The Geodesic Properties of the Hypercylindrical Spacetime

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Abstract

We study the geodesic properties of the static hypercylindrical spacetimes in (4+1) dimensions. The effective potential analysis and gravitational lensing effects are studied in the static spacetime. We give the marginal orbits by the effective potential in timelike case. This article is prepared for the proceedings of The Nineteenth Workshop on General Relativity and Gravitation in Japan (JGRG19), 2009.

1 Introduction

Our study is geodesic properties about 5-dimensional hypercylindrical spacetimes. It is a static spacetime. The spacetime is described by two ADM parameters the mass and tension densities. As the parameters changes, the properties of the solution shows many different behaviors. Most of this article is based on [1].

2 Hypercylindrical Spacetime

The static hypercylindrical spacetime is a class of solutions parameterized by 5-dimensional string tension and mass per length [2–4]. The metric is written as,

$$\begin{aligned} ds^2 &= -F(\rho)dt^2 + G(\rho)(d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta d\phi^2) \\ &\quad + H(\rho)dz^2 , \\ F &= \left(1 - \frac{K_a}{\rho}\right)^s \left(1 + \frac{K_a}{\rho}\right)^{-s} , \\ G &= \left(1 - \frac{K_a}{\rho}\right)^{2-\frac{(1+a)s}{2-a}} \left(1 + \frac{K_a}{\rho}\right)^{2+\frac{(1+a)s}{2-a}} , \\ H &= \left(1 - \frac{K_a}{\rho}\right)^{\frac{(-1+2a)s}{2-a}} \left(1 + \frac{K_a}{\rho}\right)^{\frac{(1-2a)s}{2-a}} , \end{aligned} \tag{1}$$

where a is the string tension to mass ratio, $s = \frac{2(2-a)}{\sqrt{3(1-a+a^2)}}$, and $K_a = \sqrt{\frac{1-a+a^2}{3}} G_5 \zeta$. Hypercylindrical spacetime gives 5-dimensional cylindrical solution in which the spherically symmetric 3+1 dimensions are orthogonal to the 1-dimensional line like 5^{th} coordinate. For specific values of a, we have well known solutions. For $a = 0.5$, the metric (1) becomes the Schwarzschild solution, and for $a = 2$ a Kaluza-Klein bubble solution. The position of horizon is given by K_a in the Schwarzschild case. Except for $a = 0.5$, the $\rho = K_a$ corresponds to a naked singularity [5]. In the effective potential level, the naked singularities can be classified by two types, weakly and strongly naked singularity [6, 7].

The physical importance of these facts is the observable quantities of the weakly naked singularity in $-1 \le a \le 2$ which has the maximum point in the effective potential which gives the unstable circular

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orbits and Schwarzschild case. The observation quantities such as the deflection angles of lightlike orbits are not easy to distinguish one from the other in this case.

On the other hand, the strongly naked singularity shows qualitatively different behaviors in $a \geq -1$ or $2 \leq a$. These behaviors can be shown in the effective potential of lightlike case comparing to the weakly naked singularity.

The effective potential can be obtained from the geodesic Lagrangian. The metric components are only dependent on coordinate ρ. It gives conserved quantities E for time, L for angle, and W for 5th direction momentum. Without loss of generality, we choose the coordinate θ equals to $\frac{\pi}{2}$ using spherical symmetry of the metric. The effective potential is given [1],

$$V_{eff} = \frac{1}{2}\frac{F(\rho)L^2}{\rho^2 G(\rho)} . \tag{2}$$

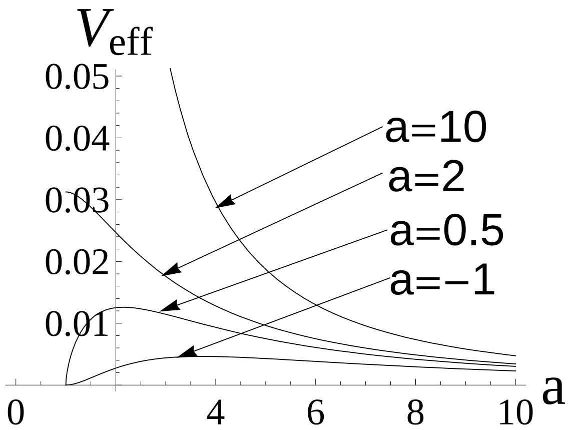

Figure 1: The effective potential at specific value of parameter a.

