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Two new Painlevé-integrable (2+1) and (3+1)-dimensional KdV equations with constant and time-dependent coefficients

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Abstract

In this work we develop two new (2+1) and (3+1)-dimensional KdV equations with constant and time-dependent coefficients. The integrability of each established equation is investigated via using the Painlevé test. We also examine the compatibility conditions to ensure the integrability for each model. The Hirota's method is used to derive multiple-soliton solutions for these equations. We establish the dispersion relation and the phase shifts for each case.

1. Introduction

Constructions of completely integrable equations are flourishing and gaining a mass volume of useful studies and promising findings. The existence of Lax pair, solvable by the inverse scattering transform method, passing the Painlevé test, having infinitely many symmetries, and other properties, are some of the criteria related to the development of the nonlinear integrable equations. The field of integrable equations is an active multidisciplinary area of research due to the fact that integrable equations describe the real features and reveal the mysterious nature of the nonlinearity in science and engineering applications [1–16].

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In recent years, the study of integrable equations has become a hot research area in nonlinear mathematical physics and wave propagations. This originates from the fact that these equations give qualitative and quantitative features of several scientific aspects of many unrelated phenomena such as plasmas, fluids, lattice vibrations of a crystal at low temperature, propagation of waves in shallow water area, traffic flow, soliton propagation in nonlinear transmission lines, pulse propagation in optical fibers and wave guides, plasma-laser interaction, convection in pure and binary fluids, nonlinear excitations of ultra-cold atoms in Bose-Einstein condensates, large amplitude waves (rogue waves), etc. [16–28].

The soliton, and more precisely multiple soliton solutions, are important features of nonlinear integrable equations. The existence of multiple soliton solutions often implies the integrability of the considered differential equations, but this is not sufficient, and other schemes such as the Painlevé test, the Lax pair, and other techniques are necessary to confirm the integrability of the equation. Studies of completely integrable equations are flourishing both theoretically and experimentally in the literature [1–20]. Integrable models describe the real features of scientific and engineering phenomena.

The (2+1)-dimensional KdV equation reads

$$v_t + v_{xxx} + 3(v\partial_y^{-1}v_x)_x = 0, \tag{1}$$

or equivalently

$$u_{ty} + u_{xxxy} + 3(u_y u_x)_x = 0, \tag{2}$$

obtained by using

$$v(x, y, t) = u_y(x, y, t). \tag{3}$$

This equation was firstly obtained by Boiti et al. [1] by using the idea of the weak Lax pair. Eq. (1) is also called the Boiti—Leon—Manna—Pempinelli equation and has been shown to possess Lax pair, an infinite number of conservation laws, integrability properties, and multiple soliton solutions. Other solutions have been studied widely by using different methods [1–28]. In addition, rich localized excitations of this equation were derived for this equation [10–16]. Moreover, this equation will reduce to the standard (1+1)-dimensional KdV equation in the case of $y = x$.

In this work, we aim to extend our work in [2] and develop two new (2+1)-dimensional and (3+1)-dimensional KdV equations, with constant coefficients and time dependent coefficients for each type. We first introduce a (2+1)-dimensional KdV equation with constant coefficients that takes the form

$$v_t + v_{xxx} + \alpha(v\partial_y^{-1}v_x)_x + \beta(\partial_y^{-1}v_{xx}) + \gamma(\partial_y^{-1}v_{yy}) = 0, \tag{4}$$

where α, β, and γ are constants, by adding the two terms $\beta(\partial_y^{-1}v_{xx}), \gamma(\partial_y^{-1}v_{yy})$ to Eq. (2). Moreover, we introduce the (2+1)-dimensional KdV equation with time-dependent coefficients

$$v_t + f_1(t)v_{xxx} + f_2(t)(v\partial_y^{-1}v_x)_x + f_3(t)(\partial_y^{-1}v_{xx}) + f_4(t)(\partial_y^{-1}v_{yy}) = 0, \tag{5}$$

where $f_i(t), 1 \le i \le 4$ are time-dependent coefficients. For $\alpha = 3, \beta = 0$, and $\gamma = 0$, Eq. (4) will be reduced to Eq. (1). Similarly, for $y = x$, Eq. (1) will be reduced to the KdV equation.

It is well known that equations with time-dependent coefficients are more closely related to real scientific applications. This type of equations has attracted a lot of research works and has been observed in optical fibers, wave propagations, rogue waves, and many other physical systems. Over the years, researchers have discovered many powerful methods, such as the inverse scattering method, Hirota's bilinear method, Darbous transformation method, classical Lie symmetries, and many others.

