsto} \begin{pmatrix} -C^{(2)} \\ B^{(2)} \end{pmatrix}.$$

Taking this together and recalling that the F-string couples to the NS two-form and the D-1 brane to the D1-brane, this result implies that F-strigs at strong couling must behave as D1-branes in weak couling and vice versa. The same must carry over the the dual objects - the NS5-brane and the D5-brane. The D3-brane plays a special role, because the self-dual four-form is invariant under S, and the D3-brane looks like a fixed point under this map.

With this $SL(2,\mathbf{R})$ symmetry at hand, we may wonder how it acts on the objects. We made some general statemenst on dualistic behavious, lets take it a but more serious.

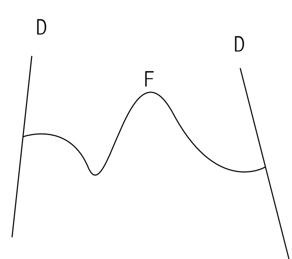

Figure 1.1: *In a picture it is easy to pick out the D- and F-string. But is there a symmetry that relates these to similar objects?*

We saw how the F-string is the propagtor of the $B^{(2)}$ NS form, and let it have one unit of charge, denoted as $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ Likewise, let de D1-brane carry one R-charge, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ Or, in general,

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \#F \\ \#D \end{pmatrix}$$

We may boldly propose that this doublet of IIB objects transforms under the $SL(2,\mathbf{R})$ or, for this matter, the integer subgroup $SL(2,\mathbf{Z})$. The presence of such D-strings breaks the symmetry group

$$SL(2,\mathbf{R}) \to SL(2,\mathbf{Z}).$$

Note that the charges of the F- and D-strings themselves are integer, since the theory admits in mobth sections dual 5-branes carrying magnetic charges, giving a generalized Dirac quantization condition.
First of all, in case of the F-string, we generally have:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \stackrel{SL(2,\mathbf{Z})}{\to} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$$

Now a well educated guess might be that the D-string is the $SL(2,\mathbf{Z})$ partner of the F-string. This is quickly justified by noting that indeed the D-string has not got NS-charge, and one R-charge, and for the F-string it is vice versa. Moreover, the bosonic degrees of freedom of the two objects equal, since it are precisely the massless modes of the F-string ending on a D-string that generate these degrees of freedom, that is oscillations of the D-string are generated by the degrees of freedom of the F-string. Supersymmertry requires both the fermionc degrees of freedom to be the same too.
Now lets fouces our attention to the most general string doublet, with m F-strings and n D-strings. For the sake of argument we will focus on the case with $m > (n-1)$ and $gcd(m,n) = 1$. The requirement $m > (n-1)$ is needed s that every pair of D-strings can get connected by a string, and we take m and n relatively prime to focus on stable states only. Were there a greatest common divisor, then the configuration might split up in two pathces or sectors, equal to this commong divisor. We want to rule this possibility out.
Now as the branes are tied up by strings, their separation length will tend

to get small, in fact of the order of the string-lenght scale. We must know what happen with Dp-branes in general at such scales and beyond.
For that matter, lets stack up N Dp-branes, parallel, and lets have them approach each-other. Let us denote coordiantes of the Dp-brane as $\{x^0, x^1, \ldots, x^p, \bar{x}^{p+1}, \ldots, \bar{x}^9\}$ where a bar over a coordinates means that the coordinate is identified with a Dirichelet boundary condition, that is it is constant, and for one Dp-brane we might as well set $\bar{x} = 0$.
Now each of these Dp-branes naturally carries a $U(1)$ gauge-group, so the total symmetry for a stack of N Dp-branes is $U(1)^N$. As these branes approach, as indeed is the case when they are part of a network and tied up by strings, letting the string length go to zero, this gauge-group is enhanced to $U(1)^N \to U(N)$, and we are in the case of a field theory of $9-(p+1)$ gauge fields Φ_j with precisely this gauge-group. We know how to write down a field-theory for such a configuration: it is a Yang-Mills theory with $U(N)$. In any such theory the fields are subject to a self-interaction:

$$L_{self} = g_{YM}^2 \sum_{i,j=p+1}^{9} [\Phi_i, \Phi_j]^2 \tag{1.-2}$$

This contribution vanishes in its vacuum, when both Φ_i, Φ_j are in the center of the general linear group, $\Phi_i, \Phi_j \in C(GL)$, and this centrum is generated by the diagonal matrices. Thus the ground state is completely described by $9-(p+1)$ N-dimensional diagonal matrices. Thus one such field equals

$$\Phi_j = diag(\phi_j^1, \phi_j^2, \ldots, \phi_j^N)$$

But for each field Φ_j we also have N degrees of freedom describing the $j-$th coordinate of the N branes. So up to permutation at least, they must be the same, that is, the diagonal matrices describe the coordinates of the branes, and we rename $\Phi_j \to X_j$. The other way around: The coordinates of the stack of branes get promoted to matrices, where in the ground-state this interpretation is really straightforward. The interesting and central question of this document will be how to interpret the fields as they are not in their ground-state, and off-diagonal terms will emerge.
Now an $U(N)$ gauge group behaves in many ways as a direct product of a $SU(N)$ and a $U(1)$. Observe that indeed in the vacuum, the center of mass coordinate is proportional to the trace TrX^j and this part is described by the $U(1)$ mentioned above. This center of mass position should be arbitrary,

as we'd like a global translation invariance. There is here some similarity with 't Hooft's large N conjecture which states that an $SU(N)$ Yang-Mills gauge theory converges towards a string theory as N goes to infinity. 't Hooft was led to this conjecture by inspecting Feynman diagrams for the $SU(N)$ case (a quark theory with N colors) and observed how in the large N limit only diagrams with spherical topology survived, with a 't Hooft Yang-Mills coupling proportional to $1/N$, a first order correction coming from torus-like topologies. In the case at hand, we are actually describing a Yang-Mills theory in $SU(N)$ gauge, identifying it with the study of D-branes, but we will touch this topic into more detail later on.

1.2 T-duality

We may wonder wether the theories described above are really distinct, or actually in some way different realizations of the same under- or overlying theory. There is a lot of strong evidence that the latter statement indeed is the case, something which is very apparent when investigating the Type-II theory with some more scrutinity, and trying to make contact between the A- and B-side.

1.2.1 T-duality from the states

First of all, The occurrence of the Dp-brane found it's origin in a duality in the bosonic string. When trying to make contact with reality, we may decide to compactify a (spatial) direction, that is choose one dimension to be projected onto a compact manifold with some characteristic length-scale. The easiest example of such a manifold is a circle, but in trying to compactify more spatial dimensions at once we can come across more exotic manifolds. Hence we take $X^i \equiv X^i + 2\pi R$ where R^i denotes the radius of the circle. Now in this compactified picture evaluation of the spectrum shows a discrete symmetry taking $R^i \leftrightarrow 1/R^i$ while simultaneously swapping the number of times the string wraps itself around the radius, n, with the quantum of momentum w - a quantum number which is immediately obtained due to the compactification.

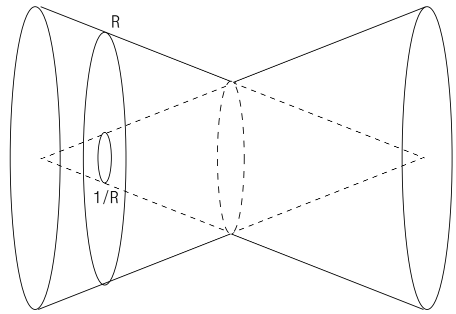

Figure 1.2: *The T-dual map. The left-and right outer circles represent circles of infinite radius (the vertical axis is scaled hyperbolically). The smaller circle on the left is a typical compactification circle with it's smaller, dual circle inside. In the middle is the self-dual circle. By duality we might as well focus on the left half of the cone-like figure above*

Taking this *T-duality* - $R \leftrightarrow 1/R^i, n^i \leftrightarrow w^i$, reverses the sign of the rightmovers: $X_+^i \mapsto -X_+^i$. We can construct the fermionic picture thus, as to preserve full supersymmetry of the full action, by simultaneously demanding that $\chi_+^i \mapsto -\chi_+^i$. Since, rather peculiarly, the zero mode in the Ramond sector of this chiral spinor described the Clifford-algebra, we have for the full Clifford-algebra under T-duality:

$$\begin{aligned} \Gamma^\mu &\xrightarrow{T} \Gamma^\mu \quad (\mu \neq i) \\ \Gamma^i &\xrightarrow{T} -\Gamma^i \\ \prod_{k=1}^{9} \Gamma_k \equiv \Gamma^{11} &\xrightarrow{T} -\Gamma^{11} \end{aligned}$$

As the chirality is the eigenvalue of the Γ^{11} we see how T-dulaity reverses chirality since $P_\pm \xrightarrow{T} P_\mp$. Now the difference between the Type-IIA and Type-IIB theory lies in the choice of relative chirality between the left- and rightmovers. IIA has opposite chirality, while IIB has the same chiralty. We choose things so as to take the left sector with a positive chirality for both theories, and only choosing specific right-moving chiralty. As the projection-operator $P_\pm = \frac{1}{2}(1 \pm \Gamma^{11})$, acting on the right-moving sector, changes sign, we see how a T-duality, on the abstract level for sure, gives

$$\text{Type-IIA} \overset{T}{\leftrightarrow} \text{Type-IIB} \tag{1.-5}$$

The flip in the Γ^{11} plays a rather central role in the T-duality. Take for instance Type-IIA, where we hace a left chiral spinor times a right chiral spinor both in the **8** but in total: $\mathbf{8}_L \times \mathbf{8}_R$. To make the Clebsch-Gordan decomposition we make spinor bi-linears from the full algebra generated by the gamma's (that is, $\{1, \Gamma^{\mu}, \Gamma^{\mu_1\mu_2}, \ldots, \Gamma^{11}\}$).
With some abusive notation we may write $\mathbf{8}_L \times \mathbf{8}_R = |\chi_R\rangle\langle\chi_L|$.

Thus we notice how the Dp-branes IIA are sent to D(p-1)-branes IIB, if the compactification is done parallel to the D-brane. Let's get this listed up in a table:

	Parallel T	Transverse T
Dp-brane in IIA	D(p-1) in IIB	D(p+1) in IIB

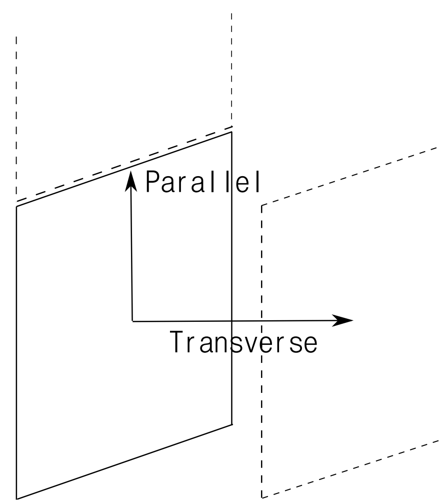

Figure 1.3: *A sketch of the distinction between a parallel and a transverse T-duality. The vectors have a length of* $2\pi R$*, with* R *the radius of compactification*

1.3 M-theory

Already at this point we have seen some hints that not only the string theories are related by dualities, but that they might in fact be embedded into some bigger thing. The first hint was the suspected presence of a 9- and 8-brane. But also the dilaton is quite suspect. We have some reason to suspect the dilaton to be some kind of a Kaluze-Klein degree of freedom, emerging from a compactification from a higher manifold of extension. Also we noticed how IIA and IIB are $\mathcal{N} = 2, D = 10$ sugra's, while it is a famous statement in sugra that the highest dimension for which such a sugra is consistent is D=11.

