



Vacuum energy in asymptotically flat $2 + 1$ gravity



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ARTICLE INFO

Article history:

Received 23 October 2016

Received in revised form 26 January 2017

Accepted 2 February 2017

Available online 8 February 2017

Editor: M. Cvetič

ABSTRACT

We compute the vacuum energy of three-dimensional asymptotically flat space based on a Chern–Simons formulation for the Poincaré group. The equivalent action is nothing but the Einstein–Hilbert term in the bulk plus half of the Gibbons–Hawking term at the boundary. The derivation is based on the evaluation of the Noether charges in the vacuum. We obtain that the vacuum energy of this space has the same value as the one of the asymptotically flat limit of three-dimensional anti-de Sitter space.

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1. Introduction

Asymptotically flat spacetimes are one of the most intuitive classes of systems that exist in gravity. We expect that, for localized matter distributions, the Einstein equations will have solutions asymptotically matching Minkowski space, far away from the source. In four dimensions, even outside matter distributions, the vacuum Einstein equations can accommodate solutions with non-zero Riemann curvature, as is seen for example in the case of the Schwarzschild black hole. The Riemann curvature there tends to the Minkowski flat value of zero at large distances, parameterized by the radial coordinate.

The picture in three dimensions, however, is different as gravity is topological in nature. The Riemann tensor here has only six independent components, and is linearly related with the Einstein tensor. The Einstein equation necessarily gives vacuum solutions which are locally Riemann flat. So, if the metric is to be the field that describes an isolated mass distribution in three dimensions, the information about the mass can only manifest as topological properties of the spacetime. Various schemes towards this end exist. For example, in spacetimes with cosmological constant $\Lambda = 0$, conical singularities generated by isolated mass particles have the mass encoded in an angular deficit of the azimuthal periodicity in the metric, which becomes less than 2π [1]. Also, depending on the parameter enumerating angular deficit, one can even get solutions which are angular excesses, though they do not represent physical solutions. On the other hand, identification of points along

the curves of a Killing vector comprising a linear combination of Lorentz boosts and a translation along a spatial direction has been carried out in flat space leading to *flat-space cosmologies* [2]. These topological identifications were inspired by the ones in AdS_3 leading to the BTZ black hole. In fact, following [3], the whole class of solutions in $(2 + 1)$ -dimensional flat space is classified by two free, dimensionless, parameters μ and j . With G being the three-dimensional gravitational constant, the parameter $\mu = 8GM$ is related to mass, while $j = 4GJ$ is related to angular momentum.

Due to the existence of these various solutions, all of which must return to the 3D Minkowski solution in appropriate limits of the parameters describing the respective topological deformations corresponding to the 3D vacuum, the role of physical properties of the vacuum itself becomes quite important. We focus here on the vacuum energy of 3D Minkowski space. Adopting a field-theory approach and using an off-shell equivalence between three-dimensional Einstein–Hilbert gravity and Chern–Simons action for Poincaré gauge group, we calculate the mass as the Noether charge for spacetime diffeomorphisms, which is on-shell equivalent to gauge transformations. We do this for two classes of physically admissible solutions, the conical singularity and flat-space cosmologies.

Spacetimes whose parameters lie in a negative interval $-1 < \mu = -\alpha^2 < 0$ possess a conical defect of magnitude $2\pi(1 - |\alpha|)$. These are, in general, spacetimes of a spinning particle.¹ The static sector of a massive point particle is given by [1]

$$ds^2 = -dt^2 + r^{-\beta} (dr^2 + r^2 d\theta^2), \quad 0 \leq \theta < 2\pi. \quad (1.1)$$

¹ In AdS_3 space, the spinning particles are nothing but the BTZ black hole with negative mass [4].

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The value of G is fixed by the usual pre-factor in the Einstein equation $G_{\mu\nu} = 8\pi G T_{\mu\nu}$, where the speed of light has been set to unity. To see that this solution is locally flat, it is convenient to make a coordinate transformation $(r, \theta) \rightarrow (\rho, \phi)$,

$$\rho = \frac{r^\alpha}{\alpha}, \quad \phi = \alpha \theta, \quad (1.2)$$

with $\alpha = \frac{2-\beta}{2}$, which leads to the transformed flat metric

$$ds^2 = -dt^2 + d\rho^2 + \rho^2 d\phi^2, \quad 0 \leq \phi < 2\pi\alpha. \quad (1.3)$$

