

Phases of $N = \infty$ QCD-like gauge theories on $S^3 \times S^1$ and nonperturbative orbifold-orientifold equivalences

Mithat Ünsal^{1*}

¹ *SLAC, Stanford University, Menlo Park, CA 94025*

Physics Department, Stanford University, Stanford, CA, 94305

ABSTRACT: We study the phase diagrams of $N = \infty$ vector-like, asymptotically free gauge theories as a function of volume, on $S^3 \times S^1$. The theories of interest are the ones with fermions in two index representations [adjoint, (anti)symmetric, and bifundamental abbreviated as QCD(adj), QCD(AS/S) and QCD(BF)], and are interrelated via orbifold or orientifold projections. The phase diagrams reveal interesting phenomena such as disentangled realizations of chiral and center symmetry, confinement without chiral symmetry breaking, zero temperature chiral transitions, and in some cases, exotic phases which spontaneously break the discrete symmetries such as C, P, T as well as CPT. In a regime where the theories are perturbative, the deconfinement temperature in SYM, and QCD(AS/S/BF) coincide. The thermal phase diagrams of thermal orbifold QCD(BF), orientifold QCD(AS/S), and $\mathcal{N} = 1$ SYM coincide, provided charge conjugation symmetry for QCD(AS/S) and \mathbb{Z}_2 interchange symmetry of the QCD(BF) are not broken in the phase continuously connected to \mathbb{R}^4 limit. When the S^1 circle is endowed with periodic boundary conditions, the (nonthermal) phase diagrams of orbifold and orientifold QCD are still the same, however, both theories possess chirally symmetric phases which are absent in $\mathcal{N} = 1$ SYM. The match and mismatch of the phase diagrams depending on the spin structure of fermions along the S^1 circle is naturally explained in terms of the necessary and sufficient symmetry realization conditions which determine the validity of the nonperturbative orbifold orientifold equivalence.

KEYWORDS: $1/N$ Expansion, Spontaneous Symmetry Breaking.

*unsal@slac.stanford.edu

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1. Introduction and results

Understanding the dynamics of the strongly coupled, vectorlike gauge theories is an outstanding problem of contemporary physics. Various methods have been developed to come to grips with QCD and QCD-like theories. Among such lattice gauge theory and effective field theory have been the most successful and reliable, and supersymmetry has been useful in addressing some questions. A more qualitative method is the $1/N$ expansion.

In this paper, we examine the dynamics and phase diagrams of certain asymptotically free, confining vectorlike gauge theories (without fundamental scalars) in the infinite number of color limit by benefiting from the available techniques mentioned above. Since $N = \infty$ is a thermodynamic limit, it is not necessary to take the volume to infinity to attain phase transitions [1–3]. In fact, dialing the volume (and temperature), which acts as an external parameter, leads to rich phase diagrams. We work on the compact space $Y^3 \times S^1$, where Y^3 is some three manifold. Our primary choice of the three manifold is S^3 for reasons explained below. We will discuss the decompactification limits $\mathbb{R}^3 \times S^1$, $\mathbb{R} \times S^3$ and \mathbb{R}^4 where the hardest problems of the QCD-like theories lie. The study of the dynamics of the QCD-like theories as a function of radii reveals new phenomena such as disentangled realization of chiral and center symmetry, confinement without chiral symmetry breaking, zero temperature chirally symmetric phases, as well as the appearance of chirally asymmetric phases at weak coupling. Most of these phenomena are testable on the lattice.

The QCD-like theories that will be of interest are the ones with fermions in double index (or tensor) representations. The virtue of the double index representation is that unlike the fundamental quarks, a finite number of the tensor flavors are not kinematically suppressed in the large N limit. In fact, too many tensor quarks may overwhelm the asymptotic freedom which our discussion assumes. Therefore, we restrict the number of flavors to at most five, $n_f \leq 5$. In the determination of the vacuum structures, thermodynamics and other properties, the double index quarks are as important as gluons and sometimes, as we will see, the fermionic contributions play the decisive role. There are three classes of double index representations for the complex $U(N)$ color gauge group: adjoint, antisymmetric, symmetric. For the product gauge group $U(N) \times U(N)$, there is also two index bifundamental representation. We abbreviate these theories as QCD(adj), QCD(AS/S) and QCD(BF). At first glance, it may sound strange to include a product gauge group. There are two good reasons for doing

so: First, it is a vectorlike gauge theory, and second, more interestingly, its phase diagram turns out to be identical to QCD(AS/S).¹

The QCD-like theories formulated on \mathbb{R}^4 (or $\mathbb{R}^{3,1}$), unlike conformal field theories such as $\mathcal{N} = 4$ SYM, do not have a tunable coupling constant. However, formulating this theory on $S^3 \times S^1$, where the radius of either S^3 or S^1 is much smaller than the strong confinement scale Λ^{-1} , i.e, $\min(R_{S^3}, R_{S^1}) \ll \Lambda^{-1}$, we can benefit from asymptotic freedom. In this regime, the coupling constant is weak $g^2(1/\min(R_{S^3}, R_{S^1})) \ll 1$, and perturbative methods are reliable on the scale of the smaller radius.

Formulating the theory on small $S^3 \times S^1$, as shown in [4–6] has the advantage of making the center symmetry realizations accessible to perturbation theory. (Also see [7–9] for related work.) Since the three-sphere S^3 is simply-connected just like \mathbb{R}^3 , its spin structure is fixed. On the other hand, depending on the choice of the boundary conditions for the fermions in the S^1 direction (antiperiodic or periodic), the partition function will either be the usual thermal ensemble $Z = \text{tr} e^{-\beta H}$ or a twisted partition function $\tilde{Z} = \text{tr} e^{-\beta H} (-1)^F$. F is the fermion number operator. Depending on the choice of the boundary conditions for fermions on S^1 , the center symmetry should be viewed as temporal \mathcal{G}_t for antiperiodic or spatial center symmetry \mathcal{G}_s for periodic boundary conditions. Accordingly, there are two distinct classes of center symmetry changing transitions, one associated with temporal Wilson lines (thermal Polyakov loop) and the other associated with the spatial Wilson lines. The temporal center symmetry realization forms a criterion for the deconfinement, confinement transition, and the change in symmetry realization is triggered by thermal fluctuations. The transition manifests itself as an abrupt change in the thermodynamic free energy density \mathcal{F} . \mathcal{F} may be extracted from the minimum of the effective potential for thermal Polyakov loops, hence from the partition function $\mathcal{F} = -\frac{1}{\beta V_{S^3}} \log Z$ where V_{S^3} and β are respectively volume of the three sphere and inverse temperature. But the spatial Wilson lines do not constitute a deconfinement criterion. The change in the spatial symmetry realization is due to zero temperature quantum mechanical fluctuations (rather than thermal) and the phase transition is accompanied by an abrupt change in the vacuum energy density. Therefore it is more naturally viewed as a quantum phase transition. The vacuum energy density may be extracted from the effective potential of the spatial Wilson line and the twisted partition function $\mathcal{E} = -\frac{1}{V_{S^3} \times V_{S^1}} \log \tilde{Z}$ where $V_{S^3} \times V_{S^1}$ is the volume of the four manifold $S^3 \times S^1$.² The reader should keep in

¹As a matter of convention, by the n_f flavor vectorlike (or QCD-like) gauge theory, we refer to n_f Dirac fermions for QCD(AS/S/BF) since (anti)symmetric and bifundamental representations are complex, and n_f Majorana (or Weyl) adjoint fermion for QCD(adj) because the adjoint representation is real. Since a gauge invariant mass term is allowed in any of these theories, they are all vectorlike by the standard definition. The ratio of the number of fermionic degrees of freedom to the bosonic degrees of freedom is just n_f for n_f flavor QCD(adj) and QCD(BF), and is $n_f(1 \mp \frac{1}{N})$ for n_f flavor QCD(AS/S).

²Strictly speaking, on $S^3 \times S^1$, if one wants to have an operator interpretation, the only way to define a transfer matrix (or Hamiltonian) is to interpret S^1 as temporal (and there is no second choice.) [10]. On the other hand, on $\mathbb{R}^3 \times S^1$ limit, one can define the transfer matrix for either choices of S^1 , temporal or spatial. The two choices of the S^1 have different physical interpretations. The case for temporal S^1 (which is widely studied in literature) is the finite temperature field theory on \mathbb{R}^3 . In the case $S^3 \times$ (temporal S^1), one can define a transfer matrix. If S^1 is a spatial circle, then this should be viewed (on $\mathbb{R}^3 \times S^1$) as a zero temperature field theory on a space with one compact direction, where zero temperature axis corresponds to one of the

mind that the realizations and physical interpretations of the temporal and spatial center symmetries for a given theory are different, and the existence of one do not imply the other, see for example [11–13]

Phases of $N = \infty$ QCD(adj) and QCD(AS/S/BF) on $S^3 \times S^1$: For thermal $\mathcal{N} = 1$ SYM and QCD(AS/S/BF), we analytically demonstrate the temporal center symmetry changing confinement deconfinement transitions in a perturbative regime of the theory. These transitions are associated with a jump in the free energy density from $\mathcal{O}(N^2)$ in the deconfined plasma phase to being $\mathcal{O}(1)$ in the confined phase. In the regime where the radii of S^3 and S^1 are small compared to the strong confinement scale Λ^{-1} , $\max(R_{S^3}, \beta) \ll \Lambda^{-1}$ and a perturbative one loop analysis is reliable, we find that the transition occurs at exactly the same temperature:

$$T_d^{\text{SYM}} = T_d^{\text{QCD(AS/S)}} = T_d^{\text{QCD(BF)}} \quad (1.1)$$

At first glance, this result is rather surprising and its underlying reason, based on nonperturbative large N orbifold/orientifold equivalence, will be explained below. We do not know how to calculate the deconfinement temperature in the strongly coupled, nonperturbative regime of these theories. Therefore, a priori there is no way to tell whether Eq.1.1 will hold in the nonperturbative regime. The implication of the large N orbifold/orientifold equivalence is that Eq.1.1 should be true in the $N = \infty$ vectorlike gauge theories even in the nonperturbative regime. The matching of the deconfinement temperatures in Eq.1.1 generalizes easily to the multiflavor case as well where T_d^{SYM} is replaced by $T_d^{\text{QCD(adj)}}$.

If S^1 is endowed with periodic boundary conditions, we do not observe any spatial center symmetry changing transition in SYM and its multiflavor generalization QCD(adj). This is by itself surprising and just mimics the behaviour of these theories on $\mathbb{R}^3 \times S^1$ where spatial center symmetry realization is independent of the size of the S^1 circle [12], and is unbroken at any radius. On the other hand, we do observe a center symmetry changing transition in QCD(AS/S/BF) occurring at exactly the same radius of S^1 .

$$R_{S^1,s}^{\text{QCD(AS/S)}} = R_{S^1,s}^{\text{QCD(BF)}} \quad (1.2)$$

This transition is associated with the breaking of the spatial center symmetry \mathcal{G}_s and is accompanied with a change in the vacuum energy density. In the \mathcal{G}_s symmetry broken phase, the vacuum energy density is $\mathcal{O}(N^2)$ while in the unbroken phase, it is $\mathcal{O}(1)$.

noncompact \mathbb{R} . Or in Minkowski space, one can think of an Hamiltonian formulation of a gauge theory on $\mathbb{R}^{2,1} \times \text{spatial } S^1$. Therefore, a quantum mechanical interpretation of the correlators in the functional integral is possible. The reader may be worried that the Euclidean theory on $S^3 \times (\text{spatial } S^1)$ does not have an operator interpretation (or even any temperature related interpretation) in the sense of the last statement at any finite value of S^3 . Nonetheless, we can define an effective action for the spatial Wilson lines by compactifying the functional integral representation of $\text{tr} e^{-\beta H} (-1)^F$ on $\mathbb{R}^3 \times S^1$ into $S^3 \times S^1$. The effective action defined in this way turns out to be a very useful tool as we will see in the course of the study of phases. Therefore, we regard the effective action (for spatial Wilson lines) on $S^3 \times S^1$ as an “auxiliary” device, and borrow its terminology such as “ground state, correlation function, vacuum energy” from its decompactification limit, where a quantum mechanical interpretation makes sense. The twisted partition function, though lacking any operator interpretation on $S^3 \times S^1$, turns out to be beneficial due to the smooth volume dependence conjecture for (spatial) center symmetry realization that we will discuss momentarily.

The chiral properties of these vectorlike theories are equally interesting. Upon thermal compactification, we find similar behaviour for all the QCD-like theories. At high temperatures or high curvatures (i.e, at small thermal S^1 or small S^3), there is no condensate and the theory is chirally symmetric. The phase at small radius of S^3 but low temperature, shows confinement without chiral symmetry breaking. When $\min(R_{S^3}, \beta) \gg \Lambda^{-1}$, the chiral symmetry is broken. The $n_f = 1$ QCD-like theories possess a discrete chiral symmetry \mathbb{Z}_{2h} where $h = N$ for QCD(BF) and SYM and $h = N \mp 2$ for QCD(AS/S). The chiral symmetry is spontaneously broken to \mathbb{Z}_2 leading to h isolated vacua which are distinguished by the phase of the condensate. The $n_f > 1$ case is accompanied with both discrete and continuous chiral symmetry breaking, therefore leading to both domain walls and Goldstone bosons. There are h isolated vacuum manifolds distinguished by the phase of the determinant of the rank n_f chiral condensate matrix. The isolated vacua in the $n_f = 1$ case are replaced by suitable coset spaces. The chiral properties are discussed in detail in subsections 3.4, 4.4, and 5.6.

The case where S^1 is endowed with the periodic boundary condition is drastically different between QCD(adj) and QCD(AS/S/BF). The chiral properties of QCD(AS/S/BF) are essentially the same as in the thermal case, i.e., a chirally symmetric phase at high curvature and on a small S^1 circle $\min(R_{S^1}, R_{S^3}) \ll \Lambda^{-1}$, and a chirally asymmetric phase in the opposite limit. However, the underlying physical reasons behind the restoration of chiral symmetry on small S^1 differs from the thermal case (where it is due to tree level antiperiodic boundary conditions for fermions), as it is a one loop quantum effect. However for SYM (or QCD(adj)), we find the chiral symmetry realization is independent of the size of the S^1 circle, and it only depends on the curvature of S^3 . At large curvature, the theory is chirally symmetric while at small curvature, the chiral symmetry is broken. The physical reasons why the chiral symmetry realizations are so different between QCD(adj) and QCD(BF/AS/S) and why the former (latter) has a chirally asymmetric (symmetric) phase at small S^1 is explained in 3.4, and 4.4.

Large N equivalences, small n_f universalities: The results of the phase diagrams would be completely mysterious were it not for the nonperturbative large N equivalence. The QCD-like theories described above, for each value of n_f , are related to each other via a chain of orbifold and orientifold projections (see refs. [14–20] and references therein). The case where $n_f = 1$ is particularly interesting, as it relates supersymmetric $\mathcal{N} = 1$ SYM theory to nonsupersymmetric theories such as QCD(BF/AS/S). Ref. [14,15] demonstrated that *as long as the discrete symmetries defining the neutral sectors are unbroken, there is a nonperturbative large N equivalence among the neutral sectors of the theories related to each other via orbifold and orientifold projections.*³ The large N equivalence does not make any reference to properties such as supersymmetry, conformal symmetry, spacetime dimension, or the topology

³In [14], the necessary criteria for the validity of the equivalence is derived by comparing the generalized loop equations (the Schwinger-Dyson equations for the correlators of the gauge invariant operators) on lattice regularized parent and daughter theories. If the symmetries defining the neutral sectors is unbroken (broken), then the loop equation do (not) coincide. This demonstrates the necessity of the symmetry realizations. However, the loop equation are nonlinear and typically has multiple solutions. Therefore, coinciding loop equations do not constitute a proof outside the strong coupling, large mass phase of the lattice gauge theory where one can prove uniqueness of the solution. The more abstract coherent state approach plugs this hole and demonstrates that the unbroken symmetry is the sufficient condition for large N equivalence [15].

of the spacetime manifold or other details. Its validity only relies on the realizations of symmetries defining the respective neutral sectors of our vectorlike theories.

If the symmetries defining neutral sectors are unbroken, then the $N = \infty$ neutral sector dynamics coincide. This implies a well defined mapping between the expectation values of Wilson lines and Wilson loops, the expectation values of the chiral condensates, the nonperturbative particle spectra such as glueball and meson masses, the multibody decay amplitudes, as well as the thermodynamics. In other words, the large N equivalence provides a powerful $N = \infty$ duality without solving neither theory (which is hard) [14, 15].

In our case, the neutral sectors amount to the bosonic subsectors of the $U(N)$ QCD(adj) theory (which is neutral under the fermion number modulo two $\mathbb{Z}_2 = (-1)^F$), the charge conjugation symmetry \mathcal{C} even subsector of $U(N)$ QCD(AS/S) and \mathbb{Z}_2 exchange even subsector of $U(N) \times U(N)$ QCD(BF). Therefore, the validity of orbifold and orientifold equivalences is crucially tied to their respective unbroken \mathbb{Z}_2 symmetries. If the corresponding \mathbb{Z}_2 is unbroken, a prediction of large N equivalence is that the phase diagrams of n_f flavor large N QCD(adj) and QCD(AS/S/BF) should be the same. In the thermal case, where the properties of QCD(AS/S/BF) and SYM coincide, the \mathbb{Z}_2 charge conjugation symmetry for QCD(AS/S) and the \mathbb{Z}_2 exchange symmetry for QCD(BF) are unbroken in any phase of these theories. (More precisely, there are three phases and in two of them, we can analytically demonstrate that these symmetries are unbroken. In the third phase where $\min(R_{S^3}, \beta) \gg \Lambda^{-1}$, the current knowledge is consistent with unbroken \mathbb{Z}_2 .) In the nonthermal case, where the phase diagram of QCD(adj) is different from QCD(AS/S/BF), there exists a phase in which the \mathbb{Z}_2 symmetries for QCD(AS/S/BF) are spontaneously broken. Outside that phase, the large N equivalence is still valid. Therefore, coinciding phase diagrams in the thermal case among our QCD-like gauge theories may naturally be viewed as a consequence of large N orbifold/orientifold equivalence. The differences between the phase diagrams of QCD(adj) and QCD(AS/S/BF) in the nonthermal case may be viewed as an eminent demonstration of the symmetry realization conditions.

A conjecture on the spatial and temporal center symmetry realization: The examination of the center symmetry realizations of QCD-like theories on $S^3 \times S^1$ leads us to a smooth volume dependence conjecture for the asymptotically free, confining gauge theories.

In confining vectorlike, asymptotically free large N gauge theories formulated on $S^3 \times S^1$, the (spatial and temporal) center symmetry realizations are independent of the size of the S^3 sphere in the following sense: If a perturbative center symmetry changing transition exists in small $S^3 \times S^1$, it smoothly interpolates into a nonperturbative transition in $\mathbb{R}^3 \times S^1$. If there is no change in center symmetry realizations on small S^3 , the theory on $\mathbb{R}^3 \times S^1$ does not undergo a center symmetry changing transition as a function of S^1 volume either.

In QCD(AS/S/BF), there is a change in the spatial and temporal center symmetry realization on $S^3 \times S^1$, and we expect this to continue to $\mathbb{R}^3 \times S^1$. For thermal QCD(adj), there is a change in the temporal center symmetry realization on small $S^3 \times S^1$ which is expected to interpolate all the way to $\mathbb{R}^3 \times S^1$. However, for QCD(adj) with periodic boundary conditions, the spatial center symmetry is unbroken on $S^3 \times S^1$, and this result also holds on $\mathbb{R}^3 \times S^1$.

This conjecture is a natural generalization of the one by Aharony et.al. [6] in the case of pure Yang-Mills theory.

2. Generalities of vector-like gauge theories

Let us first review some generalities of the asymptotically free, confining vectorlike gauge theories, and set the tools for the analysis. Throughout this paper, we will examine both zero temperature and finite temperature phases of certain large N vectorlike theories on Euclidean spacetime manifolds, $Y^3 \times S^1$. The three manifolds may be chosen to be $Y^3 = \{T^3, \mathbb{R}^3, S^3\}$. Our primary choice is S^3 even though we will comment on the decompactification limit \mathbb{R}^3 , and the flat three space T^3 .

The reason for choosing $S^3 \times S^1$ is, as illustrated in [4–6], the ability to observe a confinement deconfinement transition in the perturbative regime of the theory if S^1 is interpreted as a thermal circle. As we will observe momentarily, $S^3 \times S^1$ also admits other center symmetry changing transitions which are not associated with abrupt change in the free energy density, but vacuum energy density. The corresponding center symmetry should be viewed as spatial. We will observe the existence of both spatial and temporal types of center symmetry changing transitions for QCD-like theories.

Strictly speaking, a theory at finite spatial volume and at finite N , with a finite number of local fields cannot have a phase transition and spontaneous symmetry breaking, but just rapid crossovers. The thermodynamic limit in the theories of interest is attained in the $N \rightarrow \infty$ limit, and phase transitions at $N = \infty$ are perfectly sensible [1]. In the $N = \infty$ limit, the radii of the $S^3 \times S^1$ space take the roles of external tunable parameters. The size of the $S^3 \times S^1$ also tells us whether the theory is strongly or weakly coupled due to asymptotic freedom. In the regime where $\min(R_{S^3}, R_{S^1}) \geq \Lambda^{-1}$, the theory is strongly coupled and we lack analytical tools to establish the details of the dynamics and phase transitions. However, by benefiting from the existing lattice result in the strongly coupled regime, we will be able to infer the qualitative structure of the phase diagrams.

2.1 Thermal and twisted partition functions

We wish to analyze the phase diagrams of the QCD-like theories (the n_f flavor QCD(AS/S/BF) and QCD(adj)) on $S^3 \times S^1$ in the $N = \infty$ limit. To do so, we need to write the partition function of the theory on $S^3 \times S^1$. The spin structure on S^3 is fixed. Depending on the choice of the boundary conditions of fermions on S^1 circle (antiperiodic or periodic), the partition function will either be the usual thermal ensemble $Z = \text{tr} e^{-\beta H}$ or “a twisted partition function” $\tilde{Z} = \text{tr} e^{-\beta H} (-1)^F$. The operator F is the fermion number operator. In particular, the twisted partition function \tilde{Z} can be used to probe phase transitions, for example, the ones associated with spatial center symmetry, or chiral symmetry. Since it depends on the phase of the theory, it is not an index.⁴ It is useful to express the formulae for these partition

⁴The twisted partition function is conceptually more profound than supersymmetric index. It is a useful tool to study the phases of all of the QCD-like theories, not only the supersymmetric ones. In fact, among all the QCD-like theories examined in this paper, there is only one whose underlying Lagrangian is supersymmetric.

functions to understand them better. Let \mathcal{B} and \mathcal{F} be the bosonic and fermionic subsectors of the physical Hilbert space graded according to the fermion number operator. Physical states in \mathcal{B} and \mathcal{F} will satisfy

$$(-1)^F |B\rangle = |B\rangle, \quad (-1)^F |F\rangle = -|F\rangle \quad (2.1)$$

grading. Therefore, the explicit form of the thermal partition function and twisted partition function are

$$\begin{aligned} Z &= \text{tr} e^{-\beta H} = \sum_{\mathcal{B}} e^{-\beta E_n} + \sum_{\mathcal{F}} e^{-\beta E_n} = \int dE [\rho_B(E) + \rho_F(E)] e^{-\beta E} = Z_B + Z_F \\ \tilde{Z} &= \text{tr} e^{-\beta H} (-1)^F = \sum_{\mathcal{B}} e^{-\beta E_n} - \sum_{\mathcal{F}} e^{-\beta E_n} = \int dE [\rho_B(E) - \rho_F(E)] e^{-\beta E} = Z_B - Z_F \end{aligned} \quad (2.2)$$

where $\rho_B(E)$ and $\rho_F(E)$ are bosonic and fermionic density of states. Z_B and Z_F are defined as the contribution to partition function from the bosonic \mathcal{B} and fermionic \mathcal{F} subsectors of the physical Hilbert space, respectively. Referring to \tilde{Z} as twisted partition function is justified since the sum over all fermionic states contributes with a minus sign as opposed to the usual partition function where the sum is over all states without alternating, hence the moniker “twisted”. Throughout this paper, we will benefit from both partition functions to establish phase diagrams of the QCD-like theories. In particular, the free energy can be derived from the thermal partition function

$$\mathcal{F} = -\frac{1}{\beta V_{S^3}} \log Z \quad (2.3)$$

where β is inverse temperature, the radius of the thermal S^1 . Similarly, the vacuum energy density can be extracted from the twisted partition function

$$\mathcal{E} = -\frac{1}{R_{S^1} V_{S^3}} \log \tilde{Z} \quad (2.4)$$

where the R_{S^1} is just the radius of a spatial S^1 .

2.2 Temporal and spatial center symmetry

The manifold $S^3 \times S^1$ has the fundamental group $\pi_1(S^3 \times S^1) = \pi_1(S^1) \equiv \mathbb{Z}$ just like the $\pi_1(\mathbb{R}^3 \times S^1) \equiv \mathbb{Z}$. The existence of compact directions for which the fundamental group is

This is QCD(adj) with $n_f = 1$ or $\mathcal{N} = 1$ SYM. Even for the supersymmetric theory, the $\tilde{Z} = \text{tr} [e^{-\beta H} (-1)^F]$ is not generally a supersymmetric index on curved backgrounds. Formulating $\mathcal{N} = 1$ SYM theory on a curved manifold breaks the supersymmetry. (Because of the absence of the covariantly constant spinors on curved spaces, one can not define global supersymmetry.) At a more mundane level, we will explicitly see in sections 3.1, and 3.4 that $\mathcal{N} = 1$ SYM on $S^3 \times (\text{spatial } S^1)$ has at least two phases: one with unique vacuum and the other with N vacua, therefore \tilde{Z} depends on the phase, which means it can not be an index. If the three manifold S^3 is replaced by a flat space such as T^3 or \mathbb{R}^3 where global supersymmetry is restored, then the $\text{tr} [e^{-\beta H} (-1)^F]$ corresponds to a supersymmetric Witten index [21]. Therefore, in the large radius limit i.e., $R_{S^3} \Lambda \gg 1$ where we can approximate S^3 with \mathbb{R}^3 , the leading large N behaviour of $\text{tr} e^{-\beta H} (-1)^F$ may be viewed as an index. The main point is \tilde{Z} reduces to a supersymmetric index only under extremely special circumstances.

nontrivial generates a global symmetry called center symmetry, \mathcal{G} . The center symmetry is the invariance of the gauge theory under gauge transformations which are only periodic up to an element of the center of the gauge group. Since both \mathbb{R}^3 and S^3 are simply connected, the center symmetry realization will be determined by the Wilson line winding around the S^1 circle.

