Higher order contributions to the effective action of $\mathcal{N} = 2$ and 4 supersymmetric Yang-Mills theories from heat kernel techniques in superspace

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Contents

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Abstract

The one-loop effective action for $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supersymmetric Yang-Mills theories are computed to order F^5 and F^6 respectively by the use of heat kernel techniques in $\mathcal{N} = 1$ superspace. The computations are carried out via the introduction of a new method for computing DeWitt-Seeley coefficients in the coincidence limit.

To order F^5, the bosonic components of both $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supersymmetric Yang-Mills theories are extracted and compared with the existing literature. For $\mathcal{N} = 4$ super Yang-Mills theories the F^5 terms are found to be consistent with the non-Abelian Born-Infeld action computed to this order by superstring methods and various other means of computing deformations of supersymmetric Yang-Mills theory. The result proved to be the final piece of a puzzle, leaving little doubt that there exists a unique deformation of maximally symmetric super Yang-Mills theories at this order. The F^6 terms will be of importance for comparison with superstring calculations, including direct tests of the AdS/CFT conjecture.

The bosonic components of $\mathcal{N} = 2$ supersymmetric Yang-Mills are also shown to be consistent with existing literature, and will be of importance for testing of generalizations of the AdS/CFT conjecture.

Chapter 1

Introduction

Supersymmetry was discovered a little over thirty five years ago, and continues to be an area of intense theoretical scrutiny. It has an interesting and rather unique history [1, 2] in that is was independently discovered in various guises by four different research teams [3, 4, 5, 6, 7, 8], all motivated by purely aesthetic and theoretical considerations, but still remains to be observed in nature. One of the major goals of the next generation of particle accelerators is the detection of supersymmetry. Despite this lack of experimental evidence, supersymmetry plays an increasingly valuable and dominant role in much of modern theoretical high energy physics. It is the cornerstone of superstring theory [9, 10, 11, 12], the most notable of recent attempts to unify general relativity and quantum field theory.

The discovery of supersymmetry simultaneously realized the theoretical desires of finding a non-trivial extension of the Poincaré algebra, and an underlying relationship between fermions and bosons, the two classes of previously unrelated fundamental particles in Yang-Mills theories and in particular the standard model. It brought with it a powerful new set of tools and ideas in addition to an array of other appealing features, and of course great promise for movement towards a quantum theory of gravity.

An incredibly productive area of research in recent years relates to the connection between string theories and Yang-Mills theories. Initially prompted by the observation that low energy effective actions for massless degrees of freedom in open string theories are generalizations of Yang-Mills theories [13, 14], the interface between the two has yielded valuable insights on both sides. Indeed, the overlap between the two has proven to be far richer than one may have initially guessed based on the requirement that string theory must, if it does indeed provide the foundation for a unified theory of all four fundamental interactions, reproduce all of the successes of the standard model. More recently, added impetus has come from the discovery of

D-branes [15, 16] and the fact that their low energy effective actions are also extensions of supersymmetric Yang-Mills theory. Perhaps the most notable example of this connection is the Maldacena AdS/CFT conjecture [17, 18, 19] which identifies two seemingly unrelated theories. Namely, it identifies a theory possessing gravitational degrees of freedom, type IIB superstring theory formulated in an $AdS_5 \times S^5$ background, with one which has none at all, namely four-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. For details see the reviews [20, 21, 22, 23, 24, 25, 26] and references therein.

Maldacena's original arguments have since been generalized to include a conjectured duality between certain $\mathcal{N} = 2$ superconformal supersymmetric Yang-Mills theories and superstring theories in special backgrounds [27, 28, 29, 30, 31, 32], and more recently, by the discovery of supergravity dual for the $\mathcal{N} = 1$ superconformal β-deformation of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory [33].

Accordingly, detailed tests of these conjectures will require comparison of the effective action for specific $\mathcal{N} = 2$ and 4 supersymmetric Yang-Mills theories with results derived from superstring calculations.

This interface between superstring theory and supersymmetric gauge theories provides much of the motivation behind the work presented in this thesis, the precise details of which are given below. The bulk of the research carried out is presented in chapters 5 and 6, which are based solely on the single author publications [34, 35]. In short, higher order contributions of the effective actions of $\mathcal{N} = 2$ and $\mathcal{N} = 4$ non-Abelian super Yang-Mills theories were obtained, with the intent of testing, or laying some of the groundwork for future tests, of various links between superstring theory and super Yang-Mills theory. The present work therefore sits entirely on the supersymmetric quantum field theory side. Heat kernel techniques, $\mathcal{N} = 1$ superfields and the background field formalism are the primary tools, and along the way a new method for efficiently computing DeWitt-Seeley coefficients is revealed.

1.1 Background

It is now well known the low energy effective action for the massless vector degrees of freedom in open string theories is a deformation of the Yang-Mills action, and admits an expansion in powers of the string tension α' [13, 14]. It is a functional of the background field strength F_{ab} and its covariant derivatives $\nabla_{a_1} \dots \nabla_{a_n} F_{bc}$, with a generic structure of the form $\sum_{n=0}^{\infty} (\alpha')^n \, c_n \, F^{n+2}$, where here and in all that follows (unless specified otherwise) F^n denotes terms of mass dimension $2n$ in F

and its covariant derivatives[1]. The corrections proportional to powers of α' arise due to additional interactions induced by virtual massive superstring modes. The non-derivative terms in the effective action were computed to all orders in α' in type I string theory [36, 37] using path integral techniques and are given by the Born-Infeld action [38]

$$S_{BI} = \int \mathrm{d}^{10}x \, \sqrt{\det(\delta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu})}. \tag{1.1.1}$$

Direct computation of a four point string scattering amplitude yields the term proportional to $(\alpha')^2$ in the expansion of the Born-Infeld action [37, 39]. Derivative corrections to the Born-Infeld action were first considered in [40] (also see [41, 42, 43]), where terms involving four F's and four derivatives were extracted from the four point superstring scattering amplitude (terms involving two derivatives vanish). Contributions with four derivatives and arbitrary numbers of F's were computed via string sigma-model loop calculations in the boundary state operator formalism in [44]. This work revealed a fascinating link between the derivative corrections and the curvature tensor for a nonsymmetric metric on the sigma model target space. These four-derivative calculations have been extended to superspace in [45] (see also [46]).

In a not completely unrelated development, with the discovery of D-branes [15, 16] it was quickly established that the low energy effective action describing their massless degrees of freedom are also deformed maximally supersymmetric Yang-Mills theories, which are themselves a considerable source of interest [47, 48, 49, 50, 51, 52, 53, 54, 55, 56]; for a review see [57]. For D-brane probes in the background of a stack of D-branes, the expansion parameter is not α', but is determined by the vacuum expectation values of scalar fields which specify the separation of the probe from the stack.

For a single Dp-brane, the lowest order terms in the derivative expansion of the effective action for the massless modes (corresponding to constant field strength) are known to all orders in α', and are given by a ten dimensional supersymmetric Born-Infeld action dimensionally reduced to $p+1$ dimensions [36, 37, 58, 59, 60, 61, 62, 63, 64]. In the case when there are N coincident D-branes the gauge group becomes U(N) [65] and the leading order term in the α' expansion of the effective action is $D = 10$ supersymmetric U(N) Yang-Mills theory dimensionally reduced to $p+1$ dimensions. The first detailed investigation was carried out in [66], but the precise structure of this effective action still remains largely unknown. In contrast

[1]More specifically, terms of the form $\nabla^{2m}F^{n-m}$, with $n > m$, consisting of $n-m$ field strengths and $2m$ covariant derivatives.

to the Abelian case, its determination is complicated by the Bianchi identity

$$[F_{ab}, F_{cd}] = 2\mathrm{i}\,\nabla_{[a}\nabla_{b]}F_{cd}, \tag{1.1.2}$$

and so it not possible to unambiguously separate the effective action into derivative-independent and derivative-dependant terms[2]. In the non-Abelian case therefore one cannot truncate to constant field strength, and derivative terms must be considered [57]. Currently only a few terms in the α' expansion are known.

Of particular relevance is the fact that supersymmetry seems to be a sufficiently strong constraint to uniquely specify the form (up to field redefinitions) of the deformation of a maximally supersymmetric Yang-Mills theory to some order in F^n [51, 52]. For example, it has been shown in the *Abelian* case that supersymmetry uniquely fixes deformations to fourth order [48, 49] and sixth order [50] in $D = 10$. More recently it was proven [71, 72] that in the case of constant field strength, Abelian Yang-Mills theory in $D = 10$ has a unique deformation given by the Born-Infeld action. The proof is based on the assumption that certain BPS solutions should exist to the equations of motion. It is argued in [71] that since BPS configurations are intimately related to supersymmetry, the result lends support to the idea that supersymmetry is constraining the form of the deformation.

In the *non-Abelian* case, if supersymmetry is sufficient to fix the form of the deformation to some order in F^n, then *any* means to compute a supersymmetric deformation of the non-Abelian Yang-Mills action at this order must yield the terms in the non-Abelian D-brane effective action at this order. This in turn means that the structure of the D-brane effective action can be examined using a number of techniques, and not just by direct computation. In particular, the effective action for non-Abelian super Yang-Mills theory at this order should correspond to non-Abelian D-brane energy effective action, since the former is a deformation of classical Yang-Mills action. However, since it will be dependent on the choice of gauge, with a change of gauge inducing a field redefinition[3], direct comparison of low-energy effective actions with deformations obtained by other means is potentially non-trivial. It was argued in [37] (also see [73]) that field redefinitions only effect terms

[2]This ambiguity ultimately gives rise to some freedom in how one may choose to define a 'non-Abelian Born-Infeld action'. A proposal to resolve the ambiguity was suggested in [66], where all commutators of field strengths are to considered as belonging to the derivative-dependent part, and the non-Abelian Born-Infeld action is defined in terms of symmetrized traces of products of field strengths. Although this definition does not seem to be universally adopted within the literature, or at least there appears to be some confusion, this will not concern the present work. For discussions and results relating to this and the now well know symmetrized trace prescription see [57, 67, 66, 68, 69, 70] and references therein.

[3]For related matters see [73, 37].

in the effective action which contain covariant derivatives of the field strength, and so the non-derivative terms should be the same for all methods of calculation if the deformation is unique.

The single strongest piece of evidence to suggest that maximal supersymmetry is sufficient to fix the form of the deformation at order F^5 comes from [74], where the authors claim to have found a single and unique supersymmetric deformation to $D = 10$, $\mathcal{N} = 1$ supersymmetric Yang-Mills theory at this order. Up to a single multiplicative constant κ, the bosonic field strength contribution was found to be:

$$\kappa\,\mathrm{tr}_{\mathrm{Ad}}\bigg(F^{ab}(\nabla_a F^{cd})(\nabla^e F_{bc})F_{de} - (\nabla^e F^{ab})F^{cd}(\nabla_a F_{bc})F_{de} - \frac{1}{2}(\nabla^e F^{ab})(\nabla_e F_{ca})F_{bd}F^{dc} - \frac{1}{2}(\nabla^e F^{ab})F_{da}(\nabla_e F_{bc})F^{cd} + \frac{1}{8}(\nabla^e F^{ab})F^{cd}(\nabla_e F_{ab})F_{cd} - \mathrm{i}F^{ab}F_{bc}F^{cd}F^{e}{}_{a}F_{de} + \mathrm{i}F^{ab}F^{cd}F^{e}{}_{a}F_{bc}F_{de} + \frac{\mathrm{i}}{2}F^{ab}F^{de}F_{bc}F^{c}{}_{a}F_{de}\bigg). \quad (1.1.3)$$

Prior to this, extending the method of [71] (for a review see [75]), an order $(\alpha')^3$ deformation[4] in $D = 10$ was calculated in [78], and was found to be precisely (1.1.3) up to some overall normalization. Furthermore, a number of tests have successfully been applied to confirm that expression (1.1.3) is consistent with string theoretic predictions [79, 80]. A string theory calculation of the full five-point scattering amplitude for gluons has been carried out [81, 82, 83, 84], from which it is inferred that the corresponding low-energy effective action has again precisely the order $(\alpha')^3$ terms (1.1.3). Other approaches also provided information on the Born-Infeld action at this order [54, 56, 55, 69].

Prior to the publication [34], which forms the basis of chapter 5 of this thesis, there was evidence that the F^5 deformation was not unique. As noted in [78, 74, 81], the results of a computation of one-loop effective action for $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions using supergraphs [85, 86] yielded different non-derivative F^5 terms to the $D = 4$ version of (1.1.3). If the F^5 contributions (1.1.3) are to be uniquely specified by maximal supersymmetry, they should be found in the one-loop effective action for $D = 10$, $\mathcal{N} = 1$ supersymmetric Yang-Mills theory[5]. Consequently one should expect that in restricting this result to $D = 4$, the F^5 contributions to the one-loop effective action for $\mathcal{N} = 4$, $D = 4$ supersymmetric Yang-Mills theory should be generated [87, 88]. The work [85, 86] was inconsistent

[4]Partial results at order $(\alpha')^3$ had previously been obtained in [76], [77] and [43]. Partial F^5 terms in ten dimensional super Yang-Mills were provided by [76].

[5]This statement rests on the tacit assumption that the one-loop effective action generates the entire F^5 contribution. This is generally believed to be the case, for example see the review [51] and references therein.

with this.

In [34], the one-loop effective action for non-Abelian $\mathcal{N} = 4$ supersymmetric Yang-Mills theory is calculated in $\mathcal{N} = 1$ superfield form through to order F^6 in the absence of a chiral scalar background. Prior to this the effective action was known in superfield form only up to order F^4 (see for example [89, 90]). The technique employed is a modification of that developed in [91, 92] based on the properties of 'moments' of heat kernels[6]. At order F^5, extraction of components from the resulting superfield expression yields, up to an overall multiplicative factor, perfect agreement with (1.1.3).

As already noted, the F^5 results in [85, 86] were originally found to be in disagreement with the result (1.1.3). However, soon after the release of [34], the authors of [85, 86] discovered and corrected an error[7] to yield consistent results [85]. This finally cleared up the matter.

Taken together, the fact that all of these results [78, 81, 79, 80, 74, 34, 85] computed by four independent means yield the same expression, (1.1.3), now leaves little doubt that this F^5 deformation of maximally supersymmetric Yang-Mills theory is in fact a unique deformation at this order.

The one-loop non-Abelian F^6 terms in the effective action of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory computed in [34] are potentially important for comparison with string theoretic results at this order, as are the recently computed two-loop Abelian F^6 terms [100]. Koerber and Sevrin [101] have computed deformations of supersymmetric Yang-Mills theory to order F^6 based on the approach of requiring certain BPS solutions to the equations of motion [78]. Order $(\alpha')^4$ terms extracted from five gluon [84] and six gluon [102, 103, 104, 105] scattering amplitudes calculated in superstring theory will allow the calculation of F^6 terms in the effective action for the massless modes in superstring theory, but this is yet to be completed. Comparison of the F^6 structures from $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, BPS arguments and superstring theory will be important in establishing whether supersymmetry is a sufficiently strong constraint to uniquely determine the non-Abelian D-brane effective action at this order.

Evidence which suggests that this uniqueness does not extend to all orders, and that in general one should not expect a direct correspondence between the non-Abelian D-brane effective action and quantized super Yang-Mills at higher orders, comes from the fact that the F^8 terms in the one-loop Abelian $\mathcal{N} = 4$ super Yang-Mills effective action [106] differ from the F^8 terms in the Born-Infeld action [87].

[6] For an alternative techniques see for example [93, 94, 95, 96, 97, 98, 99] and references therein.
[7] Effectively pointed out in footnote 8 of [34].

For a detailed discussion see [106, 107].

It should be mentioned that the issue of derivative corrections to both the Abelian and non-Abelian Born-Infeld action has also been approached from the viewpoint of non-commutative geometry [108, 109, 110, 111, 112, 113, 114, 115].

In [35], on which chapter 6 is based, the results and methods of [34] (chapter 5) are extended to compute the one-loop effective action for arbitrary non-Abelian $\mathcal{N} = 2$ super Yang-Mills theories to order F^5. Technically, this proves at least as challenging as the computation of the one-loop effective action of $\mathcal{N} = 4$ super Yang-Mills to order F^6. The results of both [34] and [35] should be useful for future direct tests of the AdS/CFT conjecture and its generalizations relating to the existence of supergravity duals for certain superconformally invariant non-Abelian $\mathcal{N} = 2$ super Yang-Mills theories [17, 116, 117, 118, 119, 120, 121].

As already mentioned, the one-loop computations in superspace in this thesis are approached using heat kernel techniques, for which the associated mathematics and physics literature is vast (see for example [93, 94, 96, 99] and references therein). The technique adopted is a modification of what shall be referred to as the *Gaussian approach* [91, 92], which is closely related to other well know approaches [122, 123, 124] (and for example see [125]) for computing the asymptotic expansion of heat kernels which employ plane wave expansions of the delta function and Gaussian integration identities. For a recent guide to the literature and a detailed review of heat kernel expansions and an array of approaches see [99].

1.2 Outline

Since the $\mathcal{N} = 1$ superfield formalism is used throughout, chapter 2 is dedicated to a review of this formalism. In particular, some of the fundamental ingredients will be discussed, as will classical $\mathcal{N} = 2$ and 4 super Yang-Mills theories. Chapter 3 is concerned with the quantization of Yang-Mills theories in the context of the background field formalism, which is described in general before being applied to case of $\mathcal{N} = 2$ and 4 super Yang-Mills theories in superspace.

Chapter 4 will begin with a brief review of heat kernels and some associated computational techniques. This will include their relation to the one-loop effective action and to propagators, as well as their asymptotic expansion. The Gaussian differential equation approach [91, 92] will then be described in some detail.

Chapters 5 and 6 deal with the computation of contributions to the one-loop effective actions of non-Abelian $\mathcal{N} = 4$ and $\mathcal{N} = 2$ super Yang-Mills theories respectively. A modified version of the Gaussian approach is developed in the process, and a new technique for computing DeWitt-Seeley coefficients in the coincidence limit is described. A detailed comparison with existing literature is carried out, and we conclude with some brief remarks concerning the extension of the computational technique employed.

The three appendices are concerned with the conventions and notion used, the computerization of the computational techniques employed, and the details of some derivations. For the curious reader, and for completeness, a compact disc has also been attached. The contents of the CD are not necessary for the reading of the thesis, and merely document the computerized calculations in full detail.

Chapter 2

Field theory in $\mathcal{N} = 1$ superspace

In this chapter we will briefly review classical field theory in flat $\mathcal{N} = 1$ superspace, with certain emphasis being placed on the $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supersymmetric Yang-Mills theories. Additionally it serves to familiarize the reader with the notation and conventions adopted in this thesis, which are based primarily on those of the textbooks [126] and [127]. Further details of these can be found in appendix A.

The reader is assumed to have some familiarity with the $\mathcal{N} = 1$ superfield formalism, some supermathematics, standard classical field theory and group theory. For a more thorough and systematic treatment of superfield theories and supersymmetry in general, the reader is directed to some of the pioneering articles [3, 4, 5, 6, 7, 8, 128, 129, 130], textbooks [126, 127, 89, 131], and review articles and lecture notes [132, 133, 134, 135, 136, 137, 138, 139, 140]. For details on supermathematics the textbooks [141, 142, 143, 144] are recommended.

In this work we will be interested in field theories formulated in flat $\mathcal{N} = 1$ superspace, which is an extension of Minkowski space $\mathbb{R}^{3,1}$, augmented by four fermionic or anticommuting coordinates. Field theories formulated in this space display many desirable features both classically and at the quantum level. They exhibit manifest global $\mathcal{N} = 1$ supersymmetry and provide a powerful set of tools permitting one to perform calculations and obtain results which would otherwise be intractable in the component formalism. Ultimately, however, such theories must at some stage be reduced to field theories in $\mathbb{R}^{3,1}$ since this is currently the only means to physically interpret them. Superfield theories encode a great deal of information and rather trivial looking actions in superspace describe comparatively more complicated actions in Minkowski space.

We begin our review by recalling the coset construction of $\mathcal{N} = 1$ superspace.

2.1 $\mathcal{N} = 1$ superspace and superfields

2.1.1 Superspace

The notion of (real) superspace was first introduced by Salam and Strathdee [145, 146] (also see [5, 6]) just after the work of Wess and Zumino [7]. It is an application of the coset space method proposed originally by Cartan in 1946 [147] and later rediscovered independently in a more physical context [148, 149, 150, 151, 152]. Just as Minkowski space can be realized as the coset space $\Pi(1,3)/SO(1,3)$ (the Poincaré group modulo its Lorentz subgroup), Salam and Strathdee's idea was to identify a new space with the coset space $\mathcal{S}\Pi(1,3)/\mathcal{S}SO(1,3)$ (the $\mathcal{N} = 1$ super Poincaré group $\mathcal{S}\Pi(1,3)$, modulo a subgroup $\mathcal{S}SO(1,3)$, the Grassmann shell of the Lorentz group) and consider field theories on this space. Since $\mathcal{S}\Pi(1,3)$ is a generalization of $\Pi(1,3)$, this procedure leads to a generalization of Minkowski space, a supermanifold which necessarily includes anticommuting coordinates.

One may also consider the complex extension of the $\mathcal{N} = 1$ super Poincaré group, and the notion of complex superspace which is parameterized by coset space coordinates associated with the complex extension of the $\mathcal{N} = 1$ super Poincaré group modulo the complex extension of the Grassmann shell of the Lorentz group. In doing so one may view real superspace as a hypersurface in complex superspace, a notion heavily related to the so-called chiral and antichiral subspaces of superspace which play a major role in the $\mathcal{N} = 1$ superfield formulation.

In general the coset approach provides a systematic way for constructing field representations of a particular group realized on the coordinates of a coset space of that group. The procedure includes the identification of vierbiens, spin connections and covariant derivatives. For a brief summary see [153]. We now state a few results from this construction pertinent to $\mathcal{N} = 1$ superfield theories.

Real superspace, denoted by $\mathbb{R}^{4|4} = \mathbb{R}^4_c \times \mathbb{R}^4_a$, is parameterized by the coset space coordinates $z^A = (x^a, \theta^\alpha, \bar{\theta}_{\dot{\alpha}})$, where x^a are real c-numbers (ie $x^a \in \mathbb{R}_c$) and $(\theta^\alpha)^* = \bar{\theta}^{\dot{\alpha}}$ are complex a-numbers (ie $\theta^\alpha, \bar{\theta}_{\dot{\alpha}} \in \mathbb{C}_a$)). These coordinates possess the following well-know transformation properties under the action of the super Poincaré group. Under spacetime translations

$$x'^a = x^a + c^a \qquad \theta'^\alpha = \theta^\alpha \qquad \bar{\theta}'_{\dot{\alpha}} = \bar{\theta}_{\dot{\alpha}}; \tag{2.1.1}$$

under Lorentz transformations

$$
\begin{aligned}
x'^a &= \exp(\lambda)^a{}_b x^b \\
\theta'^\alpha &= \left(\exp\left(\frac{1}{2}\lambda^{ab}\sigma_{ab}\right)\right)^\alpha{}_\beta \theta^\beta \\
\bar\theta'_{\dot\alpha} &= \left(\exp\left(\frac{1}{2}\lambda^{ab}\tilde\sigma_{ab}\right)\right)_{\dot\alpha}{}^{\dot\beta} \bar\theta_{\dot\beta};
\end{aligned}
\tag{2.1.2}
$$

and under supersymmetry transformations

$$
x'^a = x^a - \mathrm{i}\epsilon\sigma^a\bar\theta + \mathrm{i}\theta\sigma^a\bar\epsilon \qquad \theta'^\alpha = \theta^\alpha + \epsilon^\alpha \qquad \bar\theta'_{\dot\alpha} = \bar\theta_{\dot\alpha} + \bar\epsilon_{\dot\alpha}. \tag{2.1.3}
$$

It is clear that the θ^α and $\bar\theta_{\dot\alpha}$ are left and right Weyl spinors, together forming a Majorana spinor.

The most general super Poincaré transformation on $\mathbb{R}^{4|4}$ is found to be

$$
\begin{aligned}
x'^a &= \exp(\lambda)^a{}_b x^b + c^a - \mathrm{i}\epsilon\sigma^a\bar\theta + \mathrm{i}\theta\sigma^a\bar\epsilon \\
\theta'^\alpha &= \left(\exp\left(\frac{1}{2}\lambda^{ab}\sigma_{ab}\right)\right)^\alpha{}_\beta \theta^\beta + \epsilon^\alpha \\
\bar\theta'_{\dot\alpha} &= \left(\exp\left(\frac{1}{2}\lambda^{ab}\tilde\sigma_{ab}\right)\right)_{\dot\alpha}{}^{\dot\beta} \bar\theta_{\dot\beta} + \bar\epsilon_{\dot\alpha}.
\end{aligned}
\tag{2.1.4}
$$

The antichiral and chiral subspaces are parameterized by $z_+^A = (x_+, \theta_+^\alpha)$ and $z_-^A = (x_-^a, \bar\theta_{-\dot\alpha})$ respectively, which are defined by

$$
x_+^a = x^a + \mathrm{i}\theta\sigma^a\bar\theta \tag{2.1.5}
$$

$$
\theta_+^\alpha = \theta^\alpha. \tag{2.1.6}
$$

and

$$
x_-^a = x^a - \mathrm{i}\theta\sigma^a\bar\theta = (x_+^a)^* \tag{2.1.7}
$$

$$
\bar\theta_{-\dot\alpha} = \bar\theta_{\dot\alpha} = (\theta_{+\alpha})^*, \tag{2.1.8}
$$

Both are closed with respect to transformations by the super Poincaré group (2.1.4).

2.1.2 Tensor superfields and covariant derivatives

Field theories formulated on superspace make use of tensor superfields: field representations of the group $\mathcal{S}\Pi(1,3)$ which are defined on the coset coordinates z^A and classified by the irreducible representations of the subgroup $\mathcal{S}SO(1,3)$.

All irreducible representations of the Lorentz group $SO(1,3)$ (and hence $\mathcal{S}SO(1,3)$) are described in section A.1 of appendix A . Tensor superfield representations of the super Poincaré group of Lorentz type $(\frac{n}{2}, \frac{m}{2})$ are superfields which carry n undotted and m dotted spinor indices (simultaneously symmetrized in both) and transform under the action of the super Poincaré group as:

$$V'(z') = e^{\frac{1}{2}\lambda^{ab}M_{ab}} V(z). \tag{2.1.9}$$

Here the (external) tensor indices have been suppressed and it is understood that the generator of the associated Lorentz representation M_{ab} acts on these indices. The coordinates $z, z' \in \mathbb{R}^{4|4}$ are related by a super Poincaré transformation. One can rewrite such a representation using differential operators as

$$V'(z) = g(c, \epsilon, \bar{\epsilon}, \lambda) V(g^{-1}z) \tag{2.1.10}$$

with $z' = g^{-1}z$ being related by a super Poincaré transformation corresponding to the group element g^{-1} and

$$g(c, \epsilon, \bar{\epsilon}, \lambda) = \exp\Big(\mathrm{i}(-c^a \mathbf{P}_a + \frac{1}{2}\lambda^{ab}\boldsymbol{\mathcal{L}}_{ab} + \epsilon^\alpha \mathbf{Q}_\alpha + \bar{\epsilon}_{\dot{\alpha}} \bar{\mathbf{Q}}^{\dot{\alpha}})\Big). \tag{2.1.11}$$

Here the operators are easily identified by examining (2.1.4). They are:

$$\mathbf{P}_a = \mathrm{i}\partial_a \tag{2.1.12}$$

$$\boldsymbol{\mathcal{L}}_{ab} = \mathrm{i}(x_b\partial_a - x_a\partial_b + (\sigma_{ab})^{\alpha\beta}\theta_\alpha\partial_\beta - (\tilde{\sigma}_{ab})^{\dot{\alpha}\dot{\beta}}\bar{\theta}_{\dot{\alpha}}\bar{\partial}_{\dot{\beta}} - M_{ab}) \tag{2.1.13}$$

$$\mathbf{Q}_\alpha = \mathrm{i}\partial_\alpha + (\sigma^a)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_a \tag{2.1.14}$$

$$\bar{\mathbf{Q}}_{\dot{\alpha}} = -\mathrm{i}\partial_{\dot{\alpha}} - (\sigma^a)_{\alpha\dot{\alpha}}\theta^\alpha\partial_a. \tag{2.1.15}$$

Again the Lorentz generator M_{ab} acts only on the field's Lorentz indices.

The action of an infinitesimal supersymmetry transformations on an arbitrary superfield $V(z)$ can then be expressed as

$$\delta V(z) = \mathrm{i}(\epsilon^\alpha \mathbf{Q}_\alpha + \bar{\epsilon}_{\dot{\alpha}}\bar{\mathbf{Q}}^{\dot{\alpha}})V(z). \tag{2.1.16}$$

The derivative of a tensor superfield is itself not a tensor superfield since the derivatives ∂_A do not (anti)commute with all of the generators (2.1.12)-(2.1.15). In particular $[\partial_A, \mathbf{Q}_\alpha\}$ and $[\partial_A, \bar{\mathbf{Q}}_{\dot{\alpha}}\}$, where the graded commutator $[\,.\,,\,.\,\}$ is defined by

$$[\mathbf{X}, \mathbf{Y}\} = \mathbf{X}\cdot\mathbf{Y} - (-1)^{\varepsilon(\mathbf{X})\varepsilon(\mathbf{Y})}\mathbf{Y}\cdot\mathbf{X}, \tag{2.1.17}$$

do not vanish. Naturally one is interested in finding covariant derivatives $D_A = (D_a, D_\alpha, \bar{D}^{\dot{\alpha}})$ which map tensors to tensors,

$$[D_A, \mathbf{P}_a\} = [D_A, \mathbf{Q}_\alpha\} = [D_A, \bar{\mathbf{Q}}_{\dot{\alpha}}\} = 0. \tag{2.1.18}$$

They can be determined systematically via the coset construction, and are given by

$$D_a = \partial_a \qquad D^a = \eta^{ab} D_b = \partial^a \tag{2.1.19}$$

$$D_\alpha = \partial_\alpha + \mathrm{i}\bar\theta^{\dot\alpha}(\sigma^a)_{\alpha\dot\alpha}\partial_a \qquad D^\alpha = \varepsilon^{\alpha\beta}D_\beta \qquad D^\alpha = -\partial^\alpha - \mathrm{i}\bar\theta_{\dot\alpha}(\tilde\sigma_a)^{\dot\alpha\alpha}\partial^a \tag{2.1.20}$$

$$\bar D^{\dot\alpha} = \bar\partial^{\dot\alpha} + \mathrm{i}\theta_\alpha(\tilde\sigma^a)^{\dot\alpha\alpha}\partial^a \qquad \bar D_{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}\bar D^{\dot\beta} \qquad \bar D_{\dot\alpha} = -\bar\partial_{\dot\alpha} - \mathrm{i}\theta^\alpha(\sigma_a)_{\alpha\dot\alpha}\partial_a. \tag{2.1.21}$$

They satisfy the (anti)commutation relations

$$\{D_\alpha, D_\beta\} = \{\bar D^{\dot\alpha}, \bar D^{\dot\beta}\} = [D_\alpha, \partial_a] = [\bar D^{\dot\alpha}, \partial_a] = 0 \tag{2.1.22}$$

$$\{D_\alpha, \bar D_{\dot\alpha}\} = -2\mathrm{i}(\sigma^a)_{\alpha\dot\alpha}\partial_a, \tag{2.1.23}$$

the latter indicating that flat superspace possesses torsion.

One may also establish

$$\begin{aligned} &D_A(UV) = D_A(U)V + (-1)^{\varepsilon(U)\varepsilon_A} U D_A(V) \\ &\varepsilon(D_A V) = \varepsilon_A + \varepsilon(V) \pmod 2 \end{aligned} \tag{2.1.24}$$

and that the spinor covariant derivatives, D_α and $\bar D_{\dot\alpha}$, are related under complex conjugation

$$(D_\alpha V)^* = (-1)^{\varepsilon(V)}\bar D_{\dot\alpha} V^* \qquad (D^2 V)^* = \bar D^2 V^* \tag{2.1.25}$$

where V and U are arbitrary superfields and we have introduced the notation

$$D^2 = D^\alpha D_\alpha \qquad \bar D^2 = \bar D_{\dot\alpha}\bar D^{\dot\alpha}. \tag{2.1.26}$$

One may expand a tensor superfield in a finite Taylor series with respect to its odd coordinates, the coefficients of which, called component fields, are supersmooth in $x^a \in \mathbb{R}_c$. Such a series terminates since $\theta_\alpha\theta_\beta\theta_\gamma = \bar\theta_{\dot\alpha}\bar\theta_{\dot\beta}\bar\theta_{\dot\gamma}=0$. Collectively the component fields are referred to as a supermultiplet. It is often most convenient to define component fields so that they correspond to taking multiple covariant derivatives followed by setting $\theta_\alpha = \bar\theta_{\dot\alpha} = 0$. Following [127], the act of setting $\theta_\alpha = \bar\theta_{\dot\alpha} = 0$ will be referred to as a space projection, and denoted as $|$. Accordingly, for a real but otherwise unconstrained superfield $V(z) = V(z)^*$, the components can be defined as

$$\begin{gathered} A(x) = V(z)\Big| \qquad \psi_\alpha(x) = D_\alpha V(z)\Big| \qquad \bar\psi_{\dot\alpha}(x) = \bar D_{\dot\alpha} V(z)\Big| \\ F(x) = -\frac{1}{4}D^2 V(z)\Big| \qquad \bar F(x) = -\frac{1}{4}\bar D^2 V(z)\Big| \qquad C_{\alpha\dot\alpha}(x) = \frac{1}{2}[D_\alpha, \bar D_{\dot\alpha}]V(z)\Big| \\ \chi_\alpha(x) = -\frac{1}{4}D_\alpha \bar D^2 V(z)\Big| \qquad \bar\chi_{\dot\alpha}(x) = -\frac{1}{4}\bar D_{\dot\alpha} D^2 V(z)\Big| \\ H(x) = \frac{1}{32}\{D^2, \bar D^2\}V(z)\Big|. \end{gathered} \tag{2.1.27}$$

As is very well known, as a consequence of (2.1.4), (2.1.9), and (2.1.18), the components play the role of ordinary tensor fields when one reduces a field theory in superspace to one on Minkowski space. The supersymmetry transformations manifest themselves at the component level by relating some of these ordinary tensor fields. One can readily establish how the components transform under the action of an infinitesimal supersymmetry transformation by taking various numbers of covariant derivatives of (2.1.16) followed by space projection. In the process one uses (2.1.18), (2.1.27) and exploits the similarity of the supersymmetry generators (2.1.14), (2.1.15) and the covariant derivatives.

One may also consider superfields defined on the subspace of superspace with coordinates $z_+^A = (x_+, \theta_+^\alpha)$ or $z_-^A = (x_-^{\underline{a}}, \bar{\theta}_{-\dot{\alpha}})$, which are known as chiral and antichiral superfields respectively, as first introduced by Ferrara, Wess and Zumino [154]. This idea turns out to be of great significance in superfield theory primarily due to the fact chiral superfields have a much shorter Taylor expansion, and therefore yield smaller multiplets than the unconstrained superfield $V(z)$ described above. For example, a chiral superfield $\Phi(z_+)$ possess only three component fields:

$$\Phi(z_+) = A(x_+) + \theta^\alpha \psi_\alpha(x_+) + \theta^2 B(x_+). \tag{2.1.28}$$

or more fully

$$\begin{aligned} \Phi(z) =& A(x) + \theta^\alpha \psi_\alpha(x) + \theta^2 F(x) + \mathrm{i}\theta\sigma^a\bar{\theta}\partial_a A(x) \\ &+ \frac{\mathrm{i}}{2}\theta^2\bar{\theta}\tilde{\sigma}^a\partial_a\psi(x) + \frac{1}{4}\theta^2\bar{\theta}^2\partial^a\partial_a A(x). \end{aligned} \tag{2.1.29}$$

Equivalently, one may consider chiral and antichiral superfields as superfields satisfying constraints. A chiral superfield $\Phi(z)$ satisfies the constraint

$$\bar{D}^{\dot{\alpha}}\Phi(z) = 0, \tag{2.1.30}$$

the solution to which is given in (2.1.29). Similarly an antichiral superfield $\bar{\Phi}(z)$ satisfies

$$D_\alpha\bar{\Phi}(z) = 0. \tag{2.1.31}$$

The constraints are really just those of holomorphicity (for example see [127, 153]), implying these fields are defined on the chiral subspaces. The constraints are integrable because $\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = 0$.

Finally, it is useful to note that it is possible to recast many of the properties of superspace integration, delta functions (see section A.2) and component field extraction in terms of the covariant derivatives. In particular, for an arbitrary superfield $V(z)$, one has:

$$\int \mathrm{d}^8 z\, D_A(V(z)) = 0 \tag{2.1.32}$$

$$\int d^6z\, V(z) = -\frac{1}{4}\int d^4x \bar{D}^2 V(z) = -\frac{1}{4}\int d^4x \bar{D}^2 V(z)\Big| \tag{2.1.33}$$

$$\int d^6\bar{z}\, V(z) = -\frac{1}{4}\int d^4x D^2 V(z) = -\frac{1}{4}\int d^4x D^2 V(z)\Big| \tag{2.1.34}$$

$$\begin{aligned}\int d^8z\, V(z) &= -\frac{1}{4}\int d^6z \bar{D}^2 V(z) = \frac{1}{16}\int d^4x D^2\bar{D}^2 V(z) = \frac{1}{16}\int d^4x D^2 \bar{D}^2 V(z)\Big| \\ &= -\frac{1}{4}\int d^6\bar{z} D^2 V(z) = \frac{1}{16}\int d^4x \bar{D}^2 D^2 V(z) = \frac{1}{16}\int d^4x D^2\bar{D}^2 V(z)\Big| \\ &= \frac{1}{16}\int d^4x D^\alpha \bar{D}^2 D_\alpha V(z)\Big| = \frac{1}{16}\int d^4x \bar{D}_{\dot\alpha} D^2 \bar{D}^{\dot\alpha} V(z)\Big|. \end{aligned} \tag{2.1.35}$$

The (anti)chiral delta function can be expressed as

$$\delta_+(z-z) = -\frac{1}{4}\bar{D}^2\delta^{(8)}(z-z) \qquad \delta_-(z-z) = -\frac{1}{4}D^2\delta^{(8)}(z-z). \tag{2.1.36}$$

2.1.3 Actions and equations of motion

As with all classical field theories, the object of central interest in theories formulated in $\mathcal{N}=1$ superspace is the action functional. More precisely, superfield theories are expressed in terms of an action which when extremized yields the dynamical equations of motion. Generally speaking this means defining, for a given set of tensor superfields $v^M(z)$ and their derivatives, a super Poincaré invariant action (super)functional $S : v^M(z) \mapsto \mathbb{R}_c$ as follows:

$$\begin{aligned} S[v^M(z)] &= \int d^8z\, \mathcal{L}(v^M, D_A v^M, \ldots, D_{A_1}\ldots D_{A_k} v^M) \\ &\quad + \left\{ \int d^6z\, \mathcal{L}_c(v^M, D_A v^M, \ldots, D_{A_1}\ldots D_{A_k} v^M) + c.c. \right\} \end{aligned} \tag{2.1.37}$$

where *c.c.* denotes the complex conjugate. Here the super Lagrangian density $\mathcal{L}$ is simply a scalar superfield (ie a Lorentz scalar combination of $v^M(z)$ and its derivatives), and $\mathcal{L}_c$ a chiral scalar superfield subject to the constraint $\bar{D}_{\dot\alpha}\mathcal{L}_c = 0$. Actions constructed in this manner are manifestly invariant under super Poincaré transformations (for complete details see [89, 127]).

The component form of any such action, by which it is meant the action cast in terms of the superfield's components in Minkowski space, is in general found by putting together the rules for reducing superspace integrals to Minkowski space and the definition of the component fields. Usually one works with the covariant derivative identities (2.1.35) and (2.1.27). This prescription yields

$$\begin{aligned} S[v] &= \int d^8z\, \mathcal{L}(v) + \left\{ \int d^6z\, \mathcal{L}_c(v) + c.c. \right\} \\ &= \int d^4x \left(\frac{1}{32}\{D^2, \bar{D}^2\}\mathcal{L}(v) - \left(\frac{1}{4}D^2\mathcal{L}_c(v) + c.c.\right)\right)\Bigg|. \end{aligned} \tag{2.1.38}$$

Unlike field theories expressed in ordinary spacetime we now have an added complication due to the existence of chiral and antichiral subspace. Accordingly one is required to generalize the notion of the functional derivative.

Consider a dynamical system of tensor superfields $v^M(z) = (V^m(z), \Phi^\mu(z), \bar{\Phi}^{\dot{\mu}}(z))$, where $V^m(z) = (V^m(z))^*$, $\bar{D}_{\dot{\alpha}}\Phi^\mu(z) = 0$ and $\bar{\Phi}^{\dot{\mu}} = (\Phi^\mu(z))^*$. Under an arbitrary infinitesimal variation of these fields, $v^M(z) \to v^M(z) + \delta v^M(z)$, the variation of the action δS is expressed as

$$\begin{aligned}\delta S[v] &= S[v+\delta v] - S[v] \\ &= S[V+\delta V, \Phi+\delta\Phi, \bar{\Phi}+\delta\bar{\Phi}] - S[V,\Phi,\bar{\Phi}] \\ &= \int \mathrm{d}^8 z\, \delta V^m(z) \frac{\delta S}{\delta V^m(z)} + \int \mathrm{d}^6 z\, \delta\Phi^\mu(z) \frac{\delta S}{\delta \Phi^\mu(z)} + \int \mathrm{d}^6 \bar{z}\, \delta\bar{\Phi}^{\dot{\mu}}(z) \frac{\delta S}{\delta \bar{\Phi}^{\dot{\mu}}(z)}\end{aligned} \tag{2.1.39}$$

and $\delta S/\delta V^m(z)$, $\delta S/\delta\Phi^\mu(z)$ and $\delta S/\delta\bar{\Phi}^{\dot{\mu}(z)}$ are the left (super)functional derivatives of S at $V^m(z)$, $\Phi^\mu(z)$ and $\bar{\Phi}^{\dot{\mu}}(z)$ respectively. The classical equations of motion are then expressed through the stationary action principle, i.e. solutions are the set of fields which satisfy the equations

$$\frac{\delta S}{\delta V^m} = \frac{\delta S}{\delta \Phi^\mu} = \frac{\delta S}{\delta \bar{\Phi}^{\dot{\mu}}} = 0. \tag{2.1.40}$$

Finally it should be pointed out that (at least currently) no physical meaning is usually ascribed to superfields or superspace. Its usage is ordinarily viewed as an elegant and useful mathematical book-keeping device for the different component fields in a supermultiplet. Consequently all physical meaning of any superfield theory is obtained purely through the underlying component action in Minkowski space.

2.2 Superfield theories

In this section we begin by examining the simplest $\mathcal{N} = 1$ superfield theory, the so-called Wess-Zumino models, which illustrate the important features of $\mathcal{N} = 1$ superfield theories. Discovered and discussed in detail by Wess and Zumino [155], these models where the first examples of four dimensional supersymmetric field theories with linearly realized supersymmetry. Afterward we will consider the generalization to include local gauge invariance, namely supersymmetric Yang-Mills theories.

2.2.1 Extended Wess-Zumino models

From a physical point of view, the most general action involving n chiral fields $\Phi^i(z) = (\bar{\Phi}_i(z))^*$, $i = 1, \ldots, n$, with $\bar{D}_{\dot{\alpha}}\Phi^i = D_\alpha \bar{\Phi}_i = 0$, is

$$S[\Phi, \bar{\Phi}] = \int \mathrm{d}^8 z \, \bar{\Phi}_i \Phi^i + \int \mathrm{d}^6 z \, \mathcal{L}_c(\Phi) + \int \mathrm{d}^6 \bar{z} \, \bar{\mathcal{L}}_c(\bar{\Phi}) \tag{2.2.41}$$

where $\mathcal{L}_c(\Phi)$ and $\bar{\mathcal{L}}_c(\bar{\Phi})$ are finite polynomials in Φ and $\bar{\Phi}$ respectively. This will be referred to as the extended Wess-Zumino model, since the case where $n = 1$ and $\mathcal{L}_c$ is at most a third order polynomial is generally known as just the Wess-Zumino model.

Recalling that the power series expansion of a chiral superfield is given by equation (2.1.29), we can express the components as

$$A^i(x) = \Phi^i(z)\Big| \qquad \psi^i_\alpha(x) = D_\alpha \Phi^i(z)\Big| \qquad F^i(x) = -\frac{1}{4} D^2 \Phi^i(z)\Big| \tag{2.2.42}$$

and similarly for $\bar{\Phi}_i(z)$. The component action is easily found to be:

$$\begin{aligned} S &= \int \mathrm{d}^8 z \, \bar{\Phi}_i \Phi^i + \left(\int \mathrm{d}^6 z \, \mathcal{L}_c(\Phi) + c.c. \right) \\ &= \int \mathrm{d}^4 x \, \Bigg(A^i(x) \partial^a \partial_a \bar{A}_i(x) - \frac{\mathrm{i}}{2} \psi^{i\alpha}(x) \partial_{\alpha\dot{\alpha}} \bar{\psi}(x)^{\dot{\alpha}}_i + F^i(x) \bar{F}_i(x) \\ &\qquad + \Big\{ \Big(-\frac{1}{4} \frac{\partial^2 \mathcal{L}_c(A(x))}{\partial A^i(x) \partial A^j(x)} \psi^{i\alpha}(x) \psi^j_\alpha(x) + \frac{\partial \mathcal{L}_c(A(x))}{\partial A^i(x)} F^i(x) \Big) + c.c. \Big\} \Bigg). \end{aligned} \tag{2.2.43}$$

Consequently at the component level the extended Wess-Zumino action (2.2.41) describes a system of n complex scalar fields $\Phi^i(x)$, n complex scalar auxiliary fields $F^i(x)$ and n Majorana spinors.

Using the techniques described in subsection 2.1.2, one finds that under infinitesimal supersymmetry transformations with parameters $\epsilon_\alpha = (\bar{\epsilon}_{\dot{\alpha}})^*$ the component fields transform linearly as follows:

$$\begin{aligned} \delta A^i(x) &= -\epsilon \psi^i(x) \\ \delta \psi^i_\alpha(x) &= -2\epsilon_\alpha F^i(x) - 2\mathrm{i}\bar{\epsilon}^{\dot{\alpha}} \partial_{\alpha\dot{\alpha}} A^i(x) \\ \delta F^i(x) &= -\mathrm{i}\bar{\epsilon}^{\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \psi^{i\alpha}(x). \end{aligned} \tag{2.2.44}$$

At this stage one may determine the equations of motion for the component fields by either using the principle of least action on the component action, or by deriving the superfield equations of motion. It is generally easier to take the latter route, and obtain the component equations of motion by operating on the superfield equations of motion with various numbers of covariant derivatives followed by space projections.

Having eliminated the auxiliary fields through their equations of motion, one finds that the resulting component action possess non-linear, on-shell supersymmetry which is dependent on the particular Wess-Zumino model under consideration. This is a typical feature of $\mathcal{N} = 1$ superfield theories.

As a final remark, examination of the quantum properties of the extended Wess-Zumino models shows that in fact the most general renormalizable model is one where $\mathcal{L}_c(\Phi)$ is at most cubic having the general form

$$\mathcal{L}_c(\Phi) = a + b_i\Phi^i + m_{ij}\Phi^i\Phi^j + \lambda_{ijk}\Phi^i\Phi^j\Phi^k \tag{2.2.45}$$

Here m_{ij} is the mass matrix, and cubic terms generate interactions, in complete analogy to standard quantum field theory.

2.2.2 Supersymmetric Yang-Mills theories

In superspace

Examination of the first term in the extended Wess-Zumino action (2.2.41),

$$\int d^8z\, \bar{\Phi}_i\Phi^i \tag{2.2.46}$$

immediately reveals a global symmetry. More specifically, consider a finite dimensional (perhaps reducible) representation $\mathcal{R}$ of a compact connected Lie group G, generated by a set of Hermitian generators $(\mathcal{T}^I)^i{}_j$ which satisfy

$$[\mathcal{T}^I, \mathcal{T}^J] = \mathrm{i}f^{IJK}\mathcal{T}^K \qquad (\mathcal{T}^I)^\dagger = \mathcal{T}^I \qquad I = 1, 2, \dots, \dim G \tag{2.2.47}$$

where the structure constants f^{IJK} will be assumed to be antisymmetric. The action (2.2.46) is then invariant under the global symmetry transformations

$$\Phi'^i(z) = \big(\mathrm{e}^{\mathrm{i}\xi^I\mathcal{T}^I}\big)^i{}_j\Phi^j(z) \qquad \bar{\Phi}'_i(z) = \bar{\Phi}_j(z)\big(\mathrm{e}^{-\mathrm{i}\xi^I\mathcal{T}^I}\big)^j{}_i \qquad \xi^I \in \mathbb{R}_c. \tag{2.2.48}$$

which will extend to the entire action provided $\mathcal{L}_c$ is also invariant.

Promoting this symmetry to a local or gauge symmetry leads to supersymmetric Yang-Mills models. The scalar field Φ^i satisfies the chirality constraint $\bar{D}_{\dot{\alpha}}\Phi = 0$, which cannot be maintained under local transformations in which the gauge parameter is an arbitrarily superfield, ie

$$\bar{D}_{\dot{\alpha}}(\Phi'(z)) = \bar{D}_{\dot{\alpha}}(e^{\mathrm{i}\xi^I(z)\mathcal{T}^I}\Phi(z)) \neq 0 \tag{2.2.49}$$

for arbitrary superfield $\xi^I(z)$. For consistency we must demand that $\xi^I(z)$ itself be chiral, which immediately ruins the invariance of $\bar{\Phi}\Phi$.

The solution to this problem is to introduce a real but otherwise arbitrary 'compensating' or gauge superfield that takes values in the Lie algebra, $V(z) = V^I(z)\mathcal{T}^I$, and has the following gauge transformation properties[1]

$$e^{2V'(z)} = e^{\mathrm{i}\bar{\Lambda}(z)}e^{2V(z)}e^{-\mathrm{i}\Lambda(z)} \tag{2.2.50}$$

where

$$\begin{aligned} \Lambda(z) = \Lambda^I(z)\mathcal{T}^I \qquad & \bar{D}_{\dot{\alpha}}\Lambda^I(z) = 0 \\ \bar{\Lambda}^I(z) = \bar{\Lambda}^I(z)\mathcal{T}^I \qquad & D_\alpha\bar{\Lambda}^I(z) = 0 \\ (\Lambda^I(z))^* = \bar{\Lambda}^I(z). & \end{aligned} \tag{2.2.51}$$

The gauge superfield V, also known as the Yang-Mills superfield, is used to yield the gauge invariant action

$$\int \mathrm{d}^8 z\, \bar{\Phi} e^{2V} \Phi \tag{2.2.52}$$

where the gauge transformations are now:

$$\Phi'(z) = e^{\mathrm{i}\Lambda(z)}\Phi(z) \qquad \bar{\Phi}'(z) = \bar{\Phi}(z)e^{-\mathrm{i}\bar{\Lambda}(z)} \tag{2.2.53}$$

with Λ and $\bar{\Lambda}$ subject to (2.2.51). The component form of this action describes the coupling of a Yang-Mills field to spin 1/2 matter.

The covariant derivatives D_A do not preserve the gauge transformation properties. In analogy to ordinary gauge theories it is natural to look for *gauge* covariant derivatives, and attempt to identify torsion and curvature in terms of them. Accordingly one introduces a set set of gauge covariant derivatives, $\mathcal{D}_A^{(+)} = (\mathcal{D}_a^{(+)}, \mathcal{D}_\alpha^{(+)}, \bar{\mathcal{D}}^{\dot{\alpha}(+)})$, which satisfy

$$(\mathcal{D}_A^{(+)}\Phi) \rightarrow (\mathcal{D}_A^{(+)}\Phi)' = e^{\mathrm{i}\Lambda}(\mathcal{D}_A^{(+)}\Phi) \qquad \Rightarrow \qquad \mathcal{D}_A^{'(+)} = e^{\mathrm{i}\Lambda}\mathcal{D}_A^{(+)}e^{-\mathrm{i}\Lambda}. \tag{2.2.54}$$

As usual one achieves this by introducing gauge connections taking values in the Lie algebra, in this case the superfields $\Gamma_A^{(+)}(z) = \Gamma_A^{I(+)}(z)\mathcal{T}^I$, such that

$$\mathcal{D}_A^{(+)} = D_A + \mathrm{i}\Gamma_A^{(+)}(z) \qquad (\mathcal{D}_A^{(+)}\Phi) = (D_A\Phi(z)) + \mathrm{i}\Gamma_A^{(+)}(z)\Phi(z). \tag{2.2.55}$$

This immediately implies that the connection transforms as

$$\Gamma_A^{'(+)}(z) = e^{\mathrm{i}\Lambda}\Gamma_A^{(+)}(z)e^{-\mathrm{i}\Lambda} - \mathrm{i}(e^{\mathrm{i}\Lambda}D_A e^{-\mathrm{i}\Lambda}) \tag{2.2.56}$$

under gauge transformations. Examining (2.2.54) and the transformations properties (2.2.50) of the gauge field $V(z)$ introduced above, one particular choice of a

[1] The choice of $2V(z)$ instead of just $V(z)$ is merely a matter of convenience.

gauge connection is:

$$\begin{aligned}
\mathcal{D}^{(+)}_\alpha &= e^{-2V} D_\alpha e^{2V} & \bar{\mathcal{D}}^{\dot\alpha(+)} &= \bar{D}^{\dot\alpha} \\
\mathcal{D}^{(+)}_a &= -\frac{1}{2}(\tilde\sigma_a)^{\dot\alpha\alpha}\mathcal{D}^{(+)}_{\alpha\dot\alpha} = -\frac{\mathrm{i}}{4}(\tilde\sigma_a)^{\dot\alpha\alpha}\{\mathcal{D}^{(+)}_\alpha, \bar{\mathcal{D}}^{(+)}_{\dot\alpha}\} \\
\Gamma^{(+)}_\alpha &= -\mathrm{i}(e^{-2V} D_\alpha e^{2V}) \qquad \Gamma^{(+)}_{\dot\alpha} = 0 \qquad \Gamma^{(+)}_a = -\frac{1}{4}(\tilde\sigma_a)^{\dot\alpha\alpha}\bar{D}_{\dot\alpha}(e^{-2V} D_\alpha e^{2V}).
\end{aligned} \tag{2.2.57}$$

The supertorsion $H_{AB}{}^C$ and supercurvature $F^{(+)}_{AB} = F^{I(+)}_{AB}\mathcal{T}^I$ tensors are then identified from their definitions through the (anti)commutation relations

$$\begin{aligned}
[\mathcal{D}^{(+)}_A, \mathcal{D}^{(+)}_B\} &= \mathcal{D}^{(+)}_A\mathcal{D}^{(+)}_B - (-1)^{\varepsilon_A\varepsilon_B}\mathcal{D}^{(+)}_B\mathcal{D}^{(+)}_A \\
&= H_{AB}{}^C\mathcal{D}^{(+)}_C + \mathrm{i}F^{(+)}_{AB}
\end{aligned} \tag{2.2.58}$$

of the gauge covariant derivatives, which satisfy the following algebra [156]

$$\begin{aligned}
\{\mathcal{D}^{(+)}_\alpha, \mathcal{D}^{(+)}_\beta\} &= \{\bar{\mathcal{D}}^{(+)}_{\dot\alpha}, \bar{\mathcal{D}}^{(+)}_{\dot\beta}\} = 0 \\
\{\mathcal{D}^{(+)}_\alpha, \bar{\mathcal{D}}^{(+)}_{\dot\alpha}\} &= -2\mathrm{i}\mathcal{D}^{(+)}_{\alpha\dot\alpha} = -2\mathrm{i}(\sigma^a)_{\alpha\dot\alpha}\mathcal{D}^{(+)}_a \\
[\mathcal{D}^{(+)}_\alpha, \mathcal{D}^{(+)}_{\beta\dot\beta}] &= 2\mathrm{i}\varepsilon_{\alpha\beta}\bar{W}^{(+)}_{\dot\beta} \qquad \bar{W}^{(+)}_{\dot\beta} = \bar{W}^{I(+)}_{\dot\beta}\mathcal{T}^I \\
[\bar{\mathcal{D}}^{(+)}_{\dot\alpha}, \mathcal{D}^{(+)}_{\beta\dot\beta}] &= 2\mathrm{i}\varepsilon_{\dot\alpha\dot\beta}W^{(+)}_\beta \qquad W^{(+)}_\beta = W^{I(+)}_\beta\mathcal{T}^I \\
[\mathcal{D}^{(+)}_{\alpha\dot\alpha}, \mathcal{D}^{(+)}_{\beta\dot\beta}] &= (\sigma^a)_{\alpha\dot\alpha}(\sigma^b)_{\beta\dot\beta}G^{(+)}_{ab} = -\varepsilon_{\alpha\beta}(\bar{\mathcal{D}}^{(+)}_{\dot\alpha}\bar{W}^{(+)}_{\dot\beta}) - \varepsilon_{\dot\alpha\dot\beta}(\mathcal{D}^{(+)}_\alpha W^{(+)}_\beta).
\end{aligned} \tag{2.2.59}$$

Contraction of the spinor indices in the last commutator yields the Bianchi identity $(\mathcal{D}^{(+)\alpha}W^{(+)}_\alpha) = (\bar{\mathcal{D}}^{(+)}_{\dot\alpha}\bar{W}^{(+)\dot\alpha})$.

The supercurvature superfields (or gauge superfield strengths)

$$\begin{aligned}
W^{(+)}_\alpha &= -\frac{1}{8}\bar{D}^2(e^{-2V}D_\alpha e^{2V}) \\
\bar{W}^{(+)}_{\dot\alpha} &= \frac{1}{8}e^{-2V}D^2(e^{2V}\bar{D}_{\dot\alpha}e^{-2V})e^{2V},
\end{aligned} \tag{2.2.60}$$

are used to construct the 'pure' or kinetic part of the gauge superfield's action:

$$\frac{1}{4g^2C(\mathcal{R})}\mathrm{tr}_\mathcal{R}\left(\int \mathrm{d}^6z\, W^{\alpha(+)}W^{(+)}_\alpha + \int \mathrm{d}^6\bar{z}\,\bar{W}^{(+)}_{\dot\alpha}\bar{W}^{\dot\alpha(+)}\right). \tag{2.2.61}$$

Here g is a coupling constant, $\mathrm{tr}_\mathcal{R}$ indicates the trace over the gauge indices of the representation $\mathcal{R}$, and $C(\mathcal{R})$ is a Casimir coefficient defined by $\mathrm{tr}_\mathcal{R}(T^IT^J) = C(\mathcal{R})\delta^{IJ}$. One can readily establish that this part of the action is invariant under gauge transformations due to the cyclic properties of the trace and the fact that the superfield strengths have the following homogeneous gauge transformation properties

$$\begin{aligned}
W^{(+)}_\alpha &\to W'^{(+)}_\alpha = e^{\mathrm{i}\Lambda}W^{(+)}_\alpha e^{-\mathrm{i}\Lambda} \\
\bar{W}^{(+)}_{\dot\alpha} &\to \bar{W}'^{(+)}_{\dot\alpha} = e^{\mathrm{i}\bar\Lambda}\bar{W}^{(+)}_{\dot\alpha}e^{-\mathrm{i}\bar\Lambda}.
\end{aligned} \tag{2.2.62}$$

Alternatively one may have sought gauge covariant derivatives $\mathcal{D}_A^{(-)}$ which maintained the gauge transformation property

$$(\mathcal{D}_A^{(-)}\bar{\Phi}^{\mathrm{T}}) \to (\mathcal{D}_A^{(-)}\bar{\Phi}^{\mathrm{T}})' = e^{\mathrm{i}\bar{\Lambda}}(\mathcal{D}_A^{(-)}\bar{\Phi}^{\mathrm{T}}) \qquad \Rightarrow \quad \mathcal{D}_A'^{(-)} = e^{\mathrm{i}\bar{\Lambda}}\mathcal{D}_A^{(-)}e^{-\mathrm{i}\bar{\Lambda}}. \tag{2.2.63}$$

The difference is merely a matter of representation, and the + and − representations (the chiral and antichiral representations respectively) are explicitly related by

$$\mathcal{D}_A^{(+)} = e^{-2V}\mathcal{D}_A^{(-)}e^{2V}. \tag{2.2.64}$$

The subsequent field strengths are related through similar identities, for example

$$W_\alpha^{(+)} = e^{-2V}W_\alpha^{(-)}e^{2V}. \tag{2.2.65}$$

Often one expresses the chiral part of the action (2.2.52) in terms of gauge covariantly chiral scalar fields

$$\bar{\Phi}e^{2V}\Phi = \bar{\Phi}^{(+)}\Phi^{(+)} = \bar{\Phi}^{(-)}\Phi^{(-)} \tag{2.2.66}$$

which satisfy

$$\Phi^{(+)} = \Phi \qquad \bar{\Phi}^{(+)} = \bar{\Phi}e^{2V} \qquad \bar{\mathcal{D}}_{\dot{\alpha}}^{(+)}\Phi^{(+)} = \mathcal{D}_\alpha^{(+)}\bar{\Phi}^{(+)\mathrm{T}} = 0 \tag{2.2.67}$$

$$\Phi^{(-)} = e^{2V}\Phi \qquad \bar{\Phi}^{(-)} = \bar{\Phi} \qquad \bar{\mathcal{D}}_{\dot{\alpha}}^{(-)}\Phi^{(-)} = \mathcal{D}_\alpha^{(-)}\bar{\Phi}^{(-)\mathrm{T}} = 0. \tag{2.2.68}$$

Clearly in these representations the gauge covariantly chiral and antichiral scalars are not directly related by Hermitian conjugation. An 'in-between' or 'central' representation which maintains this property, is defined by splitting the vector field as

$$e^{2V} = e^{w}e^{\bar{w}} \qquad \bar{w}^\dagger = w = w^I\mathcal{T}^I. \tag{2.2.69}$$

The corresponding gauge covariant derivatives, field strengths and gauge covariantly (anti)chiral superfields are then defined by

$$\begin{aligned}\mathcal{D}_\alpha = e^{-w}D_\alpha e^{w}, \qquad \bar{\mathcal{D}}_{\dot{\alpha}} = e^{\bar{w}}\bar{D}_{\dot{\alpha}}e^{-\bar{w}}\\ \mathcal{D}_a = -\frac{1}{2}(\tilde{\sigma}_a)^{\dot{\alpha}\alpha}\mathcal{D}_{\alpha\dot{\alpha}} = -\frac{\mathrm{i}}{4}(\tilde{\sigma}_a)^{\dot{\alpha}\alpha}\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot{\alpha}}\}\\ [\bar{\mathcal{D}}_{\dot{\alpha}}, \mathcal{D}_{\beta\dot{\beta}}] = 2\mathrm{i}\varepsilon_{\dot{\alpha}\dot{\beta}}W_\beta, \qquad [\mathcal{D}_\alpha, \mathcal{D}_{\beta\dot{\beta}}] = 2\mathrm{i}\varepsilon_{\alpha\beta}\bar{W}_{\dot{\beta}}.\end{aligned} \tag{2.2.70}$$

The action is representation independent and from here onward a particular representation will only be specified as necessary.