We will give some general remarks on such an 11-D sugra, and will look at what happens when we compactify the extra dimension. We will try and identify the compactified objects with string objects.

First of all we are to have a graviton in $D = 11$. This is represented by the $SO(9)$ and has the **44**. Furthermore as a SUSY the graviton needs a super-symmetric fermionic partner, the gravitino Ψ_M. It should obey a Dirac field equation and hence it enters the action as:

$$S \propto \int d^1 1x \bar{\Psi}_M \Gamma^{MNP} \partial_N \Psi_P$$

Here $\Gamma^{MNP} = \Gamma^{[M}\Gamma^N\Gamma^{P]}$ with Γ an element of the Clifford algebra in eleven dimensions. This anti-symmetric three-field is to be augmented with an anti-symmetric three-form $A^{(3)}$ to preserve supersymmetry. As a gauge-field, $A^{(3)}$ is defined up to a two-form, as has Field-strength $F^{(4)} = dA^{(3)}$

Now the unique 11-D sugra has an bosonic action:

$$S = \frac{2\pi}{(2\pi\ell)^9} \int d^{11}x\sqrt{-G}(R - \frac{1}{2 \times 4!} F_{\mu_1\mu_2\mu_3\mu_4} F^{\mu_1\mu_2\mu_3\mu_4})$$

Where the lengthy tensor-product should be read as $(F^{(4)})^2$.

Let's dare and interpret this action as one of some kind of 11-dimensional string theory. What is most interesting is that the only form that contributes to the action, is a 3-form, which couples electrically to a two-dimensional surface - the M2-brane -, and magnetically to a five-dimensional surface - the M5-brane.

What we'd like to see is how to construct the Type-IIA and -IIB theories by compactifying such an 11-dimensional theory. We ask ourselves what happens if we take the extra coordinate of the 11-D sugra to by periodic, that is take $x^{11} \to x^{11} + 2\pi R_{11}$. Intuitively we can predict the interpretation of an F-string in such a compatified field. Namely, let an M2-brane have one of its coordinates stretched out along the axis of x^{11}, that this compactifies to a string, like in figure (2.1):

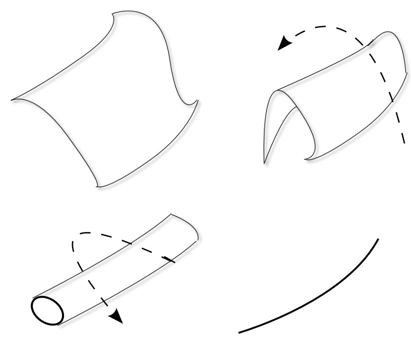

Figure 1.4: *An M2-brane compactifies to a string*

This way construction is easily extend to other objects: the D2-brane gets the interpretation of an unwrapped M2-brane in the compactified 11-D theory (an M2-brane that did not lay along the axis of compactification), and the D4-brane is like the M2-case, an M5-brane that has one dimension wrapped into a small compact manifold. The unwrapped M5-brane becomes, in the 10-D, an NS5-brane. This way we see how a lot of the objects of Type-IIA theory come into play. We are only looking for interpretation of the Dp-branes for $p = 0, 6, 8$. These constructions are a bit more technical, although the D0-brane is rather straightforwardly interpreted as a so-called Kaluza-Klein state - a degree of freedom is a residue in compatification schemes.

The statement that, upon compactification on a circle, IIA becomes IIB, translates naturally to M-theory in the following array:

$$\begin{array}{rl} \mathrm{M} \overset{\mathbf{S}^1}{\to} \textsc{Type-IIA} \overset{\mathbf{S}^1}{\to} & \textsc{Type-IIB} \\ & \Downarrow \\ \mathrm{M} \overset{\mathbf{T}}{\to} & \textsc{Type-IIB} \end{array}$$

Here $\mathbf{S^1}$ is the circle and $\mathbf{T}$ the torus. From this point of view the symmetries of IIB become rather appealing: they become equivalent with the

symmetries of the torus. To see this, note that the torus is parameterized with a $\tau = \tau_1 \mathbf{x_1} + \tau_2 \mathbf{x_2}$ with $\mathbf{x_{1,2}}$ linear independent normal vectors. We can map this onto a complex torus taking $\tau \rightarrow \tau_1 + i\tau_2$. Both parameterizations have the following symmetries: $\tau \mapsto \tau + 1$ where we normalized the radii of the torus, and $\tau \mapsto -1/\tau$. The group generated by these operations over an arbitrary field $\mathbf{F}$ is precisely the group $SL(2, \mathbf{F})$. We can even see how the $SL(2\mathbf{Z})$ dual of the F-string is indeed the the D-string as follows:
Let the targetspace coordinates of 11-D M-theory be $\{x_0, x_1, \ldots, x_9, x_{11}\}$ where we skip x_{10} and denote it with x_{11} to remind us that this is the 11-th M-direction. Now consider again an M2-brane with one of its sides stretched out along the x^{11}, and one along the, say, x^1 coordinate
Upon a reduction of the 11-th coordinate this becomes the IIA-string, and after a T-duality along the 9-th direction this becomes the IIB-string.
On the other hand, suppose the M-brane was strechted out along $\{x_1, x_9\}$ and we would apply the same reduction scheme. Then after the x^{11} reduction we would end up with and IIA D2-brane, and after the T-duality along x^9 we would find the IIB D1-brane. In summing up:

$$\begin{aligned} \mathrm{M2}_{1,11} &\stackrel{S^1_{11}}{\rightarrow} \text{IIA F-STRING} \stackrel{S^1_9}{\rightarrow} \text{IIB F-STRING} \\ \mathrm{M2}_{1,9} &\stackrel{S^1_{11}}{\rightarrow} \text{D2-BRANE} \stackrel{S^1_9}{\rightarrow} \text{D1-BRANE} \end{aligned} \tag{1.-8}$$

Where the last line is equivalent to:

$$\mathrm{M2}_{1,11} \stackrel{S^1_9}{\rightarrow} \text{D2-BRANE} \stackrel{S^1_{11}}{\rightarrow} \text{D1-BRANE} \tag{1.-8}$$

Thus the two lines in 1.-8 are related by a flip $x^{11} \leftrightarrow x^9$. Thinking about IIB as a direct compactification from M-theory on a torus, this is the modular switch $\tau = \tau_1 + i\tau_2 \leftrightarrow \tau_2 + i\tau_1$ which is a simple reparameterization of the torus, in fact an $SL(2, \mathbf{Z})$ transformation. Now we see how indeed the D-string is the IIB $SL(2, \mathbf{Z})$ dual of the F-string.
Now actually the bound state (m, n) of m charges of the F-string and n charges of the D-string, gets a clear interpretation. It is the M2-brane, wrapped m times around the x^{11} and n times around the x^9.
This wrapping procedure also yields that the D3-brane in IIB must behave as a fixed point under the $SL(2, \mathbf{Z})$, because an D3-brane is constructed by taking the M5-brane and wrapping two of its directions on the torus. In this procedure, it doesn't matter what modular transformation we apply to the torus of compactification, as the D3-brane is wrapped on both moduli of the

torus.

1.4 F-theory

This construction of making the IIB by wrapping M-theoretic coordinates on a torus, has the disadventage that it does not reveal the "pre-theory" of full ten dimensional Type-IIB. The symmetries of IIB have a beautiful geometric generalization known under the name of F-theory. It think the F here stands for Fibercation, but I couldn't find evidence for this. Its appearance follows most naturally when seeking 7-branes in the $SL(2\mathbf{Z})$ duality of Type-IIB. To this ends we introduce a set of coordinates: $\{x^0, x^1, \dots, x^7, z\}$, $z = x^8 + ix^9$. Reconsider the sugra action of Type-IIB and take in account only the scalar fields:

$$S_{scalar} = \frac{1}{k^2} \int d^{10}x\sqrt{-G}(R + \frac{1}{4}Tr[\partial_\mu \mathcal{M} \partial^\mu \mathcal{M}]) = \frac{1}{k^2} \int d^{10}x\sqrt{-G}(R - \frac{\partial\tau\bar{\partial}\bar{\tau}}{2(\Im\tau)^2}) \qquad (1.\text{-}8)$$

Variation with respect to $\bar{\tau}$ yields the equation of motion

$$\partial\bar{\partial}\tau + \frac{2\partial\tau\bar{\partial}\tau}{\bar{\tau} - \tau} = 0 \qquad (1.\text{-}8)$$

A reasonable Ansatz is to look for holomorphic functions $\partial\tau = 0$, for example

$$\tau(z) = \frac{1}{2\pi i} Ln(z - z_i).$$

Before we proceed, recall the way $SL(2, \mathbf{Z})$ acted on τ. We have noticed how this Möbius group was freely generated over $\mathbf{Z}$ by the two generators defined by $S : \tau \mapsto -1/\tau, T : \tau \mapsto \tau + 1$. This defines a region in the complex plane of inequivalent tori, called the fundamental domain.

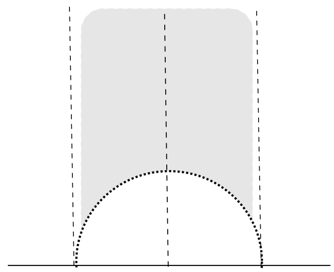

Figure 1.5: *The shaded area is* $\mathcal{F} = \{z \in \mathbf{C} || z| \geq 1, -1 < \Re z < 1\}$

Now to return to the scalar action, and its Ansantz recall that the R-potential $C^{(0)}$ is electrically coupled to the D-instanton, and magnetically to the D7-brane. Furthermore we took this scalar in a doublet with the dilaton, and previously we assumed the dilaton to be constant over the manifold. We now allow the dilaton and hence the modular parameter to vary over the base. With these idea in mind lets try to make the ideas a bit more rigorous.

