

PAPER • OPEN ACCESS

Relativistic wave equations of arbitrary spin in quantum mechanics and field theory, example spin $s=2$

To cite this article: Volodimir Simulik 2017 *J. Phys.: Conf. Ser.* **804** 012040

View the [article online](#) for updates and enhancements.

Related content

- [Link between the relativistic canonical quantum mechanics of arbitrary spin and the corresponding field theory](#)
Volodimir Simulik
- [Generalizations of the Dirac equation in covariant and Hamiltonian form](#)
R-K Loide, I Ots and R Saar
- [Impressionist views of quantum engineers](#)
Alain Aspect

Relativistic wave equations of arbitrary spin in quantum mechanics and field theory, example spin $s=2$

Volodimir Simulik

Principal research associate, Institute of Electron Physics, National Academy of Sciences of Ukraine, 21 Universitetska Str., Uzhgorod 88000, Ukraine

E-mail: vsimulik@gmail.com

Abstract. Relativistic wave equations for the particles of arbitrary spin suggested by Bhabha, Bargmann–Wigner, Rarita–Schwinger (for spin $s = 3/2$) and other authors are under consideration. The comparison with the equations introduced recently by the author in Ukr. J. Phys., Vol. 60, N 10. 985–1006 (2015) and J. Phys.; Conf. Ser. Vol. 670, 012047(1–16) (2016) is given. The advantages of the new equations are considered briefly. The three level consideration (relativistic canonical quantum mechanics, canonical Foldy–Wouthuysen type field theory, locally covariant field theory) is presented. The operator link between the relativistic canonical quantum mechanics and locally covariant field theory of arbitrary spin is found. This link is given by the extended Foldy–Wouthuysen transformation between the $2(2s+1)$ -component local field theory and the corresponding relativistic canonical quantum mechanics. The important partial example of spin $s=2$ case is considered in details. The 10-component Dirac-like wave equation for the spin $s=(2,2)$ particle-antiparticle doublet is suggested. The link with 10-component Maxwell-like equation for this doublet is considered briefly. The hypothesis on the spin $s=5/2$ particle is discussed.

1. Introduction

The investigation of a particle equations for an arbitrary spin is the interesting problem of contemporary theoretical physics. The big zoo of known elementary particles (and corresponding fields) prefers a systematic description, which can be given by a suitable field equation for an arbitrary spin. Therefore, the particle equations for an arbitrary spin are the subject for careful investigations from the start, given by Paul Dirac [1] in 1936, until today.

Of course, the first-order partial differential equations are preferable. The canonical nonlocal pseudo-differential representations of such equations are under consideration as well. Thus, the second order equations (like the Klein–Gordon–Fock equation) are not the subject of the given below investigation.

Different approaches to the description of the field theory of an arbitrary spin can be found in [1–16]. Our contribution is presented in [15, 16]. Here and in [15, 16] only the approach started in [4] is the basis for further application. Other results given in [1–3, 5–14] are not used here. Below we continue the investigations of [16].

Note only some general deficiencies of the known equations for arbitrary spin [1–3, 5–14]. The consideration of the partial cases, when the substitution of the fixed value of spin is fulfilled, is



not successful in all approaches. For example, for the spin $s > 1$ existing equations have the redundant components and should be complemented by some additional conditions. Indeed, the known equations [17, 18] for the spin $s=3/2$ (and their confirmation in [19]) should be essentially complemented by the additional conditions. The main difficulty in the models of an arbitrary spin is the interaction between the fields of higher-spin. Even the quantization of higher-spin fields generated the questions. These and other deficiencies of the known equations for higher-spin are considered, e. g., in [20–31] (a brief review of deficiencies see in [24]).

Even this brief analysis makes us sure in the prospects of the investigations started in [32, 33] and continued in [15, 16]. The successful description of the arbitrary spin field models is not the solved problem today. There is a place for new approaches among the known equations for an arbitrary spin.

Equations suggested in [32, 33] and [15, 16] are free of the mentioned above deficiencies. The equations from [32, 33] have been derived directly from the well-defined equations of relativistic canonical quantum mechanics. Therefore, the partial differential equations for arbitrary spin in [15, 16, 32, 33] are without redundant components. It is the advantage of equations from [15, 16] in comparison with equations considered in [17–31]. Indeed, the Rarita–Schwinger equation for spin $s=3/2$ has 16 components, our equation for spin $s=(3/2, 3/2)$ particle-antiparticle doublet has 8 components. The Bargman–Wigner equation in partial case $s=3/2$ has 12 components. Bhabha itself [34] have analyzed the partial case $s=3/2$ for his equation [2]. He have found [34] that in this case his equation [2] coincides with the Rarita–Schwinger equation, i. e. has 16 components. Therefore, in [32, 33] and [15, 16] the new equation for arbitrary spin has been suggested. The partial case of spin $s=(3/2, 3/2)$ particle-antiparticle doublet has been considered in [15, 16]. Below the partial case of spin $s=(2, 2)$ particle-antiparticle doublet is under consideration.