Moreover, we introduce a (3+1)-dimensional KdV equation with constant coefficients takes the form

$$v_t + v_{xxx} + \alpha(v\partial_y^{-1}v_x)_x + \beta(\partial_y^{-1}v_{xx}) + \gamma(\partial_y^{-1}v_{yy}) + \delta(\partial_y^{-1}v_{yz}) = 0, \tag{6}$$

where α, β, γ and δ are constants. Note that this equation is obtained by adding only the term $\partial_y^{-1}v_{yz}) = v_z$ to Eq. (4). We also present another (3+1)-dimensional KdV equation with time-dependent coefficients as

$$v_t + g_1(t)v_{xxx} + g_2(t)(v\partial_y^{-1}v_x)_x + g_3(t)(\partial_y^{-1}v_{xx}) + g_4(t)(\partial_y^{-1}v_{yy}) + g_5(t)(\partial_y^{-1}v_{yz}) = 0, \tag{7}$$

where $g_i(t), 1 \leq i \leq 5$ are time-dependent coefficients. For $\alpha = 3, \beta = 0, \gamma = 0$, and $\delta = 0$, Eq. (6) will be reduced to Eq. (1).

First, we first aim to test the two extended equations (4), (5) and similarly (6), (7) via using the Painlevé for the integrability of each developed model. Moreover, the second goal is to derive multiple soliton solutions for each equation by using the simplified Hirota's method.

2. The (2+1)-dimensional KdV equation with constant coefficients

The new (2+1)-dimensional KdV equation, with constant coefficients, reads

$$u_{ty} + u_{xxxy} + \alpha(u_y u_x)_x + \beta u_{xx} + \gamma u_{yy} = 0, \tag{8}$$

obtained upon using the potential

$$v(x, y, t) = u_y(x, y, t), \tag{9}$$

in Eq. (4).

In this section, we aim to show that the new (2+1)-dimensional KdV equation (8) is Painlevé integrable. Moreover, we will use the simplified Hirota's algorithm to formally derive multiple real and multiple complex soliton solutions.

2.1. Painlevé analysis

To emphasize the integrability of the equation (8), which is assumed to have a solution as a Laurent expansion about a singular manifold $\psi = \psi(x, y, t)$ as

$$u(x, y, t) = \sum_{k=0}^{\infty} u_k(x, y, t)\psi^{k-\gamma}, \tag{10}$$

where $u_k(x, y, t), k = 0, 1, 2, \ldots$, are the functions of x, y and t. On substitution of (10) in equation (8), then equating the most dominant terms we get, $\gamma = 1$ and

$$u_0(x, y, t) = \psi_(x, y). \tag{11}$$

Putting this value of γ in (10) yields

$$u(x, y, t) \cong \psi_x \psi^{-1} + u_k(x, y, t)\psi^{k-1}. \tag{12}$$

Further, using (12) and equation (11) in equation (8), we get characteristic equation for resonances with one branch with two resonances at $k = -1, 1, 4$, and 6. However, as usual, the resonance at $k = -1$ corresponds to the arbitrariness of singular manifold $\psi(x, y, t) = 0$. The next step is to determine the coefficients u_2, u_3, and u_5. After detailed computations, we observed explicit expressions for u_2, u_3, and u_5, and we found that u_1, u_4, and u_6 turn out to be arbitrary functions, and thus compatibility conditions, for $k = 1, 4, 6$ are satisfied identically.

2.2. Multiple soliton solutions

We first substitute

$$u(x, y, t) = e^{k_i x + r_i y - \omega_i t}, \tag{13}$$

into the linear terms of (8), where we find the dispersion relation ω_i takes the form

$$\omega_i = \frac{k_i^3 r_i + \beta k_i^2 + \gamma r_i^2}{r_i}, i = 1, 2, 3. \tag{14}$$

As a result, the phase variables are given as

$$\theta_i = k_i x + r_i y - \frac{k_i^3 r_i + \beta k_i^2 + \gamma r_i^2}{r_i}) t, i = 1, 2, 3. \tag{15}$$

To determine the single soliton solution, we use the transformation

$$u(x, y, t) = \frac{6}{\alpha}(\ln f(x, y, t))_x, \tag{16}$$

where the auxiliary function $f(x, y, t)$, for the single soliton solution is given by

$$f(x, y, t) = 1 + e^{\theta_1} = 1 + e^{k_1 x + r_1 y - \frac{k_1^3 r_1 + \beta k_1^2 + \gamma r_1^2}{r_1} t}. \tag{17}$$