The complete action in $\mathcal{N} = 1$ superspace describing non-Abelian Yang-Mills fields V coupled to matter Φ is:

$$S[\Phi, \bar{\Phi}, V] = \int \mathrm{d}^8 z\, \bar{\Phi}e^{2V}\Phi + \Big(\int \mathrm{d}^6 z\, \mathcal{L}_c(\Phi) + c.c\Big) + \frac{1}{2g^2C(\mathcal{R})}\int \mathrm{d}^6 z\, \mathrm{tr}_{\mathcal{R}}(W^2) \tag{2.2.71}$$

having used

$$\int \mathrm{d}^6 z \, \mathrm{tr}_{\mathcal{R}}(W^2) = \int \mathrm{d}^6 \bar{z} \, \mathrm{tr}_{\mathcal{R}}(\bar{W}^2) \tag{2.2.72}$$

modulo total derivative terms. This action will be gauge invariant provided the coefficients in the powers series expansion of $\mathcal{L}_c$ are gauge invariant tensors of the gauge group.

Yang-Mills theories in components

We will now briefly examine (2.2.71) at the component level and demonstrate that it is in fact a supersymmetric Yang-Mills theory.

Before we proceed we must first examine exactly how V transforms under gauge transformations. Recalling (2.2.50), with a little work one may obtain the following expression, which describes how V transforms under an infinitesimal gauge transformation:

$$\begin{aligned}\delta V &= -\frac{\mathrm{i}}{2} L_V(\bar{\Lambda} + \Lambda) + \frac{\mathrm{i}}{2} L_V \coth(L_V)(\bar{\Lambda} - \Lambda) \\ &= \frac{\mathrm{i}}{2}(\bar{\Lambda} - \Lambda) - \frac{\mathrm{i}}{2}[V, \bar{\Lambda} + \Lambda] + \frac{\mathrm{i}}{6}[V[V, \bar{\Lambda} - \Lambda]] + \mathcal{O}(V^4)\end{aligned} \tag{2.2.73}$$

where $L_A(B) = [A, B]$, or more explicitly

$$L_A^0(B) = B \qquad\qquad L_A^n(B) = [A, L_A^{n-1}(B)]. \tag{2.2.74}$$

In deriving (2.2.73) we have used the identities

$$\mathrm{e}^{A+\lambda B} = \mathrm{e}^A \Big(1 + \int_0^1 \mathrm{d}\xi \, \mathrm{e}^{-\xi A} \lambda B \mathrm{e}^{\xi A}\Big) = \Big(1 + \int_0^1 \mathrm{d}\xi \, \mathrm{e}^{\xi A} \lambda B \mathrm{e}^{-\xi A}\Big) \mathrm{e}^A \tag{2.2.75}$$

and

$$\mathrm{e}^A B \mathrm{e}^{-A} = \sum_{n=0}^{\infty} \frac{1}{n!} L_A^n(B) = \mathrm{e}^{L_A} B \tag{2.2.76}$$

for operators A and B, and infinitesimal parameter λ.

In order to simplify the discussion we will consider the special case where the real gauge field $V^I(z)$ has the following component content:

$$\begin{aligned} &V^I(z)\Big| = D_\alpha V^I(z)\Big| = D^2 V^I(z)\Big| = 0 && V^I_{\alpha\dot{\alpha}}(x) = \frac{1}{2}[D_\alpha, \bar{D}_{\dot{\alpha}}] V^I(z)\Big| \\ &\chi^I_\alpha(x) = -\frac{1}{4} D_\alpha \bar{D}^2 V^I(z)\Big| && \bar{\chi}^I_{\dot{\alpha}}(x) = -\frac{1}{4} \bar{D}_{\dot{\alpha}} D^2 V^I(z)\Big| \\ &N^I(x) = \frac{1}{32}\{D^2, \bar{D}^2\} V^I(z)\Big|. \end{aligned} \tag{2.2.77}$$

which implies $V^3 = 0$. Due to the fact that the component field $V_a(x)$ turns out to be the Yang-Mills vector potential, V is in general known as a vector multiplet. This simplified component content can be achieved by eliminating some of the component fields in $V(z)$ through a gauge choice known as the Wess-Zumino gauge [157]. This choice of gauge unfortunately breaks supersymmetry, but it can be restored by accompanying the supersymmetry transformations by a compensating gauge transformation. See [89, 127] for details.

The most general gauge transformation consistent with the gauge choice (2.2.77) is found by examining δV at the component level. For example, a space projection of (2.2.73) along with the definition of the components of V immediately yields

$$\delta V\Big| = \frac{\mathrm{i}}{2}(\bar{\Lambda} - \Lambda)\Big| = 0 \qquad \Rightarrow \quad \bar{\Lambda}\Big| = \Lambda\Big|. \tag{2.2.78}$$

Proceeding in this way, one finds that the most general gauge transformation consistent with the Wess-Zumino gauge is given by

$$\Lambda(z) = e^{\mathrm{i}\mathcal{H}}\xi(x) \qquad\qquad \xi = \xi^I \mathcal{T}^I = \xi^\dagger \tag{2.2.79}$$

which acts on V as

$$\delta V = \frac{\mathrm{i}}{2}(\bar{\Lambda} - \Lambda) - \frac{\mathrm{i}}{2}[V, \bar{\Lambda} + \Lambda]. \tag{2.2.80}$$

Operating on this and (2.2.79) with covariant derivatives, we can project out the transformation properties of the components. In doing so we establish that they all transform according to standard non-Abelian Yang-Mills transformation laws

$$\begin{aligned} \delta V_a^I &= \partial_a \xi^I - \xi^K f^{KJI} V_a^J \\ \chi_\alpha^I &= -\xi^K f^{KJI} \chi_\alpha^J \\ N^I &= -\xi^K f^{KJI} N^J. \end{aligned} \tag{2.2.81}$$

Similarly the components of Φ, defined by (2.2.42), transform as

$$\begin{aligned} \delta A^i &= \mathrm{i}\xi^K (\mathcal{T}^K)^i{}_j A^j \\ \delta \psi_\alpha^i &= \mathrm{i}\xi^K (\mathcal{T}^K)^i{}_j \psi_\alpha^j \\ \delta F^i &= \mathrm{i}\xi^K (\mathcal{T}^K)^i{}_j F^j. \end{aligned} \tag{2.2.82}$$

It is a straight forward exercise, using the techniques described earlier, to extract the component action. The pure Yang-Mills part of the action reduces to:

$$\begin{aligned} &\frac{1}{2g^2 C(\mathcal{R})} \int \mathrm{d}^6 z \, \mathrm{tr}_{\mathcal{R}}(W^2) \\ &\qquad = \frac{1}{g^2 C(\mathcal{R})} \int \mathrm{d}^4 x \, \mathrm{tr}_{\mathcal{R}} \Big(-\frac{1}{4} F^{ab} F_{ab} - \mathrm{i}\chi \sigma^a \nabla_a \bar{\chi} + 2N^2 \Big) \end{aligned} \tag{2.2.83}$$

where

$$\begin{aligned} \nabla_a &= \partial_a - \mathrm{i}V_a, & [\nabla_a, \nabla_b] &= -\mathrm{i}F_{ab} \\ F_{ab} &= \partial_a V_b - \partial_b V_a - \mathrm{i}[V_a, V_b], & \nabla_a \bar{\chi}^{\dot{\alpha}} &= \partial_a \bar{\chi}^{\dot{\alpha}} - \mathrm{i}[V_a, \bar{\chi}^{\dot{\alpha}}] \\ V_a &= V_a^I \mathcal{T}^I \qquad N = N^I \mathcal{T}^I & \chi_\alpha &= \chi_\alpha^I \mathcal{T}^I. \end{aligned} \tag{2.2.84}$$

The component field N is auxiliary. The component action for the remainder of the Yang-Mills action is found to be

$$\begin{aligned}
&\int \mathrm{d}^8 z\, \bar{\Phi} e^{2V} \Phi + \Big\{ \int \mathrm{d}^6 z\, \mathcal{L}_c(\Phi) + c.c \Big\} \\
&= \int \mathrm{d}^4 x \Bigg(-(\nabla^a \bar{A}^i)(\nabla_a A_i) + 2\bar{A}_i N^I (T^I)^i{}_j A^j - \frac{\mathrm{i}}{2}\psi \sigma^a \nabla_a \bar{\psi}(x)_i - \bar{\psi}_i \bar{\chi}^I (T^I)^i{}_j A^j \\
&\qquad - \bar{A}_i \chi^I (T^I)^i{}_j \psi^j + F^i(x) \bar{F}_i(x) \\
&\qquad + \Big\{ \Big(-\frac{1}{4} \frac{\partial^2 \mathcal{L}_c(A(x))}{\partial A^i(x) \partial A^j(x)} \psi^{i\alpha}(x) \psi^i_\alpha(x) + \frac{\partial \mathcal{L}_c(A(x))}{\partial A^i(x)} F^i(x) \Big) + c.c. \Big\} \Bigg)
\end{aligned} \tag{2.2.85}$$

where

$$\begin{aligned}
\nabla_a A^i &= \partial_a A^i - \mathrm{i} V^I_a (\mathcal{T}^I)^i{}_j A^j \\
\nabla_a \chi^i_\alpha &= \partial_a \chi^i_\alpha - \mathrm{i} V^I_a (\mathcal{T}^I)^i{}_j \chi^j_\alpha
\end{aligned} \tag{2.2.86}$$

and (2.2.71) is just the off-shell supersymmetric extension of a field theory with n complex scalar fields $A^i(x)$ coupled to a Yang-Mills field $V^I_a(x)$.

The most general renormalizable $\mathcal{N} = 1$ Yang-Mills action is then described by the action (2.2.71) where $\mathcal{L}$ is given by (2.2.45), provided the coefficients m_{ij} and λ_{ijk} are invariant tensors of the gauge group.

The equations of motion in superspace

To complete this discussion we will derive the superspace equations of motion for the general super Yang-Mills action (2.2.71). The most direct approach is as follows. For an alternative for pure super Yang-Mills theory see [127].

Under an arbitrary variation δV of the gauge superfield, the variation of the action (2.2.71) is given by

$$\begin{aligned}
\delta_V S =& S[\Phi, \bar{\Phi}, V + \delta V] - S[\Phi, \bar{\Phi}, V] \\
=& \int \mathrm{d}^8 z\, \bar{\Phi} \delta \mathrm{e}^{2V} \Phi + \frac{1}{g^2 C(\mathcal{R})} \Big(\mathrm{tr}_{\mathcal{R}} \int \mathrm{d}^6 z\, (\delta W^\alpha) W_\alpha \Big) \\
=& \int \mathrm{d}^8 z\, \bar{\Phi} \delta \mathrm{e}^{2V} \Phi + \frac{1}{2g^2 C(\mathcal{R})} \Big(\mathrm{tr}_{\mathcal{R}} \int \mathrm{d}^8 z\, \delta(\mathrm{e}^{-2V} D^\alpha \mathrm{e}^{2V}) W^{(+)}_\alpha \Big)
\end{aligned} \tag{2.2.87}$$

having used the definition of $W^{(+)}_\alpha$ (2.2.60). Writing $\mathrm{e}^{2(V+\delta V)}$ in the form (2.2.75), integration by parts and the cyclic property of the trace lead to

$$\begin{aligned}
\delta_V S = \int \mathrm{d}^8 z \Bigg(& 2\bar{\Phi} \int_0^1 \mathrm{d}\xi \mathrm{e}^{2\xi V} \delta V \mathrm{e}^{-2\xi V} \mathrm{e}^{2V} \Phi \\
& - \frac{1}{g^2 C(\mathcal{R})} \mathrm{tr}_{\mathcal{R}} \int_0^1 \mathrm{d}\xi \mathrm{e}^{-2\xi V} \delta V \mathrm{e}^{2\xi V} (\mathcal{D}^{(+)\alpha} W^{(+)}_\alpha) \Bigg)
\end{aligned} \tag{2.2.88}$$

where

$$(\mathcal{D}^{(+)\alpha}W^{(+)}_\alpha) = D^\alpha W^{(+)}_\alpha + \{(\mathrm{e}^{-2V}D^\alpha \mathrm{e}^{2V}), W^{(+)}_\alpha\}. \tag{2.2.89}$$

The identity

$$L^n_A \mathcal{T}^I = (-1)^n(\check{A}^n)^{IJ}\mathcal{T}^J \qquad A = A^I\mathcal{T}^I \qquad \check{A} = A^I t^I, \tag{2.2.90}$$

where $(t^I)^{JK} = -\mathrm{i}f^{IJK}$ are the generators of the adjoint representation, gives

$$\int_0^1 \mathrm{d}\xi \mathrm{e}^{-2\xi V}\delta V \mathrm{e}^{2\xi V} = \delta V^I \left(\frac{\mathrm{e}^{2\check{V}}-1}{2\check{V}}\right)^{IJ}\mathcal{T}^J \tag{2.2.91}$$

and one readily obtains

$$\delta_V S = \int \mathrm{d}^8 z\, \delta V^I \Bigg(2\left(\frac{1-\mathrm{e}^{-2\check{V}}}{2\check{V}}\right)^{IJ} \bar{\Phi}\,\mathcal{T}^J \mathrm{e}^{2V}\Phi \\ - \frac{1}{g^2}\left(\frac{\mathrm{e}^{2\check{V}}-1}{2\check{V}}\right)^{IJ}(\mathcal{D}^{(+)\alpha}W^{(+)}_\alpha)^J\Bigg). \tag{2.2.92}$$

This leads to the equations of motion

$$\frac{\delta S}{\delta V^I} = 2\left(\frac{1-\mathrm{e}^{-2\check{V}}}{2\check{V}}\right)^{IJ}\bar{\Phi}\,\mathcal{T}^J\mathrm{e}^{2V}\Phi - \frac{1}{g^2}\left(\frac{\mathrm{e}^{2\check{V}}-1}{2\check{V}}\right)^{IJ}(\mathcal{D}^{(+)\alpha}W^{(+)}_\alpha)^J = 0. \tag{2.2.93}$$

Using

$$\mathrm{e}^A\mathcal{T}^I\mathrm{e}^{-A} = (\mathrm{e}^{-\check{A}})^{IJ}\mathcal{T}^J \qquad A = A^I\mathcal{T}^I \qquad \check{A} = A^I t^I \tag{2.2.94}$$

the Bianchi identity $\mathcal{D}^{(+)\alpha}W^{(+)}_\alpha = \bar{\mathcal{D}}^{(+)}_{\dot\alpha}\bar{W}^{(+)\dot\alpha}$, and recalling the definitions of $\Phi^{(+)}$ and $\bar{\Phi}^{(+)}$ (2.2.67), the equations become

$$(\mathcal{D}^{(+)\alpha}W^{(+)}_\alpha)^I = (\bar{\mathcal{D}}^{(+)}_{\dot\alpha}\bar{W}^{(+)\dot\alpha})^I = 2g^2\bar{\Phi}^{(+)}\mathcal{T}^I\Phi^{(+)}. \tag{2.2.95}$$

It follows that

$$(\mathcal{D}^\alpha W_\alpha)^I = (\bar{\mathcal{D}}_{\dot\alpha}\bar{W}^{\dot\alpha})^I = 2g^2\bar{\phi}\mathcal{T}^I\phi \tag{2.2.96}$$

in any representation, where the scalars are gauge covariantly chiral: $\bar{\mathcal{D}}_{\dot\alpha}\phi = \mathcal{D}_\alpha\bar{\phi}^{\mathrm{T}} = 0$.

The equations of motion for Φ and $\bar{\Phi}$ are comparatively much simpler to derive. Under an arbitrary variation of the chiral fields one finds

$$\begin{aligned}\delta_\Phi S =& S[\Phi+\delta\Phi, \bar{\Phi}+\delta\bar{\Phi}, V] - S[\Phi,\bar{\Phi},V] \\ =& -\frac{1}{4}\int \mathrm{d}^6 z\, \bar{D}^2\big(\bar{\Phi}_i(\mathrm{e}^{2V})^i{}_j\big)\delta\Phi^j - \frac{1}{4}\int \mathrm{d}^6\bar{z}\,\delta\bar{\Phi}_i D^2\big((\mathrm{e}^{2V})^i{}_j\Phi^j\big) \\ &+ \int \mathrm{d}^6 z\,\delta\Phi^i\frac{\partial\mathcal{L}_c(\Phi)}{\partial\Phi^i} + \int \mathrm{d}^6\bar{z}\,\delta\bar{\Phi}_i\frac{\partial\bar{\mathcal{L}}_c(\bar{\Phi})}{\partial\bar{\Phi}_i},\end{aligned} \tag{2.2.97}$$

which immediately gives

$$\frac{\delta S}{\delta \Phi^i} = -\frac{1}{4}\bar{D}^2\big(\bar{\Phi}_j(\mathrm{e}^{2V})^j{}_i\big) + \frac{\partial \mathcal{L}_c(\Phi)}{\partial \Phi^i} = 0 \qquad (2.2.98)$$

$$\frac{\delta S}{\delta \bar{\Phi}_i} = -\frac{1}{4}D^2\big((\mathrm{e}^{2V})^i{}_j\Phi^j\big) + \frac{\partial \bar{\mathcal{L}}_c(\bar{\Phi})}{\partial \bar{\Phi}_i} = 0. \qquad (2.2.99)$$

2.2.3 $\mathcal{N} = 2, 4$ super Yang-Mills

In this section we will briefly describe $\mathcal{N} = 2$ and the maximally supersymmetric $\mathcal{N} = 4$ super Yang-Mills theories. The latter is in fact nothing more than a special case of the former, and can be obtained in component form by dimensional reduction of $\mathcal{N} = 1$ supersymmetric Yang-Mills theory from $D = 10$ [158, 159]. It was the first known four dimensional field theory that was ultraviolet finite to all orders in perturbation theory. When formulated in $\mathcal{N} = 1$ superspace, only one of the supersymmetries is manifest and linearly realized, the remaining supersymmetries being nonlinearly realized. Extended supersymmetry transformations in $\mathcal{N} = 1$ superspace have been considered in [160, 89, 161]. For complete details [89].

$\mathcal{N} = 2$

The most general $\mathcal{N} = 2$ super Yang-Mills action, cast in $\mathcal{N} = 1$ superfield form, consists of two parts,

$$S_{\mathcal{N}=2} = S_{\text{pure}} + S_{\text{hyper}}, \qquad (2.2.100)$$

the 'pure' and 'hypermultiplet' parts. The pure $\mathcal{N} = 2$ super Yang-Mills action is given by:

$$S_{\text{pure}} = \frac{1}{g^2 C(\mathcal{R})}\mathrm{tr}_{\mathcal{R}}\left(\int \mathrm{d}^8 z\; e^{-2V}\bar{\Phi}e^{2V}\Phi + \frac{1}{2}\int \mathrm{d}^6 z\; W^2\right). \qquad (2.2.101)$$

The chiral superfields transform in the adjoint representation of the gauge group, and have been expressed in (2.2.101) in the form $\Phi = \Phi^I \mathcal{T}^I$, with $(\mathcal{T}^I)^i{}_j$ the Hermitian generators of an arbitrary representation $\mathcal{R}$ of the gauge group. Similarly $V = V^I\mathcal{T}^I$. The standard form for the super Yang-Mills action (2.2.71) is achieved by exploiting (2.2.94), which leads to

$$\frac{1}{C(\mathcal{R})}\mathrm{tr}_{\mathcal{R}}\left(e^{-2V}\bar{\Phi}e^{2V}\Phi\right) = \bar{\Phi}^I(e^{2\check{V}})^{IJ}\Phi^J. \qquad (2.2.102)$$

Needless to say, the pure part of the action (2.2.101) is invariant under the following gauge transformations

$$e^{2V'} = e^{\mathrm{i}\bar{\Lambda}}e^{2V}e^{-\mathrm{i}\Lambda} \qquad \Phi' = e^{\mathrm{i}\Lambda}\Phi e^{-\mathrm{i}\Lambda} \qquad (2.2.103)$$

$$V = V^\dagger = V^I\mathcal{T}^I \qquad \Lambda = \Lambda^I\mathcal{T}^I = \bar{\Lambda}^\dagger \qquad D_\alpha\Lambda^I = 0. \qquad (2.2.104)$$

The hypermultiplet part of the action is

$$S_{\text{hyper}} = \int d^8z \big(\bar{Q}e^{2V}Q + \bar{\tilde{Q}}(e^{-2V})^{\mathrm{T}}\tilde{Q}\big) + \sqrt{2}\int d^6z\, \tilde{Q}^{\mathrm{T}}\Phi\, Q + \sqrt{2}\int d^6\bar{z}\, \bar{Q}\bar{\Phi}\bar{\tilde{Q}}^{\mathrm{T}} + \mathcal{M}\int d^6z\, \tilde{Q}^{\mathrm{T}}\, Q + \mathcal{M}\int d^6\bar{z}\, \bar{\tilde{Q}}\bar{Q}^{\mathrm{T}} \quad (2.2.105)$$

where $\mathcal{M}$ the mass of the chiral scalars Q and $\tilde{Q}$, which transform respectively in a representation R and its conjugate R_c of the gauge group. These two chiral scalars are the $\mathcal{N} = 1$ components of an $\mathcal{N} = 2$ multiplet called the hypermultiplet.

S_{hyper} is invariant under the gauge transformations

$$e^{2V'} = e^{\mathrm{i}\bar{\Lambda}}e^{2V}e^{-\mathrm{i}\Lambda} \qquad \Phi' = e^{\mathrm{i}\Lambda}\Phi e^{-\mathrm{i}\Lambda} \quad (2.2.106)$$

$$Q' = e^{\mathrm{i}\Lambda}Q \qquad \tilde{Q}' = (e^{-\mathrm{i}\Lambda})^{\mathrm{T}}\tilde{Q} \quad (2.2.107)$$

$$V = V^\dagger = V^I T^I \qquad \Lambda = \Lambda^I T^I = \bar{\Lambda}^\dagger \qquad D_\alpha \Lambda^I = 0 \quad (2.2.108)$$

where T^I are the generators of the representation R. This can be written as in the previous section by making the following redefinition of the chiral scalars

$$\begin{pmatrix} Q \\ \tilde{Q} \end{pmatrix} = A\chi \qquad A = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1} & -\mathrm{i}\mathbb{1} \\ \mathbb{1} & \mathrm{i}\mathbb{1} \end{pmatrix} \qquad \chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \quad (2.2.109)$$

with $\mathbb{1}$ the identity matrix in the representation R. Under this redefinition S_{hyper} becomes

$$S_{\text{hyper}} = \int d^8z\, \bar{\chi}e^{2V}\chi + \frac{\mathcal{M}}{2}\left\{\int d^6z\, \chi^{\mathrm{T}}\chi + c.c.\right\} + \frac{1}{\sqrt{2}}\left\{\int d^6z\, \chi^{\mathrm{T}}\mathbb{Z}\Phi\chi + c.c.\right\} \quad (2.2.110)$$

where now $V = V^I\tilde{T}^I$, $\Phi = \Phi^I\tilde{T}^I$ and

$$\mathbb{Z} = \begin{pmatrix} 0 & -\mathrm{i}\mathbb{1} \\ \mathrm{i}\mathbb{1} & 0 \end{pmatrix}. \quad (2.2.111)$$

The new generators $\tilde{T}^I$ are defined by

$$\tilde{T}^I = \begin{pmatrix} T^I_- & -\mathrm{i}T^I_+ \\ \mathrm{i}T^I_+ & T^I_- \end{pmatrix} \qquad T^I_\pm = \frac{1}{2}\big(T^I \pm (T^I)^{\mathrm{T}}\big), \quad (2.2.112)$$

are Hermitian and antisymmetric,

$$(\tilde{T}^I)^\dagger = \tilde{T}^I \qquad (\tilde{T}^I)^{\mathrm{T}} = -\tilde{T}^I, \quad (2.2.113)$$

thereby generating a real representation of the gauge group equivalent to $R \oplus R_c$.

From an $\mathcal{N}=2$ superfield perspective, the field strength W^α and the covariantly chiral version ϕ of the chiral scalar Φ are leading $\mathcal{N}=1$ components of a covariantly chiral $\mathcal{N}=2$ superfield strength $\mathcal{W}$. In $\mathcal{N}=1$ superspace this 'second supersymmetry' manifests itself by mixing ϕ and W^α in a form which is analogous to the mixing of the components of an $\mathcal{N}=1$ chiral scalar under $\mathcal{N}=1$ supersymmetry transformations (ie see (2.2.42) and (2.2.44)). As noted above, the two chiral scalars Q and $\tilde{Q}$ are the $\mathcal{N}=1$ components of the hypermultiplet, and also become mixed under the 'second supersymmetry' transformations.

For example, in the massless case, writing all fields in the central representation, $\Phi_c = \mathrm{e}^{\bar{w}}\Phi\mathrm{e}^{-\bar{w}}$, $Q_c = \mathrm{e}^{\bar{w}}Q$, $\tilde{Q}_c = (\mathrm{e}^{-\bar{w}})^{\mathrm{T}}\tilde{Q}$ and so on, the action (2.2.100) is invariant under the following $\mathcal{N}=2$ supersymmetry transformations:

$$\delta W_\alpha = -\frac{1}{4}\epsilon_\alpha\bar{\mathcal{D}}^2\bar{\Phi}_c + \mathrm{i}\bar{\epsilon}^{\dot{\alpha}}\mathcal{D}_{\alpha\dot{\alpha}}\Phi_c \qquad \delta\bar{W}_{\dot{\alpha}} = -\frac{1}{4}\bar{\epsilon}_{\dot{\alpha}}\mathcal{D}^2\Phi_c + \mathrm{i}\epsilon^\alpha\mathcal{D}_{\alpha\dot{\alpha}}\bar{\Phi}_c \tag{2.2.114}$$

$$\delta\Phi_c = \epsilon^\alpha W_\alpha \qquad \delta\bar{\Phi}_c = \bar{\epsilon}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}} \tag{2.2.115}$$

$$\delta Q_c = -\mathrm{i}\chi\bar{\Phi}_c Q_c + \frac{1}{4}\bar{\mathcal{D}}^2(\chi\bar{\tilde{Q}}_c) \qquad \delta\tilde{Q}_c = -\mathrm{i}\chi\bar{\Phi}_c\tilde{Q}_c - \frac{1}{4}\bar{\mathcal{D}}^2(\chi\bar{Q}_c) \tag{2.2.116}$$

$$\delta\bar{Q}_c = \mathrm{i}\chi\bar{Q}_c\Phi_c - \frac{1}{4}\mathcal{D}^2(\chi\tilde{Q}_c^{\mathrm{T}}) \qquad \delta\bar{\tilde{Q}}_c = \mathrm{i}\chi\bar{\tilde{Q}}_c\Phi_c + \frac{1}{4}\mathcal{D}^2(\chi Q_c^{\mathrm{T}}) \tag{2.2.117}$$

$$\chi = \lambda(\theta) + \bar{\lambda}(\bar{\theta}). \tag{2.2.118}$$

Here chiral and antichiral parameters λ and $\bar{\lambda} = \lambda^*$ respectively are independent of spacetime, and posses the expansion $\lambda = \gamma + \theta^\alpha\epsilon_\alpha + \theta^2(\beta_1 + \mathrm{i}\beta_2)$, where γ parameterizes central charge transformations, and β_1, β_2 parameterize $SU(2)/U(1)$. See [89] for complete details.

$\mathcal{N}=4$

Taking the special case where $\mathcal{M}=0$ and the representation R as the adjoint, we obtain $\mathcal{N}=4$ super Yang-Mills. Defining $\Phi^I_i = (\Phi^I, Q^I/g, \tilde{Q}^I/g)$ this is usually written in the form

$$S = \frac{1}{g^2C(\mathcal{R})}\mathrm{tr}_{\mathcal{R}}\Bigg(\Bigg(\int\mathrm{d}^8z\, e^{-2V}\bar{\Phi}^i e^{2V}\Phi_i + \frac{1}{4}\int\mathrm{d}^6z\, W^2 + \frac{1}{4}\int\mathrm{d}^6\bar{z}\,\bar{W}^2 + \Bigg(\frac{\sqrt{2}}{3!}\int\mathrm{d}^6z\,\epsilon^{ijk}[\Phi_i,\Phi_j]\Phi_k + c.c.\Bigg)\Bigg)\Bigg). \tag{2.2.119}$$

with $\epsilon^{123}=1$ for the totally antisymmetric ϵ^{ijk} tensor. Besides the manifest $\mathcal{N}=1$ supersymmetry and $SU(3)$ symmetry, this action possess the following three additional non-linearly realized supersymmetries (parameterized by ϵ^a_i, $i=1,2,3$):

$$\delta W_\alpha = -\frac{1}{4}\epsilon_{i\alpha}\bar{\mathcal{D}}^2\bar{\Phi}^i_c + \mathrm{i}\bar{\epsilon}^{i\dot{\alpha}}\mathcal{D}_{\alpha\dot{\alpha}}\Phi_{ci} \qquad \delta\bar{W}_{\dot{\alpha}} = -\frac{1}{4}\bar{\epsilon}^i_{\dot{\alpha}}\mathcal{D}^2\Phi_{ci} + \mathrm{i}\epsilon^\alpha_i\mathcal{D}_{\alpha\dot{\alpha}}\bar{\Phi}^i_c \tag{2.2.120}$$

$$\delta\Phi_{ci} = \epsilon^\alpha_i W_\alpha \qquad \delta\bar{\Phi}^i_c = \bar{\epsilon}^i_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}. \tag{2.2.121}$$

A note on superconformal field theories

As a final remark we note that, analogous to massless field theories formulated on $\mathbb{R}^{3,1}$ which are invariant under the group of conformal transformations (which include the Poincaré transformations), massless field theories formulated on $\mathbb{R}^{4|4}$ are invariant under the so-called $\mathcal{N} = 1$ superconformal group, having amongst its subgroups the $\mathcal{N} = 1$ super Poincaré group. From the perspective of the AdS/CTF correspondence, the existence of this symmetry is of key significance. For present purposes it is sufficient to merely note that the massless $\mathcal{N} = 2$ (and therefore $\mathcal{N} = 4$) non-Abelian super Yang-Mills theories are indeed $\mathcal{N} = 1$ superconformally invariant. For further details relating to this matter and this symmetry group, see [89, 127].

Chapter 3

Quantization of non-Abelian super Yang-Mills theories

The functional approach to the quantization of superfield theories is conceptually identical to that for ordinary field theories, with only minor modifications necessary to deal with additional complications due to the existence of the chiral and antichiral subspaces. In this chapter we will briefly review the quantization procedure for both field and superfield theories in the context of the background field formulation. The first section is devoted to a general discussion of this formulation including generating functionals and the background field effective action. In the second section we explicitly treat the cases of $\mathcal{N} = 2$ and 4 super Yang-Mills theories at one-loop order.

3.1 Generalities

3.1.1 The background field formalism

Generally speaking, the background field formalism is concerned with explicitly maintaining the classical gauge invariance at the quantum level. Ordinarily this gauge invariance is lost in the gauge fixing process and preserving it can vastly simplify calculations and analysis.

More specifically, traditionally one would calculate the gauge invariant elements of the S-matrix by computing Green's functions from a generating functional. In order to generate well defined propagators, one must make a choice of gauge which results in a gauge dependent generating functional. The generating functional itself needn't be gauge invariant since it not physically observable, however all computations must ultimately yield gauge invariant physical observables. Consequently, during the course of computing the S-matrix, one is forced to handle non-invariant quantities whose final gauge invariance is assured by the application of a renormalization scheme which satisfies the generalized Ward identities [162, 163, 164, 165, 166]. Such identities are typically very complicated.

One way around such complications is to adopt the background field method, which more fully exploits the classical gauge invariance. In this scheme the generalized Ward identities are trivially satisfied, in that the classical gauge invariance is manifest at all stages during computations. For this reason the technique is used extensively in modern quantum field theory, superfield theories, including supergravity theories, and recent attempts in quantizing gravity, including string theory.

The procedure was first introduced by B. DeWitt [167, 93, 168, 169, 170, 171, 172, 173, 174] in the context of ordinary gauge field theories. Although initially applied at one-loop order in the loop expansion of effective action, the procedure was soon generalized, first by 't Hooft, to include multiloop processes [175, 176, 177, 178]. For a review see [179].

Being universal enough to admit superspace generalizations, the approach was soon applied to superfield and supergravity theories [180, 181, 182, 89, 183, 184]. In applying the approach to super Yang-Mills theories formulated in superspace we roughly follow the works [180, 89].

3.1.2 Condensed Notation

The background field technique is identical in its application to field theories and superfield theories, and can be simultaneously summarized by adopting a two-fold condensed DeWitt notation [93, 185] (also see [127]). Accordingly, an index adorned with a hat will indicate a DeWitt index, doing double duty as a discrete label for the field components and as a continuous label for the points in either D dimensional spacetime or $\mathcal{N} = 1$ superspace. Summation over repeated DeWitt indices thereby implicitly includes integration over the appropriate space. More explicitly, given a (super)field theory with field content $v^M(y)$, we will write

$$v^{\hat{M}} = v^M(y) \tag{3.1.1}$$

where y is either a point in flat D dimensional spacetime, $y \in \mathbb{R}^{D-1,1}$, *or* $\mathcal{N} = 1$ superspace, $y \in \mathbb{R}^{4|4}$. In the case of ordinary field theory v^M simply enumerates all fields and components in the dynamical theory, all of which are considered to be real $v^M = (v^M)^*$. In the superfield case v^M will denote all unconstrained, chiral and antichiral fields: $v^M = (V^m, \Phi^\mu, \bar{\Phi}^{\dot{\mu}})$, where $V^m = (V^m)^*$, $\bar{D}_{\dot{\alpha}}\Phi^\mu = 0$, and $\bar{\Phi}^{\dot{\mu}} = (\Phi^\mu)^*$. In general the field v^M posses Grassmann parity $\varepsilon(v^M) = \varepsilon_M$.

To illustrate the usage of this notation consider coupling of the (super)fields $v^{\hat{M}}$ to their sources $j_{\hat{M}}$, which is in general now simply written as $j_{\hat{M}}v^{\hat{M}}$. Expanding the condensed notation

$$j_{\hat{M}}v^{\hat{M}} = \int \mathrm{d}^{(M)}y\, j_M(y)v^M(y), \tag{3.1.2}$$

where $\mathrm{d}^{(M)}y$ denotes the appropriate integration measure(s) for the theory under consideration. For ordinary field theories in D dimensional spacetime we have

$$\mathrm{d}^{(M)}y = \mathrm{d}^D x \qquad\qquad j_{\hat{M}}v^{\hat{M}} = \int \mathrm{d}^D x\, j_M(x)v^M(x), \tag{3.1.3}$$

and for superfield theories

$$\mathrm{d}^{(M)}y = \mathrm{d}^{(M)}z \qquad\qquad \mathrm{d}^{(M)}z = \begin{cases} \mathrm{d}^8 z & M = m \\ \mathrm{d}^6 z & M = \mu \\ \mathrm{d}^6 \bar{z} & M = \dot{\mu} \end{cases} \tag{3.1.4}$$

$$\begin{aligned} j_{\hat{M}}v^{\hat{M}} &= \int \mathrm{d}^{(M)}z\, j_M(z)v^M(z) \\ &= \int \mathrm{d}^8 z\, J_m(z)V^m(z) + \int \mathrm{d}^6 z\, \Theta_\mu(z)\Phi^\mu(z) + \int \mathrm{d}^6 \bar{z}\, \bar{\Theta}_{\dot{\mu}}(z)\bar{\Phi}^{\dot{\mu}}(z). \end{aligned} \tag{3.1.5}$$

In the latter case the sources $j_M = (J_m, \Theta_\mu, \bar{\Theta}_{\dot{\mu}})$ are respectively unconstrained, chiral and antichiral superfields. We will also find a need to introduce delta functions $\delta^{\hat{M}}{}_{\hat{N}}$, where for field theory we have

$$\delta^{\hat{M}}{}_{\hat{N}} = \delta^M{}_N \delta^D(x, x'); \tag{3.1.6}$$

and superfield theory

$$\delta^{\hat{M}}{}_{\hat{N}} = \delta^M{}_N \delta^{(M)}(z, z') \qquad\qquad \delta^{(M)}(z, z') = \begin{cases} \delta^8(z, z') & M = m \\ \delta_+(z, z') & M = \mu \\ \delta_-(z, z') & M = \dot{\mu} \end{cases} \tag{3.1.7}$$

such that in general

$$\delta^{\hat{M}}{}_{\hat{N}} v^{\hat{N}} = v^{\hat{M}}. \tag{3.1.8}$$

We will also adopt the following conventions for left and right functional derivatives of a functional $F[\Psi]$ with respect to fields $\Psi^{\hat{M}}$:

$${}_{\hat{M}\ldots\hat{N}},F,{}_{\hat{P}\ldots\hat{Q}}[\Psi] = \frac{\overrightarrow{\delta}}{\delta\Psi^{\hat{M}}} \cdots \frac{\overrightarrow{\delta}}{\delta\Psi^{\hat{N}}} F[\Psi] \frac{\overleftarrow{\delta}}{\delta\Psi^{\hat{P}}} \cdots \frac{\overleftarrow{\delta}}{\delta\Psi^{\hat{Q}}}. \tag{3.1.9}$$

In the event that a functional has additional arguments the meaning should be clear from the context.

Finally, in this notation one may write the functional supertrace sTr of a given linear differential operator Δ as

$$\mathrm{sTr}\Delta = (-1)^{\varepsilon_M}\Delta^{\hat{M}}{}_{\hat{M}} = \int \mathrm{d}^{(M)}y\,(-1)^{\varepsilon_M}\Delta^{M}{}_{M}(y,y). \tag{3.1.10}$$

where the (super)functional supermatrix $\Delta^{\hat{M}}{}_{\hat{N}} = \Delta^{M}{}_{N}(y,y')$, is obtained by operating on y and first index of the δ-functions defined above:

$$\Delta^{\hat{M}}{}_{\hat{N}} = \Delta\delta^{\hat{M}}{}_{\hat{N}}. \tag{3.1.11}$$

In turn one may then formally define the functional superdeterminant sDet of an operator as:

$$\mathrm{sDet}\Delta = \exp(\mathrm{sTr}(\ln\Delta)). \tag{3.1.12}$$

3.1.3 The background field effective action

Background-quantum splitting

The object of primary interest in modern quantum field theory is the effective action, from which the S-matrix is uniquely determined. The effective action has a long history (for example see [186, 187]) but appears to have been formally introduced first perturbatively by J. Goldstone, A. Salam and S. Weinberg [188] and then later nonperterbatively [189, 93]. The ultimate goal of the background field method is the computation of an effective action which retains the gauge symmetries of the classical action. In what follows we restrict attention to those (super)field theories with irreducible and closed gauge algebras, to which class the super Yang-Mills theories belong.

Suppose that we are given such a theory with classical action $S_0[v]$ which is invariant under the infinitesimal gauge transformations

$$\delta v^{\hat{M}} = R^{\hat{M}}{}_{\hat{\mathcal{K}}}[v]\zeta^{\hat{\mathcal{K}}} \qquad\qquad S_{0},_{\hat{M}}[v]R^{\hat{M}}{}_{\hat{\mathcal{K}}}[v] = 0 \tag{3.1.13}$$

for some functional supermatrix $R^{\hat{M}}{}_{\hat{\mathcal{K}}}[v]$, with $\zeta^{\hat{\mathcal{K}}}$ being the gauge parameters[1]. Closure of the gauge algebra means that the so-called generators of the gauge transformations, $R^{\hat{M}}{}_{\hat{\mathcal{K}}}[v]$, satisfy

$$R^{\hat{M}}{}_{\hat{\mathcal{J}},\hat{N}}[v]R^{\hat{N}}{}_{\hat{\mathcal{K}}}[v] - (-1)^{\varepsilon_{\mathcal{J}}\varepsilon_{\mathcal{K}}}R^{\hat{M}}{}_{\hat{\mathcal{K}},\hat{N}}[v]R^{\hat{N}}{}_{\hat{\mathcal{J}}}[v] = R^{\hat{M}}{}_{\hat{\mathcal{L}}}[v]C^{\hat{\mathcal{L}}}{}_{\hat{\mathcal{J}}\hat{\mathcal{K}}}[v], \tag{3.1.14}$$

[1]The index $\mathcal{K}$ runs over all gauge parameters which possess Grassmann parity $\varepsilon_{\mathcal{K}}$. In the superfield case they are in general unconstrained superfields, chiral and antichiral superfields.

for some functionals $C^{\hat{\mathcal{L}}}{}_{\hat{\mathcal{J}}\hat{\mathcal{K}}}[v]$, called the structure coefficients of the gauge algebra, which satisfy

$$C^{\hat{\mathcal{L}}}{}_{\hat{\mathcal{J}}\hat{\mathcal{K}}}[v] = (-1)^{\varepsilon_{\mathcal{J}}\varepsilon_{\mathcal{K}}} C^{\hat{\mathcal{L}}}{}_{\hat{\mathcal{K}}\hat{\mathcal{J}}}[v]. \tag{3.1.15}$$

Irreducibility requires that $R^{\hat{M}}{}_{\hat{\mathcal{K}}}[v_0]\zeta^{\hat{\mathcal{K}}} = 0$ has no non-trivial solutions with compact support in y (ie in superspace or spacetime), where v_0 satisfies the classical equations of motion $S_{0,\hat{M}}[v_0] = 0$.

The basic procedure one follows when applying the background field technique is to split all of the fields $v^{\hat{M}}$ which appear in the classical action into a quantum piece $v_Q^{\hat{M}}$ and a background piece $v_B^{\hat{M}}$. In general this splitting may be written as

$$v^{\hat{M}} = \Omega^{\hat{M}}(v_B, v_Q). \tag{3.1.16}$$

We will consider a splitting which is subject to the constraints

$$v^{\hat{M}} = \Omega^{\hat{M}}(v, 0) = \Omega^{\hat{M}}(0, v). \tag{3.1.17}$$

which are imposed to ensure that the original action is recovered in the absence of a background, and so that to leading order in a quantum field expansion one obtains the original action as a functional of the background field. As a power series these conditions imply that

$$\Omega^{\hat{M}}(v_B, v_Q) = v_Q^{\hat{M}} + v_B^{\hat{M}} + \text{non-linear terms}. \tag{3.1.18}$$

The splitting function $\Omega^{\hat{M}}$ is chosen such that the spilt action $S_0[\Omega^M(v_B, v_Q)]$ possesses a gauge symmetry where the background fields $v_B^{\hat{M}}$ enjoy the same gauge transformation properties as the original field $v^{\hat{M}}$. The quantum fields $v_Q^{\hat{M}}$ are to be the integration variables in the subsequent functional integrals. For the purpose of constructing a gauge invariant effective action, it is often sufficient to consider a linear splitting for all fields, and $v^{\hat{M}}$ is simply replaced by $v_B^{\hat{M}} + v_Q^{\hat{M}}$ in the classical action. Such is the case for Yang-Mills theories formulated on Minkowski space. For non-Abelian superfield theories however, a linear splitting is inadequate primarily due to the non-linearity of the gauge transformations (2.2.73).

The original gauge transformations (3.1.13) can now be 'distributed' between the background and quantum fields. In particular the split action $S_0[\Omega(v_B, v_Q)]$ will maintain its original gauge invariance, but the separate variation is ambiguous. As a consequence the gauge transformations have two significant interpretations.

The first interpretation, which guides our choice of splitting, is the background picture where the background fields posses the original transformation properties:

$$\delta v_B^{\hat{M}} = R^{\hat{M}}{}_{\hat{\mathcal{K}}}[v_B]\zeta^{\hat{\mathcal{K}}}. \tag{3.1.19}$$

This implies that

$$\delta v_Q^{\hat{M}} = \frac{\delta v_Q^{\hat{M}}}{\delta \Omega^{\hat{N}}} \left(R^{\hat{N}}{}_{\hat{\mathcal{K}}}[\Omega(v_B, v_Q)] - \frac{\delta \Omega^{\hat{N}}}{\delta v_B^{\hat{P}}} R^{\hat{P}}{}_{\hat{\mathcal{K}}}[v_B] \right) \zeta^{\hat{\mathcal{K}}} \tag{3.1.20}$$

for $S_0[\Omega^M(v_B, v_Q)]$ to remain invariant. It is essential under this interpretation that the infinitesimal variation of $v_Q^{\hat{M}}$ be independent of the background field, and desirable that it be linear. We will assume that such a splitting can be found, and that equation (3.1.20) becomes

$$\delta v_Q^{\hat{M}} = \mathcal{Q}^{\hat{M}}{}_{\hat{\mathcal{K}}}[v_Q] \zeta^{\hat{\mathcal{K}}} \tag{3.1.21}$$

which explicitly takes the form

$$d v_Q^{M} = \mathcal{Q}^{M}{}_{N\mathcal{K}} v_Q^N \zeta^{\mathcal{K}} \tag{3.1.22}$$

with some constants $\mathcal{Q}^{M}{}_{N\mathcal{K}}$. In practice this usually means that the quantum gauge field transforms covariantly, as a non-gauge field would prior to splitting. The existence of this interpretation will eventually guarantee that the effective action be manifestly invariant under the original transformations.

The second interpretation is the quantum picture, where the background fields remain invariant

$$\delta v_B^{\hat{M}} = 0 \qquad \Rightarrow \qquad \delta v_Q^{\hat{M}} = \frac{\delta v_Q^{\hat{M}}}{\delta \Omega^{\hat{N}}} R^{\hat{N}}{}_{\hat{\mathcal{K}}}[\Omega(v_B, v_Q)] \zeta^{\hat{\mathcal{K}}}. \tag{3.1.23}$$

It is this interpretation which must be considered when gauge fixing.

Background Gauge Fixing

After splitting, the theory is gauge fixed by carefully selecting background dependent gauge fixing functions $\chi^{\hat{\mathcal{K}}}[v_B, v_Q]$, $\varepsilon(\chi^{\mathcal{K}}) = \varepsilon_{\mathcal{K}}$, which transform covariantly in the background picture, (3.1.19) and (3.1.21). This gauge fixing is chosen to break the quantum gauge invariance (3.1.23) whilst yielding a gauge fixed action which will remain invariant in the background picture.

The background field generating functional $Z[j, v_B]$ is then given by

$$Z[j, v_B] \; = \; \mathrm{N} \int [\mathrm{d}v_Q] \delta(F^{\hat{\mathcal{K}}}) \,\mathrm{sDet}\Big(\frac{\delta \chi^{\hat{\mathcal{J}}}[v_B, v_Q]}{\delta \zeta^{\hat{\mathcal{K}}}} \Big) \exp\Big\{ \mathrm{i}(S_0[\Omega(v_B, v_Q)] + j_{\hat{M}} v_Q^{\hat{M}}) \Big\}, \tag{3.1.24}$$

having applied the well known Faddeev-Popov prescription, the validity of which requires that (3.1.14) be satisfied, and so the gauge algebra is closed. In the above expression: N is a normalization constant[2]; $[\mathrm{d}v_Q]$ denotes the functional integration measure of the quantum fields; the sources $j_{\hat{M}}$ are coupled only to the quantum fields; δ indicates a functional delta function, the presence of which breaks the gauge invariance; $F^{\hat{\mathcal{K}}}$ is given by

$$F^{\hat{\mathcal{K}}}[v_B, v_Q] = \chi^{\hat{\mathcal{K}}}[v_B, v_Q] - f^{\hat{\mathcal{K}}}, \tag{3.1.25}$$

the fields $f^{\hat{\mathcal{K}}}$ being arbitrary external fields; and the argument of the superdeterminant is the variation of $\chi^{\hat{\mathcal{K}}}[v_B, v_Q]$ with respect to the gauge parameters in the quantum picture.

Since the generating functional is independent of the choice of 'gauge slice' and hence of the superfields $f^{\hat{\mathcal{K}}}$, one is free to functionally integrate the generating functional with respect to them. In the process one may weight the integral by an arbitrary term, but in doing so must maintain normalization, ie in the generating functional one includes a term

$$\begin{gathered} 1 = \int [\mathrm{d}f][\mathrm{d}a] \exp\left\{\frac{\mathrm{i}}{2\gamma}\left(f^{\hat{\mathcal{J}}} Y_{\hat{\mathcal{J}}\hat{\mathcal{K}}} f^{\hat{\mathcal{K}}} + a^{\hat{\mathcal{J}}} \grave{Y}_{\hat{\mathcal{J}}\hat{\mathcal{K}}} a^{\hat{\mathcal{K}}}\right)\right\} \\ \varepsilon(Y_{\hat{\mathcal{J}}\hat{\mathcal{K}}}) = \varepsilon(\grave{Y}_{\hat{\mathcal{J}}\hat{\mathcal{K}}}) = \varepsilon_{\mathcal{J}} + \varepsilon_{\mathcal{K}} \qquad \varepsilon(a^{\mathcal{K}}) = \varepsilon(f^{\mathcal{K}}) + 1 \pmod 2 \\ Y_{\hat{\mathcal{J}}\hat{\mathcal{K}}} = (-1)^{\varepsilon_{\mathcal{J}}+\varepsilon_{\mathcal{K}}+\varepsilon_{\mathcal{J}}\varepsilon_{\mathcal{K}}} Y_{\hat{\mathcal{K}}\hat{\mathcal{J}}} \qquad \grave{Y}_{\hat{\mathcal{J}}\hat{\mathcal{K}}} = -(-1)^{\varepsilon_{\mathcal{J}}\varepsilon_{\mathcal{K}}} \grave{Y}_{\hat{\mathcal{K}}\hat{\mathcal{J}}} \end{gathered} \tag{3.1.26}$$

In this expression γ is a gauge parameter chosen for computational convenience, and $Y_{\hat{\mathcal{J}}\hat{\mathcal{K}}}$ an arbitrary functional supermatrix (which may possess background field dependance) chosen also for convenience subject to $f^{\hat{\mathcal{J}}} Y_{\hat{\mathcal{J}}\hat{\mathcal{K}}} f^{\hat{\mathcal{K}}}$ being invariant under background field transformations. The superfields $a^{\hat{\mathcal{K}}}$ are the Nielsen-Kallosh ghosts [190, 191, 192, 193], introduced purely to maintain normalization, and they therefore possesses opposite statistics to $f^{\hat{\mathcal{K}}}$ to ensure cancellation of superdeterminants. More specifically we require

$$\int [\mathrm{d}f] \exp\left\{\frac{\mathrm{i}}{2\gamma} f^{\hat{\mathcal{J}}} Y_{\hat{\mathcal{J}}\hat{\mathcal{K}}} f^{\hat{\mathcal{K}}}\right\} = \left(\int [\mathrm{d}a] \exp\left\{\frac{\mathrm{i}}{2\gamma} a^{\hat{\mathcal{J}}} \grave{Y}_{\hat{\mathcal{J}}\hat{\mathcal{K}}} a^{\hat{\mathcal{K}}}\right\}\right)^{-1} \tag{3.1.27}$$

and the operator $\grave{Y}_{\hat{\mathcal{J}}\hat{\mathcal{K}}}$ is chosen to ensure this. The operators Y and $\grave{Y}$ differ trivially, in the sense that all entries are equal 'up to sign', which is merely a consequence of our choice to introduce the symmetrized operator in $f^{\hat{\mathcal{J}}} Y_{\hat{\mathcal{J}}\hat{\mathcal{K}}} f^{\hat{\mathcal{K}}}$.

In ordinary Yang-Mills theories $Y_{\hat{\mathcal{J}}\hat{\mathcal{K}}}$ is commonly chosen to be a functional identity supermatrix, in which case these ghosts decouple and may be ignored. However in many situations, as with super Yang-Mills theories, their presence is significant

[2]The irrelevant normalization constant in all subsequent path integral formulae will generically be denoted by N, although in general they will differ.

due to the fact that the inclusion of a background dependent operator $Y_{\mathcal{J}\mathcal{K}}$ significantly simplifies the theory. Employing the identities

$$\text{sDet}\Delta = \text{N}\int [\text{d}\xi][\text{d}\xi'] \exp\left\{\text{i}\xi'_{\hat{M}}\Delta^{\hat{M}}{}_{\hat{N}}\xi^{\hat{N}}\right\} \qquad \varepsilon(\xi'_M) = \varepsilon(\xi^M) = 1 \tag{3.1.28}$$

$$(\text{sDet}\Delta)^{-1} = \text{N}\int [\text{d}\xi][\text{d}\xi'] \exp\left\{\text{i}\xi'_{\hat{M}}\Delta^{\hat{M}}{}_{\hat{N}}\xi^{\hat{N}}\right\} \qquad \varepsilon(\xi'_M) = \varepsilon(\xi^M) = 0, \tag{3.1.29}$$

equation (3.1.26) and

$$\frac{\delta\chi^{\hat{\mathcal{J}}}}{\delta\zeta^{\hat{\mathcal{K}}}} = \frac{\delta\chi^{\hat{\mathcal{J}}}}{\delta v_Q^{\hat{M}}}\frac{\delta v_Q^{\hat{M}}}{\delta\zeta^{\hat{\mathcal{K}}}} = \frac{\delta\chi^{\hat{\mathcal{K}}}}{\delta\Omega^{\hat{N}}}R^{\hat{N}}{}_{\hat{\mathcal{K}}}[\Omega(v_B, v_Q)], \tag{3.1.30}$$

the generating functional (3.1.24) becomes

$$Z[j, v_B] = \text{N}\int [\text{d}v_Q][\text{d}c'][\text{d}c][\text{d}a] \exp\left\{\text{i}(S[v_B, v_Q, c', c, a] + j_{\hat{M}}v_Q^{\hat{M}})\right\} \tag{3.1.31}$$

where

$$S[v_B, v_Q, c', c, a] = S_0[\Omega(v_B, v_Q)] + S_{\text{gf}}[v_B, v_Q] + S_{\text{gh}}[v_B, v_Q, c', c'a], \tag{3.1.32}$$

having defined

$$S_{\text{gf}}[v_B, v_Q] = \frac{1}{2\gamma}\chi^{\hat{\mathcal{J}}}Y_{\hat{\mathcal{J}}\hat{\mathcal{K}}}\chi^{\hat{\mathcal{K}}} \tag{3.1.33}$$

$$S_{\text{gh}}[v_B, v_Q, c', c'a] = \frac{1}{2\gamma}a^{\hat{\mathcal{J}}}\grave{Y}_{\hat{\mathcal{J}}\hat{\mathcal{K}}}a^{\hat{\mathcal{K}}} + c'_{\hat{\mathcal{J}}}\frac{\delta\chi^{\hat{\mathcal{J}}}}{\delta\Omega^{\hat{N}}}R^{\hat{N}}{}_{\hat{\mathcal{K}}}[\Omega(v_B, v_Q)]c^{\hat{\mathcal{K}}}. \tag{3.1.34}$$

The fields $c^{\mathcal{K}}$ and $c'_{\mathcal{K}}$ are the Faddeev-Popov ghosts, and satisfy $\varepsilon(c^{\mathcal{K}}) = \varepsilon(c'_{\mathcal{K}}) = \varepsilon(v_Q^M) + 1$.

In complete analogy to the conventional case, the generating functional for connected background Green's functions $W[j, v_B]$ is related to the background generating functional $Z[j, v_B]$ by

$$W[j, v_B] = -\text{i}\ln Z[j, v_B]. \tag{3.1.35}$$

Defining the background mean field $\psi^{\hat{M}}$

$$\psi^{\hat{M}} = \frac{\delta W[j, v_B]}{\delta j_{\hat{M}}}, \tag{3.1.36}$$

then the generalized background field effective action is obtained by making the Legendre transform

$$\Gamma[\psi, v_B] = W[j, v_B] - j_{\hat{M}}\psi^{\hat{M}} \tag{3.1.37}$$

where $j_{\hat{M}}$ is expressed in terms of $\psi^{\hat{M}}$ via (3.1.36). As usual we note that functional differentiation with respect to the mean field yields the source

$${}_{\hat{M}},\Gamma[\psi, v_B] = (-1)^{\varepsilon_M}\,\Gamma,_{\hat{M}}[\psi, v_B] = -j_{\hat{M}}. \tag{3.1.38}$$

As a consequence of all this, the background field effective action $\Gamma[0, v_B]$ is explicitly gauge invariant under the same transformations as the original action. Physically it is the sum over all one-particle-irreducible diagrams with background fields on external legs and quantum fields inside loops. We will now demonstrate that it possesses the desired symmetries and is equivalent to the conventional effective action evaluated in an unusual background field dependent gauge.

Symmetries of the background field effective action

Endowing the sources with the gauge transformation properties

$$\delta j_M = -(-1)^{\varepsilon_M \varepsilon_\mathcal{K}} j_N \mathcal{Q}^N{}_{M\mathcal{K}} \zeta^\mathcal{K}, \tag{3.1.39}$$

it follows, by construction, that the background field generating functional $Z[j, v_B]$ as given in (3.1.24) will be invariant under the transformations

$$\delta v_B^{\hat{M}} = R^{\hat{M}}{}_{\hat{\mathcal{K}}}[v_B] \zeta^{\hat{\mathcal{K}}}. \tag{3.1.40}$$

This is simply proven by simultaneously invoking the change of integration variable

$$v_Q^M \to v_Q^M + \mathcal{Q}^M{}_{N\mathcal{K}} v_Q^N \zeta^\mathcal{K}. \tag{3.1.41}$$

As a consequence one can easily establish that the generalized background field effective action $\Gamma[\psi, v_B]$ is invariant under (3.1.40) and

$$\delta \psi^M = \mathcal{Q}^M{}_{N\mathcal{K}} \psi^N \zeta^\mathcal{K}. \tag{3.1.42}$$

The background field effective action $\Gamma[0, v_B]$ is therefore invariant under (3.1.40) alone, the same gauge symmetry as the classical action.