The start of our consideration is taken from [4], where the main foundations of the relativistic canonical quantum mechanics (RCQM) are formulated. In the articles [32, 33] and [15, 16] the results of [4] are generalized and extended. The operator link between the results of [35] and [4] (between the canonical Foldy–Wouthuysen type field theory and the relativistic canonical quantum mechanics) is suggested. Note that the cases $s=3/2$ and $s=2$ are not presented in [4], especially in explicit demonstrative forms.

The main idea and main new feature of our approach to the field theory of arbitrary spin is the start from the arbitrary spin relativistic canonical quantum mechanics. Therefore on every level ((i) relativistic canonical quantum mechanics, (ii) the canonical Foldy–Wouthuysen type field theory, (iii) locally covariant field theory) we are dealing with the equations without redundant components.

The concepts, definitions and notations here are the same as in [16]. For example, in the Minkowski space-time

$$M(1, 3) = \{x \equiv (x^\mu) = (x^0 = t, \vec{x} \equiv (x^j))\}; \quad \mu = \overline{0, 3}, j = 1, 2, 3, \quad (1)$$

x^μ denotes the Cartesian (covariant) coordinates of the points of the physical space-time in the arbitrary-fixed inertial reference frame. We use the system of units with $\hbar = c = 1$. The metric tensor is given by

$$g^{\mu\nu} = g_{\mu\nu} = g_\nu^\mu, (g_\nu^\mu) = \text{diag}(1, -1, -1, -1); \quad x_\mu = g_{\mu\nu}x^\nu, \quad (2)$$

and summation over the twice repeated indices is implied.

Minimal necessary information on the square-root operator $\hat{\omega} \equiv \sqrt{\hat{\vec{p}}^2 + m^2} = \sqrt{-\Delta + m^2} \geq m > 0$, where $\hat{\vec{p}} \equiv (\hat{p}^j) = -i\nabla$, $\nabla \equiv (\partial_\ell)$, has been considered in [16], see the section 2 in [16].

2. Briefly on our equations for an arbitrary spin

This section represents main information on the three level arbitrary spin model given in [16, 33].

2.1. Level of relativistic canonical quantum mechanics

The **arbitrary spin Schrödinger–Foldy equation of motion** is given by

$$(i\partial_0 - \hat{\omega})f(x) = 0, \quad (3)$$

where the wave function has the form

$$f \equiv \text{column}(f^1, f^2, \dots, f^N), \quad N = 2s + 1, \quad f \in H^{3,N}. \quad (4)$$

Note that here the operator $\hat{\omega} \equiv \sqrt{-\Delta + m^2}$ is the relativistic analog of the energy operator (Hamiltonian) of nonrelativistic quantum mechanics, $H^{3,N}$ is the Hilbert space of N -component functions. The step from the particle singlet of arbitrary spin to the corresponding particle-antiparticle doublet is evident.

Thus, for the arbitrary spin particle-antiparticle doublet the system of two N -component equations $(i\partial_0 - \hat{\omega})f(x) = 0$ and $(i\partial_0 + \hat{\omega})f(x) = 0$ is used. Therefore, the corresponding Schrödinger–Foldy equation is given by (3), where the $2N$ -component wave function is the direct sum of the particle and antiparticle wave functions, respectively. Due to the historical tradition of the physicists the antiparticle wave function is put in the down part of the $2N$ -column.

The **general solution** of the Schrödinger–Foldy equation of motion (3) (in the case of particle-antiparticle arbitrary spin doublet) has the form

$$f(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k e^{-ikx} a^{2N}(\vec{k}) d_{2N}, \quad kx \equiv \omega t - \vec{k} \cdot \vec{x}, \quad \omega \equiv \sqrt{\vec{k}^2 + m^2}, \quad (5)$$

where the orts of the $2N$ -dimensional Cartesian basis are given by

$$d_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}, \quad d_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}, \quad \dots, \quad d_{2N} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 1 \end{pmatrix}. \quad (6)$$

All necessary information on the relativistic canonical quantum mechanics of an arbitrary spin can be found in section 4 of [16].

2.2. Level of the canonical Foldy–Wouthuysen type field theory of an arbitrary spin

The operator, which transform the relativistic canonical quantum mechanics of the arbitrary spin particle-antiparticle multiplet into the corresponding canonical particle-antiparticle field, is given by

$$v_{2N} = \begin{pmatrix} I_N & 0 \\ 0 & \hat{C}I_N \end{pmatrix}, \quad v_{2N}^{-1} = v_{2N}^\dagger = v_{2N}, \quad v_{2N}v_{2N} = I_{2N}, \quad N = 2s + 1, \quad (7)$$

where $\hat{C}I_N$ is the $N \times N$ operator of complex conjugation. Indeed, the operator (7) translates any operator from canonical field FW representation into the relativistic canonical quantum mechanics representation and vice versa:

$$v_{2N} \hat{q}_{cf}^{\text{anti-Herm}} v_{2N} = \hat{q}_{qm}^{\text{anti-Herm}}, \quad v_{2N} \hat{q}_{qm}^{\text{anti-Herm}} v_{2N} = \hat{q}_{cf}^{\text{anti-Herm}}. \quad (8)$$