Let $\mathcal{B}$ be a base manifold of real dimension d and let $\mathcal{M}$ be the eliptically fibered manifold obtained by adjoining a torus $\mathbf{T}^2$ at every point on $\mathcal{B}$ ($dim(\mathcal{M}) = d + 2$). Of course, $\mathcal{B}$ will be the manifold on which a IIB theory will be projected while $\mathcal{M}$ will play the role of the manifold on which the envelopping theory is to be defined.
Endow $\mathcal{B}$ with a coordinate system $\vec{z}$ and let, as in the above, the modular structure of $\mathbf{T}^2$ be denoted by $\tau(\vec{z})$ where $\tau \in \mathcal{F}$ is in the fundamental domain.
For this fibration to work we need the modular invariance to be manifest on the base manifold, that is:

$$\oint_\gamma \tau dz = g \cdot \tau$$

with $g \in SL(2, \mathbf{Z})$ and $\gamma \in \mathcal{B}$ a closed on the base manifold.
F-theory is defined as:
Type IIB string theory compactifed on $\mathcal{B}$ with $\lambda(\vec{z}) = \tau(\vec{z})$, with λ the scalar doublet of Type-IIB, $\lambda = C^0 + ie^{-\Phi}$.
For consistency, we need F-theory to be compatible with M-theory dualities

explained in the previous chapter. Recall that we constructed IIB from M-theory by compactifying the latter on $\mathbf{T}^2$. We obtained in this construction IIB on S^1 and regained the IIB itself in nine dimension by examining the limit where we let the radius of this $\mathbf{S}^1$ go to zero. But before this limit, the relation is summed up as:
M-theory on $\mathbf{T}^2 \leftrightarrow$ IIB on $\mathbf{S}^1$.
We want this equivalence to prevail. For this consider, in all its generality, F-theory on $\mathcal{M} \times \mathbf{S}^1$, that is IIB on $\mathcal{B} \times \mathbf{S}^1$. The above consideration asks for the equivalence:
F-theory on $\mathcal{M} \times \mathbf{S}^1 \leftrightarrow$ M-theory om $\mathcal{M}$.
Aiming not only for IIB-type dualities we more generally work with the following philosophy, where we write a caligraphic subscript to denote a manifold of definition:

$$
\begin{array}{rcl}
[\text{F-THEORY}]_{\mathcal{M}} & \leftrightarrow & [\text{STRING THEORY}]_{\mathcal{K}} \\
 & \Downarrow & \\
\text{M-THEORY}_{\mathcal{M}} & \leftrightarrow & [\text{STRING THEORY}]_{\mathcal{K}\times\mathbf{S}^1} \\
 & \Downarrow & \\
\text{TYPE-IIB}_{\mathcal{M}} & \leftrightarrow & [\text{STRING THEORY}]_{\mathcal{K}\times\mathbf{T}^2}
\end{array}
$$

The entry String Theory refers to any of the five admissable string theories. This buisines is indeed rather abstract. To make this program suitable for actual compactifications, an explicit construction is made for the fibration of the torus. As the topic of F-theory is beyond the scope of this document I will not go into this in great detail, but refer to e.g. [5]

1.5 Large N field theory and the Gauge/Gravity correspondance

One of the objectives of string theory is to get a better understanding of how gravity fits into the framework of physics as we know it. Microscopic physics is has a stunning exact decription in terms of quantum field theories, and still the most celebrated physical theory whith the highest predictive power is the Standard Model. As gravity has no role whatsover in the Standard

Model, we must adapt either our percetion of this quantum field theories, or our understanding of gravity. We cannot stick with the idea that one sector - the macroscopic - is described by General Relativity while the microscopic is a pure QFT, since this would for example violate general Heisenberg inequalites at an overlap region between te two sectors. String theory, altough not complete or completely rigorous, unifies quantum fields with the gravitational field.
There is a surprinsing result that relates (Conformal) field theories with string theories, known as Gauge/Gravity dualities. This idea really find its roots in a propsal made by 't Hooft in the early development of string theory: the large N limit of Yang-Mills theory.

In ordinary $SU(3)$ QFT - Quantum Chromodynamics (QCD) - it is extremely hard to do actual computations. The best we can do at the moment is to do so called lattice computations. That this is so hard and that it realies mainly on computational and algorimic skill is easily seen by looking at how this technique works. One namely makes local patches in space times discrete, for example one represents such a patch in terms of a lattice with N^3 points, and between any such points one leaves open the possibility of an interaction J_i. The configuration space one needs to model is thus of the order $2^{(N^3)}$, generally a number even a large computer has problems dealing with.
One would like to find another way of dealing with the problems, or even try and construct an approximation scheme. With these motivations, 't Hooft proposed to investigate genreal $SU(N)$ YM theories.
We think of $SU(N)$ QFT as a QCD with N colors. It admits one fundamental mass cut-off scale M_{QCD} and the interactions and vertices are weighted by the Yang-Mills coupling g_{YM}. As any parameter in QFT's, this couling depends on the typical energy of the system (it admits a runnig coupling constant).
QCD has some divergent one-loop diagrams such as the gluon self-energy. To regularize these divergences we should renormalize the QCD coupling. Briefly, this is done - with the aid of dimensional regularization - by:

$$g_{YM} = \mu^{-\epsilon/2} Z_g g_{s,Ren.}(\mu)$$

where $Z_g = 1 + \frac{1}{\epsilon}\alpha_{s,Ren.}\mu\frac{\beta_0}{4\pi} + \ldots$, $\epsilon = d - 4$ with d the total space-time dimension, and $\beta_0 = \frac{11}{3}C - \frac{2}{3}N$, C the adjoint Casimir of $SU(N)$ and N

the number of colors in the theory. The parameter $\alpha_{s,Ren.}$ obeys the beta equation

$$\mu\frac{d}{d\mu}\alpha_{s,Ren.}(\mu) = -\frac{\beta_0}{2\pi}\alpha^2_{s,Ren.}(\mu)$$

with a typlical solution

$$\alpha_{s,Ren.} = \frac{4\pi}{\beta_0 Ln(\mu^2/M^2_{QCD})}$$

where μ dentoes the typical energy scale and M_{QCD}. From this solution, asymptotic freedom is clear, even for the limit $N \to \infty$. This is the 't Hooft limit:

$$N \to \infty, Ng^2_{YM} fixed$$

so that the leading terms stay of the same order.
The Feynman rules are straightforward, and can be obtained by looking at the $SU(N)$ invariant Lagrangian. In terms of the two parameters declared in the 't Hooft limit, Vertices (V) are weighted with a factor of $\frac{N}{\lambda}$, propagators (E) with λN and loops (F) with a factor of N. So in total a typical Feynaman diagram had a couplling of $N^{V-E+F}\lambda^{E-V} = N^{(2-2g)}\lambda^{E-V}$, where g denots the number of handles of the manifold the Feynman diagram represents. Thus the diagram with spherical - $g = 0$ - topology are the most important in the 't Hooft limit. This diagrammatic limit is called planar. The next to leading order has the toroidal - $g = 1$ - topology, etcetera.
Indeed, this is in close analogy with the string theory, where the string Feynman diagrams have the same perturbative behaviour.

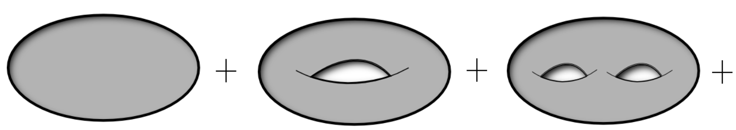

Figure 1.6: *The perturbative expansion of a string theory in terms of its geni of tori*

This leads to proposing that in fact a $SU(N)$ QCD becomes a string theory in the large N limit. We will see some evidence for explicit constructions

later on.
More generally one would like to know how to construct such dualities, and to know what kind of string theory we have on the other side of the field theory. A famous example of such an equivalence in the AdS/CFT correspondance[26], which originally proposed that a stack of N D3 branes has a dual descripiton in terms of a conformal field theory. More generally, lets look at a typical SU(N) supersymmetric Yang-Mills theory in, say, four dimensions. The supersymmetry is maximal for $\mathcal{N} = 4$ supercharges. We take this example becomes it has so many symmetries: it is in fact on of the scarce examples of a conformal theory with conformal group $SO_{4,2}$. We thus have $N^2 - 1$ gluons, four fermions and some six Higgs fields. Globally the fermions have an $SU(4)$ rotation symmetry.
Now to think of a dual string theory, we should preserve some sense of the conformal symmetry group $SO_{4,2}$. We could, for this purpouse, look for a Riemannian manifold $\mathcal{M}$ with a maximal AdS_5 metric (the group of isometries of this metric is indeed isomorphic to $SO_{4,2}$). So we try and project a string theory on a manifold $AdS_5 \times V$, with V some five dimensional manifold if we think of the original susy's. We could also think of a six-dimensional manifold if our purpose was to compare the CFT with an M-theory.
Now motivated by the compactification prcedures, it looks naturally to choose $V = \mathbf{S}^5$, a propolsal that finds back-up bearing in mind the $SU(4) \simeq SO(6)$ symmetry group on the ferminoc sector of the SYM.
This are hints that $mathcalN = 4$ SU(N) SYM might be the same as one of the five susy string theories on $AdS_5 \times S^5$, hence establishing a gauge/gravity duality. The surprising interpretation of the result is that the de facto physics on the string side is realized in the AdS_5, which is five-dimensional, wheras the CFT lives in four dimenisons.
The proper AdS/CFT correspondance is more explicit and is the first rigorous example of the Holographic Principle.

Chapter 2

Matrix theory

Whether or not M-theory is exact: it proves to serve as a fine and luminous guideline in getting the picture of the web of dualities. It is a bit like having a map of the world that works perfectly fine for all the places where you once have been, but you can't really tell whether it will be a good guide to the places you haven't seen yet.
Anyway, it would be sufficient to have some idea or feeling for the fundamental degrees of freedom of this theory, since for now it merely is an effective one. These degrees of freedom are expected to open up only at some very high energy scale, presumably around or below the string length-scale, where it is suspected that elementary concepts such as space and time may be different.
We have seen that for a stack of N D-branes, coordinates get interpreted as the diagonal entries of coordinate *matrices*. We discovered this while investigating the symmetries of a Type-IIB theory, arranging fields and scalars in doublets. Now let's run back to the arguments and reconsider the case for a D(-1)-brane, the instanton brane. Recalling the Yang-Mills action, we can write down the field theory for a collection of N such instantons, it is simply:

$$S = \sum_{i<j} [X_i, X_j]^2 + \sum_{i,a,b} \Gamma^i_{ab} Tr \chi^a [X^i, \chi^b] \tag{2.0}$$

We decided to include the coordinate-fermion interaction - which is readily taken up from the Yang-Mills reduction procedure. Now first of all, note how all the defining coordinates of the instanton are matrix-values. Amplitudes are, in this Euclidian measure, schemetcally of the form:

$$\mathcal{A} \propto \int [\mathcal{D}X][\mathcal{D}\psi]e^{-S} \tag{2.0}$$

We immediately see that the trivial vacuum, described by pairwise commuting matrices, is the only critcal point os S. A non-susy stable vacuum seizes to exist, because when scaling the fields $X \to \lambda X$, the action scales as $X \to \lambda^4 X$, so we always need $S = 0$ for critical points.
From the picture of M-theory, Type-IIB is obtained by compactifying the 11-D sugra on a torus. In order to get hold of some of the degrees of freedom of M-theory itself these instantons look promising. Now recall that at the same time IIB can be obtained by dualizing IIA on a circle, and to obtain the instanton, we need to take a D0-brane and compactify the time-like direction. As it is most instructive to think of M-theory as the strong limit of IIA, we are led to focussing out interest on these D0-branes, in way the precursors of the promising instantonic modes. Now we might boldly state that also in the IIA theory the D-branes are described by matrices, and the D0-brane is again fully matrix-valued. To make it even more bold, we could propose that these fully-matrix valued D0-branes stay matrix-valued in the strong M-regime, and there could play the role of fundamental degree of freedom, with an action described by a SYM-equation. This does makes sense as the D0-branes are part of the M-theory spectrum, where we recall that the IIA interpretation of these particles are in the Kaluza-Klein modes.
This is in fact the matrix proposal, some subtleties to be treated below beside.
To describe IIB fully in terms of a matrix quantum mechanics, where the particles are the instantons, is proposed in [28]. The top-down procedure mentioned above is built up in [7] and has led to the area sometimes referred to as M(atrix) theory. For now we will be interested in the latter.