Whether this center symmetry should be regarded as spatial or temporal depends on the spin structure on the S^1 circle. For the antiperiodic boundary condition for fermions around the S^1 circle, the center symmetry is naturally interpreted as the temporal center symmetry \mathcal{G}_t . The corresponding QCD-like theory on $S^3 \times S^1$ may be regarded as a finite temperature field theory on S^3 . In the decompactification limit of the S^3 , we obtain the theory on $\mathbb{R}^3 \times S^1$, a thermal field theory on \mathbb{R}^3 .

If the fermions are endowed with periodic boundary conditions on the S^1 circle, then the theory on $S^3 \times S^1$ does not have any thermal interpretation at finite S^3 . In the limit where S^3 decompactifies to \mathbb{R}^3 , the theory becomes formulated on $\mathbb{R}^3 \times S^1$. Here, S^1 is a compact spatial dimension unlike the thermal case where it is viewed as a temporal (or thermal) dimension. The QCD-like theory on $\mathbb{R}^3 \times S^1$ may be viewed as a zero temperature field theory where the decompactified thermal circle is viewed as one of the infinite dimensions in \mathbb{R}^3 . Therefore, the center symmetry along the S^1 circle is a spatial center symmetry, which we denote by \mathcal{G}_s .

On $\mathbb{R}^3 \times S^1$ where S^1 is temporal, the behaviour of thermal Wilson-Polyakov loops forms a criteria for deconfinement and is related to the static quark-antiquark potential. If the S^1 circle is spatial, the spacelike Wilson loops, even though they can be used as order parameters of the (spatial) center symmetry, do not constitute a deconfinement criteria. They merely measure the correlations of the spatial gauge fields [22].

Lattice formulations of the vectorlike gauge theories usually deal with a discretized four torus T^4 which is not simply connected, and for which the fundamental group is $\pi_1(T^3 \times S^1) \equiv (\mathbb{Z})^4$. This means on T^4 , the topology permits noncontractible Wilson lines winding around each direction separately. In this case, more elaborate breaking patterns probing $(\mathcal{G}_s)^3 \times \mathcal{G}_t$ symmetry realizations should occur as in pure Yang-Mills theory on T^4 [2, 3].⁵ In this regard, $S^3 \times S^1$ topology seems to capture an important ingredient of infinite volume theory on $\mathbb{R}^3 \times S^1$, i.e, the only order parameter measuring the center symmetry is the one winding around the S^1 circle. On the other hand, if the size of the three manifolds T^3 and S^3 is much larger than the strong confinement scale, we may approximate both manifolds as \mathbb{R}^3 and the physics of the theory should become the one on $\mathbb{R}^3 \times S^1$.

2.3 Symmetry and effective action

Our interest is in $U(N)$ color gauge theories with dynamical fermions in the two index representation. The center symmetry of a gauge theory is in general different from the center of the gauge group (unless the matter is in the adjoint representation), and is determined by the charges of its dynamical fermions under the center of the group. The center for the group $U(N)$ is $U(1)$.

⁵For the pure gauge theory on T^4 or $S^3 \times S^1$, the distinction between \mathcal{G}_s and \mathcal{G}_t disappears due to the absence of fermions.

A dynamical fundamental representation fermion has charge +1 under the $U(1)$ center of the group. The symmetric and antisymmetric representation fermions, which carry two upper indices has charge +2, and the adjoint representation, which carries one upper and one lower index has charge 0 under the center. Let $a \in \mathbb{Z}$ be the charge of an external, nondynamical test quark. It is easy to see that a fundamental fermion can screen any external charge without any cost, the AS/S fermion can screen any even charge, and convert any odd charge to +1, while an adjoint fermion cannot screen any. Therefore, we can define the equivalence classes for static external charges as

$$a \cong a + q_d \quad (2.5)$$

where q_d refers to the charge of the dynamical quark under the center. The quotient of the charge lattice of the static test quarks to the conjugacy class of the dynamical quarks is essentially the center symmetry of the theory. The quotient groups are respectively, $\mathbb{Z}/(q_d\mathbb{Z}) \cong \{1, \mathbb{Z}_2, \mathbb{Z}\}$ in the presence of the fundamental, (anti)symmetric and adjoint representation dynamical quarks. Therefore, for QCD with fundamental quarks, center symmetry is absent, for QCD(AS/S) it is \mathbb{Z}_2 and QCD(adj), it is $U(1)$.⁶

The QCD(BF) is a vectorlike theory with gauge group $U(N) \times U(N)$ and with bifundamental fermions. The center of the group is therefore $U(1) \times U(1)$. However, the massless bifundamental fermions have charges under the center given by

$$(\lambda_1)^i{}_j : (1, -1), \quad (\lambda_2)^j{}_i : (-1, 1). \quad (2.6)$$

This just means the dynamical bifundamental fermions can screen any test charge whose charge is an integer multiple of $(1, -1)$. Suppose an external test quark has doublet of charge (a, b) on the center charge lattice $\mathbb{Z} \times \mathbb{Z}$. The equivalence class of (a, b) is defined by

$$(a, b) \cong (a, b) + (1, -1) \quad (2.7)$$

The charge lattice modulo this congruence represents the charges that cannot be screened in the QCD(BF) theory and is given by $(\mathbb{Z} \times \mathbb{Z})/\mathbb{Z} \cong \mathbb{Z}$. Therefore, the center symmetry of QCD(BF) is $U(1)$, just like QCD(adj).⁷

The form of the effective action is dictated by the symmetries of the fundamental theory. Since center symmetry is a symmetry of the original QCD-like theories, it has to be a symmetry of our effective theories. In the subsequent section, we will explicitly construct the one loop effective potentials for both temporal and spatial Wilson lines wrapping the S^1 circles. Before doing so, we may express what we should expect on symmetry grounds. The effective action should be

$$S[U] = \sum_{\mathcal{R}} \sum_{n=1}^{\infty} a_n(Y^3 \times S^1, \mathcal{R}) \text{tr}_{\mathcal{R}} U^n \quad (2.8)$$

⁶The center of the group $SU(N)$ is \mathbb{Z}_N . If N is odd, dynamical quark with N -ality 2 can screen any external test quark. Therefore, no center symmetry remains just like when one introduces fundamental quarks. If N is even, then the odd test charges cannot be screened and the center symmetry of the QCD(AS/S) theory is just \mathbb{Z}_2 as above.

⁷If one consider an $SU(N) \times SU(N)$ quiver gauge theory with bifundamental fermions, then the charge lattice becomes $\mathbb{Z}_N \times \mathbb{Z}_N$. Since the dynamical bifundamental fermion can screen any integer multiple of $(1, -1)$, the center symmetry reduce to $(\mathbb{Z}_N \times \mathbb{Z}_N)/\mathbb{Z}_N \cong \mathbb{Z}_N$.

The coefficients $a_n(Y^3 \times S^1, \mathcal{R})$ depend on the detailed structure of the underlying theory, such as the topology of the three-manifold S^3 or \mathbb{R}^3 , the matter content, the representation \mathcal{R} under the color gauge group, as well as the Lorentz symmetry group and the statistics of particles (fermions or bosons), and finally on the boundary conditions imposed on the S^1 circle, periodic or antiperiodic. Whether the action Eq.2.8 should be interpreted as the effective action for the spatial or temporal Wilson line depends on the spin structure of fermions on the S^1 circle. For the periodic (antiperiodic) choice of boundary conditions, Eq.2.8 is the effective action for spatial (temporal) Wilson line.

The trace in the effective action Eq.2.8 corresponds to the specific representation of the fields and in fact, makes the center symmetries manifest. For example, for one and two index representations of the color gauge group, we have

$$\text{tr}_{\text{fund.}} U = \text{tr} U, \quad \text{tr}_{\text{adj}} U = \text{tr} U \text{ tr} U^\dagger, \quad \text{tr}_{\text{S/AS}} U = \frac{1}{2}(\text{tr} U \text{ tr} U \pm \text{tr} U^2). \quad (2.9)$$

As expected, the $U(1)$ and \mathbb{Z}_2 center symmetry invariance, acting as $U \rightarrow e^{i\alpha} U$ and $U \rightarrow -U$ is manifest for the adjoint and AS/S representations respectively. Introducing fundamental representation fermions in the original theory destroy center symmetry completely, and this is also manifest in the effective action.⁸ Similarly, the trace over the bifundamental representation of a $U(N)_1 \times U(N)_2$ gauge group is

$$\text{tr}_{\text{BF}} U = \text{tr} U_1 \text{ tr} U_2^\dagger \quad (2.10)$$

where the invariance under the $U(1)$ center symmetry $U_i \rightarrow e^{i\alpha} U_i$ is manifest.

The one loop effective potential for the (spatial or temporal) Wilson line is just $V_{\text{eff}}[U] = \frac{S[U]}{V_{S^3 \times S^1}}$. In the regime where both S^1 and S^3 are small and of comparable size, the short wavelength degrees of freedoms can be integrated out perturbatively. Then effective potential may be conveniently regarded as a $d = 0+0$ dimensional unitary matrix model. Our analytical calculations in the subsequent section will demonstrate the presence (or absence) of weak coupling transitions in the regime where $\beta \sim R_{S^3}$ for temporal and $R_{S^1} \sim R_{S^3}$ spatial center symmetry.

If the $R_{S^3} \gg \Lambda^{-1}$ while R_{S^1} (or β) $\ll \Lambda^{-1}$, then one can perturbatively integrate out the heavy modes in the theory. These are the Kaluza-Klein modes along the S^1 circle. If S^1 is thermal, then this theory may be regarded as a canonical textbook example of thermal field theory formulated on $\mathbb{R}^3 \times S^1$. The theory at the scale of the small radius (which corresponds to high temperature) is perturbative. On the other hand, the long distance physics (at scales much larger than compactification radius β) is intrinsically nonperturbative, and understanding its dynamics requires relevant effective field theories. We will not pursue such a goal here. On the other hand, we will explicitly demonstrate that the well-known effective thermal one loop potentials for Wilson lines on $\mathbb{R}^3 \times S^1$ can be recovered by taking

⁸The effective action (in the presence of fundamental fermions in the original theory) does not possess center symmetry just like Gross-Witten model [1]. Schnitzer demonstrates that the theory undergoes a third order phase transition [7,8]. Such large N phase transitions are not associated with spontaneous breaking of any symmetry. Also see [23,24]

the appropriate decompactification limit of S^3 . The effective one loop potential obtained in this way is $d = 3 + 0$ dimensional, and depends on $U(\mathbf{y})$ where $\mathbf{y} \in \mathbb{R}^3$ is a slowly varying function of coordinates, and just coincides with the thermal one loop potential in [22]. The one loop effective potential in $\mathbb{R}^3 \times S^1$ can only tell us the realization of the center symmetry, whether it is broken or not. But, unlike the perturbative case above, it can not be used to demonstrate the phase transition as the temperature is lowered. In such cases, we rely on existing lattice results.

2.4 Chiral anomaly at large N

In the limit where the size of the base space is sufficiently large, $\min(R_{S^3}, R_{S^1}) \gg \Lambda^{-1}$, we may regard the theory as it is on \mathbb{R}^4 to a good approximation. It is important to recall some basics of the axial chiral anomaly relation for two index fermions. One feature of double index representation fermions which is different from fundamental fermions on \mathbb{R}^4 , is that the anomaly does not vanish in the infinite number of color limit. The anomaly relation is [25] (see [26] for a review)

$$\partial^\mu J_\mu^5 = \frac{g^2 n_f}{16\pi^2} \text{tr}_{\mathcal{R}} F_{\mu\nu} \tilde{F}^{\mu\nu} = \frac{g^2 h n_f}{16\pi^2} \begin{cases} \text{tr} F_{\mu\nu} \tilde{F}^{\mu\nu} & \text{QCD(adj/AS/S)} \\ \text{tr} F_1 \tilde{F}_1 + \text{tr} F_2 \tilde{F}_2 & \text{QCD(BF)} \end{cases} \quad (2.11)$$

where the $\text{tr}_{\mathcal{R}}$ in the first trace denotes the representation of the fermions forming the chiral current J_μ^5 . For $U(N)$ gauge group, $\text{tr}_{\mathcal{R}} F \tilde{F} = h \text{tr} F \tilde{F}$ where in the latter the trace is over defining representation. Here,

$$h = \{N, N, N-2, N+2\}, \quad \text{for QCD(adj), QCD(BF), QCD(AS/S)}$$

respectively, and the fact that it is proportional to N is the reason why the anomaly is not suppressed in the large N limit for two index representation fermions. For fundamental fermions, $h = 1$ and therefore, in 't Hooft's large N limit where $g^2 N = \text{fixed}$, the coefficient of the right hand side is $\mathcal{O}(n_f/N)$, and anomaly vanishes [27] unless n_f scales with N . For two index fermions, there is no suppression and the classical axial $U(1)_A$ symmetry at the quantum level is \mathbb{Z}_{2hn_f} . Note that by taking the number of fundamental representation flavors n_f scale with N by keeping n_f/N fixed as $N \rightarrow \infty$, one can also keep the fundamental quarks “alive” in the $N = \infty$ limit [28], as well as the anomaly. But we will not pursue it here.

The discrete symmetry \mathbb{Z}_{2hn_f} is determined by finding the number of fermionic zero modes in a one instanton background for the given gauge group and representation \mathcal{R} of fermions. $2h$ is the number of fermionic zero modes for each flavor, in the representation \mathcal{R} . Therefore, the simplest nonvanishing fermionic correlator in a one instanton background has to have $2hn_f$ fermionic insertions, so that the Grassmann integral over the zero modes will not trivially be zero. Let us call the operator with $2hn_f$ fermionic insertions \mathcal{O} . Under a generic $U(1)_A$ transformation, $\mathcal{O} \rightarrow e^{i\alpha 2Ncn_f} \mathcal{O}$. The nonvanishing of expectation value of \mathcal{O} requires $e^{i\alpha 2Ncn_f} = e^{2\pi i k}$, where k is an integer. Hence the phase α must take values in discrete group \mathbb{Z}_{2hn_f} . This implies, for the quantum theory the discrete axial symmetry is \mathbb{Z}_{2hn_f} .

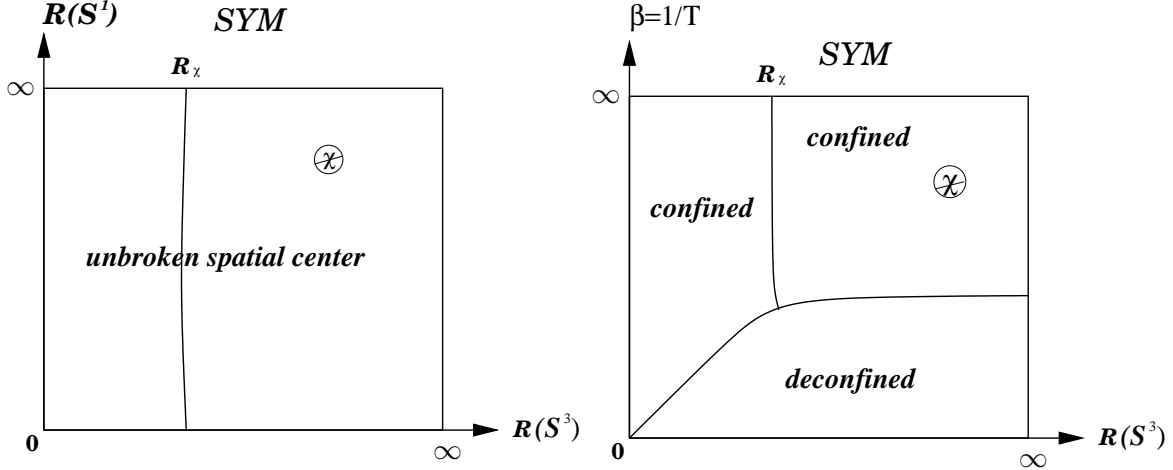


Figure 1: Schematic phase diagrams of $U(N)$ $\mathcal{N}=1$ SYM as a function of the radii $R_{S^1}(\beta)$ and R_{S^3} . The left figure is the nonthermal case where fermions obey periodic boundary conditions along S^1 , and the right one is the thermal case corresponding to anti-periodic boundary conditions for fermions. Thermal $\mathcal{N}=1$ super-Yang-Mills has two confining low temperature phases in which temporal center symmetry is unbroken. The discrete chiral symmetry is spontaneously broken at sufficiently large radius $R_{S^3} > R_\chi$ while small radius $R_{S^3} < R_\chi$ shows confinement without chiral symmetry breaking. The theory also possesses a deconfined high temperature phase with unbroken chiral symmetry. The nonthermal compactification respects spatial center symmetry throughout the whole phase diagram. Regardless of the size of the S^1 circle, for sufficiently small radius, $R_{S^3} < R_\chi$, the theory is in a chirally symmetric phase. For sufficiently large radius, $R_{S^3} > R_\chi$, the discrete chiral symmetry is believed to be spontaneously broken. These sketches depict the simplest scenario in which only a single phase transition separates various phases; more complicated scenarios with distinct deconfinement, chiral restoration transitions are also possible.

3. The $\mathcal{N}=1$ SYM on $S^3 \times S^1$

3.1 Phases as a function of volume

To study the theory as a function of the volume of $S^3 \times S^1$ (not temperature), we use the twisted partition function $\tilde{Z} = \text{tr} e^{-\beta H} (-1)^F$. In the Euclidean functional integral, this means employing periodic boundary conditions for fermions on the S^1 circle. Notice that this boundary condition is supersymmetry preserving in the decompactification limit $\mathbb{R}^3 \times S^1$. If either of the radii R_{S^3} or R_{S^1} or both is much smaller than the strong confinement scale, the theory is amenable to a perturbative one loop analysis. In particular, we are searching a transition in the regime where $R_{S^1} \sim R_{S^3}$, where perturbative analysis can demonstrate its presence or absence [6]. By working in the background of a constant gauge field configuration U , the group valued Wilson line $U \equiv e^{i \int A_0}$, we evaluate the one loop effective potential following [4, 5]. The twisted partition function of the theory on $S^3 \times S^1$ and the one loop effective potential is given in terms of the spatial Wilson line as

$$\tilde{Z}(x) = \int dU \exp -S[x, U], \quad S[x, U] = \sum_{n=1}^{\infty} \frac{1}{n} \left\{ (-z_V(x^n) + z_f(x^n)) \left(\text{tr}(U^n) \text{tr}(U^{\dagger n}) \right) \right\} \quad (3.1)$$

where $x \equiv e^{-R_{S^1}/R_{S^3}}$, dU is the Haar measure and $S[U]$ is the effective one loop action for the spatial Wilson line. The functional form of the action ($\text{tr}(U^n)\text{tr}(U^{\dagger n})$) is dictated by the symmetries of the original theory, whereas the coefficients ($-z_V(x^n) + z_f(x^n)$) are due to matter content and the topology of underlying manifold. Recall that the $\mathcal{N}=1$ SYM on $S^3 \times S^1$ has a global $U(1)$ center symmetry associated with the Wilson lines in the S^1 direction and this is manifest in the Eq.3.1 as the invariance $U \rightarrow e^{i\alpha}U$. The coefficients for vectors and spinors are given by (the so called “single particle partition function”)

$$z_V(x) = \frac{6x^2 - 2x^3}{(1-x)^3}, \quad z_f(x) = \frac{4x^{3/2}}{(1-x)^3}. \quad (3.2)$$

It is convenient to express the Haar measure in the eigenvalue basis and combine the Jacobian with the one loop effective action. On symmetry grounds, the Jacobian has to be a linear combination of center symmetry invariant double-trace operators and should not depend on the volume of the manifold $S^3 \times S^1$, i.e $\sum_n c_n \text{tr}(U^n)\text{tr}(U^{\dagger n})$ where c_n is a pure number. The Haar measure is

$$\int dU = \int \prod_i dv_i J[v_i] = \int \prod_i dv_i \prod_{i < j} \sin^2 \left[\frac{v_i - v_j}{2} \right] = \int \prod_i dv_i \exp \left\{ \sum_{i \neq j} \log \left| \sin \left[\frac{v_i - v_j}{2} \right] \right| \right\} \quad (3.3)$$

Using $\log |2 \sin \frac{x}{2}| = -\sum_{n=1}^{\infty} \frac{1}{n} \cos nx$, the Van der Monde determinant (or Jacobian) may be rewritten as

$$\ln J[U] = -\sum_{n=1}^{\infty} \sum_{i,j} \frac{1}{n} \cos[n(v_i - v_j)] = -\sum_{n=1}^{\infty} \frac{1}{n} |\text{tr} U^n|^2 \quad (3.4)$$

The Jacobian, in either form Eq.3.3 or Eq.3.4, can be regarded as a potential among eigenvalues and in effect, provides a repulsive interaction among them.

At this stage, we may express the twisted partition function in the eigenvalue basis of the spatial Wilson line as

$$\tilde{Z}(x) = \int \prod_i dv_i e^{\ln J[U] - S[x,U]} \quad (3.5)$$

where $\prod_i dv_i$ is the flat measure over the compact eigenvalues v_i . The Jacobian of the measure can be absorbed into the definition of the action and we may define the “effective action” with a flat measure. This gives

$$S_{\text{eff}}[x, U] \equiv S[x, U] - \ln J[U] = \sum_{n=1}^{\infty} \frac{1}{n} \left\{ (1 - z_V(x^n) + z_f(x^n)) (|\text{tr}(U^n)|^2) \right\} \quad (3.6)$$

$S_{\text{eff}}[x, U]$ governs the dynamics of the eigenvalues, hence center symmetry realizations. For any value of $x \in [0, 1)$, we have the inequality (or positivity in the second form)

$$(1 - z_V(x^n) + z_f(x^n)) \geq 1 \quad \text{or} \quad -z_V(x^n) + z_f(x^n) \geq 0 \quad \text{for } x \in [0, 1) \quad (3.7)$$

and moreover, $(1 - z_V(x^n) + z_f(x^n))$ is a monotonically increasing function of x . This means, the effective action S_{eff} is always larger than the Jacobian contribution to the action:

$$S_{\text{eff}}[x, U] \geq -\ln J[U], \quad \text{for } x \in [0, 1) \quad (3.8)$$

The net combined effect of the action $S_{\text{eff}}[x, U]$ is to provide a repulsive interaction among eigenvalues, and to force them to be maximally apart from one another. The minima of the action is located at $U = e^{i\alpha} \text{Diag}(1, e^{i2\pi/N}, \dots, e^{i2\pi(N-1)/N})$ where α is some phase, which arises due to the fact that gauge group is $U(N)$ rather than $SU(N)$. Consequently, the Wilson line eigenvalues distribute uniformly over the unit circle, the expectation values of Wilson line

$$\langle \text{tr } U \rangle = 0. \quad (3.9)$$

and the spatial center symmetry \mathcal{G}_s is unbroken. Therefore, there is no phase transitions associated with spatial center symmetry realizations around $R_{S^1} \sim R_{S^3} \ll \Lambda^{-1}$ where our analysis is reliable. This is unlike the thermal case which has a deconfinement transition associated with the change in the temporal center symmetry realization \mathcal{G}_t around $\beta \sim R_{S^3}$, and two distinct phases at weak coupling. The N scaling of the “vacuum energy density”, or the expectation value $\langle S_{\text{eff}} \rangle$ obtained by using the functional integral Eq.3.1, is given by

$$\mathcal{E}^{\text{SYM}} = N^2(0 + \mathcal{O}(1/N^2)) = \mathcal{O}(1) \quad (3.10)$$

where the leading zero piece is the classical, and the subleading is due to the fluctuations. A relevant analysis in the decompactification limit $\mathbb{R}^3 \times S^1$ where S^1 is small in units of the strong scale demonstrates that the spatial center symmetry remains unbroken. We also expect, on $(\text{large } S^1) \times (\text{small } S^3)$ unbroken center symmetry.

The inequality Eq.3.8 is a consequence of the matter content of $\mathcal{N}=1$ SYM and the choice of periodic boundary conditions for fermions. In particular, for pure YM theory, there is no such constraint. In fact, for pure YM, $1 - z_V(x)$ is monotonically decreasing function of x and it becomes negative at some critical x_c leading to a change in the symmetry realization, and confinement deconfinement transition. In the case at hand, the fermionic contribution overwhelms the vector boson contribution and leads to the positivity Eq.3.8. As we will see, there is no analogous inequality for the thermal compactification of $\mathcal{N}=1$ SYM, or for the other thermal ensembles that we will examine such as QCD(AS/S/BF). In fact, all such thermal ensembles show characteristics similar to pure YM and possess a deconfinement transition.

However, even for the QCD-like theories in which the fermions satisfy periodic boundary conditions on S^1 (for example the nonsupersymmetric theories related to $\mathcal{N}=1$ SYM via orbifold/orientifold projections) such as QCD(AS/S/BF), the inequality fails to hold and there are phase transitions associated with spatial center symmetry breaking. We will see such examples. The last statement seems to favor supersymmetric theories. This is not so. In particular, adding multiflavor adjoint Majorana fermions into SYM (hence a nonsupersymmetric QCD with n_f adjoint representation fermions) also satisfies the inequality (which now reads $(1 - z_V(x^n) + n_f z_f(x^n)) \geq 1$) and does not undergo a spatial center symmetry changing transition.