Relation to the conventional effective action

To establish the background field effective action is equivalent to the conventional effective action when computing the S-matrix, consider slightly modifying $Z[j, v_B]$ as given in (3.1.31) to become

$$\begin{aligned} Z[j, v_B] = \mathrm{N} \int & [\mathrm{d}v_Q][\mathrm{d}c'][\mathrm{d}c][\mathrm{d}a] \\ & \exp\left\{ \mathrm{i}\Big(S[v_B, v_Q, c', c, a] + j_{\hat{M}} \big(\Omega^{\hat{M}}(v_B, v_Q) - \Omega^{\hat{M}}(v_B, 0) \big) \Big) \right\}. \end{aligned} \tag{3.1.43}$$

with $S[v_B, v_Q, c', c, a]$ still being given by (3.1.32). Here the sources are not just coupled to the quantum fields, but rather

$$\Omega^{\hat{M}}(v_B, v_Q) - \Omega^{\hat{M}}(v_B, 0) = v_Q^{\hat{M}} + \text{non-linear terms} \tag{3.1.44}$$

for reasons which will become clear. For the purposes of computing the S-matrix the higher order coupling terms are irrelevant, and so this definition of $Z[j, v_B]$ is, for all practical purposes, equivalent to the previous one. Introducing the inverse splitting function

$$v_Q^{\hat{M}} = \Upsilon^{\hat{M}}(v_B, v) \qquad\qquad v^{\hat{M}} = \Omega^{\hat{M}}(v_B, \Upsilon(v_B, v)). \tag{3.1.45}$$

we now make the following change of integration variable in (3.1.43):

$$v^{\hat{M}} = \Omega^{\hat{M}}(v_B, v_Q). \tag{3.1.46}$$

This gives rise to a superdeterminant factor $\mathrm{sDet}(\delta v_Q/\delta v)$ in the path integral, which is equal to unity in the superfield case, or yields an irrelevant factor to be absorbed into the normalization in the standard field theory case. Such a change of integration variable leads to

$$Z[j, v_B] = \tilde{Z}[j] \exp\left(-\mathrm{i} j_{\hat{M}} v_B^{\hat{M}}\right) \tag{3.1.47}$$

where $\tilde{Z}[j]$ is just the conventional generating functional

$$\tilde{Z}[j] = \mathrm{N} \int [\mathrm{d}v][\mathrm{d}c'][\mathrm{d}c][\mathrm{d}a] \exp\left\{\mathrm{i}(S'[v_B, v, c', c, a] + j_{\hat{M}} v^{\hat{M}})\right\}, \tag{3.1.48}$$

with gauge fixed action

$$\begin{aligned} S'[v_B, v, c', c, a] &= S[v_B, \Upsilon(v_B, v), c', c, a] \\ &= S_0[v] + \frac{1}{2\gamma}\left(\tilde{\chi}^{\hat{\mathcal{J}}} Y_{\hat{\mathcal{J}}\hat{\mathcal{K}}} \tilde{\chi}^{\hat{\mathcal{K}}} + a^{\hat{\mathcal{J}}} \grave{Y}_{\hat{\mathcal{J}}\hat{\mathcal{K}}} a^{\hat{\mathcal{K}}}\right) \\ &\qquad\qquad + c'_{\hat{\mathcal{J}}} \frac{\delta \tilde{\chi}^{\hat{\mathcal{J}}}}{\delta v^{\hat{N}}} R^{\hat{N}}{}_{\hat{\mathcal{K}}}[v] c^{\hat{\mathcal{K}}}, \end{aligned} \tag{3.1.49}$$

having chosen $Y_{\hat{\mathcal{J}}\hat{\mathcal{K}}}$ as before, along with non-conventional background dependant gauge fixing conditions

$$\tilde{\chi}^{\hat{\mathcal{K}}}[v_B, v] = \chi^{\hat{\mathcal{K}}}[v_B, \Upsilon(v_B, v)]. \tag{3.1.50}$$

From the conventional definitions

$$\tilde{W}[j] = -\mathrm{i} \ln \tilde{Z}[j] \tag{3.1.51}$$

$$\tilde{\psi}^{\hat{M}} = \frac{\delta \tilde{W}[j]}{\delta j_{\hat{M}}} \qquad\qquad \tilde{\Gamma}[\tilde{\psi}] = \tilde{W}[j] - j_{\hat{M}} \tilde{\psi}^{\hat{M}} \tag{3.1.52}$$

and noting that (3.1.47) leads to

$$W[j, v_B] = \tilde{W}[j] - j_{\hat{M}} v_B^{\hat{M}} \tag{3.1.53}$$

and therefore

$$\psi^{\hat{M}} = \tilde{\psi}^{\hat{M}} - v_B^{\hat{M}}, \tag{3.1.54}$$

we observe that

$$\begin{aligned}\Gamma[\psi, v_B] &= W[j, v_B] - j_{\hat{M}}\psi^{\hat{M}} \\ &= \big(\tilde{W}[j] - j_{\hat{M}}v_B^{\hat{M}}\big) - j_{\hat{M}}\big(\tilde{\psi}_B^{\hat{M}} - v_B^{\hat{M}}\big) \\ &= \tilde{\Gamma}[\tilde{\psi}] = \tilde{\Gamma}[\psi + v_B].\end{aligned} \tag{3.1.55}$$

Setting $\psi = 0$ gives the desired result

$$\Gamma[0, v_B] = \tilde{\Gamma}[v_B]. \tag{3.1.56}$$

Computing the background field effective action

In practice one computes $\Gamma[0, v_B]$ perturbatively, often expanding it in a functional power series in $\hbar$, the so-called loop expansion. Reinserting factors of $\hbar$, and putting together (3.1.31), (3.1.35), (3.1.37) and (3.1.38) one obtains:

$$\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\Gamma[0,v_B]} = \mathrm{N}\int [\mathrm{d}v_Q][\mathrm{d}b] \exp\left\{\frac{\mathrm{i}}{\hbar}\big(S[v_B, v_Q, b] - \hbar^{\frac{1}{2}}\ {}_{\hat{M}},\Gamma[0, v_B]v_Q^{\hat{M}}\big)\right\}, \tag{3.1.57}$$

where for simplicity all ghosts (Faddeev-Popov and Neilson-Kallosh) are now denoted by $b = (c', c, a)$, and their total integration measure by $[\mathrm{d}b]$. Bearing in mind that $S_{\mathrm{gh}}[v_B, v_Q, b]$ (as given in (3.1.32)) is quadratic in ghosts, one now makes the field redefinitions $v_Q \to \hbar^{\frac{1}{2}}v_Q$ and $b \to \hbar^{\frac{1}{2}}b$, and expands $S[v_B, v_Q, b]$ in a power series in $\hbar^{\frac{1}{2}}$ (or equivalently in v_Q) as

$$S[v_B, v_Q, b] = S_0[v_B] + \hbar\, S_{\mathrm{gh}}[v_B, 0, b] + \sum_{n=1}^{\infty}\frac{\hbar^{\frac{n}{2}}}{n!}\, S_{,\hat{M}_1\dots\hat{M}_n} S[v_B, 0, b]\, v_Q^{\hat{M}_1}\dots v_Q^{\hat{M}_n}. \tag{3.1.58}$$

Inserting this expansion into (3.1.57) yields

$$\begin{aligned}e^{\frac{\mathrm{i}}{\hbar}(\Gamma[0,v_B]-S_0[v_B])} = \mathrm{N}\int [\mathrm{d}v_Q][\mathrm{d}b] \exp\Bigg\{&\frac{\mathrm{i}}{2}v_Q^{\hat{M}}\ {}_{\hat{M}},S_{,\hat{N}}\,[v_B, 0, b]v_Q^{\hat{N}} + \mathrm{i}S_{\mathrm{gh}}[v_B, 0, c] \\ &+ \sum_{n=3}^{\infty}\frac{\hbar^{\frac{n}{2}-1}}{n!}\, S_{,\hat{M}_1\hat{M}_2\dots\hat{M}_n}\,[v_B, 0, b]v_Q^{\hat{M}_1}v_Q^{\hat{M}_2}\dots\, v_Q^{\hat{M}_n} \\ &- \hbar^{-\frac{1}{2}}\Big(\,\Gamma_{,\hat{M}}\,[0, v_B] -\ S_{,\hat{M}}\,[v_B, 0, b]\Big)v_Q^{\hat{M}}\Bigg\}.\end{aligned} \tag{3.1.59}$$

Anticipating the loop expansion we write

$$\Gamma[0, v_B] = S_0[v_B] + \sum_{n=1}^{\infty}\hbar^n\Gamma^{(n)}[0, v_B] \tag{3.1.60}$$

which may be inserted into (3.1.59) to perturbatively compute $\Gamma[0, v_B]$. $\Gamma^{(n)}[0, v_B]$ is the n-loop contribution to the background field effective action. We note here that for each term in this expansion to be separately invariant under the background symmetries, we require that the total gauge fixed action (including ghosts) have each term in a power series expansion in the quantum fields being separately invariant. This places restrictions on the possible choices of splitting function. For our purposes it is sufficient to note that demanding that the transformation (3.1.21) be homogenous satisfies this need for all the Yang-Mills theories.

To one-loop order, which is the present case of interest, one immediately finds

$$e^{\mathrm{i}\Gamma^{(1)}[0,v_B]} = \mathrm{N} \int [\mathrm{d}v_Q][\mathrm{d}b] \exp\left\{ \frac{\mathrm{i}}{2} v_Q^{\hat{M}} {}_{\hat{M}}, S,_{\hat{N}} [v_B, 0, b] v_Q^{\hat{N}} + \mathrm{i} S_{\mathrm{gh}}[v_B, 0, b] \right\}, \quad (3.1.61)$$

and, as is well known, is just a product of functional superdeterminants of the operators in the quantum quadratic part of the action.

As a final remark, one must of course regulate and renormalize as usual in the process of computing $\Gamma[0, v_B]$.

3.2 Quantizing super Yang-Mills theories

3.2.1 Quantum-background splitting

In this section we will apply the background field method to $\mathcal{N} = 2$ super Yang-Mills theories, with the ultimate goal of computing contributions to the one-loop effective action. For simplicity, and for reasons which will be described later, we will not be interested in the full background field effective action, and will only concern ourselves with the case where the gauge superfield V acquires a background expectation value. All of the chiral scalar superfields are therefore to be interpreted as quantum fields to be integrated over. We will briefly comment on the effects and complications of providing these scalars with a non-zero background.

First recall the $\mathcal{N} = 2$ super Yang-Mills action[3] (2.2.100):

$$S = \frac{1}{g^2} \mathrm{tr}_{\mathcal{R}} \left(\int \mathrm{d}^8 z \, e^{-2V} \bar{\Phi} e^{2V} \Phi + \frac{1}{2} \int \mathrm{d}^6 z \, W^2 \right) + \int \mathrm{d}^8 z \big(\bar{Q} e^{2V} Q + \bar{\tilde{Q}} (e^{-2V})^{\mathrm{T}} \tilde{Q} \big)$$
$$+ \sqrt{2} \int \mathrm{d}^6 z \, \tilde{Q}^{\mathrm{T}} \Phi \, Q + \sqrt{2} \int \mathrm{d}^6 \bar{z} \, \bar{Q} \bar{\Phi} \bar{\tilde{Q}}^{\mathrm{T}} + \mathcal{M} \int \mathrm{d}^6 z \, \tilde{Q}^{\mathrm{T}} Q + \mathcal{M} \int \mathrm{d}^6 \bar{z} \, \bar{\tilde{Q}} \bar{Q}^{\mathrm{T}}. \quad (3.2.62)$$

[3]From here onward we assume that the generators of the gauge group have been normalized such that $\mathrm{tr}_{\mathcal{R}}(\mathcal{T}^I \mathcal{T}^J) = \delta^{IJ}$.

As described earlier in some detail, this action is invariant under the following gauge transformations:

$$\begin{aligned} e^{2V'} &= e^{i\bar{\Lambda}} e^{2V} e^{-i\Lambda} \qquad \Phi' = e^{i\Lambda}\Phi e^{-i\Lambda} \\ Q' &= e^{i\Lambda} Q \qquad \tilde{Q}' = (e^{-i\Lambda})^{\mathrm{T}} \tilde{Q} \end{aligned} \tag{3.2.63}$$

where

$$V = V^\dagger = V^I \mathcal{T}^I \qquad \Lambda = \Lambda^I \mathcal{T}^I = \bar{\Lambda}^\dagger \qquad D_\alpha \Lambda^I = 0 \tag{3.2.64}$$

To facilitate usage of the 'central' representation described briefly in subsection 2.2.2, we first rewrite the gauge field V in the following way

$$e^{2V} = e^w e^{\bar{w}} \qquad \bar{w}^\dagger = w = w^I \mathcal{T}^I. \tag{3.2.65}$$

In doing so, we have introduced an additional gauge freedom, which manifests itself in the transformations

$$e^{w'} = e^{i\bar{\Lambda}} e^w e^{ik} \qquad e^{\bar{w}'} = e^{-ik} e^{\bar{w}} e^{-i\Lambda} \qquad k = k^I T^I \quad k^\dagger = k \tag{3.2.66}$$

for a real but otherwise unconstrained superfield k.

A suitable background-quantum splitting scheme for the field V is then defined by

$$e^w = e^{w_B} e^{w_Q} \tag{3.2.67}$$

or equivalently

$$e^{2V} = e^{w_B} e^{w_Q} e^{\bar{w}_Q} e^{\bar{w}_B} = e^{w_B} e^{2V_Q} e^{\bar{w}_B}, \tag{3.2.68}$$

where as usual the subscripts B and Q denote background and quantum pieces respectively. Other possible splittings include $e^{2V} \to e^{2V_B} e^{2V_Q}$ or $e^{2V} \to e^{2V_Q} e^{2V_B}$ and naturally lend themselves to the background chiral and antichiral representations respectively, but come at the cost of losing convenient conjugation properties.

Under the proposed splitting (3.2.67), the resulting action will be gauge invariant under the transformations

$$\begin{aligned} e^{w'} &= e^{w'_B} e^{w'_Q} = e^{i\bar{\Lambda}} e^{w_B} e^{w_Q} e^{ik} \\ e^{\bar{w}'} &= e^{\bar{w}'_Q} e^{\bar{w}'_B} = e^{-ik} e^{\bar{w}_Q} e^{\bar{w}_B} e^{-i\Lambda} \end{aligned} \tag{3.2.69}$$

The quantum interpretation of this is:

$$\begin{aligned} w'_B &= w_B \qquad e^{w'_Q} = e^{-w_B} e^{i\bar{\Lambda}} e^{w_B} e^{w_Q} e^{ik} = e^{i\bar{\Omega}} e^{w_Q} e^{ik} \\ \bar{w}'_B &= \bar{w}_B \qquad e^{\bar{w}'_Q} = e^{-ik} e^{\bar{w}_Q} e^{\bar{w}_B} e^{-i\Lambda} e^{-\bar{w}_B} = e^{-ik} e^{\bar{w}_Q} e^{-i\Omega} \end{aligned} \tag{3.2.70}$$

or

$$V'_B = V_B \qquad e^{2V'_Q} = e^{i\bar{\Omega}} e^{2V_Q} e^{-i\Omega} \tag{3.2.71}$$

where new 'background gauge parameters' have been defined:

$$e^{\mathrm{i}\Omega} = e^{\bar{w}_B} e^{\mathrm{i}\Lambda} e^{-\bar{w}_B} \qquad\qquad e^{\mathrm{i}\bar{\Omega}} = e^{-w_B} e^{\mathrm{i}\bar{\Lambda}} e^{w_B}. \tag{3.2.72}$$

This interpretation must be adopted to correctly gauge fix the theory.

The background picture is:

$$\begin{aligned} e^{w'_B} &= e^{\mathrm{i}\bar{\Lambda}} e^{w_B} e^{\mathrm{i}k} & \qquad e^{w'_Q} &= e^{-\mathrm{i}k} e^{w_Q} e^{\mathrm{i}k} \\ e^{\bar{w}'_B} &= e^{-\mathrm{i}k} e^{\bar{w}_B} e^{-\mathrm{i}\Lambda} & \qquad e^{\bar{w}'_Q} &= e^{-\mathrm{i}k} e^{\bar{w}_Q} e^{\mathrm{i}k} \end{aligned} \tag{3.2.73}$$

or

$$e^{2V'_B} = e^{\mathrm{i}\bar{\Lambda}} e^{2V_B} e^{-\mathrm{i}\Lambda} \qquad\qquad V'_Q = e^{-\mathrm{i}k} V_Q e^{\mathrm{i}k}. \tag{3.2.74}$$

As desired, in the background picture V_B transforms exactly as the original gauge field V, and V_Q transforms linearly, homogenously and independently of the background. This will provide the symmetry at quantum level provided it is not violated by our choice of gauge.

In describing the effect of this splitting on the action, it is convenient to introduce background gauge covariant derivatives. With this splitting scheme the original gauge covariant derivatives in the central representation (2.2.2) split as

$$\mathcal{D}_\alpha = e^{-w} D_\alpha e^{w} = e^{-w_Q} e^{-w_B} D_\alpha e^{w_B} e^{w_Q} = e^{-w_Q} \mathcal{D}_{B\alpha} e^{w_Q} \tag{3.2.75}$$

$$\bar{\mathcal{D}}_{\dot{\alpha}} = e^{\bar{w}} \bar{D}_{\dot{\alpha}} e^{-\bar{w}} = e^{\bar{w}_Q} e^{\bar{w}_B} \bar{D}_{\dot{\alpha}} e^{-\bar{w}_B} e^{-\bar{w}_Q} = e^{\bar{w}_Q} \bar{\mathcal{D}}_{B\dot{\alpha}} e^{-\bar{w}_Q} \tag{3.2.76}$$

where background gauge covariant derivatives are defined by

$$\mathcal{D}_{B\alpha} = e^{-w_B} D_\alpha e^{w_B}, \qquad \bar{\mathcal{D}}_{B\dot{\alpha}} = e^{\bar{w}_B} \bar{D}_{\dot{\alpha}} e^{-\bar{w}_B} \tag{3.2.77}$$

and satisfy the algebra

$$\begin{aligned} \mathcal{D}_{Ba} &= -\frac{1}{2}(\tilde{\sigma}_a)^{\dot{\alpha}\alpha} \mathcal{D}_{B\alpha\dot{\alpha}} = -\frac{\mathrm{i}}{4}(\tilde{\sigma}_a)^{\dot{\alpha}\alpha} \{\mathcal{D}_{B\alpha}, \bar{\mathcal{D}}_{B\dot{\alpha}}\} \\ [\bar{\mathcal{D}}_{B\dot{\alpha}}, \mathcal{D}_{B\beta\dot{\beta}}] &= 2\mathrm{i}\varepsilon_{\dot{\alpha}\dot{\beta}} W_{B\beta}, \qquad [\mathcal{D}_{B\alpha}, \mathcal{D}_{B\beta\dot{\beta}}] = 2\mathrm{i}\varepsilon_{\alpha\beta} \bar{W}_{B\dot{\beta}} \end{aligned} \tag{3.2.78}$$

with $\bar{W}_{B\dot{\alpha}}$ and $W_{B\beta}$ the background superfield strengths (compare with the algebra (2.2.2)). These superfields, and the background gauge covariant derivatives which define them, have the following gauge transformation properties in the background picture:

$$\mathcal{D}'_{Ba} = e^{-\mathrm{i}k} \mathcal{D}_{Ba} e^{\mathrm{i}k} \qquad \mathcal{D}'_{B\alpha} = e^{-\mathrm{i}k} \mathcal{D}_{B\alpha} e^{\mathrm{i}k} \qquad \bar{\mathcal{D}}'_{B\dot{\alpha}} = e^{-\mathrm{i}k} \bar{\mathcal{D}}_{B\dot{\alpha}} e^{\mathrm{i}k} \tag{3.2.79}$$

$$W'^{\alpha}_B = e^{-\mathrm{i}k} W^{\alpha}_B e^{\mathrm{i}k} \qquad\qquad \bar{W}'^{\dot{\alpha}}_B = e^{-\mathrm{i}k} \bar{W}^{\dot{\alpha}}_B e^{\mathrm{i}k}. \tag{3.2.80}$$

Notice that they are background covariantly chiral, as are the background gauge parameters defined earlier, ie $\mathcal{D}_{B\alpha}\bar{W}^{\dot{\alpha}}_{B} = \bar{\mathcal{D}}_{B\dot{\alpha}}W^{\alpha}_{B} = \mathcal{D}_{B\alpha}\bar{\Omega}^{I} = \bar{\mathcal{D}}_{B\dot{\alpha}}\Omega^{I} = 0$.

It is also convenient to introduce the background covariantly chiral fields ϕ, φ and $\tilde{\varphi}$ in place of the chiral fields Φ, Q and $\tilde{Q}$:

$$\begin{aligned} \phi &= e^{\bar{w}_B}\Phi e^{-\bar{w}_B} & \qquad \bar{\mathcal{D}}_{\dot{\alpha}}\phi &= 0 \\ \varphi &= e^{\bar{w}_B}Q & \qquad \bar{\mathcal{D}}_{\dot{\alpha}}\varphi &= 0 \\ \tilde{\varphi} &= (e^{-\bar{w}_B})^{\mathrm{T}}\tilde{Q} & \qquad \bar{\tilde{\mathcal{D}}}_{\dot{\alpha}}\tilde{\varphi} &= 0. \end{aligned} \tag{3.2.81}$$

The last of these, $\tilde{\varphi}$, is covariantly chiral in the conjugate representation $\tilde{\mathcal{D}}_{\alpha} = (e^{\bar{w}_B})^{\mathrm{T}}D_{\alpha}(e^{-\bar{w}_B})^{\mathrm{T}}$, as indicated. These new fields fields transform covariantly in the background picture,

$$\phi' = e^{-\mathrm{i}k}\phi e^{\mathrm{i}k} \qquad \varphi' = e^{-\mathrm{i}k}\varphi \qquad \tilde{\varphi}' = e^{\mathrm{i}k}\tilde{\varphi}. \tag{3.2.82}$$

At this stage it is worth pointing out that a linear background-quantum splitting of the chiral scalars ϕ, φ and $\tilde{\varphi}$ (or equivalently Φ, Q and $\tilde{Q}$) is sufficient if one wishes to compute the full background effective action. Such a splitting has the desirable effect of generating new mass-like terms for the quantum fields, at the cost of introducing cross (or quantum field interaction) terms in the quadratic part of the quantum action. Usually such terms are removed by choosing appropriate gauge fixing conditions, the superfield equivalent of the R_ξ gauge [194, 195], which although consistent with the background field approach are inherently non-local. For a detailed treatment see [196, 197].

Under the proposed background splitting, the original superfield strength W_α splits as

$$\begin{aligned} W_\alpha &= \frac{1}{8}[\bar{\mathcal{D}}^{\dot{\alpha}}, \{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot{\alpha}}\}] \\ &= \frac{1}{8}[e^{\bar{w}_Q}\bar{\mathcal{D}}^{\dot{\alpha}}_{B}e^{-\bar{w}_Q}, \{e^{-w_Q}\mathcal{D}_{B\alpha}e^{w_Q}, e^{\bar{w}_Q}\bar{\mathcal{D}}_{B\dot{\alpha}}e^{-\bar{w}_Q}\}] \\ &= \frac{1}{8}e^{\bar{w}_Q}[\bar{\mathcal{D}}^{\dot{\alpha}}_{B}, \{e^{-2V_Q}\mathcal{D}_{B\alpha}e^{2V_Q}, \bar{\mathcal{D}}_{B\dot{\alpha}}\}]e^{-\bar{w}_Q} \\ &= \frac{1}{8}e^{\bar{w}_Q}[\bar{\mathcal{D}}^{\dot{\alpha}}_{B}, \bar{\mathcal{D}}_{B\dot{\alpha}}(e^{-2V_Q}\mathcal{D}_{B\alpha}e^{2V_Q}) + \{\mathcal{D}_{B\alpha}, \bar{\mathcal{D}}_{B\dot{\alpha}}\}]e^{-\bar{w}_Q} \\ &= e^{\bar{w}_Q}\left(\frac{1}{8}\bar{\mathcal{D}}^{2}_{B}(e^{-2V_Q}\mathcal{D}_{B\alpha}e^{2V_Q}) + W_{B\alpha}\right)e^{-\bar{w}_Q}. \end{aligned} \tag{3.2.83}$$

The superfield strength piece of the original action therefore splits as

$$
\begin{aligned}
&\mathrm{tr}_{\mathcal{R}} \int \mathrm{d}^6 z \, \frac{1}{2} W^2 \\
&= \mathrm{tr}_{\mathcal{R}} \int \mathrm{d}^6 z \, \frac{1}{2} \left(\frac{1}{8} \bar{\mathcal{D}}_B^2 (e^{-2V_Q} \mathcal{D}_B^\alpha e^{2V_Q}) + W_B^\alpha \right) \left(\frac{1}{8} \bar{\mathcal{D}}_B^2 (e^{-2V_Q} \mathcal{D}_{B\alpha} e^{2V_Q}) + W_{B\alpha} \right) \\
&= \mathrm{tr}_{\mathcal{R}} \int \mathrm{d}^6 z \, \frac{1}{2} \left(\frac{1}{64} \bar{\mathcal{D}}_B^2 (e^{-2V_Q} \mathcal{D}_B^\alpha e^{2V_Q}) \bar{\mathcal{D}}_B^2 (e^{-2V_Q} \mathcal{D}_{B\alpha} e^{2V_Q}) \right. \\
&\qquad\qquad\qquad\qquad \left. + \frac{1}{4} W_B^\alpha \, \bar{\mathcal{D}}_B^2 (e^{-2V_Q} \mathcal{D}_{B\alpha} e^{2V_Q}) + W_B^2 \right) \\
&= -\mathrm{tr}_{\mathcal{R}} \int \mathrm{d}^8 z \, \left(\frac{1}{32} (e^{-2V_Q} \mathcal{D}_B^\alpha e^{2V_Q}) \bar{\mathcal{D}}_B^2 (e^{-2V_Q} \mathcal{D}_{B\alpha} e^{2V_Q}) \right. \\
&\qquad\qquad \left. + \frac{1}{2} W_B^\alpha (e^{-2V_Q} \mathcal{D}_{B\alpha} e^{2V_Q}) \right) + \mathrm{tr}_{\mathcal{R}} \int \mathrm{d}^6 z \, \frac{1}{2} W_B^2, \qquad (3.2.84)
\end{aligned}
$$

where, having used chirality, all but the purely background contribution has been lifted to an expression on full superspace.

Thus the original action (3.2.62) becomes:

$$
\begin{aligned}
S_{\mathrm{split}} = \frac{1}{g^2} \mathrm{tr}_{\mathcal{R}} \Big\{ &\int \mathrm{d}^8 z \, e^{-2V_Q} \bar{\phi} e^{2V_Q} \phi - \frac{1}{32} \int \mathrm{d}^8 z \, \Big((e^{-2V_Q} \mathcal{D}^\alpha e^{2V_Q}) \bar{\mathcal{D}}^2 (e^{-2V_Q} \mathcal{D}_\alpha e^{2V_Q}) \\
&- 16 W^\alpha (e^{-2V_Q} \mathcal{D}_\alpha e^{2V_Q}) \Big) + \frac{1}{2} \int \mathrm{d}^6 z \, W^2 \Big\} + \int \mathrm{d}^8 z \big(\bar{\varphi} e^{2V_Q} \varphi + \bar{\tilde{\varphi}} (e^{-2V_Q})^{\mathrm{T}} \tilde{\varphi} \big) \\
&+ \sqrt{2} \int \mathrm{d}^6 z \, \tilde{\varphi}^{\mathrm{T}} \phi \, \varphi + \sqrt{2} \int \mathrm{d}^6 \bar{z} \, \bar{\varphi} \, \bar{\phi} \, \bar{\tilde{\varphi}}^{\mathrm{T}} + \mathcal{M} \int \mathrm{d}^6 z \, \tilde{\varphi}^{\mathrm{T}} \varphi + \mathcal{M} \int \mathrm{d}^6 \bar{z} \, \bar{\tilde{\varphi}} \, \bar{\varphi}^{\mathrm{T}}, \qquad (3.2.85)
\end{aligned}
$$

where here, and subsequently, we unambiguously omit the B subscript on field strengths and gauge covariant derivatives since they are all background.

3.2.2 Gauge fixing

Adopting the notation $\mathcal{K} = (\kappa, \dot{\kappa})$, which runs over chiral and antichiral fields respectively, suitable gauge fixing conditions are given by [194, 195]

$$
F^{\hat{\mathcal{K}}} = \chi^{\hat{\mathcal{K}}} - f^{\hat{\mathcal{K}}} \qquad \chi^{\mathcal{K}} = \begin{cases} \chi^I = -\frac{1}{4} (\mathrm{e}^{-\bar{w}_B})^{IJ} (\bar{\mathcal{D}}^2 V_Q)^J & \mathcal{K} = \kappa \\ \bar{\chi}^I = -\frac{1}{4} (\mathrm{e}^{w_B})^{IJ} (\mathcal{D}^2 V_Q)^J & \mathcal{K} = \dot{\kappa} \end{cases} \qquad (3.2.86)
$$

where the background generating functional is weighted by

$$
\exp \left\{ \frac{\mathrm{i}}{2g\gamma} \big(f^{\hat{\mathcal{J}}} Y_{\hat{\mathcal{J}}\hat{\mathcal{K}}} f^{\hat{\mathcal{K}}} + a^{\hat{\mathcal{J}}} \grave{Y}_{\hat{\mathcal{J}}\hat{\mathcal{K}}} a^{\hat{\mathcal{K}}} \big) \right\} \qquad \varepsilon(a^{\mathcal{K}}) = 1, \qquad (3.2.87)
$$

with $Y_{\hat{\mathcal{J}}\hat{\mathcal{K}}}$ chosen to be

$$
Y_{\hat{\mathcal{J}}\hat{\mathcal{K}}} = \begin{cases} -\frac{1}{4} \bar{D}^2 \big((\mathrm{e}^{-2V_B})^{IJ} \delta_-(z, z') \big) & \mathcal{J} = \kappa, \, \mathcal{K} = \dot{\kappa} \\ -\frac{1}{4} D^2 \big((\mathrm{e}^{2V_B})^{IJ} \delta_+(z, z') \big) & \mathcal{J} = \dot{\kappa}, \, \mathcal{K} = \kappa \end{cases} \qquad (3.2.88)
$$

and therefore

$$\grave{Y}_{\hat{\mathcal{J}}\hat{\mathcal{K}}} = \begin{cases} +\frac{1}{4}\bar{D}^2\big((\mathrm{e}^{-2V_B})^{IJ}\delta_-(z,z')\big) & \mathcal{J}=\kappa,\, \mathcal{K}=\dot{\kappa} \\ -\frac{1}{4}D^2\big((\mathrm{e}^{2V_B})^{IJ}\delta_+(z,z')\big) & \mathcal{J}=\dot{\kappa},\, \mathcal{K}=\kappa \end{cases} \tag{3.2.89}$$

and all other components in both operators vanishing. To elucidate the meaning of such a weighting, one can easily show by using the (anti)chirality of $\chi^{\hat{\mathcal{K}}}$ and $a^{\hat{\mathcal{K}}}$, integration by parts, and lifting the integrals to full superspace, that this has the effect of contributing the following gauge fixing term S_{gf} and Neilson-Kallosh ghost term S_{NK} to the total generating functional:

$$S_{\mathrm{gf}} = \frac{1}{32g^2\gamma}\mathrm{tr}_{\mathcal{R}}\int \mathrm{d}^8z\, V_Q\{\mathcal{D}^2,\bar{\mathcal{D}}^2\}V_Q \tag{3.2.90}$$

$$S_{\mathrm{NK}} = \frac{1}{g^2\gamma}\mathrm{tr}_{\mathcal{R}}\int \mathrm{d}^8z\, \bar{\mathrm{a}}\mathrm{a}. \tag{3.2.91}$$

Here the Neilson-Kallosh ghosts belong to the adjoint representation of the gauge group, and are background (anti)chiral ie $\mathrm{a}' = \mathrm{e}^{\bar{w}_B}a'\mathrm{e}^{-\bar{w}_B}$ and $\bar{\mathcal{D}}_{\dot{\alpha}}\mathrm{a} = 0$.

Noting that V_Q varies with respect the gauge parameters $\zeta^{\hat{\mathcal{K}}} = (\Lambda^I(z), \bar{\Lambda}^I(z))$ in (3.2.71) as (also see (3.2.72) and (2.2.73))

$$\delta V_Q = -\frac{\mathrm{i}}{2}L_{V_Q}(\bar{\Omega}+\Omega) + \frac{\mathrm{i}}{2}L_{V_Q}\coth(L_{V_Q})(\bar{\Omega}-\Omega) \qquad \bar{\mathcal{D}}_{\dot{\alpha}}\Omega = \mathcal{D}_\alpha\bar{\Omega} = 0 \tag{3.2.92}$$

and the gauge fixing functions vary with respect to V_Q as

$$\delta\chi^{\mathcal{K}} = \begin{cases} -\frac{1}{4}(\mathrm{e}^{-\bar{w}_B})^{IJ}(\bar{\mathcal{D}}^2\delta V_Q)^J & \mathcal{K}=\kappa \\ -\frac{1}{4}(\mathrm{e}^{w_B})^{IJ}(\mathcal{D}^2\delta V_Q)^J & \mathcal{K}=\dot{\kappa} \end{cases} \tag{3.2.93}$$

it then follows that the Faddeev-Popov ghost part of the action becomes

$$\begin{aligned} S_{\mathrm{FP}} &= c'_{\hat{\mathcal{J}}}\frac{\delta\tilde{\chi}^{\hat{\mathcal{J}}}}{\delta\zeta^{\hat{\mathcal{K}}}}c^{\hat{\mathcal{K}}} \\ &= \int \mathrm{d}^6z\, c'^I\delta\chi^I + \int \mathrm{d}^6\bar{z}\, \bar{c}'^I\delta\bar{\chi}^I \\ &= -\frac{1}{4}\int \mathrm{d}^6z\, c'^I(\mathrm{e}^{\bar{w}_B})^{IJ}\bar{\mathcal{D}}^2(\delta V_Q^J)) - \frac{1}{4}\int \mathrm{d}^6\bar{z}\, \bar{c}'^I(\mathrm{e}^{-w_B})^{IJ}\mathcal{D}^2\delta V_Q^J) \\ &= \mathrm{tr}_{\mathcal{R}}\Big(\int \mathrm{d}^8z\, (\mathrm{C}'+\bar{\mathrm{C}}')\delta V_Q)\Big). \end{aligned} \tag{3.2.94}$$

This has been simplified using the (anti)chirality of ghosts $c'^{\hat{\mathcal{K}}} = (c'^I(z), \bar{c}'^I(z))$, and we have defined background (anti)chiral ghosts belonging to the adjoint representation: $C' = \mathrm{e}^{\bar{w}_B}c'\mathrm{e}^{-\bar{w}_B}$, $\bar{C}' = \mathrm{e}^{-w_B}\bar{c}'\mathrm{e}^{w_B}$, $\bar{\mathcal{D}}_{\dot{\alpha}}C' = \mathcal{D}_\alpha\bar{C}' = 0$, where in the last three lines of this expression it is to be understood that δV_Q is to be replaced by the expression (3.2.92), which in turn has Ω and $\bar{\Omega}$ replaced by the background (anti)chiral ghosts C and $\bar{C}$ respectively. This finally gives

$$S_{\mathrm{FP}} = -\frac{\mathrm{i}}{2}\mathrm{tr}_{\mathcal{R}}\bigg(\int \mathrm{d}^8 z\, (C' + \bar{C}') L_{V_Q} \left[\bar{C} + C - \coth(L_{V_Q})(\bar{C} - C) \right] \bigg) \qquad (3.2.95)$$

and illustrates the highly non-trivial interaction between the Faddeev-Popov ghosts and the quantum gauge field.

Putting this all together we obtain the background generating functional in the absence of sources:

$$Z[0, w_B, \bar{w}_B] = \mathrm{N} \int [\mathrm{d}V_Q][\mathrm{d}\phi][\mathrm{d}\varphi][\mathrm{d}\tilde{\varphi}][\mathrm{d}c][\mathrm{d}c'][\mathrm{d}a] e^{\mathrm{i}S_{\mathrm{total}}}, \qquad (3.2.96)$$

the total action being

$$S_{\mathrm{total}} = S_{\mathrm{split}} + S_{\mathrm{FP}} + S_{\mathrm{NK}} + S_{\mathrm{gf}} \qquad (3.2.97)$$

with S_{split}, S_{gf}, S_{FP}, and S_{NK} being given by (3.2.85), (3.2.90), (3.2.95) in (3.2.91) respectively.

3.2.3 Quantum field expansion

Since we are interested in only one-loop contributions to the background effective action, the total action needs only to be expanded up to terms quadratic in quantum fields. All such quadratic contributions, with the exception of V_Q and Faddeev-Popov ghosts, can be trivially read off from S_{total}.

To extract the Faddeev-Popov ghost contribution one uses

$$L_{V_Q} \left[\bar{C} + C - \coth(L_{V_Q})(\bar{C} - C) \right] = C - \bar{C} + [V_Q, \bar{C} + C] + \mathcal{O}(V_Q^2), \qquad (3.2.98)$$

and so

$$\begin{aligned} S_{\mathrm{FP}} &= -\frac{\mathrm{i}}{2}\mathrm{tr}_{\mathcal{R}}\bigg(\int \mathrm{d}^8 z\, (C' + \bar{C}')(C - \bar{C}) \bigg) + \text{cubic terms} \\ &= -\frac{\mathrm{i}}{2}\mathrm{tr}_{\mathcal{R}}\bigg(\int \mathrm{d}^8 z\, (\bar{C}'C - C'\bar{C}) \bigg) + \text{cubic terms} \end{aligned} \qquad (3.2.99)$$

having used the ghosts' (anti)chirality.

Noting that

$$(e^{-2V_Q}\mathcal{D}_\alpha e^{2V_Q}) = 2(\mathcal{D}_\alpha V_Q) - 2[V_Q, (\mathcal{D}_\alpha V_Q)] + \mathcal{O}(V_Q^3) \qquad (3.2.100)$$

then the total quadratic V_Q contribution, which comes from the split field strength (3.2.84) and $S_{\rm gf}$, is given by

$$
\begin{aligned}
&\frac{1}{32g^2}\mathrm{tr}_{\mathcal{R}}\int \mathrm{d}^8 z\,\Big(-(e^{-2V_Q}\mathcal{D}^\alpha e^{2V_Q})\bar{\mathcal{D}}^2(e^{-2V_Q}\mathcal{D}_\alpha e^{2V_Q}) \\
&\qquad\qquad - 16W^\alpha(e^{-2V_Q}\mathcal{D}_\alpha e^{2V_Q}) + \frac{1}{\gamma}V_Q\{\mathcal{D}^2,\bar{\mathcal{D}}^2\}V_Q\Big) \\
&= \frac{1}{32g^2}\mathrm{tr}_{\mathcal{R}}\int \mathrm{d}^8 z\,\Big(-4(\mathcal{D}^\alpha V_Q)\bar{\mathcal{D}}^2\mathcal{D}_\alpha V_Q - 32W^\alpha\mathcal{D}_\alpha V_Q \\
&\qquad\qquad + 32W^\alpha[V_Q,(\mathcal{D}_\alpha V_Q)] + \frac{1}{\gamma}V_Q\{\mathcal{D}^2,\bar{\mathcal{D}}^2\}V_Q\Big) + \mathcal{O}(V_Q^3) \\
&= \frac{1}{8g^2}\mathrm{tr}_{\mathcal{R}}\int \mathrm{d}^8 z\,\Big(V_Q\mathcal{D}^\alpha\bar{\mathcal{D}}^2\mathcal{D}_\alpha V_Q - 8W^\alpha\mathcal{D}_\alpha V_Q \\
&\qquad\qquad + 8V_Q[W^\alpha,(\mathcal{D}_\alpha V_Q)] + \frac{1}{4\gamma}V_Q\{\mathcal{D}^2,\bar{\mathcal{D}}^2\}V_Q\Big) + \mathcal{O}(V_Q^3),
\end{aligned}
$$

having integrated by parts, and used the cyclic property of the trace.

Re-scaling $g \to \sqrt{2}g$, followed by $V_Q \to gV_Q$, $\phi \to g\phi$, $\mathrm{a} \to g\mathrm{a}$, $C \to 2\mathrm{i}C$, and dropping terms linear in quantum fields since they will not contribute to the effective action, the total action $S_{\rm total}$ up to terms quadratic in quantum fields is

$$S_{\rm total} = S_0[V_B] + S_2. \tag{3.2.101}$$

The classical action $S_0[V_B]$ is just

$$S_0[V_B] = \frac{1}{4g^2}\mathrm{tr}_{\mathcal{R}}\int \mathrm{d}^6 z\, W^2, \tag{3.2.102}$$

and the action quadratic in quantum fields is

$$
\begin{aligned}
S_2 =& \frac{1}{16}\int \mathrm{d}^8 z\, V_Q\Big(\mathcal{D}^\alpha\bar{\mathcal{D}}^2\mathcal{D}_\alpha + 8W^\alpha\mathcal{D}_\alpha + \frac{1}{4\gamma}\{\mathcal{D}^2,\bar{\mathcal{D}}^2\}\Big)V_Q \\
&+ \int \mathrm{d}^8 z\big(\bar\phi\phi + \bar\varphi\varphi + \bar{\tilde\varphi}\tilde\varphi\big) + \mathcal{M}\int \mathrm{d}^6 z\,\tilde\varphi^{\rm T}\varphi + \mathcal{M}\int \mathrm{d}^6\bar z\,\bar{\tilde\varphi}\,\bar\varphi^{\rm T} \\
&+ \int \mathrm{d}^8 z\big(C'\bar C + \bar C' C\big) + \int \mathrm{d}^8 z\,\frac{1}{2\gamma}\bar{\mathrm{a}}\mathrm{a}.
\end{aligned}
\tag{3.2.103}
$$

Notice that there is no longer a trace over the gauge indices in S_2, since V_Q, ϕ and the ghosts are now expressed as a column vectors with respect to their gauge indices, and W^α is contracted with generators of the adjoint representation of the gauge group.

By playing with the covariant derivatives (3.2.78), one can prove the identity,

$$
\begin{aligned}
\Box &= \mathcal{D}^a\mathcal{D}_a - W^\alpha\mathcal{D}_\alpha + \bar W_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha} \\
&= -\frac{1}{8}\mathcal{D}^\alpha\bar{\mathcal{D}}^2\mathcal{D}_\alpha + \frac{1}{16}\{\mathcal{D}^2,\bar{\mathcal{D}}^2\} - W^\alpha\mathcal{D}_\alpha - \frac{1}{2}(\mathcal{D}^\alpha W_\alpha) \\
&= -\frac{1}{8}\bar{\mathcal{D}}_{\dot\alpha}\mathcal{D}^2\bar{\mathcal{D}}^{\dot\alpha} + \frac{1}{16}\{\mathcal{D}^2,\bar{\mathcal{D}}^2\} + \bar W_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha} + \frac{1}{2}(\bar{\mathcal{D}}_{\dot\alpha}\bar W^{\dot\alpha})
\end{aligned}
\tag{3.2.104}
$$

where $\Box$ is called the super d'Alembertian. Using this, the vector part of S_2 simply becomes

$$-\frac{1}{2}\int \mathrm{d}^8 z\, V_Q\Big[\Box + \frac{1}{16}\Big(\frac{1}{2\gamma}+1\Big)\{\mathcal{D}^2, \bar{\mathcal{D}}^2\}\Big]V_Q. \tag{3.2.105}$$

This takes its simplest form in the supersymmetric Fermi-Feynman gauge [198, 180, 89] where the gauge parameter $\gamma = -1/2$, which from here onward is the gauge we choose.

The resulting theory describes a single massless vector multiplet V_Q and six chiral scalars all propagating the presence of a Yang-Mills background. Four of these chiral scalars are massless and are in the adjoint representation of the gauge group, three of which are ghosts and posses odd statistics. The final two chiral scalars each have mass $\mathcal{M}$, and together transform in the some real representation of the gauge group $R \oplus R_c$.

3.2.4 Chiral fields coupled to external Yang-Mills

In the following chapters we will compute the super Yang-Mills background one-loop effective action by employing heat kernel techniques. As it stands, the operator in the vector part of S_2 is of Laplace type, but in order to handle chiral fields in the action using heat kernel techniques, and to correctly cast the chiral contribution in the form of superdeterminants, we must convert it into a more usable form. This is achieved as follows.

In the absence of any interactions, the most general classical action describing a massive chiral scalar field in a real representation $\mathcal{R}$ coupled to a super Yang-Mills background is given by:

$$S[\Phi, \bar{\Phi}, V_B] = \int \mathrm{d}^8 z\, \bar{\Phi} e^{2V_B}\Phi + \frac{m}{2}\Big\{\int \mathrm{d}^6 z\, \Phi^{\mathrm{T}}\,\Phi + c.c.\Big\}. \tag{3.2.106}$$

In deriving the background one-loop effective action for this theory, $\Gamma^{(1)}_{\Phi,m,\mathcal{R}}$, we are faced with evaluating

$$e^{\mathrm{i}\Gamma^{(1)}_{\Phi,m,\mathcal{R}}} = \int [\mathrm{d}\Phi] e^{\mathrm{i}S[\Phi,\bar{\Phi},V_B]}. \tag{3.2.107}$$

First we re-write the classical action in the more desirable form using (anti)chirality and the antisymmetry of the group generators:

$$S[\Phi, \bar{\Phi}, V_B] = \frac{1}{2}\mathbf{\Phi}^{\hat{M}}\mathcal{H}_{\hat{M}\hat{N}}(m)\mathbf{\Phi}^{\hat{N}} \tag{3.2.108}$$

where

$$\mathbf{\Phi}^{\hat{M}} = \begin{pmatrix} \Phi(z) \\ \bar{\Phi}(z) \end{pmatrix} \tag{3.2.109}$$

and

$$\mathcal{H}_{\hat{M}\hat{N}}(m) = \begin{pmatrix} m\mathbb{1}_{\mathcal{R}}\delta_+(z,z') & -\frac{1}{4}\bar{D}^2\mathrm{e}^{-2V_B}\delta_-(z,z') \\ -\frac{1}{4}D^2\mathrm{e}^{2V_B}\delta_+(z,z') & m\mathbb{1}_{\mathcal{R}}\delta_-(z,z') \end{pmatrix}. \tag{3.2.110}$$

This latter is of course a massive generalization of $Y_{\hat{\mathcal{J}}\hat{\mathcal{K}}}$ given in equation (3.2.88).

The one-loop effective action is therefore just

$$\Gamma^{(1)}_{\Phi,m,\mathcal{R}} = \frac{\mathrm{i}}{2}\ln \mathrm{sDet}(\mathcal{H}(m)) \tag{3.2.111}$$

We now employ the 'doubling trick' (see for example [89, 127]), which involves noting that the change in integration variable of $\Phi \to \mathrm{i}\Phi$ in (3.2.107) leaves the functional integration measure unchanged, but has the overall effect of redefining the mass $m \to -m$. Consequently

$$\mathrm{sDet}(\mathcal{H}(m)) = \mathrm{sDet}(\mathcal{H}(-m)) \tag{3.2.112}$$

and so we are free to write

$$\Gamma^{(1)}_{\Phi,m,\mathcal{R}} = \frac{\mathrm{i}}{4}\ln \mathrm{sDet}(\mathcal{H}(m)\mathcal{H}(-m)). \tag{3.2.113}$$

Computing $\mathcal{H}(m)\mathcal{H}(-m)$ one finds

$$\mathcal{H}(m)\mathcal{H}(-m) = \begin{pmatrix} \Box_+^{(+)} - m^2\mathbb{1}_{\mathcal{R}} & 0 \\ 0 & \Box_-^{(-)} - m^2\mathbb{1}_{\mathcal{R}} \end{pmatrix} \begin{pmatrix} \mathbb{1}_{\mathcal{R}}\,\delta_+(z,z') \\ \mathbb{1}_{\mathcal{R}}\,\delta_-(z,z') \end{pmatrix} \tag{3.2.114}$$

where we have introduced the background (anti)chiral d'Alembertian operators $\Box_\pm$ in terms of background gauge covariant derivatives in (anti)chiral representations

$$\Box_+^{(+)} = \frac{1}{16}\bar{\mathcal{D}}^{(+)2}\mathcal{D}^{(+)2} \quad \mathcal{D}_\alpha^{(+)} = e^{-2V_B}D_\alpha e^{2V_B} \qquad \bar{\mathcal{D}}^{\dot{\alpha}(+)} = \bar{D}^{\dot{\alpha}} \tag{3.2.115}$$

$$\Box_-^{(-)} = \frac{1}{16}\mathcal{D}^{(-)2}\bar{\mathcal{D}}^{(-)2} \quad \mathcal{D}_\alpha^{(-)} = D_\alpha \qquad \bar{\mathcal{D}}^{\dot{\alpha}(-)} = e^{2V_B}\bar{D}^{\dot{\alpha}}e^{-2V_B}. \tag{3.2.116}$$

The operators $\Box_+$ and $\Box_-$ act on background covariantly chiral and antichiral scalars respectively (in the appropriate representations). One can readily show by applying both sides of equation (3.2.104) to such scalars, and by using the Bianchi identities $\mathcal{D}^\alpha W_\alpha = \bar{\mathcal{D}}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}$, that in any representation

$$\Box_+\,\Psi = \frac{1}{16}\bar{\mathcal{D}}^2\mathcal{D}^2\Psi = (\mathcal{D}^a\mathcal{D}_a - W^\alpha\mathcal{D}_\alpha - \frac{1}{2}(\mathcal{D}^\alpha W_\alpha))\Psi \qquad \bar{\mathcal{D}}_{\dot{\alpha}}\Psi = 0 \tag{3.2.117}$$

$$\Box_-\,\bar{\Psi} = \frac{1}{16}\mathcal{D}^2\bar{\mathcal{D}}^2\bar{\Psi} = (\mathcal{D}^a\mathcal{D}_a + \bar{W}_{\dot{\alpha}}\bar{\mathcal{D}}^{\dot{\alpha}} + \frac{1}{2}(\bar{\mathcal{D}}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}))\bar{\Psi} \qquad \mathcal{D}_\alpha\bar{\Psi} = 0. \tag{3.2.118}$$

Alternatively, one may prove these identities by pushing the outer derivative through to annihilate the scalar and in doing so use the covariant derivative algebra. Consequently we find that these are Laplace-type operators, which are necessary to define the heat kernels.

The effective action then becomes[4]

$$\Gamma^{(1)}_{\Phi,m,\mathcal{R}} = \frac{\mathrm{i}}{4}\ln \mathrm{sDet}(\Box^{(+)}_{+} - m^2 \mathbb{1}_{\mathcal{R}}) + \frac{\mathrm{i}}{4}\ln \mathrm{sDet}(\Box^{(-)}_{-} - m^2 \mathbb{1}_{\mathcal{R}}). \qquad (3.2.119)$$

3.2.5 $\mathcal{N}=2$ super Yang-Mills to one-loop

Returning now to the problem of computing the $\mathcal{N}=2$ super Yang-Mills one-loop effective action, $\Gamma^{(1)}[w_B, \bar{w}_B]$, we find

$$e^{\mathrm{i}\Gamma^{(1)}[w_B,\bar{w}_B]} = \mathcal{N}\int [\mathrm{d}V_Q][\mathrm{d}\phi][\mathrm{d}\chi][\mathrm{d}c][\mathrm{d}c'][\mathrm{d}a]e^{\mathrm{i}S_2} \qquad (3.2.120)$$

with

$$S_2 = -\frac{1}{2}\int \mathrm{d}^8 z\, V_Q \Box V_Q + \int \mathrm{d}^8 z\big(\bar{\phi}\phi + \bar{\chi}\chi\big) + \mathcal{M}\Big\{\int \mathrm{d}^6 z\, \chi^{\mathrm{T}}\chi + c.c.\Big\} + \int \mathrm{d}^8 z\big(C'\bar{C} + \bar{C}'C\big) - \int \mathrm{d}^8 z\, \bar{\mathrm{a}}\mathrm{a} \qquad (3.2.121)$$

where the field χ, which transforms in a real representation of the gauge group equivalent to $R \oplus R_c$, has been introduced through a field redefinition of φ and $\tilde{\varphi}$ similar to that performed in subsection 2.2.3. Ultimately the effective action at the one-loop level (3.2.120) reduces to a linear combination of two types of contribution [89, 196]:

$$\Gamma^{(1)}[w_B, \bar{w}_B] = \frac{\mathrm{i}}{2}\ln \mathrm{sDet}\Box - 3\Gamma^{(1)}_{\Phi,0,\mathrm{Ad}} + \Gamma^{(1)}_{\Phi,0,\mathrm{Ad}} + \Gamma^{(1)}_{\Phi,\mathcal{M},R\oplus R_c}. \qquad (3.2.122)$$

In this expression the second term on the right originates from the three massless ghosts, the third from the massless chiral scalar ϕ, and the fourth from the hypermultiplet scalars.

3.2.6 $\mathcal{N}=4$ super Yang-Mills to one-loop

The case of $\mathcal{N}=4$ super Yang-Mills is obtained from the $\mathcal{N}=2$ case by setting the hypermultiplet mass $\mathcal{M}=0$ and the representation R (and R_c) to be the adjoint. The one-loop effective action (3.2.122) simply becomes

$$\begin{aligned}\Gamma^{(1)}[w_B, \bar{w}_B] &= \frac{\mathrm{i}}{2}\ln \mathrm{sDet}\Box - 3\Gamma^{(1)}_{\Phi,0,\mathrm{Ad}} + \Gamma^{(1)}_{\Phi,0,\mathrm{Ad}} + 2\Gamma^{(1)}_{\Phi,0,\mathrm{Ad}} \\ &= \frac{\mathrm{i}}{2}\ln \mathrm{sDet}\Box, \end{aligned} \qquad (3.2.123)$$

a result which was first established by [180] (also see [199]).

[4]Results like this also have some validity where the mass terms come from chiral scalar backgrounds living in the Cartan subalgebra, for a discussion see [197].

Chapter 4

Heat kernel techniques

In this chapter we will briefly survey the role of heat kernels in quantum field theories, with particular emphasis on the Schwinger-DeWitt technique, before introducing some recent refinements in the so-called *Gaussian approach*.

4.1 The Schwinger-DeWitt technique

Heat kernels where first introduced into quantum physics by Fock [200] and Schwinger [187], completely elaborated for applications in quantum field theory in curved space by DeWitt [201, 167, 202, 93], and continue to be an active area of research in both mathematics and physics. The literature is vast, and it would be impossible to give a complete list of references here. Instead we direct the reader to [203, 204, 96, 99], and references therein.

For physicists, heat kernels, their expansions and associated computational techniques provide a very powerful and versatile set of tools for analyzing various aspects of quantum dynamics. These include the study of effective actions, divergences, renormalisation, and anomalies. The techniques are universal enough to admit application to a wide range of theories on manifolds with and without boundaries, including supermanifolds, quite often regardless of group structure and spin. They are particularly useful when computing one-loop contributions to the effective action. Again we direct the reader to [99] for a detailed review of heat kernel expansions and the array of possible approaches.

Although generalizations of the following discussions are well known [93], for the current purposes it is sufficient to consider only Laplace-type operators in flat space ($\mathbb{R}^{3,1}$ or $\mathbb{R}^{4|4}$) which posses a mass term[1]. As is standard practice, any massless

[1] By which we mean operators that are at most second order and of the form: $\partial^A \partial_A + V^A \partial_A + P - m^2$.

operators, which may give rise to infrared divergences, are regulated via the inclusion of a mass (or infrared regulator) which is set to zero at the end of calculations.

Given such an operator H, which in general will possess some background dependence on background fields, the associated Green's function $G^{\hat{M}}{}_{\hat{N}} = G^M{}_N(y, y')$ is defined by

$$H^{\hat{M}}{}_{\hat{N}} G^{\hat{N}}{}_{\hat{P}} = -\delta^{\hat{M}}{}_{\hat{P}}. \tag{4.1.1}$$

This admits the Fock-Schwinger or proper time representation

$$G^{\hat{M}}{}_{\hat{N}} = \mathrm{i} \int_0^\infty \mathrm{d}s \, K^{\hat{M}}{}_{\hat{N}}(s) \tag{4.1.2}$$

where $K^{\hat{M}}{}_{\hat{N}}(s) = K(y, y'; s)$ is the heat kernel associated with the operator H. It is defined as the solution to the Schrodinger type or pseudo-heat equation

$$\mathrm{i} \frac{\partial}{\partial s} K^{\hat{M}}{}_{\hat{P}}(s) + H^{\hat{M}}{}_{\hat{N}} \, K^{\hat{N}}{}_{\hat{P}}(s) = 0, \tag{4.1.3}$$

subject to the initial condition

$$\lim_{s \to +0} K^{\hat{M}}{}_{\hat{N}}(s) = \delta^{\hat{M}}{}_{\hat{N}}. \tag{4.1.4}$$

Equivalently it may be formally defined as[2]

$$K^{\hat{M}}{}_{\hat{N}}(s) = \left(e^{\mathrm{i}s(H + \mathrm{i}\epsilon)}\right)^{\hat{M}}{}_{\hat{N}}, \qquad \epsilon \to +0. \tag{4.1.5}$$

In the well known Schwinger-DeWitt technique, one seeks a solution to (4.1.3) of the form (suppressing gauge indices and restricting to $D = 4$ in the standard field theory case):

$$K(y, y'; s) = \frac{\mathrm{i}}{(4\pi \mathrm{i}s)^2} e^{\mathrm{i}\sigma(y,y')/2s - \mathrm{i}sm^2} F(y, y'; s) \tag{4.1.6}$$

where $\sigma(y, y')$ is half the square of the proper distance between x and x', ie $\sigma(x, x') = \frac{1}{2}(x - x')^2$ in Minkowski space. In superspace and associated subspaces, suitable generalizations are straightforward (for details see for example [127]). The function $F(y, y')$ has an asymptotic expansion in the limit $s \to 0$ of the form:

$$F(y, y'; s) = \sum_{n=0}^{\infty} (\mathrm{i}s)^n a_n(y, y'). \tag{4.1.7}$$

The $a_n(y, y')$ are known as heat kernel or DeWitt-Seeley coefficients, and can be computed in a variety of ways [99]. The best known method is the recursive DeWitt technique [203, 93], where the general solution (4.1.6) with asymptotic expansion (4.1.7), is inserted back into the heat equation (4.1.3). One then identifies terms carrying the same powers of s and in doing so obtains a recursion relation between the coefficients.

[2]The convergence factor of iϵ shall be dropped in subsequent equations and its presence implied.

4.2 The heat kernel and one-loop effective action

It is well known that the one-loop effective action is simply related to the functional supertrace of a heat kernel. The former is given by

$$\Gamma^{(1)}[v_B] = \frac{\mathrm{i}}{2}\mathrm{sTr}\ln H \tag{4.2.8}$$

where H is the operator which appears in the part of the action quadratic in quantum (super)fields. The heat kernel appears via use of the identity

$$\mathrm{sTr}\ln H = -\int_0^\infty \mathrm{d}s\, s^{-1}\,\mathrm{sTr}\, e^{\mathrm{i}sH} \tag{4.2.9}$$

up to some additive numerical constant, which can be established by separately considering the variation of the left and right hand side, and using

$$H^{-1} = -\mathrm{i}\int_0^\infty \mathrm{d}s\, e^{\mathrm{i}sH} \tag{4.2.10}$$

and (2.2.75). Equation (4.2.9) then immediately leads to

$$\Gamma^{(1)}[v_B] = -\frac{\mathrm{i}}{2}\int_0^\infty \mathrm{d}s\; s^{-1}\,\mathrm{sTr}\, e^{\mathrm{i}sH} = -\frac{\mathrm{i}}{2}\int_0^\infty \mathrm{d}s\, s^{-1}\, K(s) \tag{4.2.11}$$

having used (4.1.5), and where $K(s)$ denotes the functional supertrace of the heat kernel

$$K(s) = (-1)^{\varepsilon_M} K^{\hat{M}}{}_{\hat{M}}(s). \tag{4.2.12}$$

This expression for the one-loop effective has a potential divergence at the lower limit of the proper time integration (this is a UV divergence) and can be regulated using the following scheme:

$$\Gamma^{(1)}_\rho = \frac{\mu^{2\rho}}{2}\int_0^\infty \mathrm{d}s\,(\mathrm{i}s)^{\rho-1} K(s), \tag{4.2.13}$$

where μ and ρ are the renormalization point and regularization parameter respectively.

4.3 The Gaussian approach

We will now describe the main heat kernel technique used in this thesis, the Gaussian approach [91, 92], which is closely related to other more well known approaches [122, 205, 206, 123, 124] which employ plane wave expansions of the delta function and Gaussian integration identities. The key observation is that the covariant derivatives in the operator in the heat kernel occur only in commutators (giving rise to field strengths and derivatives of field strengths) when computing DeWitt-Seeley coefficients in even dimensions. The Gaussian approach attempts to optimize this expansion of the heat kernel in nested commutators.

In short, this technique can be used for computing heat kernels to some order in their asymptotic expansion, or in some simple cases cases the entire heat kernel to all orders. In contrast to iterative methods, where one computes successive DeWitt-Seeley coefficients, the procedure provides a means by which one can collectedly compute all of the coefficients up to some desired order. The technique is universal enough to be adapted to superfield theories, and proves very useful in computing the effective action in the derivative expansion.

At the early stages of the work presented in this thesis, the Gaussian approach had only been applied to a few cases [91, 92], and a more general application was yet to be attempted. Originally it was hoped, which was the motivation for following this path, that this approach would provide a relatively simple means of computing higher order DeWitt-Seeley coefficients in $\mathcal{N} = 4$ super Yang-Mills formulated in $\mathcal{N} = 1$ superspace. Unfortunately, as will be discussed later, it turns out that one can not proceed without modifying the approach. After doing so one obtains with yet another method for computing DeWitt-Seeley coefficients.

The basic idea behind the Gaussian approach is to attempt to solve a differential equation satisfied by the heat kernel by using various identities similar to those which may be employed when attempting to compute moments of Gaussian integrals. One either attempts to solve the differential equation to some order in an expansion, or in some simple cases to all orders.

The easiest way to describe the approach is through a simple example. The ideas are easily extended to more complicated cases.

4.3.1 The non-supersymmetric case

The simplest possible case which displays sufficient complication is an ordinary non-supersymmetric field theory, with operator of the form

$$H = \nabla^a \nabla_a - m^2, \tag{4.3.14}$$

and where the background field strength F_{ab} defined by the covariant derivatives

$$[\nabla_a, \nabla_b] = -\mathrm{i}F_{ab}, \tag{4.3.15}$$

is covariantly constant:

$$[\nabla_c, F_{ab}] = (\nabla_c F_{ab}) = 0. \tag{4.3.16}$$

This example proves simple enough to fully compute the heat kernel $K(x, x'; s)$ to all orders in its asymptotic expansion, as we now illustrate, first in the coincidence limit.

Starting with a Fourier representation for the delta function in Minkowski space

$$\delta^{(4)}(x - x') = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} e^{\mathrm{i}k_a(x^a - x'^a)}, \tag{4.3.17}$$

and the definition

$$K^{\hat{M}}{}_{\hat{P}}(s) = \left(e^{\mathrm{i}sH}\right)^{\hat{M}}{}_{\hat{N}} \tag{4.3.18}$$

of the heat kernel where H is given by (4.3.14), one obtains

$$\begin{aligned} K(x, x'; s) &= \mathrm{e}^{\mathrm{i}sH} \mathbb{1} \delta^{(4)}(x - x') \\ &= \int \frac{\mathrm{d}^4 k}{(2\pi)^4} e^{\mathrm{i}sH} e^{\mathrm{i}k^a(x_a - x'_a)} \\ &= \int \frac{\mathrm{d}^4 k}{(2\pi)^4} e^{-\mathrm{i}sm^2} e^{\mathrm{i}k^a(x_a - x'_a)} \left(e^{-\mathrm{i}k^a(x_a - x'_a)} e^{\mathrm{i}s\nabla^a \nabla_a} e^{\mathrm{i}k^a(x_a - x'_a)}\right) \end{aligned} \tag{4.3.19}$$

The term in parentheses can be evaluated as

$$e^{-\mathrm{i}k^a(x_a - x'_a)} e^{\nabla^a \nabla_a} e^{\mathrm{i}k^a(x_a - x'_a)} = e^{\mathrm{i}sX^a X_a} \tag{4.3.20}$$

where X_a is just a 'shifted' covariant derivative

$$X_a = \nabla_a + \mathrm{i}k_a, \tag{4.3.21}$$

which satisfies

$$[X_a, X_b] = -\mathrm{i}F_{ab}. \tag{4.3.22}$$

The procedure thus far is well known and has been around for some time [123, 124], to the extent that it can now be found in textbooks (for example see [125]). Strictly speaking, however, when applying this approach in the context of the background field technique one should really write the delta-function as

$$\delta^{(4)}(x - x') \mathbb{1} = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} e^{\mathrm{i}k_a(x^a - x'^a)} \mathcal{I}(x, x') \tag{4.3.23}$$

where the additional factor $\mathcal{I}(x, x')$ is a functional of the background fields, satisfies $\mathcal{I}(x, x) = \mathbb{1}$, and is included to ensure the correct gauge transformation properties of the heat kernel[3] at x and x' . This factor is the parallel displacement propagator, for complete details see [207] and references therein. To one-loop order in the background effective action, where one needs only to consider the heat kernel in the coincidence limit, this factor can be ignored.

[3]Typically the Green's function, operator H and therefore the heat kernel should have indices which transform covariantly. Consequently so must the delta function.

The coincidence limit

In the coincidence limit the kernel becomes

$$K(x;s) = \lim_{x'\to x} K(x,x';s) = e^{-\mathrm{i}sm^2}\int \frac{\mathrm{d}^4k}{(2\pi)^4} e^{\mathrm{i}sX^aX_a}. \tag{4.3.24}$$

From this perspective it is not immediately apparent that this quantity is itself not a differential operator, and it is of some significance to note that it is not. The heat kernel possess an asymptotic expansion, the coefficients of which are the DeWitt-Seeley coefficients in the coincidence limit, all of which are functionals of field strengths and covariant derivatives thereof. As a consequence one can expect to be able expand the exponential in (4.3.24) in a power series in s and arrange the terms into commutators at each order.