2.1 M(atrix) Theory

Although at times taken implicitly, from now on we will describe target space in terms of *light-cone* coordinates
$\{X^-, X^1, \ldots, X^{D-1}, X^+\}$ and X^+ will serve as time.
With the remarks made above in mind, the Matrix proposal is as follows:

Full, uncompactified, eleven-dimensional M-theory is completely captured by a Matrix quantum-mechanics - Super-Yang-Mills (SYM) - theory of N D0-branes in the limit $N \to \infty$, in the infinite momentum frame (IMF)

This is a very strong statement and needs thorough checking. To begin with all the objects of M-theory need to be recovered from the SYM, and we will try and do so. Actually, all the Type-IIA objects need to be recovered by appropriate compactification of the SYM.
We will make this proposal somewhat more explicit by concretely setting up the Lagrangian and the Hamiltonian of the SYM, and then try and find the M- and string-theoretic objects it should encompass.
We give here the Lagrangian we will work with, that described precisely the dimensional reduced SYM stated above. The construction really hinges on the de facto reduction procedure and will be discussed in the next section.

$$L = \int \frac{1}{2R_{11}} Tr\Big[D_0X^iD_0X^i + \frac{1}{2}[X_i, X^j]^2 + \theta^T(i\dot{\theta} - \gamma_i[X^i, \theta])\Big] \qquad (2.0)$$

This is the Lagrangian of the BFSS Matrix Theory proposal. The X^i are eight scalar fields represented by $N \times N$ matrices. The fermionic fields θ^α and $\theta^{\dot{\alpha}}$ are likewise $N \times N$ matrices. This Lagrangian will be a guide for many that follows. As the theory is formulated in the IMF, we can formulate it in an Hamiltonain picutre as well. We will do so at times when we need this. In the Lagrangian, R_{11} is the M-theory radius of compactification that is to yield Type-IIA theory. The statement is that this Lagrnagian is to describe M-theory itself, so the limit $R_{11} \to \infty$ is to be taken, as well as $N/R_{11} \to \infty$. So in a way this is a large N field theory duality.

2.2 Dimensional reduction of a SYM

To describe this matrix quantum mechanics of a D0-brane or particle, we are to start with the full ten-dimensional SYM action and dimensionally reduce it all the way down to $1 + 0$ dimensions. We will start on the most general ground: a bosonic part decribed by the field-strenght, and a fermionic part that resembles an action which - upon variation - yields a Dirac equation:

$$L_{SYM} = L_B + L_F$$
$$L_B = -\frac{1}{4}\int d^{10}x Tr(F^2)$$
$$L_F = \frac{i}{2}\int d^{10}x Tr(\bar{\lambda}\Gamma \cdot D\lambda) \tag{2.-1}$$

where the bosonic trace is over the lie-structure indices and the fermionic trace over the spinor indices.
Now the procedure of dimensional reduction is straightforward. We expand both the field-strength's gauge-field and the spinor in a Fourier series:

$$A^a_\mu(x^\nu) = \sum_{(n)} A^a_{\mu,(n)}(x^0) exp(i\sum_j \frac{n_j x^j}{R_j}) \tag{2.0}$$

$$\lambda^a(x^\nu) = \sum_{(n)} \lambda^a_{(n)} exp(i\sum_j \frac{n_j x^j}{R_j}) \tag{2.1}$$

Here (n) is the vector of all quanta of compactification, $(n) = (n_1, \ldots, n_9)$. Reality of A is expressed in the condition $A^*_{(n)} = A_{(-n)}$ Now to plug the mode expansion into the fieldstrength we simply recall that $F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc}A^b_\mu A^c_\nu$. What comes into the expression is the term F^2, that is:

$$F^2 = 2(\partial_\mu A^a_\nu)^2 - 2(\partial_\mu A^a_\nu)(\partial^\nu A^{\mu a}) + 4f^{abc}(\partial_\mu A^a_\nu)A^{\mu b}A^{\nu c} + f^{abc}f^{ade}A^b_\mu A^c_\nu A^{\mu d}A^{\nu e}$$

Now just plug in the Foerier expansion, pick up a volume factor $(2\pi)^9 R_1 R_2 \cdots R_9$. Now each factor gets a mass factor $M^2_{(n)} = n^2_j/R^2_j$. We compactify on the nine-torus an shrink it down to zero volume at fixed shape and all these masses yield infinity for $(n) \neq 0$ so these modes decouple from the mass-gap, so we only need to take into account the zero-array. We end up with

$$-\frac{1}{4}F^2 = g^{-1}\int \left[(\dot{A}^a_{0(0)})^2 - 2f^{abc}\dot{A}^a_{\nu(0)}A^{0b}_{(0)}A^{\nu c}_{(0)} - \frac{1}{2}f^{abc}f^{ade}A^b_{\mu(0)}A^c_{\nu(0)}A^{\mu d}_{(0)}A^{\nu e}_{(0)}\right]$$

Here the reduced Yang-Mills coupling carries the toridal volume. Drop the zero-subscript and recall how $A^i = X^i, A^0 = A$ and introduce the covariant derivate $D_t = \partial_t + A$ to find

$$S_{bos} = -\frac{1}{g}\int \left[(D_t X^a_i)^2 + \frac{1}{2}f^{abc}f^{ade}X^b_i X^c_j X^{id}X^{je}\right] \tag{2.1}$$

which is in terms of matrix commutator:

$$S_{bos} = \frac{1}{g}\int Tr\Big[(D_t X_i)^2 + \frac{1}{2}[X_i, X_j]^2\Big] \tag{2.1}$$

with the covariant derivative in the adjoint, $D_t = \partial_t + [A, \cdot]$ The fermionic case is done is exactly the same way.

2.3 The M2-brane

One of the objects we need to recover is the brane that couples electrically to the sugra 3-form of M-theory, the M2-brane. Its construction from Matrix theory is rather entertaining, and is perhaps best motivated by starting with the M2-brane action itself, to try to recast is into a form that fits in the Matrix proposal.
For this, lets consider the world-volume action of the M2-brane:

$$S = -T\int d^3\sigma\sqrt{-\gamma}(\gamma^{ab}\partial_a X^\mu \partial_b X_\mu - \Lambda) \tag{2.1}$$

The shift Λ is included to pertain some sense of scale invariance.
First of all, variation with respect to γ gives as equation of motion for this metric:

$$\gamma_{ab} = \partial_a X^\mu \partial_b X_\mu \equiv h_{ab}$$

Note that in the metric γ^{ab}, the indices now run over $a, b = 0, 1, 2$ which will diminish our ability of gauge-fixing as many terms as in the Polyakov action for the string: we have six independent components for γ and only three symmetries: two diffeomorphisms and one scale-invariance.
First we will show that we can perceive this action in a symplectic way, writing it down in terms of Poisson-brackets. Then we will forge these brackets into Lie-brackets, and map them onto a matrix representation, yielding standard matrix-commutators. Although the procedure is straightforward it is not trivial.
First of all lets set $\gamma_{0i} = 0$, $i = 1, 2$. Furthermore take $\gamma_{00} = -\frac{4}{\nu^2}|\partial_i X^\mu \partial_j X_\mu| \equiv -\frac{4}{\nu^2}h$. This way the metric has the form:

$$\gamma_{\mu\nu} = \begin{pmatrix} -\frac{4}{\nu^2}h & 0 & 0 \\ 0 & h_{11} & h_{12} \\ 0 & h_{12} & h_{22} \end{pmatrix}$$

In terms of this gauge the action becomes simply

$$S - \frac{T\nu}{4} \int d^3\sigma (\dot{X}^\mu)^2 - \frac{4}{\nu^2}|h|$$

To obtain the symplectic form, we introduce

$$\{X, Y\} = \varepsilon^{ij} \partial_i X \partial_j Y \tag{2.1}$$

with ε^{ij} the totally antisymmetric tensor in two indices.

A straightforward calculation shows how these brackets allow one to express the action as

$$S = \frac{T\nu}{4} \int d^3\sigma \left[(\dot{X}^\mu)^2 - \frac{2}{\nu^2}\{X^\mu, X^\nu\}^2 \right] \tag{2.1}$$

The objective is to forge the bracket of functions into a commutator of matrices. This might smell like the quantum mechanical procedure of replacing the classical Poisson-brackets of functions on a sympletctic manifold with a defining Hamiltonian with operators in commutators, it has got nothing to do with that.
So we will look at functions X^μ on $\Sigma \times \mathbf{R}$ where in case of the M2-brane, $dim(M) = 2$. For the sake of argument we will take the M2-brane to be spherical, $\Sigma = \mathbf{S}^2$. We have not discussed such an object, but remind that only globally this will alter some properties. Locally this will behave as a flat membrane. In fact the procedure described below can be done for M2-branes with the structure of any Riemaniann manifold: $\mathbf{S}^N, \mathbf{T}^N$ or $\mathbf{CP}^N$.