Substituting (17) into (16) gives the single soliton solution as

$$u(x, y, t) = \frac{6 k_1 e^{k_1 x + r_1 y - \frac{k_1^3 r_1 + \beta k_1^2 + \gamma r_1^2}{r_1} t}}{1 + e^{k_1 x + r_1 y - \frac{k_1^3 r_1 + \beta k_1^2 + \gamma r_1^2}{r_1} t}}. \tag{18}$$

For the two soliton solutions we set the auxiliary function as

$$f(x, y, t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1 + \theta_2}, \tag{19}$$

where the phase variables $\theta_i, i = 1, 2, 3$ are given earlier in (15), and a_{12} is the phase shift that will be determined. Substituting (19) and (16) in (8) and solving for the phase shift a_{12}, we find

$$a_{12} = \frac{k_1 k_2 r_1 r_2 (3(k_1 - k_2)(r_1 - r_2) + 2\beta) - \beta(k_1^2 r_2^2 + k_2^2 r_1^2)}{k_1 k_2 r_1 r_2 (3(k_1 + k_2)(r_1 + r_2) + 2\beta) - \beta(k_1^2 r_2^2 + k_2^2 r_1^2)}, \tag{20}$$

which can be generalized to

$$a_{ij} = \frac{k_i k_j r_i r_j \left(3(k_i - k_j)(r_i - r_j) + 2\beta\right) - \beta(k_i^2 r_j^2 + k_j^2 r_i^2)}{k_i k_j r_i r_j \left(3(k_i + k_j)(r_i + r_j) + 2\beta\right) - \beta(k_i^2 r_j^2 + k_j^2 r_i^2)}, 1 \le i < j \le 3. \tag{21}$$

Substituting (19)–(20) into (16) gives the two soliton solutions.

For the three soliton solutions, we set the auxiliary function by

$$f(x,t) = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12}e^{\theta_1+\theta_2} + a_{13}e^{\theta_1+\theta_3} + a_{23}e^{\theta_2+\theta_3} + b_{123}e^{\theta_1+\theta_2+\theta_3}. \quad (22)$$

Proceeding as before, we find

$$b_{123} = a_{12}a_{23}a_{13}. \quad (23)$$

The three soliton solutions are obtained by substituting (22) into (16).

3. The (2+1)-dimensional KdV equation with time-dependent coefficients

The new (2+1)-dimensional KdV equation, with time-dependent coefficients takes the form

$$u_{ty} + f_1(t)u_{xxxy} + f_2(t)(u_y u_x)_x + f_3(t)u_{xx} + f_4(t)u_{yy} = 0, \quad (24)$$

obtained upon using the potential

$$v(x,y,t) = u_y(x,y,t), \quad (25)$$

in Eq. (5).

In this section we will investigate the integrability of this equation and to derive multiple soliton solutions.

3.1. Painlevé analysis

To determine the compatibility condition to guarantee the integrability of the new KdV equation (24), which is assumed to have a solution as a Laurent expansion about a singular manifold $\psi = \psi(x,t)$ as

$$u(x,y,t) = \sum_{k=0}^{\infty} u_k(x,y,t)\psi^{k-\gamma}, \quad (26)$$

where $u_k(x,y,t)'s$ $(k = 0, 1, 2, ...)$ are the functions of x, y, and t. Substituting (26) in equation (29), then equating the most dominant terms we get, $\gamma = 1$ and

$$u_0(x,y,t) = \psi_{x,y}. \quad (27)$$

Putting this value of γ in (26) yields

$$u(x,y,t) \cong \psi_x \psi^{-1} + u_k(x,y,t)\psi^{k-1}. \quad (28)$$

Further, using (28) and equation (27) in equation (24), characteristic equation for resonances has been obtained, to get one branch with three resonances at $k = -1, 1, 4$, and 6. The resonance at $k = -1$ corresponds to the arbitrariness of singular manifold $\psi(x,y,t) = 0$. The next step is to determine the coefficients u_2, u_3, and u_5, from the recursion relation and to verify the compatibility conditions for the existence of the free functions u_1, u_4, and u_6. After detailed computations, we observed explicit expressions for u_2 and u_3, and u_5. Moreover, we found that compatibility condition to ensure integrability requires that $f_1(t) = f_2(t) = f_3(t) = f(t)$, and $f_4(t) = g(t)$, where $f(t)$ and $g(t)$ remain any differentiable functions. Having these two conditions, gives u_1, u_4, and u_6 to be arbitrary functions and this implies that equation (29) passes the Painlevé test for complete integrability.