To prove this assertion [208, 207], one can easily show that $K(x;s)$ is not a differential operator by demonstrating that it commutes with an arbitrary function $f(x)$. This is achieved by taking a Fourier decomposition of the function, and operating on it:

$$\begin{aligned} K(x;s)f(x) &= K(x;s)\int \frac{\mathrm{d}^4\tilde{k}}{(2\pi)^4} e^{\mathrm{i}\tilde{k}x} f(\tilde{k}) \\ &= \int \frac{\mathrm{d}^4\tilde{k}}{(2\pi)^4} e^{\mathrm{i}\tilde{k}x} f(\tilde{k}) \int \frac{\mathrm{d}^4k}{(2\pi)^4} e^{-\mathrm{i}sm^2} e^{\mathrm{i}s(\nabla+\mathrm{i}k+\mathrm{i}\tilde{k})^2} \\ &= f(x)K(x;s) \end{aligned} \tag{4.3.25}$$

where in the last line a change of variable $k \to k - \tilde{k}$ has been made.

Consider now the massless case

$$\tilde{K}(x;s) = \int \frac{\mathrm{d}^4k}{(2\pi)^4} e^{\mathrm{i}sX^aX_a}. \tag{4.3.26}$$

We now explain how to compute $\tilde{K}(x;s)$ using the Gaussian approach. What makes the approach different for other plane wave approaches, is that we now different with respect to s to obtain

$$\frac{\mathrm{d}\tilde{K}(x;s)}{\mathrm{d}s} = \mathrm{i}\tilde{K}^a{}_a(x;s), \tag{4.3.27}$$

where we have introduced the notation

$$\tilde{K}_{a_1\ldots a_n}(x;s) = \int \frac{\mathrm{d}^4k}{(2\pi)^4} X_{a_1}\ldots X_{a_n} e^{\mathrm{i}sX^aX_a}. \tag{4.3.28}$$

This is the cornerstone of the scheme, and it is this differential equation which one attempts to solve in the Gaussian approach. In particular, if we can express $\tilde{K}^a{}_a(x;s)$

in terms of $\tilde{K}(x;s)$, then (4.3.27) is a linear differential equation for $\tilde{K}(x;s)$ which is easily solved. One achieves this by making repeated use of the identity

$$0 = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\partial}{\partial k_{a_n}} \left(X_{a_1} \dots X_{a_{n-1}} e^{\mathrm{i}s X^a X_a} \right). \tag{4.3.29}$$

This is reminiscent of the method one may use to determine the moments

$$\int \frac{\mathrm{d}^4 k}{(2\pi)^4} k_{a_1} \dots k_{a_{n-1}} e^{-k^2} \tag{4.3.30}$$

of a Gaussian integral, in terms of the Gaussian integral itself. Accordingly, the objects $\tilde{K}_{a_1 \dots a_n}(x;s)$ defined in (4.3.28) shall be referred to as moments of the kernel[4].

Choosing the case $n = 2$ in (4.3.29), one obtains

$$\begin{aligned} 0 &= \mathrm{i}\delta_a{}^b \tilde{K}(x;s) + \int \frac{\mathrm{d}^4 k}{(2\pi)^4} X_a \frac{\partial}{\partial k_b} e^{\mathrm{i}s X^2} \\ &= \mathrm{i}\delta_a{}^b \tilde{K}(x;s) - 2s \int \frac{\mathrm{d}^4 k}{(2\pi)^4} X_a \left(\int_0^1 \mathrm{d}\xi \; e^{\mathrm{i}s\xi X^2} X^b e^{-\mathrm{i}s\xi X^2} \right) e^{\mathrm{i}s X^2} \end{aligned} \tag{4.3.31}$$

where the identity

$$[A, e^B] = \int_0^1 \mathrm{d}\xi \; e^{\xi B} [A, B] e^{(1-\xi)B}, \tag{4.3.32}$$

has been applied.

In the present case, the quantity

$$\int_0^1 \mathrm{d}\xi \; e^{\mathrm{i}s\xi X^2} X^b e^{-\mathrm{i}s\xi X^2} = \sum_{n=0}^{\infty} \frac{s^n}{(n+1)!} L^n_{\mathrm{i}X^2}(X^b) \quad \text{with} \quad L_A(B) = [A, B] \tag{4.3.33}$$

can be exactly evaluated due to the simplifying assumption that the field strength is covariantly constant, $[X_a, [X_b, X_c]] = 0$. Ultimately this allows one to solve the differential equation (4.3.27) exactly. In the general case one is forced to solve the differential equation merely to some order in s.

Since

$$L^n_{\mathrm{i}X^2}(X^b) = (-2)^n (F^n)^b{}_a X^a \tag{4.3.34}$$

where

$$(F^0)^a{}_b = \delta^a{}_b \qquad\qquad (F^n)^a{}_b = F^a{}_{c_1} F^{c_1}{}_{c_2} F^{c_2}{}_{c_3} \dots F^{c_{n-1}}{}_b, \tag{4.3.35}$$

it follows that

$$\int_0^1 \mathrm{d}\xi \; e^{\mathrm{i}s\xi X^2} X^b e^{-\mathrm{i}s\xi X^2} = B^b{}_c(s) X^c \tag{4.3.36}$$

where

$$B^b{}_c(s) = \left(\frac{e^{-2sF} - \mathbf{1}}{-2sF} \right)^b{}_c . \tag{4.3.37}$$

[4]Occasionally this term will be used to collectively refer to all moments including $\tilde{K}(x;s)$ itself.

Noting that $[\nabla_a, B^b{}_c(s)] = 0$, equation (4.3.31) reduces to

$$0 = \mathrm{i}\delta^b_a \tilde{K}(x;s) - 2sB^b{}_c(s)\tilde{K}_a{}^c(x;s), \tag{4.3.38}$$

and since $B^a{}_b(s)$ is invertible

$$\tilde{K}_a{}^b(x;s) = -\mathrm{i}\left(\frac{F}{e^{-2sF} - \mathbf{1}}\right)^b{}_a \tilde{K}(x;s). \tag{4.3.39}$$

Insertion of this into the differential equation (4.3.27) yields

$$\frac{\mathrm{d}\tilde{K}(x;s)}{\mathrm{d}s} = \mathrm{tr}\left(\frac{F}{e^{-2sF} - \mathbf{1}}\right)\tilde{K}(x;s), \tag{4.3.40}$$

which is the desired linear differential equation for $\tilde{K}(x;s)$. This can be rewritten as

$$\frac{\mathrm{d}\ln\tilde{K}(x;s)}{\mathrm{d}s} = \mathrm{tr}\left(\frac{F}{e^{-2sF} - \mathbf{1}}\right). \tag{4.3.41}$$

Integrating both sides gives

$$\ln\tilde{K}(x;s) = \mathrm{tr}\ln\left(\frac{e^{2sF} - \mathbf{1}}{2F}\right)^{-\frac{1}{2}} + c, \tag{4.3.42}$$

c being an integration constant independent of s. This gives

$$\tilde{K}(x;s) = C\det\left(\frac{e^{2sF} - \mathbf{1}}{2F}\right)^{-\frac{1}{2}} \tag{4.3.43}$$

with $C = e^c$. This constant can be determined in a variety of ways. The simplest and most direct approach is to match the leading order terms in the asymptotic power series in s of (4.3.43) and its original definition (4.3.26) (for an alternative approach see [91]). In the latter case one simply expands the exponential $e^{\mathrm{i}sX^aX_a}$ after rescaling $k^a \to s^{-\frac{1}{2}}k^a$ as

$$\tilde{K}(x;s) = \frac{1}{s^2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{-\mathrm{i}k^2}(1+\mathrm{i}s\nabla^2+\frac{2\mathrm{i}}{\sqrt{s}}k^a\nabla_a+\ldots) = -\frac{\mathrm{i}}{(4\pi s)^2}+\mathcal{O}(s^{-1}). \tag{4.3.44}$$

Expression (4.3.43) gives

$$\tilde{K}(x;s) = \frac{C}{s^2} + \mathcal{O}(s^{-1}) \tag{4.3.45}$$

and comparison of the two provides the constant. This in turn yields the standard result [187, 209, 210, 211]:

$$\tilde{K}(x;s) = -\frac{\mathrm{i}}{(4\pi)^2}\det\left(\frac{e^{2sF} - \mathbf{1}}{2F}\right)^{-\frac{1}{2}} = -\frac{\mathrm{i}}{16\pi^2 s^2}\det\left(\frac{sF}{\sinh sF}\right)^{\frac{1}{2}} \tag{4.3.46}$$

From here one can proceed to compute the one-loop effective action as described in section 4.2.

Non-coincident points in spacetime

The above exercise can be repeated to compute the full heat kernel at non-coincident points, which only slightly complicates the calculation. This is useful in the case of higher loop contributions to the effective action, where one requires the full propagator in the presence of background fields.

Defining

$$\tilde{K}(x,x';s) = \int \frac{\mathrm{d}^4k}{(2\pi)^4} e^{\mathrm{i}k(x-x')} e^{\mathrm{i}sX^2} \mathcal{I}(x,x') \tag{4.3.47}$$

the differential equation is now

$$\frac{\mathrm{d}\tilde{K}(x,x';s)}{\mathrm{d}s} = \mathrm{i}\tilde{K}^a{}_a(x,x';s) \tag{4.3.48}$$

where

$$\tilde{K}_{a_1\dots a_n}(x,x';s) = \int \frac{\mathrm{d}^4k}{(2\pi)^4} X_{a_1}\dots X_{a_n} e^{\mathrm{i}k.(x-x')} e^{\mathrm{i}sX^2} \mathcal{I}(x,x'). \tag{4.3.49}$$

Using

$$0 = \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\partial}{\partial k_b} e^{\mathrm{i}k(x-x')} X_a e^{\mathrm{i}sX^2} \mathcal{I}(x,x') \tag{4.3.50}$$

which reduces to

$$\begin{aligned}\tilde{K}_a{}^b(x,x';s) = -\mathrm{i}\left(\frac{F}{e^{-2sF}-\mathbf{1}}\right)^b{}_a \tilde{K}(x,x';s) \\ - \mathrm{i}(x-x')^c \left(\frac{F}{e^{-2sF}-\mathbf{1}}\right)^b{}_c \tilde{K}_a(x,x',s),\end{aligned} \tag{4.3.51}$$

we are forced to deal with the moment $\tilde{K}_a(x,x',s)$ which was absent in the coincidence limit.

Using the identify

$$0 = \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\partial}{\partial k_b} e^{\mathrm{i}k(x-x')} e^{\mathrm{i}sX^2} \mathcal{I}(x,x') \tag{4.3.52}$$

it follows that

$$\tilde{K}_a(x,x';s) = -\mathrm{i}(x-x')^b \left(\frac{F}{e^{-2sF}-\mathbf{1}}\right)_{ab} \tilde{K}(x,x';s), \tag{4.3.53}$$

and so equation (4.3.51) can be written as

$$\begin{aligned}\tilde{K}_{ab}(x,x';s) = -\Bigg\{\mathrm{i}\left(\frac{F}{e^{-2sF}-\mathbf{1}}\right)_{ba} \\ + (x-x')^c(x-x')^d \left(\frac{F}{e^{-2sF}-\mathbf{1}}\right)_{bc} \left(\frac{F}{e^{-2sF}-\mathbf{1}}\right)_{ad}\Bigg\}\tilde{K}(x,x';s)\end{aligned} \tag{4.3.54}$$

The differential equation is then

$$\frac{\mathrm{d}\ln\tilde{K}(x,x';s)}{\mathrm{d}s} = \mathrm{tr}\left(\frac{F}{e^{-2sF}-\mathbf{1}}\right) + \mathrm{i}(x-x')^a(x-x')^b\left(\frac{F^2}{(e^{2sF}-\mathbf{1})(e^{-2sF}-\mathbf{1})}\right)_{ab} \quad . \tag{4.3.55}$$

which integrates to give

$$\tilde{K}(x,x';s) = -\frac{\mathrm{i}}{16\pi^2 s^2}\det\left(\frac{sF}{\sinh sF}\right)^{\frac{1}{2}} e^{\frac{\mathrm{i}}{4}(x-x')^a(F\coth sF)_{ab}(x-x')^b}C(x,x'), \tag{4.3.56}$$

where the integration 'constant' $C(x,x')$ must satisfy the boundary condition $C(x,x)=1$. The constant is determined by imposing

$$\tilde{K}(x,x';s\to+0) = \delta^{(4)}(x-x') \tag{4.3.57}$$

on the solution, where one finds $C(x,x')=\mathcal{I}(x,x')$. In more complicated examples $\mathcal{I}(x,x')$ plays a far more active role [207].

In the next chapter, when we treat Yang-Mills theories in superspace, it will not be possible to solve the differential equation exactly. Instead of attempting to solve it to some order, it turns out to be more economical to modify the approach. In doing so we will develop a general algorithm for computing the DeWitt-Seeley coefficients in the coincidence limit.

Chapter 5

One-loop effective action for $\mathcal{N} = 4$ super Yang-Mills theory

Having developed all of the necessary background material we will now proceed to compute contributions to the one-loop effective action for non-Abelian $\mathcal{N} = 4$ super Yang-Mills theories in $\mathcal{N} = 1$ superspace. This will be achieved through application of the Gaussian approach, which first needs to be modified due to the presence of a non-Abelian gauge group.

This chapter is based on the original work [34], which was primarily concerned with testing the conjectured correspondence between the effective action of $\mathcal{N} = 4$ super Yang-Mills and non-Abelian D-brane effective action from superstring theory. This correspondence was based on the notion that maximally supersymmetry might provide a sufficiently strong constraint to uniquely determine the allowed deformations of super Yang-Mills theories. See chapter 1 for greater details.

The one-loop effective action of $\mathcal{N} = 4$ super Yang-Mills was computed in [34] for the first time to order F^6 in superspace. Here we provide the full details of this calculation. The bosonic component of the F^5 terms are extracted, and the results are compared to existing literature. To facilitate this comparison a detailed treatment of the F^5 field strength structures is necessary. The results are shown to be consistent with the form of the non-Abelian D-brane effective action computed to this order by superstring methods and various other means of computing deformations of maximally supersymmetric Yang-Mills theory, leaving little doubt that there is a unique deformation at this order. Some improvements and updates have also been made since the publication of [34].

5.1 The heat kernel

Recall that we wish to compute the regulated $\mathcal{N} = 4$ super Yang-Mills one-loop effective action via

$$\Gamma^{(1)}_{\rho} = \frac{\mu^{2\rho}}{2}\int_0^\infty \mathrm{d}s\,(\mathrm{i}s)^{\rho-1}\mathrm{e}^{\mathrm{i}m^2 s}K(s). \tag{5.1.1}$$

In his expression, μ and ρ are the renormalization point and regularization parameter respectively; m is an infrared regulator; and $K(s)$ is the functional supertrace of the heat kernel

$$K(s) = \mathrm{tr}_{\mathrm{Ad}}\int \mathrm{d}^8 z \lim_{z'\to z} \mathrm{e}^{\mathrm{i}s\Box}\delta^{(8)}(z,z') \equiv \mathrm{tr}_{\mathrm{Ad}}\int \mathrm{d}^8 z \lim_{z'\to z} K(z,z';s); \tag{5.1.2}$$

and $\delta^{(8)}(z,z')$ the full superspace delta function.

$$\delta^{(8)}(z,z') = \delta^{(4)}(x,x')\delta^{(2)}(\theta-\theta')\delta^{(2)}(\bar\theta-\bar\theta'). \tag{5.1.3}$$

The operator $\Box$ is the (super) d'Alembertian

$$\Box = \mathcal{D}^a\mathcal{D}_a - W^\alpha\mathcal{D}_\alpha + \bar W_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha} \tag{5.1.4}$$

where the background gauge covariant derivatives satisfy the algebra

$$\begin{gathered}
\{\mathcal{D}_\alpha,\mathcal{D}_\beta\} = \{\bar{\mathcal{D}}_{\dot\alpha},\bar{\mathcal{D}}_{\dot\beta}\} = 0\\
\{\mathcal{D}_\alpha,\bar{\mathcal{D}}_{\dot\alpha}\} = -2\mathrm{i}\mathcal{D}_{\alpha\dot\alpha} = -2\mathrm{i}(\sigma^a)_{\alpha\dot\alpha}\mathcal{D}_a\\
[\mathcal{D}_\alpha,\mathcal{D}_{\beta\dot\beta}] = 2\mathrm{i}\varepsilon_{\alpha\beta}\bar W_{\dot\beta} \qquad \bar W_{\dot\beta} = \bar W^I_{\dot\beta}\mathcal{T}^I\\
[\bar{\mathcal{D}}_{\dot\alpha},\mathcal{D}_{\beta\dot\beta}] = 2\mathrm{i}\varepsilon_{\dot\alpha\dot\beta}W_\beta \qquad W_\beta = W^I_\beta\mathcal{T}^I\\
[\mathcal{D}_{\alpha\dot\alpha},\mathcal{D}_{\beta\dot\beta}] = (\sigma^a)_{\alpha\dot\alpha}(\sigma^b)_{\beta\dot\beta}G_{ab} = -\varepsilon_{\alpha\beta}(\bar{\mathcal{D}}_{\dot\alpha}\bar W_{\dot\beta}) - \varepsilon_{\dot\alpha\dot\beta}(\mathcal{D}_\alpha W_\beta).
\end{gathered} \tag{5.1.5}$$

Introducing a plane wave basis for the delta functions

$$\delta^{(4)}(x,x') = \int\frac{\mathrm{d}^4 k}{(2\pi)^4}\,\mathrm{e}^{\mathrm{i}k^a\omega_a} \tag{5.1.6}$$

$$\delta^{(2)}(\theta-\theta') = 4\int \mathrm{d}^2\kappa\;\mathrm{e}^{\mathrm{i}\kappa^\alpha(\theta-\theta')_\alpha}, \qquad \delta^{(2)}(\bar\theta-\bar\theta') = 4\int \mathrm{d}^2\bar\kappa\;\mathrm{e}^{\mathrm{i}\bar\kappa_{\dot\alpha}(\bar\theta-\bar\theta')^{\dot\alpha}} \tag{5.1.7}$$

where

$$\omega_a = x_a - x'_a - \mathrm{i}\theta\sigma_a\bar\theta' + i\theta'\sigma_a\bar\theta, \tag{5.1.8}$$

and defining

$$\int \mathrm{d}\eta = 16\int\frac{\mathrm{d}^4 k}{(2\pi)^4}\int\mathrm{d}^2\kappa\int\mathrm{d}^2\bar\kappa, \tag{5.1.9}$$

one finds, analogous to the example in the previous chapter, that $K(z,z';s)$ has the form

$$K(z,z';s) = \int\mathrm{d}\eta\;\mathrm{e}^{\mathrm{i}k^a\omega_a}\mathrm{e}^{\mathrm{i}\kappa^\alpha(\theta-\theta')_\alpha}\mathrm{e}^{\mathrm{i}\bar\kappa_{\dot\alpha}(\bar\theta-\bar\theta')^{\dot\alpha}}\mathrm{e}^{\mathrm{i}s\Delta} \tag{5.1.10}$$

with

$$\Delta = X^a X_a - W^\alpha X_\alpha - \bar{W}^{\dot\alpha}\bar{X}_{\dot\alpha}. \tag{5.1.11}$$

As before the X's are shifted covariant derivatives defined by

$$\begin{aligned} X_a &= \mathcal{D}_a + ik_a \\ X_\alpha &= \mathcal{D}_\alpha + i\kappa_\alpha - k_{\alpha\dot\alpha}(\bar\theta - \bar\theta')^{\dot\alpha} \\ \bar{X}_{\dot\alpha} &= \bar{\mathcal{D}}_{\dot\alpha} + i\bar\kappa_{\dot\alpha} + k_{\alpha\dot\alpha}(\theta - \theta')^\alpha, \end{aligned} \tag{5.1.12}$$

and satisfy

$$\begin{gathered} \{X_\alpha, X_\beta\} = \{\bar{X}_{\dot\alpha}, \bar{X}_{\dot\beta}\} = 0, \quad \{X_\alpha, \bar{X}_{\dot\alpha}\} = -2\mathrm{i}X_{\alpha\dot\alpha}, \\ [X_\alpha, X_{\beta\dot\beta}] = 2\mathrm{i}\varepsilon_{\alpha\beta}\bar{W}_{\dot\beta}, \qquad [\bar{X}_{\dot\alpha}, X_{\beta\dot\beta}] = 2\mathrm{i}\varepsilon_{\dot\alpha\dot\beta}W_\beta \\ [X_a, X_b] = G_{ab}. \end{gathered} \tag{5.1.13}$$

Taking the limit $z' \to z$ in (5.1.10), which we implicitly take from this point onward, one obtains

$$K(z;s) \equiv \lim_{z'\to z} K(z,z';s) = \int \mathrm{d}\eta \; \mathrm{e}^{\mathrm{i}s\Delta}. \tag{5.1.14}$$

The leading term in the asymptotic expansion of $K(z;s)$ is of order s^2, a fact which can be seen by making the re-scaling $k_a \to s^{-\frac{1}{2}}k_a$, and by observing that the integral over the fermionic parameters κ_α and $\bar\kappa_{\dot\alpha}$ will bring down at least four factors of s. Defining the DeWitt-Seeley coefficients $a_n(z)$ in the usual manner,

$$K(z;s) = \frac{\mathrm{i}}{(4\pi\mathrm{i}s)^2}\sum_{n=0}^{\infty}(\mathrm{i}s)^n a_n(z) \qquad a_i(z) = 0, \quad i = 0,1,2,3, \tag{5.1.15}$$

then the one-loop effective action then takes the form

$$\Gamma^{(1)}_{\mathcal{N}=4} = \frac{1}{32\pi^2}\sum_{n=4}^{\infty}\frac{(n-3)!}{m^{2n-4}}\int \mathrm{d}^8z \, \mathrm{tr}_{\mathrm{Ad}}(a_n), \tag{5.1.16}$$

which is an expansion in inverse powers of the infrared regulator m, and is free of ultra-violet divergences. Here the DeWitt-Seeley coefficients are real but otherwise unconstrained superfields. At the component level the non-trivial coefficients, a_n for $n \geq 4$, contain bosonic field strength terms of the form F^n, and we see that the mass m plays a similar role to α' in string theory in that it effectively keeps track of such terms.

In the present case computing the one-loop effective action to all orders is not possible, which will be illustrated in the next section. Our goal therefore is to extend the known results by computing a_5 and a_6. The first non-trivial coefficient a_4 is well-known (see for example [89, 90]), and at the component level its bosonic term corresponds exactly with the associated term in the non-Abelian Born Infeld action [66]. This coefficient can easily be obtained by simply expanding the exponential $e^{is\Delta}$ in (5.1.14). After re-scaling k^a one finds

$$K(z;s) = \frac{s^2}{4!}\int d\eta\ e^{-ik^2}(W\kappa - \bar{W}\bar{\kappa})^4 + \mathcal{O}(s^3) \tag{5.1.17}$$

and so

$$\mathrm{tr}_{\mathrm{Ad}}(a_4) = \frac{1}{3}\,\mathrm{tr}_{\mathrm{Ad}}(2\,W^2\bar{W}^2 - W^\alpha \bar{W}_{\dot{\alpha}} W_\alpha \bar{W}^{\dot{\alpha}}). \tag{5.1.18}$$

It is easy to see that computing the DeWitt-Seeley coefficients by the direct process of expanding the exponential quickly becomes very laborious and cumbersome. It is appears to be more efficient than the recursive DeWitt method when computing the first non-trivial coefficient, but is not particularly tidy or systematic for computing higher coefficients. In the next section we will make some modifications to the Gaussian approach, in that we will not actually attempt to directly solve the differential equation satisfied by the heat kernel. A more systematic method which amounts to expanding the exponential and collecting terms into commutators is developed.

5.2 The modified Gaussian approach

As before we begin by differentiating $K(z;s)$ with respect to s, generating the differential equation

$$\frac{dK(z;s)}{ds} = iK^a{}_a(z;s) - iW^\alpha K_\alpha(z;s) - i\bar{W}^{\dot{\alpha}} K_{\dot{\alpha}}(z;s), \tag{5.2.19}$$

where again the notation

$$K_{A_1 A_2 \dots A_n}(z,t) = \int d\eta\ X_{A_1} X_{A_2} \dots X_{A_n} e^{is\Delta} \tag{5.2.20}$$

has been adopted, and the integration measure is defined in (5.1.9). Using

$$0 = \int d\eta\ \frac{\partial}{\partial k_b}\left(X_a e^{is\Delta}\right) \tag{5.2.21}$$

and

$$[A, e^B] = \int_0^1 d\xi\ e^{\xi B}[A,B]e^{(1-\xi)B}, \tag{5.2.22}$$

it follows that

$$0 = i\delta_a{}^b K(z;s) - 2s\int d\eta\ X_a \sum_{n=0}^{\infty} \frac{(is)^n}{(n+1)!} L^n_\Delta(J^b)\ e^{is\Delta} \tag{5.2.23}$$

where

$$J^a = X^a - \frac{i}{2}W\sigma^a(\bar{\theta} - \bar{\theta}') - \frac{i}{2}(\theta - \theta')\sigma^a \bar{W}. \tag{5.2.24}$$

After contracting the spacetime indices this can be written as

$$K_a{}^a(z;s) = \frac{2\mathrm{i}}{s}K(z,t) - \int \mathrm{d}\eta\; X_a \sum_{n=1}^{\infty} \frac{(\mathrm{i}s)^n}{(n+1)!} L_\Delta^n(J^a)\; \mathrm{e}^{\mathrm{i}s\Delta}. \tag{5.2.25}$$

Similarly,

$$0 = \int \mathrm{d}\eta \frac{\partial}{\partial \kappa_\beta}\left(X_\alpha \mathrm{e}^{\mathrm{i}s\Delta}\right) \tag{5.2.26}$$

and

$$0 = \int \mathrm{d}\eta \frac{\partial}{\partial \bar{\kappa}_{\dot\beta}}\left(\bar{X}_{\dot\alpha} \mathrm{e}^{\mathrm{i}s\Delta}\right) \tag{5.2.27}$$

lead to

$$W^\alpha K_\alpha(z;s) = \frac{2\mathrm{i}}{s}K(z;s) - (\mathcal{D}^\alpha W_\alpha)K(z;s) + \int \mathrm{d}\eta\; X_\alpha \sum_{n=1}^{\infty} \frac{(\mathrm{i}s)^n}{(n+1)!} L_\Delta^n(W^\alpha)\; \mathrm{e}^{\mathrm{i}s\Delta} \tag{5.2.28}$$

and

$$\bar{W}^{\dot\alpha} K_{\dot\alpha}(z;s) = \frac{2\mathrm{i}}{s}K(z;s) + (\bar{\mathcal{D}}_{\dot\alpha} \bar{W}^{\dot\alpha})K(z;s) + \int \mathrm{d}\eta\; \bar{X}_{\dot\alpha} \sum_{n=1}^{\infty} \frac{(\mathrm{i}s)^n}{(n+1)!} L_\Delta^n(\bar{W}^{\dot\alpha})\; \mathrm{e}^{\mathrm{i}s\Delta} \tag{5.2.29}$$

respectively.

Inserting the expression (5.2.25), (5.2.28) and (5.2.29) into the differential equation (5.2.19), one obtains:

$$\begin{aligned}
\frac{\mathrm{d}K(z;s)}{\mathrm{d}s} - \frac{2}{s}K(z;s) = &-\mathrm{i}\int \mathrm{d}\eta\; X_a \sum_{n=1}^{\infty} \frac{(\mathrm{i}s)^n}{(n+1)!} L_\Delta^n(J^a)\; \mathrm{e}^{\mathrm{i}s\Delta} \\
&-\mathrm{i}\int \mathrm{d}\eta\; X_\alpha \sum_{n=1}^{\infty} \frac{(\mathrm{i}s)^n}{(n+1)!} L_\Delta^n(W^\alpha)\; \mathrm{e}^{\mathrm{i}s\Delta} \\
&-\mathrm{i}\int \mathrm{d}\eta\; \bar{X}_{\dot\alpha} \sum_{n=1}^{\infty} \frac{(\mathrm{i}s)^n}{(n+1)!} L_\Delta^n(\bar{W}^{\dot\alpha})\; \mathrm{e}^{\mathrm{i}s\Delta},
\end{aligned} \tag{5.2.30}$$

where the $K(z;s)$ pieces have been brought to the left hand side and the Bianchi identity, $\mathcal{D}^\alpha W_\alpha = \bar{\mathcal{D}}_{\dot\alpha}\bar{W}^{\dot\alpha}$, has been used.

The significance of this expression is seen in terms of the asymptotic expansion (5.1.15), which we first rewrite as

$$K(z;s) = \frac{\mathrm{i}}{(4\pi)^2}\sum_{n=4}^{\infty}(\mathrm{i}s)^{n-2}a_n. \tag{5.2.31}$$

The left hand side of (5.2.30) is then

$$\frac{\mathrm{d}K(z;s)}{\mathrm{d}s} - \frac{2}{s}K(z;s) = -\frac{1}{16\pi^2}\sum_{n=4}^{\infty}(n-4)(\mathrm{i}s)^{n-3}a_n = \frac{s^2 a_5}{16\pi^2} + \frac{2\mathrm{i}s^3 a_6}{16\pi^2} + \cdots \tag{5.2.32}$$

It is now clear that in this particular combination of the kernel and its derivative the first non-trivial coefficient a_4 is absent[1]. Exploiting this fact, the objective now becomes to determine the DeWitt-Seeley coefficients by expanding the right hand side of (5.2.30) in a power series in s, and identifying it with the right hand side of (5.2.32). In contrast to the Gaussian approach as applied earlier, we make no attempt to solve the differential equation 5.2.30) by expressing the right hand side in terms of $K(z;s)$. This turns out to be the most efficient way to proceed, and as will been seen shortly, it is far from obvious whether, to some given order in s, the right hand side of (5.2.30) is even expressible in terms of $K(z;s)$.

The background has been arbitrary to this point. However from now on it will be placed on-shell, $\mathcal{D}^\alpha W_\alpha = \bar{\mathcal{D}}_{\dot\alpha}\bar{W}^{\dot\alpha} = 0$ (see (2.2.96) in the absence of scalar background), since this is sufficient for the purpose of making comparisons with the literature and computing the S-matrix.

Since the summation on the right hand side of (5.2.30) involves the repetitive calculation of commutators, it is first useful to establish the following relations:

$$
\begin{aligned}
[\Delta, X_a] &= 2G^b{}_a X_b + (\mathcal{D}_a W^\alpha) X_\alpha + (\mathcal{D}_a \bar{W}^{\dot\alpha}) \bar{X}_{\dot\alpha} \\
[\Delta, X_\alpha] &= (\mathcal{D}_\alpha W^\beta) X_\beta \\
[\Delta, \bar{X}_{\dot\alpha}] &= (\bar{\mathcal{D}}_{\dot\alpha} \bar{W}^{\dot\beta}) \bar{X}_{\dot\beta} \\
[\Delta, Y] &= (\mathcal{D}^a \mathcal{D}_a Y) + 2(\mathcal{D}^a Y) X_a - W^\alpha (\mathcal{D}_\alpha Y) - \bar{W}^{\dot\alpha} (\bar{\mathcal{D}}_{\dot\alpha} Y) \\
&\quad -(-1)^{\varepsilon(Y)} \left[W^\alpha, Y\right\} X_\alpha - (-1)^{\varepsilon(Y)} \left[\bar{W}^{\dot\alpha}, Y\right\} \bar{X}_{\dot\alpha}.
\end{aligned}
\tag{5.2.33}
$$

These are not entirely trivial to establish and are significantly simplified by working on-shell. For example, in the process one is required to establish that on-shell $(\mathcal{D}^a G_{ab}) = \mathrm{i}(\sigma_b)_{\alpha\dot\alpha}\{W^\alpha, \bar{W}^{\dot\alpha}\}$. See appendix B for full details.

From the commutation relations (5.2.33) it is clear that the summations on the right hand side of (5.2.30) will generate a series of moments of the form $K_{A_1\ldots A_i}(z;s)$ as defined in (5.2.20). Furthermore, it is not difficult to show that to order n in this summation, the moments generated have at most $(n+1)$ indices. It is convenient to always place these indices in a specific order: first undotted, then dotted, then spacetime. This can be achieved through the commutation relations (5.1.13). With such an ordering, the leading term in a moment's asymptotic power series has the

[1] This feature is not particular to the current example. Differential equations of the form (5.2.30), where the first non-trivial coefficient is absent, arise naturally when applying these techniques to heat kernels associated with 'reasonable' operators. This is most obvious in ordinary D dimensional spacetime with Laplace-type operators.

following behaviour[2]

$$K_{A_1 \dots A_{p+q}}(z;s) \sim \frac{1}{s^2}\left(\frac{1}{s}\right)^{\left[\frac{p}{2}\right]} s^{4-q} = s^{2-q-\left[\frac{p}{2}\right]} \qquad q \leq 4 \tag{5.2.34}$$

where $K_{A_1 \dots A_{p+q}}(z;s)$ has p spacetime indices, q spinor indices and $[\frac{p}{2}]$ denotes the largest integer part of $\frac{p}{2}$. Moments with more than two undotted or dotted indices vanish as $X_\alpha X_\beta X_\gamma = \bar{X}_{\dot{\alpha}} \bar{X}_{\dot{\beta}} \bar{X}_{\dot{\gamma}} = 0$.

From these considerations it is now quite clear that the summations in the differential equation (5.2.30) cannot be done explicitly, since they all quickly lead to a proliferation of more and more complicated terms, and moments with ever increasing numbers of indices. Expressing each of these moments in terms of $K(z;s)$ through identities such as (5.2.21), (5.2.26) and (5.2.27) would be a formidable task, and it is much easier to expand each of these moments into a power series.

Given the above power series arguments, and from comparison with equation (5.2.32), the summation in equation (5.2.30) can be seen to truncate at $n = 2k - 5$ when evaluating a_k for $k \geq 5$. Moreover, it turns out that after tracing over gauge indices the last term in this truncated summation always vanishes due to the cyclic property of this trace, making it necessary to sum only to $n = 2k - 6$. In particular this means that to evaluate a_5 and a_6 one is permitted to truncate at $n = 4$ and 6 respectively. To show this, we note that in evaluating a_k, the terms in the summation coming from $n = 2k - 5$ are always of the form

$$s^{2k-5} T^{\alpha\beta\dot{\alpha}\dot{\beta} b_1 \dots b_{2k-8}} K_{\alpha\beta\dot{\alpha}\dot{\beta} b_1 \dots b_{2k-8}}(z;s) \qquad k \geq 5 \tag{5.2.35}$$

to the order of interest. The coefficient T in this expression is in general a graded commutator, and the moment is only ever required to leading order. To leading order this moment is proportional to the identity matrix in its group indices, and consequently we are left with terms which consists solely of a graded commutator. In performing the trace over the gauge indices such terms vanish.

[2]To clarify the terminology adopted here and in what follows: the expressions 'leading term' or 'leading order' refer to the first (expected) non-trivial term in the asymptotic expansion of the moment, ie $K(z;s)$ has a leading term of order s^2. Analogously 'subleading order' refers to the second (expected) non-trivial term and so on.

5.3 The F^5 terms

5.3.1 Evaluating a_5

Before proceeding to compute a_5 and a_6 it is instructive to examine the differential equation (5.2.30) in a little more detail. Since

$$L^n_A(BC) = \sum_{m=0}^{n} \frac{n!}{m!(n-m)!}\, L^{n-m}_A(B)\, L^m_A(C), \tag{5.3.36}$$

if A has even Grassmann parity, then

$$\begin{aligned} L^n_\Delta(J^a - X^a) &= \frac{\mathrm{i}}{2}(\sigma^a)_{\alpha\dot\alpha} L^n_\Delta((\bar\theta - \bar\theta')^{\dot\alpha} W^\alpha - (\theta-\theta')^\alpha \bar W^{\dot\alpha}) \\ &= \frac{\mathrm{i}}{2}(\sigma^a)_{\alpha\dot\alpha}\big((\bar\theta - \bar\theta')^{\dot\alpha} ad^n_\Delta(W^\alpha) - (\theta-\theta')^\alpha L^n_\Delta(\bar W^{\dot\alpha})\big) + M^a_n \end{aligned} \tag{5.3.37}$$

where

$$M^a_n = \frac{\mathrm{i}}{2}(\sigma^a)_{\alpha\dot\alpha} \sum_{m=0}^{n-1} \frac{n!}{m!(n-m)!}\big(L^{n-m-1}_\Delta(\bar W^{\dot\alpha})\, L^m_\Delta(W^\alpha) + L^{n-m-1}_\Delta(W^\alpha)\, L^m_\Delta(\bar W^{\dot\alpha})\big).$$

The differential equation (5.2.30) can then be expressed in the form

$$\begin{aligned} \frac{\mathrm{d}K(z;s)}{\mathrm{d}s} - \frac{2}{s}K(z;s) = &- \mathrm{i}\int \mathrm{d}\eta\; X_a \sum_{n=1}^{\infty} \frac{(\mathrm{i}s)^n}{(n+1)!} L^n_\Delta(X^a)\; \mathrm{e}^{\mathrm{i}s\Delta} \\ &- \mathrm{i}\int \mathrm{d}\eta\; X_\alpha \sum_{n=1}^{\infty} \frac{(\mathrm{i}s)^n}{(n+1)!} L^n_\Delta(W^\alpha)\; \mathrm{e}^{\mathrm{i}s\Delta} \\ &- \mathrm{i}\int \mathrm{d}\eta\; \bar X_{\dot\alpha} \sum_{n=1}^{\infty} \frac{(\mathrm{i}s)^n}{(n+1)!} L^n_\Delta(\bar W^{\dot\alpha})\; \mathrm{e}^{\mathrm{i}s\Delta} \\ &- \mathrm{i}\int \mathrm{d}\eta\; X_a \sum_{n=1}^{\infty} \frac{(\mathrm{i}s)^n}{(n+1)!} M^a_n\; \mathrm{e}^{\mathrm{i}s\Delta}. \end{aligned} \tag{5.3.38}$$

This will simplify the computation for two reasons. Firstly, to order n in the summation, the last of the four terms on the right hand side will generate moments with at most n indices (whereas the first three generate moments with at most $n+1$). By investigating its powers series behaviour, one ultimately finds that this last term will not contribute when computing a_5. Secondly, the form (5.3.38) allows us to take advantage of the fact, which was merely noted and not exploited fully in [34], that there is significant cancellation between several terms generated by the summation in the first three of the four terms on the right hand side of (5.3.38). To see why this is so, and to remove the need for calculating such terms, we employ the identity (5.3.36) as

$$BL^n_A(C) = L^n_A(BC) - \sum_{m=1}^{n} \frac{n!}{m!(n-m)!}\, L^m_A(B)\, L^{n-m}_A(C). \tag{5.3.39}$$

This yields

$$X_a L^n_\Delta(X^a) + X_\alpha L^n_\Delta(W^\alpha) + \bar{X}_{\dot\alpha} L^n_\Delta(\bar{W}^{\dot\alpha}) =$$

$$L^n_\Delta(X^a X_a) - L^n_\Delta(W^\alpha X_\alpha) - L^n_\Delta(\bar{W}^{\dot\alpha}\bar{X}_{\dot\alpha}) - \sum_{m=1}^{n} \frac{n!}{m!(n-m)!} \Big\{ L^m_\Delta(X_a)\, L^{n-m}_\Delta(X^a)$$
$$+ L^m_\Delta(X_\alpha)\, L^{n-m}_\Delta(W^\alpha) + L^m_\Delta(\bar{X}_{\dot\alpha})\, L^{n-m}_\Delta(\bar{W}^{\dot\alpha}) \Big\} \quad (5.3.40)$$

having used the equations of motion. The first three terms on the second line then give $L^n_\Delta(\Delta) = 0$, and so ultimately (5.3.38) reduces to

$$\frac{\mathrm{d}K(z;s)}{\mathrm{d}s} - \frac{2}{s}K(z;s) = -\mathrm{i}\int\! \mathrm{d}\eta\ X_a \sum_{n=1}^{\infty} \frac{(\mathrm{i}s)^n}{(n+1)!} M^a_n\, \mathrm{e}^{\mathrm{i}s\Delta}$$
$$+\mathrm{i}\int\! \mathrm{d}\eta\ \sum_{n=1}^{\infty}\sum_{m=1}^{n} \frac{(\mathrm{i}s)^n}{m!(n-m)!(n+1)} \Big\{ L^m_\Delta(X_a)\, L^{n-m}_\Delta(X^a)$$
$$+ L^m_\Delta(X_\alpha)\, L^{n-m}_\Delta(W^\alpha) + L^m_\Delta(\bar{X}_{\dot\alpha})\, L^{n-m}_\Delta(\bar{W}^{\dot\alpha}) \Big\}\, \mathrm{e}^{\mathrm{i}s\Delta}. \quad (5.3.41)$$

Determining the coefficient a_5 now only involves computing

$$\mathrm{i}\int\! \mathrm{d}\eta\ \sum_{n=1}^{4}\sum_{m=1}^{n} \frac{(\mathrm{i}s)^n}{m!(n-m)!(n+1)} \Big\{ L^m_\Delta(X_a)\, L^{n-m}_\Delta(X^a) +$$
$$L^m_\Delta(X_\alpha)\, L^{n-m}_\Delta(W^\alpha) + L^m_\Delta(\bar{X}_{\dot\alpha})\, L^{n-m}_\Delta(\bar{W}^{\dot\alpha}) \Big\}\, \mathrm{e}^{\mathrm{i}s\Delta}. \quad (5.3.42)$$

To illustrate how one proceeds in this modified approach we will consider the $n = 1$ contribution. Using the commutation relations (5.2.33), the $n = 1$ term gives

$$-\frac{s}{2!}\int\! \mathrm{d}\eta \big([\Delta, X_a]X^a + [\Delta, X_\alpha]W^\alpha + [\Delta, \bar{X}_{\dot\alpha}]\bar{W}^{\dot\alpha}\big)\, \mathrm{e}^{\mathrm{i}s\Delta}$$
$$= -\frac{s}{2}\Big(G^{ab}K_{ab}(z;s) + (\mathcal{D}^a W^\alpha)K_{\alpha a}(z;s) + (\mathcal{D}^a \bar{W}^{\dot\alpha})K_{\dot\alpha a}(z;s)$$
$$- (\mathcal{D}_\alpha W^\beta)W^\alpha K_\beta(z;s) - (\bar{\mathcal{D}}_{\dot\alpha}\bar{W}^{\dot\beta})\bar{W}^{\dot\alpha}K_{\dot\beta}(z;s)$$
$$+ \Big((\mathcal{D}_\alpha W^\beta)(\mathcal{D}_\beta W^\alpha) + (\bar{\mathcal{D}}_{\dot\alpha}\bar{W}^{\dot\beta})(\bar{\mathcal{D}}_{\dot\beta}\bar{W}^{\dot\alpha})\Big)K(z;s)\Big). \quad (5.3.43)$$

This expression is further simplified by noting that

$$2G^{ab}K_{ab}(z;s) = G^{ab}G_{ab}K(z;s) \quad (5.3.44)$$

and that terms of the form $s\,K(z,t) \sim \mathcal{O}(s^3)$ do not contribute to the order of interest.

There now only remains the problem of expanding the contributing moments to the required order in s. In the present situation, computing a_5, this does not generate any additional difficulties since it is only necessary to expand all surviving moments to leading order in their power series expansion in powers of s. As described earlier such expressions are readily obtained by directly expanding the exponential. For instance:

$$K_\alpha(z;s) = \int \mathrm{d}\eta\, X_\alpha \mathrm{e}^{\mathrm{i}s\Delta} = \frac{s}{24\pi^2}(W_\alpha \bar{W}^2 + \bar{W}^2 W_\alpha - \bar{W}_{\dot\alpha} W_\alpha \bar{W}^{\dot\alpha}) + \mathcal{O}(s^2). \quad (5.3.45)$$

Later a more systematic approach will be described.

Carrying out this procedure for the entire right hand side of equation (5.3.38) for $n = 1$ to 4, $\mathrm{tr}_{\mathrm{Ad}}(a_5)$ can be identified. A complete list of the required moments computed to leading order is given below (also see[3] appendix C). The numerical factor common to all moments is $H = \mathrm{i}(4\pi\mathrm{i})^{-2}$.

$$K_{\alpha\beta\dot\alpha\dot\beta ab}(z;s) = -\frac{2\mathrm{i}}{s^3}\,\varepsilon_{\alpha\beta}\,\varepsilon_{\dot\alpha\dot\beta}\,\eta_{ab}\,\mathbb{1}\,H \quad (5.3.46)$$

$$K_{\alpha\beta\dot\alpha\dot\beta}(z;s) = -\frac{4}{s^2}\,\varepsilon_{\alpha\beta}\,\varepsilon_{\dot\alpha\dot\beta}\,\mathbb{1}\,H \quad (5.3.47)$$

$$K_{\alpha\beta\dot\alpha ab}(z;s) = \frac{2}{s^2}\,\varepsilon_{\alpha\beta}\,\eta_{ab}\,\bar{W}_{\dot\alpha}\,H \quad (5.3.48)$$

$$K_{\alpha\dot\alpha\dot\beta ab}(z;s) = \frac{2}{s^2}\,\varepsilon_{\dot\alpha\dot\beta}\,\eta_{ab}\,W_\alpha\,H \quad (5.3.49)$$

$$K_{\alpha\beta\dot\alpha\dot\beta a}(z;s) = 0 \quad (5.3.50)$$

$$K_{\alpha\beta\dot\alpha}(z;s) = -\frac{4\mathrm{i}}{s}\,\varepsilon_{\alpha\beta}\,\bar{W}_{\dot\alpha}\,H \quad (5.3.51)$$

$$K_{\alpha\dot\alpha\dot\beta}(z;s) = -\frac{4\mathrm{i}}{s}\,\varepsilon_{\dot\alpha\dot\beta}\,W_\alpha\,H \quad (5.3.52)$$

$$K_{\alpha\beta ab}(z;s) = \frac{\mathrm{i}}{s}\,\varepsilon_{\alpha\beta}\,\eta_{ab}\,\bar{W}^2\,H \quad (5.3.53)$$

$$K_{\alpha\dot\alpha ab}(z;s) = \frac{\mathrm{i}}{s}\eta_{ab}\big(W_\alpha \bar{W}_{\dot\alpha} - \bar{W}_{\dot\alpha} W_\alpha\big)H \quad (5.3.54)$$

$$K_{\alpha\beta ab}(z;s) = -\frac{\mathrm{i}}{s}\,\varepsilon_{\alpha\beta}\,\eta_{ab}\,W^2\,H \quad (5.3.55)$$

$$K_{\alpha\beta\dot\alpha a}(z;s) = -\frac{2\mathrm{i}}{s}\,\varepsilon_{\alpha\beta}\,(\mathcal{D}_a \bar{W}_{\dot\alpha})\,H \quad (5.3.56)$$

$$K_{\alpha\dot\alpha\dot\beta a}(z;s) = -\frac{2\mathrm{i}}{s}\,\varepsilon_{\dot\alpha\dot\beta}\,(\mathcal{D}_a W_\alpha)\,H \quad (5.3.57)$$

$$K_{\alpha\beta}(z;s) = 2\varepsilon_{\alpha\beta}\bar{W}^2\,H \quad (5.3.58)$$

$$K_{\alpha\dot\alpha}(z;s) = 2\big(W_\alpha \bar{W}_{\dot\alpha} - \bar{W}_{\dot\alpha} W_\alpha\big)H \quad (5.3.59)$$

$$K_{\dot\alpha\dot\beta}(z;s) = -2\varepsilon_{\dot\alpha\dot\beta}W^2\,H \quad (5.3.60)$$

[3] In particular see subsection C.2.2. Note that this appendix involves knowledge of *Mathematica* syntax. The usage of *Mathematica* will be described later.

$$K_{\alpha ab}(z;s) = -\frac{1}{3}\big(W_\alpha \bar{W}^2 + \bar{W}^2 W_\alpha - \bar{W}_{\dot\alpha} W_\alpha \bar{W}^{\dot\alpha}\big)\eta_{ab}\, H \tag{5.3.61}$$

$$K_{\dot\alpha ab}(z;s) = \frac{1}{3}\big(W^2 \bar{W}_{\dot\alpha} + \bar{W}_{\dot\alpha} W^2 - W^\alpha \bar{W}^{\dot\alpha} W_\alpha\big)\eta_{ab}\, H \tag{5.3.62}$$

$$K_{\alpha\beta a}(z;s) = \frac{2}{3}\,\varepsilon_{\alpha\beta}\big(\bar{W}_{\dot\alpha}(\mathcal{D}_a \bar{W}^{\dot\alpha}) + 2(\mathcal{D}_a \bar{W}_{\dot\alpha})\bar{W}^{\dot\alpha}\big) H \tag{5.3.63}$$

$$\begin{aligned} K_{\alpha\dot\alpha a}(z;s) = \frac{2}{3}\big(&W_\alpha(\mathcal{D}_a \bar{W}_{\dot\alpha}) - \bar{W}_{\dot\alpha}(\mathcal{D}_a W_\alpha) \\ &+ 2(\mathcal{D}_a W_\alpha)\bar{W}_{\dot\alpha} - 2(\mathcal{D}_a \bar{W}_{\dot\alpha})W_\alpha\big) H \end{aligned} \tag{5.3.64}$$

$$K_{\dot\alpha\dot\beta a}(z;s) = -\frac{2}{3}\,\varepsilon_{\dot\alpha\dot\beta}\big(W^\alpha(\mathcal{D}_a W_\alpha) + 2(\mathcal{D}_a W^\alpha)W_\alpha\big) H \tag{5.3.65}$$

$$K_\alpha(z;s) = -\frac{2\mathrm{i}s}{3}\big(\bar{W}_{\dot\alpha} W_\alpha \bar{W}^{\dot\alpha} - \bar{W}^2 W_\alpha - W_\alpha \bar{W}^2\big) H \tag{5.3.66}$$

$$K_{\dot\alpha}(z;s) = \frac{2\mathrm{i}s}{3}\big(W^\alpha \bar{W}_{\dot\alpha} W_\alpha - W^2 \bar{W}_{\dot\alpha} - \bar{W}_{\dot\alpha} W^2\big) H \tag{5.3.67}$$

$$\begin{aligned} K_{\dot\alpha a}(z;s) = \frac{\mathrm{i}s}{6}\big(&W^\alpha \bar{W}_{\dot\alpha}(\mathcal{D}_a W_\alpha) - W^2(\mathcal{D}_a \bar{W}_{\dot\alpha}) - \bar{W}_{\dot\alpha} W^\alpha(\mathcal{D}_a W_\alpha) \\ &+ 2W^\alpha(\mathcal{D}_a \bar{W}_{\dot\alpha})W_\alpha - 2W^\alpha(\mathcal{D}_a W_\alpha)\bar{W}_{\dot\alpha} - 2\bar{W}_{\dot\alpha}(\mathcal{D}_a W^\alpha)W_\alpha \\ &+ 3(\mathcal{D}_a W^\alpha)\bar{W}_{\dot\alpha} W_\alpha - 3(\mathcal{D}_a W^\alpha)W_\alpha \bar{W}_{\dot\alpha} - 3(\mathcal{D}_a \bar{W}_{\dot\alpha})W^2\big) H \end{aligned} \tag{5.3.68}$$

$$\begin{aligned} K_{\alpha a}(z;s) = -\frac{\mathrm{i}s}{6}\big(&\bar{W}_{\dot\alpha} W_\alpha(\mathcal{D}_a \bar{W}^{\dot\alpha}) - \bar{W}^2(\mathcal{D}_a W_\alpha) - W_\alpha \bar{W}_{\dot\alpha}(\mathcal{D}_a \bar{W}^{\dot\alpha}) \\ &+ 2\bar{W}_{\dot\alpha}(\mathcal{D}_a W_\alpha)\bar{W}^{\dot\alpha} - 2\bar{W}_{\dot\alpha}(\mathcal{D}_a \bar{W}^{\dot\alpha})W_\alpha - 2W_\alpha(\mathcal{D}_a \bar{W}_{\dot\alpha})\bar{W}^{\dot\alpha} \\ &+ 3(\mathcal{D}_a \bar{W}_{\dot\alpha})W_\alpha \bar{W}^{\dot\alpha} - 3(\mathcal{D}_a \bar{W}_{\dot\alpha})\bar{W}^{\dot\alpha} W_\alpha - 3(\mathcal{D}_a W_\alpha)\bar{W}^2\big) H \end{aligned} \tag{5.3.69}$$

To summarize, we see that the problem of computing the original kernel to subleading order has been reduced to computing several moments to leading order.

The final result, $\mathrm{tr}_{\mathrm{Ad}}(a_5)$, is dramatically simplified by recognizing that all terms may be reduced to a linear combination of only two types of structures, which serve as a basis of tensor structures. This fact can be established through dimensional analysis and use of the algebra (5.1.5), integration by parts, the cyclic property of the trace, and by applying various on-shell identities such as[4]

$$\begin{gathered} \mathcal{D}_\alpha W_\beta = \mathcal{D}_\beta W_\alpha, \qquad \bar{\mathcal{D}}_{\dot\alpha} \bar{W}_{\dot\beta} = \bar{\mathcal{D}}_{\dot\beta} \bar{W}_{\dot\alpha}, \qquad \mathcal{D}_\alpha \mathcal{D}_\beta W_\gamma = \bar{\mathcal{D}}_{\dot\alpha} \bar{\mathcal{D}}_{\dot\beta} \bar{W}_{\dot\gamma} = 0, \\ \mathcal{D}_\alpha \mathcal{D}_\beta \bar{\mathcal{D}}_{\dot\alpha} \bar{W}_{\dot\beta} = 4\varepsilon_{\alpha\beta}\{\bar{W}_{\dot\alpha}, \bar{W}_{\dot\beta}\}, \qquad \mathcal{D}_\alpha \bar{\mathcal{D}}_{\dot\alpha} \mathcal{D}_\beta W^\alpha = -4\{\bar{W}_{\dot\alpha}, W_\beta\}, \\ (\mathcal{D}^a G_{ab}) = \mathrm{i}(\sigma_b)_{\alpha\dot\alpha}\{W^\alpha, \bar{W}^{\dot\alpha}\}, \qquad (\mathcal{D}^a \mathcal{D}_a W^\alpha) = [W^\beta, \mathcal{D}_\beta W^\alpha]. \end{gathered} \tag{5.3.70}$$

The latter is easily established by using (3.2.104). The basis of tensor structures which most naturally presents itself in this superfield approach consists of: terms with two contracted spacetime covariant derivatives acting in various ways on two chiral and two antichiral superfield strengths (for example $(\mathcal{D}^a W^\alpha)(\mathcal{D}_a W_\alpha)\bar{W}^2$);

[4] See appendix B for further details regarding these identities.

and a single spinor covariant derivative with five superfield strengths (for example $(\mathcal{D}_\alpha W^\beta)W^\alpha W_\beta \bar{W}^2$).

We choose to bring terms of the first type into a form where the two contracted spacetime covariant derivatives act separately on the first two superfield strengths[5], and terms of the second type into a form where the spinor covariant derivatives acts on the first superfield strength. The final result is:

$$\begin{aligned}\mathrm{tr}_{\mathrm{Ad}}(a_5) = \frac{1}{30}\mathrm{tr}_{\mathrm{Ad}}\big(&(\mathcal{D}^a W^\alpha)(\mathcal{D}_a W_\alpha)\bar{W}^2 + (\mathcal{D}^a W^\alpha)(\mathcal{D}_a \bar{W}_{\dot{\alpha}})\bar{W}^{\dot{\alpha}} W_\alpha \\ &- (\mathcal{D}^a W^\alpha)(\mathcal{D}_a \bar{W}_{\dot{\alpha}})W_\alpha \bar{W}^{\dot{\alpha}} - 3(\mathcal{D}_\alpha W^\beta)W^\alpha W_\beta \bar{W}^2 \\ &- (\mathcal{D}_\alpha W^\beta)W^\alpha \bar{W}_{\dot{\alpha}} W_\beta \bar{W}^{\dot{\alpha}}\big) + c.c., \end{aligned} \quad (5.3.71)$$

where the complex conjugate of any term is effectively obtained by replacing all undotted spinor indices (and unbarred objects) by dotted spinor indices (and barred objects) and vice-versa. The corresponding piece of the one-loop effective action can immediately be deduced by insertion into equation (5.1.16).

The above computation may readily be carried out by hand. However, at higher orders it is useful to use *Mathematica* to carry out much of the symbolic manipulation. See appendix C for the details of this process.

5.3.2 a_5 at the component level

We are now in a position to extract the component form of $\mathrm{tr}_{\mathrm{Ad}}(a_5)$. For the purposes of comparison with existing literature, it is only necessary to consider the contribution containing the field strength F_{ab} and its covariant derivatives. Extraction of this component is most easily achieved by setting all[6] but the vector component to zero in the vector multiplet's component expansion,

$$V(z) = \theta\sigma^a\bar{\theta}V_a, \quad (5.3.72)$$

which leads to

$$\begin{aligned} W_\alpha &= \frac{\mathrm{i}}{2}(\sigma^a)_{\alpha\dot{\alpha}}(\tilde{\sigma}^b)^{\dot{\alpha}\beta}\theta_\beta F_{ab} + \mathcal{O}(\theta^2\bar{\theta}) \\ \mathcal{D}_c W_\alpha &= \frac{\mathrm{i}}{2}(\sigma^a)_{\alpha\dot{\alpha}}(\tilde{\sigma}^b)^{\dot{\alpha}\beta}\theta_\beta(\nabla_c F_{ab}) + \mathcal{O}(\theta^2\bar{\theta}) \\ \mathcal{D}_\alpha W_\beta &= -\frac{\mathrm{i}}{2}(\sigma^a)_{\beta\dot{\alpha}}(\tilde{\sigma}^b)^{\dot{\alpha}}{}_\alpha F_{ab} + \mathcal{O}(\theta\bar{\theta}) \end{aligned} \quad (5.3.73)$$

[5]Terms of the first type where the two contracted spacetime covariant derivatives act on the same superfield strength can be expressed as a linear combinations of terms of the second type by using the last of the on-shell relations in (5.3.70).

[6]The auxiliary field vanishes on shell.

or equivalently

$$W_\alpha\Big| = 0 \qquad\qquad \mathcal{D}_\alpha W_\beta\Big| = -\frac{\mathrm{i}}{2}(\sigma^a)_{\beta\dot\alpha}(\tilde\sigma^b)^{\dot\alpha}{}_\alpha F_{ab} \tag{5.3.74}$$

$$\mathcal{D}_a W_\alpha\Big| = 0 \qquad\qquad \mathcal{D}_\beta\mathcal{D}_c W_\alpha\Big| = -\frac{\mathrm{i}}{2}(\sigma^a)_{\beta\dot\alpha}(\tilde\sigma^b)^{\dot\alpha}{}_\alpha(\nabla_c F_{ab}) \tag{5.3.75}$$

and similar expressions for antichiral superfield strengths. Higher order terms in these expansions will not contribute given the form of the result (5.3.71). As before, here

$$\nabla_a = \partial_a - \mathrm{i}V_a, \qquad\qquad [\nabla_a, \nabla_b] = -\mathrm{i}F_{ab}$$
$$F_{ab} = \partial_a V_b - \partial_b V_a - \mathrm{i}[V_a, V_b]. \tag{5.3.76}$$

Simplifying the component result is rather straightforward but very time consuming[7]. In doing so it is necessary to use the σ matrix identities (A.1.17), (A.1.18), (A.1.19), (A.1.20) and

$$\varepsilon^{abcd}\varepsilon_{efgh} = -4!\,\delta_e^{[a}\delta_f^b\delta_g^c\delta_h^{d]}. \tag{5.3.77}$$

In the end, the contribution to $\int \mathrm{d}^2\theta\,\mathrm{d}^2\bar\theta\ \mathrm{tr}_{\mathrm{Ad}}(a_5)$ coming solely from the vector component of the super Yang-Mills background, which will be denoted by $\mathrm{tr}_{\mathrm{Ad}}(a_5)|_{\mathrm{v}}$, is found to be

$$\begin{aligned}\mathrm{tr}_{\mathrm{Ad}}(a_5)\Big|_{\mathrm{v}} = \frac{1}{30}\mathrm{tr}_{\mathrm{Ad}}\Bigg(2\Big(&(\nabla^e F^{ab})(\nabla_e F_{bc})F^{cd}F_{da} + (\nabla^e F^{ab})(\nabla_e F^{cd})F_{bc}F_{da} \\ &+ (\nabla^e F^{ab})(\nabla_e F_{ca})F_{bd}F^{dc}\Big) \\ -\frac{1}{2}\Big((\nabla^e F^{ab})(\nabla_e F_{ab})F^{cd}F_{cd} &+ (\nabla^e F^{ab})(\nabla_e F^{cd})F_{ab}F_{cd} + (\nabla^e F^{ab})(\nabla_e F^{cd})F_{cd}F_{ab}\Big) \\ + 2\mathrm{i}\Big(F^{ab}F_{bc}F^{cd}F^e{}_aF_{de} &- 2F^{ab}F^{cd}F_{bc}F^e{}_aF_{de} + F^{ab}F^{cd}F^e{}_aF_{bc}F_{de}\Big)\Bigg).\end{aligned} \tag{5.3.78}$$

Comparison of this result with existing expressions in the literature is a highly non-trivial exercise, being complicated by the fact that in the non-Abelian case there are many possible field strength tensor structures, all of which are not independent. This is in contrast to the relatively transparent situation in superspace (where we found that there where only a few basis structures). To proceed further we will need to examine these F^5-type structures in greater detail. As a final point of interest, it turns out that the bosonic component given above is itself not expressed minimally (this is also the case for some other results in the literature, for example [85]), in that one of the terms may be eliminated in favour of some of the others. This point will be addressed in the next section, the detailed analysis of which was first published in an appendix of [35].

[7] See appendix B or the attached CD for a computerized approach.

5.3.3 A basis for F^5 structures

The array of possible structures of the form F^n are not independent since they can be related by: the antisymmetric property of the field strength $F_{ab} = -F_{ba}$; integration by parts; the cyclic property of the trace over the gauge indices; the Bianchi identity

$$\nabla_a F_{bc} + \nabla_c F_{ab} + \nabla_b F_{ca} = 0; \tag{5.3.79}$$

the equations of motion

$$\nabla^a F_{ab} = 0. \tag{5.3.80}$$

As a consequence of these it is also useful to establish the 'non-Abelian' identity

$$[F_{ab}, F_{cd}] = 2\mathrm{i}\nabla_{[a}\nabla_{b]}F_{cd}, \tag{5.3.81}$$

and

$$\nabla^2 F_{ab} = 2\mathrm{i}[F_a{}^c, F_{cb}], \tag{5.3.82}$$

where the last expression in (5.3.70) is the superspace analogue of the latter.