Any function on the $\mathbf{S}^2$ M2-brane depends on three coordinates,

$$F : \Sigma \to \Sigma, F = F(\xi_1, \xi_2, \xi_3)$$

subject to the constraint $\xi_1^2 + \xi_2^2 + \xi_3^2 = 1$ - we take the radius to be unitary as we can always find a diffeomorphism $\phi(\Sigma)$ that fixes this. Furthermore the Σ itself is subjected to a symplectic structure by means of $\{\xi_A, \xi_B\} = \varepsilon_{ABC}\xi_C$. Now consider the Lie-algebra $SU(2)$ with it's $N-$dimensional representation, with generators J_A, $A = 1, 2, 3$. The Lie structure is defined by $[J_A, J_B] = i\varepsilon_{ABC} J_C$. Now we wish to map the symplectic structure in a bijective way onto the Lie-structure, by mapping

$$\pi : Symp(\Sigma) \to Mat_N(SU(2)), \;\; \pi(\xi_A) = \frac{2}{N} J_A \tag{2.1}$$

For this map to be well defined, we first of all need $\pi(\{\xi_A, \xi_B\}) = [J_A, J_B]$ - which is true (upto a factor of i that will be absorbed into the map) because of the explicit expressions of both the brackets - and we need to check that the Jacobi-identity on the Lie-side is satisfied on the symplectic manifold, that is, we need to check if $\{\{\xi_A, \xi_B\}, \xi_C\} + \{\{\xi_C, \xi_A\}, \xi_B\} + \{\{\xi_B, \xi_C\}, \xi_A\} = 0$, which is easily checked.
With these remarks we construct the map such that

$$\pi(\{f, g\}) = \frac{-iN}{2}[F, G]$$

for polynomials functions f, g on Σ and formal polynomial series in the polynomial ring $R[J_{A_1}, J_{A_2}, \ldots, J_{A_N}]$, $J_{A_i} \in Mat_N(SU(2))$, elements of the N-dimensional matrix representation of the $SU(2)$. In our example, $\Sigma = \mathbf{S^2}$ and functions are expanded in terms of spherical harmonics:

$$f(\xi_1, \xi_2, \xi_3) = \sum_{l,m} c_{lm} Y_l^m(\xi_1, \xi_2, \xi_3)$$

where in turn the harmonics are expanded as:

$$Y_l^m(\xi_1, \xi_2, \xi_3) =$$

Now using the map π as defined above, we can map this function onto an element of the polynomial ring, finding

$$\mathbf{Y_l^m} = (\frac{2}{N})^l \sum$$

Now we establish the matrix form of the function of the membrane:

$$\pi(f) = F = \sum c_{lm} \mathbf{Y}_l^m$$

Hence we can map the Poisson bracket in (2.3) onto a matrix commutator:

$$\{X, X\}^2 \rightarrow [X, X]^2$$

Now on the symplectic side, we may consider the average of a function f, on the Σ with $Vol(\Sigma) = 4\pi$. This average is

$$< f >= \frac{1}{4\pi} \int d^2\sigma f$$

At the same the Lie-algebra function F has this sense of an average as

$$< F >= \frac{1}{N} Tr(F)$$

We will identify these two in a large N limit, or in general:

$$\frac{1}{N} Tr \underset{N\to\infty}{\rightarrow} \frac{1}{Vol(\Sigma)} \int d^2\sigma$$

2.4 5-branes in Matrix theory

We have seen that the construction of the M2-brane is from a matrix point of view rather hands-on. The M5-brane plays a somewhat more peculiar role, and in fact it in some polarization it seems to be missing in fundamental matrix theory at all. To fix this, additional degrees of freedom are introduced to fix this. This procedure is quite unnatural and is one of the hints that Matrix theory is not by itself the final and complete description of full M-theory. We will some back to more discrepancies later on, but lets first investigate the five-brane. We really need the five-brane in order to impose a Dirac quantization condition in Matrix theory.
As we formulated matrix theory in lightfront coordinates $\{X^+, X^-, X^1, \ldots, X^9\}$, we can distinguish between two different types of 5-branes: one of which has two of its sides stretched along the lightcone axis $X^{\pm}$, called the longitudinal 5-brane (L5-brane), and one which has every side transverse to these two axes - the transverse 5-brane (T5-brane).

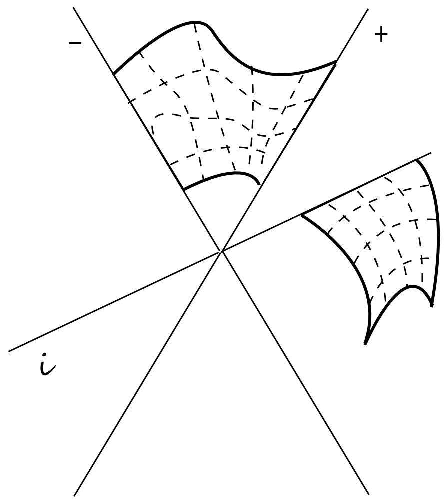

Figure 2.1: *The difference between an longitudinal and a transverse 5-brane. The longitudinal 5-brane has two of its sides along the* $\pm$ *axis. The transverse brane hangs on the axis labeled by* i*. In fact this axis is an 8-dimensional manifold.*

We try and look for this object in Matrix theory.
To do so describe we divide space-time coordinates into $\{X^m, X^a\}$ with $1 \leq m \leq 4, 5 \leq a \leq 9$ with X^m describing the five brane's spatial coordinates. Lets have the brane on the hyspersurfave $X^a = x^a_0$. Hence the introdution of the brane globally breaks

$$SO(9) \rightarrow SO(4) \times SO(5) = SO(4)_{\|} \times SO(5)_{\perp}$$

By the same token, let $(\rho, \dot{\rho})$ denote the spinor indices on the brane, that is in the $SO(4)$, and $(\alpha, \dot{\alpha})$ denote perpendicular spinor indices, in the $SO(5)$. As the 5-brane and the choice for it alligning introduces a system of prefereation, it brakes half of the supersymmetries on worldsheet. Without loss of generality say that $(\eta^{\rho}_{\alpha}, \bar{\eta}^{\dot{\rho}}_{\alpha})$ are left unbroken while complementrary $(\eta^{\dot{\rho}}_{\alpha}, \bar{\eta}^{\rho}_{\alpha})$ refrain from having susy partners.
As the five-brane does not seem to be embedded into the framework of Matrix theory like the membrane was, the idea is to put it in by hand, that is to say to introduce a new set of bosons and ferminons with the sole purpouse

of describing the 5-brane. This procedure, which does work as is explained below, is indeed a bit awkward, ans says more about the incompleteness off the model then the nature of the five-brane.

Thus we introduce a set of superspace variables $(v^{\rho\dot{\rho}}, \chi^{\alpha\dot{\rho}})$ a complex boson and fermion respectively, and rather in analogy with the dimensional reduction procudre described above, we now wrap the full Lagrangina down to six dimension in order to write down the additional five-brane Lagrangian. Recalling the position of the brane at $X^a = x_0$ the Lagrangian reads:

$$L_{5-brane} = |D_t v^{\rho\dot{\rho}}|^2 + \chi D_t \chi - v_{\rho r\dot{h}o}(X^a - x_0^a)^2 v^{\rho\dot{\rho}} - \chi_\alpha^{\dot{\rho}}(X^a - x_0^a)\gamma_a^{\alpha\beta}\chi_{\beta\dot{\rho}} - v_{\rho\dot{\rho}}(\theta - \theta_0)_\alpha^\rho \chi^{\alpha\dot{\rho}} + v_{\rho\dot{\rho}}[X^m, X^n]\sigma_{mn}^{\rho\sigma} v_\sigma^{\dot{\rho}} - |v|^4 \quad (2.1)$$

This additional Lagrangian might look cumbersome, but the first two parts are simply the kinetic terms of the new variables themselves supplemented with two covariant corrections with repspect to the five-brane itself. Then there is a coupling between the bosonic and fermionic terms, weighted by θ, and then there is a mass term and a v.e.v. term respectively.
To check whether this approach works properly it should be checked if the known interactions and processes that worked in original matrix theory, still work with a five-brane in the background, so we check if the altered zero-brane dynamics and interactions still coincide with the supergravity.

2.4.1 Dirac Quantization in M(atrix)

The presence of the M2 and M5 brane in M-theory assures a qunatization of their charges because of the Dirac quantization condition (recall that M2 couples electrically to the 11D-SUGRA three-form, and the M5 brane couples magnetically). In the case of Matrix theory we have seen that the five-brane has a different and somewhat less manifest description then the two-brane, so we need to inverstigate the quantization of charges a bit more carefull. To this end we consider a two-brane in the background (2.2). Lets take for the membrane matrix-coordinates (X^5, X^6) with the other coordinates in the ground-state diagonal form with a scalar behaviour. We choose to represent the brane-coordinates as $X^5 = R_5 P, X^6 = R_6 Q$ and rotate the matrices P and Q such that $[P, Q] = 2\pi i$. To avoid confusion, bare in mind that this is

not a second-quantization procedure.
Now as this membrane is embedded in a five-brane background, from an M-theoretic point of view it should feel the presence of the five-brane trough the magnetic flux of the three form A_{MNP}. In concreto, we have a potential $B_\mu = \int dX^5 dX^6 A_{56\mu}$.
The clever idea of [19] is to seek this effect in Matrix theory in a Berry phase due to the fermion zero modes. For this consider the membran motion $X_(t)$ with $\partial_t X(t) \ll 0$ so that a Born-Oppenheimer approximation allows for a Berry fase one the vacuum:

$$\langle X; 0| \frac{\partial}{\partial X_\mu} |X; 0\rangle.$$

The Berry phase is a quantity in quantum mechanics that arises when slowly varying one or more parameters in the moduli space around a loop. In this example, for absoluteness the five-brane wavefunction that is to pick up a phase should know have a sense of direction of the X^μ, and so only the fermionic part χ in 2.2 will contribute. Now we have a two-brane in a five-bran background, and lets take the coordinates $\vec{X} = (X^7, X^8, X^8)$ transverse to both these objects. Furthermore pick for the two-brane coordintes in the explicit representation $X^5 = 2\pi i R_5 \frac{d}{d\sigma}, X^5 = R_6 \sigma$ and have the fermionic part of the Lagrangian (2.2) as:

$$H = \int d\sigma \bar{\chi} (\gamma_5 2\pi i R_5 \frac{\partial}{\partial \sigma}) + \gamma_6 R_6 \sigma + \vec{\gamma} \cdot \vec{X}$$

Where we for rhe moment surpressed spinor indices and $\vec{\gamma} = -i\gamma_5\gamma_6\vec{\tau}$ with τ the vector of Pauli matrices. Now for chiral zero-modes under the $\gamma_5\gamma_6$ this system is indeed identical to that of a spin-$\frac{1}{2}$ particle in a magnetic field if we identify $\vec{X}$ with a magnetic field $\vec{B}$ and the Hamiltonian reduces for these modes to

$$H = \chi_0 \vec{B} \cdot \vec{\tau} \chi_0.$$

This Berry phase establishes the Dirac quntization condition and tthe couling between the two branes. That the coulping is not spoiled by higher chiral modes is because each higher chiral mode comes with two Berry phases of opposite chiralty, cancelling eachother.

2.5 Matrix interactions

Do do computations on physical systems, e.g. boson scattering, we give a more explicit form of the potential, in preperation for a computation on 2 boson scattering.