The point to note here is the altered range of the angular coordinate ϕ , modulated by the parameter α , which describes an angular deficit or excess, when $\alpha \neq 1$. For $\beta > 2$, α becomes negative and the original point $r = 0$ containing the mass is mapped by (1.2) to $\rho = \infty$, thus destroying the physical picture and asymptotics. Indeed, Ashtekar et al. [5] noted that the points $\rho = \infty$ are at a *finite* geodesic distance away from any point in the interior. This shows the breakdown of asymptotic flatness as the concept of being ‘far away’ from an isolated source. On the other hand, for $\beta < 0$, or in an interval of parameters $\mu = -\alpha^2 < -1$, the angular range exceeds 2π and instead of a deficit, we have an *excess*, describing a hyperbolic geometry similar to lettuce leaves, which is not necessarily asymptotically flat. Thus the range of parameters accommodating asymptotical flatness is

$$0 < \alpha \leq 1 \Leftrightarrow 0 \leq \beta < 2, \quad (1.4)$$

where $\alpha = 1$ or $\beta = 0$ gives us the 3-dimensional Minkowski spacetime. In this range, the deficit angle is related to the mass of the particle, m , measured with respect to the Minkowski vacuum,² through $\beta = 8Gm$. This deficit angle is always present, at any distance from the source including at infinity, and thus the spacetime is never asymptotically Minkowskian, unless the mass is zero. This is an important distinction from four dimensions as, even in the leading order of an asymptotic expansion, the spacetime is not Minkowskian and carries information about the mass.

Investigations of spacetimes with such asymptotics give interesting results. Ashtekar et al. [5] considered generic asymptotically flat spacetimes whose boundary behavior matches that of the conical singularity (1.1) and demonstrated that the bound on the range of β translated to the Hamiltonian being bounded *both* from above and below. Their starting point was the usual Einstein–Hilbert action, adopting a Regge–Teitelboim [6] approach of adding necessary surface terms to the Hamiltonian, which gives the conserved quantities. The energy corresponded to a Hamiltonian that generates time translations only for $\beta < 2$, with the value of energy being positive and lying in the range $[0, 1/4G]$. The energy of the Minkowski vacuum turns out to be zero.

Later, Marolf et al. in [7] consider a finite, differentiable action consisting of the Einstein–Hilbert term in the bulk and the Gibbons–Hawking term at the boundary for the same asymptotics that leads to a Hamiltonian with the same behavior of energy being bounded from both above and below. However, the energy appears now shifted and found to be negative, lying in the range $[-1/4G, 0]$ with the energy of the Minkowski vacuum set to $-1/4G$. Both approaches were in the metric formulation. In contrast, Corichi et al. in [8] adopted a first-order Hamiltonian formulation and showed that the results in both references [5] and [7] could be reproduced.

On the other hand, Barnich et al. in [3] calculated the vacuum energy as the flat limit of the cosmological AdS solutions and they found that it coincides with the value of the AdS vacuum.

The class of conical singularities described by eq. (1.4) are supplemented by two other classes of spacetimes, depending on other choices of the parameters μ and j . As mentioned above, the defect $\mu = -\alpha^2 < 0$ corresponds to a space with an angular deficit ($\alpha^2 < 1$) or excess ($\alpha^2 > 1$). On the other hand, when $\mu = \alpha^2 > 0$, these geometries can be interpreted as cosmological spacetimes. For completeness, we note that there exist the null orbifold when $\mu = 0 = j$, but we will not consider it here. Among the cases we consider, the Minkowski space for $|\alpha| = 1$ and $j = 0$, is accessible as a limit for conical singularities and angular excesses, or discretely from flat space cosmologies. We shall discuss all asymptotics with $\mu \neq 0$.

In order to clarify the controversy in the literature about the value of the vacuum energy of Minkowski space, we adopt a field theory approach where we consider an action for 3D flat gravity given by a Chern–Simons (CS) form for Poincaré group. The CS action naturally comes equipped with a boundary term that is one half of the usual Gibbons–Hawking term. This is along the line of a similar proposal for the AdS case, discussed earlier in [9–11]. It has been recently pointed out in [12] that the addition of half of the Gibbons–Hawking term on top of the Einstein–Hilbert action has a well defined variational principle in the asymptotically flat case.

The CS gravity action has some advantages with respect to the Einstein–Hilbert one. For example, it is more suitable for construction of flat space supergravity through a direct supersymmetrization of a gauge group [13,14]; high-spin theory in 3D is described by the CS action for $SL(n, \mathbb{R}) \times SL(n, \mathbb{R})$ [15]; spin-3 action in 2D can be obtained via reduction of CS flat action with a boundary [16]; 3D conformal gravity is a CS theory [17], etc. On the other hand, some applications of the CS action in 3D include a tunneling from flat space to flat space cosmology [18] and logarithmic corrections to entropy [19].

2. Poincaré Chern–Simons gravity

General Relativity on a $2 + 1$ dimensional manifold $\mathcal{M}$ can be written as a Chern–Simons gauge theory invariant under local the Poincaré group [20]

$$I_{CS}[A] = \frac{k}{4\pi} \int_{\mathcal{M}} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right), \quad (2.1)$$

where the constant k is called the level of the theory and $\langle \dots \rangle$ is the trace of group generators. The gauge connection 1-form $A = A_\mu(x) dx^\mu$ takes values in the Poincaré algebra $\mathfrak{iso}(2, 1)$ as $A = \frac{1}{2} \omega^{AB} J_{AB} + e^A P_A$. Here $\omega^{AB} = \omega_\mu^{AB}(x) dx^\mu$ and $e^A = e_\mu^A(x) dx^\mu$ are the gauge field 1-forms – the spin connection and the vielbein, respectively. The Greek indices $\mu, \nu, \dots = (t, r, \theta)$ label the spacetime coordinates, and the Latin ones $A, B, \dots = 0, 1, 2$ are the Lie algebra indices. Furthermore, J_{AB}, P_A are the $\mathfrak{iso}(2, 1)$ generators obeying the $2 + 1$ dimensional Poincaré algebra