An independent set of such tensor structures forms a basis, and different bases are used throughout the literature since different calculational procedures naturally select different bases. For example, we have just seen in the previous section that the structures arising at order F^5 were almost completely determined by the use of superspace: in superspace and hence at the component level, both covariant derivatives act on adjacent field strengths and are always contracted with one another. Comparing various results in the literature [212, 79, 80], we see completely different structures, such as those where covariant derivatives are contracted with field strengths[8].

Furthermore, in the present case of F^5 structures in four dimensional spacetime, the analysis is further complicated by the fact that some structures are related in a much less obvious way, which in general depends crucially on value of n in F^n and the spacetime dimension. This is elaborated upon below.

To simplify this discussion we will first introduce the following notation (where, since the discussion is independent of the gauge group, the trace is over an arbitrary

[8] An additional complication which arises when comparing results manifests itself under field redefinitions of the vector potential [73, 37]. We avoid such complications by working on-shell $\nabla^a F_{ab} = 0$, since at this order (and lower orders) tensor structures which can be removed by, or are sensitive to field redefinitions, vanish on-shell. This can be seen explicitly in [78].

representation $\mathcal{R}$):

$$
\begin{aligned}
s_{0,0} &= \mathrm{tr}_{\mathcal{R}}(F^{ab}F_{bc}F^{cd}F_{de}F^{e}_{\ a}) & s_{0,1} &= \mathrm{tr}_{\mathcal{R}}(F^{ab}F_{bc}F^{cd}F^{e}_{\ a}F_{de}) \\
s_{0,2} &= \mathrm{tr}_{\mathcal{R}}(F^{ab}F^{cd}F_{bc}F^{e}_{\ a}F_{de}) & s_{0,3} &= \mathrm{tr}_{\mathcal{R}}(F^{ab}F^{cd}F^{e}_{\ a}F_{bc}F_{de}) \\
s_{0,4} &= \mathrm{tr}_{\mathcal{R}}(F^{ab}F_{bc}F^{c}_{\ a}F^{de}F_{de}) & s_{0,5} &= \mathrm{tr}_{\mathcal{R}}(F^{ab}F^{de}F_{bc}F^{c}_{\ a}F_{de}) \\
s_{1,0} &= \mathrm{tr}_{\mathcal{R}}((\nabla^e F^{ab})(\nabla_e F_{ab})F^{cd}F_{cd}) & s_{1,1} &= \mathrm{tr}_{\mathcal{R}}((\nabla^e F^{ab})(\nabla_e F^{cd})F_{ab}F_{cd}) \\
s_{1,2} &= \mathrm{tr}_{\mathcal{R}}((\nabla^e F^{ab})(\nabla_e F^{cd})F_{cd}F_{ab}) & s_{1,3} &= \mathrm{tr}_{\mathcal{R}}((\nabla^e F^{ab})(\nabla_e F_{bc})F^{cd}F_{da}) \\
s_{1,4} &= \mathrm{tr}_{\mathcal{R}}((\nabla^e F^{ab})(\nabla_e F_{ca})F_{bd}F^{dc}) & s_{1,5} &= \mathrm{tr}_{\mathcal{R}}((\nabla^e F^{ab})(\nabla_e F^{cd})F_{bc}F_{da}) \\
s_{1,6} &= \mathrm{tr}_{\mathcal{R}}((\nabla^e F^{ab})F_{da}(\nabla_e F_{bc})F^{cd}) & s_{1,7} &= \mathrm{tr}_{\mathcal{R}}((\nabla^e F^{ab})F^{cd}(\nabla_e F_{ab})F_{cd}) \\
s_{1,8} &= \mathrm{tr}_{\mathcal{R}}((\nabla^e F^{ab})F_{bc}(\nabla_e F^{cd})F_{da}) & s_{1,9} &= \mathrm{tr}_{\mathcal{R}}((\nabla^e F^{ab})F_{ab}(\nabla_e F^{cd})F_{cd}) \\
s_{1,10} &= \mathrm{tr}_{\mathcal{R}}((\nabla^e F^{ab})F^{cd}(\nabla_e F_{cd})F_{ab}) & s_{1,11} &= \mathrm{tr}_{\mathcal{R}}((\nabla^e F^{ab})F^{dc}(\nabla_e F_{cb})F_{da}) \\
s_{2,1} &= \mathrm{tr}_{\mathcal{R}}(F^{ab}(\nabla_a F^{cd})(\nabla^e F_{bc})F_{de}) & s_{2,3} &= \mathrm{tr}_{\mathcal{R}}((\nabla^a F^{ef})(\nabla_b F_{ef})F^{ac}F_{cb}) \\
s_{2,4} &= \mathrm{tr}_{\mathcal{R}}((\nabla^a F^{ef})(\nabla_b F_{ef})F^{bc}F_{ca}) & s_{2,5} &= \mathrm{tr}_{\mathcal{R}}((\nabla^a F^{ef})F^{cb}(\nabla_b F_{ef})F_{ac}) \\
s_{2,6} &= \mathrm{tr}_{\mathcal{R}}((\nabla_b F^{ef})F^{cb}(\nabla^a F_{ef})F_{ac}) & s_{2,7} &= \mathrm{tr}_{\mathcal{R}}((\nabla^b F^{cd})(\nabla_c F^{ea})F_{de}F_{ab}) \\
s_{2,9} &= \mathrm{tr}_{\mathcal{R}}((\nabla_e F^{bc})(\nabla^a F^{de})F_{ab}F_{cd}) & s_{2,10} &= \mathrm{tr}_{\mathcal{R}}((\nabla^a F^{de})(\nabla^b F_{ec})F_{ab}F^{c}_{\ d}) \\
s_{2,11} &= \mathrm{tr}_{\mathcal{R}}((\nabla_a F^{de})(\nabla^c F_{be})F^{ab}F_{cd}) & s_{2,12} &= \mathrm{tr}_{\mathcal{R}}((\nabla^a F^{de})(\nabla_e F^{cb})F_{ab}F_{cd}) \\
s_{2,13} &= \mathrm{tr}_{\mathcal{R}}((\nabla_e F^{bc})F_{cd}(\nabla^d F^{ea})F_{ab}) & s_{2,14} &= \mathrm{tr}_{\mathcal{R}}((\nabla^a F^{cd})(\nabla^b F_{de})F^{e}_{\ c}F_{ab}) \\
s_{2,15} &= \mathrm{tr}_{\mathcal{R}}((\nabla_a F^{cd})(\nabla^e F^{ab})F_{cb}F_{ed}) & s_{2,16} &= \mathrm{tr}_{\mathcal{R}}(F^{de}(\nabla_e F^{cb})F_{ab}(\nabla^a F_{cd})) \\
s_{2,17} &= \mathrm{tr}_{\mathcal{R}}(F_{ab}(\nabla_e F^{bc})F^{ed}(\nabla^a F_{dc})) & s_{2,18} &= \mathrm{tr}_{\mathcal{R}}(F_{ab}(\nabla_e F^{bc})F_{cd}(\nabla^a F^{de})).
\end{aligned}
$$

In the notation adopted here, the first index on $s_{i,j}$ takes the values 0, 1 or 2: terms without covariant derivatives (ie pure[9] F^5 terms) if 0; terms with contracted covariant derivatives if 1; and two covariant derivatives which are not contracted with one another otherwise. The second index is arbitrary, and serves to enumerate different structures for a given value of the first index. This list is of course not complete, and excludes terms which can obviously be reduced to a linear combination of the above (modulo integration by parts and the equations of motion).

In $D = 4$ the following set provides a basis for all such possible tensor structures:

$$\{s_{0,0}, s_{0,1}, s_{0,2}, s_{0,3}, s_{1,0}, s_{1,1}, s_{1,2}, s_{1,3}, s_{1,4}, s_{2,3}\}, \tag{5.3.83}$$

which consists of four pure F^5 structures, five structures with contracted covariant derivatives acting on adjacent field strengths, and a single structure with two covariant derivatives which are not contracted. We will now prove this fact. Other

[9]The prefix ‘pure’ will be used to refer to such terms to avoid confusion with the generic usage of F^5.

bases in various dimensions can be found elsewhere in the literature, for example see [78, 74, 101]. See [101] for a discussion (without explicit detail) on an alternative F^5 basis which is perhaps the most economical for expressing the non-Abelian D-brane effective action. As will be explained, the basis we chose here turns out to be more useful for comparison with the literature.

Pure F^5 terms

One can establish that the six F^5 terms: $s_{0,0}$, $s_{0,1}$, $s_{0,2}$, $s_{0,3}$, $s_{0,4}$ and $s_{0,5}$, are not linearly independent in four dimensions. Any two may be eliminated by generating two independent identities relating all six. This is a consequence of the simple fact that for any $n \times n$ matrix A, the trace of A^{n+1} may be expressed in terms of the product of the traces of lower powers of A. For example, given a traceless (antisymmetric) 4×4 matrix A:

$$\mathrm{tr}(A^5) = \frac{5}{6}\mathrm{tr}(A^2)\mathrm{tr}(A^3). \tag{5.3.84}$$

Such identities can be derived by considering the power series expansion of

$$\det(\mathbf{1} + \lambda A) = \exp \mathrm{tr} \ln(\mathbf{1} + \lambda A) \tag{5.3.85}$$

in λ, given the knowledge that for an $n \times n$ matrix, the right hand side will terminate at order λ^n. To be clear, the analogy here is to consider the field strength F as a 4×4 matrix in its spacetime indices.

In the case at hand the presence of the additional gauge index complicates such derivations, and it is far simpler to use the more novel approach which makes use of the rather trivial σ matrix identity

$$\mathrm{tr}(\sigma^a \tilde{\sigma}^b \sigma^c \tilde{\sigma}^d \sigma^e \tilde{\sigma}^f) = \mathrm{tr}(\sigma^b \tilde{\sigma}^a \sigma^f \tilde{\sigma}^e \sigma^d \tilde{\sigma}^c). \tag{5.3.86}$$

This is proven by a simple rearrangement of indices, and is of course particular to $D = 4$. The identity implies

$$\Big(\mathrm{tr}(\sigma^a \tilde{\sigma}^b \sigma^c \tilde{\sigma}^d \sigma^e \tilde{\sigma}^f)\mathrm{tr}(\tilde{\sigma}^g \sigma^h \tilde{\sigma}^i \sigma^j) + c.c\Big)$$
$$\mathrm{tr}_{\mathcal{R}}(F_{ab}F_{cd}F_{ef}F_{gh}F_{ij} + F_{ab}F_{ef}F_{cd}F_{gh}F_{ij}) = 0, \tag{5.3.87}$$

which, using the other σ matrix identities, reduces to

$$s_{0,1} - 2s_{0,2} + s_{0,3} + \frac{1}{2}s_{0,4} + \frac{3}{2}s_{0,5} = 0. \tag{5.3.88}$$

Similarly one can use (5.3.86) to establish

$$\Big(\mathrm{tr}(\sigma^a \tilde{\sigma}^b \sigma^c \tilde{\sigma}^d \sigma^e \tilde{\sigma}^f)\mathrm{tr}(\tilde{\sigma}^g \sigma^h \tilde{\sigma}^i \sigma^j) + c.c\Big)$$
$$\mathrm{tr}_{\mathcal{R}}(F_{ab}F_{cd}F_{gh}F_{ef}F_{ij} + F_{ab}F_{ef}F_{gh}F_{cd}F_{ij}) = 0, \tag{5.3.89}$$

which becomes

$$s_{0,0} - 2s_{0,1} - s_{0,2} + \frac{3}{2}s_{0,4} - \frac{1}{2}s_{0,5} = 0. \tag{5.3.90}$$

Equations (5.3.88) and (5.3.90) allow two of the six F^5 structures to be expressed in terms of the other four. We choose to treat $s_{0,4}$ and $s_{0,5}$ as dependent:

$$s_{0,4} = -\frac{3}{5}s_{0,0} + s_{0,1} + s_{0,2} + \frac{1}{5}s_{0,3} \tag{5.3.91}$$

$$s_{0,5} = \frac{1}{5}s_{0,0} - s_{0,1} + s_{0,2} + \frac{3}{5}s_{0,3}. \tag{5.3.92}$$

D^2F^4 terms

Using the equations of motion, the Bianchi identity, integration by parts, the cyclic property of the trace and the non-Abelian identity each of the terms $s_{1,i}$ with $6 \leq i \leq 11$ are quite readily expressed in terms of the proposed basis:

$$s_{1,6} = -2\mathrm{i}s_{0,1} + 2\mathrm{i}s_{0,2} - s_{1,4} - s_{1,5} \tag{5.3.93}$$

$$s_{1,7} = 4\mathrm{i}s_{0,5} - 2s_{1,1} = \frac{4\mathrm{i}}{5}s_{0,0} - 4\mathrm{i}s_{0,1} + 4\mathrm{i}s_{0,2} + \frac{12\mathrm{i}}{5}s_{0,3} - 2s_{1,1} \tag{5.3.94}$$

$$s_{1,8} = -2\mathrm{i}s_{0,0} + 2\mathrm{i}s_{0,1} - 2s_{1,3} \tag{5.3.95}$$

$$s_{1,9} = 4\mathrm{i}s_{0,4} - s_{1,0} - s_{1,2} = -\frac{12\mathrm{i}}{5}s_{0,0} + 4\mathrm{i}s_{0,1} + 4\mathrm{i}s_{0,2} + \frac{4\mathrm{i}}{5}s_{0,3} - s_{1,0} - s_{1,2} \tag{5.3.96}$$

$$s_{1,10} = 4\mathrm{i}s_{0,4} - s_{1,0} - s_{1,2} = -\frac{12\mathrm{i}}{5}s_{0,0} + 4\mathrm{i}s_{0,1} + 4\mathrm{i}s_{0,2} + \frac{4\mathrm{i}}{5}s_{0,3} - s_{1,0} - s_{1,2} \tag{5.3.97}$$

$$s_{1,11} = -2\mathrm{i}s_{0,1} + 2\mathrm{i}s_{0,2} - s_{1,4} - s_{1,5}. \tag{5.3.98}$$

$D_aD_bF^4$ terms

Again using these properties one can generate the following independent equations:

$$s_{2,1} = \mathrm{i}s_{0,2} - \mathrm{i}s_{0,3} + \frac{1}{4}s_{1,1} - \frac{1}{2}s_{2,5} - s_{2,9} \tag{5.3.99}$$

$$s_{2,9} = -\mathrm{i}s_{0,1} + \mathrm{i}s_{0,2} - s_{1,4} - \frac{1}{2}s_{2,3} + \frac{1}{2}s_{2,6} \tag{5.3.100}$$

$$s_{2,4} = -\mathrm{i}s_{0,4} + \mathrm{i}s_{0,5} + \frac{1}{2}s_{1,2} - \frac{1}{2}s_{1,10} + s_{2,3} \tag{5.3.101}$$

$$s_{2,5} = s_{2,6} \tag{5.3.102}$$

$$s_{2,7} = s_{2,9} \tag{5.3.103}$$

$$s_{2,13} = s_{2,18} = -\mathrm{i}s_{0,0} + \mathrm{i}s_{0,1} - s_{1,3} \tag{5.3.104}$$

$$s_{2,12} = - s_{2,15} \tag{5.3.105}$$

$$s_{2,12} = - s_{1,5} + s_{2,7} \tag{5.3.106}$$

$$s_{2,10} = s_{2,14} = -s_{2,11} - s_{2,12} \tag{5.3.107}$$

$$s_{2,10} = - s_{2,1} + s_{2,9} \tag{5.3.108}$$

$$s_{2,16} = - s_{2,17} = \frac{\mathrm{i}}{2} s_{0,4} - \frac{\mathrm{i}}{2} s_{0,5} - s_{2,13} + s_{2,14}. \tag{5.3.109}$$

Furthermore, two more independent equations can be produced by again using identity (5.3.86). For example

$$\Big(\mathrm{tr}(\sigma^a \tilde{\sigma}^b \sigma^c \tilde{\sigma}^d \sigma^e \tilde{\sigma}^f)\mathrm{tr}(\tilde{\sigma}^g \sigma^h \tilde{\sigma}^i \sigma^j) + c.c\Big) \Big(\mathrm{tr}_{\mathcal{R}}((\nabla_b F_{cd})\nabla_a(F_{gh})F_{ij}F_{ef} - (\nabla_a F_{fe})\nabla_b(F_{gh})F_{ij}F_{dc})\Big) = 0, \tag{5.3.110}$$

and

$$\Big(\mathrm{tr}(\sigma^a \tilde{\sigma}^b \sigma^c \tilde{\sigma}^d \sigma^e \tilde{\sigma}^f)\mathrm{tr}(\tilde{\sigma}^g \sigma^h \tilde{\sigma}^i \sigma^j) + c.c\Big) \Big(\mathrm{tr}_{\mathcal{R}}((\nabla_a F_{gj})\nabla_f(F_{hi})F_{bc}F_{ed} - (\nabla_b F_{gj})\nabla_c(F_{hi})F_{af}F_{de})\Big) = 0, \tag{5.3.111}$$

reduce to

$$s_{1,0} + s_{1,1} - 4s_{1,4} + 4s_{2,1} + 2s_{2,4} = 0 \tag{5.3.112}$$

and

$$s_{1,1} - s_{1,2} + 4s_{1,3} - 4s_{1,4} + 8s_{1,5} - 2s_{2,3} + 2s_{2,4} - 4s_{2,9} + 4s_{2,12} - 8s_{2,15} = 0 \tag{5.3.113}$$

respectively. The latter identities prove quite difficult to establish via other means. This completely exhausts all possible combinations, and brings us to a final set independent relations.

The basis

All tensor structures can now be expressed in this basis (5.3.83). Introducing the condensed notation

$$\{a, b, c, d, e, f, g, h, i, j\} \equiv a s_{0,0} + b s_{0,1} + c s_{0,2} + d s_{0,3} + e s_{1,0} + f s_{1,1} + g s_{1,2} + h s_{1,3} + i s_{1,4} + j s_{2,3},$$

we list for completeness all terms expressed in this basis:

$$
\begin{array}{rcl}
s_{0,4} = & \{ & -\tfrac{3}{5}, \; 1, \; 1, \; \tfrac{1}{5}, \; 0, \; 0, \; 0, \; 0, \; 0, \; 0 \; \} \\
s_{0,5} = & \{ & \tfrac{1}{5}, \; -1, \; 1, \; \tfrac{3}{5}, \; 0, \; 0, \; 0, \; 0, \; 0, \; 0 \; \} \\
s_{1,5} = & \{ & 2\mathrm{i}, \; -5\mathrm{i}, \; 0, \; -\mathrm{i}, \; \tfrac{3}{4}, \; \tfrac{3}{4}, \; \tfrac{3}{4}, \; 1, \; -3, \; 0 \; \} \\
s_{1,6} = s_{1,11} = & \{ & -2\mathrm{i}, \; 3\mathrm{i}, \; 2\mathrm{i}, \; \mathrm{i}, \; -\tfrac{3}{4}, \; -\tfrac{3}{4}, \; -\tfrac{3}{4}, \; -1, \; 2, \; 0 \; \} \\
s_{1,7} = & \{ & \tfrac{4\mathrm{i}}{5}, \; -4\mathrm{i}, \; 4\mathrm{i}, \; \tfrac{12\mathrm{i}}{5}, \; 0, \; -2, \; 0, \; 0, \; 0, \; 0 \; \} \\
s_{1,8} = & \{ & -2\mathrm{i}, \; 2\mathrm{i}, \; 0, \; 0, \; 0, \; 0, \; 0, \; -2, \; 0, \; 0 \; \} \\
s_{1,9} = s_{1,10} = & \{ & -\tfrac{12\mathrm{i}}{5}, \; 4\mathrm{i}, \; 4\mathrm{i}, \; \tfrac{4\mathrm{i}}{5}, \; -1, \; 0, \; -1, \; 0, \; 0, \; 0 \; \} \\
s_{2,1} = & \{ & -\mathrm{i}, \; 2\mathrm{i}, \; \mathrm{i}, \; 0, \; -\tfrac{1}{2}, \; -\tfrac{1}{4}, \; -\tfrac{1}{2}, \; 0, \; 1, \; -\tfrac{1}{2} \; \} \\
s_{2,4} = & \{ & 2\mathrm{i}, \; -4\mathrm{i}, \; -2\mathrm{i}, \; 0, \; \tfrac{1}{2}, \; 0, \; 1, \; 0, \; 0, \; 1 \; \} \\
s_{2,5} = s_{2,6} = & \{ & \mathrm{i}, \; -\mathrm{i}, \; -\mathrm{i}, \; -\mathrm{i}, \; \tfrac{1}{2}, \; \tfrac{1}{2}, \; \tfrac{1}{2}, \; 0, \; 0, \; 1 \; \} \\
s_{2,7} = s_{2,9} = & \{ & \tfrac{\mathrm{i}}{2}, \; -\tfrac{3\mathrm{i}}{2}, \; \tfrac{\mathrm{i}}{2}, \; -\tfrac{\mathrm{i}}{2}, \; \tfrac{1}{4}, \; \tfrac{1}{4}, \; \tfrac{1}{4}, \; 0, \; -1, \; 0 \; \} \\
s_{2,10} = s_{2,14} = & \{ & \tfrac{3\mathrm{i}}{2}, \; -\tfrac{7\mathrm{i}}{2}, \; -\tfrac{\mathrm{i}}{2}, \; -\tfrac{\mathrm{i}}{2}, \; \tfrac{3}{4}, \; \tfrac{1}{2}, \; \tfrac{3}{4}, \; 0, \; -2, \; \tfrac{1}{2} \; \} \\
s_{2,11} = & \{ & 0, \; 0, \; 0, \; 0, \; -\tfrac{1}{4}, \; 0, \; -\tfrac{1}{4}, \; 1, \; 0, \; -\tfrac{1}{2} \; \} \\
s_{2,12} = -s_{2,15} = & \{ & -\tfrac{3\mathrm{i}}{2}, \; \tfrac{7\mathrm{i}}{2}, \; \tfrac{\mathrm{i}}{2}, \; \tfrac{\mathrm{i}}{2}, \; -\tfrac{1}{2}, \; -\tfrac{1}{2}, \; -\tfrac{1}{2}, \; -1, \; 2, \; 0 \; \} \\
s_{2,13} = s_{2,18} = & \{ & -\mathrm{i}, \; \mathrm{i}, \; 0, \; 0, \; 0, \; 0, \; 0, \; -1, \; 0, \; 0 \; \} \\
s_{2,16} = -s_{2,17} = & \{ & \tfrac{21\mathrm{i}}{10}, \; -\tfrac{7\mathrm{i}}{2}, \; -\tfrac{\mathrm{i}}{2}, \; -\tfrac{7\mathrm{i}}{10}, \; \tfrac{3}{4}, \; \tfrac{1}{2}, \; \tfrac{3}{4}, \; 1, \; -2, \; \tfrac{1}{2} \; \}
\end{array}
$$

As noted in the previous section, the component results obtained for a_5 were not expressed in any basis in $D = 4$. In the notation introduced here, the component result (5.3.78) is

$$
\mathrm{tr}_{\mathrm{Ad}}(a_5)\Big|_{\mathrm{v}} = \frac{1}{30}\bigg(2\Big(s_{1,3} + s_{1,5} + s_{1,4}\Big) - \frac{1}{2}\Big(s_{1,0} + s_{1,1} + s_{1,2}\Big) + 2\mathrm{i}\Big(s_{0,1} - 2s_{0,2} + s_{0,3}\Big)\bigg). \quad (5.3.114)
$$

where it is understood that the trace in the definition of the $s_{i,j}$ above is over the adjoint representation. The term $s_{1,5}$ is not a member of the basis (5.3.83) and is eliminated to give

$$
\mathrm{tr}_{\mathrm{Ad}}(a_5)\Big|_{\mathrm{v}} = \frac{1}{30}\{4\mathrm{i}, -8\mathrm{i}, -4\mathrm{i}, 0, 1, 1, 1, 4, -4, 0\}. \quad (5.3.115)
$$

5.3.4 Comparison with literature

Making comparisons with F^5 expressions found in the literature now becomes a rather simple exercise. At the time of the publication of [34], the F^5 contributions to the non-Abelian D-brane effective action had been calculated in full by several different methods (see the introduction for details). A direct computation of the $\mathcal{N} = 4$ super Yang-Mills effective action to this order in $D = 4$ using supergraph techniques had also been attempted in [85, 86].

As detailed in the introduction, Koerber and Sevrin [78] used an approach based on the requirement that certain BPS solutions should exist to the equations of motion derived from the non-Abelian D-brane effective action[10], extending earlier use of this method for the Abelian case [71]. At order $(\alpha')^3$ in $D = 10$ this approach yields[11] (1.1.3).

A number of successful tests have been applied to this result, confirming that the expression (1.1.3) is indeed consistent with string theoretic predictions [79, 80] and a unique deformation of $D = 10$, $\mathcal{N} = 1$ supersymmetric Yang-Mills theory [74]. Most recently a direct string theory calculation of the full five-point scattering amplitude for gluons has been carried out [81], from which it is inferred that the corresponding low energy effective action has precisely the order $(\alpha')^3$ terms (1.1.3).

After restricting the expression (1.1.3) to $D = 4$, and expressing it in the basis adopted here via the conversion table given above, it agrees exactly with the above $\mathcal{N} = 4$ super Yang-Mills results $\mathrm{tr}_{\mathrm{Ad}}(a_5)|_{\mathrm{v}}$ (5.3.78) and (5.3.115) (up to an overall normalized factor).

As noted earlier, this agreement was first seen in [34]. Published prior to this, the computation [85, 86] of the one-loop $\mathcal{N} = 4$ super Yang-Mills effective action to order F^5 using supergraph techniques (which gave bosonic component results only) disagreed with results from string theory. In particular, the pure F^5 terms where found to be different. However, soon after the release of [34], the authors of [85, 86] corrected an error, yielding consistent results [85].

The fact that all of these results [78, 81, 79, 80, 74, 34, 85] computed by four independent means yield the same result, (1.1.3), leads one to confidently conclude that the F^5 deformation of maximally supersymmetric Yang-Mills theory is in fact uniquely given by this expression. Evidence which suggests that this uniqueness does not extend to all orders, and that in general one should not expect a direct correspondence between the non-Abelian D-brane effective action and quantized super Yang-Mills at higher orders, comes from the fact that the F^8 terms in the one-loop Abelian $\mathcal{N} = 4$ super Yang-Mills effective action [106] differ from the F^8 terms in the Born-Infeld action [87]. For a detailed discussion see [106, 107].

It is perhaps worthy to note that due to the expected correspondence between the ultraviolet divergent part of the one-loop effective action of super Yang-Mills theory and the singular piece of the one-loop effective action in open superstring theory in the limit $\alpha' \to 0$ (see [214, 76]), the one-loop results computed here should

[10] For other approaches see [70, 213] and references therein.

[11] Partial results at order $(\alpha')^3$ had previously been obtained in [76], [77] and [43]. F^5 terms in ten dimensional super Yang-Mills were given in [76].

provide information about the F^5 terms of the latter.

Finally it should also be pointed out, which was apparently overlooked by many authors, that with a little work the bosonic one-loop effective action to order F^5 for $\mathcal{N}=4$ super Yang-Mills theory could have been extracted from pre-existing literature. In particular the bosonic DeWitt-Seeley coefficients associated with scalars, vectors or spinors in the presence of a non-Abelian background Yang-Mills field in arbitrary spacetime dimension had effectively been computed to order F^5 some time ago [87, 88, 215, 76].

From the tables in [76] and [87], one can assemble the total bosonic component of the DeWitt-Seeley coefficients associated with a theory possessing N_1 vectors, N_0 scalars and $N_{1/2}$ spinors all in the adjoint representation, coupled to a Yang-Mills background by using[12]

$$\mathbf{a}_n^{\text{tot}} = N_1\,\mathbf{a}_n(\Delta_1) + (N_0 - 2N_1)\,\mathbf{a}_n(\Delta_0) - \frac{N_{1/2}}{\gamma}\,\mathbf{a}_n(\Delta_{1/2}) \tag{5.3.116}$$

where $\gamma = 1, 2, 4$ for Dirac, Majorana and Majorana-Weyl spinors respectively, and $\mathbf{a}_n(\Delta_s)$ $(s = 0, 1, \frac{1}{2})$ denotes the contribution generated by the presence of second order (scalar, vector, spinor) operators in the original action.

The general $\mathbf{a}_4$ and $\mathbf{a}_5$ coefficients provided in [76] were obtained from one-loop counterterms computed in [215], however the pure F^5 contributions had not been extracted. From these one readily generates the following on-shell bosonic component for the $D=4$, $\mathcal{N}=4$ super Yang-Mills theory DeWitt-Seeley coefficient (see the $D=10$, $\mathcal{N}=1$ expression in [76]):

$$\text{tr}_{\text{Ad}}(\mathbf{a}_5) = \frac{2}{5!}(4s_{1,3} - 4s_{1,6} - s_{1,1} + s_{1,9}) + (F^5\text{ terms}). \tag{5.3.117}$$

Despite its incompleteness, and regardless of the fact that there exists the possibility of 'communication' between pure F terms and those containing derivatives via (5.3.81), a comparison between this expression and the result (5.3.115) can be made without any additional work. That such a comparison of partial results is even possible is a consequence of the particular form of the chosen basis[13]. Specifically, in casting any expression in our chosen basis, any pure F^5 terms will be expressed as a linear combination of the pure F^5 structures $s_{0,0} - s_{0,3}$ only. The omitted pure F^5 terms in (5.3.117) will therefore not contribute terms containing derivatives when expressed in this basis, and so a comparison of derivative terms becomes possible. Using the conversion table provided in subsection 5.3.3 one finds

[12] In the notation used in [76] and [87] $\mathbf{a}_n = b_{2n}$.

[13] In contrast, the scheme described in [101] yields a basis which is not useful for this purpose.

that (5.3.117) becomes

$$\mathrm{tr}_{\mathrm{Ad}}(\mathbf{a}_5) = \frac{1}{30}\left\{\frac{14\mathrm{i}}{5}, -4\mathrm{i}, -2\mathrm{i}, -\frac{8\mathrm{i}}{5}, 1, 1, 1, 1, 4, -4, 0\right\} + (F^5 \text{ terms}). \qquad (5.3.118)$$

Comparing this to with (5.3.115), one indeed finds exact agreement of the derivative terms.

5.4 The F^6 terms

5.4.1 Expanding moments

Employing the procedure outlined above to compute a_k for $k > 5$ will necessarily involve asymptotically expanding moments to higher than leading order. We will therefore require some sort of prescription if this scheme is to be generalized. As one may expect, it is possible to appeal to a set of techniques similar to those already used. More specifically, to evaluate any moment to arbitrary order one proceeds iteratively by using the following generalizations of the identities (5.2.21), (5.2.26) and (5.2.27):

$$0 = \int \mathrm{d}\eta \, \frac{\partial}{\partial k_b}\left(X_{A_1}\dots X_{A_n}\mathrm{e}^{\mathrm{i}s\Delta}\right) \qquad (5.4.119)$$

$$0 = \int \mathrm{d}\eta \, \frac{\partial}{\partial \kappa_\beta}\left(X_{A_1}\dots X_{A_n}\mathrm{e}^{\mathrm{i}s\Delta}\right) \qquad (5.4.120)$$

$$0 = \int \mathrm{d}\eta \, \frac{\partial}{\partial \bar{\kappa}_{\dot\beta}}\left(X_{A_1}\dots X_{A_n}\mathrm{e}^{\mathrm{i}s\Delta}\right). \qquad (5.4.121)$$

Occasionally differentiation with respect to s will also be useful, as in (5.2.19):

$$\frac{\mathrm{d}^m K_{A_1\dots A_n}(z;s)}{\mathrm{d}s^m} = \int \mathrm{d}\eta \; X_{A_1}\dots X_{A_n}(\mathrm{i}\Delta)^m \mathrm{e}^{\mathrm{i}s\Delta}. \qquad (5.4.122)$$

A combination of the two procedures is also useful, as in (5.2.30). Of course, none of this actually computes the moment directly, but is used with the intention of expressing it in terms of other moments with the same number or more indices, which are generally easier to compute. In this procedure, expanding a moment to some order will usually require knowledge of the expansion of several other moments to the same or lower order. Consequently at some point it will be necessary to evaluate at least one moment directly by expanding the exponential.

5.4.2 The moment hierarchy and a_6

Computing a_6 involves summing from $n = 1$ to 6 on the right hand side of (5.3.41), which generates a hierarchy of moments, a partial list being given below (all but

$K(z;s)$ required to subleading order):

$$K_{\alpha\beta\dot{\alpha}\dot{\beta}ab}(z;s)$$

$$K_{\alpha\beta\dot{\alpha}\dot{\beta}a}(z;s) \quad K_{\alpha\beta\dot{\alpha}ab}(z;s) \quad K_{\alpha\dot{\alpha}\dot{\beta}ab}(z;s) \quad K_{\alpha\beta\dot{\alpha}\dot{\beta}}(z;s)$$

$$K_{\alpha\beta\dot{\alpha}a}(z;s) \quad K_{\alpha\dot{\alpha}\dot{\beta}a}(z;s) \quad K_{\alpha\beta ab}(z;s) \quad K_{\alpha\dot{\alpha}ab}(z;s) \quad K_{\dot{\alpha}\dot{\beta}ab}(z;s) \quad K_{\alpha\beta\dot{\alpha}}(z;s) \quad K_{\alpha\dot{\alpha}\dot{\beta}}(z;s)$$

$$K_{\alpha\beta a}(z;s) \quad K_{\alpha\dot{\alpha}a}(z;s) \quad K_{\dot{\alpha}\dot{\beta}a}(z;s) \quad K_{\alpha ab}(z;s) \quad K_{\dot{\alpha}ab}(z;s) \quad K_{\alpha\beta}(z;s) \quad K_{\alpha\dot{\alpha}}(z;s) \quad K_{\dot{\alpha}\dot{\beta}}(z;s)$$

$$\vdots$$

$$K_{ab}(z;s) \quad K_{\alpha a}(z;s) \quad K_{\dot{\alpha}a}(z;s) \quad K_{\alpha}(z;s) \quad K_{\dot{\alpha}}(z;s)$$

$$K(z;s)$$

Generally speaking, the following structure is present in the hierarchy: from top to bottom the moments decrease in the number of indices, increase in difficulty of expansion, and the exponent of s in the leading order term increases (each row contains moments with the same leading order behaviour). From left to right, the moments decrease in their difficulty of expansion, and clearly many are related by complex conjugation.

In the prescription outlined above, the expansion of any moment generally hinges on having computed the expansion of a number of those next to or above it in the hierarchy, so naturally one begins at the top and works down. However one needn't actually attempt to identify the order in which the moments need to be computed prior to attempting to determine a DeWitt-Seeley coefficient, since this becomes self evident during the process. Moreover, one should note that there is a certain amount of freedom to the order in which one can compute these moments, and so no concrete ordering can be given.

To be more explicit about this entire process, consider the following examples which cover all important points.

Example 1

To give the general flavour of the method we will consider a simple example in full detail. The leading order moments given earlier, (5.3.46)-(5.3.69), have been listed in approximately hierarchical order, so consider computing a typical moment such as $K_{\alpha\beta a}(z;s)$ to leading order. As previously mentioned, this moment can readily be computed by directly expanding the exponential; however if the moments above it in the hierarchy are known, it is simpler to proceed as follows.

Firstly we note that one is free to compute this moment by choosing either of the following identities:

$$0 = \int d\eta \, \frac{\partial}{\partial \bar{\kappa}_{\dot{\beta}}} \left(X_\alpha X_\beta \bar{X}_{\dot{\alpha}} X_a e^{is\Delta} \right) \tag{5.4.123}$$

or

$$0 = \int d\eta \, \frac{\partial}{\partial \kappa_a} \left(X_\alpha X_\beta e^{is\Delta} \right). \tag{5.4.124}$$

Choosing the former, which after the contraction of $\dot{\alpha}$ and $\dot{\beta}$, leads directly to

$$K_{\alpha\beta a}(z; s) = \frac{1}{2} \int d\eta \; X_\alpha X_\beta \bar{X}_{\dot{\alpha}} X_a \sum_{n=0}^{\infty} \frac{(is)^{n+1}}{(n+1)!} L^n_\Delta(\bar{W}^{\dot{\alpha}}) \, e^{is\Delta}. \tag{5.4.125}$$

Knowledge of the commutation relations (5.2.33) and power series behaviour of moments (5.2.34) reveals that the summation in (5.4.125) truncates at $n = 2$ when computing $K_{\alpha\beta a}(z; s)$ to leading order. Explicitly one therefore obtains

$$\begin{aligned} K_{\alpha\beta a}(z; s) =& \frac{1}{2} \int d\eta \; X_\alpha X_\beta \bar{X}_{\dot{\alpha}} X_a \sum_{n=0}^{2} \frac{(is)^{n+1}}{(n+1)!} L^n_\Delta(\bar{W}^{\dot{\alpha}}) \, e^{is\Delta} \\ =& \frac{1}{2}(is) \int d\eta \; X_\alpha X_\beta \bar{X}_{\dot{\alpha}} X_a \bar{W}^{\dot{\alpha}} \, e^{is\Delta} \\ &+ \frac{1}{2}\frac{(is)^2}{2!} \int d\eta \; X_\alpha X_\beta \bar{X}_{\dot{\alpha}} X_a [\Delta, \bar{W}^{\dot{\alpha}}] \, e^{is\Delta} \\ &+ \frac{1}{2}\frac{(is)^3}{3!} \int d\eta \; X_\alpha X_\beta \bar{X}_{\dot{\alpha}} X_a [\Delta, [\Delta, \bar{W}^{\dot{\alpha}}]] \, e^{is\Delta}. \end{aligned} \tag{5.4.126}$$

With a certain level of familiarity with the commutation relations and power series behaviour of moments, it becomes rather obvious that the only possible terms which may contribute to the order of interest will be

$$\begin{aligned} K_{\alpha\beta a}(z; s) = & \frac{1}{2}(is) \left(-(\mathcal{D}_a \bar{W}^{\dot{\alpha}}) K_{\alpha\beta\dot{\alpha}}(z; s) - \bar{W}^{\dot{\alpha}} K_{\alpha\beta\dot{\alpha}a}(z; s) \right) \\ &+ \frac{1}{2}\frac{(is)^2}{2!} \int d\eta \; X_\alpha X_\beta \bar{X}_{\dot{\alpha}} X_a \left(2(\mathcal{D}^b \bar{W}^{\dot{\alpha}}) X_b + \{\bar{W}^{\dot{\beta}}, \bar{W}^{\dot{\alpha}}\} \bar{X}_{\dot{\beta}} \right) e^{is\Delta} \\ &+ \frac{1}{2}\frac{(is)^3}{3!} \int d\eta \; X_\alpha X_\beta \bar{X}_{\dot{\alpha}} X_a \Big(\{\bar{W}^{\dot{\beta}}, 2(\mathcal{D}^b \bar{W}^{\dot{\alpha}})\} \bar{X}_{\dot{\beta}} X_b \\ &\qquad + 2\mathcal{D}^b(\{\bar{W}^{\dot{\beta}}, \bar{W}^{\dot{\alpha}}\}) X_b \bar{X}_{\dot{\beta}} \Big) e^{is\Delta} \\ = & \frac{1}{2}(is) \left(-(\mathcal{D}_a \bar{W}^{\dot{\alpha}}) K_{\alpha\beta\dot{\alpha}}(z; s) - \bar{W}^{\dot{\alpha}} K_{\alpha\beta\dot{\alpha}a}(z; s) \right) \\ &+ \frac{1}{2}\frac{(is)^2}{2!} \Big(-2(\mathcal{D}^b \bar{W}^{\dot{\alpha}}) K_{\alpha\beta\dot{\alpha}ab}(z; s) + \{\bar{W}^{\dot{\beta}}, \bar{W}^{\dot{\alpha}}\} K_{\alpha\beta\dot{\alpha}\dot{\beta}a}(z; s) \\ &\qquad + \mathcal{D}_a(\{\bar{W}^{\dot{\beta}}, \bar{W}^{\dot{\alpha}}\}) K_{\alpha\beta\dot{\alpha}\dot{\beta}}(z; s) \Big) \\ &+ \frac{1}{2}\frac{(is)^3}{3!} \Big(2\{\bar{W}^{\dot{\beta}}, (\mathcal{D}^b \bar{W}^{\dot{\alpha}})\} K_{\alpha\beta\dot{\alpha}\dot{\beta}ab}(z; s) \\ &\qquad + 2\mathcal{D}^b(\{\bar{W}^{\dot{\beta}}, \bar{W}^{\dot{\alpha}}\}) K_{\alpha\beta\dot{\alpha}\dot{\beta}ab}(z; s) \Big). \end{aligned} \tag{5.4.127}$$

Inserting the previously computed expansions for $K_{\alpha\beta\dot{\alpha}\dot{\beta}ab}(z;s)$, $K_{\alpha\beta\dot{\alpha}\dot{\beta}}(z;s)$, $K_{\alpha\beta\dot{\alpha}ab}(z;s)$, $K_{\alpha\beta\dot{\alpha}\dot{\beta}a}(z;s)$, $K_{\alpha\beta\dot{\alpha}}(z;s)$, $K_{\alpha\beta\dot{\alpha}a}(z;s)$ as given higher up in the leading order hierarchy by equations (5.3.46), (5.3.47), (5.3.48), (5.3.50), (5.3.51) and (5.3.56) respectively, leads to

$$\begin{aligned} K_{\alpha\beta a}(z;s) = \varepsilon_{\alpha\beta}\left(2(\mathcal{D}_a\bar{W}_{\dot{\alpha}})\bar{W}^{\dot{\alpha}} + \bar{W}_{\dot{\alpha}}(\mathcal{D}_a\bar{W}^{\dot{\alpha}})\right) H \\ - \varepsilon_{\alpha\beta}(\mathcal{D}_a\bar{W}_{\dot{\alpha}})\bar{W}^{\dot{\alpha}}H - \frac{1}{3}\varepsilon_{\alpha\beta}\{\bar{W}_{\dot{\alpha}},(\mathcal{D}_a\bar{W}^{\dot{\alpha}})\}H. \end{aligned} \tag{5.4.128}$$

Expanding this yields (5.3.63) as required. One can check this result by starting with (5.4.124) rather than (5.4.123).

Example 2

Returning now to the problem of computing the moment hierarchy of a_6, the moment $K_{\alpha\beta\dot{\alpha}\dot{\beta}}(z;s)$ turns out to be a rather important object in this hierarchy in that all other moments may be expressed in terms of it, at least to the order of interest. Its power series to subleading order is not difficult to compute by directly expanding the exponential, and takes the simple form:

$$K_{\alpha\beta\dot{\alpha}\dot{\beta}}(z;s) = -\frac{4\mathrm{i}}{(4\pi \mathrm{i}s)^2}\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}} + \mathcal{O}(s^0), \tag{5.4.129}$$

where the subleading s^{-1} term vanishes due to the equations of motion. Alternatively, one may prove without detailed computation that the subleading term vanishes by considering the differential equation for $K_{\alpha\beta\dot{\alpha}\dot{\beta}}(z;s)$ itself, as follows. We first note that this moment has the general power series behavior

$$K_{\alpha\beta\dot{\alpha}\dot{\beta}}(z;s) = \frac{1}{s^2}A_{\alpha\beta\dot{\alpha}\dot{\beta}} + \frac{1}{s}B_{\alpha\beta\dot{\alpha}\dot{\beta}} + \mathcal{O}(s^0) \tag{5.4.130}$$

for some $A_{\alpha\beta\dot{\alpha}\dot{\beta}}$ and $B_{\alpha\beta\dot{\alpha}\dot{\beta}}$, which implies that

$$\frac{\mathrm{d}K_{\alpha\beta\dot{\alpha}\dot{\beta}}(z;s)}{\mathrm{d}s} = -\frac{2}{s^3}A_{\alpha\beta\dot{\alpha}\dot{\beta}} - \frac{1}{s^2}B_{\alpha\beta\dot{\alpha}\dot{\beta}} + \mathcal{O}(s^0). \tag{5.4.131}$$

Since

$$\begin{aligned} \frac{\mathrm{d}K_{\alpha\beta\dot{\alpha}\dot{\beta}}(z;s)}{\mathrm{d}s} &= \mathrm{i}\int \mathrm{d}\eta\; X_\alpha X_\beta \bar{X}_{\dot{\alpha}}\bar{X}_{\dot{\beta}}\Delta \mathrm{e}^{\mathrm{i}s\Delta} \\ &= \mathrm{i}K_{\alpha\beta\dot{\alpha}\dot{\beta}a}{}^{a}(z;s) - \mathrm{i}\int \mathrm{d}\eta\; X_\alpha X_\beta \bar{X}_{\dot{\alpha}}\bar{X}_{\dot{\beta}} W^\gamma X_\gamma \mathrm{e}^{\mathrm{i}s\Delta}, \end{aligned} \tag{5.4.132}$$

using the identity

$$0 = \int \mathrm{d}\eta\, \frac{\partial}{\partial k_a}\left(X_\alpha X_\beta \bar{X}_{\dot{\alpha}}\bar{X}_{\dot{\beta}} X_a \mathrm{e}^{\mathrm{i}s\Delta}\right) \tag{5.4.133}$$

one can establish

$$\begin{aligned}
\mathrm{i}K_{\alpha\beta\dot{\alpha}\dot{\beta}a}{}^{a}(z;s) = &-\frac{2}{s}K_{\alpha\beta\dot{\alpha}\dot{\beta}} + \frac{\mathrm{i}}{2s}\int \mathrm{d}\eta\; X_\alpha X_\beta \left((\sigma^a)_{\gamma\dot{\alpha}}(\theta-\theta')^\gamma\right)\bar{X}_{\dot{\beta}}X_a \mathrm{e}^{\mathrm{i}s\Delta} \\
&-\frac{\mathrm{i}}{2s}\int \mathrm{d}\eta\; X_\alpha X_\beta \left((\sigma^a)_{\gamma\dot{\beta}}(\theta-\theta')^\gamma\right)\bar{X}_{\dot{\alpha}}X_a \mathrm{e}^{\mathrm{i}s\Delta} \\
&-\mathrm{i}\int \mathrm{d}\eta\; X_\alpha X_\beta \bar{X}_{\dot{\alpha}}\bar{X}_{\dot{\beta}}X_a (J^a - X^a)\mathrm{e}^{\mathrm{i}s\Delta} \\
&-\mathrm{i}\int \mathrm{d}\eta\; X_\alpha X_\beta \bar{X}_{\dot{\alpha}}\bar{X}_{\dot{\beta}}X_a \sum_{n=1}^{\infty}\frac{(\mathrm{i}s)^n}{(n+1)!}L_\Delta^n(J^a)\;\mathrm{e}^{\mathrm{i}s\Delta}. \qquad (5.4.134)
\end{aligned}$$

Insertion of this into the differential equation (5.4.132) yields

$$\begin{aligned}
\frac{\mathrm{d}K_{\alpha\beta\dot{\alpha}\dot{\beta}}(z;s)}{\mathrm{d}s} + \frac{2}{s}K_{\alpha\beta\dot{\alpha}\dot{\beta}} = &\frac{\mathrm{i}}{2s}\int \mathrm{d}\eta\; X_\alpha X_\beta \left((\sigma^a)_{\gamma\dot{\alpha}}(\theta-\theta')^\gamma\right)\bar{X}_{\dot{\beta}}X_a \mathrm{e}^{\mathrm{i}s\Delta} \\
&\frac{\mathrm{i}}{2s}\int \mathrm{d}\eta\; X_\alpha X_\beta \left((\sigma^a)_{\gamma\dot{\beta}}(\theta-\theta')^\gamma\right)\bar{X}_{\dot{\alpha}}X_a \mathrm{e}^{\mathrm{i}s\Delta} \\
&-\mathrm{i}\int \mathrm{d}\eta\; X_\alpha X_\beta \bar{X}_{\dot{\alpha}}\bar{X}_{\dot{\beta}}X_a (J^a - X^a)\mathrm{e}^{\mathrm{i}s\Delta} \\
&-\mathrm{i}\int \mathrm{d}\eta\; X_\alpha X_\beta \bar{X}_{\dot{\alpha}}\bar{X}_{\dot{\beta}}X_a \sum_{n=1}^{\infty}\frac{(\mathrm{i}s)^n}{(n+1)!}L_\Delta^n(J^a)\;\mathrm{e}^{\mathrm{i}s\Delta} \\
&-\mathrm{i}\int \mathrm{d}\eta\; X_\alpha X_\beta \bar{X}_{\dot{\alpha}}\bar{X}_{\dot{\beta}}W^\gamma X_\gamma \mathrm{e}^{\mathrm{i}s\Delta}. \qquad (5.4.135)
\end{aligned}$$

We deduce from (5.4.130) and (5.4.132) that the left hand side of (5.4.135) has the power series expansion

$$\frac{\mathrm{d}K_{\alpha\beta\dot{\alpha}\dot{\beta}}(z;s)}{\mathrm{d}s} + \frac{2}{s}K_{\alpha\beta\dot{\alpha}\dot{\beta}} = \frac{1}{s^2}B_{\alpha\beta\dot{\alpha}\dot{\beta}} + \mathcal{O}(s^{-1}). \qquad (5.4.136)$$

Without the need for further computation, an inspection of the power series behaviour of the right and side of (5.4.135) reveals that $B_{\alpha\beta\dot{\alpha}\dot{\beta}}$ must vanish.

Example 3

Computing $K_{\alpha\beta\dot{\alpha}}(z;s)$ involves the identity

$$0 = \int \mathrm{d}\eta\; \frac{\partial}{\partial\bar{\kappa}_{\dot{\gamma}}}\left(X_\alpha X_\beta \bar{X}_{\dot{\alpha}}\bar{X}_{\dot{\beta}}\mathrm{e}^{\mathrm{i}s\Delta}\right) \qquad (5.4.137)$$

which, after the contraction of $\dot{\beta}$ and $\dot{\gamma}$, leads to

$$K_{\alpha\beta\dot{\alpha}}(z;s) = \int \mathrm{d}\eta\; X_\alpha X_\beta \bar{X}_{\dot{\alpha}}\bar{X}_{\dot{\beta}} \sum_{n=0}^{\infty}\frac{(\mathrm{i}s)^{n+1}}{(n+1)!}L_\Delta^n(\bar{W}^{\dot{\beta}})\;\mathrm{e}^{\mathrm{i}s\Delta}. \qquad (5.4.138)$$

To leading or subleading order, the summation can be truncated at $n = 0$ or 2 respectively. Alternatively one may have chosen to start with the identity

$$0 = \int \mathrm{d}\eta\; \frac{\partial}{\partial k_b}\left(X_\alpha X_\beta \bar{X}_{\dot{\alpha}}X_a\;\mathrm{e}^{\mathrm{i}s\Delta}\right) \qquad (5.4.139)$$

to obtain an expression for $K_{\alpha\beta\dot{\alpha}}(z;s)$, but this ends up being far more complicated primarily due to the k_a dependence of X_α and $\bar{X}_{\dot{\alpha}}$. In general, if the moment in question has less than four spinor indices, it is more convenient to choose the identities (5.4.120) or (5.4.121) rather than (5.4.119). However, if there are four spinor indices there is no choice and (5.4.119) must be used.

Summing from $n = 0$ to 2 in (5.4.138) one finds that to subleading order $K_{\alpha\beta\dot{\alpha}}(z;s)$ can be expressed in terms of

$$s^3 K_{\alpha\beta\dot{\alpha}\dot{\beta}ab}(z;s), \quad s^2 K_{\alpha\beta\dot{\alpha}\dot{\beta}}(z;s), \quad sK_{\alpha\beta\dot{\alpha}\dot{\beta}}(z;s) \quad \text{and} \quad sK_{\alpha\beta\dot{\alpha}}(z;s),$$

where only $K_{\alpha\beta\dot{\alpha}\dot{\beta}}(z;s)$ is actually required to subleading order. Notice that $K_{\alpha\beta\dot{\alpha}}(z;s)$ itself appears in this list (multiplied by s). This is a typical feature of this approach, and one can either rely on the fact that $K_{\alpha\beta\dot{\alpha}}(z;s)$ is already known to leading order, or bring it to the left hand side and premultiply both sides by an inverse operator (to appropriate order) to generate an new expression for $K_{\alpha\beta\dot{\alpha}}(z;s)$ in terms of only $K_{\alpha\beta\dot{\alpha}\dot{\beta}ab}(z;s)$ and $K_{\alpha\beta\dot{\alpha}\dot{\beta}}(z;s)$.

Example 4

As a final example, consider expanding the moment $K_{\alpha\beta}(z;s)$ to subleading order. In this case it is far more convenient to differentiate with respect to s. The power series expansion of $K_{\alpha\beta}(z;s)$ will look like

$$K_{\alpha\beta}(z;s) = A + s\,B + \mathcal{O}(s^2), \tag{5.4.140}$$

and so,

$$\frac{\mathrm{d}K_{\alpha\beta}(z;s)}{\mathrm{d}s} = B + \mathcal{O}(s). \tag{5.4.141}$$

After a little work one can establish

$$\frac{\mathrm{d}K_{\alpha\beta}(z;s)}{\mathrm{d}s} = \mathrm{i}K_{\alpha\beta a}{}^{a}(z;s) - \mathrm{i}\bar{W}^{\dot{\alpha}} K_{\alpha\beta\dot{\alpha}}(z;s). \tag{5.4.142}$$

So if both $K_{\alpha\beta ab}(z;s)$ and $K_{\alpha\beta\dot{\alpha}}(z;s)$, which are higher up the hierarchy, are known to subleading order (to order unity in s), $K_{\alpha\beta}(z;s)$ can immediately be evaluated to subleading order (ie identification of B). Additionally this generates the leading order identity

$$K_{\alpha\beta a}{}^{a}(z;s) = \bar{W}^{\dot{\alpha}} K_{\alpha\beta\dot{\alpha}}(z;s), \tag{5.4.143}$$

which serves as a useful consistency check. More specifically, inspection of (5.3.53) and (5.3.51) show that

$$K_{\alpha\beta a}{}^{a}(z;s) = \bar{W}^{\dot{\alpha}} K_{\alpha\beta\dot{\alpha}}(z;s) = \frac{4\mathrm{i}}{s}\,\varepsilon_{\alpha\beta}\,\bar{W}^2\,H \tag{5.4.144}$$

and indeed (5.4.143) is satisfied. For another detailed example of moment expansion see section C.3 of appendix C.

As can be seen from these examples, one particular advantage offered by this approach stems from the fact that there is often a freedom to choose between one of the identities (5.4.119), (5.4.120), (5.4.121) and (5.4.122) when expanding a particular moment. This provides one with a readily available means for checking intermediate results along the way to computing a DeWitt-Seeley coefficients. Alternatively, since the different choices often generate expressions which relate a particular moment to different sets of moments above it in the hierarchy, the freedom of choice can be used to generate consistency conditions like (5.4.143) amongst members in the hierarchy.

To illustrate this point, consider expanding $K_{\alpha\dot{\alpha}}(z;s)$ by using either

$$0 = \int \mathrm{d}\eta \, \frac{\partial}{\partial \kappa_\beta} \left(X_\alpha X_\beta \bar{X}_{\dot{\alpha}} \mathrm{e}^{\mathrm{i}s\Delta} \right) \tag{5.4.145}$$

or

$$0 = \int \mathrm{d}\eta \, \frac{\partial}{\partial \bar{\kappa}_{\dot{\beta}}} \left(X_\alpha \bar{X}_{\dot{\alpha}} \bar{X}_{\dot{\beta}} \mathrm{e}^{\mathrm{i}s\Delta} \right). \tag{5.4.146}$$

To first order, the former leads to

$$K_{\alpha\dot{\alpha}}(z;s) = \mathrm{i}sW^\beta K_{\alpha\beta\dot{\alpha}}(z;s) + \frac{s^2}{2}\{\bar{W}^{\dot{\beta}}, W^\beta\} K_{\alpha\beta\dot{\alpha}\dot{\beta}}(z;s), \tag{5.4.147}$$

while the latter leads to

$$K_{\alpha\dot{\alpha}}(z;s) = -\mathrm{i}s\bar{W}^{\dot{\beta}} K_{\alpha\dot{\alpha}\dot{\beta}}(z;s) - \frac{s^2}{2}\{\bar{W}^{\dot{\beta}}, W^\beta\} K_{\alpha\beta\dot{\alpha}\dot{\beta}}(z;s). \tag{5.4.148}$$

Equating these yields the first order relation

$$\bar{W}^{\dot{\beta}} K_{\alpha\dot{\alpha}\dot{\beta}}(z;s) + W^\beta K_{\alpha\beta\dot{\alpha}}(z;s) = \mathrm{i}s\{\bar{W}^{\dot{\beta}}, W^\beta\} K_{\alpha\beta\dot{\alpha}\dot{\beta}}(z;s), \tag{5.4.149}$$

which the reader can easily verify is satisfied by the first order moments (5.3.46)-(5.3.69) given earlier.

Ultimately one can generate a web of consistency checks and tests that actually makes it rather difficult to miss any errors in a calculation. To leading order these checks appear rather trivial, however at higher orders they become more complicated and prove to be useful in providing confidence in the final result.