The central potential $V = -\frac{1}{2}Tr[X^i, X^j]^2$ can be worked out as follows. First of all note the the summation over i, j is pairwise, so let's focus on one term, e.g. $V_1 = -Tr\frac{1}{2}[X, Y]$. Now first of all lets look at a 2×2 example. We notate the matrices as

$$X = \begin{pmatrix} x_1 & \alpha^x_{12} \\ \alpha^x_{21} & x_2 \end{pmatrix}, Y = \begin{pmatrix} y_1 & \alpha^y_{12} \\ \alpha^y_{21} & y_2 \end{pmatrix} \tag{2.1}$$

and we introduce the handy variables $\Delta_x = x_2 - x_1$, $\Delta_y = y_2 - y_1$. We take the matrices to be Hermitian, $\alpha_{12} = \alpha^*_{21}$ so in this example we can drop the subscripts. Now the potential reads

$$V(\Delta_x, \Delta_y, \alpha_x, \alpha_y) = 4|\alpha_x \alpha^*_y|^2 + |\alpha_x \Delta_y - \alpha_x \Delta_y|^2 \tag{2.1}$$

Indeed this is a system of two non-linearaly coupled anharmonic oscillators. More generally, for N particles, we should split up our matrices in terms of its diagonal part and its off-diagonal part (a $U(1) \times SU(N)$ division) and calculate the potential. Now the matrix looks like

$$X = \begin{pmatrix} x_1 & & & & \\ & x_2 & & (\alpha^x_{ij}) & \\ & & x_3 & & \\ & (\alpha^x_{ji}) & & x_4 & \\ & & & & x_5 \end{pmatrix}$$

and we introduce $\Delta^x_{ij} = x_j - x_i$. We do the same for the other coordinates $Y, Z, \ldots$. Now the potential is readily generalized:

$$V = Det(\Lambda)^2 - \sum_{i,j,n,m} (\alpha^m_{ij}\Delta^n_{ij} + \alpha^n_{ij}\Delta^m_{ij})^2 \tag{2.1}$$

with $\Lambda = 4(\alpha_{ij})$. To convince ourselves that M(atrix) theory is in a way a proper framework to describe the degrees of freedom of an eleven-dimensional SUGRA, we should match objects, fields and interactions. For this section

we will study a system the system of two gravitions and check if the description mathces the SUGRA side. With the remarks made above in mind, we consider a two-boson collision and by comparision to the oscillator we determine its potential.
Like in GR, we do not expect to be able to immideiatly do an exact calculation so we work in a background field method and we'll make some approximations along the way. Starting out with the lagrnaginan (2.1) we express the bosonic fields in terms of a background plus fluctuations

$$X^i = B^i + H^i$$

with $|H^i| \ll 1$, and here $|H|$ denotes the entry-wise absolute value (so that the condition $|H| \ll 1$ surely caries over to the determinant). Now recalling the YM origin, only the fields X^i were interpreted as matrix coordinates, the ter $X^0 = A$ is still a potential to be cast in the covariant expression of the derivative. In the background approach we thus have a covariant derivative with respect to this background, $D^{bg}_\mu A^\mu = \partial_t A - i[B^i, X^i]$ and it proves convenient to impose a Lorentz-like gauge for this:

$$D^{bg}_\mu A^\mu = 0 \tag{2.1}$$

Now this gauge fixing procedure deserves some care. Recall from the Fadeev-Popov procedure the philosophy of gauge fixing actions:
In calculating partition functions,

$$\mathcal{Z} = \int [\mathcal{D}A^\mu] e^{iS[A]}$$

we integrate over too many gague equavalent configurations. Instead, if G is the gauge group, we want to integrate over only one represetant of the group, that is only one gauge configuration. The set of gauge inequivalences is denoted $[G/H]$ the set generated by unique representants of iets ideal. So we really need:

$$\mathcal{Z} = \int_{[G/H]} [\mathcal{D}A^\mu] e^{iS[A]}$$

To gauge-fix the term we are certainly to work with the above expression. Analytically we can impose a gauge fixing in any Lagraninging by introducing a Lagrange multiplier with gauge fixing term $S_{g.f.}$, so that the equations of motion of that very multiplier yield the gauge condition. Analytically this is fine at the cost of a Fadeev-Popov determinant:

$$\mathcal{Z} = \int [\mathcal{D}A^\mu] \underset{FP}{\Delta} e^{i(S[A]+S_{g.f.})}$$

The clever trick due to those Fadeev and Popov is to represent the determinant as a Grassmann expononent so that an extra term is added to the action, the ghost term S_{ghost}:

$$\mathcal{Z} = \int [\mathcal{D}A^\mu][\mathcal{D}C][\mathcal{D}\bar{C}] e^{i(S[A]+S_{g.f.}+S_{ghost})}$$

In the example above, the additional ghost action yields

$$S_{ghost} = \partial_\tau C \partial_\tau \bar{C} - [B^i, \bar{C}][B^i, C].$$

Now we are ready to analyze the full action with respect to the gauge condition (2.5). Plugging in the background expanions into the Larganginan, inserting the ghosts and taking Euclidian time ($\tau = it, A \to -iA$), yields a full action:

$$\begin{aligned} S = \frac{1}{2R} \int Tr \Big[& (\partial_\tau B^i)^2 + (\partial_\tau Y^i)^2 + (\partial_\tau A)^2 + (\partial_\tau C)^2 + \\ & \frac{1}{2}[B^i, B^j]^2 - [B^i, Y^j]^2 - [B^i, A]^2 - [B^i, C]^2 \\ & -2i\dot{B}^j[A, Y^j] + \theta^T\dot{\theta} - \theta^T\gamma_i[B^i, \theta] \Big] + \mathcal{O}(A^3, C^3, Y^3, \theta^3) \end{aligned} \tag{2.0}$$

In this expressions, squares involving the ghosts are actually to be read as $(\partial_\tau C)^2 = (\partial_\tau \bar{C})(\partial_\tau C)$ and $[B^i, C]^2 = [B^i, \bar{C}][B^i, C]$.
To take a toy model, lets look at a two-boson system - a system that describes two gravitons in M(atrix)-tehory, and place it in its centre of mass frame, with an impact paramter $b > 0$. Let the motion be along X^1 and the impact paramter be paremetrized along X^2 so the system is represented in the background:

$$B^1 = \begin{pmatrix} vt & 0 \\ 0 & -vt \end{pmatrix}, \;\; B^2 = \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}, \;\; B^i = 0, i > 2 \tag{2.0}$$

This is a 2×2 system, hence the matrices are in $U(2)$, with generators $[t^a, t^b] = -\epsilon^{abc}t^c = f^{abc}t^c$ for example $t^a = \frac{i}{2}\sigma^a$ the Pauli matrices. In order to calculate the potentials we need such an algebra for explicity ($X^i = X^i_a t^a$). Now the procedure is straightforward: the background field is plugged into (2.2). Note that all the kinetic parts in de Lagrangian should be partially integrated to yield terms like $-\partial_\tau^2 Y$. Well, in terms of this lie-algebra valued matrices, we obtain the action

$$S = \sum_{a=1}^{2} \left\{ (\frac{1}{2}Y^i_a + \frac{1}{2}A_a + \bar{C}_a)(\partial_\tau^2 - r^2)(\frac{1}{2}Y^i_a + \frac{1}{2}A_a + C_a) \right\} +$$
$$(Y^i_3 + A_3 + \bar{C}_3)\partial_\tau^2(Y^i_3 + A_3 + C_3) + 4v(A_1Y_2 - A_2Y_1) + \theta_+^T(\partial_\tau - vt\gamma^1 - b\gamma^2)\theta_- + \quad (2.0)$$

Here $r^2 = (v\tau)^2 + b^2$.
Now we work by the philosophy of the introduction of the section and realte this system to that of the harmonical oscillator. We need to determine the masses of the oscillators, which is done by standard procedure: we write the energy in terms of the oscillator matrix and determine the eigenvalues.
What we see is that each element of the set $\{A, Y^i, C, \psi)$ contributes to the oscillator. We have two ghosts C bot with mass r^2, eight bosons Y^i with mass r^2,

In a ground-state approximantion we are left with an effective potential

$$V_{eff} = \sum_b \omega_b - \sum_f \omega_f - \sum_{gh} \omega_{gh} =$$
$$\left(8 \cdot r + \sqrt{r^2 + 2v} + \sqrt{r^2 - 2v}\right)_b - \left(2 \cdot r\right)_g - \left(8 \cdot \sqrt{r^2 + v} + 8 \cdot \sqrt{r^2 - v}\right)_f \quad (2.0)$$

expaniding this potential in a series in v/r^2, all the odd powers in v cancel, and in this full supersymmetry we find

$$V = -\frac{15}{16}\frac{v^4}{r^7} + \mathcal{O}(\frac{v^6}{r^{11}}) \quad (2.0)$$

In case of lower less susy, for example in fewer dimensions, the lowest order contribution is $V = -\frac{7}{8}\frac{v^2}{r^5}$.

2.6 Matrix String Theory

Matrix theory allows for a picturesque way of describing string interactions. In fact we must get an idea of how string theory emerges from Matrix theory. [8] While doing so, we will encounter a richer spectrum of symmetries,

as described below.
Recall how the world-sheet of a string is described by the coordinates $X^\mu(\sigma,\tau)$ where, in the case of the superstring, $\mu = 0,...,9$. Preferring a light-cone gauge we introduce $X^\pm = \frac{1}{2}\sqrt{2}(X^0 \pm X^{D-1})$. Of these fields, X^- will serve as a time-component and X^+ gets compactified on a circle of radius R, while the eight fields $X^i, i = 1,...,8$ will be coined the transverse fields. Thus our theory is embedded in $S^1 \times \mathcal{M}^8$, $\mathcal{M}^8$ a Riemanian manifold. The compactification gives rise to a winding number N, so that to the time-like field X^+ a momentum p^+ is associated, with $p^+ = N/R$. The winding number is interpreted as the number of times the string has wrapped itself along the direction of compactification.
Thus, in units of R, the spatial world-sheet coordinate σ runs in the half-open interval $[0, N[\equiv A_N$. In order to describe te joining and splitting of strings we decide to make a partition of this interval:

$$A_N = \bigcup_{I=1}^{N} A_I, \quad A_I = [I-1, I[\tag{2.0}$$

At the same time we introduce a partition of the transverse fields:

$$X^i_I = X^i|_{A_I} \tag{2.0}$$

That is to say, X^i_I is the restriction of X^i to the partition cell A_I. In each partition field, σ now runs in $[0,1[$ and the have the identification:

$$X^i_I(\sigma+1,\tau) = X^i_{I+1}(\sigma,\tau). \tag{2.0}$$

We can think of this as if, instead of having one string, we now have N strings glued together. Now interpret these restricted fields as eigenvalues of an $N \times N$ matrix, so that the field itself gets a matrix description:

$$X^i = \begin{pmatrix} X^i_1 & & & & \\ & X^i_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & X^i_N \end{pmatrix} \tag{2.0}$$

We decomposed, at least locally: $\mathbf{R}^8 \mapsto (\mathbf{R}^8)^N$. The previous chapters have proven the justification of thus construction.