$$\begin{aligned} [J_{AB}, J_{CD}] &= \eta_{AD} J_{BC} - \eta_{AC} J_{BD} + \eta_{BC} J_{AD} - \eta_{BD} J_{AC}, \\ [P_A, J_{BC}] &= \eta_{AB} P_C - \eta_{AC} P_B, \\ [P_A, P_B] &= 0. \end{aligned} \quad (2.2)$$

We use the signature $\eta_{AB} = \text{diag}(-, +, +)$. The trace of the above generators defines the invariant tensor of the Lie algebra and it has the form $\langle J_{AB} P_C \rangle = \epsilon_{ABC}$, while $\langle J_{AB} J_{CD} \rangle = 0 = \langle P_A P_B \rangle$. With this construction for the gauge connection A_μ , we see the action (2.1) transforming to

$$I_{CS} = \frac{k}{4\pi} \int_{\mathcal{M}} \epsilon_{ABC} R^{AB} \wedge e^C - \frac{k}{8\pi} \int_{\partial\mathcal{M}} \epsilon_{ABC} \omega^{AB} \wedge e^C, \quad (2.3)$$

² The mass m of the point particle is shifted so that the Minkowski space, $\mu = -1$, corresponds to $m = 0$, that is, $\mu = -(1 - 4Gm)^2$.

where $R^{AB} = \frac{1}{2} R^{AB}{}_{\mu\nu} dx^\mu dx^\nu = d\omega^{AB} + \omega^A_C \wedge \omega^{CB}$. The first term is exactly the Einstein–Hilbert action, once we realize that the localized gauge fields ω^{AB} and e^A are nothing but the spin connection and triad frame fields of first-order gravity,

$$I_{EH} = \frac{1}{16\pi G} \int d^3x \sqrt{-g} R = \frac{1}{32\pi G} \int \epsilon_{ABC} R^{AB} \wedge e^C, \quad (2.4)$$

identifying the level of the Chern–Simons theory with the gravitational constant G by $k = \frac{1}{4G}$.

The second term in (2.3) is a boundary term defined on the boundary $\partial\mathcal{M}$. We take a radial Gaussian foliation of the space-time in the coordinates $x^\mu = (x^1, x^i) = (r, x^i)$, $i = 0, 2$,

$$ds^2 = N^2(r) dr^2 + h_{ij}(r, x) dx^i dx^j, \quad (2.5)$$

so that the boundary is placed at constant radius $r = r_B$. Here, h_{ij} is the induced metric on the boundary.

We work in first-order formulation where the fundamental fields are the vielbein $e^A = e^A_\mu dx^\mu$ and the Lorenz connection $\omega^{AB} = \omega^{AB}_\mu dx^\mu$. One possible choice of the vielbein in the foliation (2.5), where the Poincaré indices split as $A = (1, a)$, is

$$\begin{aligned} e^1 &= N dr, \\ e^a &= e^a_i dx^i. \end{aligned} \quad (2.6)$$

The boundary vielbein e^a_i is related to the induced metric by $h_{ij} = \eta_{ab} e^a_i e^b_j$, and the extrinsic curvature of the boundary is

$$K_{ij} = -\frac{1}{2N} \partial_r h_{ij}. \quad (2.7)$$

The components of ω^{AB} are calculated from $de^A + \omega^{AB} \wedge e_B = 0$, leading to

$$\begin{aligned} \omega^{1a} &= K^a, \\ \omega^{ab} &= \omega^{ab}_i dx^i + e^{i[a} \partial_r e^{b]}_i dr, \end{aligned} \quad (2.8)$$

where $K^a = K^a_i dx^i = e^{aj} K_{ij} dx^i$ is the extrinsic curvature 1-form and the antisymmetrization of indices in $e^{i[a} \partial_r e^{b]}_i$ includes the factor $\frac{1}{2}$. The Lorentz connection corresponds to the spacetime metric, $\omega^{ab}(g)$, on the l.h.s. of the equality, and to the boundary metric, $\omega^{ab}(h)$, on the r.h.s.. The induced metric h_{ij} and its inverse h^{ij} raise and lower the boundary world indices, whereas the boundary vielbein e^a_i and its inverse e^i_a projects the world indices $i, j, \dots$ to the Lorentz ones $a, b, \dots$, and vice versa.