Here $\hat{q}_{\text{qm}}^{\text{anti-Herm}}$ is an arbitrary operator from the relativistic canonical quantum mechanics of the 2N-component particle-antiparticle doublet in the anti-Hermitian form, e. g., the operator $(\partial_0 + i\hat{\omega})$ of equation of motion (3), the operator of spin taken in anti-Hermitian form, etc., $\hat{q}_{\text{cf}}^{\text{anti-Herm}}$ is an arbitrary operator from the canonical field theory of the 2N-component particle-antiparticle doublet in the anti-Hermitian form. Thus, the only warning is that operators here must be taken in anti-Hermitian form, see section 9 in [32] for the details and see [36, 37] for the mathematical correctness of anti-Hermitian operators application.

Note briefly that on this basis we are able to find the general transformation, which gives the direct relationship between the relativistic canonical quantum mechanics and the Dirac model, see, e. g., the formulae (41)–(45) in [38] and section 6 of [38] in general. In such approach the Dirac Hamiltonian has an anti-Hermitian form. Thus we have suggested extended non-Hermitian Foldy–Wouthuysen transformation and not only the Foldy–Wouthuysen transformation for non-Hermitian Hamiltonians as in [39]. For an arbitrary spin such extended non-Hermitian Foldy–Wouthuysen transformation has been suggested in [32, 33] as well.

Further, the operator (7) translates

$$\phi = v_{2N}f, \quad f = v_{2N}\phi, \quad (9)$$

the general solution (5) of the Schrödinger–Foldy equation (3) into the **general solution**

$$\phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \left[e^{-ikx} a^N(\vec{k}) d_N + e^{ikx} a^{*\check{N}}(\vec{k}) d_{\check{N}} \right], \quad (10)$$

$N = 1, 2, \dots, N$, $\check{N} = N + 1, N + 2, \dots, 2N$, of the **Foldy–Wouthuysen type equation for an arbitrary spin**

$$(i\partial_0 - \Gamma_{2N}^0 \hat{\omega})\phi(x) = 0, \quad \Gamma_{2N}^0 \equiv \sigma_{2N}^3 = \begin{vmatrix} I_N & 0 \\ 0 & -I_N \end{vmatrix}, \quad (11)$$

$\hat{\omega} \equiv \sqrt{-\Delta + m^2}$, $N = 2s + 1$, and *vice versa*. Note that 2N-dimensional Cartesian basis is not changed under the transformation (9). Thus, the transformation (7), (8) translates the matrices Γ_{2N}^0 and

$$\Gamma_{2N}^j = \begin{vmatrix} 0 & \Sigma_N^j \\ -\Sigma_N^j & 0 \end{vmatrix}, \quad j = 1, 2, 3, \quad (12)$$

into the relativistic canonical quantum mechanics representation and *vice versa*

$$\bar{\Gamma}_{2N}^{\bar{\mu}} = v_{2N} \Gamma_{2N}^{\bar{\mu}} v_{2N}, \quad \Gamma_{2N}^{\bar{\mu}} = v_{2N} \bar{\Gamma}_{2N}^{\bar{\mu}} v_{2N}. \quad (13)$$

In (12) Σ_N^j are the $N \times N$ Pauli matrices. Matrices Γ_{2N}^{μ} (11), (12) together with matrix $\Gamma_{2N}^4 \equiv \Gamma_{2N}^0 \Gamma_{2N}^1 \Gamma_{2N}^2 \Gamma_{2N}^3$ satisfy the anticommutation relations of the Clifford–Dirac algebra.

All necessary information on the canonical Foldy–Wouthuysen type field theory of an arbitrary spin can be found in subsection 5.1 of [16].

2.3. Level of the locally covariant arbitrary spin particle-antiparticle field

The operator, which transform the canonical (Foldy–Wouthuysen type) model of the arbitrary spin particle-antiparticle field into the corresponding locally covariant particle-antiparticle field, is the generalized FW operator and is given by

$$V^{\mp} = \frac{\mp \vec{\Gamma}_{2N} \cdot \vec{p} + \hat{\omega} + m}{\sqrt{2\hat{\omega}(\hat{\omega} + m)}}, \quad V^- = (V^+)^{\dagger}, \quad V^- V^+ = V^+ V^- = I_{2N}, \quad N = 2s + 1, \quad (14)$$

where Γ_{2N}^j are known from (12) and Σ_N^j are the $N \times N$ Pauli matrices.

Note that in formulas (14) and in all formulas before the end of the subsection the values of N are only even. Therefore, the canonical field equation (11) describes the larger number of multiplets than the generalized Dirac equation (15) given below.

The formulas (15), (16) below are found from the corresponding formulas (11) and (10) of canonical field model on the basis of the operator (14).