Having summed the right hand side of (5.3.38) from $n = 1$ to 6 and expanded all surviving terms to order s^3, a_6 can finally be identified (and of course a_5 is also recovered). This computation is a very laborious task, and to this order so many terms are generated that it is no longer practical to compute by hand. The final result is calculated with the aid of the symbolic mathematical program *Mathematica*, which in itself is still a laborious task. The details are outlined in appendix C, and for full details see the attached CD. The final expression for $\mathrm{tr}_{\mathrm{Ad}}(a_6)$ is given below, and is a somewhat simplified version of that presented in [34]. The further simplifications are due mainly to integration by parts. Because of the size of the expression, and the fact that there are many equivalent ways of expressing the result, it is a significant challenge to find the most compact and symmetric looking form. By extensive use of commutation relations, equations of motion and the cyclicity of the trace, the result is brought into a manifestly real form involving only seven distinct types of terms, each listed schematically below (where G_{ab} was defined in (5.1.5)):

$$W^2\,\bar{W}^2\,\mathcal{D}_a^4,\qquad W^2\,\bar{W}^2\,G_{ab}\,\mathcal{D}_a^2,\qquad W^3\,\bar{W}^2\,\mathcal{D}_\alpha\,\mathcal{D}_a^2,\qquad W^2\,\bar{W}^3\,\bar{\mathcal{D}}_{\dot{\alpha}}\,\mathcal{D}_a^2,$$
$$W^3\,\bar{W}^3\,\mathcal{D}_\alpha\,\bar{\mathcal{D}}_{\dot{\alpha}},\qquad W^4\,\bar{W}^2\,\mathcal{D}_\alpha^2,\qquad W^2\,\bar{W}^4\,\bar{\mathcal{D}}_{\dot{\alpha}}^2\ .$$

Here, for example, $W^2\,\bar{W}^2\,\mathcal{D}_a^4$ is taken to mean terms which contain (some specific permutation and contraction of) two chiral superfield strengths, two antichiral superfield strengths and four spacetime covariant derivatives. The final result is

$$
\begin{aligned}
&\mathrm{tr}_{\mathrm{Ad}}(a_6) = \\
&\tfrac{1}{2}\tfrac{1}{7!}\Big(80(\mathcal{D}^a\mathcal{D}^bW^\alpha)\Big((\mathcal{D}_aW_\alpha)(\mathcal{D}_b\bar{W}_{\dot{\alpha}})\bar{W}^{\dot{\alpha}}+(\mathcal{D}_a\bar{W}_{\dot{\alpha}})(\mathcal{D}_b\bar{W}^{\dot{\alpha}})W_\alpha-(\mathcal{D}_a\bar{W}_{\dot{\alpha}})(\mathcal{D}_bW_\alpha)\bar{W}^{\dot{\alpha}}\Big)\\
&-24(\mathcal{D}^a\mathcal{D}^bW^\alpha)\Big((\mathcal{D}_aW_\alpha)\bar{W}_{\dot{\alpha}}(\mathcal{D}_b\bar{W}^{\dot{\alpha}})+(\mathcal{D}_a\bar{W}_{\dot{\alpha}})\bar{W}^{\dot{\alpha}}(\mathcal{D}_bW_\alpha)-(\mathcal{D}_a\bar{W}_{\dot{\alpha}})W_\alpha(\mathcal{D}_b\bar{W}^{\dot{\alpha}})\Big)\\
&-112(\mathcal{D}^a\mathcal{D}^bW^\alpha)\Big(W_\alpha(\mathcal{D}_a\bar{W}_{\dot{\alpha}})(\mathcal{D}_b\bar{W}^{\dot{\alpha}})+\bar{W}_{\dot{\alpha}}(\mathcal{D}_a\bar{W}^{\dot{\alpha}})(\mathcal{D}_bW_\alpha)-\bar{W}_{\dot{\alpha}}(\mathcal{D}_aW_\alpha)(\mathcal{D}_b\bar{W}^{\dot{\alpha}})\Big)\\
&+152G^{ab}W^\alpha\Big((\mathcal{D}_aW_\alpha)(\mathcal{D}_b\bar{W}_{\dot{\alpha}})\bar{W}^{\dot{\alpha}}+(\mathcal{D}_a\bar{W}_{\dot{\alpha}})(\mathcal{D}_b\bar{W}^{\dot{\alpha}})W_\alpha-(\mathcal{D}_a\bar{W}_{\dot{\alpha}})(\mathcal{D}_bW_\alpha)\bar{W}^{\dot{\alpha}}\Big)\\
&+256G^{ab}W^\alpha\Big((\mathcal{D}_aW_\alpha)\bar{W}_{\dot{\alpha}}(\mathcal{D}_b\bar{W}^{\dot{\alpha}})+(\mathcal{D}_a\bar{W}_{\dot{\alpha}})\bar{W}^{\dot{\alpha}}(\mathcal{D}_bW_\alpha)-(\mathcal{D}_a\bar{W}_{\dot{\alpha}})W_\alpha(\mathcal{D}_b\bar{W}^{\dot{\alpha}})\Big)\\
&-40G^{ab}W^\alpha\Big(W_\alpha(\mathcal{D}_a\bar{W}_{\dot{\alpha}})(\mathcal{D}_b\bar{W}^{\dot{\alpha}})+\bar{W}_{\dot{\alpha}}(\mathcal{D}_a\bar{W}^{\dot{\alpha}})(\mathcal{D}_bW_\alpha)-\bar{W}_{\dot{\alpha}}(\mathcal{D}_aW_\alpha)(\mathcal{D}_b\bar{W}^{\dot{\alpha}})\Big)\\
&+(\mathcal{D}_\alpha W^\beta)\Big(8(\mathcal{D}^aW^\alpha)(\mathcal{D}_aW_\beta)\bar{W}^2+40(\mathcal{D}^a\bar{W}_{\dot{\alpha}})(\mathcal{D}_aW^\alpha)W_\beta\bar{W}^{\dot{\alpha}}+92(\mathcal{D}^a\bar{W}_{\dot{\alpha}})(\mathcal{D}_a\bar{W}^{\dot{\alpha}})W^\alpha W_\beta\\
&\qquad+140(\mathcal{D}^aW^\alpha)(\mathcal{D}_a\bar{W}_{\dot{\alpha}})\bar{W}^{\dot{\alpha}}W_\beta-48(\mathcal{D}^aW^\alpha)(\mathcal{D}_a\bar{W}_{\dot{\alpha}})W_\beta\bar{W}^{\dot{\alpha}}-132(\mathcal{D}^a\bar{W}_{\dot{\alpha}})(\mathcal{D}_aW^\alpha)\bar{W}^{\dot{\alpha}}W_\beta\Big)\\
&+(\mathcal{D}_\alpha W^\beta)\Big(-8(\mathcal{D}^aW^\alpha)W_\beta(\mathcal{D}_a\bar{W}_{\dot{\alpha}})\bar{W}^{\dot{\alpha}}+8(\mathcal{D}^a\bar{W}_{\dot{\alpha}})W^\alpha(\mathcal{D}_aW_\beta)\bar{W}^{\dot{\alpha}}+36(\mathcal{D}^a\bar{W}_{\dot{\alpha}})\bar{W}^{\dot{\alpha}}(\mathcal{D}_aW^\alpha)W_\beta\\
&\qquad+36(\mathcal{D}^aW^\alpha)\bar{W}_{\dot{\alpha}}(\mathcal{D}_a\bar{W}^{\dot{\alpha}})W_\beta-44(\mathcal{D}^a\bar{W}_{\dot{\alpha}})W^\alpha(\mathcal{D}_a\bar{W}^{\dot{\alpha}})W_\beta\Big)\\
&+(\mathcal{D}_\alpha W^\beta)\Big(68W^\alpha(\mathcal{D}^aW_\beta)(\mathcal{D}_a\bar{W}_{\dot{\alpha}})\bar{W}^{\dot{\alpha}}+40\bar{W}_{\dot{\alpha}}(\mathcal{D}^aW^\alpha)(\mathcal{D}_aW_\beta)\bar{W}^{\dot{\alpha}}-68\bar{W}_{\dot{\alpha}}(\mathcal{D}^a\bar{W}^{\dot{\alpha}})(\mathcal{D}_aW^\alpha)W_\beta\\
&\qquad+40W^\alpha(\mathcal{D}^a\bar{W}_{\dot{\alpha}})(\mathcal{D}_a\bar{W}^{\dot{\alpha}})W_\beta-108W^\alpha(\mathcal{D}^a\bar{W}_{\dot{\alpha}})(\mathcal{D}_aW_\beta)\bar{W}^{\dot{\alpha}}+28\bar{W}_{\dot{\alpha}}(\mathcal{D}^aW^\alpha)(\mathcal{D}_a\bar{W}^{\dot{\alpha}})W_\beta\Big)\\
&+(\mathcal{D}_\alpha W^\beta)\Big(-156W^\alpha W_\beta(\mathcal{D}^a\bar{W}_{\dot{\alpha}})(\mathcal{D}_a\bar{W}^{\dot{\alpha}})+40\bar{W}_{\dot{\alpha}}W^\alpha(\mathcal{D}^aW_\beta)(\mathcal{D}_a\bar{W}^{\dot{\alpha}})+64\bar{W}^2(\mathcal{D}^aW^\alpha)(\mathcal{D}_aW_\beta)\\
&\qquad-52W^\alpha\bar{W}_{\dot{\alpha}}(\mathcal{D}^a\bar{W}^{\dot{\alpha}})(\mathcal{D}_aW_\beta)+116W^\alpha\bar{W}_{\dot{\alpha}}(\mathcal{D}^aW_\beta)(\mathcal{D}_a\bar{W}^{\dot{\alpha}})-104\bar{W}_{\dot{\alpha}}W^\alpha(\mathcal{D}^a\bar{W}^{\dot{\alpha}})(\mathcal{D}_aW_\beta)\Big)\\
&+(\mathcal{D}_\alpha W^\beta)\Big(-24(\mathcal{D}^aW^\alpha)W_\beta\bar{W}_{\dot{\alpha}}(\mathcal{D}_a\bar{W}^{\dot{\alpha}})-8(\mathcal{D}^a\bar{W}_{\dot{\alpha}})W^\alpha W_\beta(\mathcal{D}_a\bar{W}^{\dot{\alpha}})+32(\mathcal{D}^a\bar{W}_{\dot{\alpha}})\bar{W}^{\dot{\alpha}}W^\alpha(\mathcal{D}_aW_\beta)\\
&\qquad+32(\mathcal{D}^aW^\alpha)\bar{W}_{\dot{\alpha}}W_\beta(\mathcal{D}_a\bar{W}^{\dot{\alpha}})-24(\mathcal{D}^a\bar{W}_{\dot{\alpha}})W^\alpha\bar{W}^{\dot{\alpha}}(\mathcal{D}_aW_\beta)\Big)\\
&+(\mathcal{D}_\alpha W^\beta)\Big(-20W^\alpha(\mathcal{D}^aW_\beta)\bar{W}_{\dot{\alpha}}(\mathcal{D}_a\bar{W}^{\dot{\alpha}})+8\bar{W}_{\dot{\alpha}}(\mathcal{D}^aW^\alpha)W_\beta(\mathcal{D}_a\bar{W}^{\dot{\alpha}})+48\bar{W}_{\dot{\alpha}}(\mathcal{D}^a\bar{W}^{\dot{\alpha}})W^\alpha(\mathcal{D}_aW_\beta)\\
&\qquad+36W^\alpha(\mathcal{D}^a\bar{W}_{\dot{\alpha}})\bar{W}^{\dot{\alpha}}(\mathcal{D}_aW_\beta)+12W^\alpha(\mathcal{D}^a\bar{W}_{\dot{\alpha}})W_\beta(\mathcal{D}_a\bar{W}^{\dot{\alpha}})-56\bar{W}_{\dot{\alpha}}(\mathcal{D}^aW^\alpha)\bar{W}^{\dot{\alpha}}(\mathcal{D}_aW_\beta)\Big)\\
&+(\mathcal{D}_\alpha W^\beta)(\bar{\mathcal{D}}_{\dot{\alpha}}\bar{W}^{\dot{\beta}})\Big(-112W^\alpha W_\beta\bar{W}^{\dot{\alpha}}\bar{W}_{\dot{\beta}}+16\bar{W}^{\dot{\alpha}}W^\alpha W_\beta\bar{W}_{\dot{\beta}}+60\bar{W}^{\dot{\alpha}}\bar{W}_{\dot{\beta}}W^\alpha W_\beta\\
&\qquad+28W^\alpha\bar{W}^{\dot{\alpha}}\bar{W}_{\dot{\beta}}W_\beta+4W^\alpha\bar{W}^{\dot{\alpha}}W_\beta\bar{W}_{\dot{\beta}}+4\bar{W}^{\dot{\alpha}}W^\alpha\bar{W}_{\dot{\beta}}W_\beta\Big)\\
&+(\mathcal{D}_\alpha W^\beta)W^\alpha(\bar{\mathcal{D}}_{\dot{\alpha}}\bar{W}^{\dot{\beta}})\Big(-70W_\beta\bar{W}^{\dot{\alpha}}\bar{W}_{\dot{\beta}}-128\bar{W}^{\dot{\alpha}}W_\beta\bar{W}_{\dot{\beta}}-58\bar{W}^{\dot{\alpha}}\bar{W}_{\dot{\beta}}W_\beta\Big)\\
&+(\mathcal{D}_\alpha W^\beta)\bar{W}^{\dot{\alpha}}(\bar{\mathcal{D}}_{\dot{\alpha}}\bar{W}^{\dot{\beta}})\Big(50W^\alpha W_\beta\bar{W}_{\dot{\beta}}+156W^\alpha\bar{W}_{\dot{\beta}}W_\beta+50\bar{W}_{\dot{\beta}}W^\alpha W_\beta\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+(\mathcal{D}_\alpha W^\beta)\Big(-48W^\alpha W_\beta(\bar{\mathcal{D}}_{\dot\alpha}\bar{W}^{\dot\beta})\bar{W}^{\dot\alpha}\bar{W}_{\dot\beta} + 8\bar{W}^{\dot\alpha}\bar{W}_{\dot\beta}(\bar{\mathcal{D}}_{\dot\alpha}\bar{W}^{\dot\beta})W^\alpha W_\beta - 32W^\alpha\bar{W}^{\dot\alpha}(\bar{\mathcal{D}}_{\dot\alpha}\bar{W}^{\dot\beta})\bar{W}_{\dot\beta}W_\beta \\
&\quad +56\bar{W}^{\dot\alpha}W^\alpha(\bar{\mathcal{D}}_{\dot\alpha}\bar{W}^{\dot\beta})\bar{W}_{\dot\beta}W_\beta + 52\bar{W}^{\dot\alpha}W^\alpha(\bar{\mathcal{D}}_{\dot\alpha}\bar{W}^{\dot\beta})W_\beta\bar{W}_{\dot\beta} - 36W^\alpha\bar{W}^{\dot\alpha}(\bar{\mathcal{D}}_{\dot\alpha}\bar{W}^{\dot\beta})W_\beta\bar{W}_{\dot\beta}\Big) \\
&+(\mathcal{D}_\alpha W^\beta)(\mathcal{D}_\gamma W^\alpha)\Big(-939W_\beta W^\gamma\bar{W}^2 - 13W^\gamma W_\beta\bar{W}^2 - \bar{W}_{\dot\alpha}W_\beta W^\gamma\bar{W}^{\dot\alpha} + 557\bar{W}_{\dot\alpha}W^\gamma W_\beta\bar{W}^{\dot\alpha} \\
&\quad -295\bar{W}^2W_\beta W^\gamma + 15\bar{W}^2W^\gamma W_\beta - 61W_\beta\bar{W}^2W^\gamma + 85W^\gamma\bar{W}^2W_\beta + 59W_\beta\bar{W}_{\dot\alpha}W^\gamma\bar{W}^{\dot\alpha} \\
&\qquad -125W^\gamma\bar{W}_{\dot\alpha}W_\beta\bar{W}^{\dot\alpha} - 9\bar{W}_{\dot\alpha}W_\beta\bar{W}^{\dot\alpha}W^\gamma - 13\bar{W}_{\dot\alpha}W^\gamma\bar{W}^{\dot\alpha}W_\beta\Big) \\
&+(\mathcal{D}_\alpha W^\beta)W_\beta(\mathcal{D}_\gamma W^\alpha)\Big(875W^\gamma\bar{W}^2 + 59\bar{W}^2W^\gamma + 31\bar{W}_{\dot\alpha}W^\gamma\bar{W}^{\dot\alpha}\Big) \\
&+(\mathcal{D}_\alpha W^\beta)W^\gamma(\mathcal{D}_\gamma W^\alpha)\Big(-87W_\beta\bar{W}^2 - 231\bar{W}^2W_\beta - 3\bar{W}_{\dot\alpha}W_\beta\bar{W}^{\dot\alpha}\Big) \\
&+(\mathcal{D}_\alpha W^\beta)\bar{W}_{\dot\alpha}(\mathcal{D}_\gamma W^\alpha)\Big(67W_\beta W^\gamma\bar{W}^{\dot\alpha} - 13W^\gamma W_\beta\bar{W}^{\dot\alpha} + 107\bar{W}^{\dot\alpha}W_\beta W^\gamma - 69\bar{W}^{\dot\alpha}W^\gamma W_\beta \\
&\quad -169W_\beta\bar{W}^{\dot\alpha}W^\gamma + 485W^\gamma\bar{W}^{\dot\alpha}W_\beta\Big)\Big) + c.c.
\end{aligned}
$$

Again the corresponding contribution to the effective action can be obtained by inspection. Extraction of the component form of a_6 is now in principle straightforward, and contains F^6-type field strength terms. These are potentially important for comparison with recent string theoretic results [101, 102, 103], as are the recently computed two-loop Abelian F^6 terms [100]. A detailed comparison remains to be carried out.

5.4.3 Comparison with literature

It is possible, however, to perform a quick yet highly non-trivial test on the F^6 results. The form of the one-loop effective action for $\mathcal{N} = 4$ super Yang-Mills theory is known in the Abelian case in the constant field strength approximation [87, 88, 216, 217], and the coefficient of F^6 is zero. Inspection of $\mathrm{tr}_{\mathrm{Ad}}(a_6)$ reveals that in the Abelian limit, F^6 contributions with constant field strength can come only from terms of the form: $W^4\ \bar{W}^2\ \mathcal{D}^2_\alpha$ and $W^2\ \bar{W}^4\ \bar{\mathcal{D}}^2_{\dot\alpha}$, which encompass the last seven lines in $\mathrm{tr}_{\mathrm{Ad}}(a_6)$ as it is given. The terms of the form $W^3\ \bar{W}^3\ \mathcal{D}_\alpha\ \bar{\mathcal{D}}_{\dot\alpha}$ (for example $(\mathcal{D}_\alpha W^\beta)(\bar{\mathcal{D}}_{\dot\alpha}\bar{W}^{\dot\beta})W^\alpha W_\beta\bar{W}^{\dot\alpha}\bar{W}_{\dot\beta}$) all clearly vanish on-shell since in the Abelian case

$$
W_\alpha W_\beta = \frac{1}{2}\varepsilon_{\alpha\beta}W^2 \qquad\qquad \bar{W}_{\dot\alpha}\bar{W}_{\dot\beta} = -\frac{1}{2}\varepsilon_{\alpha\beta}\bar{W}^2. \tag{5.4.150}
$$

Employing the identities (5.4.150), in the Abelian limit with constant field strength $\mathrm{tr}_{\mathrm{Ad}}(a_6)$ explicitly reduces to

$$
\begin{aligned}
\frac{1}{4}\frac{1}{7!}(\mathcal{D}_\alpha W^\beta)(\mathcal{D}_\beta W^\alpha)W^2\bar{W}^2\Big[&(939 - 13 + 1 + 557 + 295 + 15 + 85 \\
&\qquad + 61 + 59 + 125 - 9 + 13) \\
&+ (-875 - 59 + 31)
\end{aligned}
$$

$$+ (-87 - 231 + 3)$$
$$+ (-67 - 13 - 107 - 69 - 169 - 485)\Big] + c.c.$$
$$= \frac{1}{4}\frac{1}{7!}(\mathcal{D}_\alpha W^\beta)(\mathcal{D}_\beta W^\alpha)W^2\bar{W}^2\Big[2128 - 903 - 315 - 910\Big] + c.c. \qquad (5.4.151)$$

The terms in the square parentheses indeed sum to zero, and one finds non-trivial cancellation consistent with [87, 88, 217].

Finally, we can attempt to make some contact with the results from string theory and the $(\partial F)^4$ derivative corrections of the Born-Infeld Lagrangian. In [40], and later found by other means [44, 101], it was shown that up to a multiplicative constant τ these corrections are given by

$$\tau\left((\partial^e F^{ab})(\partial_e F_{ba})(\partial^f F^{cd})(\partial_f F_{dc}) + 2(\partial^e F^{ab})(\partial^f F_{ba})(\partial_e F^{cd})(\partial_f F_{dc})\right.$$
$$\left. -4(\partial^e F^{ab})(\partial^f F_{bc})(\partial_e F^{cd})(\partial_f F_{da}) - 8(\partial^e F^{ab})(\partial_e F_{bc})(\partial^f F^{cd})(\partial_f F_{da})\right), \qquad (5.4.152)$$

modulo terms proportional to the equations of motion $\partial^a F_{ab} = 0$ (ie terms removable by field redefinitions). These four terms, up to terms proportional to the equations of motion, integration by parts and the Bianchi identity, provide a basis for all possible $(\partial F)^4$ tensor structures [44].

Examination of $\mathrm{tr}(a_6)$ reveals that, in the Abelian limit, only those terms given in the first three lines will contribute $(\partial F)^4$ terms to the effective action of $\mathcal{N} = 4$ super Yang-Mills theory at one-loop. In this limit the corresponding pieces of $\mathrm{tr}(a_6)$ reduce to:

$$-\frac{1}{180}(\partial^a\partial^b W^\alpha)\Big(W_\alpha(\partial_a \bar{W}_{\dot\alpha})(\partial_b \bar{W}^{\dot\alpha}) + 2(\partial_a W_\alpha)(\partial_b \bar{W}_{\dot\alpha})\bar{W}^{\dot\alpha}\Big) + c.c. \qquad (5.4.153)$$

Projecting out the bosonic field strength part of this gives

$$-\frac{1}{16(6!)}\Big(\mathrm{tr}(\sigma^a\tilde\sigma^b\sigma^c\tilde\sigma^d)\mathrm{tr}(\tilde\sigma^e\sigma^f\tilde\sigma^g\sigma^h) + c.c\Big)$$
$$(\partial^m\partial^n F^{ab})\Big(F_{cd}(\partial_m F_{ef})(\partial_n F_{gh}) + 2(\partial_m F_{cd})(\partial_n F_{ef})F_{gh}\Big) \qquad (5.4.154)$$

which reduces to[14]

$$-\frac{1}{360}(\partial^e\partial^f F^{ab})\left(8F_{bc}(\partial_e F^{cd})(\partial_f F_{da}) + 4(\partial_e F_{bc})F^{cd}(\partial_f F_{da})\right.$$
$$\left. -F_{ba}(\partial_e F^{cd})(\partial_f F_{dc}) - 2(\partial_e F_{ba})(\partial_f F^{cd})F_{dc}\right). \qquad (5.4.155)$$

[14]See the attached CD for a derivation.

It is a simple matter to show using the equations of motion, the Bianchi identity and integration by parts that [44]

$$(\partial^e\partial^f F^{ab})F_{bc}(\partial_e F^{cd})(\partial_f F_{da}) = -\frac{1}{2}(\partial^e F^{ab})(\partial^f F_{bc})(\partial_e F^{cd})(\partial_f F_{da}) \qquad (5.4.156)$$

$$(\partial^e\partial^f F^{ab})(\partial_e F_{bc})F^{cd}(\partial_f F_{da}) = \frac{1}{2}(\partial^e F^{ab})(\partial^f F_{bc})(\partial_e F^{cd})(\partial_f F_{da}) - (\partial^e F^{ab})(\partial_e F_{bc})(\partial^f F^{cd})(\partial_f F_{da}) \qquad (5.4.157)$$

$$(\partial^e\partial^f F^{ab})F_{ba}(\partial_e F^{cd})(\partial_f F_{dc}) = \frac{1}{2}(\partial^e F^{ab})(\partial_e F_{ba})(\partial^f F^{cd})(\partial_f F_{dc}) - (\partial^e F^{ab})(\partial^f F_{ba})(\partial_e F^{cd})(\partial_f F_{dc}) \qquad (5.4.158)$$

$$(\partial^e\partial^f F^{ab})(\partial_e F_{ba})(\partial_f F^{cd})F_{dc} = -\frac{1}{2}(\partial^e F^{ab})(\partial_e F_{ba})(\partial^f F^{cd})(\partial_f F_{dc}) \qquad (5.4.159)$$

and so (5.4.155) reduces to

$$-\frac{1}{6!}\big((\partial^e F^{ab})(\partial_e F_{ba})(\partial^f F^{cd})(\partial_f F_{dc}) + 2(\partial^e F^{ab})(\partial^f F_{ba})(\partial_e F^{cd})(\partial_f F_{dc}) -4(\partial^e F^{ab})(\partial^f F_{bc})(\partial_e F^{cd})(\partial_f F_{da}) - 8(\partial^e F^{ab})(\partial_e F_{bc})(\partial^f F^{cd})(\partial_f F_{da})\big)\,. \qquad (5.4.160)$$

Up to an overall multiplicative constant this is precisely the expression (5.4.152), the $(\partial F)^4$ corrections to the Born-Infeld lagrangian [40, 44, 101]. As noted in [57], which is clear from this work, that this $(\partial F)^4$ term forms part of supersymmetric invariant, and has emerged in other contexts [218, 219].

If one assumes that maximal supersymmetry provides a sufficiently strong constraint for uniquely fixing such terms in the full effective action for $\mathcal{N} = 4$ super Yang-Mills theory at order F^6, the above comparison would suggest that either (up to field redefinitions): there are no higher loop $(\partial F)^4$ contributions to the effective action of Abelian $\mathcal{N} = 4$ super Yang-Mills theory; or that the total $(\partial F)^4$ contribution from higher loops is proportional to precisely the linear combination of superfield structures given in (5.4.153).

Chapter 6

One-loop effective action for $\mathcal{N} = 2$ super Yang-Mills theory

This chapter is based to the published work [35], and deals with extending the $\mathcal{N} = 4$ results computed in the previous chapter, to evaluate the one-loop effective action to order F^5 for arbitrary $\mathcal{N} = 2$ super Yang-Mills theories. The computation parallels that of the $\mathcal{N} = 4$ case, being an application of the modified Gaussian approach with a few adjustments for application to chiral subspace. At the component level the results are shown to be in agreement with existing literature, and will be useful for future tests of the AdS/CFT correspondence for theories other than $\mathcal{N} = 4$ super Yang-Mills theories which possess supergravity duals. For related discussions, material and different approaches to analogous problems also see [220, 221, 222, 223, 199, 224, 225, 106, 97, 226, 227, 228, 197]

6.1 The chiral heat kernel

We saw earlier in subsection 3.2.5, that to evaluate the one-loop effective action for arbitrary $\mathcal{N} = 2$ super Yang-Mills theories, we need to compute

$$\Gamma^{(1)}[w_B, \bar{w}_B] = \frac{\mathrm{i}}{2} \ln \mathrm{sDet}\Box - 2\Gamma^{(1)}_{\Phi,0,\mathrm{Ad}} + \Gamma^{(1)}_{\Phi,\mathcal{M},R\oplus R_c} \tag{6.1.1}$$

where $\ln \mathrm{sDet}\Box$ was computed to order F^6 in the last chapter, and

$$\Gamma^{(1)}_{\Phi,m,\mathcal{R}} = \frac{\mathrm{i}}{4} \ln \mathrm{sDet}(\Box_+ - m^2 \mathbb{1}_\mathcal{R}) + \frac{\mathrm{i}}{4} \ln \mathrm{sDet}(\Box_- - m^2 \mathbb{1}_\mathcal{R}) \tag{6.1.2}$$

with $\Box_+$ and $\Box_-$ given by (3.2.117). We therefore need only compute, $\Gamma^{(1)}_{\Phi,m,\mathcal{R}}$, the one-loop effective action for chiral fields in the presence of a super Yang-Mills background in some real representation $\mathcal{R}$ of the gauge group, to assemble the effective action for $\mathcal{N}=2$ super Yang-Mills to a given order. Here we work to order F^5. In terms of heat kernels, this amounts to computing

$$\Gamma^{(1)}_{\Phi,m,\mathcal{R}} = \frac{\mu^{2\rho}}{4}\int_0^\infty \mathrm{d}s\,\mathrm{i}s^{\rho-1}e^{\mathrm{i}m^2 s}\big(K_+(s)+K_-(s)\big). \qquad (6.1.3)$$

In this expression: μ and ρ are the renormalization point and regularization parameter respectively; the mass m is either an explicit mass (as in the case of the hypermultiplet mass $\mathcal{M}$ in (3.2.121)), or a infrared regulator for massless chiral scalars (as in the case of ϕ in (3.2.121) and the ghosts)[1]; and $K_+(t)$ and $K_-(t)$ are the functional traces of the chiral and antichiral heat kernels respectively, which are defined by:

$$K_\pm(s) = \mathrm{tr}_\mathcal{R}\int \mathrm{d}^6 z_\pm \int \mathrm{d}^6 z'_\pm \delta_\pm(z,z')\; e^{\mathrm{i}s\Box_\pm}\delta_\pm(z,z') \equiv \mathrm{tr}_\mathcal{R}\int \mathrm{d}^6 z_\pm\, K_\pm(z;s). \quad (6.1.4)$$

Here $\delta_\pm(z,z')$ are the (anti)chiral delta functions,

$$\delta_+(z,z') = -\frac{1}{4}\,\bar{\mathcal{D}}^2\mathbb{1}\delta^{(8)}(z,z') \qquad (6.1.5)$$

$$\delta_-(z,z') = -\frac{1}{4}\,\mathcal{D}^2\mathbb{1}\delta^{(8)}(z,z') \qquad (6.1.6)$$

$$\delta^{(8)}(z,z') = \delta^{(4)}(x,x')\delta^{(2)}(\theta-\theta')\delta^{(2)}(\bar\theta-\bar\theta'), \qquad (6.1.7)$$

and for brevity $\mathrm{d}z_\pm$ has been used to denote the integration measure over the (anti)chiral subspace of full superspace.

One can show that (for example see [127])

$$K_+(t) = K_-(t) \qquad (6.1.8)$$

and so

$$\Gamma^{(1)}_{\Phi,m,\mathcal{R}} = \frac{\mu^{2\rho}}{2}\int_0^\infty \mathrm{d}s\,(\mathrm{i}s)^{\rho-1}e^{\mathrm{i}m^2 s}K_+(s). \qquad (6.1.9)$$

requiring computation of only the chiral kernel.

For our purposes, as before, it suffices to consider an on-shell background, $\mathcal{D}^\alpha W_\alpha = \bar{\mathcal{D}}_{\dot\alpha}\bar W^{\dot\alpha} = 0$, so that

$$\Box_+ = \frac{1}{16}\bar{\mathcal{D}}^2\mathcal{D}^2 = \mathcal{D}^a\mathcal{D}_a - W^\alpha\mathcal{D}_\alpha \qquad (6.1.10)$$

acting on chiral superfields.

[1] One may also have introduced a background for the adjoint scalar Φ in (2.2.100) taking values in the Cartan subalgebra, which then generates masses for the vector multiplet, the hypermultiplet and ghosts. For a discussion see [197].

The chiral kernel's asymptotic expansion in s is expressed in terms of the DeWitt-Seeley coefficients $a_n^+(z)$, which are chiral superfields and at the component level contain bosonic field strength terms of the form F^n:

$$K_+(z;s) = \frac{\mathrm{i}}{(4\pi\mathrm{i}s)^2}\sum_{n=0}^{\infty}(\mathrm{i}s)^n a_n^+(z), \qquad a_0^+(z) = a_1^+(z) = 0. \tag{6.1.11}$$

Prior to the publication [35], only the first non-trivial coefficient a_2^+ was known in superfield form in the non-Abelian case [229, 230, 231, 232]:

$$a_2^+ = W^2. \tag{6.1.12}$$

Evaluating $\Gamma^{(1)}_{\chi,m,\mathcal{R}}$ therefore amounts to computing the DeWitt-Seeley coefficients:

$$\Gamma^{(1)}_{\chi,m,\mathcal{R}} = \frac{1}{32\pi^2\rho}\int \mathrm{d}^6z\, \mathrm{tr}_{\mathcal{R}}(a_2^+) + \frac{1}{32\pi^2}\sum_{n=3}^{\infty}\frac{(n-3)!}{m^{2n-4}}\int \mathrm{d}^6z\, \mathrm{tr}_{\mathcal{R}}(a_n^+). \tag{6.1.13}$$

The coefficient a_2^+ clearly provides information about divergences of the theory, and we see from (3.2.122) that $\mathcal{N}=2$ super Yang-Mills will be ultra-violet finite at one-loop provided

$$2\,\mathrm{tr}_{\mathrm{Ad}}\, W^2 = \mathrm{tr}_{R\oplus R_c}\, W^2. \tag{6.1.14}$$

This is the well known result [233].

It turns out that the coefficients a_n^+ with $n \geq 3$ are expressible in terms of $\bar{\mathcal{D}}^2$ acting on field strengths and their covariant derivatives, and so this allows the second term on the right hand side of (6.1.13) to be lifted to a gauge-invariant superfunctional on full superspace. This is easily proven as follows.

By differentiating the kernel $K_+(z;s)$ with respect to s, one observes that:

$$\begin{aligned}
\frac{\mathrm{d}K_+(z;s)}{\mathrm{d}s} &= \frac{\mathrm{i}}{16}\int \mathrm{d}^6z'_+\,\delta_+(z,z')\bar{\mathcal{D}}^2\mathcal{D}^2\mathrm{e}^{\mathrm{i}s\Box_+}\delta_+(z,z') \\
&= \frac{\mathrm{i}}{16}\bar{\mathcal{D}}^2\left(\int \mathrm{d}^6z'_+\,\delta_+(z,z')\mathcal{D}^2\mathrm{e}^{\mathrm{i}s\Box_+}\delta_+(z,z')\right) \\
&= \frac{\mathrm{i}}{16}\bar{\mathcal{D}}^2\left(\lim_{z'\to z}\mathcal{D}^2\mathrm{e}^{\mathrm{i}s\Box_+}\delta_+(z,z')\right),
\end{aligned} \tag{6.1.15}$$

since

$$\bar{\mathcal{D}}_{\dot{\alpha}}\delta_+(z,z') = 0. \tag{6.1.16}$$

On the other hand, (6.1.11) yields

$$\frac{\mathrm{d}K_+(z;s)}{\mathrm{d}s} = -\frac{1}{16\pi^2}\sum_{n=3}^{\infty}(n-2)(\mathrm{i}s)^{n-3}a_n^+(z). \tag{6.1.17}$$

Comparison of (6.1.15) and (6.1.17) demonstrates that the DeWitt-Seeley coefficients other than a_2^+ are expressible in the desired form.

At this stage it is convenient to introduce a new set of coefficients by writing $\lim_{z'\to z} \mathcal{D}^2 \mathrm{e}^{\mathrm{i}s\Box_+}\delta_+(z,z')$ as an asymptotic series,

$$\lim_{z'\to z} \mathcal{D}^2 \mathrm{e}^{\mathrm{i}s\Box_+}\delta_+(z,z') = \frac{\mathrm{i}}{(4\pi \mathrm{i}s)^2}\sum_{n=0}^{\infty}(\mathrm{i}s)^n c_n(z), \tag{6.1.18}$$

where a relatively simple computation reveals the first non-trivial coefficients are given by

$$c_0 = -4\,\mathbb{1}, \qquad c_1 = 0, \tag{6.1.19}$$

whilst comparison of (6.1.15), (6.1.17) and (6.1.18) yields

$$a_n^+(z) = \frac{1}{16(n-2)}\bar{\mathcal{D}}^2(c_{n-1}(z)) \qquad n \geq 3. \tag{6.1.20}$$

The effective action can then be written as

$$\Gamma^{(1)}_{\chi,m,\mathcal{R}} = \frac{1}{32\pi^2\rho}\int \mathrm{d}^6 z\, \mathrm{tr}_{\mathcal{R}}(W^2) - \frac{1}{128\pi^2}\sum_{n=3}^{\infty}\frac{(n-3)!}{(n-2)m^{2n-4}}\int \mathrm{d}^8 z\, \mathrm{tr}_{\mathcal{R}}(c_{n-1}), \tag{6.1.21}$$

the second term now being expressed on full superspace. At the component level c_n for $n \geq 2$ contains bosonic field strengths of the form F^{n+1}.

The determination of the effective action is thereby reduced to computing the new coefficients c_n, which can of course be obtained by the same techniques used for computing DeWitt-Seeley coefficients. If desired, the DeWitt-Seeley coefficients themselves can be recovered through identity (6.1.20), which is nothing more than a projection onto the chiral subspace. Computationally it turns out to be more efficient to compute the new coefficients, rather than attempt to compute the a_n^+ and rearrange them into expressions on full superspace, which in itself is a non-trivial task.

Since we wish to compute the effective action to order F^5, we will be required to evaluate c_2, c_3 and c_4. As a brief look ahead, recall that the expansion of the heat kernel in the $\mathcal{N}=4$ super Yang-Mills case was

$$K(z;s) = -\frac{\mathrm{i}s^2 a_4}{(4\pi)^2} + \frac{s^3 a_5}{(4\pi)^2} + \ldots \tag{6.1.22}$$

Computing the F^5 terms (ie a_5) in that case merely involved the determination of the second non-trivial coefficient of $K(z;s)$. In the present chiral case, where the power series behaviour of the associated kernel has the form (6.1.18), one is forced to compute the fourth non-trivial coefficient to acquire contributions to the same order. For this reason one may anticipate that the computation of the F^5 terms is much more difficult in the latter situation, which is in fact exactly what happens. It turns out that the computation of c_3 and c_4 is of comparable difficultly to the computation of a_5 and a_6 respectively as presented in the previous chapter. At first glance this seems quite surprising given the very simple basis we saw for the F^4 and F^5 structures in superspace[2] (see equations ((5.1.18)) and ((5.3.71))).

6.2 Details of the computation

6.2.1 The general scheme

Introducing a plane wave basis for the chiral delta function[3],

$$\delta_+(z,z') = 4\mathbb{1}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{e}^{\mathrm{i}k^a\omega_a}\int\mathrm{d}^2\kappa\;e^{\mathrm{i}\kappa^\alpha(\theta-\theta')_\alpha} \tag{6.2.23}$$

where[4]

$$\omega_a = x_a - x'_a - i\theta\sigma_a\bar{\theta}' + i\theta'\sigma_a\bar{\theta}. \tag{6.2.24}$$

In the coincidence limit, $\mathcal{D}^2\mathrm{e}^{\mathrm{i}s\Box_+}\delta_+(z,z')$ becomes

$$\lim_{z'\to z}\mathcal{D}^2\mathrm{e}^{\mathrm{i}s\Box_+}\delta_+(z,z') = K_+{}^\alpha{}_\alpha(z;s) = \int\mathrm{d}\eta_+\;\tilde{X}^\alpha\tilde{X}_\alpha\mathrm{e}^{\mathrm{i}s\Delta}, \tag{6.2.25}$$

with the definitions

$$\tilde{X}_a = \mathcal{D}_a + \mathrm{i}k_a \qquad\qquad \tilde{X}_\alpha = \mathcal{D}_\alpha + \mathrm{i}\kappa_\alpha \tag{6.2.26}$$

$$\Delta_+ = \tilde{X}^a\tilde{X}_a - W^\alpha\tilde{X}_\alpha \tag{6.2.27}$$

$$\int\mathrm{d}\eta_+ = 4\int\frac{\mathrm{d}^4k}{(2\pi)^4}\int\mathrm{d}^2\kappa \tag{6.2.28}$$

and with the usual notation

$$K_{+A_1A_2\ldots A_n}(z;s) = \int\mathrm{d}\eta_+\;\tilde{X}_{A_1}\tilde{X}_{A_2}\ldots\tilde{X}_{A_n}e^{\mathrm{i}s\Delta_+} \tag{6.2.29}$$

for the moments of the kernel. Note that there is also a shift $-k_{\alpha\dot{\alpha}}(\bar{\theta}-\bar{\theta}')^{\dot{\alpha}}$ in $\mathcal{D}_\alpha$ which always vanishes in the coincidence limit since there are no $\bar{\mathcal{D}}_{\dot{\alpha}}$ operators present. The $\tilde{X}$'s satisfy the algebra

$$\{\tilde{X}_\alpha,\tilde{X}_\beta\} = 0, \qquad [\tilde{X}_a,\tilde{X}_b] = G_{ab}, \qquad [\tilde{X}_\alpha,\tilde{X}_a] = \mathrm{i}(\sigma_a)_{\alpha\dot{\alpha}}\bar{W}^{\dot{\alpha}}. \tag{6.2.30}$$

[2]Unmentioned in the preceding chapter, due to its relative unimportance there, is the remarkable fact the two terms $\mathrm{tr}_{\mathcal{R}}(W^2\bar{W}^2)$ and $\mathrm{tr}_{\mathcal{R}}(W^\alpha\bar{W}_{\dot{\alpha}}W_\alpha\bar{W}^{\dot{\alpha}})$ (modulo total derivative terms) form a basis for on-shell F^4 structures in superspace, (as do pure F^4 turns in $\mathbb{M}^4$). This is merely a consequence of the identities (5.3.70).

[3]From here onward we work in the chiral representation.

[4]Note that although ω_a is not itself chiral, $\bar{\mathcal{D}}_{\dot{\alpha}}(\omega_a) = -\mathrm{i}(\sigma_a)_{\alpha\dot{\alpha}}(\theta-\theta')^\alpha$, the entire delta function is annihilated by $\bar{\mathcal{D}}_{\dot{\alpha}}$ since $(\theta-\theta')^3 = 0$.

In this notation the power series (6.1.18) is:

$$K_{+}{}^{\alpha}{}_{\alpha}(z;s) = \frac{\mathrm{i}}{(4\pi \mathrm{i}s)^2} \sum_{n=0}^{\infty} (\mathrm{i}s)^n c_n(z). \tag{6.2.31}$$

Differentiating $K_{+}{}^{\alpha}{}_{\alpha}(z;s)$ with respect to s yields the differential equation

$$\frac{\mathrm{d}K_{+}{}^{\alpha}{}_{\alpha}(z;s)}{\mathrm{d}s} = \mathrm{i}K_{+}{}^{\alpha}{}_{\alpha}{}^{a}{}_{a}(z;t). \tag{6.2.32}$$

As before we use the identity

$$0 = \int \mathrm{d}\eta_+ \, \frac{\partial}{\partial k_b} \left(\tilde{X}_\alpha \tilde{X}_\beta \tilde{X}_a \mathrm{e}^{\mathrm{i}s\Delta_+} \right) \tag{6.2.33}$$

to establish

$$K_{+}{}^{\alpha}{}_{\alpha}{}^{a}{}_{a}(z;s) = \frac{2\mathrm{i}}{s} K_{+}{}^{\alpha}{}_{\alpha}(z;s) - \int \mathrm{d}\eta_+ \, \tilde{X}^\alpha \tilde{X}_\alpha \tilde{X}^a \sum_{n=1}^{\infty} \frac{(\mathrm{i}s)^n}{(n+1)!} L^n_{\Delta_+}(\tilde{X}_a) \, \mathrm{e}^{\mathrm{i}s\Delta_+}. \tag{6.2.34}$$

Thus the differential equation (6.2.32), becomes:

$$\frac{\mathrm{d}K_{+}{}^{\alpha}{}_{\alpha}(z;s)}{\mathrm{d}s} + \frac{2}{s} K_{+}{}^{\alpha}{}_{\alpha}(z;s) = -\mathrm{i} \int \mathrm{d}\eta_+ \, \tilde{X}^\alpha \tilde{X}_\alpha \tilde{X}^a \sum_{n=1}^{\infty} \frac{(\mathrm{i}s)^n}{(n+1)!} L^n_{\Delta_+}(\tilde{X}_a) \, \mathrm{e}^{\mathrm{i}s\Delta_+}, \tag{6.2.35}$$

where the left hand side in terms of the expansion (6.2.31) is

$$\frac{\mathrm{d}K_{+}{}^{\alpha}{}_{\alpha}(z;s)}{\mathrm{d}s} + \frac{2}{s} K_{+}{}^{\alpha}{}_{\alpha}(z;s) = -\frac{1}{16\pi^2} \sum_{n=0}^{\infty} n (\mathrm{i}s)^{n-3} c_n(z). \tag{6.2.36}$$

As expected the differential equation yields an expansion where the first non-trivial coefficient $c_0(z)$ is absent. The objective now becomes to determine the coefficients $c_n(z)$ by expanding the right hand side of (6.2.35) in a power series in s, and identifying it with the right hand side of (6.2.36).

We will make repeated use of the following commutation relations:

$$\begin{aligned}
[\Delta_+, \tilde{X}_a] &= 2G^b{}_a \tilde{X}_b + (\mathcal{D}_a W^\alpha)\tilde{X}_\alpha + \mathrm{i}(\sigma_a)_{\alpha\dot{\alpha}} \bar{W}^{\dot{\alpha}} W^\alpha \\
[\Delta_+, \tilde{X}_\alpha] &= (\mathcal{D}_\alpha W^\beta)\tilde{X}_\beta - 2\mathrm{i}(\sigma^a)_{\alpha\dot{\alpha}} \bar{W}^{\dot{\alpha}} \tilde{X}_a \\
[\Delta_+, Y] &= (\mathcal{D}^a \mathcal{D}_a Y) + 2(\mathcal{D}^a Y)\tilde{X}_a - W^\alpha (\mathcal{D}_\alpha Y) - (-1)^{\varepsilon(Y)} \left[W^\alpha, Y \right\} \tilde{X}_\alpha.
\end{aligned} \tag{6.2.37}$$

Comparing these with the equivalent relations in the previous chapter (5.2.33), it is clear that that the absence of an operator like $\tilde{\bar{X}}_{\dot{\alpha}}$ introduces complication rather than any simplification.

As before these relations indicate that summation will generate a series of moments of the form $K_{+A_1 \ldots A_i}(z;s)$ as defined in (6.2.29). Furthermore, it is not difficult to show that to order n in this summation, the moments generated have at most $(n+3)$ indices. In this case it is convenient to place these indices in a specific order: first spinor, then spacetime. This can be achieved through the commutation relations (2.2.59). With such an ordering the leading term in a moment's asymptotic power series has the following behaviour:

$$K_{+A_1\ldots A_{q+p}}(z;s) \sim \frac{1}{s^2}\left(\frac{1}{s}\right)^{\left[\frac{p}{2}\right]} s^{2-q} = s^{-q-\left[\frac{p}{2}\right]} \qquad q \leq 2 \tag{6.2.38}$$

where $K_{+A_1\ldots A_{q+p}}(z;s)$ has q undotted spinor indices, p vector indices and $[\frac{p}{2}]$ denotes the largest integer part of $\frac{p}{2}$. Moments with greater than two undotted spinor indices vanish since $\tilde{X}_\alpha \tilde{X}_\beta \tilde{X}_\gamma = 0$.

From these considerations, and by comparison with (6.2.36), the summation in (6.2.35) truncates at $n = 2k-1$ when evaluating $c_k(z)$ for $k \geq 2$. Moreover, it turns out the last term in this truncated summation always vanishes due to the fact that it takes the form

$$-\,\mathrm{i}\frac{(2\mathrm{i}s)^{2k-1}}{(2k)!}(\mathcal{D}^{a_1}\mathcal{D}^{a_2}\ldots\mathcal{D}^{a_{2k-2}}G^{a_{2k-1}a_{2k}})K_{+\ \ \alpha a_1 a_2\ldots a_{2k}}^{\ \ \alpha}(z;s) \qquad k \geq 2, \tag{6.2.39}$$

where the moment is only ever required to leading order in its power series in s. To this order the moment is always totally symmetric in its spacetime indices, whereas G is antisymmetric. Consequently all such terms vanish[5], and when evaluating $c_k(z)$ the summation truncates at $n = 2k-2$.

Computing moments which result from the summation in (6.2.35) to appropriate order in s is achieved either through direct expansion of the moment's exponential, or iteratively through the use of the identities

$$0 = \int \mathrm{d}\eta_+ \frac{\partial}{\partial k_b}\left(\tilde{X}_{A_1}\ldots\tilde{X}_{A_n}\mathrm{e}^{\mathrm{i}s\Delta_+}\right), \tag{6.2.40}$$

$$0 = \int \mathrm{d}\eta_+ \frac{\partial}{\partial \kappa_\alpha}\left(\tilde{X}_{A_1}\ldots\tilde{X}_{A_n}\mathrm{e}^{\mathrm{i}s\Delta_+}\right) \tag{6.2.41}$$

and

$$\frac{\mathrm{d}^m K_{+A_1\ldots A_n}(z;t)}{\mathrm{d}s^m} = \int \mathrm{d}\eta_+ \, \tilde{X}_{A_1}\ldots\tilde{X}_{A_n}(\mathrm{i}\Delta_+)^m e^{\mathrm{i}s\Delta_+}. \tag{6.2.42}$$

As before these are used to express the desired moment in terms of moments which are easier to compute, which lie higher up the 'moment hierarchy'.

As noted above, there is a rough correspondence in the amount of work involved in computing the coefficients $a_n(z)$ of the last chapter, and the coefficients $c_{n-2}(z)$ defined here (where $n \geq 5$). At a practical level the reason for this is twofold:

[5] Alternatively, to this order the moment is proportional to the identity matrix in its group indices and the coefficient therefore vanishes under integration by parts.

the presence of the three differential operators, $\tilde{X}^\alpha \tilde{X}_\alpha \tilde{X}^a$, which sit in front of the summation in the differential equation (6.2.35) (compared to just one in previous chapter); and the fact that the moments involved need to be computed higher order than previously. It is impractical (and unnecessary) to provide the full details of the c_3 and c_4 calculations here, however to give the reader a feel for the vast size of the computations involved a few specific details pertinent to the computation c_3 and c_4 will now be given[6].

6.2.2 Computing c_3

To compute c_3 one is required to evaluate the truncated summation from equation (6.2.35):

$$-\,\mathrm{i}\int \mathrm{d}\eta_+ \; \tilde{X}^\alpha \tilde{X}_\alpha \tilde{X}^a \sum_{n=1}^{4} \frac{(\mathrm{i}s)^n}{(n+1)!} L^n_{\Delta_+}(\tilde{X}_a)\; \mathrm{e}^{\mathrm{i}s\Delta_+}. \tag{6.2.43}$$

A detailed investigation of the resulting c_3 moment hierarchy reveals that it contains ten moments, four of which need to be computed to subleading order, while all others are required to leading order. The expansion of these moments is achieved through the use of the methods described above and in the previous chapter. A complete list is given in hierarchical order below (although there is much freedom in the ordering). The numerical factor common to all moments is $H = \mathrm{i}(4\pi\mathrm{i})^{-2}$.

$$K_+{}^\alpha{}_\alpha(z;s) = -\frac{4}{s^2}\,\mathbb{1}\,H \tag{6.2.44}$$

$$K_+{}^\alpha{}_{\alpha ab}(z;s) = -\frac{2}{s^3}(\mathrm{i}\eta_{ab}\,\mathbb{1} + sG_{ab})\,H \tag{6.2.45}$$

$$K_+{}^\alpha{}_{\alpha abcd}(z;s) = \frac{1}{s^4}(\eta_{ab}\eta_{cd} + \eta_{ad}\eta_{bc} + \eta_{ac}\eta_{bd})\,\mathbb{1}\,H \tag{6.2.46}$$

$$K_+{}^\alpha{}_{\alpha a}(z;s) = -\frac{2}{3s}(\sigma_a)_{\alpha\dot\alpha}(W^\alpha \bar{W}^{\dot\alpha} + 4\bar{W}^{\dot\alpha}W^\alpha)H \tag{6.2.47}$$

$$K_+{}^\alpha{}_{\alpha abc}(z;s) = -\frac{1}{3s^3}\Big(\mathrm{i}\big(\eta_{ab}(\sigma_c)_{\alpha\dot\alpha} + \eta_{bc}(\sigma_a)_{\alpha\dot\alpha} + \eta_{ac}(\sigma_b)_{\alpha\dot\alpha}\big)\big(W^\alpha\bar{W}^{\dot\alpha} + 4\bar{W}^{\dot\alpha}W^\alpha\big) + 4(\mathcal{D}_aG_{bc}) + 4(\mathcal{D}_bG_{ac})\Big)H \tag{6.2.48}$$

$$K_{+\alpha abcd}(z;s) = \frac{\mathrm{i}}{2s^3}(\eta_{ab}\eta_{cd} + \eta_{ad}\eta_{bc} + \eta_{ac}\eta_{bd})W_\alpha\,H \tag{6.2.49}$$

$$K_{+\alpha abc}(z;s) = \frac{1}{2s^2}(\eta_{ab}(\mathcal{D}_cW_\alpha) + \eta_{ac}(\mathcal{D}_bW_\alpha) + \eta_{bc}(\mathcal{D}_aW_\alpha))H \tag{6.2.50}$$

$$K_{+\alpha ab}(z;s) = \frac{1}{s^2}\eta_{ab}W_\alpha\,H \tag{6.2.51}$$

[6] Alternatively see the attached CD which fully details the computation of a_6 from the previous chapter and is, in fact, significantly simpler.

$$K_{+\alpha a}(z;s) = -\frac{\mathrm{i}}{s}(\mathcal{D}_a W_\alpha)\, H \tag{6.2.52}$$

$$K_{+\alpha}(z;s) = -\frac{2\mathrm{i}}{s} W_\alpha\, H \tag{6.2.53}$$

Some of the moments have been simplified. For example, the prescription for evaluating ${K_+}^{\alpha}{}_{\alpha a}(z;s)$ first yields

$${K_+}^{\alpha}{}_{\alpha a}(z;s) = -\frac{2}{s}\left(\mathrm{i}(\mathcal{D}^\alpha \mathcal{D}_a W_\alpha) + \frac{2\mathrm{i}}{3}\mathcal{D}^b G_{ba} + (\sigma_a)_{\alpha\dot\alpha}\bar W^{\dot\alpha} W^\alpha\right) H \tag{6.2.54}$$

which after some work gives (6.2.47).

Dropping terms which do not contribute to the order of interest, but without any further simplification, the $n = 1$ contribution to (6.2.43) is immediately found to be:

$$\begin{aligned}
\frac{s}{2}\Bigg(& 2G^{ba} {K_+}^{\alpha}{}_{\alpha ab}(z;s) + \Big(2(\mathcal{D}^b {G^a}_b) - (\mathcal{D}^\alpha \mathcal{D}^a W_\alpha) + \mathrm{i}(\sigma^a)_{\gamma\dot\gamma}\bar W^{\dot\gamma} W^\gamma\Big) {K_+}^{\alpha}{}_{\alpha a}(z;s) \\
&- (\mathcal{D}^\gamma \mathcal{D}^a \mathcal{D}_a W_\gamma) {K_+}^{\alpha}{}_{\alpha}(z;s) + 4(\mathcal{D}^\alpha G^{ba}) K_{+\alpha ab}(z;s) + (\mathcal{D}^2 \mathcal{D}^a \mathcal{D}_a W^\alpha) K_{+\alpha}(z;s) \\
&+ \Big(4(\mathcal{D}^\alpha \mathcal{D}^b {G^a}_b) + (\mathcal{D}^2 \mathcal{D}^a W^\alpha) - 2\mathrm{i}(\sigma^a)_{\gamma\dot\gamma}\bar W^{\dot\gamma}(\mathcal{D}^\alpha W^\gamma)\Big) K_{+\alpha a}(z;s) \\
&+ 2(\mathcal{D}^2 G^{ba}) K_{+ab}(z;s)\Bigg)
\end{aligned} \tag{6.2.55}$$

The reader may care to compare this expression with (5.3.43), the corresponding $n = 1$ term in the computation a_5.

It should be quite clear the calculational procedure employed here gives rise to a large number of terms; a great many combinations of covariant derivatives acting on field strengths. After summing from $n = 1$ to 4 in (6.2.35), the resulting expressions can be vastly simplified, and since the two terms $\mathrm{tr}_{\mathcal{R}}(W^2 \bar W^2)$ and $\mathrm{tr}_{\mathcal{R}}(W^\alpha \bar W_{\dot\alpha} W_\alpha \bar W^{\dot\alpha})$ (modulo total derivative terms) form a basis for such on-shell structures in superspace, much work needs to be done to bring them into this form. This lies in stark contrast to the calculation of a_4, which merely involves two or three lines (see 5.1.17 and (5.1.18)). All results will be given in in section 6.3 below.

6.2.3 Computing c_4

It cannot be overstated that despite the eventual simplicity of the results the computation of c_4 is very involved and requires a vast amount of work. As with a_6 in the previous chapter, so many terms are generated in computing c_4 that it is no longer practical to compute by hand. Again the final result is calculated with the aid of*Mathematica*.

With a little familiarity, and from the details given above regarding c_3, it is not difficult to see that in computing c_4 this process gives rise to a proliferation of terms. Indeed, simply expanding the summation from $n = 1$ to 6 in the differential equation (6.2.35) generates literally thousands of terms. In turn each of these terms contain a moment, which themselves contain tens of terms upon expansion, even after significant simplification.

In total the c_4 moment hierarchy consists about twenty moments. Some of these moments are required to be expanded to 'subsubleading' order (as opposed to at most subleading order in the previous chapter). This in itself can be a substantial task. As an exemplar, consider the moment $K_{+\alpha abc}(z;s)$, which needs to be computed to subleading order. One is free to expand this via

$$0 = \int \mathrm{d}\eta_+ \, \frac{\partial}{\partial k_c} \left(\tilde{X}_\alpha \tilde{X}_a \tilde{X}_b \mathrm{e}^{\mathrm{i}s\Delta_+} \right) \tag{6.2.56}$$

or

$$0 = \int \mathrm{d}\eta_+ \, \frac{\partial}{\partial \kappa_\beta} \left(\tilde{X}_\alpha \tilde{X}_\beta \tilde{X}_a \tilde{X}_b \tilde{X}_c \mathrm{e}^{\mathrm{i}s\Delta_+} \right), \tag{6.2.57}$$

which is again a freedom that can be used to check results. Using the first of these identities, to the order of interest one finds

$$K_{+\alpha abc}(z;s) = \mathrm{i}s \int \mathrm{d}\eta_+ \, \tilde{X}_\alpha \tilde{X}_\beta \tilde{X}_a \tilde{X}_b \tilde{X}_c \sum_{n=1}^{3} \frac{(\mathrm{i}s)^n}{(n+1)!} L^n_{\Delta_+}(W^\beta) \, \mathrm{e}^{\mathrm{i}s\Delta_+}. \tag{6.2.58}$$

In evaluating the summation it becomes clear that to fully expand this moment to subleading order, the moments $K_+{}^\alpha{}_{\alpha abcd}(z;s)$, $K_+{}^\alpha{}_{\alpha ab}(z;s)$, $K_+{}^\alpha{}_{\alpha abc}(z;s)$ are required to subleading order, whilst the moments $K_+{}^\alpha{}_{\alpha abcdef}(z;s)$, $K_+{}^\alpha{}_{\alpha abcde}(z;s)$, $K_+{}^\alpha{}_{\alpha a}(z;s)$, $K_+{}^\alpha{}_\alpha(z;s)$, $K_{+\alpha abcd}(z;s)$, $K_{+\alpha ab}(z;s)$, in addition to $K_{+\alpha abc}(z;s)$ itself are all required to leading order. After a substantial amount of work, and even after significant simplification, to subleading order this moment is found to be:

$$\begin{aligned}
K_{+\alpha abc}(z;s) = \;& \frac{1}{2s^2}\Big(\eta_{ab}(\mathcal{D}_c W_\alpha) + \eta_{ac}(\mathcal{D}_b W_\alpha) + \eta_{bc}(\mathcal{D}_a W_\alpha)\Big) H \\
&+ \frac{\mathrm{i}}{12s}\Big(6W_\alpha(\mathcal{D}_a G_{bc}) - 6W_\alpha(\mathcal{D}_b G_{ac}) + 6W_\alpha(\mathcal{D}_c G_{ab}) - 6G_{ab}(\mathcal{D}_c W_\alpha) \\
&- 6G_{ac}(\mathcal{D}_b W_\alpha) - 6G_{bc}(\mathcal{D}_a W_\alpha) - 14(\mathcal{D}_a G_{bc})W_\alpha - 2(\mathcal{D}_b G_{ac})W_\alpha \\
&- 6(\mathcal{D}_c G_{ab})W_\alpha + W^\gamma(\mathcal{D}_c \mathcal{D}_\alpha W_\gamma)\eta_{ab} - G_{dc}(\mathcal{D}^d W_\alpha)\eta_{ab} - (\mathcal{D}_\alpha W^\gamma)(\mathcal{D}_c W_\gamma)\eta_{ab} \\
&+ (\mathcal{D}_c W^\gamma)(\mathcal{D}_\alpha W_\gamma)\eta_{ab} - 3(\mathcal{D}_c \mathcal{D}_\alpha W^\gamma)W_\gamma \eta_{ab} - (\mathcal{D}^d W_\alpha)G_{dc}\eta_{ab} \\
&+ W^\gamma(\mathcal{D}_b \mathcal{D}_\alpha W_\gamma)\eta_{ac} - G_{db}(\mathcal{D}^d W_\alpha)\eta_{ac} - (\mathcal{D}_\alpha W^\gamma)(\mathcal{D}_b W_\gamma)\eta_{ac} \\
&+ (\mathcal{D}_b W^\gamma)(\mathcal{D}_\alpha W_\gamma)\eta_{ac} - 3(\mathcal{D}_b \mathcal{D}_\alpha W^\gamma)W_\gamma \eta_{ac} - (\mathcal{D}^d W_\alpha)G_{db}\eta_{ac} \\
&+ W^\gamma(\mathcal{D}_a \mathcal{D}_\alpha W_\gamma)\eta_{bc} - G_{da}(\mathcal{D}^d W_\alpha)\eta_{bc} - (\mathcal{D}_\alpha W^\gamma)(\mathcal{D}_a W_\gamma)\eta_{bc} \\
&+ (\mathcal{D}_a W^\gamma)(\mathcal{D}_\alpha W_\gamma)\eta_{bc} - 3(\mathcal{D}_a \mathcal{D}_\alpha W^\gamma)W_\gamma \eta_{bc} - (\mathcal{D}^d W_\alpha)G_{da}\eta_{bc}
\end{aligned}$$

$$
\begin{aligned}
&- 4\mathrm{i}W^\gamma \bar{W}^{\dot\gamma} W_\gamma \eta_{bc}(\sigma_a)_{\alpha\dot\gamma} + 4\mathrm{i}W^\gamma W_\gamma \bar{W}^{\dot\gamma}\eta_{bc}(\sigma_a)_{\alpha\dot\gamma} - 8\mathrm{i}\bar{W}^{\dot\gamma} W^\gamma W_\gamma \eta_{bc}(\sigma_a)_{\alpha\dot\gamma} \\
&+ 3\mathrm{i}W^\gamma \bar{W}^{\dot\gamma} W_\alpha \eta_{bc}(\sigma_a)_{\gamma\dot\gamma} - 4\mathrm{i}W^\gamma W_\alpha \bar{W}^{\dot\gamma}\eta_{bc}(\sigma_a)_{\gamma\dot\gamma} + 3\mathrm{i}\bar{W}^{\dot\gamma} W^\gamma W_\alpha \eta_{bc}(\sigma_a)_{\gamma\dot\gamma} \\
&- 4\mathrm{i}\bar{W}^{\dot\gamma} W_\alpha W^\gamma \eta_{bc}(\sigma_a)_{\gamma\dot\gamma} + 3\mathrm{i}W_\alpha W^\gamma \bar{W}^{\dot\gamma}\eta_{bc}(\sigma_a)_{\gamma\dot\gamma} - 3\mathrm{i}W_\alpha \bar{W}^{\dot\gamma} W^\gamma \eta_{bc}(\sigma_a)_{\gamma\dot\gamma} \\
&- 4\mathrm{i}W^\gamma \bar{W}^{\dot\gamma} W_\gamma \eta_{ac}(\sigma_b)_{\alpha\dot\gamma} + 4\mathrm{i}W^\gamma W_\gamma \bar{W}^{\dot\gamma}\eta_{ac}(\sigma_b)_{\alpha\dot\gamma} - 8\mathrm{i}\bar{W}^{\dot\gamma} W^\gamma W_\gamma \eta_{ac}(\sigma_b)_{\alpha\dot\gamma} \\
&+ 3\mathrm{i}W^\gamma \bar{W}^{\dot\gamma} W_\alpha \eta_{ac}(\sigma_b)_{\gamma\dot\gamma} - 4\mathrm{i}W^\gamma W_\alpha \bar{W}^{\dot\gamma}\eta_{ac}(\sigma_b)_{\gamma\dot\gamma} + 3\mathrm{i}\bar{W}^{\dot\gamma} W^\gamma W_\alpha \eta_{ac}(\sigma_b)_{\gamma\dot\gamma} \\
&- 4\mathrm{i}\bar{W}^{\dot\gamma} W_\alpha W^\gamma \eta_{ac}(\sigma_b)_{\gamma\dot\gamma} + 3\mathrm{i}W_\alpha W^\gamma \bar{W}^{\dot\gamma}\eta_{ac}(\sigma_b)_{\gamma\dot\gamma} - 3\mathrm{i}W_\alpha \bar{W}^{\dot\gamma} W^\gamma \eta_{ac}(\sigma_b)_{\gamma\dot\gamma} \\
&- 4\mathrm{i}W^\gamma \bar{W}^{\dot\gamma} W_\gamma \eta_{ab}(\sigma_c)_{\alpha\dot\gamma} + 4\mathrm{i}W^\gamma W_\gamma \bar{W}^{\dot\gamma}\eta_{ab}(\sigma_c)_{\alpha\dot\gamma} - 8\mathrm{i}\bar{W}^{\dot\gamma} W^\gamma W_\gamma \eta_{ab}(\sigma_c)_{\alpha\dot\gamma} \\
&+ 3\mathrm{i}W^\gamma \bar{W}^{\dot\gamma} W_\alpha \eta_{ab}(\sigma_c)_{\gamma\dot\gamma} - 4\mathrm{i}W^\gamma W_\alpha \bar{W}^{\dot\gamma}\eta_{ab}(\sigma_c)_{\gamma\dot\gamma} + 3\mathrm{i}\bar{W}^{\dot\gamma} W^\gamma W_\alpha \eta_{ab}(\sigma_c)_{\gamma\dot\gamma} \\
&- 4\mathrm{i}\bar{W}^{\dot\gamma} W_\alpha W^\gamma \eta_{ab}(\sigma_c)_{\gamma\dot\gamma} + 3\mathrm{i}W_\alpha W^\gamma \bar{W}^{\dot\gamma}\eta_{ab}(\sigma_c)_{\gamma\dot\gamma} - 3\mathrm{i}W_\alpha \bar{W}^{\dot\gamma} W^\gamma \eta_{ab}(\sigma_c)_{\gamma\dot\gamma} \\
&- (\mathcal{D}_a\mathcal{D}_b\mathcal{D}_c W_\alpha) - (\mathcal{D}_a\mathcal{D}_c\mathcal{D}_b W_\alpha) - (\mathcal{D}_b\mathcal{D}_a\mathcal{D}_c W_\alpha) - (\mathcal{D}_b\mathcal{D}_c\mathcal{D}_a W_\alpha) \\
&- (\mathcal{D}_c\mathcal{D}_a\mathcal{D}_b W_\alpha) - (\mathcal{D}_c\mathcal{D}_b\mathcal{D}_a W_\alpha)\Big) H
\end{aligned}
\tag{6.2.59}
$$

The size of this expression is quite typical, and is by no means an extreme example. As noted, this moment was required to be expanded to subleading order, whilst others are required to be to evaluated to the next highest order, and can be seen to contain hundreds of terms. It is clearly impractical to give full details of the calculation.