With this notation and using $\epsilon_{1ab} = -\epsilon_{ab}$, the boundary term is

$$\begin{aligned} -\frac{k}{8\pi} \int_{\partial\mathcal{M}} \epsilon_{ABC} \omega^{AB} \wedge e^C &= \frac{k}{8\pi} \int_{\partial\mathcal{M}} \epsilon_{ab} (2\omega^{1a} \wedge e^b + \omega^{ab} \wedge e^1) \\ &= \frac{k}{4\pi} \int_{\partial\mathcal{M}} d^2x \epsilon^{ij} \epsilon_{ab} K^a_i e^b_j = \frac{1}{2} B_{GH}. \end{aligned} \quad (2.9)$$

Note that $e^1 = 0$ and $\omega^{ab}(g) = \omega^{ab}(h)$ on the boundary. The Gibbons–Hawking boundary term reads

$$B_{GH} = -\frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^2x \sqrt{-h} K, \quad (2.10)$$

with $K = h^{ij} K_{ij}$ being the trace of the extrinsic curvature.

This calculation shows that the boundary term, which arises naturally in Chern–Simons Poincaré gravity, equals one-half of the

standard Gibbons–Hawking boundary term, and we will use it as our boundary piece in the gravitational action. In AdS gravity, this anomalous Gibbons–Hawking boundary term [21] has been shown to result in a finite action principle and proper values of the Noether charges [11,22].

The usual Gibbons–Hawking term provides a well-defined action principle for the Dirichlet boundary conditions on the induced metric. A change of boundary term has consequence of the boundary conditions, as well. In the next section we address this question in asymptotically flat space.

3. Boundary conditions

A suitable set of boundary conditions for the action (2.3) is the one for which the variation of the action vanishes when the equations of motion hold. The variation of the action (2.1), on-shell, gives rise to a surface term

$$\begin{aligned} \delta I_{CS} &= \frac{k}{4\pi} \int_{\partial\mathcal{M}} \langle \delta A \wedge A \rangle \\ &= \frac{k}{8\pi} \int_{\partial\mathcal{M}} \epsilon_{ABC} (\delta e^A \wedge \omega^{BC} - e^A \wedge \delta \omega^{BC}). \end{aligned} \quad (3.1)$$

In an equivalent form, in an adapted frame (2.6) which implies (2.8), we have

$$\delta I_{CS} = \frac{k}{4\pi} \int_{\partial\mathcal{M}} \epsilon_{ab} (\delta e^a \wedge \omega^{1b} - e^a \wedge \delta \omega^{1b}), \quad (3.2)$$

for $r = \text{const}$, in terms of boundary quantities.

Let us analyze the fall-off conditions in the boundary metric for a spacetime which behaves asymptotically as a spinning particle ($\mu = -\alpha^2$ in Eq. (1.1)). The boundary is parametrized by the local coordinates $x^i = (t, \theta)$, such that the induced metric behaves for large r as [5,7]

$$h_{ij} = \begin{bmatrix} -1 + \mathcal{O}(1/r) & \mathcal{O}(r^{-\frac{\beta}{2}-1}) \\ \mathcal{O}(r^{-\frac{\beta}{2}-1}) & r^{2-\beta} + \mathcal{O}(r^{1-\beta}) \end{bmatrix}. \quad (3.3)$$

One possible choice for the boundary zweibein is

$$\begin{aligned} e^0 &= A dt, \\ e^2 &= \frac{C}{r^2} dt + r^{1-\frac{\beta}{2}} B d\theta, \end{aligned} \quad (3.4)$$

where the functions $A(r, x)$, $B(r, x)$ and $C(r, x)$ are regular in the asymptotic region, such that their expansion is

$$\begin{aligned} A &= 1 + \mathcal{O}(1/r), \\ B &= 1 + \mathcal{O}(1/r), \\ C &= \mathcal{O}(1). \end{aligned} \quad (3.5)$$

In addition, the lapse function for large r has the form $N = r^{-\frac{\beta}{2}} + \mathcal{O}(r^{-\frac{\beta+1}{2}})$. The components of Levi-Civita connection $\omega^{AB}(e)$ are

$$\begin{aligned} \omega^{10} &= -\frac{r^{\frac{\beta}{2}} A'}{A} e^0 - \chi e^2, \\ \omega^{12} &= \chi e^0 - r^{\frac{\beta}{2}} \left(\frac{B'}{B} + \frac{2-\beta}{2r} \right) e^2, \end{aligned} \quad (3.6)$$

where the prime denotes radial derivative and we have defined the function

$$\chi = \frac{r^{\frac{\beta}{2}-2}}{2A} \left(\frac{CB' - BC'}{B} + \frac{(6-\beta)C}{2r} \right). \quad (3.7)$$

Asymptotically, the above function behaves as $\chi = \mathcal{O}(r^{\frac{\beta}{2}-3})$, what implies that ω^{1a} behaves as

$$\begin{aligned} \omega^{10} &= \mathcal{O}(r^{\frac{\beta}{2}-2}), \\ \omega^{12} &= -\frac{2-\beta}{2} B d\theta + \mathcal{O}(1/r). \end{aligned} \quad (3.8)$$

On the other hand, the asymptotic form of the boundary frame is

$$\begin{aligned} e^0 &= A dt, \\ e^2 &= r^{1-\frac{\beta}{2}} B d\theta + \mathcal{O}(1/r^2), \end{aligned} \quad (3.9)$$

what yields a finite variation of the action,

$$\delta I_{CS} = -\frac{k}{4\pi} \frac{2-\beta}{2} \int d^2x (A\delta B - B\delta A). \quad (3.10)$$

The action principle is satisfied if $A + \gamma B$ (with $\gamma = \text{Const.}$) vanishes on the boundary, because then $A\delta B - B\delta A = 0$ on $\partial\mathcal{M}$. This condition is fulfilled since, from eq. (3.5), $A = B$ up to the $\mathcal{O}(1/r)$ terms.