3.2. Multiple soliton solutions

Inserting the compatibility conditions derived earlier in (24) leads to

$$u_{ty} + f(t)u_{xxxy} + f(t)(u_y u_x)_x + f(t)u_{xx} + g(t)u_{yy} = 0. \tag{29}$$

We first substitute

$$u(x, y, t) = e^{k_i x + r_i y - \omega_i(t)}, \tag{30}$$

into the linear terms of (29), where we find the dispersion relation $\omega_i(t)$ takes the form

$$\omega_i(t) = \int \frac{f(t)k_i^3 r_i + f(t)k_i^2 + g(t)r_i^2}{r_i}\, dt, i = 1, 2, 3. \tag{31}$$

As a result, the phase variables are given as

$$\theta_i = k_i x + r_i y - \omega_i(t), i = 1, 2, 3. \tag{32}$$

To determine the single soliton solution, we use the transformation

$$u(x, y, t) = 6(\ln f(x, y, t))_x, \tag{33}$$

where the auxiliary function $f(x, y, t)$, for the single soliton solution is given by

$$f(x, y, t) = 1 + e^{\theta_1}. \tag{34}$$

Substituting (34) into (33) gives the single soliton solution as

$$u(x, y, t) = \frac{6k_1 e^{k_1 x + r_1 y - \omega_(t)}}{1 + e^{k_1 x + r_1 y - \omega_i(t)}}. \tag{35}$$

For the two soliton solutions we set the auxiliary function as

$$f(x, y, t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1 + \theta_2}, \tag{36}$$

where the phase variables $\theta_i, i = 1, 2, 3$ are given earlier in (32), and a_{12} is the phase shift that will be determined. Substituting (36) and (33) in (29) and solving for the phase shift a_{12}, we find

$$a_{12} = \frac{3k_1k_2r_1r_2(k_1 - k_2)(r_1 - r_2) - (k_1r_2 - k_2r_1)^2}{3k_1k_2r_1r_2(k_1 + k_2)(r_1 + r_2) - (k_1r_2 - k_2r_1)^2}, \tag{37}$$

which can be generalized to

$$a_{ij} = \frac{3k_ik_jr_ir_2(k_i - k_j)(r_i - r_j) - (k_ir_j - k_jr_i)^2}{3k_ik_jr_ir_j(k_i + k_j)(r_i + r_j) - (k_ir_j - k_2r_i)^j}, 1 \leq i < j \leq 3. \tag{38}$$

Substituting (36)–(37) into (33) gives the two soliton solutions.

For the three soliton solutions, we set the auxiliary function by

$$f(x, t) = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12}e^{\theta_1+\theta_2} + a_{13}e^{\theta_1+\theta_3} + a_{23}e^{\theta_2+\theta_3} + b_{123}e^{\theta_1+\theta_2+\theta_3}. \tag{39}$$

Proceeding as before, we find

$$b_{123} = a_{12}a_{23}a_{13}. \tag{40}$$

The three soliton solutions are obtained by substituting (39) into (33).

4. The (3+1)-dimensional KdV equation with constant coefficients

The new (3+1)-dimensional KdV equation, with constant coefficients reads

$$u_{ty} + u_{xxxy} + \alpha(u_y u_x)_x + \beta u_{xx} + \gamma u_{yy} + \delta u_{zy} = 0, \tag{41}$$

obtained upon using the potential

$$v(x, y, z, t) = u_y(x, y, z, t), \tag{42}$$

in Eq. (6).

The approach we followed for this equation is identical to the analysis presented earlier for the (2+1)-dimensional KdV equation, hence we skip details and only summarize the obtained results.

4.1. *Painlevé analysis*

To emphasize the integrability of the equation (41), we followed the procedure used earlier to get a characteristic equation for resonances with one branch with two resonances at $k = -1, 1, 4$, and 6, where we observed explicit expressions for u_2, u_3, and u_5, and we found that u_1, u_4, and u_6 turn out to be arbitrary functions.