Thus, for the general form of arbitrary spin locally covariant particle-antiparticle field **the arbitrary spin Dirac-like equation of motion** follows from the equation (11) after the transformation (14) and is given by

$$\left[i\partial_0 - \Gamma_{2N}^0 (\vec{\Gamma}_{2N} \cdot \vec{p} + m) \right] \psi(x) = 0. \quad (15)$$

The **general solution** has the form

$$\psi(x) = V^- \phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \left[e^{-ikx} a^N(\vec{k}) v_N^-(\vec{k}) + e^{ikx} a^{*\check{N}}(\vec{k}) v_N^+(\vec{k}) \right], \quad (16)$$

where amplitudes and notation \check{N} are the same as in (10); $\{v_N^-(\vec{k}), v_N^+(\vec{k})\}$ are $2N$ -component Dirac basis spinors with properties of orthonormalisation and completeness similar to 4-component Dirac spinors from [40].

The corresponding equations for **the spin $s=(3/2, 3/2)$ particle-antiparticle doublet** are the partial cases of the equations (3), (11) and (15). This partial case has been considered in [15, 16]. Below we presented another partial case, i. e. the explicit form of the equations for **the spin $s=(2, 2)$ particle-antiparticle doublet**.

3. A brief scheme of the relativistic canonical quantum mechanics of the 10-component spin $s=(2, 2)$ particle-antiparticle bosonic doublet

The Schrödinger–Foldy equation is given by

$$(i\partial_0 - \hat{\omega})f(x) = 0, \quad f \equiv \text{column}(f^1, f^2, f^3, \dots, f^{10}). \quad (17)$$

The general solution of the equation (17) has the form

$$f(x) = \begin{vmatrix} f_{\text{part}} \\ f_{\text{antipart}} \end{vmatrix} = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k e^{-ikx} g^A(\vec{k}) d_A. \quad (18)$$

The orts of the 10-component Cartesian basis are given by

$$d_1 = \text{column}(1, 0, 0, \dots, 0), \quad d_2 = \text{column}(0, 1, 0, \dots, 0), \quad \dots, \quad d_{10} = \text{column}(0, 0, 0, \dots, 1). \quad (19)$$

The space of the states is the rigged Hilbert space $S^{3,10} \subset H^{3,10} \subset S^{3,10*}$. The $SU(2)$ -spin generators have the form

$$s_{10}^1 = s_{23} = \frac{1}{2} \begin{vmatrix} 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & \sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & \sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & -\sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{6} & 0 & -\sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{6} & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \end{vmatrix},$$

$$s_{10}^2 = s_{31} = \frac{i}{2} \begin{vmatrix} 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & -\sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & -\sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & -\sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{6} & 0 & -\sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{6} & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \end{vmatrix}, \quad (20)$$

$$s_{10}^3 = s_{12} = \begin{vmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{vmatrix},$$

and satisfy the commutation relations $[s^j, s^\ell] = i\varepsilon^{j\ell n} s^n$ of SU(2) algebra generators. The Casimir operator of the SU(2)-algebra representation is given by $\vec{s}^2 = 6\mathbf{I}_{10} = 2(2+1)\mathbf{I}_{10}$. The stationary complete set of operators is as follows: $\vec{p} \equiv (\hat{p}^j) = -i\partial_j$, $s_{10}^3 \in (20)$. The equations on the momentum operator eigenvalues have the form

$$\vec{p} e^{-ikx} d_A = \vec{k} e^{-ikx} d_A, \quad A = \overline{1, 10}, \quad (21)$$

and the equations on the spin projection operator s_{10}^3 (20) eigenvalues are given by

$$s_{10}^3 d_1 = 2d_1, \quad s_{10}^3 d_2 = d_2, \quad s_{10}^3 d_3 = 0, \quad s_{10}^3 d_4 = -d_4, \quad s_{10}^3 d_5 = -2d_5, \quad (22)$$

$$s_{10}^3 d_6 = -2d_6, \quad s_{10}^3 d_7 = -d_7, \quad s_{10}^3 d_8 = 0, \quad s_{10}^3 d_9 = d_9, \quad s_{10}^3 d_{10} = 2d_{10}.$$

The interpretation of the amplitudes in the general solution (18) follows from the equations (21), (22). The functions $g^1(\vec{k})$, $g^2(\vec{k})$, $g^3(\vec{k})$, $g^4(\vec{k})$, $g^5(\vec{k})$ in the solution (18) are the momentum-spin quantum-mechanical amplitudes of the particle (boson) with the momentum \vec{p} and spin projection eigenvalues (+2, +1, 0, -1, -2), respectively, and the functions $g^6(\vec{k})$, $g^7(\vec{k})$, $g^8(\vec{k})$, $g^9(\vec{k})$, $g^{10}(\vec{k})$ are the momentum-spin quantum-mechanical amplitudes of the antiparticle with the momentum \vec{p} and spin projection eigenvalues (-2, -1, 0, +1, +2), respectively. The generators of the reducible unitary bosonic representation of the Poincaré group \mathcal{P} are given by

$$\hat{p}_0 = \hat{\omega} \equiv \sqrt{-\Delta + m^2}, \quad \hat{p}_\ell = i\partial_\ell, \quad \hat{j}_{\ell n} = x_\ell \hat{p}_n - x_n \hat{p}_\ell + s_{\ell n} \equiv \hat{m}_{\ell n} + s_{\ell n}, \quad (23)$$