Of course, the most interesting (and hardest) question at this stage is what happens as we make the volume of S^3 in $\mathbb{R} \times S^3$ and S^1 in $\mathbb{R}^3 \times S^1$ larger? With either of these two maneuvers, because of asymptotic freedom, the theory goes from a weakly coupled regime to a strongly coupled regime and there is no known controlled approximation. Below, we will

argue that, in the sense of spatial center symmetry realizations, nothing interesting happens. The theory is always in the spatial center symmetric phase. However, there are changes in chiral symmetry realization dependent on the size of S^3 , but not S^1 . We will come back to this point in the next section.

The theory in the decompactification limit $\mathbb{R}^3 \times S^1$ is relatively simpler because of the restoration of supersymmetry on flat space (zero curvature) limit. (Recall that our boundary condition for fermion is periodic, hence supersymmetry preserving.) In the supersymmetric compactification of the $\mathcal{N}=1$ SYM on $\mathbb{R}^3 \times S^1$, the one loop effective potential vanishes identically. This statement is true to all orders in perturbation theory due to supersymmetry. However, an instanton induced nonperturbative superpotential do get generated and dictates the vacuum structure of the theory. This regime is analysed in the literature in depth [29,30]. It is shown that on $\mathbb{R}^3 \times S^1$, the configuration Eq.3.9 is the minimum of the nonperturbative effective potential for $\mathcal{N}=1$ SYM. The most important consequence of the nonperturbative superpotential is unbroken spatial center symmetry \mathcal{G}_s on small S^1 circle in the $\mathbb{R}^3 \times S^1$ limit. It is strongly believed that the spatial center symmetry \mathcal{G}_s is unbroken at large radius of the S^1 circle as well. The absence of a spatial center symmetry changing transition on $S^3 \times S^1$, as well as in $\mathbb{R}^3 \times S^1$ is in favor of the conjecture made in the introduction.⁹

3.2 Finite temperature phases

Choosing antiperiodic boundary conditions for fermions in the Euclidean functional integral is equivalent to a thermal $\mathcal{N}=1$ SYM theory on S^3 . The partition function is the usual thermal ensemble $Z = \text{tr} e^{-\beta H}$. At finite temperature, the only change in Eq.3.6 is the coefficient of the fermionic contribution in the expansion. The action for the temporal Wilson line is

$$S_{\text{eff}}[x, U] = \sum_{n=1}^{\infty} \frac{1}{n} a_n(x) |\text{tr} U^n|^2, \quad a_n(x) \equiv 1 - z_V(x^n) + (-1)^n z_f(x^n) \quad (3.11)$$

where the alternating sign is due to the change in the boundary conditions of fermions. The variable $x = e^{-\beta/R_{S^3}} = e^{-1/(TR_{S^3})}$ has a temperature dependence, unlike the case of spatial S^1 where x is unrelated to temperature.

In the large volume limit where $\min(R_{S^3}, \beta) \gg \Lambda^{-1}$, the dynamics of the theory should be independent of the choice of the boundary conditions. However, in small volume, the choice of the boundary condition alters the dynamics drastically. With this modification of the boundary conditions for fermions, $a_{2n+1}(x)$ become a monotonically decreasing function of x which may become negative, whereas $a_{2n}(x)$ is still monotonically increasing due to alternating $(-1)^n$ factor in Eq.3.11. The function $a_1(x)$ is the decisive element in the phase

⁹Unbroken spatial center symmetry on $\mathbb{R}^3 \times S^1$ has important implications which may help us to gain greater understanding of the large N limit of vectorlike gauge theories. The volume independence in the large N gauge theories (generalization of the old Eguchi-Kawai reduction) can be formulated as an orbifold equivalence. This equivalence is valid for QCD(adj) on $\mathbb{R}^3 \times S^1$ so long as the spatial center symmetry is unbroken. Therefore, in large N limit, small volume QCD(adj) is equivalent to large volume theory. More precisely, the volume expansion and reduction may be formulated as orbifold projections for spatial dimensions for which the first homotopy group is nontrivial, and its validity relies on unbroken center symmetry. This implies the theory on $S^3 \times S^1$ should be independent of S^1 radius, but not S^3 . For a fuller discussion, see [31].

transition. In some loose sense, $a_1(x)$ may be interpreted as the mass of the simplest temporal Wilson line $\text{tr}U$. (Recall that $x \rightarrow 0$ is low temperature $\beta \rightarrow \infty$, and $x \rightarrow 1$ is the high temperature $\beta \rightarrow 0$ limit.)

As the temperature is increased, $a_1(x)$ becomes zero at some critical x_c and there is no action (energy) cost to have a nonvanishing expectation value of the temporal Wilson line (or thermal Polyakov loop) $\text{tr}U$. In the regime where $x > x_c$ $a_1(x)$ is negative and the action is minimized with at $\langle \text{tr}U \rangle \neq 0$. Therefore, the $U(1)$ center symmetry is spontaneously broken and the theory is in a deconfined plasma phase with a characteristic free energy density of $\mathcal{O}(N^2)$. At low temperature ($x < x_c$), all the coefficient $a_n(x)$ are positive. The action provides a repulsion among the eigenvalues of the Wilson line, and hence, eigenvalues are uniformly distributed. Therefore, $\langle \text{tr}U \rangle = 0$. This implies an unbroken center symmetry and, the theory is in the confined phase. The leading N^2 order term in free energy vanishes identically, reflecting the N independence of the number of physical gauge invariant states of the theory. The resulting free energy is just $\mathcal{O}(1)$, and it is due to fluctuations. (In other words, the spectral density of the color singlets remains $\mathcal{O}(1)$ in confined phase. This means, as a function of temperature, the theory undergoes a confinement deconfinement transition at some temperature of order $T \sim 1/R_{S^3}$. The vicinity of the $x \sim x_c$ in the case of pure YM theory is examined in detail in [5].

It is also instructive to see how the effective action on $S^3 \times S^1$ reproduce the well-known perturbative thermal field theory result in $\mathbb{R}^3 \times S^1$ limit [22]. Since temperature enters into our partition function in combination $\frac{\beta}{R_{S^3}}$, naively the high temperature is same as large S^3 . This is true for some conformal theories such as $\mathcal{N} = 4$ SYM. It is, however, a false statement for confining gauge theories. There is in fact another scale in our problem, the strong confinement scale Λ . In order to have perturbative access to the theory on large S^3 , the temperature must be much larger than the strong confinement scale, i.e., $T \gg \Lambda$. More precisely, $\beta \ll \min(R_{S^3}, \Lambda^{-1})$ has to hold in order to derive thermal one loop potential in $\mathbb{R}^3 \times S^1$ [22].

Let $\epsilon \equiv \frac{\beta}{R_{S^3}} \ll 1$. The leading results for single particle partition functions are: $z_V(x^n) = \frac{4}{\epsilon^3 n^3} + \frac{18}{\epsilon^2 n^2}$ and $z_f(x^n) = \frac{4}{\epsilon^3 n^3} - \frac{6}{\epsilon^2 n^2}$, where the subleading terms vanish in the evaluation of potential in the \mathbb{R}^3 limit. The one loop effective potential for thermal SYM $V_{\text{eff}}^{\text{SYM}}[U] = \frac{S_{\text{eff}}}{\beta V_{S^3}}$ can easily be extracted from Eq.3.11 as

$$V_{\text{eff}}^{\text{SYM}}[U] = \sum_{n=1}^{\infty} \left[\frac{1}{n\beta V_{S^3}} - \frac{2T^4}{\pi^2} \frac{1}{n^4} (1 - (-1)^n) \right] |\text{tr}U^n|^2 \quad (3.12)$$

The second term is the standard thermal one loop potential for SYM theory on \mathbb{R}^3 . The first term, due to Jacobian, provides an eigenvalue repulsion, but is identically zero in the infinite volume \mathbb{R}^3 limit, and is kept for later convenience. The potential is minimized when all the eigenvalues coincide, i.e, $U = e^{iv_0} 1$ which corresponds to a Dirac-delta distribution of eigenvalues, $\rho(v) = \delta(v - v_0)$. (The eigenvalue distribution $\rho(v)$ is normalized as $\int_0^{2\pi} dv \rho(v) = 1$.) This configuration spontaneously breaks the $U(1)$ temporal center symmetry \mathcal{G}_t . The

theory is in the deconfined phase and the leading order free energy density is

$$\mathcal{F}^{\text{SYM}} = -\frac{2N^2T^4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^4} (1 - (-1)^n) = -\frac{2N^2T^4}{\pi^2} \frac{\pi^4}{90} \left(1 + \frac{7}{8}\right) = -\frac{\pi^2}{24} N^2 T^4 \quad (3.13)$$

reflecting the characteristic of deconfined phase. This is, as expected, the Stefan-Boltzmann result for a free gas of N^2 bosonic and fermionic degrees of freedom.

3.3 Multiwinding loops and perturbative widening of eigenvalue distributions

Even though the measure can be ignored in the infinite volume limit of S^3 , it has a non-trivial role at any finite volume. Let $R_{S^3} \gg \Lambda^{-1}$ and $\beta \ll \Lambda^{-1}$. The question is, how does the eigenvalue distribution (of Wilson line) behaves as the temperature is lowered? This question is interesting for two reasons. Primarily, the dynamics of eigenvalues directly determines the center symmetry realizations. Also, lattice gauge theory (by necessity), is formulated on a finite space. The goal of the following discussion is to point out that the widening of the eigenvalue distributions on finite spaces has two sources, perturbative and nonperturbative. On $\mathbb{R}^3 \times S^1$, this widening is intrinsically nonperturbative.¹⁰

In the limit where $R_{S^3}(R_{T^3}) \gg \Lambda^{-1}$ and $\beta \ll \Lambda^{-1}$ the separation between eigenvalues is much smaller than one, the problem simplifies further and reduces to a well-know Hermitian Gaussian matrix model which is exactly solvable. In this limit, the effective action becomes

$$S_{\text{eff}}(v_i) = - \sum_{i \neq j=1}^N \log |v_i - v_j| + \frac{1}{2} \left(\frac{2\pi^2 N}{\epsilon^3} \right) \sum_{i=1}^N v_i^2 \quad (3.14)$$

where the first term is the usual repulsive eigenvalue-eigenvalue interaction and the second term provides a trapping potential for eigenvalues.

The trapping potential has anomalously large coefficient compared to the repulsive potential. Therefore, Wilson line eigenvalues are spread on a very narrow support of width w , and their distribution can be solved exactly, yielding the Wigner semi-circle distribution.

$$\rho(v) = \begin{cases} \frac{2}{\pi w^2} \sqrt{w^2 - v^2} & \text{for } v \in [-w, w] \\ 0 & \text{for } v \in (w, 2\pi - w) \end{cases} \quad (3.15)$$

The width, in terms of the radii, is given by

$$w = \frac{1}{\pi} \epsilon^{\frac{3}{2}} = \frac{1}{\pi} \left(\frac{\beta}{R_{S^3}} \right)^{\frac{3}{2}} \quad (3.16)$$

Notice that the eigenvalue distribution Eq.3.15, in the limit $\epsilon \rightarrow 0$ is the Dirac-delta function $\rho(v) \rightarrow \delta(v)$, as expected. The width of the distribution w starts to widen as we make ϵ

¹⁰Lattice gauge theory is traditionally formulated on $T^3 \times S^1$. Even though the spaces T^3 and S^3 are topologically distinct, the local dynamics of strongly coupled gauge theory should be independent of the global topology in the large volume limit. Therefore, as long as the characteristic sizes of T^3 and S^3 is much larger than the strong confinement scale Λ^{-1} , one is free to replace the volume of three sphere V_{S^3} with the volume of three torus V_{T^3} in the discussion below. In fact, sufficiently large sphere or torus is essentially on the same footing with \mathbb{R}^3 for dynamical considerations.

larger with an $\epsilon^{\frac{3}{2}}$ scaling. This widening is perturbative and is unrelated to nonperturbative physics of deconfinement on $\mathbb{R}^3 \times S^1$.

Let us investigate the temperature (or width) dependence of the Polyakov loop in this regime. The expectation values of a Wilson/Polyakov loop with winding number n is

$$\langle \frac{1}{N} \text{tr} U^n \rangle = \int_0^{2\pi} dv \rho(v) e^{inv} = \frac{2}{wn} J_1(wn) = 1 - \frac{n^2 w^2}{8} + O(w^4) \quad (3.17)$$

In the $\epsilon \rightarrow 0$ limit, since the eigenvalues coincide $[\rho(v) \rightarrow \delta(v)]$, we obtain the expected result, $\langle \frac{1}{N} \text{tr} U^n \rangle = 1$ for all n . The formula Eq.3.17 shows some interesting features as well depending on the winding number. Let $w \ll 1$ be the fixed width of the eigenvalue distribution. It is then natural to split the Wilson/Polyakov loops according to their winding numbers. The loops with low winding number ($n \ll 1/w$) show behaviour similar to single winding loop. In particular, their expectation value is of order one. However, the expectation value of loops with high winding number $n \gg 1/w$ are suppressed, and rapidly fluctuates around zero with an amplitude bounded by $\frac{1}{(nw)^{3/2}}$. In particular, in the scaling limit where $\frac{n}{N}$ is fixed as we take $N \rightarrow \infty$, we have $\langle \frac{1}{N} \text{tr} U^n \rangle \rightarrow 0$. This is not surprising. As long as there are fluctuations in the eigenvalues, the higher order Fourier coefficient will fall off. Even though the eigenvalues of the Wilson/Polyakov line are localized on a very narrow support $v_i \in [-w, +w]$, because of the large winding number of the corresponding Wilson line (when the number of winding times the width becomes of order one, i.e, $v_i n \in [-wn, +wn] \sim [-\pi, \pi]$), the expectation value turns into a sum over random phases in the $[0, 2\pi)$ interval. Explicitly, $\langle \frac{1}{N} \text{tr} U^n \rangle = \langle \frac{1}{N} \sum_{i=1}^N e^{iv_i n} \rangle$. By the random phase approximation, the averaged sum of random phases on the $[0, 2\pi]$ interval is suppressed, explaining the smallness of the multiwinding loops.

Lowering the temperature is equivalent to increasing the width. On the other hand, inverse width ($1/w$) draws a line between the Wilson/Polyakov loops which (almost) vanish and the ones which do not. In particular, for a given width w , there are roughly $1/w$ many loop classes whose expectation values are still of $\mathcal{O}(1)$. When the temperature approaches the strong scale of the theory (from the high temperature side), the region of validity of the perturbative, and narrow support approximations break down, and we cannot say anything about the regime where temperature is in the vicinity of the strong scale. (See, however ref. [32] for the recent attempts in this window.) The numerical lattice results on the other hand show that the trend continues in the nonperturbative regime, and expectation values of all the Polyakov loops vanish in the confined phase restoring the temporal center symmetry.

3.4 Disentangled chiral and center symmetry realizations on $S^3 \times S^1$

There are multiple interesting lessons arising from the consideration of these theories on $S^3 \times S^1$. One outcome is the disentanglement of chiral and center symmetry realization. We have shown that for SYM theory endowed with periodic boundary conditions for fermions there is no change in center symmetry realization when one extrapolates from large to small radius of S^3 . The spatial center symmetry \mathcal{G}_s is unbroken throughout this volume change. However, the situation for chiral symmetry is rather different. At small (large) R_{S^3} , regardless of the size of S^1 , chiral symmetry is unbroken (broken).

In order to see the absence of the chiral condensate on sufficiently small S^3 , it suffices to recall that the eigenvalue spectrum of the free Dirac operator on a three sphere has a gap of order $1/R_{S^3}$ and there are no fermionic zero modes.¹¹ The eigenfrequencies of a spinor on S^3 are given by

$$\omega_n^2 = (n + \frac{1}{2})^2 \frac{1}{(R_{S^3})^2}, \quad n = 0, 1, 2, \dots \quad (3.18)$$

with degeneracies $n(n+1)$ at the level n . There are two simple regime determined by the ratio of the amplitude of smallest eigenfrequency to the strong confinement scale. Let the radius of S^3 be much smaller than Λ^{-1} . Since the fermionic modes are heavy, i.e., $\frac{|\omega_0|}{\Lambda} \gg 1$, in the description of the long distance dynamics of the theory (at length scales much larger than the size of S^3), they can be integrated out perturbatively. Therefore, a fermionic condensate cannot form and the theory is in a chirally symmetric phase.

What happens at large radius (or at small curvature), $R_{S^3} \gg \Lambda^{-1}$? In this case, since $\frac{|\omega_0|}{\Lambda} \ll 1$, there are a large number of modes below the strong scale, therefore the theory on S^3 (at zeroth order) may be regarded as the theory on \mathbb{R}^3 . Therefore, global supersymmetry gets restored, and we can use techniques from supersymmetric theory to infer the chiral properties. At sufficiently small S^1 , it is possible to calculate the chiral condensate reliably, and the result can be extended to arbitrary size of S^1 via holomorphy. Therefore, the theory on $S^3 \times S^1$ should undergo a curvature induced chiral phase transition at some radius of R_{χ, S^3} around the strong scale Λ^{-1} of the theory. We have

$$\begin{aligned} \langle \text{tr} \lambda \lambda \rangle &= 0, & \langle \text{tr} U \rangle &= 0 \quad \text{small } S^3 \\ \langle \text{tr} \lambda \lambda \rangle &\neq 0, & \langle \text{tr} U \rangle &= 0 \quad \text{large } S^3 \end{aligned} \quad (3.19)$$

The simplest phase diagram consistent with this knowledge is illustrated in fig.1. The theory has two phases. A chirally symmetric phase in which the discrete chiral \mathbb{Z}_{2N} symmetry is unbroken, and a chirally asymmetric phase in which \mathbb{Z}_{2N} is spontaneously broken down to \mathbb{Z}_2 of $(-1)^F$ by the formation of the fermion bilinear condensate. We expect the condensate to have a smooth evolution above a critical radius and extrapolate to the condensate of the $\mathcal{N}=1$ SYM theory in the limit of zero curvature. The interesting point of this transition is that it is completely disentangled from (spatial) center symmetry realizations. The spatial center symmetry [with periodic boundary conditions for fermions] is unbroken on $S^3 \times S^1$ with arbitrary radii.

¹¹This is also the reason why the supersymmetry is explicitly broken on curved spaces. Because of the absence of covariantly constant spinors, we cannot define global supersymmetries. In certain supersymmetric theories admitting twisting, hence spin zero fermions, one can have globally defined supercharges. The $\mathcal{N}=1$ SYM does not admit a nontrivial twisting in $d=4$, and once carried to curved spaces (or discrete spaces such as lattice), does not preserve supersymmetry. On the other hand, the $\mathcal{N}=4$ SYM has nontrivial twists with nilpotent scalar supercharge, hence can be formulated in curved spacetimes (and discretized spacetimes, i.e, lattice) by preserving a subset of supersymmetries exactly. The magic in both case is the coordinate independence of the scalar supercharges, $Q^2 = 0$. The observation about curved spacetimes leads to the topological twisting of supersymmetric theories, and the latter leads to the supersymmetric lattice formulation. But the essence of the idea, i.e, coordinate independence of scalar supersymmetry, is same in both cases. For progress in supersymmetric lattices, see [33, 34].

Notice that the absence of chiral condensate at small S^3 is a consequence of the gap in the spectrum of the free Dirac operator (which is due to the topology of S^3) combined with the largeness of this gap compared to the strong scale. In contrast, on a flat space such as $T^3 \times S^1$ or $T^3 \times \mathbb{R}$ (with the periodic boundary conditions for fermions in each direction), $\mathcal{N} = 1$ SYM undergoes neither a chiral transition, nor a center symmetry changing transition. Therefore, at even small radii of the three torus, there is a chiral condensate. In fact, in the flat space, the twisted partition function $\tilde{Z} = \text{tr}[e^{-\beta H}(-1)^F]$ becomes the supersymmetric Witten index counting the number of supersymmetric vacua [21]. In this particular example, the index is $\tilde{Z} = N$.

It should be noted that the original calculation of the index was performed on a small T^3 , by performing a Born-Oppenheimer approximation. On \mathbb{R}^4 , however, the theory is strongly coupled. The common wisdom of strongly coupled gauge theories tells us that a condensate will form in the most attractive channel (MAC) [35, 36]. In this case, the condensate $\langle \text{tr} \lambda \lambda \rangle$ breaking the discrete chiral symmetry $\mathbb{Z}_{2N} \rightarrow \mathbb{Z}_2$, possess a nonzero phase equal to an integer multiple of $2\pi/N$.

There is one more interesting feature of the chiral transition on $S^3 \times S^1$. Let $R_{S^1} \ll \Lambda^{-1}$ and let us dial the size of S^3 from small to large. It is known that the chiral symmetry is broken on large S^3 (which can be approximated as \mathbb{R}^3) where S^1 is small [29], and as demonstrated above, it is unbroken on small S^3 . Therefore, it is likely that the theory undergoes a weak coupling chiral transition. The study of this regime in more detail should be useful and is left for future work. The simplest phase diagram of the nonthermal $\mathcal{N} = 1$ theory consistent with our current knowledge is shown in Fig.1.

At finite temperature, the phase diagram of the $\mathcal{N} = 1$ SYM theory on S^3 is richer and is shown in Fig.1. In particular, the use of antiperiodic boundary conditions around the thermal S^1 circle (irrespective of the size of S^3) contributes a tree level thermal mass to fermions (recall that the fermions already have a classical mass gap due to the curvature of three-sphere). Let us consider the case where $R_{S^3} \gg \Lambda^{-1}$ and $\beta \ll \Lambda^{-1}$, hence the curvature effects are negligible. In this regime, the thermal tree level masses are given by $\omega_n = 2\pi(n + \frac{1}{2})T$ where $n = 0, \pm 1, \pm 2, \dots$, and T is temperature. Hence, the frequencies are bounded, in magnitude, from below by πT . Therefore, analysing the dynamics of the theory at distances large compared to $\beta = T^{-1}$, the fermions can be viewed as a heavy Kaluza-Klein tower of particles and can be integrated out perturbatively with no formation of any nonperturbative condensate, and consequently with no breaking of chiral symmetry. (For a rigorous proof of this argument in lattice formulations of vectorlike gauge theories, see [37, 38])¹² Therefore, in this phase, the temporal center symmetry \mathcal{G}_t is broken and the chiral symmetry is unbroken.

In the limit where $R_{S^3} \ll \Lambda^{-1}$, irrespective of the value of the temperature, there cannot be any chiral condensate as discussed above. However, as we discussed in the previous section, there is a change in the temporal center symmetry realizations when $TR_{S^3} \sim 1$, from a high temperature deconfined plasma phase to a low temperature confined phase.

¹²The reader will realize that the argument on the absence of chiral condensate at high temperature is essentially same as the one we have presented above in the case of high curvature space. It therefore seems tempting to draw an analogy between temperature and curvature as they induce the same effect on chiral symmetry. However, this interpretation does not go far because of the center symmetry realizations, which clearly distinguishes curvature and temperature.

When $\min(R_{S^3}, \beta) \gg \Lambda^{-1}$, we do not expect any dependence on the boundary conditions. In this phase, chiral symmetry is expected to be spontaneously broken, but not the temporal center symmetry. (This conclusion for thermal $\mathcal{N} = 1$ SYM is also reached in ref. [5].) Therefore, there are at least three phases of the theory,

$$\begin{aligned} \langle \text{tr} \lambda \lambda \rangle &\neq 0, & \langle \text{tr} U \rangle &= 0 & \min(R_{S^3}, \beta) &> \Lambda^{-1} \\ \langle \text{tr} \lambda \lambda \rangle &= 0, & \langle \text{tr} U \rangle &= 0 & R_{S^3} < \Lambda^{-1} \text{ and } \beta > R_{S^3} \\ \langle \text{tr} \lambda \lambda \rangle &= 0, & \langle \text{tr} U \rangle &\neq 0 & \beta < \min(R_{S^3}, \Lambda^{-1}) \end{aligned} \quad (3.20)$$

The simplest possibility for the phase diagram consistent with our current knowledge is shown in Fig.1. Clearly, the strangest phase is the one in which neither chiral symmetry, nor the center symmetry is spontaneously broken which implies confinement without chiral symmetry breaking. Unfortunately, this phase is not visible in lattice simulations which uses a discretized torus as base space. On flat torus, the deconfinement/confinement transition is usually entangled to the chiral transition [39], at least they cannot be parametrically different. This is what we have seen so far in lattice simulations. The curved background intelligently disentangles the two symmetry realizations.

4. Phases of orbifold QCD(BF) theory on $S^3 \times S^1$

4.1 Twisted partition function and the phases as a function of volume

In this section, we wish to analyse the phase diagram of the orbifold theory on $S^3 \times S^1$. In order to compare with the $\mathcal{N} = 1$ SYM, we analyze the orbifold theory under the same conditions. First, we consider the case where the fermions on the S^1 circle have periodic boundary conditions. As we discussed before, the periodic boundary conditions on the Euclidean functional integral correspond to the twisted partition function $\tilde{Z} = \text{tr} e^{-\beta H} (-1)^F$. The study of the twisted partition function will reveal a rich phase structure for the orbifold theory.

The $n_f = 1$ orbifold QCD(BF) is a gauge theory with a product gauge group $U(N) \times U(N)$ with gauge bosons in adjoint representation of each group and bifundamental fermions transforming as fundamental under one and antifundamental under the other. The theory has equal number of bosonic and fermionic degrees of freedom, but it is nonsupersymmetric due to differing color quantum numbers of its elementary constituents. The mismatch of the color representation of bosons and fermions has a significant effect on the eigenvalue dynamics and spatial center symmetry realization of the orbifold theory.

