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Irregular blocks, $\mathcal{N} = 2$ gauge theory and Mathieu system

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Abstract. The Alday-Gaiotto-Tachikawa (AGT) conjecture relates 4d $\mathcal{N} = 2$, $SU(2)$ SYM theories with N_f matter hypermultiplets to 2d CFT. In case of pure 4d $\mathcal{N} = 2$, $SU(2)$ SYM there is a corresponding irregular conformal block in 2d CFT. The AGT correspondence may be extended within a certain limit (the Nekrasov-Shatashvili limit) to the correspondence between an effective twisted superpotentials of 2d $\mathcal{N} = 2$ SUSY and the Zamolodchikov's "classical" conformal blocks. When narrowed to the pure 4d $\mathcal{N} = 2$ SYM case its limit is related to an irregular classical conformal block. It will be shown that according to the triple correspondence (2dCFT/Gauge/Bethe - c.f. Piatek's talk) the irregular classical conformal block yields spectrum of Mathieu operator. The latter can be obtained as a "classical" limit of the null vector decoupling equation for three-point degenerate irregular block. It will also be shown that the Mathieu spectrum can be also obtained from the limit of the pure gauge theory as a solution of the saddle point equation as well as from the Bohr-Sommerfeld quantization of the Seiberg-Witten theory.

1. Introduction

The Mathieu equation and its solution proved useful in many fields of physics ranging from solid state physics to astrophysics and cosmology. It appeared as a result of study of the elliptic membrane oscillations [1]. The equation that describes mathematically this process is a second order elliptic partial differential equation which splits into two independent ordinary differential equations termed the Mathieu and the modified Mathieu equation both with real coefficients. For definiteness let us focus on the former, which assumes the form

$$\left(-\frac{d^2}{dx^2} + 2h^2 \cos 2x\right) \psi(x) = \lambda \psi(x). \quad (1)$$

λ and h are parameters of the equation to whose we further refer as the spectrum of Mathieu operator and the coupling constant respectively. The solution of eq. (1) has the following property

$$\psi(x + \pi) = e^{i\pi\nu} \psi(x).$$

The parameter ν is termed Floque exponent. It may assume integer and non-integer values. In what follows we are concerned with the non-integer one. For $h \ll 1$, i.e., in the weak coupling



regime the spectrum of Mathieu operator λ can be developed in terms of h and ν , namely

$$\lambda = \nu^2 + \frac{h^4}{2(\nu^2 - 1)} + \frac{(5\nu^2 + 7)h^8}{32(\nu^2 - 4)(\nu^2 - 1)^3} + \frac{(9\nu^4 + 58\nu^2 + 29)h^{12}}{64(\nu^2 - 9)(\nu^2 - 4)(\nu^2 - 1)^5} + \mathcal{O}(h^{16}). \quad (2)$$

The Mathieu equation also emerged in the context of recent studies of interrelationship between Quantum Integrable Systems (QIS), 2d Conformal Field Theories (2d CFT) and 4d $\mathcal{N} = 2$ super-Young-Mills field theories (SYM) [2]. As it turned out the Nekrasov partition function for 4d $\mathcal{N} = 2$, $SU(N)$ SYM that depend on two regularization parameters ϵ_1, ϵ_2 [3], when the two parameters tend to zero $\epsilon_1, \epsilon_2 \rightarrow 0$, approaches the exponential with an exponent having a divergent part proportional to a prepotential (for $N = 2$ this is the Seiberg-Witten prepotential [4]). The latter is uniquely determined by N moduli and their duals as well as a certain elliptic curve which is topologically equivalent to a genus $N - 1$ Riemann surface [5]. The moduli are related to $N - 1$ periods of the elliptic curve. The latter, in turn, were revealed to be related to a classical integrable system, namely, N -particle periodic Toda chain [6, 7]. Its quantization corresponds to taking the zero limit in the Nekrasov partition function only in one parameter ϵ_2 , called *the Nekrasov-Shatashvili limit* (NS limit). The divergent term in the exponent of thus obtained expression is proportional to the twisted superpotential $\mathcal{W}$ for 2d SYM. This twisted superpotential as proposed in ref. [2] plays a role of the Young-Young function, which is a statement of *Bethe gauge correspondence*. This can be summarized by the following diagram:

$$\mathcal{Z}(\hat{\Lambda}, \mathbf{a}, \epsilon_1, \epsilon_2) \xrightarrow[\text{NS limit}]{\epsilon_2 \rightarrow 0} \exp \left\{ \frac{1}{\epsilon_2} \mathcal{W}(\hat{\Lambda}/\epsilon_1, \mathbf{a}) \right\} \xrightarrow{(\epsilon_1, \epsilon_2) \rightarrow (0, 0)} \exp \left\{ \frac{1}{\epsilon_1 \epsilon_2} \mathcal{F}_{\text{SW}}(\hat{\Lambda}, \mathbf{a}) \right\}. \quad (3)$$