$$\hat{j}_{0\ell} = -\hat{j}_{\ell 0} = t\hat{p}_\ell - \frac{1}{2} \{x_\ell, \hat{\omega}\} - \left(\frac{s_{\ell n} \hat{p}_n}{\hat{\omega} + m} \equiv \check{s}_\ell \right), \quad (24)$$

whereas the spin $s=(2,2)$ SU(2) generators $\vec{s} = (s^{\ell n})$ have the form (20). The Schrödinger–Foldy equation (17) (and the set $\{f\}$ of its solutions (18)) is invariant with respect to the reducible unitary fermionic representation of the Poincaré group \mathcal{P} , whose Hermitian 10×10 matrix-differential nonlocal generators are given in (23), (24). The corresponding Casimir operators

$$p^2 = \hat{p}^\mu \hat{p}_\mu = m^2 \mathbf{I}_{10}, \quad W = w^\mu w_\mu = m^2 \vec{s}^2_{10} = 2(2+1) m^2 \mathbf{I}_{10}. \quad (25)$$

4. Canonical Foldy–Wouthuysen type field model for the 10-component spin $s=(2,2)$ particle-antiparticle doublet

The canonical Foldy–Wouthuysen type field equation and its solution are given by

$$(i\partial_0 - \Gamma_{10}^0 \hat{\omega})\phi(x) = 0, \quad \Gamma_{10}^0 = \begin{vmatrix} I_5 & 0 \\ 0 & -I_5 \end{vmatrix}, \quad \hat{\omega} \equiv \sqrt{-\Delta + m^2}, \quad (26)$$

$$\phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \left[e^{-ikx} g^A(\vec{k}) d_A + e^{ikx} g^{*B}(\vec{k}) d_B \right], \quad (27)$$

$A = 1, 2, 3, 4, 5$, $B = 6, 7, 8, 9, 10$, the Cartesian orts are known from (19). The link with the relativistic canonical quantum mechanics is given by the operator

$$v_{10} = \begin{vmatrix} I_5 & 0 \\ 0 & CI_5 \end{vmatrix}, \quad v_{10}^{-1} = v_{10}^\dagger = v_{10}, \quad v_{10}v_{10} = I_{10}. \quad (28)$$

The spin operators of the canonical field theory found from the quantum-mechanical $SU(2)$ spin (20) on the basis of the transformation (28) satisfy the commutation relations $[s^j, s^\ell] = i\varepsilon^{j\ell n} s^n$ of $SU(2)$ algebra generators and have the following form

$$\vec{s}_{10} = \begin{vmatrix} \vec{s}_5 & 0 \\ 0 & \vec{s}_5 \end{vmatrix}, \quad (29)$$

where the spin $s=2$ 5×5 $SU(2)$ generators \vec{s}_5 are given by the up 5×5 diagonal blocks in (20). The corresponding Casimir operator is given by $\vec{s}_{10}^2 = 6I_{10} = 2(2+1)I_{10}$, where I_{10} is the 10×10 - unit matrix. The stationary complete set of operators is given by the operators \vec{p} , $s_{10}^3 = s_z$ of the momentum and spin projection. The equations on the eigenvectors and eigenvalues of the operators \vec{p} and $s_{10}^3 = s_z$ have the form

$$\begin{aligned} \vec{p} e^{-ikx} d_A &= \vec{k} e^{-ikx} d_A, \quad A = \overline{1, 5}, \quad \vec{p} e^{ikx} d_B = -\vec{k} e^{ikx} d_B, \quad B = \overline{6, 10}, \\ s_{10}^3 d_1 &= 2d_1, \quad s_{10}^3 d_2 = d_2, \quad s_{10}^3 d_3 = 0, \quad s_{10}^3 d_4 = -d_4, \quad s_{10}^3 d_5 = -2d_5, \\ s_{10}^3 d_6 &= 2d_6, \quad s_{10}^3 d_7 = d_7, \quad s_{10}^3 d_8 = 0, \quad s_{10}^3 d_9 = -d_9, \quad s_{10}^3 d_{10} = -2d_{10}, \end{aligned} \quad (30)$$

and determine the interpretation of the amplitudes in the general solution (27). Note that the direct quantum-mechanical interpretation of the amplitudes $g^A(\vec{k})$, $g^B(\vec{k})$ in solution (27) should be taken from the quantum-mechanical equations (21), (22) and is given in section 3. The canonical field equation (26) and the set $\{\phi\}$ of its solutions (27) are invariant with respect to the reducible unitary bosonic representation of the Poincaré group \mathcal{P} , whose Hermitian 10×10 matrix-differential nonlocal generators are given by

$$\hat{p}^0 = \Gamma_{10}^0 \hat{\omega} \equiv \Gamma_{10}^0 \sqrt{-\Delta + m^2}, \quad \hat{p}^\ell = -i\partial_\ell, \quad \hat{j}^{\ell n} = x^\ell \hat{p}^n - x^n \hat{p}^\ell + s_{10}^{\ell n} \equiv \hat{m}^{\ell n} + s_{10}^{\ell n}, \quad (31)$$