4. Noether charge

Let $L(\phi)$ be a Lagrangian 3-form describing a configuration of fields ϕ , whose variation is $\delta L = \frac{\delta L}{\delta \phi} \delta \phi + d\Theta(\phi, \partial\phi, \delta\phi)$, and $\xi^\mu = \xi^\mu \partial_\mu$ a set of asymptotic Killing vectors. The Noether current corresponding to a diffeomorphism generated by the vector field $\xi^\mu(x)$ can be written in general as [23]

$$*J = -\Theta - i_\xi L, \quad (4.1)$$

where $*J = \frac{1}{2} \sqrt{-g} \epsilon_{\mu\nu\lambda} J^\mu dx^\nu \wedge dx^\lambda$ is the Hodge dual of the current. For the Chern–Simons action (2.1), the above procedure for the connection obeying the Chern–Simons equation of motion $F = dA + A \wedge A = 0$ yields

$$*J = \frac{k}{4\pi} d\langle Ai_\xi A \rangle. \quad (4.2)$$

The above formula is a consequence of the fact that the diffeomorphisms $\delta x^\mu = \xi^\mu(x)$ act on the fields as Lie derivatives, which satisfy the differential geometry identity $\mathcal{L}_\xi = i_\xi d + di_\xi$, where i_ξ is the contraction operator and $d = dx^\mu \partial_\mu$ is the exterior derivative. Thus, the Lie derivative acts on the 3-form Lagrangian L as a total derivative $\mathcal{L}_\xi L = d(i_\xi L)$. In consequence, invariance of the action $I[\phi] = \int L(\phi)$ under general coordinate transformation implies the conservation law $d *J = 0$. For a given system, the Noether current can always be written globally as $*J = dQ[\xi]$, as discussed in Ref. [24], such that one can obtain the Noether charge as a surface integral on the spacelike boundary $\partial\Sigma$.

The charge is then expressed as an integral over an appropriate asymptotics,

$$Q[\xi] = \frac{k}{4\pi} \int_{\partial\Sigma} \langle Ai_\xi A \rangle. \quad (4.3)$$

It is worthwhile noticing that general coordinate transformations with parameter ξ become algebraically equal, on-shell, to the Poincaré gauge transformations upon field-dependent redefinitions of gauge parameters: $\lambda^{AB} = \xi^\nu \omega_\nu^{AB}$ and $\lambda^A = \xi^\nu e_\nu^A$ for Lorentz rotations and translations, respectively. Dependence of λ on the gauge fields makes the calculation of the conserved charges associated to Poincaré transformations more complicated. A realization

of off shell equivalence between the two sets of local transformations involve trivial symmetries [25], which enables one to construct the charge (4.3) starting directly from $\text{iso}(2, 1)$.

Indeed, a general coordinate transformation acting on the gauge connection is given by the identity

$$\mathcal{L}_\xi A = Di_\xi A + i_\xi F, \quad (4.4)$$

what makes evident that the gauge transformation is on-shell equivalent to a diffeomorphic transformation. Therefore, the charges are the same. This is no longer true in higher-dimensional Chern–Simons theories [26].

In the next section, we employ the equivalence between Chern–Simons theory and gravity in $2+1$ dimensions to calculate the mass of the solutions in asymptotically flat gravity.

4.1. Conical singularity

Let us study the conical singularity in the spinless case. We recall that the metric is given by Eq. (1.1) with $\alpha > 0$, where the angular variable θ takes values $0 \leq \theta \leq 2\pi$. The angular deficit, $1 - \alpha$, is related to a mass sitting at the singularity through $\alpha = 1 - 4Gm$ and the Minkowski vacuum corresponds to $\beta = 0$ when the metric becomes identically flat with a full angular range of 2π , as discussed in (1.4).

We stress that the coordinate r used in the metric (1.1) is not the usual radial distance from the center because the perimeter at r is not $2\pi(1 - \alpha)r$. To get a locally flat metric (1.3) with the angular deficit, we have to change the coordinates as (1.2). On the other hand, the ADM form of the metric with $N = \alpha^2$ and $N_\theta = 0$ is realized in the ADM coordinates $(t', r', \theta) = (t/\alpha, \alpha\rho, \theta)$.