The twisted partition function of the theory and the effective action for the spatial Wilson lines is given by

$$\tilde{Z}(x)^{\text{QCD(BF)}} = \int dU_1 dU_2 \exp -S[x, U_1, U_2] \quad (4.1)$$

where $x = e^{-R_{S^1}/R_{S^3}}$. The Haar measure is

$$\int dU_1 dU_2 = \int \prod_i dv_i^1 dv_i^2 \prod_{i < j} \sin^2 \left[\frac{v_i^1 - v_j^1}{2} \right] \sin^2 \left[\frac{v_i^2 - v_j^2}{2} \right], \quad (4.2)$$

the decoupled product of the Haar measures of each gauge group. Therefore, it does not provide a mutual repulsion between the first cluster and second cluster of eigenvalues. However, as before, within each cluster, it provides a logarithmic repulsive potential among eigenvalues as seen in Eq.3.3.

Therefore, we may express the effective action as in Eq.3.6, in the form

$$S_{\text{eff}}[x, U_1, U_2] = S[x, U_1, U_2] - \ln J[U_1] - \ln J[U_2] \\ = \sum_{n=1}^{\infty} \frac{1}{n} \left\{ (1 - z_V(x^n)) \left(|\text{tr}(U_1^n)|^2 + |\text{tr}(U_2^n)|^2 \right) + z_f(x^n) \left(\text{tr}(U_1^n) \text{tr}(U_2^{\dagger n}) + h.c. \right) \right\} \quad (4.3)$$

The first half of the equation is due to the Jacobian, the gauge bosons (and ghosts) and originates from the two gauge group factors. It has a manifest $U(1) \times U(1)$ center symmetry, associated with global rotations $U_i \rightarrow e^{i\alpha_i} U_i$, $i = 1, 2$. The second term is due to bifundamental Dirac fermion. Recall that introducing bifundamental fermions in the original theory reduce this center symmetry to a $U(1)$ diagonal. This is also manifest in our effective action, which is only invariant under the restricted rotation with $\alpha_1 = \alpha_2 = \alpha$. The original theory is also invariant under the \mathbb{Z}_2 symmetry of the orbifold which interchanges the two gauge group factors. We will denote this \mathbb{Z}_2 symmetry with \mathcal{I} . Therefore, the symmetries of the effective action Eq.4.3 are given by

$$\begin{aligned} \mathcal{G}_s : U_1 &\rightarrow e^{i\alpha} U_1, & \mathcal{G}_s : U_2 &\rightarrow e^{i\alpha} U_2 \\ \mathcal{I} : U_1 &\leftrightarrow U_2 \end{aligned} \quad (4.4)$$

The low energy effective potential has to possess all the symmetries of the original theory (if the symmetries are non-anomalous) and this is manifest in our effective action.

In order to grasp the spatial center and orbifold shift symmetry realizations easily, it would be more helpful to express the action Eq.4.3 in terms of $\mathcal{I} \times \mathcal{G}_s$ eigenstates. [This furthermore eases the comparison with the orientifold QCD(AS/S) theory, which we will discuss later.] Let us define the basis which simultaneously diagonalize the $\mathcal{I} \times \mathcal{G}_s$ operations. The eigenfunctions and eigenvalues of spatial center and orbifold shift symmetry are given by

$$\begin{aligned} \text{tr} \Omega_{\pm}^k &= \text{tr} U_1^k \pm \text{tr} U_2^k, & k &= 1, \dots, \infty, \\ \mathcal{G}_s \text{tr} \Omega_{\pm}^k &= e^{i\alpha k} \text{tr} \Omega_{\pm}^k, & \mathcal{I} \text{tr} \Omega_{\pm}^k &= \pm \text{tr} \Omega_{\pm}^k \end{aligned} \quad (4.5)$$

The operators $\text{tr} \Omega_{\pm}^k$ are even-odd linear combination of Wilson lines of the two gauge groups with winding number k .

The reason for keeping track of the $\mathcal{I} = \mathbb{Z}_2$ interchange symmetry of the orbifold theory is tied to the nonperturbative large N equivalence. The necessary (and sufficient) condition for the validity of nonperturbative large N equivalence is unbroken $\mathcal{I} = \mathbb{Z}_2$ symmetry of the orbifold theory [14, 15]. If the symmetry \mathcal{I} is not spontaneously broken, then the dynamics of the large N QCD(BF) coincides with the dynamics of SYM in their respective neutral sectors. Therefore, while keeping track of the spatial center symmetry realization in the classification of phases, we should also understand the realizations of the orbifold interchange symmetry.

Expressing the effective action of the orbifold theory Eq.4.3 in terms of the $\mathcal{G}_s \times \mathcal{I}$ symmetry eigenstates results in

$$S_{\text{eff}}^{\text{QCD(BF)}}[\Omega_+, \Omega_-] = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} \left\{ a_n^+(x) |\text{tr}(\Omega_+^n)|^2 + a_n^-(x) |\text{tr}(\Omega_-^n)|^2 \right\} \quad (4.6)$$

where the coefficients are given by

$$\begin{aligned} a_n^+(x) &= (1 - z_V(x^n) + z_f(x^n)) \\ a_n^-(x) &= (1 - z_V(x^n) - z_f(x^n)) \end{aligned} \quad (4.7)$$

The effective potential S_{eff} Eq.4.3 determines the symmetry realizations for the spatial center symmetry \mathcal{G}_s and the $\mathcal{I} = \mathbb{Z}_2$ interchange symmetry of the orbifold QCD(BF) theory and is fairly easy to analyse. In the interval $x \in [0, 1)$ (while keeping either S^3 or S^1 much smaller than strong scale), we observe the analog of positivity Eq.3.7 for the coefficient of even modes: $a_n^+(x) \geq 1$ for all $x \in [0, 1)$. This means all the even modes have positive “mass” $a_n^+(x)$. On the other hand, the coefficients of the odd $\text{tr}\Omega_-^n$ modes, $a_n^-(x) = (1 - z_V(x^n) - z_f(x^n))$ are monotonically decreasing functions of x , and become negative at sufficiently large x . Let x_c be the locus of $a_1^-(x)$. Therefore, for $x > x_c$, $a_1^-(x)$ becomes negative and leads to spontaneous breaking of the spatial center symmetry $\mathcal{G}_s = U(1)$. Since this symmetry breaking is driven by an \mathcal{I} -odd mode, it spontaneously breaks the \mathcal{I} symmetry of the orbifold theory.

The visualization of this symmetry realization on $\mathbb{R}^3 \times S^1$ in the eigenvalue basis was given by D. Tong in [16], here we adopt his arguments. We already argued that there are two clusters of eigenvalues, associated with each gauge group. These two clusters, (each of which has N eigenvalues in them), in the symmetry broken phase, wishes to be maximally apart from each other. If the center of mass of one cluster is located at e^{iv_0} , the other cluster is located at the antipodal point, $-e^{iv_0}$. Therefore, the vacuum expectation values of the spatial Wilson lines in the two gauge groups are anti-parallel to each other. (see fig.2, small spatial). We will in the next section see that in the thermal case, the positions of the two clusters coincide, making the temporal Wilson lines parallel and restoring the \mathcal{I} symmetry and breaking just the temporal center symmetry \mathcal{G}_t . (see fig.2, small thermal)

In the limit where $x \rightarrow 1$, (small spatial circle), the width of the two clusters of eigenvalues shrinks to zero in the sense described in section 3.3, and the distributions of the eigenvalues become the Dirac-delta functions given as

$$\rho_1(v_1) = \delta(v_1 - v_0), \quad \rho_2(v_2) = \delta(v_2 - v_0 - \pi) \quad (4.8)$$

where v_0 is some phase. Therefore, the analysis of the effective potential Eq.4.6 reveals

$$\left\langle \frac{1}{N} \text{tr} \Omega_- \right\rangle = 2e^{iv_0}, \quad \left\langle \frac{1}{N} \text{tr} \Omega_+ \right\rangle = 0 \quad (4.9)$$

breaking the $\mathcal{I} \times \mathcal{G}_s$ symmetry. (More detailed account of this symmetry breaking and its consequences is discussed in section 4.3.) The vacuum energy density can be deduced from the twisted partition function in the limit $x = 1 - \epsilon = 1 - \frac{R_{S^1}}{R_{S^3}} \rightarrow 1$ easily.

$$\mathcal{E}^{\text{QCD(BF)}}(x \rightarrow 1) = -\frac{1}{R_{S^1} V_{S^3}} \log \tilde{Z}(x \rightarrow 1)$$

QCD(AS/S/BF)

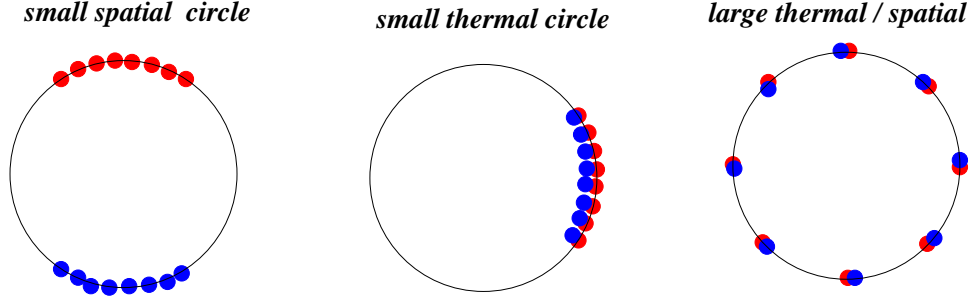


Figure 2: The distribution of the eigenvalues of the spatial (and temporal) Wilson lines for $U(N)_1 \times U(N)_2$ QCD(BF) theory. The red (blue) points label eigenvalues of the Wilson line $U_1(U_2)$. At small spatial S^1 circle, both spatial center symmetry \mathcal{G}_s and \mathcal{I} interchange symmetry of orbifold are broken. (Wilson lines U_1 and U_2 are antiparallel.) At small temporal S^1 , the temporal center symmetry \mathcal{G}_t is broken, but not \mathcal{I} . (Wilson lines are parallel.) At large radius of S^1 (either thermal or spatial), the center symmetry is restored, as well as \mathcal{I} in the case of spatial S^1 . Exactly analogously, one may regard the same picture for orientifold $U(N)$ QCD(AS/S). In this case, the role of the \mathcal{I} symmetry of orbifold is taken by charge conjugation symmetry \mathcal{C} , and the role of the Wilson line U_2 is played by the mirror image, U^* . Therefore, the red (blue) points label eigenvalues of the Wilson line U (its complex conjugate U^*). At small spatial S^1 circle, both \mathcal{G}_s and \mathcal{C} of QCD(AS/S) are broken. (antiparallel U and U^*) At small temporal S^1 , \mathcal{G}_t is broken, but not \mathcal{C} . (parallel U and U^* .) At large radius of S^1 , all symmetries mentioned above are restored. (The blue and red points are split to guide the eye.) The orientifold QCD(AS/S) will be discussed in detail in next section. We should also point that for high temperature SYM, the eigenvalues clump just like QCD(AS/S/BF). At low temperature, they delocalize and distribute uniformly. On the other hand, for spatial S^1 circle, the eigenvalues are uniformly distributed regardless of the size of the S^1 .

$$= \frac{1}{R_{S^1} V_{S^3}} \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{-8}{n^4 \epsilon^3} \right) |N(1 - (-1)^n)|^2 = -\frac{\pi^2}{24} (2N^2) \frac{1}{(R_{S^1})^4} \quad (4.10)$$

where the subleading terms in ϵ and N are neglected. The $2N^2$ is a kinematical factor reflecting that there are $2N^2$ gauge fields and bifundamental fermions in QCD(BF).

On the other hand, for $x < x_c$ while keeping $\min(R_{S^3}, R_{S^1}) \ll \Lambda^{-1}$ for the validity of the perturbative analysis, the coefficients $a_n^\pm(x)$ are all positive. This implies a net repulsive force among eigenvalues of Wilson line, and a uniform distribution. This means, the $U(1)$ spatial center symmetry and \mathbb{Z}_2 shift symmetry of the orbifold QCD(BF) theory are restored within this phase. The minimum of the effective potential is located at

$$\langle \text{tr} U_1 \rangle = \langle \text{tr} U_2 \rangle = 0, \quad \langle U_i \rangle = e^{i\alpha_i} \text{Diag} (1, e^{2\pi/N}, \dots, e^{2\pi(N-1)/N}) \quad (4.11)$$

The leading $\mathcal{O}(N^2)$ contributions to vacuum energy density vanishes identically within the symmetry restored phase, and hence the first nontrivial contribution to the N scaling of the vacuum energy come from the fluctuations:

$$\mathcal{E}^{\text{QCD(BF)}} = N^2(0 + \mathcal{O}(1/N^2)) = \mathcal{O}(1) \quad (4.12)$$

where the $\mathcal{O}(1)$ contribution is due to the quantum fluctuations. Therefore, the theory undergoes a phase transition which alters its ground state energy from $\mathcal{O}(N^2)$ in the symmetry broken phase to being $\mathcal{O}(1)$ in the unbroken phase.

The perturbative one loop analysis of the center symmetry realizations can be extended to cover the region $\min(R_{S^1}, R_{S^3}) \ll \Lambda^{-1}$ where the coupling at the scale of the smaller radius is small $g^2 \ll 1$, and one can construct the relevant effective theories. In particular, our analysis shows that the theory on $\mathbb{R}^3 \times S^1$ (or large $S^3 \times S^1$) on the limit of small spatial S^1 has a phase in which $\mathcal{I} = \mathbb{Z}_2$ interchange symmetry of orbifold is spontaneously broken. This is in agreement with D. Tong's result on $\mathbb{R}^3 \times \text{small } S^1$ [16]. In this regime, higher order corrections to the effective potential are small perturbations which can not alter the conclusion on symmetry realizations. On large spatial S^1 , since the theory is strongly coupled, we can not directly check the realization of $\mathcal{G}_s \times \mathcal{I}$ symmetry. Nevertheless, as we will discuss momentarily, for thermal compactification, the \mathcal{I} symmetry of orbifold theory is unbroken at small S^1 . Therefore, the \mathcal{I} symmetry realizations, probed by the \mathcal{I} odd-combination of the Wilson line, is *dependent* of the choice of the boundary conditions for fermions, and should indeed disappear above some critical radius of the order of the strong confinement scale, and in the large volume limit.

It is very plausible that in the sense of spatial center symmetry and the \mathcal{I} interchange symmetry realization, these are the only two phases of the theory. In particular, we expect the spatial center symmetry and shift symmetry restoring transition in the perturbative regime $R_{S^1} \sim R_{S^3} \ll \Lambda^{-1}$ on $S^3 \times S^1$ to extrapolate all the way to a strongly coupled transition on $\mathbb{R}^3 \times S^1$. There is simply no evidence indicating otherwise [40], although there are claims [41]. Given this, there should be a strong coupling $\mathcal{G}_s \times \mathcal{I}$ restoring phase transition around the confinement scale of the QCD(BF) theory on $\mathbb{R}^3 \times S^1$. Since QCD(BF) is a vectorlike gauge theory, this should be testable on the lattice.

4.2 Finite temperature phases

In this section, we wish to examine the dynamics of the finite temperature orbifold field theory on $S^3 \times S^1$. This corresponds to choosing anti-periodic boundary conditions for fermions on the S^1 circle, while keeping the periodic boundary conditions for bosons. The resulting Euclidean functional integral corresponds to the partition function $Z = \text{tr} e^{-\beta H}$ of a thermal ensemble on S^3 space with inverse temperature β .

The net effect of the change in boundary conditions for fermions is to make the coefficient of fermionic term in Eq.4.3 an alternating one:

$$z_f(x^n) \rightarrow (-1)^n z_f(x^n) \quad (4.13)$$

This, in turn, alters the prefactors in the action Eq.4.6 into

$$\begin{aligned} a_n^+(x) &= (1 - z_V(x^n) + (-1)^n z_f(x^n)) \\ a_n^-(x) &= (1 - z_V(x^n) - (-1)^n z_f(x^n)) \end{aligned} \quad (4.14)$$

and the action should be interpreted as the effective action for the temporal Wilson line (or thermal Polyakov loop).

As stated earlier, there is a pleasant twist to the story in the thermal case. In particular, $a_1^-(x)$ which leads to the spontaneous breaking of \mathbb{Z}_2 symmetry of the orbifold theory is now positive definite. On the other hand, $a_1^+(x)$ which is the mass of the \mathcal{I} even mode $\text{tr} \Omega^+$ is

now a monotonically decreasing function of x . The mass $a_1^+(x)$ of the \mathcal{I} even mode passes through zero at the locus $x = x_c$, and is negative for $x > x_c$ leading to an instability.

Therefore, at sufficiently high temperature, ($x \rightarrow 1$), the $\mathcal{G}_t = U(1)$ temporal center symmetry breaks down spontaneously, but not the $\mathbb{Z}_2 = \mathcal{I}$ symmetry of the orbifold theory. In the limit $x \rightarrow 1$, the order parameters are given by

$$\langle \frac{1}{N} \text{tr} U_1 \rangle = \langle \frac{1}{N} \text{tr} U_2 \rangle = e^{iv_0}, \quad \text{or} \quad \langle \frac{1}{N} \text{tr} \Omega_- \rangle = 0, \quad \langle \frac{1}{N} \text{tr} \Omega_+ \rangle = 2e^{iv_0}, \quad (4.15)$$

In other words, the minima of the effective potential is the parallel nonzero thermal Wilson line for the two gauge groups. Needless to say, the eigenvalue distributions are given by $\rho_1(v_1) = \delta(v_1 - v_0)$ and $\rho_2(v_2) = \delta(v_2 - v_0)$. This means the two cluster of the eigenvalues, in the thermal case, are literally on top of each other, implying unbroken \mathcal{I} symmetry of the orbifold theory. Therefore, in the high temperature phase, the breaking pattern is $U(1) \times \mathcal{I} \rightarrow \mathcal{I}$. The free energy density within the deconfined phase is order N^2 . It simplifies in the $x \rightarrow 1$ limit to the usual Stefan-Boltzman result:

$$\mathcal{F}^{\text{QCD(BF)}}(x \rightarrow 1) = -\frac{1}{\beta V_{S^3}} \log Z(x \rightarrow 1) = -\frac{\pi^2}{24} (2N^2) T^4 \quad (4.16)$$

reflecting the fact that at high temperature there are $2N^2$ gauge bosons and fermions contributing to the free energy.

At sufficiently low temperature ($x < x_c$), the discussion of the temporal center symmetry restoration is analogous to the spatial center symmetry discussion 4.1. At low temperature, the order parameters are

$$\langle \frac{1}{N} \text{tr} U_1 \rangle = \langle \frac{1}{N} \text{tr} U_2 \rangle = 0, \quad \text{or} \quad \langle \text{tr} \Omega_- \rangle = 0, \quad \langle \text{tr} \Omega_+ \rangle = 0, \quad (4.17)$$

In the confined phase, both temporal center symmetry and \mathbb{Z}_2 symmetry of orbifold theory are unbroken. The free energy density is $\mathcal{O}(1)$ reflecting the N independence of the number of color singlet single trace operators.

One can give a conceptual derivation demonstrating that \mathbb{Z}_2 symmetry in thermal orbifold QCD(BF) theory (at sufficiently high temperature or small thermal circle) cannot be spontaneously broken. Let us recall some basic lore on thermal theories: The thermal Polyakov loop, as usual, measures the excess of the free energy that an external test charge generates in the thermal medium. If the medium, via its thermal fluctuations, is unable to screen such a charge, the excess free energy is infinite, therefore, the expectation value of the Polyakov loop is zero (unbroken temporal center symmetry). This is the confined phase. On the other hand, if the thermal fluctuations of color are large enough so that the medium can screen the test charge, then the excess free energy is finite and the Polyakov loop acquires a vacuum expectation value breaking the temporal center symmetry. This is the deconfined phase.

Since the QCD(BF) gauge theory is a product gauge group $U(N) \times U(N)$, an external test quark has charges under both centers $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ as discussed in section 2.2. The $\langle \frac{1}{N} \text{tr} U_1 \rangle$ corresponds to the excess free energy of an external quark whose charge is $(1, 0)$ and $\langle \frac{1}{N} \text{tr} U_2 \rangle$ to the one whose charge is $(0, 1)$. In the confined phase, Eq.4.17 implies neither

$(1,0)$ nor $(0,1)$ can be screened. In the deconfined phase, Eq.4.15 implies that the excess of free energy to have a test charge $(1,0)$ in thermal medium is equal to the one of having $(0,1)$. On $\mathbb{R}^3 \times S^1$, the fact that $\Delta\mathcal{F}_{(1,0)} = \Delta\mathcal{F}_{(0,1)}$ should be expected. The difference of these two charges is $(1,-1)$ and the dynamical fermions (which have charges $\pm(1,-1)$) can easily convert one into the other. Since free energy is a class function, and $(1,0)$ and $(0,1)$ are in the same conjugacy class by Eq.2.7, we should indeed expect that $\Delta\mathcal{F}_{(1,0)} - \Delta\mathcal{F}_{(0,1)} = 0$, implying $\langle \text{tr}\Omega_- \rangle = 0$. This retrospectively justifies that the \mathbb{Z}_2 symmetry in thermal orbifold QCD(BF) theory cannot be spontaneously broken.

4.3 Digression: Topological classification of order parameters

As we discussed, the orbifold theory has $U(1)$ (spatial or temporal) center symmetry, and \mathbb{Z}_2 shift symmetry exchanging the two gauge group factors. These symmetries act on the Wilson lines as given in Eq.4.5.

In order to define the breaking patterns unambiguously, we wish to analyze symmetries in some more detail. First note that the action of the \mathcal{I} on the order parameter $\text{tr}\Omega_-$ negates it, and so does a $U(1)$ center symmetry action by $e^{i\alpha} = e^{i\pi}$. Therefore, the combined action of the two does not change the order parameter $\text{tr}\Omega_-$. Therefore, the vacuum expectation value of the $\text{tr}\Omega_-$ cannot break the combined \mathbb{Z}_2 gauge and \mathbb{Z}_2 shift symmetry. In order to isolate the precise symmetry breaking pattern, let us (artificially) split the center symmetry into a \mathbb{Z}_2 (global gauge rotations generated by $e^{i\pi}$), and the quotient of the $U(1)$ by \mathbb{Z}_2 . This can be done by declaring equivalences among phases $e^{iv} \in U(1)$ as $v \equiv (v + \pi)$, i.e., identifying antipodal points on the S^1 circle (the $U(1)$ group manifold). We will refer to this coset space as $\widetilde{U(1)}$. Therefore, we may write $U(1) \equiv \widetilde{U(1)} \times \mathbb{Z}_2$. Up to this point $U(1)$ center symmetry could have been either spatial or temporal. Let us first analyze the periodic compactification along the S^1 circle. In order to see the difference between the breaking patterns in nonthermal and thermal case, we introduce the notation $\mathcal{G}_s = U(1) \equiv \widetilde{U(1)} \times \widetilde{\mathcal{G}}_s$ where $\widetilde{\mathcal{G}}_s$ is the \mathbb{Z}_2 subgroup of spatial center symmetry \mathcal{G}_s .

Notice that the $\widetilde{\mathcal{G}}_s$ subgroup of spatial center symmetry is not the same as the \mathcal{I} interchange symmetry of the orbifold theory. Even though their action on $\text{tr}\Omega_-$ is the same, the $\text{tr}\Omega_+$ is singlet under interchange symmetry and negates under spatial center subgroup $\widetilde{\mathcal{G}}_s$. Alternatively, one can check their action on Wilson lines $\text{tr}U_1$ and $\text{tr}U_2$:

$$\mathcal{I} : U_1 \leftrightarrow U_2, \quad \widetilde{\mathcal{G}}_s : U_i \rightarrow -U_i, \quad i = 1, 2 \quad (4.18)$$

The symmetry of the theory may conveniently be written as

$$\widetilde{U(1)} \times (\mathbb{Z}_2)^2 = \widetilde{U(1)} \times \{1, \mathcal{I}, \widetilde{\mathcal{G}}_s, \mathcal{I}\widetilde{\mathcal{G}}_s\} \quad (4.19)$$

and the pattern of spontaneous symmetry breaking discussed in section 4.1 corresponds to

$$\widetilde{U(1)} \times (\mathbb{Z}_2)^2 \rightarrow \mathbb{Z}_{2,D} = \{1, \mathcal{I}\widetilde{\mathcal{G}}_s\} \quad (4.20)$$

The vacuum of the theory is not invariant under the discrete operations of $\{\mathcal{I}, \widetilde{\mathcal{G}}_s\}$, but is invariant under the diagonal subgroup $\mathbb{Z}_{2,D} = \{1, \mathcal{I}\widetilde{\mathcal{G}}_s\}$.

There is one important corollary we want to address in this pattern. The fact that the diagonal $\mathbb{Z}_{2,D} = \{1, \mathcal{I}\tilde{\mathcal{G}}_s\}$ is unbroken has simple, yet non-trivial implications for local \mathcal{I} -odd order parameters.

Corollary: Let $\mathcal{O}(x)$ be a local order parameter probing the \mathcal{I} interchange symmetry of the orbifold QCD(BF) theory. The phase of the theory with the breaking pattern Eq.4.20 does not admit local order parameters to acquire a vacuum expectation value.

Since $\mathcal{O}(x)$ is local, it is a singlet under center symmetry. Therefore, we have

$$(\mathcal{I}\tilde{\mathcal{G}}_s) : \mathcal{O}(x) = \mathcal{I}\mathcal{O}(x) = -\mathcal{O}(x).$$

Since the vacuum of the theory is invariant under $\mathcal{I}\tilde{\mathcal{G}}_s$, the vacuum expectation value of the operator $\mathcal{O}(x)$ must satisfy

$$\langle \mathcal{O}(x) \rangle = \langle (\mathcal{I}\tilde{\mathcal{G}}_s)\mathcal{O}(x) \rangle = -\langle \mathcal{O}(x) \rangle = 0$$

Therefore, even though the theory is in the \mathcal{I} broken phase, the *local* order parameters must have vanishing expectation values.