In particular, for $N = 2$, a case we are the most concerned with in what follows, a moduli and its dual are determined by two periods i.e., $2\pi a = \Pi(A)$, $2\pi a_D = \Pi(B)$ that take the form of the action of the classical sine-Gordon model or 2-particle periodic Toda chain

$$\Pi(\Gamma) \equiv \oint_{\Gamma} P_0(\varphi) d\varphi = \oint_{\Gamma} \sqrt{2(u - \hat{\Lambda}^2 \cos \varphi)} d\varphi, \quad \Gamma = A, B.$$

Its quantization leads to the Schrödinger equation for the sine-Gordon/2p Toda model

$$\left(-\frac{\epsilon_1^2}{2} \frac{d^2}{d\varphi^2} + \hat{\Lambda}^2 \cos \varphi \right) \psi(\varphi) = u \psi(\varphi).$$

WKB solution to the above equation [8] provides a spectrum which coincides the one in eq. (1) (see also appendix C of [9] for direct computations)

$$\Pi(\Gamma) \xrightarrow{\epsilon_1 = \hbar'} \tilde{\Pi}(\Gamma) \equiv \oint_{\Gamma} P(\varphi, \epsilon_1) d\varphi, \quad \psi_{\text{WKB}}(\varphi) = \exp \left\{ \frac{i}{\epsilon_1} \int^{\varphi} P(\rho, \epsilon_1) d\rho \right\}.$$

It also corresponds to the quantization of moduli $\mathbf{a}(a)$, which entails the quantization of Seiberg-Witten curves [10]. The twisted superpotential from middle exponent in eq. (3) is found due to the duality relation $a_D(a) = (\partial \mathcal{W} / \partial a)(a)$.

The spectrum of the Mathieu operator can be also found from the deformed critical colored Young diagram [10, 9]. It has been shown that the Nekrasov partition function for $N_f \geq 0$ $\mathcal{N} = 2$ $SU(N)$ SYM can be rewritten as a sum over profiles $f_{\mathbf{a}, \mathbf{k}}(x|\epsilon_1, \epsilon_2)$ of the deformed, colored Young diagrams [11]. For the case in question $N_f = 0$, $SU(2)$ it takes the form

$$\mathcal{Z}(\hat{\Lambda}, \mathbf{a}, \epsilon_1, \epsilon_2) = \sum_{f_{\mathbf{a}, \mathbf{k}}} \exp \left\{ -\frac{1}{4} \iint dx dy f''_{\mathbf{a}, \mathbf{k}}(x|\epsilon_1, \epsilon_2) \gamma_{\epsilon_1, \epsilon_2}(x - y; \hat{\Lambda}) f''_{\mathbf{a}, \mathbf{k}}(y|\epsilon_1, \epsilon_2) \right\}, \quad (4)$$

where $\mathbf{k} \equiv \{k_{\alpha,i}\}$, $\alpha = 1, 2$; $i \in \mathbb{N}$ and $i > j \Rightarrow k_{\alpha,j} \geq k_{\alpha,i} \geq 0$ are colored partitions (Young diagrams) with the number of boxes $|\mathbf{k}| = \sum k_{\alpha,i}$. Making use of the argument in ref. [11] within the NS limit $\epsilon_2 \rightarrow 0$, $\mathbf{k} \rightarrow \infty$ and $\epsilon_2 \mathbf{k} = \boldsymbol{\omega} < \infty$ such that the entire sum in eq. (4) can be approximated by the path integral over sequences of nonpositive real numbers $\omega_{\alpha,i}$ [9]