In the range of $-1 \leq a \leq 2$, the effective potentials have maximum point which light can make unstable circular orbit named photon sphere which partly covers their naked singularity like horizon. The other values of parameter a give no maximum point like as shown in Figure 1, so the naked singularity is exposed completely. The size of horizon is related to the capture-cross section which light cannot escape from gravity of the spacetime. In the Figure 2, the maximum value is achieved for the Schwarzschild case, and as the parameter value a becomes farther away from the Schwarzschild case, the capture-cross section becomes smaller. Briefly, the deflection angles for spacetimes with $-1 \leq a \leq 2$ are qualitatively

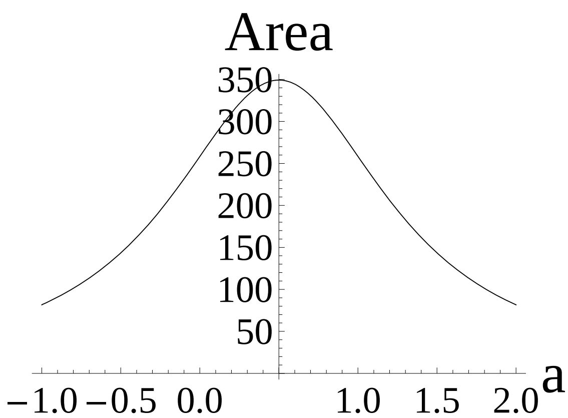

Figure 2: The effective potential at specific value of parameter a.

a	-1	-0.5	0	0.5	1	1.5
L	2.9	4.8	6.4	7	5.2	3
ρ	1.56673	4.59308	7.54371	8.38068	9.07818	5.48575

Table 1: The relations of a, L, ρ are shown. For each value of the constant a, the angular momentum L and radius ρ for marginal orbit are shown. In these analysis, objects which are not $a = \frac{1}{2}$ have different geodesic properties, which Schwarzschild black string has.

similar to each other. It means that distinguishing Schwarzschild and weakly naked singularity case is not easy. On the other hand, the strongly one gives very different values of deflection angle. The weakly naked spacetime have unstable circular orbits at the maximum point in the effective potential, capture-cross section, and similar effect in gravitational lensing as those of the Schwarzschild case. It gives same properties in [6–8].

In the timelike geodesics, there are unstable circular orbits at the maximum point of effective potential like Schwarzschild case. In the range of $-1 < a < 2$, the orbit can exist only for $r \geq 6M$ as in the Schwarzschild spacetime case if $a = \frac{1}{2}$. The orbit at $r = 6M$ is called the marginal stable circular orbit or the innermost stable circular orbit. Some examples are shown in Fig. 3.