In terms of the matrices, relation (2.6) can be written as:

$$X^i(\sigma+1) = VX^i(\sigma)V^{-1} \tag{2.0}$$

where we introduced the $N \times N$ matrix V:

$$V_N = \begin{pmatrix} & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ 1 & & & & \end{pmatrix}$$

where the subscript of V denotes it's dimensionality. Going back to the partition of the interval, we would like to have a description of a multi-string state. To do so we introduce sub-partitions of A_N.
Let n_1 and n_2 be two integers such that $n_1 + n_2 = N$. Now introduce the subpartition of A_N as:

$$A_N = A_{n_1} \cup A_{n_2},$$

with $A_{n_1} = \bigcup_{I=1}^{n_1} [I-1, I[$ and A_{n_2} it's complement. Finer subpartitions are defined inductively. A two-string state is now obtained by posing boundery conditions to the restriction of the fields to A_{n_1} and A_{n_2} indepently. That is: $X^i_{n_1} = X^i_1$ and $X^i_N = X^i_{n_1+1}$. Thus the splitting of a string of length N into two strings of lenght n_1 and n_2 is described by the splitting of an $N \times N$ matrix into two sub-matrices, $n_1 \times n_1$ and $n_2 \times n_2$ respectively. Note how the momenta of the seperate strings become $p^+_k = n_k/R$. The boundary conditions in the sub-partitions are again described by the matrix V as: $X^i(\sigma+1) = V_{n_k} X^i(\sigma) V^{-1}_{n_k}$.
Due to the periodic boundary-conditions, we are now in fact studying the partition of an interval into cycles:

$$(12\cdots N) \mapsto (12\cdots n_1)(n_1+1\cdots n_1+n_2)\cdots(N-n_k+1\cdots N) \tag{2.0}$$

Each particular partition corresponds uniquely to a conjugacy class of S_N, the symmetric group in N integers.
Now to adept group theoretic language, note that the group S_N acts on the fields X^i_I as:

$$g \cdot X^i_I = X^i_{g(I)}, \;\; g \in S_N$$

Thus, for each representative of a conjugacy class $[g]$ of the element g we observe:

$$X(\sigma + 1) \simeq gX(\sigma) \tag{2.0}$$

We might as well have looked at the field $Y = hX$, with h any element of S_N, (since it doesn't matter how we arrange the matrix (2.6)), with the boundary conditions $Y(\sigma+1) = hgh^{-1}Y(\sigma)$, so the equivalence is well defined only for conjugates of g. The total manifold of our theory becomes the quotient $(\mathbf{R}^8)^N/S_N$, called an orbifold. We now are in the position to say more on the Hilbert space of this theory. First of all, let H_g be the hilbert space of states subject to the boundary condition (2.6). Now the symmetric group acts on thuis Hilbert space as

$$h(H_g) \to H_{hgh^{-1}}$$

Now let $C_g = \{h|[h,g] = 0\}$ the centralizer subgroup. It is clear for $h \in C_g$, $h(H_g) = H_g$. So not we can decompose the total Hilbert space as H_g's wich are invariant under their own centralizer or:

$$H(M/G) = \bigoplus_{[g]} \tilde{H}_g(M) \tag{2.0}$$

with $\tilde{H}_g$ the sub Hilbert sector left invariant under actions from the centralizer subgroup.

Chapter 3

Matrix Cosmology

The scales of energy and length of the theory of strings are generally to extreme to expect that we will detect even a small hint of its presence at interactions that we observe at the earth. We do not expect any application of the string theoretic framework to the field of the Standard Model and the refining of its parameters. The only hope there is is that indeed the Standard Model, bases on he famous gauge group-product $SU(3)_{colour} \times (SU(2) \times U(1))_Y$ is in fact merely an effective theory, and that we should think of the gauge group-product as embedded in a smaller gauge-group, for example $SU(5)$, $SO(10)$ or even some of the exceptional semi-simple Lie-groups. If so, indeed string theory would become more promising, as it has sectors based on gauge-groups such as $SO(32)$ of $E_8 \times E_8$ which might be stems of a sequence of embeddings such as $SU(3) \times SU(2) \times U(1) \subset SU(5) \subset SO(10) \subset E_8$. Now even if a GUT would pop up, that is even if the Large Hadron Collider would detect high-energetic GUT particles in say $SO(10)$ then still that does not say awkwardly much about the relevance of the E_8 because this semi-simple Lie-group has much more generators (248 to be precise, each of which should resemble some kind of gauge boson). One might go around this an reason that string theory is more like an abstract theory, and that any of (and only) it's sub-theories - theories with sub groups of the relevant gauge groups - are realized in physical universes. But this is not satisfactory, really because of the nature of E_8. The eighth exceptional group is the biggest of the exceptional family and admits pretty much any Lie-group as a subgroup, hence making string theory more like a theory of *anything* instead of *everything*.

The theory of General Relativity has been varified mainly *outside* the earth atmosphere - in our planetary system and far beyond. This is so because the lenght-scales at which the curvature of space-time become relevant at length-scales beyond the terrestrial scale

The entrance of non-commuting matrices as representatives for the coordinates of a system op N Dp-branes, needed that these N branes were very close together. In any typical Big-Bang cosmology, it is proposed that around the time of the big-bang, energy densities are extremely high and length-scales extremely low. This makes us suspect that as we run back in time from now, the description of local coordinates in de universe should get more and more a matrix, non-commuting character. We will in fact investigate such a scenario to find that indeed at early times, the universe should have a strong matrix-like character.

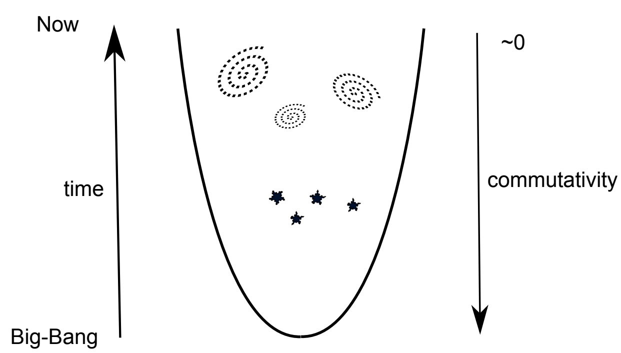

Figure 3.1: *As we run back in time, Dp-brane coordinates will tend to behave more and more like non-commuting matrices*

As the matrix proposal should be valid around and below the string length-scale, we will consider other extremely high-energetic, dense and compact objects like black holes.

3.1 A Matrix Big Bang

As Matrix theory originated by investigating physics at an below the string length scale, it is unrealisic to expect for direct evidence. However, if there is

one consequence of the matrix proposal that should become visible or have consequence, then it is the appearence of the new, off-diagonal degrees of freedom. After all, to any physical system that first had its positional degrees of freedom described by N variables, one now attaches additional degrees, more or less one degree for every pair, in a hermitian way, so this introduces about $\frac{1}{2}N^2$ new variables that behave more or less like (an)harmonic coupled oscillators, each oscillator with its own frequency and energy. We migth indeed expect a macroscopic system to be effected by these additional, in principle invisible variables, that is each macroscopic system has underlying micropscopic physics that hides the off-diagonal variables, but albeit these variables are hidden in the sense that they aer not on the diagonal and thus do not have welll-defined notion of position in space-time, they do contribute to the microscopic, and in the in to the macroscopic frequencies and energies. Whatismore, as at these extreme space-time and energy scales the concept of space itself finds a redefinition, the matrix proposal may be promising to get a hold on cosmological space-time singularities.
As the effects are so small, only the largest of closed systems should expose these underlying micropscopics. We will look at possible consequnces in two such large systems: black holes and the universe itself.

3.1.1 A specific dilaton dependence

To model a string theoretic cosmology, we have to make some remarks on the role of the dilatonic field that is present in both of the type-II theories. An interesting example of a non-constant dilaton background $\Phi = \Phi(X^\mu)$ is the choice

$$\Phi(x^\mu, X^+) = -QX^+ \tag{3.0}$$

Here X^+ denotes the lightlike lightcone coordinate and $\{x\mu\}$ is the set of coordinateds complimentary to X^+. We call this the lightlike linear dilaton. Generally, the linear dilaton is defined as $\Phi(x^\mu, X) = qX$ where now X is just a plain coordinate and again $\{x^\mu\}$ the compliment of X. It is well known that this background allows for preservation of conformal invariance. To see this, the stress-enregy tensor is easily calculated as

$$T(z) = -2(\partial X^\mu \partial X_\mu + \partial X \partial X) + q\partial^2 X$$

and the o.p.e. of $T(z)$ with itsels yields the additional term in the fourth order pole equal to $3q^2$, obtaining a central charge $c = D + 3q^2$. For origianal

bosonic theory stays conformally invariant for $c = 26$ hence for any D we can choose $k = \sqrt{\frac{26-D}{3}}$, and D becomes something like a free parameter. Normally this introduces non-critical string-theories for which indeed we can pick $D \neq 26$.
Yet in the case of the lightlike linear dilaton, (3.1.1), we should think of Q itself as a lightlike coordinate with $Q^2 = 0$, circumventing non-criticality.
In this dilaton dependence, the string couling becomes

$$g_s = e^{-QX^+},$$

the coupling is time-dependent. That the value of Q is of no physical significane stems from the scale-invariance of the lightcone coordinates. Namely, the flat ten-dimensional metric reads

$$ds^2 = -2dX^+dX^- + (dX^i)^2,$$

which is left invariant under rescalings $X^+ \to aX^+, X^- \to a^{-1}X^-$. In this string-frame the only dynamics takes place in the dilaton itself. In Einstein frame, this is not so, as

$$ds_E^2 = \frac{1}{\sqrt{g_s}} = e^{QX^+/2}ds^2 = e^{-QX^+/2}(2dX^+dX^- + (dX^i)^2)$$

Now space-time itself is a dynamical quantity and the notion of space-time originates in $X^+ \to \infty$, which we will identify with the time of a Big Bang.
Now form a matrix theory perspective, this dilaton dependence could be interpreted in two ways. First of all, our action

$$L = \int \frac{1}{2R} Tr\Big[D_0X^iD_0X^i + \frac{1}{2g_s^2}[X_i, X^j]^2 + \theta^T(i\dot{\theta} - \frac{1}{g_s}\gamma_i[X^i, \theta])\Big]$$

could be interpreted as a t theory with a time-dependent Yang-Mills coupling,

$$g_{YM} = 1/g_s = e^{QX^+}$$

due to the dynamics of the dilaton. Now we see how at late times the self-interaction blows up and the masses of the off-diagonal elemens get very large. To make the statement precise: the masses of the off-diagonal terms get generally exited above the string-theory mass gap and decouple from the physics: they can be integrated out. On the other hand, at early times these

masses become light and at or below the mass-gap.
There is a more geometric interpretation to this. For this, lets write this action with the flat metric explicitly, for example take the self interaction:

$$\frac{1}{g_s^2}[X_i, X^j]^2 = \frac{1}{g_s^2}\eta_{jm}\eta_{in}\big(X^iX^jX^mX^n + X^jX^iX^nX^m - X^iX^jX^nX^m - X^jX^iX^mX^n\big)$$

Considering rescalings of the metric η, $\eta \to f(\tau)\eta$, this might as wel be intepreted as a rescaling of the coupling: $g_s \to f^{-1}g_s$. The same is true for the fermionic part. Or the other way around, a rescaling of the couling can be interpreted as a rescaling of the metric. so, in our example, we should evaluate the SYM worldsheet at a metric

$$ds^2 = e^{Q\tau}(-d\tau^2 + d\sigma^2)$$

where we explicitely interpret X^+ as time. Now the nature of the big-bang character is more clear. At early times, say at the big bang, the metric is turned on, so time gets a preferred role over space - it becomes a modulus that tweaks the presence op space.