$$\hat{j}^{0\ell} = -\hat{j}^{\ell 0} = x^0 \hat{p}^\ell - \frac{1}{2} \Gamma_{10}^0 \{x^\ell, \hat{\omega}\} + \Gamma_{10}^0 \frac{(\vec{s}_{10} \times \vec{p})^\ell}{\hat{\omega} + m}, \quad (32)$$

where the spin $s=(2,2)$ $SU(2)$ generators $\vec{s}_{10} = (s_{10}^{\ell n})$ have the form (29). It is easy to prove by the direct verification that generators (31), (32) commute with the operator $(i\partial_0 - \Gamma_{10}^0 \hat{\omega})$ of the canonical field equation (26) and satisfy the commutation relations of the Lie algebra of the Poincaré group \mathcal{P} . The corresponding Casimir operators are given by

$$p^2 = \hat{p}^\mu \hat{p}_\mu = m^2 I_{10}, \quad W = w^\mu w_\mu = m^2 \vec{s}_{10}^2 = 2(2+1) m^2 I_{10}, \quad (33)$$

where I_{10} is the 10×10 unit matrix.

5. Locally covariant field equations for spin $s=(2,2)$ particle-antiparticle doublet

The only small difficulty is as follows. The covariant field theory equation for the spin $s=(2,2)$ particle-antiparticle doublet can not be presented in terms of Γ matrices. Indeed, the 10×10 Clifford–Dirac Γ matrices are not exist as an mathematical objects. Therefore, one need to look for the locally covariant field equations for spin $s=(2,2)$ particle-antiparticle doublet not in the terms of Γ matrices.

Consider two ways of the derivation of the covariant field equations for spin $s=(2,2)$ particle-antiparticle doublet. The first way is based on the application of the corresponding Foldy–Wouthuysen transformation directly to the canonical field equation (26). Note that there is a nonzero chance to find the appropriate transformation among a big number of so-called Foldy–Wouthuysen transformations for an arbitrary spin [41–47]. Unfortunately for our approach, many authors considered for the spin $s=1$ case the 4-vector potential (see, e. g., [47]) and not 3-component wave function $\vec{\psi} = \vec{E} - i\vec{E}$ as it is in [15, 16] and [32, 33]. Therefore, the objects for spin $s=2$ case are different as well. Indeed, our approach for arbitrary spin equations differs from other known approaches.

Hence, below another way of the derivation of the covariant field equations for spin $s=(2,2)$ particle-antiparticle doublet is realized. It is easy to construct the formalism of the relativistic canonical quantum mechanics of the spin $s=(2,0,2,0)$ particle-antiparticle multiplet. The corresponding procedure is similar to the above given in section 3. Further, the canonical Foldy–Wouthuysen type model of the spin $s=(2,0,2,0)$ particle-antiparticle field can be formulated. It is similar to the given in section 4 consideration. Moreover, both models are presented already in sections 16 and 26 of [32]. After that the 12×12 operator transformation (14) should be applied for the canonical Foldy–Wouthuysen type equation of the spin $s=(2,0,2,0)$ particle-antiparticle multiplet.

Found by the above considered way (or found as the partial 12×12 case of arbitrary spin equation (15)) covariant field equation for the spin $s=(2,0,2,0)$ particle-antiparticle multiplet is given by

$$\left[i\partial_0 - \Gamma_{12}^0 (\vec{\Gamma}_{12} \cdot \vec{p} + m) \right] \psi(x) = 0, \quad \Gamma_{12}^0 = \begin{vmatrix} \mathbf{I}_6 & 0 \\ 0 & -\mathbf{I}_6 \end{vmatrix}, \quad \Gamma_{12}^j = \begin{vmatrix} 0 & \Sigma_6^j \\ -\Sigma_6^j & 0 \end{vmatrix}, \quad (34)$$

where Σ_6^j are the 6×6 Pauli matrices $\Sigma_6^1 = \begin{vmatrix} 0 & \mathbf{I}_3 \\ \mathbf{I}_3 & 0 \end{vmatrix}$, $\Sigma_6^2 = \begin{vmatrix} 0 & -i\mathbf{I}_3 \\ i\mathbf{I}_3 & 0 \end{vmatrix}$, $\Sigma_6^3 = \begin{vmatrix} \mathbf{I}_3 & 0 \\ 0 & -\mathbf{I}_3 \end{vmatrix}$. The general solution of the equation (34) is given by the formula (327) in [32].

In order to find the covariant field equations for spin $s=(2,2)$ particle-antiparticle doublet one can put equal to zero the scalar components in (34). The place of the scalar components is evident from the corresponding quantum-mechanical consideration (section 16 of [32]).