In a first-order description of the metric (1.1), we choose the triad frame fields

$$e^0 = dt, \quad e^1 = r^{-\frac{\beta}{2}} dr, \quad e^2 = r^{1-\frac{\beta}{2}} d\theta \quad (4.5)$$

which, remembering that we have a torsionless and thus Riemannian manifold, fixes the spin-connection through the triad postulate as,

$$\omega^{12} = \frac{\beta - 2}{2} d\theta. \quad (4.6)$$

We now employ the CS formulation of $2+1$ gravity. Using the expression for Noether charge corresponding to diffeomorphisms (4.3), mass is given as the charge corresponding to the time translation Killing vector field $\xi = \partial_t$,

$$Q[\partial_t] = \frac{k}{4\pi} \int_{\partial\Sigma} \left(\left(\frac{1}{2} \omega^{AB} J_{AB} + e^A P_A \right) \left(\frac{1}{2} i_\xi \omega^{AB} J_{AB} + i_\xi e^A P_A \right) \right). \quad (4.7)$$

Upon using the Poincaré algebra and the adopted trace $\langle J_{AB} P_C \rangle = 0$, we finally get

$$\begin{aligned} Q[\partial_t] &= \frac{k}{4\pi} \int_0^{2\pi} \frac{1}{2} \epsilon_{ABC} \left(\omega_\theta^{AB} e_t^C + \omega_t^{AB} e_\theta^C \right) d\theta \\ &= \frac{k(\beta - 2)}{4}. \end{aligned} \quad (4.8)$$

Thus the energy of the vacuum ($\beta = 0$) comes out to be

$$E_0 = -\frac{k}{2} = -\frac{1}{8G}. \quad (4.9)$$

4.2. Cosmological asymptotically flat metric

In the previous section, we computed the vacuum energy as the Noether charge for the conical singularity. Let us confirm that the vacuum energy does not depend on the choice of the solution. Then, we consider the cosmological asymptotically flat metric [27] which lies in a different sector of parameter space, $\mu = \alpha^2$ and $j \neq 0$,

$$ds^2 = -f^2 dt^2 + \frac{dr^2}{f^2} + r^2(d\theta + N_\theta dt)^2. \quad (4.10)$$

Here $f^2(r) = -\mu + \frac{j^2}{r^2}$ and $N_\theta(r) = \frac{j}{r^2}$.

To calculate the Noether charges, we follow a similar approach as outlined in the previous section. The triad fields are chosen as

$$e^0 = f dt, \quad e^1 = \frac{1}{f} dr, \quad e^2 = r N_\theta dt + r d\theta, \quad (4.11)$$

which results in the torsionless spin connection

$$\omega^{01} = -\frac{1}{2} r^2 N'_\theta d\theta, \quad \omega^{02} = -\frac{r N'_\theta}{2f} dr, \quad \omega^{12} = -f d\theta. \quad (4.12)$$

Using (4.3), this gives corresponding to the killing vector corresponding to time translations $\xi^t = \partial_t$ a mass

$$Q[\partial_t] = 4kGM. \quad (4.13)$$

The vacuum here is characterized by $J = 0$ and $M = -\frac{1}{8G}$, because then the metric becomes Minkowski. This results in the vacuum energy

$$E_0 = -\frac{k}{2} = -\frac{1}{8G}, \quad (4.14)$$

what matches the result (4.9).

To calculate the angular momentum, we just have to use the corresponding angular Killing vector $\xi = \partial_\theta$ in (4.3),

$$\begin{aligned} Q[\partial_\theta] &= \frac{k}{4\pi} \int_0^{2\pi} \frac{1}{2} \epsilon_{ABC} \left(\omega_\theta^{AB} e_\theta^C + \omega_\theta^{AB} e_\theta^C \right) d\theta \\ &= 4kGJ. \end{aligned} \quad (4.15)$$

Remembering that $k = \frac{1}{4G}$ leads to

$$Q[\partial_t] = M, \quad Q[\partial_\theta] = J, \quad (4.16)$$

as expected. We confirmed that the Noether charge formula (4.3) gives the correct values for the mass, M , and the angular momentum, J , of the black hole and the vacuum energy, E_0 .

At this point, we emphasize that it is the Chern–Simons form of the action which leads to the correct answer for the charges in both flat and AdS cases, what leaves no ambiguity in the choice of possible boundary terms.

5. Discussion and conclusions

An inequivalent set of boundary conditions which accounts for conical defects [1] and flat cosmologies [27] (discussed in Section 4.2) in Euclidean sector with the line element

$$ds^2 = h_{\tau\tau}(\varphi) d\tau^2 + h_{rr}(\varphi) d\rho^2 + \rho^2 d\varphi^2 \quad (5.1)$$

is given by

$$\begin{aligned} \delta g_{\varphi\varphi} &= \mathcal{O}(\rho), & \delta g_{\varphi\tau} &= \mathcal{O}(1), & \delta g_{\tau\tau} &= \delta g_{\rho\rho} = \mathcal{O}(1), \\ \delta g_{\tau\varphi} &= \mathcal{O}(1), & \delta g_{\rho\tau} &= \mathcal{O}(1/\rho), & \delta(g_{\rho\rho} g_{\tau\tau}) &= \mathcal{O}(1/\rho). \end{aligned} \quad (5.2)$$

They are a particular case of the boundary conditions which are suitable to treat asymptotically flat Einstein gravity [28] and realizes Chiral Gravity in flat space [29]. In Ref. [12] it was shown that the only way to have well-defined action principle with this set of boundary conditions is to supplement the action with a half of the Gibbons–Hawking term. From our point of view, this choice is quite natural, as it is dictated by the Chern–Simons formulation for $iso(2, 1)$, that is, Eq. (2.3). Therefore, the conserved quantities constructed in the previous section can accommodate a large class of solutions of flat gravity in three dimensions.