This curious fact arises due to intertwining of the spatial center symmetry with the \mathcal{I} symmetry as in Eq.4.20. A genuine distinction arises between the topologically trivial and nontrivial operators. The topologically trivial operators (which do not wind around the S^1 circle) are all singlet under the center symmetry. Since $\mathbb{Z}_{2,D}$ is unbroken in neither phase of the theory, and since the local \mathcal{I} odd order parameters necessarily transform nontrivially under the action of $\mathcal{I}\tilde{\mathcal{G}}_s$, the expectation value of all such operators must be equal to zero. In particular, $\mathcal{O}(x) = \frac{\text{tr}}{N}F_1^2 - \frac{\text{tr}}{N}F_2^2$ is such an operator, trivial under center symmetry and transforming under \mathcal{I} . Our result implies that

$$\langle \text{tr}F_1^2 - \text{tr}F_2^2 \rangle = 0, \quad (\text{either phase}). \quad (4.21)$$

This unambiguously demonstrates that $\langle \text{tr}F_1^2 - \text{tr}F_2^2 \rangle = 0$ also in $\mathbb{R}^3 \times S^1$ where S^1 is small. Our current knowledge on the QCD(BF) theory is consistent with the assertion that this is true at arbitrary size S^1 , in particular on \mathbb{R}^4 limit. On $\mathbb{R}^3 \times S^1$ where spatial S^1 is small, we have the breaking pattern Eq.4.20. We believe, the theory will undergo a center symmetry restoring transition around the strong scale, and restore its spatial center symmetry along with \mathcal{I} symmetry of the orbifold QCD(BF). Our results are in accord with [16, 40] and contradicts the claim in ref. [41]. On \mathbb{R}^4 , Ref. [41] presents $\langle \text{tr}F_1^2 - \text{tr}F_2^2 \rangle \neq 0$ as a prediction of string theory, see the reference therein.

The thermal case is simpler. With the analogous notation, the breaking pattern is

$$\widetilde{U(1)} \times (\mathbb{Z}_2)^2 = \widetilde{U(1)} \times \{1, \mathcal{I}, \tilde{\mathcal{G}}_t, \mathcal{I}\tilde{\mathcal{G}}_t\} \rightarrow \mathbb{Z}_2 = \{1, \mathcal{I}\} \quad (4.22)$$

in the deconfined phase. (Therefore, it is not even necessary to quotient the temporal center symmetry into subgroups, since it is fully broken in this case.) The \mathcal{I} symmetry of orbifold theory is unbroken in the high temperature phase. In confined phase, the $\widetilde{U(1)} \times (\mathbb{Z}_2)^2$ symmetries are unbroken. In neither phase of the theory, since the \mathbb{Z}_2 symmetry of the orbifold is unbroken, no \mathcal{I} odd operator can acquire an expectation value.¹³

¹³The fact that the expectation value of the topologically nontrivial order parameter $\text{tr}\Omega_-$ is sensitive to

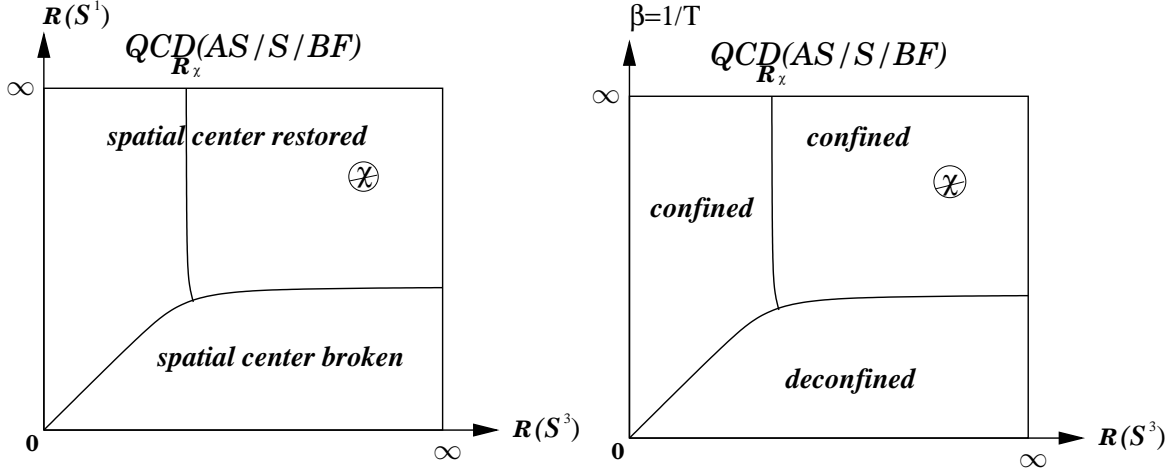


Figure 3: Schematic phase diagram of $U(N)$ QCD(AS/S/BF) with periodic boundary conditions for fermions (left) and with thermal (anti-periodic) boundary conditions (right) as a function of radii $R_{S^1}(\beta)$ and R_{S^3} . Thermal QCD(AS/S/BF) has two confining low temperature phases in which temporal center symmetry \mathcal{G}_t is unbroken. These two phases are distinguished by the discrete chiral symmetry realization. The theory is chirally symmetric at sufficiently small radius $R_{S^3} < R_\chi$ and asymmetric at large radius $R_{S^3} > R_\chi$. These theories also possess a deconfined high temperature phase with broken \mathcal{G}_t and unbroken chiral symmetry. For the nonthermal compactification where S^1 is a spatial circle, the symmetry realizations are essentially same modulo the replacement of temporal spatial symmetry with the spatial one \mathcal{G}_s . As explained in the text, the underlying reason for unbroken chiral symmetry in the spatial symmetry broken phase is a one loop quantum effect, unlike the thermal compactification where it is due to tree level mass of fermions. These sketches depict the simplest consistent scenarios. In the thermal case, the \mathcal{I} and \mathcal{C} symmetries necessary for the validity of large N equivalence are unbroken, explaining the matching of phase diagram to $\mathcal{N} = 1$ SYM. In the nonthermal case, the phase which breaks \mathcal{G}_s also breaks \mathcal{I} and \mathcal{C} , explaining the mismatch to $\mathcal{N} = 1$ SYM within that phase.

4.4 Chiral symmetries

We first examine the chiral symmetry realizations of thermal orbifold theory on S^3 , where fermions are endowed with antiperiodic boundary conditions on the thermal S^1 circle. In this case, there are no surprises and the phase diagram is essentially same as thermal $\mathcal{N} = 1$ SYM. The more interesting case is the one where fermions obey periodic boundary condition, which we will discuss afterwards.

As discussed in the $\mathcal{N} = 1$ SYM case, high curvature or high temperature does not allow a fermion condensate to form because of the same reason. In either case, the lowest fermionic Kaluza-Klein mode is much larger than the strong scale of the QCD-like theory and the fermions can be integrated out perturbatively without any formation of chiral condensate. In the limit where $\min(R_{S^3}, \beta) \gg \Lambda^{-1}$, the orbifold theory is strongly coupled. The theory has a vectorlike baryon number and axial chiral symmetry, $U(1)_B \times \mathbb{Z}_{2N}$. In vector-like gauge

the choice of the boundary conditions (non-zero with the use of periodic boundary conditions, and zero when antiperiodic boundary conditions) demonstrates that this breaking is a finite volume effect, which should disappear in the large radius limit. Nevertheless, the scale at which this transition should occur is a nonperturbative scale of the underlying theory and is physical the same way the deconfinement temperature is.

theories, the vector-like symmetries do not break down spontaneously while it is believed that the axial symmetries do [42]. The discrete (non-anomalous) axial symmetry \mathbb{Z}_{2N} acts on the bifundamental fermions as $\Psi \rightarrow e^{i\gamma_5\alpha}\Psi$, where α is an integer multiple of $2\pi/N$. The conventional wisdom tells us that a condensate will form in the most attractive channel (MAC). The MAC in this example is $\langle\bar{\Psi}\Psi\rangle$, breaking the discrete chiral symmetry as $\mathbb{Z}_{2N} \rightarrow \mathbb{Z}_2$. Therefore, we expect N vacua distinguished by the phases, $e^{i2\pi k/N}$ where $k = 0, \dots, N-1$ exactly as in SYM on \mathbb{R}^4 .

Therefore, as in the case of SYM, we expect QCD(BF) to possess at least three different phases. The simplest phase diagram of the thermal theory consistent with all the knowledge that we have gathered so far is given in fig.3.

The case where fermions are endowed with periodic boundary conditions is more interesting. The clear distinction compared to the thermal case (where fermions acquire a tree level mass term because of anti-periodic boundary condition along the thermal circle) is that the lowest fermionic mode on the S^1 circle are not classically gapped. This is purely a manifestation of the periodic boundary condition. Let us first consider the theory on $S^3 \times S^1$ where $R_{S^3} \gg \Lambda^{-1}$ is fixed, and S^1 is dialed as desired. Since the S^3 is large, we can approximate it as \mathbb{R}^3 . At tree level, the fermions have a spectrum $\omega_n^{(0)} = \frac{2\pi}{R_{S^1}}n$ where $n = 0, \pm 1, \pm 2, \dots$. Let $R_{S^1} \ll \Lambda^{-1}$ so that the theory is amenable to a perturbative treatment. It is easy to take the $\mathbb{R}^3 \times S^1$ limit of the action Eq.4.3 and obtain the one-loop effective potential for the Wilson line in the orbifold theory. The limit obtained in this way is identical to the result of [16]. The one-loop Coleman-Weinberg potential breaks the \mathcal{I} interchange symmetry which is signalled by the \mathcal{I} odd combination of the spatial Wilson line acquiring a vacuum expectation value. This quantum effect provides a mass term for fermions, by shifting all the eigenfrequencies by a half integer. It is easy to see that fermions becomes gapped due to this quantum effect. The quantum corrected eigenfrequencies are given by $\omega_n^{(1)} = \frac{2\pi}{R_{S^1}}(n + \frac{1}{2})$ where $n = 0, \pm 1, \pm 2, \dots$.

¹⁴ Notice that these eigenmodes are identical to the eigenmodes of the thermal compactification if we were to identify R_{S^1} as inverse temperature β . However, it should be kept in mind that the latter is a tree level result, and in our case, the gap is a consequence of the one loop quantum correction. The mass is set in units of inverse compactification radius, R_{S^1} . Similar to the high temperature considerations, at length scales much larger than R_{S^1} , the fermions can be regarded as a heavy Kaluza-Klein tower and therefore, can be integrated out perturbatively without the formation of a condensate. Hence chiral symmetry is restored in this phase at sufficiently small S^1 . At large S^1 , the theory is strongly coupled. At large radius, the dynamics of the theory should be independent of the choice of the boundary conditions. Therefore we expect, based on MAC argument similar to the thermal case discussed above, a chiral condensate and a chirally asymmetric phase where \mathbb{Z}_{2N} symmetry is spontaneously broken to $\mathbb{Z}_2 = (-1)^F$.

This transition is quite interesting in its own right. The reason is, the theory on $\mathbb{R}^3 \times S^1$ where fermions have periodic boundary conditions along S^1 , may be regarded as a zero

¹⁴The four dimensional mass term is prohibited by the chiral symmetry itself. The mass term generated in this case is a three dimensional mass, for the Kaluza-Klein tower of all the fermionic modes from the three dimensional point of view. This mass is allowed by the chiral symmetry

temperature field theory. Here, we are viewing one of the noncompact directions \mathbb{R} as a temperature circle, albeit decompactified, corresponding to $T = 0$. Therefore, what we see is a zero temperature chiral phase transition. We expect this transition to take place around the strong scale of the orbifold theory and be entangled with spatial center symmetry realizations.

¹⁵

Therefore, as long as $\min(R_{S^1}, R_{S^3}) \ll \Lambda^{-1}$, there should not be any chiral condensate, either due to curvature or due to quantum effects rendering fermions massive. Within this region of the phase diagram however, there is a phase transition associated with spatial center symmetry \mathcal{G}_s and \mathcal{I} interchange symmetry of the orbifold theory at around $R_{S^1} \sim R_{S^3}$. When $\min(R_{S^1}, R_{S^3}) \gg \Lambda^{-1}$, the chiral symmetry is broken, but spatial center symmetry is unbroken. The simplest phase diagram consistent with our current knowledge is shown in fig.3.

We also note that the chiral properties of orbifold QCD(BF) theory are clearly different from those of the $\mathcal{N}=1$ SYM theory on $\mathbb{R}^3 \times S^1$. The $\mathcal{N}=1$ SYM have a chiral condensate even at small radius, which by holomorphy is equal to the condensate at large radius (in units of strong scale). An implication of holomorphy is radius independence of the chiral condensate as demonstrated by Davies et.al. [29]. On the other hand, the orbifold QCD(BF) theory does not possess a chiral condensate in small radius. Ref. [43] provides a recipe relating the holomorphic quantities in $\mathcal{N}=1$ SYM to their image in the nonsupersymmetric orbifold theory. As discussed above, we expect a chiral phase transition in the orbifold theory around the strong scale. This in particular implies radius independence does not hold for the chiral condensate in QCD(BF) in the \mathcal{I} broken phase. It is, on the other hand, perfectly sensible that the condensate proposed in ref. [43], will coincide at large radius, the \mathcal{I} unbroken phase. Therefore, as long as the radius is larger than a critical size, we expect radius independence from QCD(BF) condensate, and an equivalence to $\mathcal{N}=1$ SYM.

5. Phases of orientifold QCD(AS/S) theory on $S^3 \times S^1$

5.1 The phases as a function of volume

We continue the analysis of the phases of the vector-like gauge theories with QCD(AS/S), the orientifold partner of $\mathcal{N}=1$ SYM. ¹⁶ To be able to compare with the other vector-like gauge theories examined, we work on $S^3 \times S^1$, and benefit from the spin structure of S^1 . We first study the case where fermions obey periodic boundary conditions. The fundamental quantity of interest is again the twisted partition function $\tilde{Z} = \text{tr} e^{-\beta H} (-1)^F$.

¹⁵It should be possible to see this transition on lattice simulations. The lattice should be an asymmetric torus, and with say, three long, and one short direction. The thermal boundary conditions for fermions should be used in one of the long directions, so that the theory may be regarded as a low temperature field theory. Then, by dialing the size of the short direction (or equivalently, the bare coupling of the lattice theory), this transition should be seen around strong scale.

¹⁶While this paper was in the completion process, a preprint by Hollowood and Naqvi ref. [44] appeared. The material of sections 5.1 and 5.2 has identical results with [44]. Tim Hollowood also told me that he obtained the \mathcal{I} restoring phase transitions in the case of orbifold QCD(BF) [45]. I thank them for communications related to their work.

As in the orbifold QCD(BF) theory examined in the previous section, the orientifold QCD(AS/S) is non-supersymmetric irrespective of the background space. On the other hand, in leading order in N , it has identical numbers of bosonic and fermionic degrees of freedom. The fact that the color quantum number of fermions (which is antisymmetric or symmetric representation) is different from the one of gauge boson (adjoint) affect the eigenvalue dynamics of the Wilson line, as well as the center symmetry realizations.

The twisted partition function of the QCD(AS/S) theory is given by

$$\tilde{Z}(x)^{\text{QCD(AS/S)}} = \int dU \exp(-S^{\text{QCD(AS/S)}}[x, U]) \quad (5.1)$$

where the effective action for the spatial Wilson line may be expressed as in Eq.3.6,

$$\begin{aligned} S_{\text{eff}}^{\text{QCD(AS/S)}}[x, U] &= S[x, U] - \ln J[U] \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \left\{ (1 - z_V(x^n)) |\text{tr} U^n|^2 + z_f(x^n) \left(\frac{(\text{tr} U^n)^2 \pm \text{tr} U^{2n}}{2} + \text{h.c.} \right) \right\} \end{aligned} \quad (5.2)$$

The first half of the equation is due to the Jacobian, and gauge bosons (and ghosts). It has a manifest $U(1)$ center symmetry, associated with global rotations $U \rightarrow e^{i\alpha} U$. The second term is due to the antisymmetric/symmetric representation Dirac fermion. Recall that introducing two-index representation fermions in the original theory reduced this center symmetry to a \mathbb{Z}_2 . This is manifest in our effective action, which is only invariant under the restricted rotation of $\alpha = \pi$. The QCD(AS/S) theory is also invariant under the \mathbb{Z}_2 charge conjugation symmetry which we denote by \mathcal{C} . The theory also has other discrete spacetime symmetries, but the only symmetry necessary for the validity of the nonperturbative large N equivalence between QCD(AS/S) and SYM is \mathcal{C} . Therefore, we classify phases according to charge conjugation symmetry and spatial center symmetry $\mathcal{C} \times \tilde{\mathcal{G}}_s$. (The use of $\tilde{\mathcal{G}}_s$ for spatial center symmetry rather than \mathcal{G}_s will be seen below. But, as the reader may easily guess, we will identify it with the corresponding symmetry in orbifold QCD.) The symmetries of the effective action Eq.5.2 are given by

$$\begin{aligned} \tilde{\mathcal{G}}_s : U &\rightarrow e^{i\pi} U = -U, \\ \mathcal{C} : U &\rightarrow U^* \end{aligned} \quad (5.3)$$

consistent with the symmetries of the original QCD(AS/S) theory.

Notice that the action has both double and single trace operators. The leading large N dynamics of the theory is dictated by the double trace operators with natural $\mathcal{O}(N^2)$ scaling, and the effect of the single trace operator $\text{tr} U^{2n}$ is $\mathcal{O}(N)$, thus subleading by $\frac{1}{N}$. Therefore, in the analysis of the dynamics in the $N = \infty$ limit, we neglect the $\pm \text{tr} U^{2n}$ term. This removes any distinction between QCD(AS) and QCD(S). The subleading corrections to physical correlators in QCD(AS/S) are $\mathcal{O}(1/N)$, as opposed to $\mathcal{O}(1/N^2)$ which is typical in orbifold QCD(BF).

This purely kinematical large N effect brings in a beautiful set of simplifications. In fact, exactly in the case of orbifold QCD(BF), we will be able to express the action as a sum of perfect modulus squares. Naturally enough (analogous to the orbifold example),

the completion to the sum of perfect squares occurs with the use of eigenfunctions which simultaneously diagonalize the spatial center symmetry $\widetilde{\mathcal{G}}_s$ and charge conjugation symmetry \mathcal{C} .

To do so, let us define the linear combination of the Wilson lines in orientifold QCD(AS/S) theory in the eigenbasis of the spatial center symmetry and charge conjugation symmetry $\widetilde{\mathcal{G}}_s \times \mathcal{C}$. The orientifold eigenvalue problem can be expressed as

$$\begin{aligned} \text{tr} \Omega_{\pm}^k &= \text{tr} U^k \pm \text{tr} (U^*)^k, \quad k = 1, \dots, \infty, \\ \widetilde{\mathcal{G}}_s \text{tr} \Omega_{\pm}^k &= e^{i\pi k} \text{tr} \Omega_{\pm}^k, \quad \mathcal{C} \text{tr} \Omega_{\pm}^k = \pm \text{tr} \Omega_{\pm}^k \end{aligned} \quad (5.4)$$

where we constructed \mathcal{C} even-odd combinations of Wilson lines with winding number k for a given orientation and its conjugate with opposite orientation. Notice that the action of $\widetilde{\mathcal{G}}_s$ and \mathcal{C} on eigenfunctions $\text{tr} \Omega_{\pm}^k$ is exactly identical to the action of $\widetilde{\mathcal{G}}_s$ and \mathcal{I} in the case of orbifolds, hence the notation. [Recall that in QCD(BF), the full symmetry is $\widetilde{U(1)} \times \widetilde{\mathcal{G}}_s \times \mathcal{I}$. The $\widetilde{U(1)}$ is not present in QCD(AS/S), the center symmetry that is present is just $\widetilde{\mathcal{G}}_s$.]¹⁷

Expressing the orientifold QCD(AS/S) effective actions Eq.5.2 in terms of the $\widetilde{\mathcal{G}}_s \times \mathcal{C}$ symmetry eigenstates gives

$$S_{\text{eff}}^{\text{QCD(AS/S)}}[x, \Omega_+, \Omega_-] = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n} \left\{ a_n^+(x) |\text{tr}(\Omega_+^n)|^2 + a_n^-(x) |\text{tr}(\Omega_-^n)|^2 \right\} \quad (5.5)$$

where $a_n^{\pm}(x)$ are exactly the same as in the orbifold theory given in Eq.4.7. The range of the modulus $|\text{tr} \Omega_{\pm}|$ coincides precisely in orbifold QCD(BF) and orientifold QCD(AS/S), hence the extremization problems are identical. (The extra $\widetilde{U(1)}$ symmetry present in QCD(BF) only changes the phase of $\text{tr} \Omega_{\pm}$ in QCD(BF) without altering its modulus. Since the action only depends on the modulus, the presence of $\widetilde{U(1)}$ symmetry does not interfere with the extremization of action. Nevertheless, it is significant in the discussion of quantum fluctuations, see footnote 17. The difference between the overall actions by a factor of $\frac{1}{2}$ is a purely kinematic factor (which should indeed be there) reflecting that the number of colors in $U(N) \times U(N)$ QCD(BF) is $2N^2$ and number of colors $U(N)$ in QCD(AS/S) is N^2 , and does not influence the dynamics.

¹⁷ For SYM and QCD(BF), the $U(1)$ center symmetry, in the unbroken center symmetry phase, guarantees that any topologically nontrivial Wilson line with arbitrary number of winding will have a zero vacuum expectation value. This is a consequence of unbroken symmetry. On the other hand, for QCD(AS/S), the center symmetry is just \mathbb{Z}_2 . This implies only the Wilson line with odd winding number will have a zero vacuum expectation value due to symmetry. What about the Wilson lines with even winding number? Do they acquire a (vacuum or thermal) expectation value, and if so, does this invalidate orientifold equivalence even in the large volume phase? The answer to the first question is, yes. If an operator is not protected by a symmetry, it will acquire a vacuum expectation value. However, it is possible to show that, by employing the loop equations, the vacuum expectation value of even winding number Wilson lines is suppressed in the large N , $\langle \frac{1}{N} \text{tr} U^{2n} \rangle = \mathcal{O}(1/N)$. Therefore, neither symmetry considerations, nor large N equivalence is compromised. We expect this type of behaviour, i.e., *vanishing of certain correlators without symmetry reasons*, will give us important hints about the dynamics of large N limits, and should admit a greater understanding of large N dynamics. This prediction of large N orientifold equivalence may be tested on lattice simulations.

The exact equivalence of the symmetry realization of $N = \infty$ QCD(AS/S) and QCD(BF) on $S^3 \times S^1$ is one of the main result of this paper. If \mathbb{Z}_2 symmetry of the orbifold QCD(BF) and \mathcal{C} symmetry of the orientifold QCD(AS/S) are unbroken in the $\min(R_{S^3}, R_{S^1}) \gg \Lambda^{-1}$ regime as well, there is an exact $N = \infty$ equivalence between them at arbitrary radii. Currently, there is no evidence (and neither a proof) that \mathcal{I} and \mathcal{C} are broken on large radii or on \mathbb{R}^4 . But both are strongly unlikely. Below, we will see that all the discussion of QCD(BF) can be carried verbatim to QCD(AS/S) by just replacing wherever one sees the $\mathbb{Z}_2 = \mathcal{I}$ symmetry of the orbifold theory by \mathcal{C} charge conjugation symmetry of the orientifold QCD(AS/S) theory. This, in particular means, if there is a phase in which \mathbb{Z}_2 symmetry of orbifold theory is spontaneously broken, so is \mathcal{C} in orientifold theory. If for some reason \mathbb{Z}_2 gets restored or remains unbroken in QCD(BF), so does \mathcal{C} in QCD(AS/S).¹⁸

Nevertheless, let us spell out shortly the symmetry realizations in QCD(AS/S). The effective potential S_{eff} in Eq.5.2 determines the symmetry realizations for the spatial center symmetry $\tilde{\mathcal{G}}_s$ and charge conjugation symmetry \mathcal{C} . Just above x_c , the mass of the lightest \mathcal{C} odd mode $\text{tr}\Omega_-$ becomes negative and leads to a \mathcal{C} and $\tilde{\mathcal{G}}_s$ breaking instability. The even combination $\text{tr}\Omega_+$ remains massive. In the limit $x \rightarrow 1$, the potential has a two-fold degenerate minima located at $\text{tr}U = \pm i$, or

$$\langle \frac{1}{N} \text{tr}\Omega_- \rangle = \pm 2e^{i\pi/2}, \quad \langle \frac{1}{N} \text{tr}\Omega_+ \rangle = 0 \quad (5.6)$$

spontaneously breaking the spatial center symmetry and charge conjugation symmetry to its diagonal subgroup. The meaning of this formula, in the basis of eigenvalues of the Wilson line is analogous to the orbifold example we examined. The two clusters of eigenvalues of orbifold theory is now replaced by one cluster and its mirror image. The mirror image is the effect of complex conjugation. The cluster, therefore, analogous to orbifold example, wishes to be maximally away from its image. (see fig.2.) The eigenvalue distribution may be either $\rho(v) = \delta(v - \frac{\pi}{2})$ or $\rho(v) = \delta(v + \frac{\pi}{2})$ depending on the choice of vacua. In vectorlike theories

¹⁸The simplest way to understand this companionship between \mathcal{C} in QCD(AS/S) and \mathcal{I} in QCD(BF) is to go to basics, and see how these theories are obtained by genuine projections. Let us consider a $U(2N)$ and $SO(2N)$ parent $\mathcal{N}=1$ SYM theories and apply identical projections by global \mathbb{Z}_2 gauge symmetry times fermion number modulo two $(-1)^F$, i.e, $\mathbb{Z}_2(-1)^F$. The projections are indeed identical, but have different names: orbifold projection (if one starts with unitary group) and orientifold (if one starts with orthogonal group). There are two possible orbifold projections by $\mathbb{Z}_2(-1)^F$ depending on the fermion number assignment (\pm): If one use (+) fermion number assignment, the daughter theory is a decoupled $U(N)_1 \times U(N)_2$ $\mathcal{N}=1$ SYM theory. If one uses (-), then fermions comes in bifundamental representation, i.e, fundamental under $U(N)_1$, and antifundamental under $U(N)_2$. Analogously, starting with $SO(2N)$, and doing projections by using (+) assignment, one obtains again a decoupled “product” gauge group, $U(N) \times U(N)^*$ $\mathcal{N}=1$ SYM. The $U(N)^*$ is the analog of $U(N)_2$ of the orbifold theory, and is just an auxiliary mirror image of the $U(N)$ group. Incorporating fermion number into the projection yields two index representation fermions transforming as “bifundamentals” of $U(N) \times U(N)^*$, fundamental under $U(N)$ and antifundamental under $U(N)^*$. Since the antifundamental of $U(N)^*$ is fundamental of $U(N)$, the fermions are in two index antisymmetric representation. Therefore, the \mathcal{I} symmetry which interchange the $U(N)_1$ and $U(N)_2$, is replaced by a \mathcal{C} symmetry which interchange $U(N)$ with its mirror image $U(N)^*$. If \mathcal{I} and \mathcal{C} are not spontaneously broken, there are valid parent-daughter equivalences, which in turn implies daughter-daughter equivalence. There are infinitely many order parameters which can probe the breaking of either symmetry on \mathbb{R}^4 . Therefore, the orbifold and orientifold equivalences of QCD-like theories are precisely on the same footing, and all their differences are cosmetic.

with unbroken \mathcal{C} symmetry, the eigenvalue distributions must satisfy $\rho(v) = \mathcal{C}\rho(v) = \rho(-v)$. For example, this is the case in $\mathcal{N}=1$ SYM, and as well as other QCD(adj). However, QCD(AS/S) possess a phase in which $\rho(v) \neq \rho(-v)$, and \mathcal{C} is spontaneously broken. The vacuum energy density in leading order in N is particularly simple in the $x \rightarrow 1$ limit, and is given by

$$\mathcal{E}^{\text{QCD(AS/S)}}(x \rightarrow 1) = -\frac{1}{R_{S^1} V_{S^3}} \log \tilde{Z}(x \rightarrow 1) = -\frac{\pi^2}{24(R_{S^1})^4} N(N \mp \frac{7}{15}) \quad (5.7)$$

The N^2 is a kinematical factor reflecting that there are N^2 gauge field and tensor fermions in QCD(AS/S). In leading order in N , there is a factor of two difference compared to $U(N) \times U(N)$ QCD(BF), reflecting the difference in the total number of degrees of freedom.