The similar way was demonstrated in [32] (section 23) to find the covariant Maxwell-like field equations for the spin $s=(1,1)$ particle-antiparticle doublet from the known Dirac-like equation (252) in [32] for the spin $s=(1,0,1,0)$ particle-antiparticle multiplet. Similarly to equation (275) in [32] the covariant field equation (35) (or (36)) for the spin $s=(2,2)$ particle-antiparticle doublet can be presented in terms of **curl**, **div** and **grad** operators as well.

The locally covariant field equations for 10 component spin $s=(2,2)$ particle-antiparticle doublet found by this way has the form

$$\begin{aligned} i\partial_0\psi^1 - p^3\psi^6 - m\psi^1 &= 0, \\ i\partial_0\psi^2 - p^3\psi^7 + ip^2\psi^{10} - p^1\psi^{10} - m\psi^2 &= 0, \\ i\partial_0\psi^3 - p^3\psi^8 + ip^2\psi^9 - p^1\psi^9 - m\psi^3 &= 0, \\ i\partial_0\psi^4 + p^3\psi^9 - ip^2\psi^8 - p^1\psi^8 - m\psi^4 &= 0, \end{aligned}$$

$$\begin{aligned}
i\partial_0\psi^5 + p^3\psi^{10} - ip^2\psi^7 - p^1\psi^7 - m\psi^5 &= 0, \\
i\partial_0\psi^6 - p^3\psi^1 + m\psi^6 &= 0, \\
i\partial_0\psi^7 - p^3\psi^2 + ip^2\psi^5 - p^1\psi^5 + m\psi^7 &= 0, \\
i\partial_0\psi^8 - p^3\psi^3 + ip^2\psi^4 - p^1\psi^4 + m\psi^8 &= 0, \\
i\partial_0\psi^9 + p^3\psi^4 - ip^2\psi^3 - p^1\psi^3 + m\psi^9 &= 0, \\
i\partial_0\psi^{10} + p^3\psi^5 - ip^2\psi^2 - p^1\psi^2 + m\psi^{10} &= 0, \\
-ip^2\psi^1 - p^1\psi^1 = 0, \quad -ip^2\psi^6 - p^1\psi^6 &= 0,
\end{aligned} \tag{35}$$

or the Dirac-like form

$$\left[i\partial_0 - \Gamma_8^0(\vec{\Gamma}_8 \cdot \vec{p} + m) \right] \psi(x) = 0, \quad (i\partial_0 - \sigma^1 p^3 - \sigma^3 m)\chi = 0, \quad (p^1 + ip^2)\chi = 0, \tag{36}$$

where $\psi = \text{column}(\psi^1, \psi^2, \dots, \psi^8)$, $\chi = \begin{pmatrix} \psi^9 \\ \psi^{10} \end{pmatrix}$, $\Gamma_8^0 = \begin{pmatrix} I_4 & 0 \\ 0 & -I_4 \end{pmatrix}$, $\Gamma_8^j = \begin{pmatrix} 0 & \Sigma_4^j \\ -\Sigma_4^j & 0 \end{pmatrix}$, and Σ_4^j are the 4×4 Pauli matrices $\Sigma_4^1 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$, $\Sigma_4^2 = \begin{pmatrix} 0 & -iI_2 \\ iI_2 & 0 \end{pmatrix}$, $\Sigma_4^3 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$. The form similar to (275) of [32] is found by the method similar to one presented in section 23 of [32].

6. Brief conclusions

It is demonstrated that our approach for the construction of the arbitrary spin field equations is different in comparison with other known approaches. The explicit comparison of our equation for the particles with arbitrary spin with the Bhabha and Bargman–Wigner equations is fulfilled.

The Dirac-like locally covariant field equations for 10 component spin $s=(2,2)$ particle-antiparticle doublet is introduced. The found equation does not contain the redundant components. Moreover, the equation (35) (or (36)) for the 10 component spin $s=(2,2)$ particle-antiparticle doublet can be interpreted as the equation for the classical gravitational field. Furthermore, this equation can be written in the Maxwell-like form. Massless case can be achieved very easy by putting to zero the mass term. Nevertheless, the massive case may have some special meaning as well. Additional forces (seven components) to the Newton force (three components) can be useful for the explanation of the tiny gravitational effects both in solar system and in the universe in general.

Important result is in further approbation of the effective method of deriving of arbitrary spin field equations. The three level table (relativistic canonical quantum mechanics, canonical Foldy–Wouthuysen type field, manifestly covariant field) of arbitrary spin $S=(2s+1, 2s+1)$ particle-antiparticle doublets is constructed. Thus, the equation for any spin can be written down very easy. For example, the three level model (relativistic canonical quantum mechanics, canonical field, covariant field), e. g., of the spin $s=(5/2, 5/2)$ particle-antiparticle doublet is formulated similarly to the given in [15, 16] three level spin $s=(3/2, 3/2)$ model. In the approach under consideration the particle with spin $s=5/2$ is the same theoretical reality as the particle with spin $s=1/2$, or $s=3/2$ (the particle with spin $s=2$ is the same theoretical reality as the particle with spin $s=1$). Hence, we can suggest the hypothesis that the particle with spin $s=5/2$ can exists in reality. Note that the spin $s=5/2$ case is considered in the article [14] as well. The time of life of such new particle is smaller then the time of life of spin $s=3/2$ hyperons.