3.2 Matrix Black Holes

The formulation of the BFSS proposals demands a limit of $N \to \infty$. This limit is rather un-physical in the sense that we cannot compute a generic rate of accuracy for a given large N. This rate of accuracy really depends of specific physical situations under consideration, for example graviton scattering. Now as N increases, more and more degrees of freedom are introduced into the problem - the off-diagonal terms that is. At a certain stage, these degrees do not contribute so much to the model: a lot of them will end up in a ground state. Now a legitimate question to ask is: what is a *minimum* value N_{min} for N: at what stage does the model become reliable enough in a given physical constellation?
To address this question for a specific configuration, lets consider a black hole[14], and try to describe it in terms of the matrix degrees of freedom. First of all, this minimum, or cut-off, should depend on the degrees of freedom on the black hole itself, thus on its entropy.

In the DLCQ, the coordinate X_- is compactified on a circle of radius R, thus the light-like momentum P^- is quantized in units of $1/R$: $P^- = N/R$. This N divides the lightlike coordinate X^- into sectors labeled by N, and indeed the IMF is defined by sending $N \to \infty$. Now let the black-hole be in a rest frame, that is, pick coordinates $\{X^+, X^-, X^1, \ldots, X^9\} = \{X^+, X^-, X^\perp\}$ and take an inertial frame with $\{P^+, P^-, P^\perp\} = \{M, M, 0\}$. The radius of the black hole is R_s, the Schwarzschild radius, and the black hole is embedded into the coordinate frame as in figure (3.2).

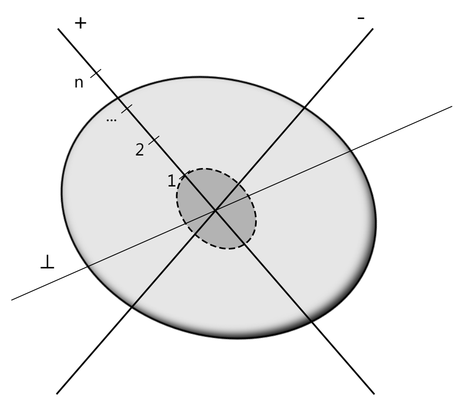

Figure 3.2: *A black hole embedded in the light-front frame. Generally, it will not fit into the inner, fundamental sector marked by the compactification radius R (the dark-grey sector). The fundamental domain really only refers to the X^+-domain.*

The black hole in general, for large enough R_s, will not fit into the fundamental sector of the light-like coordinate. But we may boost the frame in such a way that by Lorentz-contraction it will fit into the first sector. Let us make the boost such that its longitudinal momentum becomes N/R, hence giving a contraction: $\Delta X^- = \frac{M}{P^-} R_s = \frac{MR}{N} R_s$. For the black hole to fit in the longitudinal domain is $R > \Delta X^-$ thus,

$$N > MR_s = N_{min} \tag{3.0}$$

Simply extending the Schwarzschild solution of the Einstein field-equations to arbitrary dimension D, yields a Schwarzschild radius of $R_s \propto (GM)^{1/(D-3)}$, thus explicitly for the minimum N_{min}:

$$N_{min} \propto G^{\frac{1}{D-3}} M^{\frac{D-2}{D-3}} \tag{3.0}$$

And thrilling enough, the right hand side of equation (3.2) is proportional to the Beckenstein-Hawking entropy.

Chapter 4

Layman Summary

As this document is written, the group of particle physicists working at the Large Hadron Collider is gathering in Genève to discuss the results of the past year's measurements. In the time frame in which the collider has been active, it indeed only rediscovered particles that were already known to exist and even though there are some slight peaks in the data inside Higgs-window, we have not yet exclusive new results - but it has been announced that within two years we will have a conclusive result as to the existence of the Higgs-boson. Let us for the moment imagine that the Higgs would be detected. For the physics community, this would really be a celebration as it would more or less complete our understanding of a great deal of the microscopic theory of particles and their interactions - that is up to some length scale. We might compare this results with Mendeleev's proposition of the atomic table, in which he actually predicted the existence of some atoms with specific properties. For this model, the actual discovery of those atoms really was a celebration of this table and rightfully established this model as a great scientific breakthrough.

The standard model a table just like that of Mendeleev's, it only tries to table the fundamental particles at smaller scales (the sub-constituents of atoms). The current status quo is that we have discovered every single particle of the table, except the so-called Higgs boson. So detection of the Higgs would more or less confirm the way we think of elementary particles.

But Mendeleev's table and the discovery of all the atoms did not mean the end of atomic physics whatsoever. And equally the detection of the Higgs will not imply the end of particle physics at all. It is very likely that, analogously to the atom, the fundamental particles like electrons and quarks, are themselves not as elementary as we may think but are compounds of more fundamental objects. String theory is a theory that investigates such microscopics by proposing that in fact all known particles are actually constituents or vibrations of strings. This rather innocent looking proposal has rather extreme conceptual consequences. For instance it demands that the four dimensional world as we perceive it is in fact ten-dimensional, with the six missing dimensions curled up on tiny geometric objects with extremely small volumes.

One specific conceptual implications is that at the smallest length-scales the notion of position and space gets a different description. To make this extension of the description of position concrete, consider a system of two objects, say the system of two bugs in a room. For each moment in time, to completely describe the positions of this system, we must give for each bug three numbers or coordinates, $(x, y, z)_{bug1}, (x, y, z)_{bug2}$. We might as well pair these two arrays as $\{(x_1, x_2), (y_1, y_2), (z_1, z_2)\}$. It might sound trivial, but these numbers are, like any kind of numbers, subject to the property that $x \cdot y = y \cdot x$. In matrix theory, at the smallest length-scale, these numbers are replaced by bigger geometrical objects, matrices:

$$(x_1, x_2) \to X = \begin{pmatrix} x_1 & a \\ b & x_2 \end{pmatrix}$$

where a and b are interaction terms between the two bugs. If the bugs are far apart, these interaction terms are pretty much equal to zero, but when they approach eachother they are to be taken into account. A most drastic difference between matrices and numbers is that gerenally for two matrices: $X \cdot Y \neq Y \cdot X$ Only when all the interaction terms vansish, that is when the bugs are far apart, we have $X \cdot Y = Y \cdot X$ and the ordinary notion of coordinate numbers is re-established.

Indeed in everyday life, at length-scales we are used to, such description are not relevant. Yet, in huge objects that are extremely compact, the effect of this alternative description enters at the level of the energy of the system. There is a meausure of the amount in which two objects fail to obey yhe law $x \cdot y = y \cdot x$, and this object is simply the entity $x \cdot y = y \cdot x$. We call this measure the commutator and denote it with $[x, y]$. If $[x, y] = 0$, we are dealing with *regular* mathematical objects, but if not, if $[x, y] \neq 0$ care is needed. In the energy, this commutator enters in the form $\kappa[x, y]^2$, where κ is some constant. This contribtution does become relevant in such large compact systems. Examples of such systems are black holes and a universe around the time of a big-bang. This thesis investigates these effects in such systems.

Bibliography

[1] Clifford V. Johnson, *D-branes*. Cambridge Monographs on Mathematical Physics, 2nd Edition, 2006.

[2] Joseph Polchinski, *String Theory vol. II*. Cambridge Monographs on Mathematical Physics, 1998.

[3] M. B. Green, J. S. Schwarz, E. Witten, *Superstring Theory*. Cambridge University Press, 1987.

[4] K. Becker, M. Becker, J.H. Schwarz *String Theory and M-Theory*. Cambridge University Press, 2007.

[5] D.I. Olive, P.C. West (editors), *Duality and Supersymmetric Theories*. Publications of the Newton Institute, Cambridge University Press, 1999.

[6] Kostast Skenderis, *Black holes and branes in string theory*. arXiv:hep-th/9901050, 1999.

[7] T. Banks, W. Fischler, S. Shenker, L. Susskind, *M Theory As A Matrix Model: A Conjecture*. Phys.Rev.D55:5112-5128, 1997

[8] R. Dijkgraaf, E. Verlinde, H. Verlinde, *Matrix String Theory*. Nucl.Phys.B500:43-61, 1997

[9] E. Witten, *Bound States of Strings and p-Branes*. Nucl.Phys.B460:335-350, 1996

[10] E. Witten, *String Theory Dynamics in Various Dimensions*. Nucl.Phys.B443:85-126, 1995

[11] J. Polchinsky, *Dirichlet-Branes and Ramond-Ramond Charges*. Phys.Rev.Lett.75:4724-4727, 1995

[12] P.K. Townsend, *D-branes from M-branes*. Phys.Lett. B373:68-75 , 1996

[13] E. Bergshoeff, C.M. Hull, T. Ortin, *Duality in the Type–II Superstring Effective Action*. Nucl.Phys. B451:547-578, 1995

[14] T.Banks, W.Fischler, I.R.Klebanov, L.Susskind, *Schwarzschild Black Holes from Matrix Theory*. Phys.Rev.Lett.80:226-229, 1998

[15] R.H. Dijkgraaf, *Les houches lectures on fields, strings and duality*. arXiv:hep-th/9703136, 1997

[16] W. Taylor, *The M(atrix) model of M-theory*. arXiv:hep-th/0002016, 2000

[17] A. Bilal, *M(atrix) Theory: a Pedagogical Introduction*. arXiv:hep-th/9710136, 1999

[18] M. R. Douglas, D. Kabat, P. Pouliot, S. H. Shenker, *D-branes and Short Distances in String Theory*. Nucl.Phys.B485:85-127, 1997

[19] M. Berkooz, M. R. Douglas, *Five-branes in M(atrix) Theory*. Phys.Lett. B395:196-202, 1997

[20] G. 't Hooft, *Introduction to String Theory*. 2004

[21] B. Craps, S. Sethi, E. Verlinde *A Matrix Big Bang*. JHEP0510:005, 2005

[22] T.Banks, W.Fischler, I.R.Klebanov, L.Susskind *Schwarzschild Black Holes from Matrix Theory*. Phys.Rev.Lett.80:226-229, 1998

[23] Ph. A. van Reeuwijk *De Fase van Berry*. B.S. Thesis Universiteit van Amsterdam, 2010

[24] B. M. Wouters *BRST Quantization and String Theory Spectra*. M.S. Thesis Universiteit van Amsterdam, 2010

[25] M. Vonk *From string theory to matrix theory - searching for the Theory of EveryString*. M.S. Thesis Universiteit van Utrecht, 1998

[26] J. Maldacena, *Large N Fields Theories, String Thoery and Gravity*. Triest Lectures, 2001

[27] F. Hacquebord *Symmetries and Interactions in Matrix String Theory*. PhD Thesis Universiteit van Amsterdam, 1999

[28] N. Ishibashi, H. Kawai, Y. Kitazawa, and A. Tsuchiya *A Large-N Reduced Model as Superstring*. Nucl. Phys., B498:467491, 1997