$$\mathcal{Z}_{\text{inst}}(\Lambda, \mathbf{a}, \epsilon_1, \epsilon_2) \sim \int_{\mathbf{c}_0(\mathbb{R}_{\leq 0})^2} \prod_{\alpha,i} d\omega_{\alpha,i} \exp \left\{ -\frac{1}{\epsilon_2} \mathcal{H}[\rho_{\mathbf{a},\boldsymbol{\omega}}] \right\}, \quad \rho_{\mathbf{a},\mathbf{k}} := f_{\mathbf{a},\mathbf{k}} - f_{\mathbf{a},\emptyset}.$$

where $\mathbf{c}_0(\mathbb{R}_{\leq 0})^2$ is a space of such a sequences. This path integral is dominated by the term that fulfills the saddle point condition

$$(\delta \mathcal{H}[\rho_{\mathbf{a},\boldsymbol{\omega}_*}] / \delta \rho_{\mathbf{a},\boldsymbol{\omega}}^i) \delta \rho_{\mathbf{a},\boldsymbol{\omega}}^i = 0 \quad \Rightarrow \quad \hat{\Lambda}^4 \prod_{\beta=1}^2 \prod_{j \geq 1} \frac{x_{\alpha,i} - x_{\beta,j} - \epsilon_1}{x_{\alpha,i} - x_{\beta,j} + \epsilon_1} = -1,$$

where $x_{\alpha,i} := a_{\alpha} + \epsilon_1(i-1) + \omega_{\alpha,i}$, $\omega_{\alpha,i} = k_{\alpha,i}\epsilon_2$. The iterative solution to the above saddle point equation [10] yields columns $\omega_{\alpha,i}^*$ of the critical Young diagram $\boldsymbol{\omega}_*$. The twisted $SU(2)$ superpotential in terms of columns of the critical colored Young diagram reads

$$\mathcal{W}_{\text{inst}}(\hat{\Lambda}, \mathbf{a}, \epsilon_1) = \mathcal{H}[\rho_{\mathbf{a},\boldsymbol{\omega}_*}](\hat{\Lambda}, \epsilon_1) = \sum_{i \geq 1} (\hat{\Lambda}/\epsilon_1)^{4i} W_i(a, \epsilon_1)/i, \quad W_i = - \sum_{\alpha=1}^2 \sum_{j=1}^i \omega_{\alpha,j,i}^*.$$

By means of the Bethe/gauge correspondence

$$\mathcal{E}_k = \langle \text{tr} \phi^k \rangle_{\mathbf{a}}|_{\epsilon_2=0} = \frac{1}{2} \int dx x^k f''_{\mathbf{a},\boldsymbol{\omega}_*}(x|\epsilon_1),$$

one finds the spectrum for the two particle periodic Toda chain

$$\mathcal{E}_1 = 0, \quad E = 2\mathcal{E}_2 = \epsilon_1 \hat{\Lambda} \partial_{\hat{\Lambda}} \mathcal{W}(\hat{\Lambda}, \mathbf{a}, \epsilon_1) = \epsilon_1^2 \lambda,$$

with λ being a spectrum of Mathieu operator given in eq. (2).

2. Nonconformal limit of AGT relation and Mathieu equation

Alday, Gaiotto and Tachikawa have discovered the correspondence between fourdimensional $\mathcal{N} = 2, N_f = 4$ $SU(2)$ field theories and the twodimensional Liouville field theory [12]. This henceforth called AGT correspondence enabled Gaiotto the discovery of entirely new objects within the twodimensional Conformal Field Theory (2d CFT) [13]. These new states – the Gaiotto states – turned out to be the non-conformal limit of the AGT correspondence [14]. The norm of the Gaiotto state with only one parameter Λ corresponds to the Nekrasov partition function for pure gauge ($N_f = 0$) $\mathcal{N} = 2$ SYM with $SU(2)$ symmetry.

2.1. Quantum irregular conformal block

Let us consider the four-point conformal block on twodimensional Riemann sphere.¹ It takes the form

$$\mathcal{F}_{c,\Delta} \left[\begin{smallmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix} \right] (x) = x^{\Delta - \Delta_2 - \Delta_1} \tilde{\mathcal{F}}_{c,\Delta} \left[\begin{smallmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix} \right] (x),$$

¹ The extended exposition of the reasoning in this subsection can be found in ref. [14].