For $x < x_c$, the potential is positive definite, and provides a repulsive interaction among eigenvalues. Hence, the eigenvalues are uniformly distributed, and the expectation value of the Wilson line vanishes: $\langle \text{tr} U \rangle = 0$. The phase transition therefore restores both the spatial center symmetry and the charge conjugation symmetry at large radius of S^1 circle just like the orbifold QCD(BF) theory restores its $\mathcal{G}_s \times \mathcal{I}$. The vacuum energy density is of order one $\mathcal{E} \sim \mathcal{O}(1)$, and is due to the fluctuations. Therefore, the theory undergoes a phase transition which is associated with an abrupt change in its vacuum energy density from $\mathcal{O}(N^2)$ to being $\mathcal{O}(1)$, and triggered by quantum fluctuations rather than thermal ones.

5.2 Finite temperature phases

The choice of antiperiodic boundary conditions for fermions on the S^1 circle corresponds to considering QCD(AS/S) at finite temperature. The resulting Euclidean functional integral corresponds to the partition function $Z = \text{tr} e^{-\beta H}$ of a thermal ensemble on S^3 with inverse temperature β .

As usual, the change in the boundary condition is reflected as an alternating coefficient of the fermionic term in Eq.4.3.

$$z_f(x^n) \rightarrow (-1)^n z_f(x^n) \quad (5.8)$$

therefore, turning the coefficients in the effective action Eq.5.5 into Eq.4.14. The thermal effective potential is in terms of the temporal Wilson line and determines the temporal center symmetry realization $\tilde{\mathcal{G}}_t$ along with charge conjugation symmetry \mathcal{C} .

At low temperature (small x), the expectation value of the thermal Wilson line is identically zero, $\langle \frac{1}{N} \text{tr} U \rangle = 0$. Therefore, the temporal center symmetry, and charge conjugation symmetry are unbroken. The free energy density is order one reflecting the fact that spectral density of the color singlets remains $\mathcal{O}(1)$ in the large N limit. This is the characteristic of the confined phase.

As the temperature is increased, the mass of the \mathcal{C} -even mode $\text{tr} \Omega_+$ becomes negative leading to a deconfinement transition. This means, at high temperature, the temporal center symmetry \mathcal{G}_t is broken, but not the charge conjugation symmetry. In the very high temperature limit ($x \rightarrow 1$), the order parameters are

$$\langle \frac{1}{N} \text{tr} \Omega_+ \rangle = \pm 2, \quad \langle \frac{1}{N} \text{tr} \Omega_- \rangle = 0, \quad (5.9)$$

and the breaking pattern is $\{1, \tilde{\mathcal{G}}_t\} \times \{1, \mathcal{C}\} \rightarrow \{1, \mathcal{C}\}$. In terms of eigenvalues of the Polyakov loop, this means all the eigenvalues clump to either at $v = 0$ or $v = \pi$. These are the two thermal equilibrium states. The eigenvalue distribution is given by $\rho(v) = \delta(v)$ or $\rho(v) = \delta(v - \pi)$ both of which are even under the action of \mathcal{C} . This means both thermal equilibrium state lies in the \mathcal{C} even sector. From the partition function, The free energy density is particularly simple in the high temperature ($x \rightarrow 1$) limit:

$$\mathcal{F}^{\text{QCD(AS/S)}}(x \rightarrow 1) = -\frac{\pi^2}{24} T^4 N(N \mp \frac{7}{15}) \quad (5.10)$$

the expected Stefan-Boltzmann result, $-\frac{\pi^2}{45} T^4 [N^2 + \frac{7}{8} N(N \mp 1)]$

The fact that \mathcal{C} is unbroken at high temperature should not be surprising. The expectation value of the Polyakov loop (in the fundamental representation) and its charge conjugate corresponds to the excess of free energy of an external quark whose charge is $+1$ and -1 respectively. Since these charges differs by two, in the sense of center symmetry of the theory, they are in the same conjugacy class as discussed in section 2.2. This follows from the observation that the dynamical antisymmetric representation quark has charge two under the center $U(1)$ of $U(N)$, and can convert the charge by an additive factor of two with no cost. Therefore, the dynamical quarks split the charges into two equivalence classes, the even ones equivalent to zero and the odd ones equivalent to one. Since the excess of free energy is a class function taking its values in the center symmetry of the theory, we should have $\Delta\mathcal{F}_{(+1)} = \Delta\mathcal{F}_{(-1)}$ implying $\langle \text{tr} U \rangle - \langle \text{tr} U^* \rangle = 0$, and hence unbroken charge conjugation symmetry \mathcal{C} .

5.3 Local order parameters

The symmetries of the nonthermal orientifold QCD(AS/S) theory are, as discussed above, charge conjugation symmetry \mathcal{C} and spatial center symmetry $\tilde{\mathcal{G}}_s$. The symmetry breaking pattern discussed in section 5.1 corresponds to

$$(\mathbb{Z}_2)^2 = \{1, \mathcal{C}, \tilde{\mathcal{G}}_s, \mathcal{C}\tilde{\mathcal{G}}_s\} \rightarrow \mathbb{Z}_{2,D} = \{1, \mathcal{C}\tilde{\mathcal{G}}_s\} \quad (5.11)$$

leading to two degenerate vacua. These two isolated vacua are not invariant under the individual actions of \mathcal{C} and $\tilde{\mathcal{G}}_s$, but invariant under the combined $\mathcal{C}\tilde{\mathcal{G}}_s$ action. In analogy with the orbifold example where unbroken $\tilde{\mathcal{G}}_s \mathcal{I}$ has nontrivial implications for \mathcal{I} odd local order parameters, unbroken $\tilde{\mathcal{G}}_s \mathcal{C}$ has similar consequences for \mathcal{C} odd local order parameters.

Corollary: Let $\mathcal{O}(x)$ be a local order parameters odd under charge conjugation symmetry \mathcal{C} . In any phase of the theory in which $\mathcal{C}\tilde{\mathcal{G}}_s$ is unbroken, vacuum expectation value of such operators vanish.

Since $\mathcal{O}(x)$ is local, it is singlet under center symmetry. Therefore, we have

$$(\mathcal{C}\tilde{\mathcal{G}}_s)\mathcal{O}(x) = \mathcal{C}\mathcal{O}(x) = -\mathcal{O}(x). \quad (5.12)$$

Since the vacuum of the theory is invariant under $\mathcal{C}\tilde{\mathcal{G}}_s$, we obtain

$$\langle \mathcal{O}(x) \rangle = \langle (\mathcal{C}\tilde{\mathcal{G}}_s)\mathcal{O}(x) \rangle = -\langle \mathcal{O}(x) \rangle = 0 \quad (5.13)$$

as desired. Clearly, the local operators (which do not wind around the S^1 circle) are all singlets under the center symmetry. In any $\mathcal{C}\tilde{\mathcal{G}}_s$ unbroken phase of the theory, topologically trivial but \mathcal{C} odd order parameters transform nontrivially under the action of $\mathcal{C}\tilde{\mathcal{G}}_s$. Therefore, the vacuum expectation value of all such operators must vanish.

For thermal QCD(AS/S), in the high temperature deconfined phase, we found that the breaking pattern is

$$(\mathbb{Z}_2)^2 = \{1, \mathcal{C}, \tilde{\mathcal{G}}_t, \mathcal{C}\tilde{\mathcal{G}}_t\} \longrightarrow \mathbb{Z}_2 = \{1, \mathcal{C}\} \quad (5.14)$$

leading to two \mathcal{C} respecting isolated thermal equilibrium states. In the confined phase, the $(\mathbb{Z}_2)^2$ symmetry is unbroken. Therefore, no \mathcal{C} odd operator (either local or nonlocal) can acquire an expectation value regardless of the phase of the thermal theory.

Needless to say, these assertions do not show that \mathcal{C} in the case of orientifold QCD(AS/S) and \mathcal{I} in the case of orbifold QCD(BF) cannot be spontaneously broken on \mathbb{R}^4 via local order parameters. This, in principle is possible, but unlikely. Nevertheless, there is no theorem demonstrating that these two symmetries cannot be spontaneously broken on \mathbb{R}^4 , unlike the case of parity [46].

5.4 On the N dependence of \mathcal{C} breaking on $\mathbb{R}^3 \times (\text{spatial } S^1)$

For the $SU(2N+1)$ QCD(AS/S) as well as QCD with fundamental fermions, since the dynamical fermions can screen any external charge, the center symmetry is absent. Therefore, \mathcal{C} is broken as $\{1, \mathcal{C}\} \rightarrow 1$. Unlike the $U(N)$ QCD(AS/S) theory, there is no symmetry reason for local order parameters not to acquire a vacuum expectation value. We expect in this particular class of QCD-like gauge theories and choices of color gauge groups, both local and nonlocal order parameters to acquire nonzero expectation values in \mathcal{C} broken phase. For $SU(3)$ gauge theory with order few flavors, the two \mathcal{C} breaking vacua are located at $U = \exp(\pm i\frac{2\pi}{3})$, and the charge conjugation symmetry is spontaneously broken. This is unambiguously demonstrated in recent lattice studies by DeGrand and Hoffmann [13], who used a low temperature asymmetric lattice to probe \mathcal{C} breaking at small volume. These authors also demonstrated that there is a \mathcal{C} restoring transition taking place around the strong confinement scale of the theory. The more recent paper [47] also examines local order parameters in the similar setup. The authors of [47] use the component of the baryonic current along the compact direction, and nicely demonstrate that, in the \mathcal{C} broken phase, there is a persistent current correlated with imaginary part of the Wilson line $\text{tr}\Omega_-$.

We want to make few simple observations: Since the fundamental and antisymmetric representation of $SU(3)$ coincide, QCD has few natural large N generalization. One option is to use fundamental flavors in the large N , another is to antisymmetric, or a mixture of the two. The dimensions of these two representations scales differently with N , fundamental $\mathcal{O}(N)$ and antisymmetric $\mathcal{O}(N^2)$. Therefore, for a fixed number of flavors the former is kinematically suppressed in the large N limit. Nevertheless, one can take $\frac{n_f}{N}$ for fundamental fermions, and keep the total number of fermionic degrees of freedom $\mathcal{O}(N^2)$ as well [28]. Interestingly, in the presence of the fundamental fermions in $U(N)$ gauge theory, \mathcal{C} do not get broken at small spatial S^1 . The unique minimum is located at $U = -1$ which respects \mathcal{C} . (not at $U = 1$ which is the case for thermal compactification.) For $U(N)$ gauge theory with AS/S

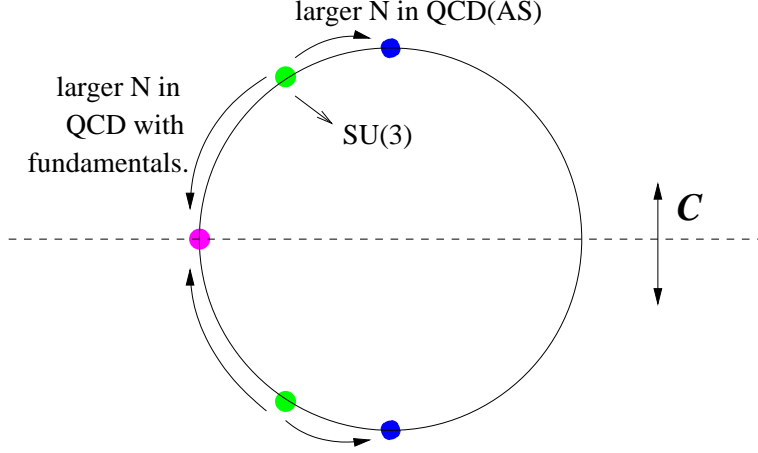


Figure 4: The N dependence of the \mathcal{C} -breaking in two natural generalization of the $SU(3)$ QCD on \mathbb{R}^3 times small spatial S^1 . For $SU(3)$ QCD with few fermions, the two minima are located at $e^{\pm i\frac{2\pi}{3}}$. For $SU(2N+1)$ QCD with fundamental fermions, the two minima gradually approach each other with increasing N . The $N = \infty$ limit has unique vacuum with unbroken \mathcal{C} . For $SU(2N+1)$ QCD(AS), the two minima approaches to antipodal points $\pm e^{i\pi/2}$ and $N = \infty$ theory breaks \mathcal{C} spontaneously, just like $SU(3)$.

representation, the two minima are located at $U = \exp(\pm i\frac{\pi}{2})$, hence breaks \mathcal{C} spontaneously. In the case of $SU(N)$, the minima of the effective potential is typically $\mathcal{O}(1/N)$ vicinity of π for fundamental fermions and within $\mathcal{O}(1/N)$ vicinity of $\pm\frac{\pi}{2}$ for AS/S representation, see figure 4:

$$\langle \text{Im} \frac{\text{tr}}{N} U \rangle^{\text{QCD(fun)}} = 0 + \mathcal{O}(1/N), \quad \langle \text{Im} \frac{\text{tr}}{N} U \rangle^{\text{QCD(AS/S)}} = \pm 1 + \mathcal{O}(1/N) \quad (5.15)$$

In other words, the \mathcal{C} breaking effects in $SU(2N+1)$ QCD is suppressed with the use of fundamental fermions, and is enhanced with AS/S representation fermions at large N . This is consistent with the fact that in large N limit, any distinction between $U(N)$ and $SU(N)$ should disappear.

Both the suppression of the \mathcal{C} breaking for fundamental fermions and the enhancement for AS/S fermions result in (at first sight counter-intuitively) the suppression of the vacuum expectation values of the local order parameters. Let $\frac{\text{tr}}{N} O(x)$ be a local order parameter probing \mathcal{C} . In QCD with fundamental fermions, this is natural immediate since \mathcal{C} breaking effects diminish with N . For QCD(AS/S), at sufficiently large N , the differences between $SU(2N+1)$ and $U(2N+1)$ is an $\mathcal{O}(1/N)$ effect. The $U(2N+1)$, unlike $SU(2N+1)$ enjoys a $\mathcal{G}_s = \mathbb{Z}_2$ spatial center symmetry. As shown in the corollary of section 5.3, the breaking of \mathcal{C} and \mathcal{G}_s are intertwined, and the two \mathcal{C} breaking vacua respects $\mathcal{C}\mathcal{G}_s$ preventing any local order parameter from acquiring vacuum expectation value. Therefore, for $SU(2N+1)$, we must have

$$\langle \frac{\text{tr}}{N} O(x) \rangle^{\text{QCD(fun/AS/S)}} = 0 + \mathcal{O}(1/N) \quad (5.16)$$

5.5 A remark on CPT and Vafa-Witten theorems

A final remark is on the details of the symmetry breaking pattern Eq.5.11, in Minkowski space $\mathbb{R}^{2,1} \times S^1$ or its Euclidean continuation $\mathbb{R}^3 \times S^1$. The latter may be regarded as a zero

temperature field theory in which one spatial direction is compactified. The analysis [12] shows that the imaginary part of the Wilson loop is not just an order parameter for spatial center symmetry $\tilde{\mathcal{G}}_s$ and charge conjugation symmetry \mathcal{C} , it also monitors parity \mathcal{P} and time reversal \mathcal{T} . Therefore, the precise breaking pattern (in the small radius limit) for the theory is $(\mathbb{Z}_2)^4 \longrightarrow (\mathbb{Z}_2)^3$:

$$\{1, \mathcal{C}\} \times \{1, \mathcal{P}\} \times \{1, \mathcal{T}\} \times \{1, \tilde{\mathcal{G}}_s\} \longrightarrow \{1, \mathcal{CP}, \mathcal{CT}, \mathcal{C}\tilde{\mathcal{G}}_s, \mathcal{PT}, \mathcal{P}\tilde{\mathcal{G}}_s, \mathcal{T}\tilde{\mathcal{G}}_s, \mathcal{CPT}\tilde{\mathcal{G}}_s\} \quad (5.17)$$

where the action of even number of operators leaves the vacuum that it acts invariant, and the action of odd number of them switches the two vacua [10]. In particular, the eight broken symmetries are

$$\{\mathcal{C}, \mathcal{P}, \mathcal{T}, \tilde{\mathcal{G}}_s, \mathcal{PTC}, \mathcal{P}\tilde{\mathcal{G}}_s\mathcal{C}, \mathcal{T}\tilde{\mathcal{G}}_s\mathcal{P}, \mathcal{CT}\tilde{\mathcal{G}}_s\} \quad (5.18)$$

The list includes spontaneously broken parity \mathcal{P} , and \mathcal{CPT} on $\mathbb{R}^{2,1} \times S^1$ at small radius. Does spontaneously broken CPT and P on this (locally) four dimensional theory clash with CPT theorem or Vafa-Witten theorem for parity [46]?

The answer is, no. CPT assumes the full Lorentz symmetry, which is lost by compactification. The parity argument is based on local order parameters on $\mathbb{R}^{3,1}$, in particular, our topologically nontrivial order parameters such as the imaginary parts of Wilson lines only emerge upon compactification.¹⁹ Therefore, there is no clash between broken CPT and broken P for QCD-like theories formulated on $\mathbb{R}^{2,1} \times S^1$ and the CPT and P theorems which are formulated on $\mathbb{R}^{3,1}$. However, it would be incorrect to make a general statement such as in vector-like gauge theories, CPT (or P) cannot be spontaneously broken. In fact, for parity, the inapplicability of the Vafa-Witten argument, which is originally formulated for gauge theories on \mathbb{R}^4 [46], in finite volumes is discussed in ref. [48]. Even though Ref. [48] did not demonstrate the existence of a P broken phase, it clearly established that there is no fundamental principle which guarantees unbroken P in finite volume. The existence of P broken phase in $U(N)$ QCD(AS/S) at any $N \geq 3$ [12], (or usual $SU(3)$ QCD with fundamental fermions [49, 50]) justifies the argument of ref. [48] by explicit demonstration.²⁰ Our discussion shows that, not only the P theorem, but also CPT theorem is inapplicable in finite volume. Both are broken at small spatial volume, and should get restored around the physical strong scale of the QCD-like theory. The existence of this zero temperature phase transition has been confirmed in recent lattice studies of by DeGrand and Hoffmann [13].

5.6 Chiral symmetries

The discussion of the chiral properties of the QCD(AS/S) theory on $S^3 \times S^1$ are verbatim identical to the orbifold QCD(BF) theory. This is true in both when S^1 is a thermal circle or just a spatial circle. The only difference is in $\mathcal{O}(1/N)$ discrepancy in the number of vacua.

¹⁹For the $SU(2N+1)$ QCD(AS/S) as well as QCD with fundamental fermions, the breaking pattern is different due to the absence of the center symmetry. In particular, we expect $\{1, \mathcal{C}\} \times \{1, \mathcal{P}\} \times \{1, \mathcal{T}\} \longrightarrow \{1, \mathcal{CP}, \mathcal{CT}, \mathcal{PT}\}$. Therefore, we expect local P-odd order parameters to acquire vacuum expectation value within this phase for this specific class of vector-like gauge theories.

²⁰We thank Tom DeGrand for pointing the refs. [49, 50] and Misha Shifman for bringing the ref. [48] to our attention.

In the case of QCD(AS/S), the chiral symmetries are \mathbb{Z}_{2N-4} and \mathbb{Z}_{2N+4} , respectively. In the regime where the radii of $S^3 \times S^1$ are large compared to the strong scale, these discrete chiral symmetries are spontaneously broken to \mathbb{Z}_2 , leading to $N - 2$ and $N + 2$ isolated vacua. The difference between broken axial symmetries \mathbb{Z}_h where $h = \{N - 2, N + 2, N, N\}$ for $U(N)$ orientifold QCD(AS/S), $\mathcal{N} = 1$ SYM theories and $U(N) \times U(N)$ QCD(BF) is unimportant in the large N limit.

The broken discrete chiral symmetry group also counts the number of ground states. In $U(N)$ SYM theory and $U(N) \times U(N)$ QCD(BF), this number is N , and for QCD(AS/S), it is respectively, $N \mp 2$. From the view point of Ref. [51], this is a property of nonsupersymmetric QCD(AS/S/BF) which may be related to the index $\text{tr}(-1)^F e^{-\beta H}$ of supersymmetric theory, at large N . Clearly, there is a match for the orbifold QCD(BF) and corrections of $\mathcal{O}(1/N)$ for QCD(AS/S). However, it should be kept in mind that each of QCD(AS/S/BF) formulated on $\mathbb{R}^3 \times S^1$ (where S^1 is spatial, implying the theory is at zero temperature) undergo a zero temperature chiral phase transition. This does not happen in $\mathcal{N} = 1$ SYM. (Ref. [51] examined these theories on $T^3 \times \mathbb{R}$ where T^3 is small, and \mathcal{C} in QCD(AS/S) and \mathcal{I} in QCD(BF) are spontaneously broken. Since the necessary symmetry realization conditions do not hold in this phase, we do not expect an equivalence. On the other hand, for large T^3 , we expect a restoration of \mathcal{C} and \mathcal{I} , and consequently the matching of the number of the ground states as described above.) A prediction of the large N orbifold and orientifold equivalence is that the condensates of these theories should coincide in the large volume limit. This is demonstrated to be so in lattice simulations [52] for QCD(AS/S) and is still undemonstrated for QCD(BF). For more on one flavor QCD-like theories, see the recent discussion [53].

6. Multiflavor generalization, n_f universalities

We may generalize the entire previous discussion of vectorlike, asymptotically free gauge theories to the case of multiple fermion flavors. Preserving the asymptotic freedom restricts the number of flavors to at most five, $n_f \leq 5$. Adding multiple adjoint representation fermions to SYM theory will yield a non-supersymmetric $U(N)$ gauge theory with $n_f > 1$ adjoint Weyl (or Majorana) fermions, which we referred to as QCD(adj). The other theories of interest are QCD(AS/S/BF) with $n_f > 1$ Dirac fermions. For a fixed n_f , these theories are related to one another via a web of orbifold and orientifold projections. If \mathcal{C} and \mathcal{I} are unbroken in QCD(AS/S) and QCD(BF) respectively, we expect complete equivalence of thermodynamics and phase diagrams of these theories to QCD(adj) in the $N = \infty$ limit [15].

As usual, we may write the thermal partition function $Z = \text{tr} e^{-\beta H}$ and the twisted partition function $\tilde{Z} = \text{tr} e^{-\beta H} (-1)^F$ for these theories. Installing appropriate changes in the discussion of effective actions (for spatial and temporal Wilson lines) just means restoring the fermionic multiplicity:

$$z_f(x^n) \rightarrow n_f z_f(x^n). \quad (6.1)$$

The phase diagrams of the vector-like gauge theories with $n_f > 1$ just mimics their $n_f = 1$ counterparts and one qualitatively reaches identical conclusions in the sense of spatial and temporal center symmetry, chiral symmetry, charge conjugation symmetry of orientifold QCD(AS/S) and interchange symmetry in orbifold QCD(BF) theories. The phase diagrams

shown in Fig.1 qualitatively depict the phases of both (nonthermal and thermal) QCD(adj). Analogously, Fig.3 describes the phases of QCD(AS/S/BF).