It is worthwhile noticing that these boundary conditions are suitable to study 3D asymptotically flat Einstein gravity at null infinity, where the asymptotic symmetries are described by the Bondi–Metzner–Sachs (BMS) group. In general, BMS boundary conditions have a wave as a solution, and are written in terms of the BMS coordinates that include retarded time, radius and angle. A BMS gauge allows to treat the flat case as the limit [3,34] of the AdS case [30], which is particularly useful to realize Flat/CFT correspondence [31,35]. Furthermore, a 2D dual theory at null infinity can be constructed starting from the CS formulation of 3D gravity [32].

The construction presented here is inspired by, but differs from, the one corresponding to Chern–Simons for AdS group. In three dimensions, a single copy of Chern–Simons for $SO(2, 2)$ group gives rise to Einstein–Hilbert action with negative cosmological constant [36] plus half of the Gibbons–Hawking term [21]. It was shown in Ref. [11] that this boundary term renders the variation of the action, at the same time, well defined and finite. The surface term in the variation of the action adopts the same form as in Eq. (3.2). At first glance, it looks like one needs to impose a Neumann boundary condition for the metric (i.e., fixing K_{ij}) for the action to be stationary [33]. A posteriori, one can see that adding half of the Gibbons–Hawking term is compatible with keeping a conformal structure at the boundary, instead of the full boundary metric h_{ij} . In particular, this can accommodate a holographic interpretation of the theory [11]. Indeed, the behavior of the fields in asymptotically AdS gravity is such that the extrinsic curvature is proportional to the boundary metric at leading order in the expansion. This accident happens only in the AdS gravity: the absence of a conformal data in the boundary metric in asymptotically flat gravity prevents a direct definition of holographic quantities in this case.

Acknowledgements

The authors would like to thank Glenn Barnich, Stephane De-tournay and Hernán González for useful comments. This work was supported by the Chilean FONDECYT Grants No. 3160139 and No. 1131075, VRIEA-PUCV grants No. 039.345/2016 (O.M.), No. 37.0/2015 (D.R.), UNAB Grant DI-1336-16/RG and CONICYT Grant DPI 20140115 (R.O.).

References

- [1] S. Deser, R. Jackiw, G. 't Hooft, Three-dimensional Einstein gravity: dynamics of flat space, *Ann. Phys.* 152 (1984) 220.
- [2] L. Cornalba, M.S. Costa, A new cosmological scenario in string theory, *Phys. Rev. D* 66 (2002) 066001, arXiv:hep-th/0203031.
- [3] G. Barnich, A. Gomberoff, H.A. Gonzalez, The Flat limit of three dimensional asymptotically anti-de Sitter spacetimes, *Phys. Rev. D* 86 (2012) 024020, arXiv:1204.3288 [gr-qc].
- [4] O. Miskovic, J. Zanelli, On the negative spectrum of the 2 + 1 black hole, *Phys. Rev. D* 79 (2009) 105011, arXiv:0904.0475 [hep-th].
- [5] A. Ashtekar, M. Varadarajan, A striking property of the gravitational Hamiltonian, *Phys. Rev. D* 50 (1994) 4944, arXiv:gr-qc/9406040.
- [6] T. Regge, C. Teitelboim, Role of surface integrals in the Hamiltonian formulation of general relativity, *Ann. Phys.* 88 (1974) 286.
- [7] D. Marolf, L. Patino, The non-zero energy of 2 + 1 Minkowski space, *Phys. Rev. D* 74 (2006) 024009, arXiv:hep-th/0604127.