where Δ_i, Δ for $i = 1, \dots, 4$ are four external and one intermediate weight respectively. c denotes the central charge of the Virasoro algebra $\mathfrak{Vir}$. The 4pt conformal block has the following expansion in terms of location x not fixed by $SL(2, \mathbb{C})$ symmetry

$$\tilde{\mathcal{F}}_{c,\Delta} \left[\begin{smallmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix} \right] (x) := 1 + \sum_{n \geq 1} x^n \mathcal{F}_{c,\Delta}^{(n)} \left[\begin{smallmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix} \right], \quad \mathcal{F}_{c,\Delta}^{(n)} \left[\begin{smallmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix} \right] := \sum_{I, J \vdash n} \gamma_\Delta \left[\begin{smallmatrix} \Delta_3 \\ \Delta_4 \end{smallmatrix} \right]_I \left[G_{c,\Delta} \right]^{IJ} \gamma_\Delta \left[\begin{smallmatrix} \Delta_2 \\ \Delta_1 \end{smallmatrix} \right]_J,$$

where $I, J \vdash n$ i.e., are partitions of n . $G_{c,\Delta}$ denotes the Gram matrix of scalar products between vectors in the highest weight representation of Virasoro algebra of generators $L_i \in \mathfrak{Vir}$, $L_{-i} = L_i^\dagger$, $L_I \equiv L_{k_1} \cdots L_{k_{\ell(I)}}$, $\mathcal{V}_{c,\Delta}^{(n)} = \text{span}\{L_{-I}|\Delta\rangle; I \vdash n\}$

$$\left[G_{c,\Delta}^{(n)} \right]_{IJ} := \langle \Delta | L_I L_{-J} | \Delta \rangle, \quad \left[G_{c,\Delta}^{(n)} \right]_{IK} \left[G_{c,\Delta}^{(n)} \right]^{KJ} = \delta_K^J. \quad (5)$$

The object γ_I – the vertex vector – takes the form

$$\gamma_\Delta \left[\begin{smallmatrix} \Delta_a \\ \Delta_b \end{smallmatrix} \right]_I \equiv \prod_{i=1}^{\ell(I)} \left(\Delta + k_i \Delta_b - \Delta_a + \sum_{i < j} k_j \right), \quad I = (k_1, \dots, k_{\ell(I)}, 0, \dots). \quad (6)$$

According to AGT correspondence the relationship between the parameters of the two theories for conformal weights parametrized as

$$\Delta_i = (Q^2 + p_i^2)/4, \quad \Delta = (Q^2 + p^2)/4 \quad \text{with} \quad c = 1 + 6Q^2, \quad Q = b + b^{-1} \quad (7)$$

reads

$$p_1 \stackrel{\text{AGT}}{=} \mu_1 - \mu_2, \quad p_2 \stackrel{\text{AGT}}{=} \epsilon_1 + \epsilon_2 - (\mu_1 + \mu_2)/2, \quad p_3 \stackrel{\text{AGT}}{=} \epsilon_1 + \epsilon_2 - (\mu_3 + \mu_4)/2, \quad p_4 \stackrel{\text{AGT}}{=} \mu_3 - \mu_4, \\ p \stackrel{\text{AGT}}{=} \sqrt{-1}a/(\epsilon_1 \epsilon_2), \quad b \stackrel{\text{AGT}}{=} \sqrt{-\epsilon_2/\epsilon_1}.$$

Let us consider the so-called *mass decoupling limit*. This is the case when all the four masses related to the four external weights by the above relation tend to infinity $\mu_i \rightarrow \infty \Rightarrow \Delta_i \rightarrow \infty$, $\Delta < \infty$ while $x\mu_1\mu_2\mu_3\mu_4 = \hat{\Lambda}^4$. In this case one obtains

$$\gamma_\Delta \left[\begin{smallmatrix} \Delta_a \\ \Delta_b \end{smallmatrix} \right]_I \stackrel{\mu_a, \mu_b \gg 1}{\sim} \left(-\frac{\mu_a \mu_b}{\epsilon_1 \epsilon_2} \right)^{\sum m_i} \leq \left(-\frac{\mu_a \mu_b}{\epsilon_1 \epsilon_2} \right)^{|I|} \delta_{I, (1^{|I|})},$$

where $I = \{k_i\}_{i \in \mathbb{N}} = \{i^{m_i}\}_{i \in \mathbb{N}}$, $I \vdash n$ and m_i is a multiplicity of i in partition I . Noting, that $|I| = n = \sum k_i = \sum i m_i$ and that $\ell(I) = \sum m_i \leq \sum i m_i = |I|$ we conclude that the above inequality is saturated provided $\ell(I) = \ell_{\max}(I) = |I|$. That is, within this limit only those vertex vectors contribute that have maximal lengths. The partition with maximal length is $(1^{|I|})$. Within the mass decoupling limit 4pt conformal block takes the form