References

- [1] Dirac P A M 1936 *Proc. R. Soc. Lond. A* **155** 447
- [2] Bhabha H J 1945 *Rev. Mod. Phys.* **17** 200
- [3] Bargman V and Wigner E P 1948 *Proc. Nat. Acad. Sci. USA.* **34** 211
- [4] Foldy L L 1956 *Phys. Rev.* **102** 568
- [5] Weaver D L, Hammer C L and Good R H Jr. 1964 *Phys. Rev.* **135** B241
- [6] Pursey D L 1964 *Nucl. Phys.* **53** 174
- [7] Pursey D L 1965 *Ann. Phys. (N.Y.)* **32** 157
- [8] Mathews P M 1966 *Phys. Rev.* **143** 978
- [9] Nelson T J and Good R H Jr. 1968 *Rev. Mod. Phys.* **40** 508
- [10] Guertin R F 1974 *Ann. Phys. (N.Y.)* **88** 504
- [11] Kraicik R A and Nieto M M 1977 *Phys. Rev. D* **15** 433
- [12] Loide R-K, Ots I and Saar R 1997 *J. Phys. A* **30** 4005
- [13] Gitman D M and Shelepin A L 2001 *Int. J. Theor. Phys.* **40** 603
- [14] Shi-Zhong H et al 2003 *Eur. Phys. J. C* **26** 609
- [15] Simulik V M 2015 *Ukr. J. Phys.* **60** 985
- [16] Simulik V M 2016 *J. Phys. Conf. Ser.* **670** 012047
- [17] Fierz M and Pauli W 1939 *Proc. R. Soc. Lond. A* **173** 211
- [18] Rarita W and Schwinger J 1941 *Phys. Rev.* **60** 61
- [19] Davydov A S 1943 *J. Exper. Theor. Phys.* **13** 313 (in Russian)
- [20] Johnson K and Sudarshan E C G 1961 *Ann. Phys. (N.Y.)* **13** 126
- [21] Velo G and Zwanziger D 1969 *Phys. Rev.* **186** 1337
- [22] Velo G and Zwanziger D 1969 *Phys. Rev.* **188** 2218
- [23] Hagen C R 1971 *Phys. Rev. D* **4** 2204
- [24] Capri A Z and Kobes R L 1980 *Phys. Rev. D* **22** 1967
- [25] Darkhosh T 1985 *Phys. Rev. D* **32** 3251
- [26] Cox W 1989 *J. Phys. A* **22** 1599
- [27] Pilling T 2005 *Int. J. Mod. Phys. A* **20** 2715
- [28] Napsuciale M, Kirchbach M and Rodriguez S 2006 *Eur. Phys. J. A* **29** 289
- [29] Kaloshin A E and Lomov V P 2011 *Particles and Nuclei, Letters* **8** 517
- [30] Red'kov V M, Kisel V V and Ovsyuk E M 2011 Particle with spin 2 and anomalous magnetic moment in external electromagnetic and gravitational fields *arXiv: 1109.1715v1* [math-ph]
- [31] Red'kov V M 2011 Particle with spin $S=3/2$ in Riemannian space-time *arXiv: 1109.3871v1* [math-ph]
- [32] Simulik V M 2014 Covariant local field theory equations following from the relativistic canonical quantum mechanics of arbitrary spin *arXiv: 1409.2766v2* [quant-ph, hep-th]
- [33] Simulik V M 2015 General form of the covariant field equations of arbitrary spin and the relativistic canonical quantum mechanics *arXiv: 1509.04630* [quant-ph, hep-th]
- [34] Bhabha H J 1951 *Proc. Indian Acad. Sci. A* **34** 335
- [35] Foldy L L and Wouthuysen S A 1950 *Phys. Rev.* **78** 29
- [36] Elliott J and Dawber P 1979 *Symmetry in physics, vol 1* (London: Macmillian Press)
- [37] Wybourne B 1974 *Classical groups for Physicists* (New York: John Wiley and sons)
- [38] Simulik V M and Krivsky I Y 2013 *Ukr. J. Phys.* **58** 1192
- [39] Alexandre J and Bender C M 2015 *J. Phys. A* **48** 185403
- [40] Bogoliubov N N and Shirkov D V 1980 *Introduction to the theory of quantized fields* (New York: John Wiley and Sons)
- [41] Garrido L M and Oliver L 1967 *Nuov. Cim. A* **LII** 588
- [42] Ramanujam G A 1974 *Nuov. Cim. A* **20** 27
- [43] Nagpal A K and Kaushal R S 1974 *Lett. Nuov. Cim.* **9** 391
- [44] Tekumalla A R, Santhanam T S 1974 *Lett. Nuov. Cim.* **10** 737
- [45] Jayaraman J 1975 *J. Phys. A* **8** L1
- [46] Leon J, Quiros M and Mittelbrunn J R 1979 *J. Math. Phys.* **20** 1068
- [47] Silenko A J 2016 *Phys. Rev. A* **94** 032104