- [8] A. Corichi, I. Rubalcava-García, Energy in first order $2 + 1$ gravity, *Phys. Rev. D* 92 (2015) 044040, arXiv:1503.03030 [gr-qc].
- [9] P. Mora, R. Olea, R. Troncoso, J. Zanelli, Finite action principle for Chern–Simons AdS gravity, *J. High Energy Phys.* 0406 (2004) 036, arXiv:hep-th/0405267.
- [10] P. Mora, R. Olea, R. Troncoso, J. Zanelli, Transgression forms and extensions of Chern–Simons gauge theories, *J. High Energy Phys.* 0602 (2006) 067, arXiv:hep-th/0601081.
- [11] O. Miskovic, R. Olea, On boundary conditions in three-dimensional AdS gravity, *Phys. Lett. B* 640 (2006) 101, arXiv:hep-th/0603092.
- [12] S. Detournay, D. Grumiller, F. Schöller, J. Simon, Variational principle and one-point functions in three-dimensional flat space Einstein gravity, *Phys. Rev. D* 89 (2014) 084061, arXiv:1402.3687 [hep-th].
- [13] H. Nishino, S.J. Gates Jr., Chern–Simons theories with supersymmetries in three-dimensions, *Int. J. Mod. Phys. A* 8 (1993) 3371.
- [14] P.S. Howe, J.M. Izquierdo, G. Papadopoulos, P.K. Townsend, New supergravities with central charges and Killing spinors in $(2 + 1)$ -dimensions, *Nucl. Phys. B* 467 (1996) 183, arXiv:hep-th/9505032.
- [15] H. Afshar, A. Bagchi, R. Fareghbal, D. Grumiller, J. Rosseel, Spin-3 gravity in three-dimensional flat space, *Phys. Rev. Lett.* 111 (2013) 121603, arXiv:1307.4768 [hep-th].
- [16] H.A. Gonzalez, M. Pino, Boundary dynamics of asymptotically flat 3D gravity coupled to higher spin fields, *J. High Energy Phys.* 1405 (2014) 127, arXiv:1403.4898 [hep-th].
- [17] H.R. Afshar, Flat/AdS boundary conditions in three dimensional conformal gravity, *J. High Energy Phys.* 1310 (2013) 027, arXiv:1307.4855 [hep-th].
- [18] A. Bagchi, S. Detournay, D. Grumiller, J. Simon, Cosmic evolution from phase transition of three-dimensional flat space, *Phys. Rev. Lett.* 111 (2013) 181301, arXiv:1305.2919 [hep-th].
- [19] A. Bagchi, R. Basu, 3D flat holography: entropy and logarithmic corrections, *J. High Energy Phys.* 1403 (2014) 020, arXiv:1312.5748 [hep-th].
- [20] E. Witten, $(2 + 1)$ -dimensional gravity as an exactly soluble system, *Nucl. Phys. B* 311 (1988) 46.
- [21] M. Banados, F. Mendez, A note on covariant action integrals in three-dimensions, *Phys. Rev. D* 58 (1998) 104014, arXiv:hep-th/9806065.
- [22] M. Banados, Global charges in Chern–Simons field theory and the $(2 + 1)$ black hole, *Phys. Rev. D* 52 (1996) 5816, arXiv:hep-th/9405171.
- [23] V. Iyer, R.M. Wald, Some properties of Noether charge and a proposal for dynamical black hole entropy, *Phys. Rev. D* 50 (1994) 846, arXiv:gr-qc/9403028.
- [24] G. Barnich, F. Brandt, Covariant theory of asymptotic symmetries, conservation laws and central charges, *Nucl. Phys. B* 633 (2002) 3, arXiv:hep-th/0111246.
- [25] R. Banerjee, D. Roy, Poincare gauge symmetries, Hamiltonian symmetries and trivial gauge transformations, *Phys. Rev. D* 84 (2011) 124034, arXiv:1110.1720 [gr-qc].
- [26] M. Banados, L.J. Garay, M. Henneaux, The dynamical structure of higher dimensional Chern–Simons theory, *Nucl. Phys. B* 476 (1996) 611, arXiv:hep-th/9605159.
- [27] G. Barnich, Entropy of three-dimensional asymptotically flat cosmological solutions, *J. High Energy Phys.* 1210 (2012) 095, arXiv:1208.4371 [hep-th].
- [28] G. Barnich, G. Compere, Classical central extension for asymptotic symmetries at null infinity in three spacetime dimensions, *Class. Quantum Gravity* 24 (2007) F15, arXiv:gr-qc/0610130.
- [29] A. Bagchi, S. Detournay, D. Grumiller, Flat-space chiral gravity, *Phys. Rev. Lett.* 109 (2012) 151301, arXiv:1208.1658 [hep-th].
- [30] O. Coussaert, M. Henneaux, P. van Driel, The asymptotic dynamics of three-dimensional Einstein gravity with a negative cosmological constant, *Class. Quantum Gravity* 12 (1995) 2961, arXiv:gr-qc/9506019.
- [31] G. Barnich, C. Troessaert, Aspects of the BMS/CFT correspondence, *J. High Energy Phys.* 1005 (2010) 062, arXiv:1001.1541 [hep-th].
- [32] G. Barnich, H.A. Gonzalez, Dual dynamics of three dimensional asymptotically flat Einstein gravity at null infinity, *J. High Energy Phys.* 1305 (2013) 016, arXiv:1303.1075 [hep-th].
- [33] C. Krishnan, A. Raju, A Neumann boundary term for gravity, arXiv:1605.01603 [hep-th].
- [34] R. Fareghbal, A. Naseh, Flat-space energy-momentum tensor from BMS/GCA correspondence, *J. High Energy Phys.* 1403 (2014) 005, arXiv:1312.2109 [hep-th].
- [35] R. Fareghbal, A. Naseh, Aspects of flat/CCFT correspondence, *Class. Quantum Gravity* 32 (2015) 135013, arXiv:1408.6932 [hep-th].
- [36] S. Deser, R. Jackiw, Three-dimensional cosmological gravity: dynamics of constant curvature, *Ann. Phys.* 153 (1984) 405.