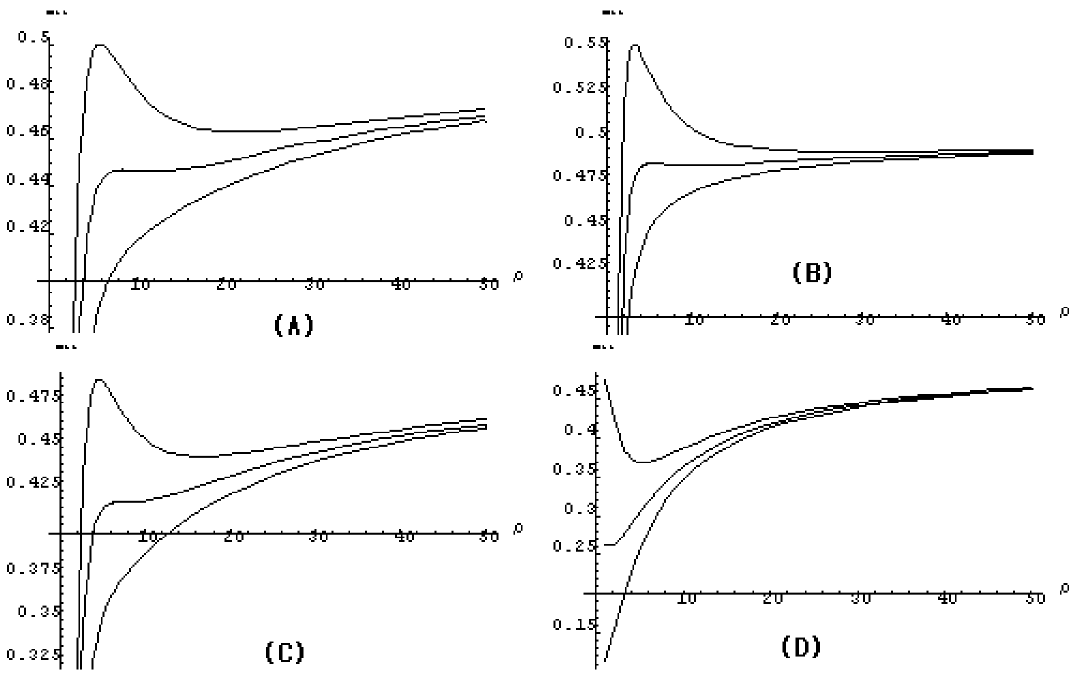

Figure 3: (A) The case of the Schwarzschild black string, $a = \frac{1}{2}$. In (A), upper curve is at $L = 8$, and the middle is at $L = 7$, and the lower is at $L = 6$(all figures are some ordering.). The marginal orbit appears at middle curve. (B) is of $a = \frac{3}{2}$. The angular momentums, L, are 4, 3, 2. (C) is for $a = 0$. From above table 1, the radius of marginal orbit is moved to left side and does not change its shape. (D) is for $a = -1$. In this case, the shape of potential is different from other case.

From the Fig. 3, we see that there are finite potential barrier due to the angular momentum for the cases of $a = 0$, $a = \frac{1}{2}$ and $a = \frac{3}{2}$. On the other hand, the potential barrier preventing a particle reaching deep inside appears appeared for the case of $a = -1$.

3 CONCLUSIONS

In this paper, we have studied the geodesic motions and the orbits of both a massive particle and light ray. The geometry of the hypercylindrical solution is dependent on single constant a, a ratio of tension and mass density. This geometry becomes that of the Schwarzschild black string for $a = \frac{1}{2}$, and the static Kaluza-Klein bubble for $a = 2$. There exist five conserved quantities corresponding to translation symmetry of time, angle, 5th dimension coordinate, and two quantities which give equatorial plane $\theta = \frac{1}{2}\pi$. The light can move around a unstable circular orbit in $-1 \leq a \leq 2$. The radial range of the unstable circular orbit is related to area of light capture. The capture cross section is formed in $-1 \leq a \leq 2$, and the largest area case is $a = \frac{1}{2}$ Schwarzschild black string. We calculate the timelike geodesic equations and the range of the constant a which gives stable circular orbits in $a < -1$. One of the characteristics of the timelike case is that there exist a marginal stable circular orbit in $-1 < a < 2$. The angular momentum and radius of this marginal orbit is numerically obtained, and the shapes of the effective potentials are similar to Schwarzschild black string in $-1 < a < 2$.

Acknowledgments

This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MEST)through the Center for Quantum Spacetime(CQUeST) of Sogang University with grant number 2005-0049409. BG is supported by the scholarship of Korea Science and Engineering Foundation (Scholarship Number:S2-2008-000-00800-1). WL was supported by the Korea Research Foundation Grant funded by the Korean Government(MOEHRD)(KRF-2007-355-C00014).

References

[1] B. Gwak, B.-H. Lee, and W. Lee, J. Korean Phys. Soc. **54**, 2202 (2009), [arXiv:0806.4320].

[2] D. Kramer, Acta Phys. Polon. B **2**, 807 (1971); A. Chodos and S. Detweiler, Gen. Rel. Grav. **14**, 879 (1982); J. Gross and M. M. Perry, Nucl. Phys. B **226**, 29 (1983); A. Davidson and D. A. Owen, Phys. Lett. B **155**, 247 (1985).

[3] C. H. Lee, Phys. Rev. D **74**, 104016 (2006), hep-th/0608167.

[4] S. Yun, arXiv:0905.4338 [gr-qc].

[5] I. Cho, G. Kang, S. P. Kim and C. H. Lee; J. Korean Phys. Soc. **53**, 1089 (2008), [arXiv:0709.1021]; *in preparation.*

[6] K. S. Virbhadra and G. F. R. Ellis, Phys. Rev. D **65**, 103004 (2002).

[7] C. M. Claudel, K. S. Virbhadra and G. F. R. Ellis, J. Math. Phys. **0142**, 818 (2001) [arXiv:gr-qc/0005050].

[8] K. S. Virbhadra, Phys. Rev. D **79**, 083004 (2009) [arXiv:0810.2109 [gr-qc]].