4.2. *Multiple soliton solutions*

We first substitute

$$u(x, y, z, t) = e^{k_i x + r_i y + s_i z - \omega_i t}, \tag{43}$$

into the linear terms of (41), where we find the dispersion relation ω_i takes the form

$$\omega_i = \frac{k_i^3 r_i + \beta k_i^2 + \gamma r_i^2 + \delta r_i s_i}{r_i}, i = 1, 2, 3, \tag{44}$$

that gives the phase variables as

$$\theta_i = k_i x + r_i y - \frac{k_i^3 r_i + \beta k_i^2 + \gamma r_i^2 + \delta r_i s_i}{r_i} t, i = 1, 2, 3. \tag{45}$$

To determine the single soliton solution, we use the transformation

$$u(x, y, z, t) = \frac{6}{\alpha}(\ln f(x, y, z, t))_x, \tag{46}$$

where the auxiliary function $f(x, y, z, t)$, for the single soliton solution is given by

$$f(x, y, z, t) = 1 + e^{\theta_1} = 1 + e^{k_1 x + r_1 y - \frac{k_1^3 r_1 + \beta k_1^2 + \gamma r_1^2 + \delta r_1 s_1}{r_1} t}. \tag{47}$$

Substituting (47) into (46) gives the single soliton solution as

$$u(x, y, z, t) = \frac{6 k_1 e^{k_1 x + r_1 y - \frac{k_1^3 r_1 + \beta k_1^2 + \gamma r_1^2 + \delta r_1 s_1}{r_1} t}}{1 + e^{k_1 x + r_1 y - \frac{k_1^3 r_1 + \beta k_1^2 + \gamma r_1^2 + \delta r_1 s_1}{r_1} t}}. \tag{48}$$

For the two soliton solutions we obtained the same phase shift as derived earlier in (20).

5. The (3+1)-dimensional KdV equation with time-dependent coefficients

The new (3+1)-dimensional KdV equation, with time-dependent coefficients takes the form

$$u_{ty} + g_1(t)u_{xxxy} + g_2(t)(u_y u_x)_x + g_3(t)u_{xx} + g_4(t)u_{yy} + g_5(t)u_{yz} = 0, \tag{49}$$

obtained upon using the potential

$$v(x, y, z, t) = u_y(x, y, z, t), \tag{50}$$

in Eq. (7).

In this section we will investigate the integrability of this equation and to derive multiple real and complex soliton solutions.

5.1. Painlevé analysis

For integrability test, we obtained characteristic equation for resonances three resonances at $k = -1, 1, 4$, and 6. Moreover, we found that compatibility condition to ensure integrability requires that $g_1(t) = g_2(t) = g_3(t) = f(t)$, $g_4(t) = g(t)$, and $g_5(t) = h(t)$, where $f(t)$, $g(t)$ and $h(t)$ remain any differentiable functions. Having these three conditions, gives u_1, u_4, and u_6 to be arbitrary functions and this implies that equation (51) passes the Painlevé test for complete integrability.

5.2. Multiple soliton solutions

Inserting the compatibility conditions derived earlier in (49) leads to

$$u_{ty} + f(t)u_{xxxy} + f(t)(u_y u_x)_x + f(t)u_{xx} + g(t)u_{yy} + h(t)u_{yz} = 0. \tag{51}$$

We first substitute

$$u(x, y, z, t) = e^{k_i x + r_i y + s_i z - \omega_i(t)}, \tag{52}$$

into the linear terms of (51), where we find the dispersion relation $\omega_i(t)$ takes the form

$$\omega_i(t) = \int \frac{f(t)k_i^3 r_i + f(t)k_i^2 + g(t)r_i^2 + h(t)r_i s_i}{r_i}\, dt, i = 1, 2, 3. \tag{53}$$

This is the only change. However, the phase shifts remain the same as

$$a_{ij} = \frac{3k_i k_j r_i r_2(k_i - k_j)(r_i - r_j) - (k_i r_j - k_j r_i)^2}{3k_i k_j r_i r_j(k_i + k_j)(r_i + r_j) - (k_i r_j - k_2 r_i)^j}, 1 \le i < j \le 3. \tag{54}$$

6. Concluding remarks

Two new (2+1)-dimensional and (3+1)-dimensional KdV equations, each with constant and time-dependent coefficients were developed. The established models were emphasized as integrable equations via using the Painlevé test. The compatibility conditions for constant coefficient models, and for time-dependent coefficients equations were investigated to ensure the integrability for these equations. Multiple solitons solutions were formally derived for each of the integrable developed models.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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