To make the phase diagrams as quantitatively clear as possible (at least in the perturbative regime of phase transitions), let us understand how the spatial and temporal center symmetry changing phase boundaries move with n_f . In the perturbative regime of the vector-like theories on $S^3 \times S^1$, the temperature of the confinement deconfinement transition (changing the temporal center symmetry realization) and the radius of R_{S^1} at which spatial center symmetry realizations change can be calculated. At leading order (in the coupling constant), it is given by

$$\begin{aligned} \beta_d^{\text{QCD(AS/S/BF/adj)}} &= R_{S^1,s}^{\text{QCD(AS/S/BF)}} = \{1.66, 1.88, 2.04, 2.17, 2.27\} R_{S^3}, \\ R_{S^1,s}^{\text{QCD(adj)}} &= \text{none} \quad \text{for } n_f = 1, \dots, 5 \end{aligned} \quad (6.2)$$

respectively. As n_f increases, the slope of the center symmetry changing phase boundary shown in Fig.5 increases. Eq.6.2 is true within the domain of validity of the one loop analysis, and can only be altered in small amounts by higher order perturbative corrections. However, in the $\mathbb{R}^3 \times S^1$ limit of these QCD-like theories, currently, there are no analytical tools which may be used to determine the radius of (spatial or temporal) S^1 at which a phase transition occurs. The change in temporal center symmetry realization is associated with the deconfinement confinement transition. The results, based on numerical lattice simulations unambiguously demonstrate [39] that the transition occurs around the strong confinement scale of the theory. (For $SU(3)$, numerically, this is reported as $\sim 200\text{Mev.}$) More recently, the spatial center symmetry realization of the QCD(AS/S) is examined on lattice simulations and spatial center symmetry changing transition for QCD is observed around the strong scale of the theory [13, 52]. We should also note that, QCD(adj) does not undergo a spatial center symmetry changing transition on $\mathbb{R}^3 \times S^1$ limit. At small S^1 , there is indeed a one-loop effective potential (for $2 \leq n_f \leq 5$) which, in effect, generates a repulsive interactions among the eigenvalues of the Wilson line just like a nonperturbative potential in $\mathcal{N} = 1$ SYM does. Consequently, all QCD(adj) theories, whether they are supersymmetric ($n_f = 1$) or not ($n_f \neq 1$), shows identical behaviour in the sense of \mathcal{G}_s .

Eq.6.2, true within the perturbative regime of these theories on $S^3 \times S^1$, does not demonstrate that the confinement/deconfinement transition temperature should coincide in the non-perturbative regime on $\mathbb{R}^3 \times S^1$. On the other hand, generic correlators of these theories are dependent on the phase. As long as the phase transitions in the QCD(adj) and QCD(AS/S/BF) theories are *not* driven by the \mathcal{C} (or \mathcal{I}) odd order parameters but anything else, the phase diagrams of these vectorlike theories must coincide precisely based on nonperturbative large N orbifold/orientifold equivalence.²¹ In particular, this implies the strong coupling phase

²¹Recall that the nonperturbative large N equivalence is valid provided the ground (or thermal equilibrium) states of both theories lie in their neutral sector. If a \mathcal{C} or \mathcal{I} odd order parameter triggers a phase transition, then the ground (or thermal equilibrium) states of the corresponding theories will move to the non-neutral sector. In the thermal phase diagram of QCD(AS/S/BF/Adj), all three phases lie in the neutral sector, and transition are driven by neutral sector operators. In contrast, in the nonthermal QCD(AS/S/BF/), there exist a phase in which \mathcal{C} and \mathcal{I} are broken, and the vacua are in the non-neutral (twisted) sector.

transition on $\mathbb{R}^3 \times S^1$ must occur at the same temperature in QCD(AS/S/BF/adj). This is a prediction of large N orbifold equivalence which may be tested on the lattice.

A watered-down version of why such universal behaviour is plausible may be surmised in perturbation theory. All of our vectorlike theories with a given number of flavors n_f possess identical renormalization group beta functions (for the 't Hooft couplings) in the $N = \infty$ limit. Let $\alpha(\mu) = \lambda(\mu)/4\pi$, then

$$\frac{d\alpha}{d\log\mu} = -\frac{b_0}{2\pi}\alpha^2 - \frac{b_1}{4\pi^2}\alpha^3 + \mathcal{O}\left(\frac{1}{N}, \alpha^4\right), \quad \text{where } b_0 = \frac{11}{3} - \frac{2}{3}n_f, \quad b_1 = \frac{17}{3} - \frac{8}{3}n_f, \quad (6.3)$$

The typical corrections in renormalization group flow of couplings is $\mathcal{O}(1/N)$ for QCD(AS/S) and $\mathcal{O}(1/N^2)$ for QCD(BF). In the large N limit, these subleading terms can be safely neglected. (See [54], for example.) Therefore, within a specific perturbative scheme for the QCD(AS/S/BF/adj) theories with a given number of flavor n_f , the scale at which the theory becomes strong is independent of what the underlying theory is. This is an obvious manifestation of the perturbative matching between the orbifold and orientifold partners. The result also reflects that the scale at which these theories become strongly coupled, say larger than $\alpha \geq 1/4$, only depends on n_f and not up on the color quantum number \mathcal{R} of the fermions. (So long as fermions are in one of the four double index representations AS/S/BF/adj.)

Let us assume for simplicity that the theories at some large scale, say M (the lattice cut-off), are assigned small couplings, $\alpha_i(M) = \alpha(M)$ independent of the number of flavors n_f in the theory. Then there is an exponential hierarchy between the scales at which strong coupling takes over. Let $Q_{n_f}^*$ denote the energy scale where the coupling constant becomes large, and the perturbative analysis is no longer reliable. To lowest order in perturbation theory, we find

$$\frac{Q_{n_f}^*}{Q_{n_f=1}^*} = \left[\frac{Q_{n_f=1}^*}{M} \right]^{\frac{2n_f-2}{11-2n_f}} \quad (6.4)$$

Since $\frac{Q_{n_f=1}^*}{M} \ll 1$, the scale at which $n_f > 1$ flavor QCD theory becomes strongly coupled is parametrically large in length (small in energies) with respect to QCD $n_f = 1$. Therefore, physical quantities such as hadron sizes, deconfinement radius (inverse temperature) in QCD with $n_f > 1$ compared to QCD with $n_f = 1$ are larger. This also implies a larger region of validity of perturbation theory with the increasing number of flavors. The cartoon of the n_f dependence of the phase boundaries for $n_f = 1$ and $n_f = 5$ is illustrated in Fig.5.

The phase diagram Fig.5 also demonstrates the chiral symmetry realizations. For thermal QCD(AS/S/BF/Adj), the chiral symmetry is unbroken at high curvature (small S^3) or at high temperature (small thermal S^1). This is also true for the nonthermal compactification of QCD(AS/S/BF) where S^1 is spatial. The QCD(adj) endowed with periodic boundary conditions for fermions is simply different. On large S^3 , (to zeroth order, the theory on $\mathbb{R}^3 \times S^1$), we expect broken chiral symmetry regardless of the the size of the spatial S^1 , just like $\mathcal{N} = 1$ SYM.

In the sense of presence or absence of chirally symmetric and asymmetric phases, Fig.5 provides a fair caricature. However, unlike the $n_f = 1$ case in which only discrete chiral symmetries are present, the $n_f > 1$ also brings in continuous chiral symmetries. The spontaneous

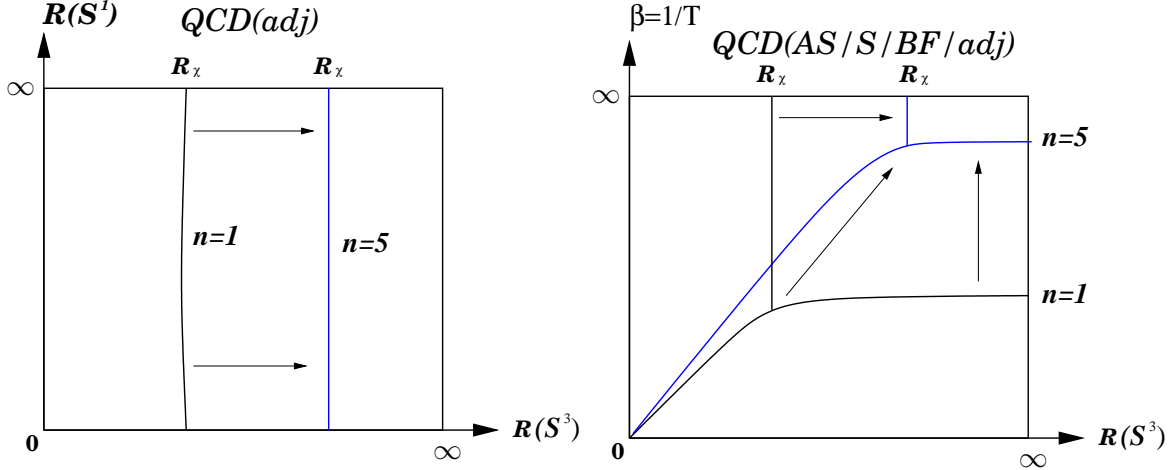


Figure 5: Multiflavor generalization of the phase diagram given in Fig.1 and Fig.3. The left figure depicts how the addition of multiple adjoint fermion flavors alters QCD(adj) with periodic boundary conditions by gradually pushing the chiral transition radius to larger values. The right figure may be thought of as corresponding to multiflavor QCD(AS/S/BF/adj) with thermal boundary conditions or QCD(AS/S/BF) with periodic boundary conditions. As seen in the figure, the net effect of adding flavors is to increase the slope of the phase transition line (associated with the appropriate center symmetry realization) in the perturbative regime and to push the nonperturbative strong confinement length to higher values (or to lower energy scales). If the theory were genuinely conformal, then there would not be such a scale and the confinement deconfinement phase boundary would not bend down around any particular scale. The phase boundaries for $n_f = 2, 3, 4$ should be between $n_f = 1$ and $n_f = 5$ in increasing order. (The $n_f \geq 6$ theories are not asymptotically free and is not examined in this paper.)

breaking of continuous chiral symmetries are associated with the Goldstone bosons. Below, we will discuss the discrete and continuous chiral symmetry breaking patterns, the associated domain walls and Goldstone bosons.

6.1 Chiral condensates in QCD(adj)

The QCD(adj) theory with n_f flavor Majorana fermions possess both continuous and discrete global symmetry given by

$$(SU(n_f) \times \mathbb{Z}_{2n_f N}) / \mathbb{Z}_{n_f} \quad (6.5)$$

The adjoint fermions, under $SU(n_f) \times \mathbb{Z}_{2n_f N}$ transform as (suppressing the flavor indices) $\psi \rightarrow V\psi$, $\psi \rightarrow e^{i2\pi/(2Nn_f)}\psi$ where $V \in SU(n_f)$. Since the \mathbb{Z}_{n_f} subgroup of the discrete chiral rotation can be undone by an element of the center of $SU(n_f)$, it should be modded out to prevent double counting.

The chiral symmetry is expected to break down to $SO(n_f)$ (times $\mathbb{Z}_2 = (-1)^F$ if n_f is odd. For even n_f , $(-1)^F$ is an element of $SO(n_f)$.) by the formation of the fermion bilinear condensate $\langle \text{tr} \psi_i \psi_j \rangle$, a symmetric tensor of $SU(n_f)$. This gives rise to a vacuum manifold with N disjoint components (islands), each of which is the coset space $SU(n_f)/SO(n_f)$. The breaking pattern of the chiral symmetry is

$$(SU(n_f) \times \mathbb{Z}_{2Nn_f}) / \mathbb{Z}_{n_f} \rightarrow SO(n_f) (\times \mathbb{Z}_2) \quad (6.6)$$

This is an interesting pattern which shows both discrete and continuous chiral symmetry breaking. The continuous symmetry breaking gives rise to $\frac{n_f(n_f+1)}{2} - 1$ Goldstone bosons. The discrete symmetry breaking is responsible for the N isolated vacuum manifolds and the domain walls interpolating among them. The determinant of the condensate, which is insensitive to flavor rotations, distinguishes the vacuum manifolds which cannot be continuously connected to each other via flavor rotation.

$$\arg \det \langle \text{tr} \psi_i \psi_j \rangle = \frac{2\pi}{N} k \in \mathbb{Z}_N, \quad k = 0, \dots, N-1 \quad (6.7)$$

Notice that for the $n_f = 1$ case, the N isolated coset spaces shrink to N isolated points, and we recover the well-known result for $\mathcal{N} = 1$ SYM theory where there are no Goldstones, just N isolated vacua with domain walls interpolating among them. It would be interesting to study the domain walls in QCD(adj) with $2 \leq n_f \leq 5$ in more detail.

At sufficiently large S^3 , where the base space may be approximated by $\mathbb{R}^3 \times S^1$, the chiral condensate must be independent of S^1 radius. This large N volume independence is discussed in detail in [31], and the necessary and sufficient condition for its validity is unbroken center symmetry. This condition is satisfied in spatial compactifications of QCD(adj) at any S^1 . In thermal compactifications, the unbroken center symmetry also implies a confined phase, therefore, the chiral condensate has to be independent of the temperature in the confined phase [31].

However, we do not expect the chiral condensate to be independent of S^3 radius. The technical reason for this, we cannot formulate the volume change of sphere $S^d, d \geq 2$ as an orbifold projection. This is unlike the case of the d-torus T^d , including S^1 [31]. In fact, if S^3 is sufficiently small, we expect a chirally symmetric phase. The independence of the chiral symmetry realization from the S^1 radius (shown in the left panel of Fig:5) in QCD(adj) (including SYM) is particularly interesting. Keeping S^1 small, and varying S^3 from small radius all the way to \mathbb{R}^3 , we observe that a chiral transition should take place in a regime where the theory is perturbative at the scale of S^1 . It may therefore be possible to investigate this chiral transition by using perturbative techniques, combined with the effective field theory considerations of ref. [55]. This is left for the future work.

6.2 Chiral condensates in QCD(AS/S/BF)

Let us consider the chiral symmetry breaking pattern of multi-flavor QCD(AS/S/BF) theories. These are all vectorlike gauge theories with n_f two index complex representation fermions. Even though the color gauge structures of these theories are different, their chiral properties are almost identical. Here, almost means up to $\mathcal{O}(1/N)$. At the classical level, both QCD(AS/S) and QCD(BF) possess a $U(n_f)_L \times U(n_f)_R$ global symmetry. At the quantum level, because of the chiral anomaly, the symmetry reduces to $SU(n_f)_L \times SU(n_f)_R \times U(1)_B \times \mathbb{Z}_{2hn_f}$ where

$$h = N-2, N+2, N \text{ for QCD(AS/S/BF)} \quad (6.8)$$

respectively. The difference in the large N limit is a negligible $\mathcal{O}(1/N)$ in the discrete chiral symmetry.

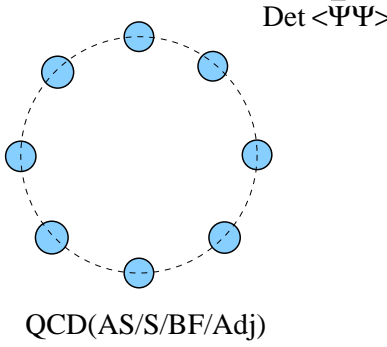


Figure 6: The vacuum structure of the $U(N)$ QCD-like gauge theories for $1 \leq n_f \leq 5$. The islands represents coset spaces associated with continuous symmetry breaking. There are $h = N - 2, N + 2, N, N$ such islands (corresponding to QCD(AS/S/BF/adj) respectively) distinguished by the phase of the determinant of the condensate, which is a \mathbb{Z}_h valued object. Since these theories show both discrete and continuous chiral symmetry breaking, there exist both Goldstone bosons (for $n_f > 1$) and domain walls $n_f \geq 1$. Notice that the coset spaces shrink to points in the case $n_f = 1$, and the Goldstone bosons disappears.

By factoring out the doubly-counted symmetries, we obtain

$$\frac{SU(n_f)_L \times SU(n_f)_R \times U(1)_B \times \mathbb{Z}_{2hn_f}}{(\mathbb{Z}_{n_f} \times \mathbb{Z}_{n_f} \times \mathbb{Z}_2)}. \quad (6.9)$$

The \mathbb{Z}_2 is common in $U(1)_B$ and \mathbb{Z}_{2hn_f} . One \mathbb{Z}_{n_f} is common to the center of the axial $A = L - R$ and \mathbb{Z}_{n_f} subgroup of axial \mathbb{Z}_{2hn_f} . The other is common to the center of the vector $V = L + R$ and \mathbb{Z}_{n_f} subgroup of vectorial $U(1)_B$. It is expected, when $\min(R_{S^3}, R_{S^1}) \gg \Lambda^{-1}$, this symmetry should be spontaneously broken by the formation of a fermion bilinear condensate down to

$$\frac{SU(n_f)_V \times U(1)_B \times \mathbb{Z}_2}{(\mathbb{Z}_{n_f} \times \mathbb{Z}_2)}. \quad (6.10)$$

This leads to a vacuum manifold with h components each of which is the coset space $[SU(N_f)_L \times SU(N_f)_R]/SU(N_f)_V$. The h condensates are distinguished by the phase of the determinant of the chiral order parameter,

$$\arg \det \langle \text{tr} \bar{\Psi}^i \Psi^j \rangle = \frac{2\pi}{h} k \in \mathbb{Z}_h, \quad k = 0, 1, \dots, h-1 \quad (6.11)$$

The continuous chiral symmetry breaking leads to $n_f^2 - 1$ Goldstone bosons. The discrete chiral symmetry breaking is responsible for the existence of the domain walls interpolating between different components of the vacuum manifolds. Notice that in the $n_f = 1$ case, there are no Goldstone bosons and the breaking pattern correctly reproduces the result derived for one-flavor theories.

In the context of large N orbifold and orientifold equivalences, there has been a certain amount of confusion about the mismatch of the number of Goldstone bosons between QCD(adj) and QCD(AS/S/BF). These points are clarified in [40] and [12] where it is shown that the equivalence only applies to symmetry invariant channels: In the case of the orientifold, this corresponds to \mathcal{C} -invariant subsector of QCD(AS/S) and in the case of orbifolds,

to the \mathcal{I} interchange symmetry invariant channel of the orbifold QCD(BF) theory. Therefore, we have to grade the Goldstone bosons according to the \mathcal{C} symmetry of orientifold and \mathcal{I} interchange symmetry of orbifold theory. The currents which may generate the corresponding Goldstone bosons are given by $J_a^{\mu 5} = \bar{\Psi}^i \gamma^\mu \gamma^5 (t_a)_{ij} \Psi^j$ where t_a are generators of the $SU(n_f)$ algebra. The even (odd) graded Goldstone can be generated by the currents $J_a^{\mu 5}$ for which t_a is a symmetric (antisymmetric) matrix. Let \mathbb{Z}_2 represent either \mathcal{C} or \mathcal{I} . The \mathbb{Z}_2 eigensystem of currents is given by

$$\mathbb{Z}_2 : J_a^{\mu 5} = \begin{cases} -J_a^{\mu 5}, & \text{for } t_a \in SO(n_f) \\ +J_a^{\mu 5}, & \text{for } t_a \in SU(n_f)/SO(n_f) \end{cases} \quad (6.12)$$

Therefore, the odd Goldstones reside in the subalgebra $SO(n_f)$ and the even ones in the complement $SU(n_f)/SO(n_f)$. In QCD(AS/S/BF), the currents corresponding to an $\frac{1}{2}n_f(n_f+1) - 1$ of the Goldstone bosons are \mathbb{Z}_2 even while those corresponding to $\frac{1}{2}n_f(n_f-1)$ of them are \mathbb{Z}_2 odd. The total number of \mathbb{Z}_2 even Goldstones in QCD(AS/S/BF) correctly reproduces the total number Goldstone bosons in QCD(adj).

Since the chiral dynamics are dictated by the symmetries of the underlying theory, and the orbifold QCD(BF) and QCD(AS/S) theories possess identical chiral symmetry, their chiral Lagrangian and dynamics should coincide. (This assumes unbroken \mathcal{C} and \mathcal{I} on \mathbb{R}^4 , both of which are unproven.) Recall that in the previous sections, we have demonstrated that the effective actions describing the spatial and temporal center symmetry realizations of QCD(AS/S) and QCD(BF) maps to identical problem in terms of \mathcal{C} and \mathcal{I} eigenstates, respectively. Therefore, for a given n_f , the infrared physics of the QCD(AS/S) and QCD(BF) seems to be remarkably close on $S^3 \times S^1$ at arbitrary radius in the large N limit. We should state that our point of view on the orbifold QCD(BF), and orbifold equivalences by and large, is opposite to the one presented so far in literature [41, 54]. In particular, we are highly optimistic that orbifold equivalences are useful source in understanding the dynamics of large N QCD-like gauge theories.

7. Structure of Large N limits, supersymmetry, and nonperturbative large N equivalence.

The study of the dynamics of vectorlike gauge theories reveals that certain aspects of these theories become indistinguishable in the $N = \infty$ limit. In particular, for a given number of flavors n_f , the bosonic $((-1)^F \text{ even})$ subsector of QCD(adj) becomes indistinguishable from the \mathcal{C} -even subsector of QCD(AS/S) and the \mathcal{I} even subsector of QCD(BF) in cases where the respective \mathbb{Z}_2 symmetries are unbroken. This is a consequence of the nonperturbative orbifold and orientifold equivalence proven rigorously by using lattice regularization in [14, 15].

The nonperturbative large N equivalence is valid provided the ground (or thermal equilibrium) states of orbifold and orientifold partners lie in their respective neutral sectors. In the thermal phase diagram of QCD(AS/S/BF/Adj), in all three phases, the ground (or thermal equilibrium) states are in the neutral sector, and all the order parameters of these transitions are neutral sector operators. (see Fig.1 and 3 right panels). Therefore, the phase diagrams must be identical in the $N = \infty$ limit.

If the $(-1)^F$, \mathcal{I} , \mathcal{C} odd order parameter probes a phase transition, respectively in SYM, QCD(BF), and QCD(AS/S), this implies the ground (or thermal equilibrium) states of the corresponding theories will move in or out of the neutral sector. For SYM, this does not happen, since $(-1)^F$ is unbroken. In contrast, the nonthermal QCD(AS/S/BF) has a phase in which the ground states are in nonneutral sector. (see Fig.1 and 3 left panels). Hence, in this phase, there is no equivalence between SYM and the others. In the other phases, there is an equivalence.

Let us make this more crisp with a simple example in the $\mathbb{R}^3 \times S^1$ limit. The matching of the free energies in the high temperature phase (small temporal S^1)

$$\lim_{N \rightarrow \infty} (N^2)^{-1} \mathcal{F}^{\text{SYM}} = \lim_{N \rightarrow \infty} (2N^2)^{-1} \mathcal{F}^{\text{QCD(BF)}} = \lim_{N \rightarrow \infty} (N^2)^{-1} \mathcal{F}^{\text{QCD(AS/S)}} = -\frac{\pi^2}{24} T^4 \quad (7.1)$$

is a consequence of the unbroken $(-1)^F$, \mathcal{C} and \mathcal{I} symmetry, respectively, and just follows from the fact that the thermal equilibrium states in these theories are in their neutral sector. On the other hand, the mismatch of ground state energies in case where S^1 is a small spatial circle

$$0 = \lim_{N \rightarrow \infty} (N^2)^{-1} \mathcal{E}^{\text{SYM}} \neq \lim_{N \rightarrow \infty} (2N^2)^{-1} \mathcal{E}^{\text{QCD(BF)}} = \lim_{N \rightarrow \infty} (N^2)^{-1} \mathcal{E}^{\text{QCD(AS/S)}} = -\frac{\pi^2}{24(R_{S^1})^4} \quad (7.2)$$

results due to broken \mathcal{I} and \mathcal{C} . In this regime of QCD(AS/S/BF), the ground states do not lie in their respective neutral sectors. This nicely illustrates the symmetry realization conditions which are both necessary and sufficient for large N orbifold and orientifold equivalence.

Certain aspects of the large N equivalence are surprising, and may teach us important lessons on the structure of large N (and hopefully smaller N) gauge theories. One aspect we did not sufficiently emphasise in the analysis of phases, yet is of importance, is vanishing of certain correlators and expectation values in the $N = \infty$ limit, not because of symmetry, but because of the dynamics of the $N = \infty$ theory. We will first allude to a few of such examples, and try to understand the underlying reasons afterwards.

7.1 Center symmetries

As explained in footnote 17, in QCD(AS/S) the center symmetry is just \mathbb{Z}_2 , unlike $\mathcal{N}=1$ SYM and QCD(BF) where the center symmetry is $U(1)$. In QCD(AS/S), the unbroken \mathbb{Z}_2 center symmetry does guarantee that the Wilson lines with odd winding number will have zero vacuum expectation value, however, it does not enforce this on the Wilson lines with even winding number. On the other hand, for QCD(BF) or $\mathcal{N}=1$ SYM, any Wilson line (with an arbitrary winding number) will have zero expectation value just due to unbroken $U(1)$ center symmetry. This just means, in QCD(AS/S), the even winding number Wilson lines must be suppressed in the large N limit, $\langle \frac{1}{N} \text{tr} U^{2k} \rangle = \mathcal{O}(1/N)$ rather than being $\mathcal{O}(1)$. Therefore, in the confining phase of QCD(AS/S) $\langle \frac{1}{N} \text{tr} U^{2k} \rangle = 0$ at $N = \infty$ without a symmetry reason. More explicitly, in the confined phase, the large N orbifold/orientifold equivalence implies

$$\langle \frac{\text{tr}}{N} U^k \rangle^{\text{SYM}} = 0, \quad \langle \frac{1}{2N} (\text{tr} U_1^k \pm \text{tr} U_2^k) \rangle^{\text{QCD(BF)}} = 0$$

$$\langle \frac{\text{tr}}{N} U^{2k+1} \rangle^{\text{QCD(AS/S)}} = 0, \quad \langle \frac{\text{tr}}{N} U^{2k} \rangle^{\text{QCD(AS/S)}} = \mathcal{O}(\frac{1}{N}), \quad (7.3)$$

where $k = 1, \dots \infty$. Lattice simulations can easily check this assertion. Identical considerations are also true for the equivalences relating the unitary, orthogonal and symplectic gauge theories. In the last two, since their center symmetries are finite groups, symmetry only implies the expectation value of the Wilson lines with odd winding number has to vanish, and the loops with even winding number are in principle $\mathcal{O}(1)$. However, large N equivalence implies the expectation value of the even-winding loops in SO/Sp theories, in confined phases, should be $\mathcal{O}(1/N)$, and hence vanish in the $N = \infty$ limit.