Again, which is probably the most time consuming part of this process, the final result can be brought into a compact form through use of integration by parts, the cyclic property of the trace, the equations of motion, the repeated use of the commutation relations (5.1.5), and application of on-shell identities such as those given in (5.3.70).

Unlike c_2 and c_3, c_4 is not obviously real after simplification and one must work a little harder to demonstrate its reality. It can be brought into a manifestly real form by using the identities (modulo total derivative terms):

$$
\mathrm{tr}_{\mathcal{R}}\Big((\mathcal{D}^a W^\alpha)(\mathcal{D}_a \bar{W}_{\dot\alpha})\bar{W}^{\dot\alpha} W_\alpha\Big) = \mathrm{tr}_{\mathcal{R}}\Big((\mathcal{D}^a \bar{W}_{\dot\alpha})(\mathcal{D}_a W^\alpha) W_\alpha \bar{W}^{\dot\alpha}\Big) \tag{6.2.60}
$$

and

$$
\begin{aligned}
&\mathrm{tr}_{\mathcal{R}}\Big((\mathcal{D}^a W^\alpha)(\mathcal{D}_a W_\alpha)\bar{W}^2 + 2(\mathcal{D}_\alpha W^\beta) W^\alpha W_\beta \bar{W}^2\Big) \\
&\qquad = \mathrm{tr}_{\mathcal{R}}\Big((\mathcal{D}^a \bar{W}_{\dot\alpha})(\mathcal{D}_a \bar{W}^{\dot\alpha}) W^2 + 2(\bar{\mathcal{D}}_{\dot\alpha} \bar{W}^{\dot\beta})\bar{W}^{\dot\alpha}\bar{W}_{\dot\beta} W^2\Big).
\end{aligned}
\tag{6.2.61}
$$

As previously discussed, unlike some other methods, one can employ a variety of checks during the course of this calculation. So it should be stressed that although a large amount of work is required here, one can be confident in the accuracy of the results.

6.3 Results

6.3.1 Superfield form

Having simplified all results, to order F^5, the one-loop effective action is found to be:

$$\Gamma^{(1)}_{\Phi,m,\mathcal{R}} = \frac{1}{32\pi^2\rho}\int \mathrm{d}^6z\,\mathrm{tr}_{\mathcal{R}}(W^2) - \frac{1}{128\pi^2m^2}\int \mathrm{d}^8z\,\mathrm{tr}_{\mathcal{R}}(c_2) - \frac{1}{256\pi^2m^4}\int \mathrm{d}^8z\,\mathrm{tr}_{\mathcal{R}}(c_3) - \frac{1}{192\pi^2m^6}\int \mathrm{d}^8z\,\mathrm{tr}_{\mathcal{R}}(c_4) \quad (6.3.62)$$

where

$$\mathrm{tr}_{\mathcal{R}}(c_2) = 0 \quad (6.3.63)$$

$$\mathrm{tr}_{\mathcal{R}}(c_3) = \frac{2}{15}\mathrm{tr}_{\mathcal{R}}(W^\alpha \bar{W}_{\dot{\alpha}} W_\alpha \bar{W}^{\dot{\alpha}} - 4W^2\bar{W}^2) \quad (6.3.64)$$

$$\begin{aligned}\mathrm{tr}_{\mathcal{R}}(c_4) = \frac{1}{105}\mathrm{tr}_{\mathcal{R}}\Big(&2(\mathcal{D}^a W^\alpha)(\mathcal{D}_a \bar{W}_{\dot{\alpha}})W_\alpha \bar{W}^{\dot{\alpha}} - 6(\mathcal{D}^a W^\alpha)(\mathcal{D}_a W_\alpha)\bar{W}^2 \\ &- 3(\mathcal{D}^a W^\alpha)(\mathcal{D}_a \bar{W}_{\dot{\alpha}})\bar{W}^{\dot{\alpha}} W_\alpha + 18(\mathcal{D}_\alpha W^\beta)W^\alpha W_\beta \bar{W}^2 \\ &+ \frac{5}{2}(\mathcal{D}_\alpha W^\beta)W^\alpha \bar{W}_{\dot{\alpha}} W_\beta \bar{W}^{\dot{\alpha}}\Big) + c.c.\end{aligned} \quad (6.3.65)$$

Prior to integration by parts this procedure actually yields

$$\mathrm{tr}_{\mathcal{R}}(c_2) = \frac{1}{3}\mathrm{tr}_{\mathcal{R}}(G^{ab}G_{ba}) = -\frac{1}{6}\mathrm{tr}_{\mathcal{R}}\left((\bar{\mathcal{D}}_{\dot{\alpha}}\bar{W}_{\dot{\beta}})(\bar{\mathcal{D}}^{\dot{\beta}}\bar{W}^{\dot{\alpha}}) + (\mathcal{D}^\alpha W^\beta)(\mathcal{D}_\beta W_\alpha)\right) \quad (6.3.66)$$

which vanishes under integration by parts since on-shell $(\mathcal{D}_\alpha \mathcal{D}_\beta W_\gamma) = 0$. It is not actually necessary that one compute this coefficient since its trivial contribution to the effective action can be deduced using dimensional reasoning and integration by parts.

6.3.2 Component form

The component form of the above expressions can be extracted through the same techniques employed previously in subsection (5.3.2). The bosonic component of $\mathrm{tr}_{\mathcal{R}}(c_3)$ is:

$$\begin{aligned}\mathrm{tr}_{\mathcal{R}}(c_3)\Big|_{\mathrm{v}} = \frac{1}{30}\mathrm{tr}_{\mathcal{R}}(&2F^{ab}F_{ab}F^{cd}F_{cd} + 3F^{ab}F^{cd}F_{ab}F_{cd} \\ &- 4F^{ab}F_{bc}F^{cd}F_{da} - 16F^{ab}F_{bc}F_{ad}F^{dc}).\end{aligned} \quad (6.3.67)$$

Using the results of subsection 5.3.3, the bosonic component of $\mathrm{tr}_{\mathcal{R}}(c_4)$ expressed in the basis (5.3.83) is:

$$\begin{aligned}\mathrm{tr}_{\mathcal{R}}(c_4)\Big|_{\mathrm{v}} =& -\frac{1}{210}\mathrm{tr}_{\mathcal{R}}\Big(19(\nabla^e F^{ab})(\nabla_e F_{ab})F^{cd}F_{cd} + 11(\nabla^e F^{ab})(\nabla_e F^{cd})F_{ab}F_{cd} \\ &+ 13(\nabla^e F^{ab})(\nabla_e F^{cd})F_{cd}F_{ab} + 32(\nabla^e F^{ab})(\nabla_e F_{bc})F^{cd}F_{da} \\ &- 60(\nabla^e F^{ab})(\nabla_e F_{ca})F_{bd}F^{dc} + \frac{261\mathrm{i}}{5}F^{ab}F_{bc}F^{cd}F_{de}F^{e}{}_{a} \\ &- 89\mathrm{i}F^{ab}F_{bc}F^{cd}F^{e}{}_{a}F_{de} - 41\mathrm{i}F^{ab}F^{cd}F_{bc}F^{e}{}_{a}F_{de} - \frac{7\mathrm{i}}{5}F^{ab}F^{cd}F^{e}{}_{a}F_{bc}F_{de}\Big) \\ =& \frac{1}{210}\{\frac{261\mathrm{i}}{5}, -89\mathrm{i}, -41\mathrm{i}, -\frac{7\mathrm{i}}{5}, 19, 11, 13, 32, -60, 0\}\end{aligned} \tag{6.3.68}$$

6.4 Comparison with literature

As mentioned in chapter 5, to date the bosonic DeWitt-Seeley coefficients associated with scalars, vectors or spinors in the presence of non-Abelian background Yang-Mills fields in arbitrary spacetime dimension have been separately computed to low order [87, 88, 215, 76]. Since at the component level the action (3.2.106) corresponds to supersymmetric matter (a set of massive scalars and their fermionic superpartners) coupled to a background non-Abelian supersymmetric Yang-Mills field, a non-trivial check of the results derived here is available.

At the component level the starting action (2.1.37) contained two scalars and two Majorana-Weyl spinors in $D = 4$, so from [76, 87] and equation (5.3.116), one generates the following on-shell bosonic components of the DeWitt-Seeley coefficients:

$$\mathrm{tr}_{\mathrm{Ad}}(\mathbf{a}_3) = 0 \tag{6.4.69}$$

$$\begin{aligned}\mathrm{tr}_{\mathrm{Ad}}(\mathbf{a}_4) = -\frac{1}{240}\mathrm{tr}_{\mathrm{Ad}}\Big(&2F^{ab}F_{ab}F^{cd}F_{cd} + 3F^{ab}F^{cd}F_{ab}F_{cd} \\ &- 4F^{ab}F_{bc}F^{cd}F_{da} - 16F^{ab}F_{bc}F_{ad}F^{dc}\Big)\end{aligned} \tag{6.4.70}$$

$$\begin{aligned}\mathrm{tr}_{\mathrm{Ad}}(\mathbf{a}_5) = \frac{1}{21}\frac{1}{5!}\mathrm{tr}_{\mathrm{Ad}}\Big(&-10(\nabla^e F^{ab})(\nabla_e F_{bc})F^{cd}F_{da} - 32(\nabla^e F^{ab})(\nabla_e F_{ca})F_{bd}F^{dc} \\ &+ 8(\nabla^a F^{ef})(\nabla_b F_{ef})F^{ac}F_{cb} + \frac{1}{2}(\nabla^e F^{ab})F_{ab}(\nabla_e F^{cd})F_{cd} \\ &- 42(\nabla^e F^{ab})F_{da}(\nabla_e F_{bc})F^{cd} + 6(\nabla_b F^{ef})F^{cb}(\nabla^a F_{ef})F_{ac} \\ &+ 6(\nabla^e F^{ab})(\nabla_e F_{ab})F^{cd}F_{cd} - \frac{19}{2}(\nabla^e F^{ab})(\nabla_e F^{cd})F_{ab}F_{cd} \\ &- 28(\nabla^a F^{de})(\nabla^b F_{ec})F_{ab}F^{c}{}_{d}\Big) + (F^5 \text{ terms})\end{aligned} \tag{6.4.71}$$

The vanishing of $\mathbf{a}_3$ is non-trivial (being non-zero in some dimensions other than $D = 4$), and as indicated only the derivative terms of $\mathbf{a}_5$ have so far been extracted from [215]. Inspection reveals immediate agreement, up to an overall numerical multiplicative constant, between $\mathbf{a}_3$, $\mathbf{a}_4$ and the bosonic components of c_2 and c_3 respectively. Taking into account the relationship between the coefficients c_n and a_n^+ in superspace, equation (6.1.20), and restricting $\mathcal{R}$ to be the adjoint representation, exact agreement is found.

To facilitate comparison with $\mathbf{a}_5$ a basis change is necessary. Using the conversion table provided in subsection 5.3.3 one finds that result (6.4.71) becomes

$$\mathrm{tr}_{\mathrm{Ad}}(\mathbf{a}_5) = \frac{1}{21}\frac{1}{5!}\{\frac{234\mathrm{i}}{5}, -32\mathrm{i}, -74\mathrm{i}, -\frac{168\mathrm{i}}{5}, 19, 11, 13, 32, -60, 0\}. \qquad (6.4.72)$$

The pure F^5 terms are or course irrelevant here, and comparing this expression with (6.3.68), and again taking into account (6.1.20), exact agreement of the derivative terms is found.

6.5 $\mathcal{N} = 2$ super Yang-Mills to one-loop

Finally, all of the previous results can be collected together to give the one-loop effective action for an arbitrary $\mathcal{N} = 2$ super Yang-Mills theory in the absence of a scalar background to order F^5 in superfield form. For the reader's convenience the result to this order is:

$$\Gamma^{(1)}[w_B, \bar{w}_B] = \frac{\mathrm{i}}{2} \ln \mathrm{sDet}(\Box - M) - 2\Gamma^{(1)}_{\Phi,m,\mathrm{Ad}} + \Gamma^{(1)}_{\Phi,\mathcal{M},R\oplus R_c} \qquad (6.5.73)$$

where

$$\Gamma^{(1)}_{\Phi,m,\mathcal{R}} = \frac{1}{32\pi^2\rho}\int \mathrm{d}^6 z\, \mathrm{tr}_{\mathcal{R}}(W^2) - \frac{1}{256\pi^2 m^4}\int \mathrm{d}^8 z\, \mathrm{tr}_{\mathcal{R}}(c_3) - \frac{1}{192\pi^2 m^6}\int \mathrm{d}^8 z\, \mathrm{tr}_{\mathcal{R}}(c_4) \qquad (6.5.74)$$

and

$$\frac{\mathrm{i}}{2}\ln \mathrm{sDet}(\Box - M^2) = \frac{1}{32\pi^2 M^4}\int \mathrm{d}^8 z\, \mathrm{tr}_{\mathrm{Ad}}(a_4) + \frac{1}{16\pi^2 M^6}\int \mathrm{d}^8 z\, \mathrm{tr}_{\mathrm{Ad}}(a_5), \qquad (6.5.75)$$

the various coefficients being given by

$$\mathrm{tr}_{\mathcal{R}}(c_3) = \frac{2}{15}\mathrm{tr}_{\mathcal{R}}(W^\alpha \bar{W}_{\dot{\alpha}} W_\alpha \bar{W}^{\dot{\alpha}} - 4W^2\bar{W}^2) \qquad (6.5.76)$$

$$\begin{aligned}\mathrm{tr}_{\mathcal{R}}(c_4) =& \frac{1}{105}\mathrm{tr}_{\mathcal{R}}\Big(2(\mathcal{D}^a W^\alpha)(\mathcal{D}_a \bar{W}_{\dot\alpha})W_\alpha \bar{W}^{\dot\alpha} - 6(\mathcal{D}^a W^\alpha)(\mathcal{D}_a W_\alpha)\bar{W}^2 \\ & - 3(\mathcal{D}^a W^\alpha)(\mathcal{D}_a \bar{W}_{\dot\alpha})\bar{W}^{\dot\alpha} W_\alpha + 18(\mathcal{D}_\alpha W^\beta)W^\alpha W_\beta \bar{W}^2 \\ & + \frac{5}{2}(\mathcal{D}_\alpha W^\beta)W^\alpha \bar{W}_{\dot\alpha} W_\beta \bar{W}^{\dot\alpha}\Big) + c.c. \end{aligned} \tag{6.5.77}$$

$$\mathrm{tr}_{\mathrm{Ad}}(a_4) = \frac{1}{3}\,\mathrm{tr}_{\mathrm{Ad}}(2\,W^2\bar{W}^2 - W^\alpha \bar{W}_{\dot\alpha} W_\alpha \bar{W}^{\dot\alpha}) \tag{6.5.78}$$

$$\begin{aligned}\mathrm{tr}_{\mathrm{Ad}}(a_5) =& \frac{1}{30}\mathrm{tr}_{\mathrm{Ad}}\big((\mathcal{D}^a W^\alpha)(\mathcal{D}_a W_\alpha)\bar{W}^2 + (\mathcal{D}^a W^\alpha)(\mathcal{D}_a \bar{W}_{\dot\alpha})\bar{W}^{\dot\alpha} W_\alpha \\ & - (\mathcal{D}^a W^\alpha)(\mathcal{D}_a \bar{W}_{\dot\alpha})W_\alpha \bar{W}^{\dot\alpha} - 3(\mathcal{D}_\alpha W^\beta)W^\alpha W_\beta \bar{W}^2 \\ & - (\mathcal{D}_\alpha W^\beta)W^\alpha \bar{W}_{\dot\alpha} W_\beta \bar{W}^{\dot\alpha}\big) + c.c. \end{aligned} \tag{6.5.79}$$

In the above expressions M and m are infrared regulators and $\mathcal{M}$ is either a physical hypermultiplet mass or is also infrared regulator[7].

Corresponding computations in string theory have yet to be carried out, and so comparisons in this respect await future developments.

[7]The regulators may all coincide, or differ depending motivations or taste [234].

Chapter 7

Conclusions

The work presented here was based on the two papers [34] and [35]. In [34] the one-loop effective action for $\mathcal{N} = 4$ super Yang-Mills theory in the absence of a chiral background was, for the first time, computed to order F^6 in full superfield form. To date the contribution computed there still stands as the only such F^6 result. The bosonic field strength components were extracted from the F^5 contributions and the results, being shown to be consistent with those obtained via other means, eventually proved to be the first correct determination of this contribution. Although a bosonic F^5 computation had been attempted prior to [34] using what proved to be a more cumbersome approach [85, 86], the results where found to contain an error. As noted in [78, 74, 81], the original results of [85, 86] where at odds with those from string theory and other approaches [78, 81, 79, 80, 74], and in particular cast some doubt on the notion that there exists a unique deformation of maximally supersymmetric Yang-Mills theory at this order. The work [34] finally cleared up this matter and when considered alongside all other results, provided the final convincing piece of evidence of a unique deformation at this order. It still remains to be determined whether supersymmetry uniquely fixes deformations of maximally supersymmetric Yang-Mills theory at higher orders.

In [35] the results of [34] where extended to include the F^5 contributions to the effective action for any $\mathcal{N} = 2$ super Yang-Mills theory. It was shown that results obtained there where consistent with existing literature. Given the large amount of work required to compute the F^5 terms in this case, and the very simple expressions which result, one is left with the distinct impression that there may exist a more economical alternative. No approach has yet presented itself as being any simpler, and this notion may be worthy of further investigation. The results of both [34] and [35] may prove useful in direct tests of the AdS/CFT conjecture and its recent generalizations.

It is now well appreciated that computing the non-Abelian D-brane effective action to some order necessarily involves field strength terms with and without covariant derivatives due to the fact that such terms are related by the Bianchi identity. As yet there still remains no clear means for uniquely truncating and obtaining partial results at a given order as one readily does in the Abelian case. Different methods of computation naturally give rise to results with very different looking field strength tensor structures, with additional complications stemming from field redefinitions. Since there seems to be no set of tensor structures which naturally stand out, a clear and detailed analysis of tensor structures needs to be carried out at each order of the expansion to facilitate a comparison of results. It was shown in chapter 5 in all its gory detail that any such comparison, even at an order as low as F^5, is a highly non-trivial exercise which requires the identification of a basis. As yet no general scheme has been found which identifies bases at a given order, or even any way of knowing the number of terms in such a basis. Some work in this direction has been done [69, 101], however explicit details regarding the F^6 basis were not provided in [101]. An obvious next step would be to clearly identity and F^6 basis (and conversions into it), up to field redefinitions, which will be necessary for the comparison of the various F^6 results [101, 102, 84] and any which become available in the future.

The bosonic F^6 contributions to the one-loop effective action for $\mathcal{N} = 4$ super Yang-Mills theory are yet to be extracted from the superfield results of [34] (a simplified version of which was given in chapter 5). This is perhaps the subject of a future work. One should not necessarily expect these results to coincide with those of the non-Abelian D-brane effective action since higher loop F^6 contributions are expected in $\mathcal{N} = 4$ super Yang-Mills theory, and have not been included. However, if one assumes or could prove a unique deformation at this order, comparison with string theory results would provide information about the higher loop contributions.

During the course of this investigation a new method for computing DeWitt-Seeley coefficients in the coincidence limit was discovered and described in detail. Although any technique for computing these coefficients could have been adopted to meet our needs, this method proved to be quite efficient and readily adaptable to computerization. If one is interested in computing multi-loop contributions to the effective action, one must compute full propagators in the presence of background fields, and therefore the associated heat kernels at non-coincident points. Directly generalizing the modified Gaussian approach to compute the full non-coincident DeWitt-Seeley coefficients is relatively straight forward. In order to maintain manifest gauge covariance, which is indeed the goal of the background field method, one must introduce a additional factor (the parallel displacement propagator) into the Fourier decomposition of the delta-function to ensure the correct gauge transformation properties of the heat kernel. This idea has been explored in the recent work [207], and a new method for computing the full non-coincident DeWitt-Seeley coefficients is described. For complete details see [207] and references there. Also see [235, 228, 236, 237, 238, 239] for recent use of this method.

As a suggestion for future work, one may care to attempt to generalize the modified Gaussian approach to harmonic superspace and or to include theories involving gravitation.

Appendix A

Notation

A.1 Two component spinors

In this body of work extensive use is made of the two component spinor formulation first introduced by B. L. van der Waerden [240, 241, 242] and since developed into a powerful formalism to supplement tensor analysis. Not only is this formalism useful for describing spinor representations of the Lorentz group, but also more generally tensors products. Utilized by Akulov and Volkov [6], and Wess and Zumino [157] (also see [154]) in the context of supersymmetry, this formalism is well adapted to supersymmetry in four spacetime dimensions in contrast to the more well known four component Dirac spinors[1]. For a detailed treatment of these ideas see [243, 127]. For the current purpose the formalism can be summarized as follows. Throughout, letters near the beginning of the Greek (Latin) alphabet will used for spinor (spacetime) indices.

In Minkowski space two inertial coordinate systems x^a and x'^a with $a = 0, 1, 2, 3$ are related by the non-homogenous transformations

$$x'^a = \Lambda^a{}_b x^b + c^b = (e^\lambda)^a{}_b x^b + c^b \tag{A.1.1}$$

which preserve the spacetime interval

$$ds^2 = \eta_{ab} dx^a dx^b = \eta_{ab} dx'^a dx'^b \qquad \eta_{ab} = \text{diag}(-1, 1, 1, 1), \tag{A.1.2}$$

η_{ab} being the Minkowski metric. This invariance yields the constraint

$$\eta_{ab} = \Lambda^c{}_a \Lambda^d{}_b \eta_{cd} \quad \Rightarrow \quad \det\Lambda = \pm 1. \tag{A.1.3}$$

The set of all transformations (A.1.1) which satisfy (A.1.3) and $\det\Lambda = 1$ is the Poincaré group $\Pi(1, 3)$, and the homogenous subset with $c^a = 0$ is the (proper)

[1] Golfand's original articles [4, 3] used Dirac spinors which necessitate the use of chiral projectors.

Lorentz group $SO(1,3)$. An infinitesimal Lorentz transformation is given by

$$\delta x^a = \lambda^a{}_b x^b. \tag{A.1.4}$$

Locally isomorphic to the Lorentz group is the group of 2×2 complex matrices with unit determinant $SL(2,\mathbb{C})$. An object ψ_α, $\alpha = 1,2$, transforming in the fundamental representation of $SL(2,\mathbb{C})$,

$$\psi'_\alpha = \Lambda_\alpha{}^\beta \psi_\beta \qquad\qquad \Lambda \,\epsilon\, SL(2,\mathbb{C}) \tag{A.1.5}$$

is called a two-component left-handed Weyl spinor. The representation is denoted $(\frac{1}{2},0)$ and is known as the left-handed spinor representation of the Lorentz group. Right-handed Weyl spinors are those objects $\bar{\psi}_{\dot\alpha}$, $\dot\alpha = 1,2$, which transform according to the conjugate representation

$$\bar{\psi}'_{\dot\alpha} = \bar{\Lambda}_{\dot\alpha}{}^{\dot\beta} \bar{\psi}_{\dot\beta} \qquad\qquad (\Lambda_\alpha{}^\beta)^* = \bar{\Lambda}_{\dot\alpha}{}^{\dot\beta} \tag{A.1.6}$$

which is denoted $(0,\frac{1}{2})$, and is called the right-handed spinor representation of the Lorentz group.

Spinors may have their dotted and undotted indices raised and lowered by ε, the antisymmetric (spinor metric) tensors as follows:

$$\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta \qquad\qquad \psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta \tag{A.1.7}$$

$$\bar{\psi}^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}\bar{\psi}_{\dot\beta} \qquad\qquad \bar{\psi}_{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}\bar{\psi}^{\dot\beta} \tag{A.1.8}$$

where

$$\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha} \qquad \varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha} \qquad \varepsilon^{\alpha\beta}\varepsilon_{\beta\gamma} = \delta^\alpha{}_\gamma \qquad \varepsilon^{12} = -\varepsilon_{12} = 1 \tag{A.1.9}$$

$$\varepsilon^{\dot\alpha\dot\beta} = -\varepsilon^{\dot\beta\dot\alpha} \qquad \varepsilon_{\dot\alpha\dot\beta} = -\varepsilon_{\dot\beta\dot\alpha} \qquad \varepsilon^{\dot\alpha\dot\beta}\varepsilon_{\dot\beta\dot\gamma} = \delta^{\dot\alpha}{}_{\dot\gamma} \qquad \varepsilon^{\dot1\dot2} = -\varepsilon_{\dot1\dot2} = 1. \tag{A.1.10}$$

Infinitesimal Lorentz transformations of two component spinors are then given by

$$\delta\psi'_\alpha = \frac{1}{2}\lambda^{ab}(\sigma_{ab})_\alpha{}^\beta\psi_\beta \qquad\qquad \delta\bar{\psi}'_{\dot\alpha} = \frac{1}{2}\lambda^{ab}(\tilde{\sigma}_{ab})_{\dot\alpha}{}^{\dot\beta}\bar{\psi}_{\dot\beta} \tag{A.1.11}$$

$$\lambda^{ab} = -\lambda^{ba} = \eta^{bc}\lambda^a{}_c \tag{A.1.12}$$

with $\lambda^a{}_b$ being introduced in (A.1.1) and (A.1.4). The generators $(\sigma_{ab})_\alpha{}^\beta$ and $(\tilde{\sigma}_{ab})_{\dot\alpha}{}^{\dot\beta}$ of $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ respectively, are defined in terms of the four $(\sigma_a)_{\alpha\dot\alpha}$ matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.1.13}$$

$$(\tilde{\sigma}_a)^{\dot{\alpha}\alpha} = \varepsilon^{\alpha\beta}\varepsilon^{\dot{\alpha}\dot{\beta}}(\sigma_a)_{\beta\dot{\beta}} \tag{A.1.14}$$

via

$$(\sigma_{ab})_\alpha{}^\beta = -\frac{1}{4}(\sigma_a\tilde{\sigma}_b - \sigma_b\tilde{\sigma}_a)_\alpha{}^\beta = -(\sigma_{ba})_\alpha{}^\beta \tag{A.1.15}$$

$$(\tilde{\sigma}_{ab})^{\dot{\alpha}}{}_{\dot{\beta}} = -\frac{1}{4}(\tilde{\sigma}_a\sigma_b - \tilde{\sigma}_b\sigma_a)^{\dot{\alpha}}{}_{\dot{\beta}} = -(\tilde{\sigma}_{ba})^{\dot{\alpha}}{}_{\dot{\beta}}. \tag{A.1.16}$$

The σ matrices satisfy the following useful properties:

$$\mathrm{tr}(\sigma_a\tilde{\sigma}_b) = (\sigma_a)_{\alpha\dot{\alpha}}(\tilde{\sigma}_b)^{\dot{\alpha}\alpha} = -2\eta_{ab} \tag{A.1.17}$$

$$(\sigma^a)_{\alpha\dot{\alpha}}(\tilde{\sigma}_a)^{\dot{\beta}\beta} = -2\delta_\alpha{}^\beta\delta_{\dot{\alpha}}{}^{\dot{\beta}} \tag{A.1.18}$$

$$\sigma_a\tilde{\sigma}_b\sigma_c = (\eta_{ac}\sigma_b - \eta_{bc}\sigma_a - \eta_{ab}\sigma_c) + \mathrm{i}\varepsilon_{abcd}\sigma^d \tag{A.1.19}$$

$$\tilde{\sigma}_a\sigma_b\tilde{\sigma}_c = (\eta_{ac}\tilde{\sigma}_b - \eta_{bc}\tilde{\sigma}_a - \eta_{ab}\tilde{\sigma}_c) - \mathrm{i}\varepsilon_{abcd}\tilde{\sigma}^d \tag{A.1.20}$$

with ε_{abcd} ($\varepsilon_{0123} = -1$) the totally antisymmetric Levi-Civita symbol.

The index free notation describing the contraction of spinor indices that will be adopted here, is governed by the following rules:

$$\psi\chi = \psi^\alpha\chi_\alpha = -\psi_\alpha\chi^\alpha \qquad \psi^2 = \psi\psi \tag{A.1.21}$$

$$\bar{\psi}\bar{\chi} = \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = -\bar{\psi}^{\dot{\alpha}}\bar{\chi}_{\dot{\alpha}} \qquad \bar{\psi}^2 = \bar{\psi}\bar{\psi} \tag{A.1.22}$$

$$\psi\sigma_a\bar{\chi} = \psi^\alpha(\sigma_a)_{\alpha\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} \qquad \bar{\psi}\sigma_a\chi = \bar{\psi}_{\dot{\alpha}}(\tilde{\sigma}_a)^{\dot{\alpha}\alpha}\chi_\alpha. \tag{A.1.23}$$

Later we will also find useful that any two objects carrying (un)dotted spinor indices satisfy

$$\psi_\alpha\phi_\beta - \psi_\beta\phi_\alpha = 2\psi_{[\alpha}\phi_{\beta]} = \varepsilon_{\alpha\beta}\psi\phi \qquad \psi^\alpha\phi^\beta - \psi^\beta\phi^\alpha = 2\psi^{[\alpha}\phi^{\beta]} = -\varepsilon^{\alpha\beta}\psi\phi \tag{A.1.24}$$

$$\bar{\psi}_{\dot{\alpha}}\bar{\phi}_{\dot{\beta}} - \bar{\psi}_{\dot{\beta}}\bar{\phi}_{\dot{\alpha}} = 2\bar{\psi}_{[\dot{\alpha}}\bar{\phi}_{\dot{\beta}]} = -\varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\psi}\bar{\phi} \qquad \bar{\psi}^{\dot{\alpha}}\bar{\phi}^{\dot{\beta}} - \bar{\psi}^{\dot{\beta}}\bar{\phi}^{\dot{\alpha}} = 2\bar{\psi}^{[\dot{\alpha}}\bar{\phi}^{\dot{\beta}]} = \varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\psi}\bar{\phi}. \tag{A.1.25}$$

In this two component notation the familiar four component Dirac spinor Ψ can be expressed as a column

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \tag{A.1.26}$$

containing one left spinor ψ_α and one right spinor $\bar{\chi}^{\dot{\alpha}}$. If $(\psi^\alpha)^* = \bar{\chi}^{\dot{\alpha}}$ the Dirac spinor is a Majorana spinor, and imposing the Weyl constraint simple sets one of these two component spinors to zero.

Spacetime indices are often written in terms of a pair of spinor indices, for example an arbitrary four vector V^a is considered to belong to the $(\frac{1}{2}, \frac{1}{2})$ representation of the Lorentz group, and is written as a bi-spinor

$$V_{\alpha\dot{\alpha}} = (\sigma_a)_{\alpha\dot{\alpha}}V^a. \tag{A.1.27}$$

Finally, taking the tensor product

$$\underbrace{(\frac{1}{2},0)\otimes(\frac{1}{2},0)\otimes\cdots\otimes(\frac{1}{2},0)}_{n-\text{times}}\otimes\underbrace{(0,\frac{1}{2})\otimes(0,\frac{1}{2})\otimes\cdots\otimes(0,\frac{1}{2})}_{m-\text{times}}, \tag{A.1.28}$$

one obtains new representations of the Lorentz group which are reducible. Irreducible representations of this form act on tensors which are symmetrized (separately) in its n-undotted spinor indices an m-dotted spinor indices. Such irreducible representations are denoted $(\frac{n}{2},\frac{m}{2})$, and classify all finite-dimensional irreducible representations of the Lorentz group.

A.2 Differentiation and integration in superspace

Partial differentiation of the superspace coordinates are characterized by the following straight forward properties:

$$\partial_A = (\partial_a, \partial_\alpha, \bar{\partial}^{\dot{\alpha}}) = \frac{\partial}{\partial z^A} = \left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial\theta^\alpha}, \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}}\right) \tag{A.2.29}$$

with

$$\partial_A z^B = \delta_A{}^B \Rightarrow \begin{cases} \partial_a x^b = \delta_a{}^b & \partial_a\theta^\alpha = \partial_a\bar{\theta}^{\dot{\alpha}} = 0 \\ \partial_\alpha\theta^\beta = \delta_\alpha{}^\beta & \partial_\alpha x^a = \partial_\alpha\bar{\theta}^{\dot{\alpha}} = 0 \\ \bar{\partial}^{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} = \delta^{\dot{\alpha}}{}_{\dot{\beta}} & \bar{\partial}^{\dot{\alpha}} x^a = \bar{\partial}^{\dot{\alpha}}\theta^\alpha = 0 \end{cases} \tag{A.2.30}$$

and

$$\begin{aligned} &\varepsilon(\partial_A) = \varepsilon(z^A) = \varepsilon_A \\ &\varepsilon(\partial_A V) = \varepsilon_A + \varepsilon(V) \pmod 2 \\ &\partial_A\partial_B = (-1)^{\varepsilon(A)\varepsilon(B)}\partial_B\partial_A \\ &\partial_A(UV) = \partial_A(U)V + (-1)^{\varepsilon(U)\varepsilon_A}U\partial_A(V) \\ &(\partial_a V)^* = \partial_a V^* \qquad (\partial_\alpha V)^* = -(-1)^{\varepsilon(V)}\bar{\partial}_{\dot{\alpha}} V^* \qquad (\bar{\partial}^{\dot{\alpha}} V)^* = -(-1)^{\varepsilon(V)}\partial^\alpha V^* \end{aligned} \tag{A.2.31}$$

for arbitrary superfields U and V, and where $\varepsilon(\cdot)$ denotes Grassmann parity.

Also desirable would be

$$\partial^A z_B = \delta^A{}_B \Rightarrow \begin{cases} \partial^a x_b = \delta^a{}_b & \partial^a\theta_\alpha = \partial^a\bar{\theta}_{\dot{\alpha}} = 0 \\ \partial^\alpha\theta_\beta = \delta^\alpha{}_\beta & \partial^\alpha x_a = \partial^\alpha\bar{\theta}_{\dot{\alpha}} = 0 \\ \bar{\partial}_{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = \delta_{\dot{\alpha}}{}^{\dot{\beta}} & \bar{\partial}_{\dot{\alpha}} x_a = \bar{\partial}_{\dot{\alpha}}\theta_\alpha = 0 \end{cases} \tag{A.2.32}$$

and so we define the raising and lowering of indices as

$$\eta^{ab}\partial_b = \partial^a \qquad \varepsilon^{\alpha\beta}\partial_\beta = -\partial^\alpha \qquad \varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\partial}^{\dot{\beta}} = -\bar{\partial}_{\dot{\alpha}}. \tag{A.2.33}$$

Integration over Grassmann odd parameters was first treated by Berezin [143] and is equivalent to differentiation. The basic properties for integration over superspace and its (anti)chiral subspaces are as follows:

$$\begin{gathered}\int \mathrm{d}\theta_\alpha\, \theta^\beta = \partial_\alpha \theta^\beta = \delta_\alpha{}^\beta \\ \mathrm{d}^2\theta = \frac{1}{4}\varepsilon^{\alpha\beta}\mathrm{d}\theta_\alpha \mathrm{d}\theta_\beta \qquad \Rightarrow \int \mathrm{d}^2\theta = \frac{1}{4}\partial^\alpha\partial_\alpha \qquad \int \mathrm{d}^2\theta\,\theta^2 = 1\end{gathered} \tag{A.2.34}$$

$$\begin{gathered}\int \mathrm{d}\bar\theta^{\dot\alpha}\, \bar\theta_{\dot\beta} = \bar\partial^{\dot\alpha}\bar\theta_{\dot\beta} = \delta^{\dot\alpha}{}_{\dot\beta} \\ \mathrm{d}^2\bar\theta = \frac{1}{4}\varepsilon_{\dot\alpha\dot\beta}\mathrm{d}\bar\theta^{\dot\alpha}\mathrm{d}\bar\theta^{\dot\beta} \qquad \to \int \mathrm{d}^2\bar\theta = \frac{1}{4}\bar\partial_{\dot\alpha}\bar\partial^{\dot\alpha} \qquad \int \mathrm{d}^2\bar\theta\,\bar\theta^2 = 1\end{gathered} \tag{A.2.35}$$

$$\mathrm{d}^8 z = \mathrm{d}^4 x\, \mathrm{d}^2\theta\, \mathrm{d}^2\bar\theta \qquad \mathrm{d}^6 z = \mathrm{d}^4 x\, \mathrm{d}^2\theta \qquad \mathrm{d}^6 \bar z = \mathrm{d}^4 x\, \mathrm{d}^2\bar\theta \tag{A.2.36}$$

$$\left(\int \mathrm{d}^8 z\, V(z)\right)^* = \int \mathrm{d}^8 z\, (V(z))^* \qquad \left(\int \mathrm{d}^6 z\, V(z)\right)^* = \int \mathrm{d}^6 \bar z\, (V(z))^* \tag{A.2.37}$$

where $V(z)$ is an arbitrary superfield. The limits of integration over x^a are assumed to be $(-\infty, \infty)$ unless otherwise stated. Ignoring any contributions from the component fields in these limits, integration by parts is in general given by

$$\int \mathrm{d}^4 x\, \partial_a(V(z)) = 0 \qquad \int \mathrm{d}^2\theta\, \partial_\alpha(V(z)) = 0 \qquad \int \mathrm{d}^2\bar\theta\, \bar\partial_{\dot\alpha}(V(z)) = 0 \tag{A.2.38}$$

for arbitrary superfield $V(z)$. For example:

$$\int \mathrm{d}^8 z\, \partial_A(U(z)V(z)) = 0 \tag{A.2.39}$$

for superfields $U(z)$ and $V(z)$, implies

$$\int \mathrm{d}^8 z\, (\partial_A U(z))V(z) = -(-1)^{\varepsilon(U)\varepsilon_A} \int \mathrm{d}^8 z\, U(z)(\partial_A V(z)). \tag{A.2.40}$$

A change of integration variables $z^A \mapsto z'^A(z)$ gives rise to a Jacobian like factor known as the superdeterminant or Berezinian, $\mathrm{Ber}(\partial z'^B/\partial z^A)$ (for details see [89, 127]. We will only consider changes of coordinates consistent with super Poincaré transformations (2.1.4), and so for our purposes is sufficient to note the Berezinian is unity in this case.

Delta functions can be defined on $\mathbb{R}^{4|4}$ and its subspaces as follows:

$$\delta^{(8)}(z - z') = \delta^{(4)}(x - x')\, \delta^{(2)}(\theta - \theta')\, \delta^{(2)}(\bar\theta - \bar\theta') \tag{A.2.41}$$

$$\delta^{(2)}(\theta - \theta') = (\theta - \theta')^2 \qquad \delta^{(2)}(\bar\theta - \bar\theta') = (\bar\theta - \bar\theta')^2 \tag{A.2.42}$$

$$\delta_+(z - z') = \frac{1}{4}\bar\partial_{\dot\alpha}\bar\partial^{\dot\alpha}\delta^{(8)}(z - z') = \delta^{(4)}(x - x')\, \delta^{(2)}(\theta - \theta') \tag{A.2.43}$$

$$\delta_-(z - z') = \frac{1}{4}\partial^\alpha\partial_\alpha\delta^{(8)}(z - z') = \delta^{(4)}(x - x')\, \delta^{(2)}(\bar\theta - \bar\theta') \tag{A.2.44}$$

They have properties

$$\int \mathrm{d}^8 z \, \delta^{(8)}(z - z') V(z) = V(z') \tag{A.2.45}$$

$$\delta^{(8)}(z) = \delta^{(8)}(-z) \qquad\qquad (\delta^{(8)}(z))^2 = \delta^{(8)}(0) = 0 \tag{A.2.46}$$

$$\int \mathrm{d}^6 z \, \delta_+(z - z') \Phi(z) = \Phi(z') \qquad\qquad \bar{D}_{\dot{\alpha}} \Phi(z) = 0 \tag{A.2.47}$$

$$\int \mathrm{d}^6 \bar{z} \, \delta_-(z - z') \bar{\Phi}(z) = \bar{\Phi}(z') \qquad\qquad D_\alpha \bar{\Phi}(z) = 0. \tag{A.2.48}$$

Appendix B

Some derivations

Here we collect together the derivations of a few results which have been used extensively throughout this work. We make use of the gauge covariant derivative algebra (5.1.5), the equations of motion

$$\mathcal{D}^{\alpha} W_{\alpha} = \bar{\mathcal{D}}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} = 0, \tag{B.1.1}$$

the (anti)chirality of the field strengths

$$\mathcal{D}_{\alpha} \bar{W}_{\dot{\alpha}} = \bar{\mathcal{D}}_{\dot{\alpha}} W_{\alpha} = 0, \tag{B.1.2}$$

and the generalized Jacobi identity

$$(-1)^{\varepsilon_A \varepsilon_C} \, [A, [B, C\}\} + (-1)^{\varepsilon_C \varepsilon_B} \, [C, [A, B\}\} + (-1)^{\varepsilon_B \varepsilon_A} \, [B, [C, A\}\} = 0, \tag{B.1.3}$$

where the graded commutator $[.,.\}$ was defined in (2.1.17).

Since

$$\mathcal{D}_{\alpha} W_{\beta} - \mathcal{D}_{\beta} W_{\alpha} = \varepsilon_{\alpha\beta} \mathcal{D}^{\gamma} W_{\gamma}, \tag{B.1.4}$$

from the equations of motion it follows that

$$\mathcal{D}_{\alpha} W_{\beta} = \mathcal{D}_{\beta} W_{\alpha}. \tag{B.1.5}$$

Similarly one can show

$$\bar{\mathcal{D}}_{\dot{\alpha}} \bar{W}_{\dot{\beta}} = \bar{\mathcal{D}}_{\dot{\beta}} \bar{W}_{\dot{\alpha}}. \tag{B.1.6}$$

It is useful to establish $\mathcal{D}_{\alpha} \mathcal{D}_{\beta} W_{\gamma} = \bar{\mathcal{D}}_{\dot{\alpha}} \bar{\mathcal{D}}_{\dot{\beta}} \bar{W}_{\dot{\gamma}} = 0$, which is easily demonstrated by noting

$$\begin{aligned} \mathcal{D}_{\alpha} \mathcal{D}_{\beta} W_{\gamma} &= \mathcal{D}_{\alpha} \mathcal{D}_{\gamma} W_{\beta} = -\mathcal{D}_{\gamma} \mathcal{D}_{\alpha} W_{\beta} = -\mathcal{D}_{\gamma} \mathcal{D}_{\beta} W_{\alpha} = \mathcal{D}_{\beta} \mathcal{D}_{\gamma} W_{\alpha} = \mathcal{D}_{\beta} \mathcal{D}_{\alpha} W_{\gamma} \\ &= -\mathcal{D}_{\alpha} \mathcal{D}_{\beta} W_{\gamma}, \end{aligned} \tag{B.1.7}$$

from which the desired result follows.

A less obvious result, $(\mathcal{D}^a G_{ab}) = \mathrm{i}(\sigma_b)_{\alpha\dot\alpha}\{W^\alpha, \bar W^{\dot\alpha}\}$, which is equivalent to establishing $(\mathcal{D}^{\dot\alpha\alpha} G_{\alpha\dot\alpha\beta\dot\beta}) = 4\mathrm{i}\{W^\beta, \bar W^{\dot\beta}\}$ with $G_{\alpha\dot\alpha\beta\dot\beta} \equiv (\sigma_a)_{\alpha\dot\alpha}(\sigma_b)_{\beta\dot\beta} G_{ab}$, is proven as follows:

$$
\begin{aligned}
(\mathcal{D}^{\dot\alpha\alpha} G_{\alpha\dot\alpha\beta\dot\beta}) &= [\mathcal{D}^{\dot\alpha\alpha}, [\mathcal{D}_{\alpha\dot\alpha}, \mathcal{D}_{\beta\dot\beta}\}\} \\
&= [\mathcal{D}^{\dot\alpha\alpha}, -\varepsilon_{\alpha\beta}(\bar{\mathcal{D}}_{\dot\alpha}\bar W_{\dot\beta}) - \varepsilon_{\dot\alpha\dot\beta}(\mathcal{D}_\alpha W_\beta)\} \\
&= -\,[\mathcal{D}_{\beta\dot\alpha}, (\bar{\mathcal{D}}_{\dot\beta}\bar W^{\dot\alpha})\} - [\mathcal{D}_{\alpha\dot\beta}, (\mathcal{D}_\beta W^\alpha)\} \qquad \text{using (B.1.5), (B.1.6)} \\
&= -\,[\mathcal{D}_{\beta\dot\alpha}, [\bar{\mathcal{D}}_{\dot\beta}, \bar W^{\dot\alpha}\}\} - [\mathcal{D}_{\alpha\dot\beta}, [\mathcal{D}_\beta, W^\alpha\}\} \\
&= -\,[\bar W^{\dot\alpha}, [\mathcal{D}_{\beta\dot\alpha}, \bar{\mathcal{D}}_{\dot\beta}\}\} - [W^\alpha, [\mathcal{D}_{\alpha\dot\beta}, \mathcal{D}_\beta\}\} \quad \text{using (B.1.1), (B.1.2), (B.1.3)} \\
&= -\,[\bar W^{\dot\alpha}, 2\mathrm{i}\varepsilon_{\dot\beta\dot\alpha} W_\beta\} - [W^\alpha, 2\mathrm{i}\varepsilon_{\beta\alpha}\bar W_{\dot\beta}\} \\
&= 4\mathrm{i}\{W^\beta, \bar W^{\dot\beta}\}. \qquad \text{(B.1.8)}
\end{aligned}
$$

Using the X_A algebra (5.1.13), one can now establish the commutation relations (5.2.33):

$$
\begin{aligned}
[\Delta, X_a] &= [X^b X_b - W^\alpha X_\alpha - \bar W^{\dot\alpha}\bar X_{\dot\alpha}, X_a\} \\
&= X^b\,[X_b, X_a\} + [X^b, X_a\}\,X^b - W^\alpha\,[X_\alpha, X_a\} \\
&\quad - [W^\alpha, X_a\}\,X_\alpha - \bar W^{\dot\alpha}\,[\bar X_{\dot\alpha}, X_a\} - [\bar W^{\dot\alpha}, X_a\}\,\bar X_{\dot\alpha} \\
&= (\mathcal{D}^b G_{ba}) + 2G^b{}_a X^b - \mathrm{i}(\sigma_a)_{\alpha\dot\alpha} W^\alpha \bar W^{\dot\alpha} \\
&\quad + (\mathcal{D}_a W^\alpha) X_\alpha - \mathrm{i}(\sigma_a)_{\alpha\dot\alpha}\bar W^{\dot\alpha} W^\alpha + (\mathcal{D}_a \bar W^{\dot\alpha})\bar X_{\dot\alpha} \\
&= 2G^b{}_a X^b + (\mathcal{D}_a W^\alpha) X_\alpha + (\mathcal{D}_a \bar W^{\dot\alpha})\bar X_{\dot\alpha}, \qquad \text{(B.1.9)}
\end{aligned}
$$

where (B.1.8) has been used. Likewise,

$$
\begin{aligned}
[\Delta, X_\beta] &= [X^a X_a - W^\alpha X_\alpha - \bar W^{\dot\alpha}\bar X_{\dot\alpha}, X_\beta\} \\
&= [X^a, [X_a, X_\beta\}\} + 2\,[X^a, X_\beta\}\,X_a - W^\alpha\,[X_\alpha, X_\beta\} \\
&\quad - [W^\alpha, X_\beta\}\,X_\alpha - \bar W^{\dot\alpha}\,[\bar X_{\dot\alpha}, X_\beta\} - [\bar W^{\dot\alpha}, X_\beta\}\,\bar X_{\dot\alpha} \\
&= -\mathrm{i}(\sigma_a)_{\beta\dot\beta}\,[X^a, \bar W^{\dot\beta}\} - 2\mathrm{i}(\sigma^a)_{\beta\dot\beta}\bar W^{\dot\beta} X_a + (\mathcal{D}_\beta W^\alpha) X_\alpha + 2\mathrm{i}\bar W^{\dot\alpha}\mathcal{D}_{\beta\dot\alpha} \\
&= (\mathcal{D}_\beta W^\alpha) X_\alpha, \qquad \text{(B.1.10)}
\end{aligned}
$$

having used (B.1.1) and (B.1.2). The relation $[\Delta, \bar X_{\dot\alpha}] = (\bar{\mathcal{D}}_{\dot\alpha}\bar W^{\dot\beta})\bar X_{\dot\beta}$ is similarly established. The last of the commutation relations in (5.2.33) is straight forward to compute.

Other on-shell identities, such as those found in (5.3.70), are established by more direct means but involve similar manipulations. For example $\mathcal{D}_\alpha\mathcal{D}_\beta\bar{\mathcal{D}}_{\dot\alpha}\bar W_{\dot\beta} = 4\varepsilon_{\alpha\beta}\{\bar W_{\dot\alpha}, \bar W_{\dot\beta}\}$ is derived by simply pulling the undotted covariant derivatives through the dotted derivative to annihilate the antichiral field strength.

Appendix C

The Gaussian approach in Mathematica

This appendix details the method in which *Mathematica* has been used to aid the computation of various DeWitt-Seeley coefficients in superspace and their bosonic fieldstrength component extraction. It is presented in the style of an annotated *Mathematica* notebook using its *standard form* conventions, where: bold maths typeset is to be read as input; indented maths typeset as output (most of which is suppressed due to size); and comments are italicized. Included are all the essential ingredients for the calculation of higher order coefficients, but their usage is illustrated through the full computation of $\mathrm{tr}_{\mathrm{Ad}}(a_5)$ only[1]. An attempt has been made to maintain the previously defined notations, and any deviations from these have been highlighted.

Obvious modifications and additions to the code are necessary for the computation of chiral coefficients $a_n^+(z)$ and higher orders.

The code was originally written with the intention of checking and augmenting low order computations done by hand. As such, its usage is rather adhoc and merely mirrors the processes one follows in performing the computations manually. The procedure is therefore not fully automated and requires much user intervention and input, particularly in simplifying results. This particular hybrid approach has probably been pushed to its practical limit, and if one where to proceed to compute higher order coefficients than those presented in the body of the thesis, the method would need to be more fully automated.

[1] See the attached CD for the complete $\mathrm{tr}_{\mathrm{Ad}}(a_6)$ computation using *Mathematica*.

C.1 Code

C.1.1 Noncommutative multiplication $(\cdot)$ and c-numbers

$$A\_ \cdot (B_{\_} + C_{\_}) := A \cdot B + A \cdot C$$

$$A\_ \cdot (B_{\_} + C_{\_}) \cdot D := A \cdot B \cdot D + A \cdot C \cdot D$$

$$(A_{\_} + B_{\_}) \cdot C\_ := A \cdot C + B \cdot C$$

$$A\_ \cdot (B\_ + C\_) := A \cdot B + A \cdot C$$

$$A\_ \cdot (B\_ + C\_) \cdot D := A \cdot B \cdot D + A \cdot C \cdot D$$

$$(A\_ + B\_) \cdot C\_ := A \cdot C + B \cdot C$$

$$A_{\_} \cdot 0 := 0$$

$$A\_ \cdot 0 := 0$$

$$0 \cdot A_{\_} := 0$$

$$0 \cdot A\_ := 0$$

$$A\_ \cdot 0 \cdot B\_ := 0$$

$$A_{\_} \cdot \mathcal{I} := A$$

$$A\_ \cdot B_{\_} \cdot \mathcal{I} := A \cdot B$$

$$\mathcal{I} \cdot A_{\_} := A$$

$$\mathcal{I} \cdot A_{\_} \cdot B\_ := A \cdot B$$

$$A\_ \cdot \mathcal{I} \cdot B\_ := A \cdot B$$

$$(a_{\_} * A_{\_}) \cdot B\_ := \mathrm{If}[\mathrm{num}[a], aA \cdot B, \mathrm{If}[\mathrm{num}[A], Aa \cdot B, \mathrm{error}]]$$

$$A\_ \cdot (a_{\_} * B_{\_}) := \mathrm{If}[\mathrm{num}[a], aA \cdot B, \mathrm{If}[\mathrm{num}[B], BA \cdot a, \mathrm{error}]]$$

$$C\_ \cdot (a_{\_} * A_{\_}) \cdot B\_ := \mathrm{If}[\mathrm{num}[a], aC \cdot A \cdot B, \mathrm{If}[\mathrm{num}[A], AC \cdot a \cdot B, \mathrm{error}]]$$

$$\mathrm{num}[a_{\_}] := \mathrm{NumberQ}[a]$$

$$\mathrm{num}[a_{\_}\ b_{\_}] := \mathrm{True}/; \mathrm{num}[a] == \mathrm{True}\ \&\&\ \mathrm{num}[b] == \mathrm{True}$$

$$\mathrm{num}[s] = \mathrm{True};\ \mathrm{num}[s^{n_{\_}}] := \mathrm{True}$$

$$\mathrm{num}[H] = \mathrm{True};\ \mathrm{num}[\sigma\_] := \mathrm{True}$$

$$\mathrm{num}[\epsilon_{\alpha}\_] := \mathrm{True};\ \mathrm{num}[\eta_{\alpha}\_] := \mathrm{True}$$

$$\mathrm{num}[\delta_{\alpha}\_] := \mathrm{True}$$

C.1.2 Behaviour of operators

The following differential operators Y_a and $\mathbb{D}_a$ correspond to X_a and $\mathcal{D}_a$ respectively, as defined in chapter 5. The K's are to be read as moments (see below).

$$\mathrm{op}[X_{\alpha\_}] := 1;\ \mathrm{op}[\bar{X}_{\alpha\_}] := 1;\ \mathrm{op}[Y_{a\_}] := 1$$

$$\mathrm{op}[K[s]] := 1;\ \mathrm{op}[K_{\alpha}\_[a\_\_]] := 1$$

$$\mathrm{op}[a_{\_}] := 0$$

$$X_{\alpha\_} \cdot A\_ := \mathcal{D}_\alpha[A] + (-1)^{\varepsilon[A]} A \cdot X_\alpha /; \mathrm{op}[A] == 0$$
$$X_{\alpha\_} \cdot A\_ \cdot B\_\_ := \mathcal{D}_\alpha[A] \cdot B + (-1)^{\varepsilon[A]} A \cdot X_\alpha \cdot B /; \mathrm{op}[A] == 0$$
$$B\_\_ \cdot X_{\alpha\_} \cdot A\_ := B \cdot \mathcal{D}_\alpha[A] + (-1)^{\varepsilon[A]} B \cdot A \cdot X_\alpha /; \mathrm{op}[A] == 0$$
$$B\_\_ \cdot X_{\alpha\_} \cdot A\_ \cdot C\_\_ := B \cdot \mathcal{D}_\alpha[A] \cdot C + (-1)^{\varepsilon[A]} B \cdot A \cdot X_\alpha \cdot C /; \mathrm{op}[A] == 0$$

$$\bar{\bar{X}}_{\alpha\_} \cdot A\_ := \bar{\mathcal{D}}_\alpha[A] + (-1)^{\varepsilon[A]} A \cdot \bar{X}_\alpha /; \mathrm{op}[A] == 0$$
$$\bar{X}_{\alpha\_} \cdot A\_ \cdot B\_\_ := \bar{\mathcal{D}}_\alpha[A] \cdot B + (-1)^{\varepsilon[A]} A \cdot \bar{X}_\alpha \cdot B /; \mathrm{op}[A] == 0$$
$$B\_\_ \cdot \bar{X}_{\alpha\_} \cdot A\_ := B \cdot \bar{\mathcal{D}}_\alpha[A] + (-1)^{\varepsilon[A]} B \cdot A \cdot \bar{X}_\alpha /; \mathrm{op}[A] == 0$$
$$B\_\_ \cdot \bar{X}_{\alpha\_} \cdot A\_ \cdot C\_\_ := B \cdot \bar{\mathcal{D}}_\alpha[A] \cdot C + (-1)^{\varepsilon[A]} B \cdot A \cdot \bar{X}_\alpha \cdot C /; \mathrm{op}[A] == 0$$

$$Y_{a\_} \cdot A\_ := \mathbb{D}_a[A] + (-1)^{\varepsilon[A]} A \cdot Y_\alpha /; \mathrm{op}[A] == 0$$
$$Y_{a\_} \cdot A\_ \cdot B\_\_ := \mathbb{D}_a[A] \cdot B + (-1)^{\varepsilon[A]} A \cdot Y_\alpha \cdot B /; \mathrm{op}[A] == 0$$
$$B\_\_ \cdot Y_{a\_} \cdot A\_ := B \cdot \mathbb{D}_a[A] + (-1)^{\varepsilon[A]} B \cdot A \cdot Y_\alpha /; \mathrm{op}[A] == 0$$
$$B\_\_ \cdot Y_{a\_} \cdot A\_ \cdot C\_\_ := B \cdot \mathbb{D}_a[A] \cdot C + (-1)^{\varepsilon[A]} B \cdot A \cdot Y_\alpha \cdot C /; \mathrm{op}[A] == 0$$

$$\mathcal{D}_{\alpha\_}[a\_ * A\_] := \mathrm{If}[\mathrm{num}[a], a\mathcal{D}_\alpha[A], \mathrm{If}[\mathrm{num}[A], \mathcal{D}_\alpha[a]A, \mathrm{derror}]]$$
$$\bar{\mathcal{D}}_{\alpha\_}[a\_ * A\_] := \mathrm{If}[\mathrm{num}[a], a\bar{\mathcal{D}}_\alpha[A], \mathrm{If}[\mathrm{num}[A], \bar{\mathcal{D}}_\alpha[a]A, \mathrm{derror}]]$$
$$\mathbb{D}_{a\_}[a\_ * A\_] := \mathrm{If}[\mathrm{num}[a], a\mathbb{D}_a[A], \mathrm{If}[\mathrm{num}[A], \mathbb{D}_a[a]A, \mathrm{derror}]]$$

$$\mathcal{D}_{\alpha\_}[0] := 0;\ \bar{\mathcal{D}}_{\alpha\_}[0] := 0;\ \mathbb{D}_{a\_}[0] := 0$$
$$\mathcal{D}_{\alpha\_}[\theta^{\gamma\_}] := \delta_\alpha{}^\gamma;\ \mathcal{D}_{\alpha\_}[\bar{\theta}^{\gamma\_}] := 0$$
$$\bar{\mathcal{D}}_{\alpha\_}[\theta^{\gamma\_}] := 0;\ \bar{\mathcal{D}}_{\alpha\_}[\bar{\theta}^{\gamma\_}] := -\delta_\alpha{}^\gamma$$
$$\mathbb{D}_{a\_}[\theta^{\gamma\_}] := 0;\ \mathbb{D}_{a\_}[\bar{\theta}^{\gamma\_}] := 0$$
$$\mathcal{D}_{\alpha\_}[\delta_{\beta\_}{}^{\gamma\_}] := 0;\ \bar{\mathcal{D}}_{\alpha\_}[\delta_{\beta\_}{}^{\gamma\_}] := 0;\ \mathbb{D}_a[\delta_{\beta\_}{}^{\gamma\_}] := 0$$

C.1.3 Ordering of differential operators

The following effectively places the indices on moments in the desired order: dotted spinor, undotted spinor, then spacetime.

$$Y_{a\_} \cdot X_{\alpha\_} := X_\alpha \cdot Y_a - \mathrm{i}\sigma_{a,\alpha,e_i} \bar{W}^{e_i++}$$
$$Y_{a\_} \cdot X_{\alpha\_} \cdot A\_\_ := X_\alpha \cdot Y_a \cdot A - \mathrm{i}\sigma_{a,\alpha,e_i} \bar{W}^{e_i++} \cdot A$$
$$A\_\_ \cdot Y_{a\_} \cdot X_{\alpha\_} := A \cdot X_\alpha \cdot Y_a - A \cdot (\mathrm{i}\sigma_{a,\alpha,e_i} \bar{W}^{e_i++})$$
$$A\_\_ \cdot Y_{a\_} \cdot X_{\alpha\_} \cdot B\_\_ := A \cdot X_\alpha \cdot Y_a \cdot B - A \cdot (\mathrm{i}\sigma_{a,\alpha,e_i} \bar{W}^{e_i++}) \cdot B$$

$$Y_{a\_} \cdot \bar{X}_{\alpha\_} := \bar{X}_\alpha \cdot Y_a - \mathrm{i}\sigma_{a,\alpha,f_j} W^{f_j++}$$
$$Y_{a\_} \cdot \bar{X}_{\alpha\_} \cdot A\_\_ := \bar{X}_\alpha \cdot Y_a \cdot A - \mathrm{i}\sigma_{a,\alpha,f_j} W^{f_j++} \cdot A$$
$$A\_\_ \cdot Y_{a\_} \cdot \bar{X}_{\alpha\_} := A \cdot \bar{X}_\alpha \cdot Y_a - A \cdot (\mathrm{i}\sigma_{a,\alpha,f_j} W^{f_j++})$$
$$A\_\_ \cdot Y_{a\_} \cdot \bar{X}_{\alpha\_} \cdot B\_\_ := A \cdot \bar{X}_\alpha \cdot Y_a \cdot B - A \cdot (\mathrm{i}\sigma_{a,\alpha,f_j} W^{f_j++}) \cdot B$$

As can be seen, additional dummy indices are generated incrementally.

$$\bar{X}_{\beta\_} \cdot X_{\alpha\_} := -X_\alpha \cdot \bar{X}_\beta - 2\mathrm{i}\sigma_{g_k,\alpha,\beta} Y_{g_{k++}}$$
$$\bar{X}_{\beta\_} \cdot X_{\alpha\_} \cdot A\_\_ := -X_\alpha \cdot \bar{X}_\beta \cdot A - 2\mathrm{i}\sigma_{g_k,\alpha,\beta} Y_{g_{k++}} \cdot A$$
$$A\_\_ \cdot \bar{X}_{\beta\_} \cdot X_{\alpha\_} := -A \cdot X_\alpha \cdot \bar{X}_\beta - 2\mathrm{i}\sigma_{g_k,\alpha,\beta} A \cdot Y_{g_{k++}}$$
$$A\_\_ \cdot \bar{X}_{\beta\_} \cdot X_{\alpha\_} \cdot B\_\_ := -A \cdot X_\alpha \cdot \bar{X}_\beta \cdot B - 2\mathrm{i}\sigma_{g_k,\alpha,\beta} A \cdot Y_{g_{k++}} \cdot B$$

$$i = 1;\ j = 1;\ k = 1;$$

C.1.4 Moments and their leading order behaviour

Below, $K[s]$ denotes the heat kernel $K(z;s)$, and $K_\alpha\_[s,\{p,q,r\}]$ its moments $K_{A_1 \dots A_{p+q+r}}(z;s)$ possessing p dotted spinor, q undotted spinor, and r spacetime indices.