$$\tilde{\mathcal{F}}_{c,\Delta} \left[\begin{smallmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix} \right] (x) \xrightarrow[x\mu_1\mu_2\mu_3\mu_4 = \hat{\Lambda}^4]{\mu_1, \mu_2, \mu_3, \mu_4 \rightarrow \infty} \mathcal{F}_{c,\Delta}(\Lambda) := \sum_{n \geq 0} \Lambda^{4n} [G_{c,\Delta}]^{(1^n)(1^n)}, \quad \Lambda = \frac{\hat{\Lambda}}{\epsilon_1 \epsilon_2}. \quad (8)$$

$\mathcal{F}_{c,\Delta}(\Lambda)$ is *irregular pure gauge conformal block*. This conformal block can be built out of the Gaiotto states that are defined as

$$\mathcal{F}_{c,\Delta}(\Lambda) = \langle \Delta, \Lambda^2 | \Delta, \Lambda^2 \rangle, \quad |\Delta, \Lambda^2 \rangle := \sum_{n \geq 0} \Lambda^{2n} \sum_{|I|=n} [G_{c,\Delta}]^{(1^n)I} L_{-I} |\Delta \rangle. \quad (9)$$

The action of the generators that form $\mathfrak{Vir}$ on the Gaiotto state is

$$L_0 |\Delta, \Lambda^2 \rangle = (\Delta + (\Lambda/2) \partial_\Lambda) |\Delta, \Lambda^2 \rangle, \quad L_1 |\Delta, \Lambda^2 \rangle = \Lambda^2 |\Delta, \Lambda^2 \rangle, \quad L_n |\Delta, \Lambda^2 \rangle = 0, \quad n \geq 2.$$

The nonconformal limit of AGT relation relates, therefore, the pure gauge $4d \mathcal{N} = 2$ SYM with $SU(2)$ symmetry to the irregular pure gauge conformal block within $2d$ CFT.

2.2. Classical irregular conformal block

The classical limit within the 2d CFT Liouville theory consists of taking the limit $b \rightarrow 0$ in the correlation function or the quantum conformal block in which it can be decomposed. The dependence of 4pt conformal block on b is clear in parametrization (7). Within this limit

$$\Delta, c \rightarrow \infty, \quad \Delta/c = \text{const.} \quad \Rightarrow \quad \Delta \sim \delta b^{-2}, \quad c \sim 6b^{-2}. \quad (10)$$

The irregular conformal block within this limit is expected to behave as

$$\mathcal{F}_{c,\Delta}(\Lambda) = \sum_{n \geq 0} \left(\frac{\hat{\Lambda}}{\epsilon_1 b} \right)^{4n} [G_{c,\Delta}]^{(1^n)(1^n)} \xrightarrow{b \rightarrow 0} \exp \left\{ \frac{1}{b^2} f_\delta \left(\frac{\hat{\Lambda}}{\epsilon_1} \right) \right\}, \quad (11)$$

where $f_\delta(\hat{\Lambda}/\epsilon_1)$ is the classical pure gauge irregular block. This behavior is further corroborated by the leading order analysis in b^{-1} performed in [15]. Making use of the formula

$$f_\delta \left(\frac{\hat{\Lambda}}{\epsilon_1} \right) = \lim_{b \rightarrow 0} b^2 \log \mathcal{F}_{c,\Delta} \left(\frac{\hat{\Lambda}}{\epsilon_1 b} \right) = \sum_{n \geq 1} \left(\frac{\hat{\Lambda}}{\epsilon_1} \right)^{4n} f_\delta^{(n)},$$

and our *Mathematica* code we were able to find up to 7 of its coefficients five of which we present below

$$\begin{aligned} f_\delta^{(1)} &= \frac{1}{2\delta}, & f_\delta^{(2)} &= \frac{5\delta - 3}{16\delta^3(4\delta + 3)}, & f_\delta^{(3)} &= \frac{9\delta^2 - 19\delta + 6}{48\delta^5(4\delta + 3)(\delta + 2)}, \\ f_\delta^{(4)} &= \frac{5876\delta^5 - 16489\delta^4 - 22272\delta^3 + 17955\delta^2 + 9045\delta - 4050}{512\delta^7(\delta + 2)(4\delta + 3)^3(4\delta + 15)}, \\ f_\delta^{(5)} &= \frac{17884\delta^6 - 96187\delta^5 - 156432\delta^4 + 388737\delta^3 - 7317\delta^2 - 138348\delta + 34020}{1280\delta^9(\delta + 2)(\delta + 6)(4\delta + 3)^3(4\delta + 15)}, \quad \dots \end{aligned} \quad (12)$$