7.2 Gluon condensate and vacuum energy

Another example of this kind is the gluon condensate $\frac{1}{N} \text{tr} F^2$. For $\mathcal{N} = 1$ SYM on $\mathbb{R}^{3,1}$, the vacuum expectation value of $\frac{1}{N} \text{tr} F_{\mu\nu}^2$ is zero because of the unbroken supersymmetry. The easiest way to see this is to recall the trace anomaly as an operator identity,

$$T_\mu^\mu = \frac{3N}{16\pi^2} \begin{cases} \text{tr} F_{\mu\nu} F^{\mu\nu} & \mathcal{N} = 1 \text{ SYM} \\ (1 + \mathcal{O}(1/N)) \text{tr} F_{\mu\nu} F^{\mu\nu} & \text{QCD(AS/S)} \\ (1 + \mathcal{O}(1/N^2)) (\text{tr} F_1 F_1 + \text{tr} F_2 F_2) & \text{QCD(BF)} \end{cases} \quad (7.4)$$

In $\mathcal{N} = 1$ SYM, the energy momentum tensor can be written as $2\sigma_{\mu,\alpha\dot{\alpha}} T^{\mu\nu} = \{\bar{Q}_{\dot{\alpha}}, J_\alpha^\nu\}$ where J_α^ν is supercurrent and $Q_\alpha = \int J_\alpha^0$ is the supercharge. Since the supercharge Q annihilates the vacuum, the vacuum expectation value of the energy momentum tensor, and hence gluon condensate $\langle \text{tr} F^2 \rangle$ is identically zero for $\mathcal{N} = 1$ SYM. On the other hand, QCD(AS/S/BF) are nonsupersymmetric and there is no symmetry reason for $\langle \frac{1}{N} \text{tr} F^2 \rangle$ to vanish, therefore, it should not. For the $n_f = 1$ QCD-like theories on $\mathbb{R}^{3,1}$ (or \mathbb{R}^4), we expect

$$\langle \frac{\text{tr}}{N} F_{\mu\nu}^2 \rangle^{\text{SYM}} = 0, \quad \langle \frac{\text{tr}}{N} F_{\mu\nu}^2 \rangle^{\text{QCD(AS/S)}} = \mathcal{O}(\frac{1}{N}), \quad \langle \frac{1}{2N} (\text{tr} F_1^2 + \text{tr} F_2^2) \rangle^{\text{QCD(BF)}} = \mathcal{O}(\frac{1}{N^2}), \quad (7.5)$$

The implication of the nonperturbative large N orbifold and orientifold equivalence is that the gluon condensate is suppressed in the $N = \infty$ limit for QCD(AS/S/BF). In these $N = \infty$ one flavor QCD-like theories, the condensate should vanish identically without any symmetry reasons.

In a Lorentz invariant theory, the vacuum energy density is

$$\mathcal{E} = \langle T^{00} \rangle = \frac{1}{4} \langle T_\mu^\mu \rangle.$$

Therefore, using the trace anomaly Eq.7.4 and using the N dependence of condensates Eq.7.5, we reach $\mathcal{E}^{\text{SYM}} = 0$, $\mathcal{E}^{\text{QCD(BF)}} = \mathcal{O}(1)$, $\mathcal{E}^{\text{QCD(AS/S)}} = \mathcal{O}(N)$. This demonstrates two things: First, the leading $\mathcal{O}(N^2)$ ground state energy is zero for both SYM, and QCD(AS/S/BF), rather than its natural $\mathcal{O}(N^2)$ scale. Second, the vacuum energy density of QCD(BF) is $\mathcal{O}(N)$ better than the QCD(AS/S) theory. [The observation that vacuum energy should be $\mathcal{O}(N)$ in QCD(AS/S) is made in [54, 56].] Despite the divergent vacuum energy of QCD(AS/S), the large N orientifold equivalence is valid (assuming \mathcal{C} is unbroken in \mathbb{R}^4), since the large N

equivalence only applies to the leading $\mathcal{O}(N^2)$ contribution to vacuum energy which happens to be zero:

$$\lim_{N \rightarrow \infty} \left[(N^2)^{-1} \mathcal{E}^{\text{QCD(AS/S)}} = \frac{\mathcal{O}(N)}{N^2} \right] = \lim_{N \rightarrow \infty} \left[(2N^2)^{-1} \mathcal{E}^{\text{QCD(BF)}} = \frac{\mathcal{O}(1)}{N^2} \right] = 0 \quad (7.6)$$

Physically, a more interesting quantity to look at is the quark mass and θ angle dependence of the vacuum energy density in the presence of a small mass perturbation [40, 54]. In this case, the vacuum degeneracy between $\{N, N, N-2, N+2\}$ vacua of the SYM, QCD(BF), QCD(AS/S), do get lifted by an amount proportional to mass times the chiral condensate of the respective theory. The vacuum energy density as a function of θ has indeed its natural $\mathcal{O}(N^2)$ scaling, and is identical for SYM and QCD(BF), and deviates only at subleading $\mathcal{O}(1/N)$ level from QCD(AS/S). Explicitly,

$$\lim_{N \rightarrow \infty} \left[\frac{\mathcal{E}^{\text{SYM}}}{N^2} = \frac{\mathcal{E}^{\text{QCD(BF)}}}{2N^2} = \frac{\mathcal{E}^{\text{QCD(AS/S)}}}{N^2} \right] (\theta) = \min_k \frac{m}{\lambda} \Lambda^3 \cos \left[\frac{2\pi}{N} k + \frac{\theta}{N} \right] \quad (7.7)$$

The N degenerate vacua splits into N branches, each of which is $2\pi N$ periodic. Consequently, the vacuum energy is just 2π periodic, i.e., $\mathcal{E}(\theta) = \mathcal{E}(\theta + 2\pi)$ as expected. The level crossings takes place at $\theta = \pi$ where two states becomes degenerate.

The main point is that the vanishing of the vacuum energy in the massless limit of SYM is due to supersymmetry. On the other hand, there is no symmetry reason for the $\mathcal{O}(N^2)$ contribution to vacuum energy to vanish for the nonsupersymmetric QCD(AS/S/BF) theories. When supersymmetry is softly broken by a small mass term, then the vacuum energy regains its natural $\mathcal{O}(N^2)$ scaling for SYM, and coincides with the natural $\mathcal{O}(N^2)$ vacuum energy densities of the nonsupersymmetric QCD(AS/S/BF).

7.3 Spectral degeneracies

The nonperturbative particle spectrum of any QCD-like theory, including $\mathcal{N} = 1$ SYM is beyond the reach of our current analytical capabilities. Unfortunately, supersymmetry has not been greatly helpful to embark on this hardest front of gauge theory either. Nonetheless, unbroken supersymmetry implies spectral degeneracies among the bosonic and fermionic excitations. Only the bosonic $(-1)^F$ even particles lie in the neutral sector of SYM. In $N = \infty$ QCD(AS/S/BF), all the gauge invariant operators (with finitely many field insertions) that one can construct are necessarily bosonic. Since the large N equivalence is between bosonic subsector of the $\mathcal{N} = 1$ SYM and \mathbb{Z}_2 (\mathcal{C} or \mathcal{I}) invariant bosonic subsectors of QCD(AS/S/BF), this means that the bosonic spectrum of QCD(AS/S/BF) and SYM must match. Let \mathcal{B} denote the bosonic Hilbert space. Then the implication of the large N equivalence on the nonperturbative particle spectra is

$$\text{spec}[\mathcal{B}]^{\text{SYM}} = \text{spec}[\mathcal{B}^+]^{\text{QCD(AS/S)}} = \text{spec}[\mathcal{B}^+]^{\text{QCD(BF)}} \quad (7.8)$$

where \mathcal{B}^+ labels \mathcal{C} -even physical states in QCD(AS/S) and \mathcal{I} -even states in QCD(BF). This brings us to our next point. The spectral matching of bosons (and fermions) in SYM is due to supersymmetry, however, there is no symmetry reason for bosonic pairs to coincide in

QCD(AS/S/BF) theory.²² One can continue this list further and applications for supersymmetric theories are given in [40].

To expand on this idea a bit further, consider a supersymmetric Ward identity for $\mathcal{N} = 1$ SYM theory. Let W be the field strength superfield of one chirality, $\bar{D}W = 0$. A particular supersymmetric Ward identity is $\langle \text{tr} D^2 W^2(x) \text{tr} D^2 W^2(0) \rangle = 0$ implying $\langle (\text{tr} F^2(x) + i \text{tr} F \tilde{F}(x)) (\text{tr} F^2(0) + i \text{tr} F \tilde{F}(0)) \rangle_{\text{conn}} = 0$. Due to unbroken parity in vectorlike gauge theories [46], the cross terms in the connected correlator vanish and we obtain

$$\langle \text{tr} F^2(x) \text{tr} F^2(0) \rangle_{\text{conn}} - \langle \text{tr} F \tilde{F}(x) \text{tr} F \tilde{F}(0) \rangle_{\text{conn}} = 0. \quad (7.9)$$

Notice that this connected correlator does not vanish due to fermion-boson degeneracy, but due to same spin, (spin zero) boson-boson degeneracy of the spectrum. The unbroken supersymmetry implies that massive scalar particles in the smallest supermultiplets of $\mathcal{N} = 1$ SYM should be degenerate with spin zero pseudoscalar, as well as with spin $\frac{1}{2}$ fermions. The spin-zero, positive (negative) parity color singlet glueballs exhaust the first (second) correlator. Since this is true at arbitrary separation, it implies

$$\sum_{\text{even parity}} \frac{|\alpha_n^+|^2}{k^2 + m_{n,+}^2} = \sum_{\text{odd parity}} \frac{|\alpha_n^-|^2}{k^2 + m_{n,-}^2} \quad (7.10)$$

at arbitrary momenta k . Here, α_n^\pm is the amplitude to create a positive (negative) parity glueball state at level n and $m_{n,\pm}^2$ is the mass of parity even/odd glueball. Notice that the relation Eq.7.10 is not a sum rule, since it is true at arbitrary momentum. Therefore, the implication of the equivalence is $\frac{m_{n,+}^2}{m_{n,-}^2} = \{1, 1 + \mathcal{O}(1/N), 1 + \mathcal{O}(1/N^2)\}$ respectively for SYM, QCD(AS/S) and QCD(BF). The nonsupersymmetric QCD(AS/S/BF) theories must exhibit parity doubled scalars and pseudoscalar provided the respective \mathbb{Z}_2 symmetries are unbroken [19, 40], in the $N = \infty$ limit. The spectral degeneracies in QCD(AS/S/BF) is not protected by symmetries, but just due to the equivalence.

7.4 On the structure of large N limit

One natural question is, why is it that in the $N = \infty$ limit, distinct quantum theories with very different fundamental symmetries (the orbifold and orientifold partners of SYM, for example), behave as if they have some higher symmetry, even though in reality they do not? Does it make sense to ask whether there are accidental symmetries emerging in the $N = \infty$ limit?

The proof of large N equivalence in fact has the key ingredient. In brief, this follows from the fact that the large N is a classical limit in the sense that the root mean square quantum

²²In multiflavor case, the generalization of the nonperturbative spectrum relation among our vectorlike gauge theories is $\text{spec}[\mathcal{B}]^{\text{QCD}(\text{adj})} = \text{spec}[\mathcal{B}^+]^{\text{QCD}(\text{AS/S/BF})}$. In particular, this implies the number of the massless (or light) nonperturbative states corresponding to the Goldstone bosons in $\text{spec}[\mathcal{B}]^{\text{QCD}(\text{adj})}$ should coincide with $\text{spec}[\mathcal{B}^+]^{\text{QCD}(\text{AS/S/BF})}$. This is indeed demonstrated to be the case as discussed in section 6, reflecting the importance of the even subsector (neutral) of bosonic Hilbert space. In other words, the \mathcal{C} and \mathcal{I} odd Goldstone bosons and mesons living in $\text{spec}[\mathcal{B}^-]^{\text{QCD}(\text{AS/S/BF})}$ has no image in the Hilbert space of the QCD(adj).

fluctuations of reasonable operators are suppressed [57]. (The reasonable operators are the ones with a smooth large N limit.) At $N = \infty$, appropriately identified operators in neutral sectors of QCD(AS/S/BF/adj) satisfy *identical* Schwinger-Dyson or loop equations [14] if the symmetries defining their neutral sectors are unbroken, leading to the equivalence of the theories. (or their classical Hamiltonians and coherence algebras coincide as described in [15].) At any finite N , there are corrections to the loop equations which are different for each individual theory and such corrections do not match. A subset of these subleading corrections may be regarded as a consequence of the quantum or thermal fluctuations in the large N limit. This is most easily seen by recalling the factorization (or cluster decomposition) property that is used in obtaining the $N = \infty$ loop equations. The factorization means, if $O_i(x)$ is some reasonable gauge invariant operator, then

$$\langle O_1(x)O_2(x) \rangle - \langle O_1(x) \rangle \langle O_2(x) \rangle = \mathcal{O}(1/N^p) \quad (7.11)$$

where $p = 1, 2$ depending on the theory. Therefore, $\langle O(x)^2 \rangle - \langle O(x) \rangle^2 = \mathcal{O}(1/N^p)$ implying the suppression of the quantum fluctuations in the $N \rightarrow \infty$ limit. In this sense, the quantum theories in the $N = \infty$ limit behave as classical [57].²³ The difference in the loop equations of QCD(AS/S) and QCD(BF) and QCD(adj) essentially arises in these subleading terms. Such fluctuations are typically of $\mathcal{O}(1/N)$ for QCD(AS/S) and $\mathcal{O}(1/N^2)$ for QCD(BF/adj). At $N = \infty$, there is no difference in the neutral sector dynamics of QCD(AS/S/BF/adj) as demonstrated by using loop equations or the coherent state approach [14, 15], and dynamics of the theories coincide.

In that regard, the large N orbifold/orientifold partners with very different fundamental symmetries, enjoys in their neutral sector, the protection of the highest symmetry available in the equivalence chain. For example, the Wilson lines of QCD(AS) clearly behaves as if they are protected by a $U(1)$ center symmetry (of its partners, SYM or QCD(BF)) rather than \mathbb{Z}_2 (its own truthful symmetry). There are spectral degeneracies in the bosonic Hilbert spaces as if it is protected by a supersymmetry, i.e., $\text{spec}[\mathcal{B}^+]^{\text{QCD(AS/S/BF)}} = \text{spec}[\mathcal{B}]^{\text{SYM}}$. If a phase transition is driven by a neutral operator, it will concurrently take place in all the theories in the equivalence chain. In other words, there is a well-defined mapping among the connected correlators of all neutral sector operators. If a progress leads to the solution of one of these $N = \infty$ theories, say $\mathcal{N} = 1$ SYM, it implies solution for the neutral sectors of all. This is the essence of the nonperturbative orbifold and orientifold equivalence.

The large N theories also possess non-neutral subsectors, as well. The loop equations of operators in these subsectors are unrelated to one another, and there is no map between them.

²³The clustering property breaks down if the theory is in the mixed phase (or there are more than one vacuum) in the theory. All of our $n_f = 1$ QCD-like theories, SYM, QCD(AS/S/BF) has N isolated vacua (in leading order in large N) characterised by their chiral condensate in \mathbb{R}^4 . In such cases, one can add a small mass term for fermions and that will lift the degeneracy completely leading to unique vacuum for arbitrary θ angle (except for $\theta = \pi$). The large N cluster decomposition will be valid at any value of the mass. Analogous operation on $\mathbb{R}^3 \times (\text{thermal } S^1)$ will pick a single thermal equilibrium state. Both operations, in essence, restricts the Hilbert space of the corresponding theories over a particular ground (or thermal equilibrium) state, hence consequently the cluster decomposition property is restored. The addition of these small perturbations (which explicitly breaks chiral and center symmetry) also changes the phase transitions of the corresponding $N = \infty$ SYM, QCD(AS/S/BF) into rapid crossovers.

For example, QCD(AS/S/BF) are non-supersymmetric and do not have fermionic excitations at all. The large N equivalence do not apply to nonneutral symmetry channels. The presence of the nonneutral sector, and its dynamics do not alter the dynamics of the neutral sector so long as the symmetries defining the neutral sectors are unbroken. If the symmetries are unbroken, then the ground (and thermal equilibrium) state of the corresponding theories are in the neutral sector. Otherwise, they are outside the neutral sector, and there is no equivalence.

8. Prospects

The phase diagrams of QCD-like gauge theories with fermions in two index representations QCD(adj/AS/S/BF) shown in Figs.1, 3, 5 and understanding their interrelation via the non-perturbative orbifold and orientifold equivalences are the main results of this work. As stated in the introduction, the nonperturbative large N orbifold and orientifold equivalences are only valid if certain symmetry realizations are satisfied. In our examples, this translates into unbroken charge conjugation symmetry \mathcal{C} in $U(N)$ QCD(AS/S) and unbroken $\mathbb{Z}_2 = \mathcal{I}$ interchange symmetry in $U(N) \times U(N)$ QCD(BF) [12,15]. The phase diagrams are in part derived within the regime of applicability of perturbative analysis thanks to the work of Refs. [4,5] on $S^3 \times S^1$, and in part borrowed from lattice gauge theory in the strongly coupled regime in cases where data exists [2,13,39]. All of our findings support the smooth volume dependence conjecture for (spatial and temporal) center symmetry realizations: *In asymptotically free, confining vectorlike large N gauge theories, if a weak coupling phase transition changing the (spatial or temporal) center symmetry realization exists at small volume on $S^3 \times S^1$, it will evolve into a full-blown nonperturbative (strong coupling) phase transition in $\mathbb{R}^3 \times S^1$. Inversely, if we do not find any such transition on small $S^3 \times S^1$, there will not be a center symmetry changing transition on $\mathbb{R}^3 \times S^1$ either.* Consequently, our phase diagrams merely reflect the simplest possibilities consistent with our current knowledge of vectorlike gauge theories and in that regard, should be viewed as conjectural rather than fully demonstrated.

In particular, we were unable to demonstrate that \mathcal{C} in the orientifold QCD(AS/S), and \mathcal{I} in the case of orbifold QCD(BF) cannot be spontaneously broken in the large S^1 limit on $\mathbb{R}^3 \times S^1$. However, we were able to show that if S^1 is a thermal (or temporal) circle (endowed with antiperiodic boundary conditions for fermions), neither \mathcal{I} nor \mathcal{C} are broken at small radius. On the other hand, if S^1 is a spatial circle (where periodic boundary conditions are used for fermions), then both \mathcal{I} and \mathcal{C} are broken on small radius. Therefore, this symmetry breaking is sensitive to the choice of the boundary conditions for fermions along the S^1 circle and is clearly a finite size effect, albeit physical as the transition is expected to occur around the strong scale Λ^{-1} . In such cases, we chose the simplest consistent possibility at large radius, i.e, unbroken \mathcal{I} and \mathcal{C} .²⁴ We were also able to show that the effective potentials

²⁴Currently, there is no proof demonstrating \mathcal{I} in orbifold QCD(BF) and \mathcal{C} in QCD(AS/S) cannot be spontaneously broken on \mathbb{R}^4 or $\mathbb{R}^3 \times$ (large S^1). The essential obstacle for the proof of such argument is the existence of “real local” order parameters probing \mathcal{C} and \mathcal{I} , unlike the case of parity [46] where all local order parameters are purely “imaginary” in Euclidean formulation in \mathbb{R}^4 . We find the spontaneous breaking of \mathcal{C} and \mathcal{I} on large radius strongly unlikely and everything that we know about these theories is consistent with

for QCD(AS/S) and QCD(BF) on $S^3 \times S^1$ map to identical problems Eq.4.6 and Eq.5.5 in the respective \mathcal{C} and \mathcal{I} eigenstates of the Wilson lines. This demonstrates that within the regime of applicability of perturbation theory, all \mathcal{C} and \mathcal{I} symmetry realizations of orientifold QCD(AS/S) and orbifold QCD(BF) are identical, hopefully altering unjustified prejudice in the literature about orbifold QCD.

We have also seen some remarkable new phenomena. Some of these are artifacts of the topology of the sphere such as low temperature confining phases without chiral symmetry breaking in QCD(adj/AS/S/BF) in the small S^3 limit. It is likely that this phase is not visible in lattice simulations traditionally formulated on T^4 . On the other hand, we have also seen zero temperature chirally symmetric phases of QCD(AS/S) on $\mathbb{R}^3 \times S^1$ (where one of the \mathbb{R} is the decompactified temperature circle) which also breaks symmetries such as CPT and P. The existence of this phase for $N = 3, n_f = 4$ QCD has been demonstrated in recent lattice simulations in ref. [13, 47]. It should be emphasised that this phase transition is *not* due to thermal fluctuations, since on $\mathbb{R}^3 \times (\text{spatial } S^1)$ the temperature is zero, and rather quantum mechanical fluctuations trigger an instability. In general, the center symmetry changing phase transitions probed by working with the twisted partition function $\tilde{Z} = \text{tr}[(-1)^F e^{-\beta H}]$ can be thought of as the transitions induced by quantum fluctuations, and the transitions monitored by working with the thermal partition function $Z = \text{tr} e^{-\beta H}$ are typically induced by thermal fluctuations.

To our knowledge, there is no analysis of phase diagrams of large N , asymptotically free, confining QCD-like gauge theories formulated on $S^3 \times S^1$ as a function of volume in the string theory literature. This is true for $\mathcal{N} = 1$ SYM, as well. Realizing that the phase diagrams of $\mathcal{N} = 1$ SYM shown in Fig.1 (both thermal and nonthermal) are as rich as any other QCD-like theory, it may be worth pursuing this further. The most interesting aspect arises in the $\mathbb{R}^3 \times (\text{spatial}) S^1$ limit. In particular, we observe that for QCD-like theories with complex representation Dirac fermions, a change in spatial center symmetry and chiral symmetry realizations should occur at some critical spatial radius (not temperature) [12, 13]. On the other hand, for real representation fermions endowed with periodic boundary conditions, a spatial center symmetry changing transition does not occur. (For adjoint representation fermions and gauge groups $SU/SO/Sp$, this is demonstrated in [31] and used to demonstrate that in such QCD-like theories the dynamics is independent of the S^1 volume.) It would be interesting to see how such a phenomenon might arise in the stringy setups of Refs. [60–62].

Finally, the phase diagram of conformal $\mathcal{N} = 4$ SYM on $S^3 \times S^1$ is currently known [4, 5] at least at weak coupling and presumably at strong coupling due to AdS/CFT. Examining the phases of the simplest orbifold and orientifold partners of $\mathcal{N} = 4$ SYM where gauge-gravity duality is probably better understood (in comparison to the asymptotically free, confining gauge theories) may help us to find new gravitational phases, as well as phase transitions among different geometries. Research in this direction is ongoing.

unbroken \mathcal{C} and \mathcal{I} . [This last viewpoint is also emphasised for \mathcal{C} without providing a proof in [58, 59]. My intuition is that, the two problems are intimately connected. If it is possible to prove one rigorously, the other will follow.] We should also note that in theories involving fundamental scalars and formulated on \mathbb{R}^4 , it is fairly easy to construct theories which spontaneously break both \mathcal{C} and \mathcal{I} via local order parameters.

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A. Independence of dynamics from the boundary conditions on $\mathbb{R}^3 \times (\text{large } S^1)$

In the paper, we made an assertion that the $N = \infty$ dynamics of the QCD-like theories must be independent of the boundary condition of fermions on $\mathbb{R}^3 \times S^1$ so long as S^1 is sufficiently large. This is an immediate corollary to the volume independence of large N gauge theories [31]. Below, we give a sketch of the proof. In order to make our arguments well defined, we use a lattice regularization. The underling assumptions of the proof are *i*) a smooth large N limit, *ii*) unbroken center symmetry.

Let us consider a $U(2N)$ QCD(adj) to be specific on $\mathbb{R}^3 \times S^1$ where the lattice along S^1 has size $2L$. Assume the fermions are endowed with periodic boundary condition, $\psi(2L) = \psi(0)$ along the S^1 circle. Then, one can construct a volume reducing \mathbb{Z}_2 projection which is equivalent to imposing the constraints: $\psi(x+L) = -\psi(x)$ for fermions and $U(x+L) = U(x)$ for link fields. The outcome is $U(2N)$ QCD(adj) on S^1 with size L . The validity of this large N equivalence relies on translation symmetry and fermion number symmetry in large volume theory (which are unbroken) and center symmetry in the daughter, small volume theory. One can also “undo” the volume reduction, by a blow-up projection, which takes $U(2N)$ gauge theory on size L lattice to a $U(N)$ gauge theory on size $2L$ lattice while preserving the antiperiodic boundary conditions for fermions. (For details, see [31].) Now, small volume theory is regarded as a parent and large volume theory as the daughter. The validity of the equivalence again relies on identical symmetry realizations.

The first \mathbb{Z}_2 projection lost half of the degrees of freedoms, so did the second \mathbb{Z}_2 .

$$[U(2N), 2L] \xrightarrow{\mathbb{Z}_2} [U(2N), L] \xrightarrow{\mathbb{Z}_2} [U(N), 2L] \quad (\text{A.1})$$

Assuming the 't Hooft large N limit is smooth, the net effect of the two projection is to change the periodic boundary conditions of fermions to antiperiodic ones, while reducing the number of color by a factor of four.

The condition of the unbroken center symmetry in the small volume theory is equivalent to the requirement that the theory be in a low temperature confining phase. Therefore, so long as the temperature is lower than the deconfinement temperature, the connected correlators and expectation values are independent of the boundary conditions along the S^1 circle. Combining this boundary value independence with volume independence, we observe that in the large N limit, the spectrum of particles, chiral condensates, gluon condensate in the QCD-like theories (and in particular, in $\mathcal{N} = 1$ SYM) should be temperature, and boundary value independent.

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