$$X_{\alpha\_} \cdot K[s] := K_\alpha[s, \{1,0,0\}]$$
$$A\_\_ \cdot X_{\alpha\_} \cdot K[s] := A \cdot K_\alpha[s, \{1,0,0\}]$$
$$X_{\alpha\_} \cdot K_{\beta}\_[s, \{a\_, b\_, c\_\}] := K_{\alpha,\beta}[s, \{a+1, b, c\}]$$
$$A\_\_ \cdot X_{\alpha\_} \cdot K_{\beta}\_[s, \{a\_, b\_, c\_\}] := A \cdot K_{\alpha,\beta}[s, \{a+1, b, c\}]$$

$$\bar{X}_{\alpha\_} \cdot K[s] := K_\alpha[s, \{0,1,0\}]$$
$$A\_\_ \cdot \bar{X}_{\alpha\_} \cdot K[s] := A \cdot K_\alpha[s, \{0,1,0\}]$$
$$\bar{X}_{\alpha\_} \cdot K_{\beta}\_[s, \{a\_, b\_, c\_\}] := K_{\alpha,\beta}[s, \{a, b+1, c\}]$$
$$A\_\_ \cdot \bar{X}_{\alpha\_} \cdot K_{\beta}\_[s, \{a\_, b\_, c\_\}] := A \cdot K_{\alpha,\beta}[s, \{a, b+1, c\}]$$

$$Y_{\alpha\_} \cdot K[s] := K_\alpha[s, \{0,0,1\}]$$
$$A\_\_ \cdot Y_{\alpha\_} \cdot K[s] := A \cdot K_\alpha[s, \{0,0,1\}]$$
$$Y_{\alpha\_} \cdot K_{\beta}\_[s, \{a\_, b\_, c\_\}] := K_{\alpha,\beta}[s, \{a, b, c+1\}]$$
$$A\_\_ \cdot Y_{\alpha\_} \cdot K_{\beta}\_[s, \{a\_, b\_, c\_\}] := A \cdot K_{\alpha,\beta}[s, \{a, b, c+1\}]$$

Moments with greater than two undotted or dotted spinor indices vanish

$$K_\alpha\_[s, \{a\_, b\_, c\_\}] := 0/; a \geq 3 \vee b \geq 3$$

Leading order behaviour is made explicit (see equation (5.2.34))

$$\mathbf{kc1} = \{K_\alpha\_[s, \{a\_, b\_, c\_\}] \to K_\alpha[s, \{a, b, c\}, 2 - (a + b + \mathrm{IntegerPart}[\frac{c}{2}])],$$
$$K[s] \to K_0[s, \{0,0,0\}, 2]\}$$
$$\mathbf{kc2} = \{s^{n\_} A\_\_\_ \cdot K_\alpha\_[s, a\_\_, b\_] \to s^n A \cdot K_\alpha[s, a, b+n],$$
$$s\, A\_\_\_ \cdot K_\alpha\_[s, a\_\_, b\_] \to s^n A \cdot K_\alpha[s, a, b+1]\}$$

C.1.5 Chirality conditions and equations of motion

$$\mathcal{D}_{\gamma\_}[\bar{W}^{\alpha\_}] := 0;\ \mathcal{D}_{\gamma\_}[\bar{W}_{\alpha\_}] := 0$$
$$\bar{\mathcal{D}}_{\gamma\_}[W^{\alpha\_}] := 0;\ \bar{\mathcal{D}}_{\gamma\_}[W_{\alpha\_}] := 0$$
$$\mathcal{D}_{\gamma\_}[W^{\gamma\_}] := 0;\ \bar{\mathcal{D}}_{\gamma\_}[\bar{W}^{\gamma\_}] := 0$$
$$\mathcal{D}_{\alpha\_}[\mathcal{D}_{\beta\_}[W^{\gamma\_}]] := 0;\ \mathcal{D}_{\alpha\_}[\mathcal{D}_{\beta\_}[W_{\gamma\_}]] := 0$$
$$\bar{\mathcal{D}}_{\alpha\_}[\bar{\mathcal{D}}_{\beta\_}[\bar{W}^{\gamma\_}]] := 0;\ \bar{\mathcal{D}}_{\alpha\_}[\bar{\mathcal{D}}_{\beta\_}[\bar{W}_{\gamma\_}]] := 0$$

C.1.6 Graded commutators, $\langle\ ,\ \rangle$, and (anti)commutation relations

Below $\Delta_{a,\alpha,\dot{\alpha}}$ denotes the operator Δ defined in equation (5.1.11). For simplicity all spacetime indices are placed in the lower position, and as usual repeated indices indicate contraction.

$$\langle A\_ \cdot B\_, C\_\rangle := A \cdot \langle B, C\rangle + (-1)^{\varepsilon[B]\varepsilon[C]}\langle A, C\rangle \cdot B$$
$$\langle A\_ \cdot B\_ \cdot D\_\_, C\_\rangle := A \cdot \langle B\cdot D, C\rangle + (-1)^{\varepsilon[B\cdot D]\varepsilon[C]}\langle A, C\rangle \cdot B \cdot D$$
$$\langle A\_, B\_ \cdot C\_\rangle := \langle A, B\rangle \cdot C + (-1)^{\varepsilon[B]\varepsilon[A]} B \cdot \langle A, C\rangle$$
$$\langle A\_, B\_ \cdot C\_ \cdot D\_\rangle := \langle A, B\rangle \cdot C \cdot D + (-1)^{\varepsilon[B]\varepsilon[A]} B \cdot \langle A, C \cdot D\rangle$$
$$\langle A\_ + B\_, C\_\rangle := \langle A, C\rangle + \langle B, C\rangle$$
$$\langle A\_, B\_ + C\_\rangle := \langle A, B\rangle + \langle A, C\rangle$$
$$\langle a\_ * A\_, B\_\rangle := \text{If}[\text{num}[a], a\langle A, B\rangle, \text{If}[\text{num}[A], A\langle a, B\rangle, \text{gcerror}]]$$
$$\langle A\_, a\_ * B\_\rangle := \text{If}[\text{num}[a], a\langle A, B\rangle, \text{If}[\text{num}[B], B\langle A, a\rangle, \text{gcerror}]]$$

$$\langle \Delta_{a\_,\alpha\_,\beta\_}, Y_{b\_}\rangle := 2G_{a,b} \cdot Y_a + \mathbb{D}_b[W^\alpha] \cdot X_\alpha + \mathbb{D}_b[\bar{W}^\beta] \cdot \bar{X}_\beta$$
$$\langle \Delta_{a\_,\alpha\_,\beta\_}, X_{\gamma\_}\rangle := \mathcal{D}_\gamma[W^\alpha] \cdot X_\alpha$$
$$\langle \Delta_{a\_,\alpha\_,\beta\_}, \bar{X}_{\gamma\_}\rangle := \bar{\mathcal{D}}_\gamma[\bar{W}^\beta] \cdot \bar{X}_\beta$$
$$\langle \Delta_{a\_,\alpha\_,\beta\_}, A\_\rangle := \mathbb{D}_a[\mathbb{D}_a[A]] + 2\mathbb{D}_a[A] \cdot Y_a - W^\alpha \cdot \mathcal{D}_\alpha[A] - \bar{W}^\beta \cdot \bar{\mathcal{D}}_\beta[A]$$
$$-(-1)^{\varepsilon[A]}\langle W^\alpha, A\rangle \cdot X_\alpha - (-1)^{\varepsilon[A]}\langle \bar{W}^\beta, A\rangle \cdot \bar{X}_\beta$$
$$\langle \Delta_{a\_,\alpha\_,\beta\_}, \theta^{\gamma\_}\rangle := -W^\gamma$$
$$\langle \Delta_{a\_,\alpha\_,\beta\_}, \bar{\theta}^{\gamma\_}\rangle := \bar{W}^\gamma$$

$$L_0[A\_, B\_] := B$$
$$L_{n\_}[\Delta_{a_{-n\_},\alpha_{-n\_},\beta_{-n\_}}, A\_] := L_n[\Delta_{a_n,\alpha_n,\beta_n}, A]$$
$$= \langle \Delta_{a_n,\alpha_n,\beta_n}, L_{n-1}[\Delta_{a_{n-1},\alpha_{n-1},\beta_{n-1}}, A]\rangle$$

$RecursionLimit = ∞;

C.1.7 Assigning parity

$\varepsilon[X_{a\_}] := 1;\ \varepsilon[\bar{X}_{a\_}] := 1;\ \varepsilon[Y_{a\_}] := 0;\ \varepsilon[\Delta_{a\_\_}] := 0$

$\varepsilon[W^{\alpha}\_] := 1;\ \varepsilon[W_{\alpha\_}] := 1;\ \varepsilon[\bar{W}^{\alpha}\_] := 0;\ \varepsilon[\bar{W}_{\alpha\_}] := 0;\ \varepsilon[G_{a\_}] := 0$

$\varepsilon[\theta^{\alpha}\_] := 0;\ \varepsilon[\bar{\theta}^{\alpha}\_] := 0;\ \varepsilon[\delta_{\alpha\_}{}^{\beta}] := 0$

$\varepsilon[\mathcal{D}_{\alpha\_}[A\_]] := \mathrm{Mod}[\varepsilon[A]+1,2]$

$\varepsilon[\bar{\mathcal{D}}_{\alpha\_}[A\_]] := \mathrm{Mod}[\varepsilon[A]+1,2]$

$\varepsilon[\mathbb{D}_{a\_}[A\_]] := \varepsilon[A]$

$\varepsilon[\langle A\_, B\_\rangle] := \mathrm{Mod}[\varepsilon[A]+\varepsilon[B],2]$

$\varepsilon[A\_\cdot B\_] := \mathrm{Mod}[\varepsilon[A]+\varepsilon[B],2]$

$\varepsilon[A\_\cdot B\_\cdot c\_\_] := \mathrm{Mod}[\varepsilon[A]+\varepsilon[B\cdot C],2]$

$\varepsilon[a\_ * A\_\cdot B\_] := \mathrm{If}[\mathrm{num}[a], \varepsilon[A\cdot B], \mathrm{perror}]$

$\varepsilon[A\_ + B\_] := \varepsilon[A]$

$\varepsilon[a\_ * A\_] := \mathrm{If}[\mathrm{num}[a], \varepsilon[A], \mathrm{If}[\mathrm{num}[A], \varepsilon[a], \mathrm{perror}]]$

C.1.8 Index contraction, raising and lowering

$\mathrm{lower} = \{A\_\_\cdot W\_{}^{\beta}\_\cdot B\_\_\ \epsilon_{\alpha\_,\beta\_} \to A\cdot W_{\alpha}\cdot B,$

$A\_\_\cdot \mathcal{G}_{\_f\_}[W\_{}^{\beta}\_]\cdot B\_\_\ \epsilon_{\alpha\_,\beta\_} \to A\cdot \mathcal{G}_f[W_\alpha]\cdot B,$

$A\_\_\cdot \mathcal{H}_{\_k\_}[\mathcal{G}_{\_f\_}[W\_{}^{\beta}\_]]\cdot B\_\_\ \epsilon_{\alpha\_,\beta\_} \to A\cdot \mathcal{H}_K[\mathcal{G}_f[W_\alpha]]\cdot B$

$A\_\_\cdot \mathbb{D}_{a\_}[g\_]\cdot B\_\_\ \eta_{a\_,b\_} \to A\cdot \mathbb{D}_b[g]\cdot B$

$A\_\_\cdot \mathbb{D}_{b\_}[g\_]\cdot B\_\_\ \eta_{a\_,b\_} \to A\cdot \mathbb{D}_a[g]\cdot B$

$A\_\_\cdot \mathcal{G}_{\_f\_}[\mathbb{D}_{a\_}[g\_]]\cdot B\_\_\ \eta_{a\_,b\_} \to A\cdot \mathcal{G}_f[\mathbb{D}_b[g]]\cdot B$

$A\_\_\cdot \mathcal{G}_{\_f\_}[\mathbb{D}_{b\_}[g\_]]\cdot B\_\_\ \eta_{a\_,b\_} \to A\cdot \mathcal{G}_f[\mathbb{D}_a[g]]\cdot B,$

$A\_\_\cdot G_{a\_,b\_}\cdot B\_\_\ \eta_{a\_,c\_} \to A\cdot G_{cb}\cdot B,$

$A\_\_\cdot G_{a\_,b\_}\cdot B\_\_\ \eta_{c\_,a\_} \to A\cdot G_{cb}\cdot B,$

$A\_\_\cdot G_{a\_,b\_}\cdot B\_\_\ \eta_{b\_,c\_} \to A\cdot G_{ac}\cdot B,$

$A\_\_\cdot G_{a\_,b\_}\cdot B\_\_\ \eta_{c\_,b\_} \to A\cdot G_{ac}\cdot B,$

$G_{a\_,a\_} \to 0\};$

$\mathrm{conditionindex} = \{A\_\_\cdot W_{\_\alpha\_}\cdot B\_\_\cdot W\_{}^{\alpha}\_\cdot C\_\_ \to -A\cdot W^{\alpha}\cdot B\cdot W_{\alpha}\cdot C,$

$A\_\_\cdot \mathcal{G}_{\_a\_}[W_{\_\alpha\_}]\cdot B\_\_\cdot W\_{}^{\alpha}\_\cdot C\_\_ \to -A\cdot \mathcal{G}_a[W^{\alpha}]\cdot B\cdot W_{\alpha}\cdot C,$

$A\_\_\cdot \mathcal{H}_{\_b\_}[\mathcal{G}_{\_a\_}[W_{\_\alpha\_}]]\cdot B\_\_\cdot W\_{}^{\alpha}\_\cdot C\_\_ \to -A\cdot \mathcal{H}_b[\mathcal{G}_a[W^{\alpha}]]\cdot B\cdot W_{\alpha}\cdot C,$

$A\_\_\cdot \mathcal{G}_{\_a\_}[W_{\_\alpha\_}]\cdot B\_\_\cdot \mathcal{H}_{\_b\_}[W\_{}^{\alpha}\_]\cdot C\_\_ \to -A\cdot \mathcal{G}_a[W^{\alpha}]\cdot B\cdot \mathcal{H}_b[W_{\alpha}]\cdot C,$

$A\_\_\cdot W_{\_\alpha\_}\cdot B\_\_\cdot \mathcal{H}_{\_b\_}[\mathcal{G}_{\_a\_}[W\_{}^{\alpha}\_]]\cdot C\_\_ \to -A\cdot W^{\alpha}\cdot B\cdot \mathcal{H}_b[\mathcal{G}_a[W_{\alpha}]]\cdot C\};$

C.1.9 Cycling (via trace) to bring expressions into the desired basis

$$\phi[A\_] := 0$$
$$\phi[\mathcal{D}_{\alpha\_}[A\_] \cdot B\_\_] := 1$$
$$\phi[\bar{\mathcal{D}}_{\alpha\_}[A\_] \cdot B\_\_] := 1$$
$$\phi[\mathbb{D}_{a\_}[A\_] \cdot \mathbb{D}_{a\_}[B\_] \cdot C\_\_] := 1$$

$$\text{cycle}[a\_ * A\_ \cdot B\_] := \text{If}[\text{num}[a], a\ \text{cycle}[A \cdot B], \text{If}[\text{num}[A], A\ \text{cycle}[a \cdot B], \text{cerror}]]$$
$$\text{cycle}[a\_ * A\_ \cdot B\_ \cdot C\_\_] := \text{If}[\text{num}[a], a\ \text{cycle}[A \cdot B \cdot C], \text{If}[\text{num}[A], A\ \text{cycle}[a \cdot B \cdot C], \text{cerror}]]$$
$$\text{cycle}[A\_ \cdot B\_] := \text{If}[\phi[A \cdot B] == 1, A \cdot B, \text{cycle}[(-1)^{\varepsilon[A]\varepsilon[B]} B \cdot A]]$$
$$\text{cycle}[A\_ \cdot B\_ \cdot C\_] := \text{If}[\phi[A \cdot B \cdot C] == 1, A \cdot B \cdot C, \text{cycle}[(-1)^{\varepsilon[A]\varepsilon[B \cdot C]} B \cdot C \cdot A]]$$

C.1.10 Miscellaneous

In simplifying the results, we will later make use of the the graded commutator, (2.1.17), and properties of the covariant derivatives.

$$\text{expandall} = \{\langle A\_, B\_\rangle \to A \cdot B - (-1)^{\varepsilon[A]\varepsilon[B]} B \cdot A,$$
$$\mathbb{D}_{a\_}[A\_ + B\_] \to \mathbb{D}_a[A] + \mathbb{D}_a[B],$$
$$\mathbb{D}_{a\_}[A\_ \cdot B\_] \to \mathbb{D}_a[A] \cdot B + A \cdot \mathbb{D}_a[B],$$
$$\mathbb{D}_{a\_}[A\_ \cdot B\_ \cdot C\_\_] \to \mathbb{D}_a[A] \cdot B \cdot C + A \cdot \mathbb{D}_a[B \cdot C],$$
$$\mathcal{D}_{\alpha\_}[A\_ + B\_] \to \mathcal{D}_\alpha[A] + \mathcal{D}_\alpha[B],$$
$$\mathcal{D}_{\alpha\_}[A\_ \cdot B\_] \to \mathcal{D}_\alpha[A] \cdot B + (-1)^{\varepsilon[A]} A \cdot \mathcal{D}_\alpha[B],$$
$$\mathcal{D}_{\alpha\_}[A\_ \cdot B\_ \cdot C\_\_] \to \mathcal{D}_\alpha[A] \cdot B \cdot C + (-1)^{\varepsilon[A]} A \cdot \mathcal{D}_\alpha[B \cdot C],$$
$$\bar{\mathcal{D}}_{\alpha\_}[A\_ + B\_] \to \bar{\mathcal{D}}_\alpha[A] + \bar{\mathcal{D}}_\alpha[B],$$
$$\bar{\mathcal{D}}_{\alpha\_}[A\_ \cdot B\_] \to \bar{\mathcal{D}}_\alpha[A] \cdot B + (-1)^{\varepsilon[A]} A \cdot \bar{\mathcal{D}}_\alpha[B],$$
$$\bar{\mathcal{D}}_{\alpha\_}[A\_ \cdot B\_ \cdot C\_\_] \to \bar{\mathcal{D}}_\alpha[A] \cdot B \cdot C + (-1)^{\varepsilon[A]} A \cdot \bar{\mathcal{D}}_\alpha[B \cdot C]\}$$

Usage of the following reduces computation time.

$$\text{listify}[a\_ + b\_\_] := \{a, b\}$$
$$\text{unlistify}[\{a\_, b\_\_\}] := a + b$$
$$\text{ker}[a\_] := a \cdot K[s]$$

C.2 Computing $\mathbf{tr}_{\mathbf{Ad}}(a_5)$

We now compute $\text{tr}_{\text{Ad}}(a_5)$*, by summing from* $n = 1$ *to 4 in equation (5.3.42).*

C.2.1 The summation

$$\mathbf{i}\sum_{n=1}^{4}\sum_{m=1}^{n}\left(\frac{(\mathbf{i}s)^n}{m!(n-m)!(n+1)}\Big(L_m[\Delta_{a_m,\alpha_m,\beta_m},Y_c]\cdot L_{n-m}[\Delta_{\mu_{n-m},\nu_{n-m},\rho_{n-m}},Y_c]\right.$$
$$+L_m[\Delta_{a_m,\alpha_m,\beta_m},X_\omega]\cdot L_{n-m}[\Delta_{\mu_{n-m},\nu_{n-m},\rho_{n-m}},W^\omega]$$
$$\left.+L_m[\Delta_{a_m,\alpha_m,\beta_m},\bar{X}_{\dot\omega}]\cdot L_{n-m}[\Delta_{\mu_{n-m},\nu_{n-m},\rho_{n-m}},\bar{W}^{\dot\omega}]\Big)\right)$$

$\%/.\{G_{a\_,b\_}\cdot Y_{a\_}\cdot Y_{b\_}\to \frac{1}{2}G_{a,b}\cdot G_{a,b}\}$;

unlistify[Map[ker, listify[%]]];

%/.kc1;

%/.kc2;

Non-contributing moments are removed,

$\%/K_{\alpha\_}[s,a\_\_,b\_/;b\geq 3]\to 0$;

$\mathbf{total} = \%//.\{K_{\alpha\_}[s,a\_\_,b\_]\to K_\alpha[s,a], K_0[s,a\_\_]\to K[s]\}$

$$= -\frac{1}{2}s\mathbb{D}_c[W^{\alpha_1}]\cdot K_{\alpha_1,c}[s,\{1,0,1\}] - \frac{1}{2}s\mathbb{D}_c[W^{\beta_1}]\cdot K_{\beta_1,c}[s,\{0,1,1\}]$$
$$+\frac{1}{2}s\mathcal{D}_\omega[W^{\alpha_1}]\cdot W^\omega\cdot K_{\alpha_1}[s,\{1,0,0\}] + \frac{1}{2}s\bar{\mathcal{D}}_{\dot\omega}[\bar{W}^{\beta_1}]\cdot\bar{W}^{\dot\omega}\cdot K_{\beta_1}[s,\{0,1,0\}]$$
$$+\dots$$

where much of the result has been suppressed.

C.2.2 Hierarchy of moments to leading order

All contributing moments need only be asymptotically expanded to leading order, a complete list in approximately hierarchal order (see subsection 5.4.2) being given below. These moments may be computed by hand, or as shown below in section C.3. The numerical factor common to all is $H = \mathrm{i}(4\pi\mathrm{i})^{-2}$.

$\mathbf{moments} = \{K_{\alpha\_,\beta\_,\gamma\_,\delta\_,a\_,b\_}[s,\{2,2,2\}]\to -\frac{2\mathrm{i}}{s^3}\,\epsilon_{\alpha,\beta}\,\epsilon_{\gamma,\delta}\,\eta_{a,b}\,\mathcal{I}\,H,$

$K_{\alpha\_,\beta\_,\gamma\_,\delta\_}[s,\{2,2,0\}]\to -\frac{4}{s^2}\,\epsilon_{\alpha,\beta}\,\epsilon_{\gamma,\delta}\,\mathcal{I}\,H,$

$K_{\alpha\_,\beta\_,\gamma\_,a\_,b\_}[s,\{2,1,2\}]\to \frac{2}{s^2}\,\epsilon_{\alpha,\beta}\,\eta_{a,b}\,\bar{W}_\gamma\,H,$

$K_{\alpha\_,\beta\_,\gamma\_,a\_,b\_}[s,\{1,2,2\}]\to \frac{2}{s^2}\,\epsilon_{\beta,\gamma}\,\eta_{a,b}\,W_\alpha\,H,$

$K_{\alpha\_,\beta\_,\gamma\_,\delta\_,a\_}[s,\{2,2,1\}]\to 0$

$$K_{\alpha\_,\beta\_,\gamma\_}[s,\{2,1,0\}] \to -\frac{4\mathrm{i}}{s}\ \epsilon_{\alpha,\beta}\ \bar{W}_{\gamma}\ H,$$
$$K_{\alpha\_,\beta\_,\gamma\_}[s,\{1,2,0\}] \to -\frac{4\mathrm{i}}{s}\ \epsilon_{\beta,\gamma}\ W_{\alpha}\ H,$$
$$K_{\alpha\_,\beta\_,a\_,b\_}[s,\{2,0,2\}] \to -\frac{\mathrm{i}}{s}\ \epsilon_{\alpha,\beta}\ \eta_{a,b}\ \bar{W}^{\dot{\gamma}}\cdot\bar{W}_{\dot{\gamma}}\ H,$$
$$K_{\alpha\_,\beta\_,a\_,b\_}[s,\{1,1,2\}] \to \frac{\mathrm{i}}{s}\eta_{a,b}\big(W_{\alpha}\cdot\bar{W}_{\beta}-\bar{W}_{\beta}\cdot W_{\alpha}\big)H,$$
$$K_{\alpha\_,\beta\_,a\_,b\_}[s,\{0,2,2\}] \to -\frac{\mathrm{i}}{s}\ \epsilon_{\alpha,\beta}\ \eta_{a,b}\ W^{\gamma}\cdot W_{\gamma}\ H,$$
$$K_{\alpha\_,\beta\_,\gamma\_,a\_}[s,\{2,1,1\}] \to -\frac{2\mathrm{i}}{s}\ \epsilon_{\alpha,\beta}\ \mathbb{D}_a[\bar{W}_{\gamma}]\ H,$$
$$K_{\alpha\_,\beta\_,\gamma\_,a\_}[s,\{1,2,1\}] \to -\frac{2\mathrm{i}}{s}\ \epsilon_{\beta,\gamma}\ \mathbb{D}_a[W_{\alpha}]\ H$$
$$K_{\alpha\_,\beta\_}[s,\{2,0,0\}] \to -2\epsilon_{\alpha,\beta}\bar{W}^{\dot{\gamma}}\cdot\bar{W}_{\dot{\gamma}}\ H,$$
$$K_{\alpha\_,\beta\_}[s,\{1,1,0\}] \to 2\big(W_{\alpha}\cdot\bar{W}_{\beta}-\bar{W}_{\beta}\cdot W_{\alpha}\big)H,$$
$$K_{\alpha\_,\beta\_}[s,\{0,2,0\}] \to -2\epsilon_{\alpha,\beta}W^{\gamma}\cdot W_{\gamma}\ H,$$
$$K_{\alpha\_,a\_,b\_}[s,\{1,0,2\}] \to \frac{1}{3}\big(W_{\alpha}\cdot\bar{W}^{\dot{\gamma}}\cdot\bar{W}_{\dot{\gamma}}+\bar{W}^{\dot{\gamma}}\cdot\bar{W}_{\dot{\gamma}}\cdot W_{\alpha}-\bar{W}^{\dot{\gamma}}\cdot W_{\alpha}\cdot\bar{W}_{\dot{\gamma}}\big)\eta_{a,b}\ H,$$
$$K_{\alpha\_,a\_,b\_}[s,\{0,1,2\}] \to \frac{1}{3}\big(W^{\gamma}\cdot W_{\gamma}\cdot\bar{W}_{\dot{\alpha}}+\bar{W}_{\dot{\alpha}}\cdot W^{\gamma}\cdot W_{\gamma}-W^{\gamma}\cdot\bar{W}^{\dot{\alpha}}\cdot W_{\gamma}\big)\eta_{a,b}\ H,$$
$$K_{\alpha\_,\beta\_,a\_}[s,\{2,0,1\}] \to -\frac{2}{3}\ \epsilon_{\alpha,\beta}\big(\bar{W}^{\dot{\gamma}}\cdot\mathbb{D}_a[\bar{W}_{\dot{\gamma}}]+2\mathbb{D}_a[\bar{W}^{\dot{\gamma}}]\cdot\bar{W}_{\dot{\gamma}}\big)H,$$
$$K_{\alpha\_,\beta\_,a\_}[s,\{1,1,1\}] \to \frac{2}{3}\big(W_{\alpha}\cdot\mathbb{D}_a[\bar{W}_{\beta}]-\bar{W}_{\beta}\cdot\mathbb{D}_a[W_{\alpha}]$$
$$+2\mathbb{D}_a[W_{\alpha}]\cdot\bar{W}_{\beta}-2\mathbb{D}_a[\bar{W}_{\beta}]\cdot W_{\alpha}\big)H,$$
$$K_{\alpha\_,\beta\_,a\_}[s,\{0,2,1\}] \to -\frac{2}{3}\ \epsilon_{\alpha,\beta}\big(W^{\gamma}\cdot\mathbb{D}_a[W_{\gamma}]+2\mathbb{D}_a[W^{\gamma}]\cdot W_{\gamma}\big)H,$$
$$K_{\alpha\_}[s,\{1,0,0\}] \to \frac{2\mathrm{i}s}{3}\big(\bar{W}^{\dot{\gamma}}\cdot W_{\alpha}\cdot\bar{W}_{\dot{\gamma}}-\bar{W}^{\dot{\gamma}}\cdot\bar{W}_{\dot{\gamma}}\cdot W_{\alpha}-W_{\alpha}\cdot\bar{W}^{\dot{\gamma}}\cdot\bar{W}_{\dot{\gamma}}\big)H,$$
$$K_{\alpha\_}[s,\{0,1,0\}] \to \frac{2\mathrm{i}s}{3}\big(W^{\gamma}\cdot\bar{W}_{\alpha}\cdot W_{\gamma}-W^{\gamma}\cdot W_{\gamma}\cdot\bar{W}_{\alpha}-\bar{W}_{\alpha}\cdot W^{\gamma}\cdot W_{\gamma}\big)H,$$
$$K_{\alpha\_,a\_}[s,\{0,1,1\}] \to \frac{\mathrm{i}s}{6}\big(W^{\gamma}\cdot\bar{W}_{\alpha}\cdot\mathbb{D}_a[W_{\gamma}]-W^{\gamma}\cdot W_{\gamma}\cdot\mathbb{D}_a[\bar{W}_{\alpha}]$$
$$-\bar{W}_{\alpha}\cdot W^{\gamma}\cdot\mathbb{D}_a[W_{\gamma}]+2W^{\gamma}\cdot\mathbb{D}_a[\bar{W}_{\alpha}]\cdot W_{\gamma}-2W^{\gamma}\cdot\mathbb{D}_a[W_{\gamma}]\cdot\bar{W}_{\alpha}$$
$$-2\bar{W}_{\alpha}\cdot\mathbb{D}_a[W^{\gamma}]\cdot W_{\gamma}+3\mathbb{D}_a[W^{\gamma}]\cdot\bar{W}_{\alpha}\cdot W_{\gamma}-3\mathbb{D}_a[W^{\gamma}]\cdot W_{\gamma}\cdot\bar{W}_{\alpha}$$
$$-3\mathbb{D}_a[\bar{W}_{\alpha}]\cdot W^{\gamma}\cdot W_{\gamma}\big)H$$
$$K_{\alpha\_,a\_}[s,\{1,0,1\}] \to \frac{\mathrm{i}s}{6}\big(\bar{W}^{\dot{\gamma}}\cdot W_{\alpha}\cdot\mathbb{D}_a[\bar{W}_{\dot{\gamma}}]-\bar{W}^{\dot{\gamma}}\cdot\bar{W}_{\dot{\gamma}}\cdot\mathbb{D}_a[W_{\alpha}]$$
$$-W_{\alpha}\cdot\bar{W}^{\dot{\gamma}}\cdot\mathbb{D}_a[\bar{W}_{\dot{\gamma}}]+2\bar{W}^{\dot{\gamma}}\cdot\mathbb{D}_a[W_{\alpha}]\cdot\bar{W}_{\dot{\gamma}}-2\bar{W}^{\dot{\gamma}}\cdot\mathbb{D}_a[\bar{W}_{\dot{\gamma}}]\cdot W_{\alpha}$$
$$-2W_{\alpha}\cdot\mathbb{D}_a[\bar{W}^{\dot{\gamma}}]\cdot\bar{W}_{\dot{\gamma}}+3\mathbb{D}_a[\bar{W}^{\dot{\gamma}}]\cdot W_{\alpha}\cdot\bar{W}_{\dot{\gamma}}-3\mathbb{D}_a[\bar{W}^{\dot{\gamma}}]\cdot\bar{W}_{\dot{\gamma}}\cdot W_{\alpha}$$
$$-3\mathbb{D}_a[W_{\alpha}]\cdot\bar{W}^{\dot{\gamma}}\cdot\bar{W}_{\dot{\gamma}}\big)H\};$$

C.2.3 Simplification

At this stage we have effectively computed $a_5(z)$*, and only need simplify the result. This is essentially achieved through pattern matching, after first having decided on a basis.*

$$\textbf{total//.moments//Expand;}$$

$$\textbf{\%//.expandall//Expand;}$$

$$\textbf{\%//.lower//.conditionindex;}$$

Selective integration by parts brings any terms containing two spacetime covariant derivatives not acting on adjacent field strengths into such a form.

$$\textbf{\%//.}\{\boldsymbol{A\_} \cdot \mathbb{D}_{\boldsymbol{a\_}}[\boldsymbol{B\_}] \cdot \boldsymbol{C\_} \cdot \mathbb{D}_{\boldsymbol{b\_}}[\boldsymbol{D\_}] \to (-1)^{\varepsilon[\mathbb{D}_b[D]]\varepsilon[A\cdot\mathbb{D}_a[B]\cdot C]}\mathbb{D}_{\boldsymbol{b}}[\boldsymbol{D}] \cdot \boldsymbol{A} \cdot \mathbb{D}_{\boldsymbol{a}}[\boldsymbol{B}] \cdot \boldsymbol{C},$$
$$\mathbb{D}_{\boldsymbol{b\_}}[\boldsymbol{D\_}] \cdot \boldsymbol{A\_} \cdot \mathbb{D}_{\boldsymbol{a\_}}[\boldsymbol{B\_}] \cdot \boldsymbol{C\_} \to -\boldsymbol{D} \cdot \mathbb{D}_{\boldsymbol{b}}[\boldsymbol{A} \cdot \mathbb{D}_{\boldsymbol{a}}[\boldsymbol{B}] \cdot \boldsymbol{C}]\}\textbf{//.expandall//Expand;}$$

On-shell identities (see equation (5.3.70)) are implemented.

$$\textbf{\%//.}\{\mathbb{D}_{\boldsymbol{a\_}}[\mathbb{D}_{\boldsymbol{a\_}}[\boldsymbol{W}^{\boldsymbol{\gamma}}\_]] \to \langle \boldsymbol{W}^{\boldsymbol{\psi}}, \boldsymbol{\mathcal{D}}_{\boldsymbol{\psi}}[\boldsymbol{W}^{\boldsymbol{\gamma}}]\rangle,\ \mathbb{D}_{\boldsymbol{a\_}}[\mathbb{D}_{\boldsymbol{a\_}}[\boldsymbol{W}_{\boldsymbol{\gamma}\_}]] \to \langle \boldsymbol{W}^{\boldsymbol{\psi}}, \boldsymbol{\mathcal{D}}_{\boldsymbol{\psi}}[\boldsymbol{W}_{\boldsymbol{\gamma}}]\rangle,$$
$$\mathbb{D}_{\boldsymbol{a\_}}[\mathbb{D}_{\boldsymbol{a\_}}[\bar{\boldsymbol{W}}^{\boldsymbol{\gamma}}\_]] \to \langle \bar{\boldsymbol{W}}^{\dot{\boldsymbol{\psi}}}, \bar{\boldsymbol{\mathcal{D}}}_{\dot{\boldsymbol{\psi}}}[\bar{\boldsymbol{W}}^{\boldsymbol{\gamma}}]\rangle,\ \mathbb{D}_{\boldsymbol{a\_}}[\mathbb{D}_{\boldsymbol{a\_}}[\bar{\boldsymbol{W}}_{\boldsymbol{\gamma}\_}]] \to \langle \bar{\boldsymbol{W}}^{\dot{\boldsymbol{\psi}}}, \bar{\boldsymbol{\mathcal{D}}}_{\dot{\boldsymbol{\psi}}}[\bar{\boldsymbol{W}}_{\boldsymbol{\gamma}}]\rangle\}$$
$$\textbf{//.expandall//Expand;}$$

Using the cyclic property of the trace all terms are cycled into the standard form, where all covariant derivatives are placed on the left.

$$\textbf{unlistify[Map[cycle, listify[\%]]];}$$

All repeated spinor indices on the left are raised.

$$\textbf{\%//.conditionindex//Expand;}$$

All terms with two covariant derivatives are cast in terms of final basis structures. Here the basis terms $f_{i,j}$ *and* $\bar{f}_{i,j}$ *are related through complex conjugation,* $(f_{i,j})^* = \bar{f}_{i,j}$*.*

$$\textbf{\%//.}\{\mathbb{D}_{\boldsymbol{a\_}}[\boldsymbol{W}^{\boldsymbol{\alpha}}\_] \cdot \mathbb{D}_{\boldsymbol{a\_}}[\boldsymbol{W}_{\boldsymbol{\alpha}\_}] \cdot \bar{\boldsymbol{W}}^{\boldsymbol{\beta}}\_ \cdot \bar{\boldsymbol{W}}_{\boldsymbol{\beta}\_} \to -\boldsymbol{f}_{\mathbf{0,1}},$$
$$\mathbb{D}_{\boldsymbol{a\_}}[\boldsymbol{W}^{\boldsymbol{\alpha}}\_] \cdot \mathbb{D}_{\boldsymbol{a\_}}[\bar{\boldsymbol{W}}^{\boldsymbol{\beta}}\_] \cdot \boldsymbol{W}_{\boldsymbol{\alpha}\_} \cdot \bar{\boldsymbol{W}}_{\boldsymbol{\beta}\_} \to -\boldsymbol{f}_{\mathbf{0,2}},$$
$$\mathbb{D}_{\boldsymbol{a\_}}[\boldsymbol{W}^{\boldsymbol{\alpha}}\_] \cdot \mathbb{D}_{\boldsymbol{a\_}}[\bar{\boldsymbol{W}}^{\boldsymbol{\beta}}\_] \cdot \bar{\boldsymbol{W}}_{\boldsymbol{\beta}\_} \cdot \boldsymbol{W}_{\boldsymbol{\alpha}\_} \to -\boldsymbol{f}_{\mathbf{0,3}},$$
$$\mathbb{D}_{\boldsymbol{a\_}}[\bar{\boldsymbol{W}}^{\boldsymbol{\beta}}\_] \cdot \mathbb{D}_{\boldsymbol{a\_}}[\bar{\boldsymbol{W}}_{\boldsymbol{\beta}\_}] \cdot \boldsymbol{W}^{\boldsymbol{\alpha}}\_ \cdot \boldsymbol{W}_{\boldsymbol{\alpha}\_} \to -\bar{\boldsymbol{f}}_{\mathbf{0,1}},$$
$$\mathbb{D}_{\boldsymbol{a\_}}[\bar{\boldsymbol{W}}^{\boldsymbol{\beta}}\_] \cdot \mathbb{D}_{\boldsymbol{a\_}}[\boldsymbol{W}^{\boldsymbol{\alpha}}\_] \cdot \bar{\boldsymbol{W}}_{\boldsymbol{\beta}\_} \cdot \boldsymbol{W}_{\boldsymbol{\alpha}\_} \to -\bar{\boldsymbol{f}}_{\mathbf{0,2}},$$
$$\mathbb{D}_{\boldsymbol{a\_}}[\bar{\boldsymbol{W}}^{\boldsymbol{\beta}}\_] \cdot \mathbb{D}_{\boldsymbol{a\_}}[\boldsymbol{W}^{\boldsymbol{\alpha}}\_] \cdot \boldsymbol{W}_{\boldsymbol{\alpha}\_} \cdot \bar{\boldsymbol{W}}_{\boldsymbol{\beta}\_} \to -\bar{\boldsymbol{f}}_{\mathbf{0,3}}\}$$

The equations of motion are used to reduce the total number of terms with a single spinor covariant derivative.

$$\%//.\{\mathcal{D}_{\alpha\_}[W^{\beta\_}]\cdot A\_\_\cdot W_{\beta\_}\cdot B\_\_\cdot W^{\alpha\_}\cdot C\_\_\to \mathcal{D}_{\alpha}[W^{\beta}]\cdot A\cdot W^{\alpha}\cdot B\cdot W_{\beta}\cdot C,$$
$$\bar{\mathcal{D}}_{\alpha\_}[\bar{W}^{\beta\_}]\cdot A\_\_\cdot \bar{W}_{\beta\_}\cdot B\_\_\cdot \bar{W}^{\alpha\_}\cdot C\_\_\to \bar{\mathcal{D}}_{\alpha}[\bar{W}^{\beta}]\cdot A\cdot \bar{W}^{\alpha}\cdot B\cdot \bar{W}_{\beta}\cdot C\}$$

All terms with a single spinor covariant derivative are then cast in terms of final basis structures.

$$\%//.\{\mathcal{D}_{\alpha\_}[W_{\beta\_}]\cdot W^{\alpha\_}\cdot W^{\beta\_}\cdot \bar{W}^{\gamma\_}\cdot \bar{W}_{\gamma\_}\to -f_{1,1},$$
$$\mathcal{D}_{\alpha\_}[W_{\beta\_}]\cdot W^{\alpha\_}\cdot \bar{W}^{\gamma\_}\cdot W^{\beta\_}\cdot \bar{W}_{\gamma\_}\to -f_{1,2},$$
$$\mathcal{D}_{\alpha\_}[W_{\beta\_}]\cdot \bar{W}^{\gamma\_}\cdot \bar{W}_{\gamma\_}\cdot W^{\alpha\_}\cdot W^{\beta\_}\to -f_{1,1},$$
$$\mathcal{D}_{\alpha\_}[W_{\beta\_}]\cdot W^{\alpha\_}\cdot \bar{W}^{\gamma\_}\cdot \bar{W}_{\gamma\_}\cdot W^{\beta\_}\to -f_{1,1},$$
$$\mathcal{D}_{\alpha\_}[W_{\beta\_}]\cdot \bar{W}^{\gamma\_}\cdot W^{\alpha\_}\cdot W^{\beta\_}\cdot \bar{W}_{\gamma\_}\to -f_{1,2},$$
$$\mathcal{D}_{\alpha\_}[W_{\beta\_}]\cdot \bar{W}^{\gamma\_}\cdot W^{\alpha\_}\cdot \bar{W}_{\gamma\_}\cdot W^{\beta\_}\to -f_{1,2},$$
$$\bar{\mathcal{D}}_{\alpha\_}[\bar{W}_{\beta\_}]\cdot \bar{W}^{\alpha\_}\cdot \bar{W}^{\beta\_}\cdot W^{\gamma\_}\cdot W_{\gamma\_}\to -\bar{f}_{1,1},$$
$$\bar{\mathcal{D}}_{\alpha\_}[\bar{W}_{\beta\_}]\cdot \bar{W}^{\alpha\_}\cdot W^{\gamma\_}\cdot \bar{W}^{\beta\_}\cdot W_{\gamma\_}\to -\bar{f}_{1,2},$$
$$\bar{\mathcal{D}}_{\alpha\_}[\bar{W}_{\beta\_}]\cdot W^{\gamma\_}\cdot W_{\gamma\_}\cdot \bar{W}^{\alpha\_}\cdot \bar{W}^{\beta\_}\to -\bar{f}_{1,1},$$
$$\bar{\mathcal{D}}_{\alpha\_}[\bar{W}_{\beta\_}]\cdot \bar{W}^{\alpha\_}\cdot W^{\gamma\_}\cdot W_{\gamma\_}\cdot \bar{W}^{\beta\_}\to -\bar{f}_{1,1},$$
$$\bar{\mathcal{D}}_{\alpha\_}[\bar{W}_{\beta\_}]\cdot W^{\gamma\_}\cdot \bar{W}^{\alpha\_}\cdot \bar{W}^{\beta\_}\cdot W_{\gamma\_}\to -\bar{f}_{1,2},$$
$$\bar{\mathcal{D}}_{\alpha\_}[\bar{W}_{\beta\_}]\cdot W^{\gamma\_}\cdot \bar{W}^{\alpha\_}\cdot W_{\gamma\_}\cdot \bar{W}^{\beta\_}\to -\bar{f}_{1,2}\}$$

Finally the definition of H is implemented, and the coefficient $\mathrm{tr}_{\mathrm{Ad}}(a_5)$ is isolated (see (5.2.32)) to yield the final result (5.3.71).

$$\%\,\frac{16\pi^2}{s^2}/.\left\{H\to\frac{\mathrm{i}}{(4\pi\mathrm{i})^2}\right\}//\mathbf{Simplify}$$
$$=\frac{1}{30}\Big(f_{0,1}-f_{0,2}+f_{0,3}-3f_{1,1}-f_{1,2}+\bar{f}_{0,1}-\bar{f}_{0,2}+\bar{f}_{0,3}-3\bar{f}_{1,1}-\bar{f}_{1,2}\Big)$$

C.3 Expanding moments

In this section, through example, we briefly demonstrate how one may expand moments. The ideas extend to all orders, where again the only real complication is that of simplification.
Consider the first order moment hierarchy given above. Suppose we have already computed, to first order, all but the last moment, $K_{\alpha a}(s)$, and we wish to complete the list. In such a case the list of moments would be just as before, but missing this entry:

$$\mathbf{moments}=\mathbf{DeleteCases}[\mathbf{moments}, x:(K_{\alpha\_}[s,\{1,0,1\}]\to\_)];$$

This moment could readily be computed hand, although its position near the base of the hierarchy ensures it is one of the most difficult to compute. Alternatively one may choose proceed as follows.
Using the ideas described in the body of the present work, the identity

$$0 = \int d\eta \, \frac{\partial}{\partial \kappa_\beta} \left(X_\alpha X_\beta X_a \mathrm{e}^{\mathrm{i}s\Delta} \right) \tag{C.3.1}$$

directly leads to

$$K_{\alpha a}(s) = \int d\eta \; X_\alpha X_\beta X_a \sum_{n=0}^{\infty} \frac{(\mathrm{i}s)^{n+1}}{(n+1)!} L^n_\Delta(W^\beta) \, \mathrm{e}^{\mathrm{i}s\Delta}. \tag{C.3.2}$$

To the order of interest this is implemented as:

$$\sum_{n=0}^{3} \frac{(\mathrm{i}s)^{n+1}}{(n+1)!} X_\alpha \cdot X_\beta \cdot Y_a \cdot L_n[\Delta_{a_n,\alpha_n,\beta_n}, W^\beta]//\mathrm{Expand}$$

unlistify[Map[ker, listify[%]]];

%/.kc1;

%/.kc2;

Non-contributing moments are now removed.

$$\%/K_{\alpha\_}[s, a\_\_, b\_/; b \geq 2] \to 0;$$

$$\%//.\{K_{\alpha\_}[s, a\_\_] \to K_\alpha[s, a], K_0[s, a\_\_] \to K[s]\};$$

The previously computed moments are now used.

%//.moments//Expand;

%//.expandall//Expand;

%//.lower//.conditionindex;

From inspection of the result, we make the dotted spinor index replacement $\beta_i \to \dot{\gamma}$. The first order result follows, and is subsequently added to the list of moments.

$$\%/.\beta_{n\_} \to \dot{\gamma}//\mathrm{Simplify}$$

$$\begin{aligned} = \frac{\mathrm{i}s}{6} H \big(& \bar{W}^{\dot{\gamma}} \cdot W_\alpha \cdot \mathbb{D}_a[\bar{W}_{\dot{\gamma}}] - \bar{W}^{\dot{\gamma}} \cdot \bar{W}_{\dot{\gamma}} \cdot \mathbb{D}_a[W_\alpha] - W_\alpha \cdot \bar{W}^{\dot{\gamma}} \cdot \mathbb{D}_a[\bar{W}_{\dot{\gamma}}] \\ & + 2\bar{W}^{\dot{\gamma}} \cdot \mathbb{D}_a[W_\alpha] \cdot \bar{W}_{\dot{\gamma}} - 2\bar{W}^{\dot{\gamma}} \cdot \mathbb{D}_a[\bar{W}_{\dot{\gamma}}] \cdot W_\alpha - 2W_\alpha \cdot \mathbb{D}_a[\bar{W}^{\dot{\gamma}}] \cdot \bar{W}_{\dot{\gamma}} \\ & + 3\mathbb{D}_a[\bar{W}^{\dot{\gamma}}] \cdot W_\alpha \cdot \bar{W}_{\dot{\gamma}} - 3\mathbb{D}_a[\bar{W}^{\dot{\gamma}}] \cdot \bar{W}_{\dot{\gamma}} \cdot W_\alpha - 3\mathbb{D}_a[W_\alpha] \cdot \bar{W}^{\dot{\gamma}} \cdot \bar{W}_{\dot{\gamma}} \big) \end{aligned}$$

$$\mathrm{moments} = \mathrm{Join}[\mathrm{moments}, \{K_{\alpha\_,a\_}[s, \{1,0,1\}] \to \%\}];$$

C.4 Component extraction

It is easy to show, using (5.3.73) that the final result given above (5.3.71) will yield, at the component level, the bosonic fieldstrength contribution:

$$\mathbf{a5comp} = \frac{1}{30}\Big(\Big(\frac{1}{64}\nabla_p[F_{a,b}]\cdot\nabla_p[F_{c,d}]\cdot F_{e,f}\cdot F_{g,h} + \nabla_p[F_{a,b}]\cdot\nabla_p[F_{e,f}]\cdot F_{c,d}\cdot F_{g,h}$$
$$+\nabla_p[F_{a,b}]\cdot\nabla_p[F_{e,f}]\cdot F_{g,h}\cdot F_{c,d}\Big)\Big(\mathrm{tr}[a,b,c,d]\tilde{\mathrm{tr}}[e,f,g,h] + \mathrm{tr}[a,b,c,d]^*\tilde{\mathrm{tr}}[e,f,g,h]^*\Big)$$
$$-\frac{\mathrm{i}}{128}\Big(3F_{a,b}\cdot F_{c,d}\cdot F_{e,f}\cdot F_{g,h}\cdot F_{i,j} - F_{a,b}\cdot F_{c,d}\cdot F_{g,h}\cdot F_{e,f}\cdot F_{i,j}\Big)$$
$$\Big(\mathrm{tr}[a,b,c,d,e,f]\tilde{\mathrm{tr}}[g,h,i,j] + \mathrm{tr}[a,b,c,d,e,f]^*\tilde{\mathrm{tr}}[g,h,i,j]^*\Big)\Big);$$

*where * denotes complex conjugation, and*

$$tr[a,b,c,d] = tr(\sigma_a\tilde{\sigma}_b\sigma_c\tilde{\sigma}_d) \qquad \tilde{tr}[a,b,c,d] = tr(\tilde{\sigma}_a\sigma_b\tilde{\sigma}_c\sigma_d)$$
$$tr[a,b,c,d,e,f] = tr(\sigma_a\tilde{\sigma}_b\sigma_c\tilde{\sigma}_d\sigma_e\tilde{\sigma}_f).$$

We implement these, and a few other identities (ie see equations (A.1.17), (A.1.19) and (5.3.77)):

$$\mathrm{tr}[a\_,b\_,c\_,d\_] := -2\eta_{a,c}\eta_{b,d} + 2\eta_{b,c}\eta_{a,d} + 2\eta_{a,b}\eta_{c,d} - 2\mathrm{i}\epsilon_{a,b,c,d}$$
$$\tilde{\mathrm{tr}}[a\_,b\_,c\_,d\_] := -2\eta_{a,c}\eta_{b,d} + 2\eta_{b,c}\eta_{a,d} + 2\eta_{a,b}\eta_{c,d} + 2\mathrm{i}\epsilon_{a,b,c,d}$$
$$\sigma[a\_,b\_,c\_] := \eta_{a,c}\sigma_b + \eta_{b,c}\sigma_a - \eta_{a,b}\sigma_c + \mathrm{i}\epsilon_{a,b,c,m}\sigma^m$$
$$\tilde{\sigma}[a\_,b\_,c\_] := \eta_{a,c}\tilde{\sigma}_b + \eta_{b,c}\tilde{\sigma}_a - \eta_{a,b}\tilde{\sigma}_c - \mathrm{i}\epsilon_{a,b,c,m}\tilde{\sigma}^m$$
$$x\_^* := x/.\mathrm{Complex}[\mathrm{a\_},\mathrm{b\_}] :\to Complex[a,-b]$$

$$\mathrm{perm1} = \mathrm{Permutations}[\{e,f,g,h\}]; \quad \mathrm{perm2} = \mathrm{Permutations}[\{e,f,g\}];$$
$$\epsilon\epsilon[a\_,b\_,c\_,d\_,e\_,f\_,g\_,h\_] := \mathrm{Sum}[-\mathrm{Signature}[\mathrm{perm1}[[j]]]\,\eta_{a,\mathrm{perm1}[[j]][[1]]}$$
$$\eta_{b,\mathrm{perm1}[[j]][[2]]}\eta_{c,\mathrm{perm1}[[j]][[3]]}\eta_{d,\mathrm{perm1}[[j]][[4]]}, \{j,1,4!\}]$$
$$\epsilon\epsilon[a\_,b\_,c\_,e\_,f\_,g\_] := \mathrm{Sum}[-\mathrm{Signature}[\mathrm{perm2}[[j]]]\,\eta_{a,\mathrm{perm2}[[j]][[1]]}$$
$$\eta_{b,\mathrm{perm2}[[j]][[2]]}\eta_{c,\mathrm{perm2}[[j]][[3]]}, \{j,1,3!\}]$$

$$\mathrm{Expand}[\sigma[a,b,c]\tilde{\sigma}[d,e,f]]//.\{\sigma_{a\_}\tilde{\sigma}_{b\_} \to -2\eta_{a,b},\ \sigma_{a\_}\tilde{\sigma}^{b\_} \to -2\delta_a^b, \tilde{\sigma}_{a\_}\sigma^{b\_} \to -2\delta_a^b,$$
$$\delta_{e\_}^{m\_}\epsilon_{a\_,b\_,c\_,m\_} \to \epsilon_{a,b,c,e},\ \sigma^{m\_}\tilde{\sigma}^{n\_}\epsilon_{a\_,b\_,c\_,m\_}\epsilon_{e\_,f\_,g\_,n\_} \to -2\epsilon\epsilon[a,b,c,e,f,g]\}$$
$$//\mathrm{Expand}$$

$$= -2\eta_{a,f}\eta_{b,e}\eta_{c,d} + 2\eta_{a,e}\eta_{b,f}\eta_{c,d} - 2\eta_{a,b}\eta_{e,f}\eta_{c,d} + 2\eta_{a,f}\eta_{b,d}\eta_{c,e} - 2\eta_{a,d}\eta_{b,f}\eta_{c,e}$$
$$- 2\eta_{a,e}\eta_{b,d}\eta_{c,f} + 2\eta_{a,d}\eta_{b,e}\eta_{c,f} - 2\eta_{a,f}\eta_{b,c}\eta_{d,e} + 2\eta_{a,c}\eta_{b,f}\eta_{d,e} - 2\eta_{a,b}\eta_{c,f}\eta_{d,e}$$
$$+ 2\eta_{a,e}\eta_{b,c}\eta_{d,f} - 2\eta_{a,c}\eta_{b,e}\eta_{d,f} + 2\eta_{a,b}\eta_{c,e}\eta_{d,f} - 2\eta_{a,d}\eta_{b,c}\eta_{e,f} + 2\eta_{a,c}\eta_{b,d}\eta_{e,f}$$
$$+ 2i\eta_{e,f}\epsilon_{a,b,c,d} - 2i\eta_{d,f}\epsilon_{a,b,c,e} + 2i\eta_{d,e}\epsilon_{a,b,c,f} - 2i\eta_{b,c}\epsilon_{d,e,f,a} + 2i\eta_{a,c}\epsilon_{d,e,f,b}$$
$$- 2i\eta_{a,b}\epsilon_{d,e,f,c}$$

$$
\begin{aligned}
\mathrm{tr}[a\_, b\_, c\_, d\_, e\_, f\_] := & -2\eta_{a,f}\eta_{b,e}\eta_{c,d} + 2\eta_{a,e}\eta_{b,f}\eta_{c,d} - 2\eta_{a,b}\eta_{e,f}\eta_{c,d} + 2\eta_{a,f}\eta_{b,d}\eta_{c,e} \\
& -2\eta_{a,d}\eta_{b,f}\eta_{c,e} - 2\eta_{a,e}\eta_{b,d}\eta_{c,f} + 2\eta_{a,d}\eta_{b,e}\eta_{c,f} - 2\eta_{a,f}\eta_{b,c}\eta_{d,e} + 2\eta_{a,c}\eta_{b,f}\eta_{d,e} \\
& -2\eta_{a,b}\eta_{c,f}\eta_{d,e} + 2\eta_{a,e}\eta_{b,c}\eta_{d,f} - 2\eta_{a,c}\eta_{b,e}\eta_{d,f} + 2\eta_{a,b}\eta_{c,e}\eta_{d,f} - 2\eta_{a,d}\eta_{b,c}\eta_{e,f} \\
& +2\eta_{a,c}\eta_{b,d}\eta_{e,f} + 2i\eta_{e,f}\epsilon_{a,b,c,d} - 2i\eta_{d,f}\epsilon_{a,b,c,e} + 2i\eta_{d,e}\epsilon_{a,b,c,f} - 2i\eta_{b,c}\epsilon_{d,e,f,a} \\
& +2i\eta_{a,c}\epsilon_{d,e,f,b} - 2i\eta_{a,b}\epsilon_{d,e,f,c}
\end{aligned}
$$

The bosonic component is now simplified.

$$\mathrm{Expand}[\mathrm{a5comp}]//.\epsilon_{a\_,b\_,c\_,d\_}\epsilon_{e\_,f\_,g\_,h\_} \to \epsilon\epsilon[a,b,c,d,e,f,g,h]//\mathrm{Expand};$$

$$\mathrm{a5comp} = \%//.\,\mathrm{Join}[\mathrm{Evaluate}[\mathrm{lower}/.\,G \to F], \{\nabla_{a\_}[0] \to 0\}];$$

Having decided on what F^5 structures are distinct,

$$
\begin{aligned}
& \mathrm{structures} = \{F_{a\_,b\_}\cdot F_{b\_,c\_}\cdot F_{c\_,d\_}\cdot F_{d\_,e\_}\cdot F_{e\_,a\_} \to s_{0,0}, \\
& F_{a\_,b\_}\cdot F_{b\_,c\_}\cdot F_{c\_,d\_}\cdot F_{e\_,a\_}\cdot F_{d\_,e\_} \to s_{0,1}, \\
& F_{a\_,b\_}\cdot F_{c\_,d\_}\cdot F_{b\_,c\_}\cdot F_{e\_,a\_}\cdot F_{d\_,e\_} \to s_{0,2}, \\
& F_{a\_,b\_}\cdot F_{c\_,d\_}\cdot F_{e\_,a\_}\cdot F_{b\_,c\_}\cdot F_{d\_,e\_} \to s_{0,3}, \\
& F_{a\_,b\_}\cdot F_{b\_,c\_}\cdot F_{c\_,a\_}\cdot F_{d\_,e\_}\cdot F_{d\_,e\_} \to s_{0,4}, \\
& F_{a\_,b\_}\cdot F_{d\_,e\_}\cdot F_{b\_,c\_}\cdot F_{c\_,a\_}\cdot F_{d\_,e\_} \to s_{0,5}, \\
& \nabla_{p\_}[F_{a\_,b\_}]\cdot \nabla_{p\_}[F_{a\_,b\_}]\cdot F_{c\_,d\_}\cdot F_{c\_,d\_} \to s_{1,0}, \\
& \nabla_{p\_}[F_{a\_,b\_}]\cdot \nabla_{p\_}[F_{c\_,d\_}]\cdot F_{a\_,b\_}\cdot F_{c\_,d\_} \to s_{1,1}, \\
& \nabla_{p\_}[F_{a\_,b\_}]\cdot \nabla_{p\_}[F_{c\_,d\_}]\cdot F_{c\_,d\_}\cdot F_{a\_,b\_} \to s_{1,2}, \\
& \nabla_{p\_}[F_{a\_,b\_}]\cdot \nabla_{p\_}[F_{b\_,c\_}]\cdot F_{c\_,d\_}\cdot F_{d\_,a\_} \to s_{1,3}, \\
& \nabla_{p\_}[F_{a\_,b\_}]\cdot \nabla_{p\_}[F_{c\_,a\_}]\cdot F_{b\_,d\_}\cdot F_{d\_,c\_} \to s_{1,4}, \\
& \nabla_{p\_}[F_{a\_,b\_}]\cdot \nabla_{p\_}[F_{c\_,d\_}]\cdot F_{b\_,c\_}\cdot F_{d\_,a\_} \to s_{1,5}\};
\end{aligned}
$$

one can easily generate all obviously equivalent structures

$$
\begin{aligned}
& \mathrm{gen}[F_{a\_,b\_}\cdot e\_\_ \to h\_] := \{F_{b,a}\cdot e \to -h,\ e\cdot F_{a,b} \to h\} \\
& \mathrm{gen}[\nabla_p[F_{a\_,b\_}]\cdot e\_\_ \to h\_] := \{\nabla_p[F_{b,a}]\cdot e \to -h,\ e\cdot \nabla_p[F_{a,b}] \to h\}
\end{aligned}
$$

$$
\begin{aligned}
& \mathrm{While}[\mathrm{Length}[\mathrm{structures}] < 6\,2^5\,5 + 6\,2^4\,4, \\
& \quad \mathrm{structures} = \mathrm{Union}[\mathrm{Flatten}[\mathrm{Join}[\{\mathrm{structures}\},\ \mathrm{Map}[\mathrm{gen},\ \mathrm{structures}]]]]]
\end{aligned}
$$

and express the result in terms of them, as follows.

a5comp//.structures//Simplify

$$= -\frac{1}{60}\mathrm{i}\big(2s_{0,0} - 12s_{0,1} + 14s_{0,2} + s_{0,4} - 7s_{0,5} - \mathrm{i}s_{1,0} - \mathrm{i}s_{1,1} - \mathrm{i}s_{1,2} + 4\mathrm{i}s_{1,3} + 4\mathrm{i}s_{1,4} + 4\mathrm{i}s_{1,5}\big)$$

Using the results of section 5.3.3, we can now express the F^5 terms in our chosen basis

$$\boldsymbol{\%//.\{s_{0,5} \to \frac{s_{0,0}}{5} - s_{0,1} + s_{0,2} + \frac{3s_{0,3}}{5},}$$
$$\boldsymbol{s_{0,4} \to -\frac{3s_{0,0}}{5} + s_{0,1} + s_{0,2} + \frac{s_{0,3}}{5},}$$
$$\boldsymbol{s_{1,5} \to 2\mathrm{i}s_{0,0} - 5\mathrm{i}s_{0,1} - \mathrm{i}s_{0,3} + \frac{3}{4}s_{1,0} + \frac{3}{4}s_{1,1} + \frac{3}{4}s_{1,2} + s_{1,3} - 3s_{1,4}\}//\mathrm{Expand}}$$
//Simplify

$$= \frac{1}{30}\mathrm{i}\left(4s_{0,0} - 8s_{0,1} - 4s_{0,2} - \mathrm{i}s_{1,0} - \mathrm{i}s_{1,1} - \mathrm{i}s_{1,2} - 4\mathrm{i}s_{1,3} + 4\mathrm{i}s_{1,4}\right)$$

which is the final result (5.3.115).

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