It was also possible to establish the relation between the twisted superpotential and the classical irregular block which reads ($\epsilon_2 \rightarrow 0, b \equiv \sqrt{-\epsilon_2/\epsilon_1} \Rightarrow b \rightarrow 0$)

$$f_\delta(\hat{\Lambda}/\epsilon_1) = \frac{1}{\epsilon_1} \mathcal{W}_{\text{inst}}(\hat{\Lambda}, a, \epsilon_1), \quad \frac{1}{n\epsilon_1} W_n(a, \epsilon_1) = -f_\delta^{(n)} \quad \text{for} \quad \delta = \frac{1}{4} - \left(\frac{a}{\epsilon_1} \right)^2.$$

3. Mathieu equation from classical limit of null vector decoupling equation

It is possible to obtain the Mathieu equation directly from the 2d CFT without any reference to the AGT relation. It should be stressed that this is possible due to the discovery of Gaiotto states within 2d CFT. The Mathieu equation can be obtained by computing the matrix element of a chiral null field built out of a vertex operator with degenerated weight and inserted between the two different Gaiotto states and using the null vector decoupling condition. However, the null vector decoupling condition can be applied to the physical correlation functions with vertex operators having weights from the Kac table, while we use matrix element of the null field between the Gaiotto states which is purely chiral object. There is a variant of the null vector decoupling theorem that applies to the matrix elements with null chiral vertex operator, namely, the Feigin-Fuchs theorem

Theorem 1 (Feigin-Fuchs). *Let $i, j, k \in \{1, 2, 3\}$ be chosen such that $j \neq i, k \neq i, j \neq k$. Let us assume that*

$$(i) \quad \Delta_i = \Delta_{r,s} := \frac{Q^2}{4} - \frac{1}{4} (rb + sb^{-1})^2, \quad r, s \in \mathbb{N};$$

(ii) the vector $|\xi_i\rangle$ lies in the singular submodule generated by the null vector $|\chi_{rs}\rangle$, i.e.:
 $|\xi_i\rangle \in \mathcal{V}_{c,\Delta_{rs}(c)+rs} \subset \mathcal{V}_{c,\Delta_{r,s}(c)}$.

Then,

$$\langle \xi_1 | V_{z_3, z_2, z_1}^{\Delta_3, \Delta_2, \Delta_1}(\xi_2 | z_2) | \xi_3 \rangle = 0,$$

iff $\Delta_j = \Delta(\sqrt{-1}\beta_j) := \frac{Q^2}{4} - \frac{1}{4}\beta_j^2$ and $\Delta_k = \Delta(\sqrt{-1}\beta_k) := \frac{Q^2}{4} - \frac{1}{4}\beta_k^2$ satisfy the fusion rules

$$\beta_j - \beta_k = pb + qb^{-1},$$

where $p \in \{1-r, 3-r, \dots, r-1\}$ and $q \in \{1-s, 3-s, \dots, s-1\}$.

Thus, the above theorem says that the matrix element with the null field insertion vanishes provided the weights of states from domain and target space of the null vertex operator fulfill appropriate fusion rules.

3.1. Null vector decoupling equation

Having Feigin-Fuchs theorem at hand we can proceed to the derivation of the Mathieu equation through the following construction. First, let us choose the vertex operator to have a weight $\Delta_{r,s}$ from the Kac table (see item (i) of Feigin-Fuchs theorem 1 for definition of $\Delta_{r,s}$ and eq. (7) for definition of Q) at the second level i.e., $rs = 2$. There are two weights from which we take

$$\Delta_+ \equiv \Delta_{2,1} = -\frac{1}{2} - \frac{3}{4}b^2.$$

The reason for this choice is that this weight is ‘light’, that is, in the classical limit $b \rightarrow 0$ it becomes constant (it is opposite to the ‘heavy’ weight $\Delta_{1,2}$ which diverges to infinity within this limit). It is expected that in this limit the light field insertions in the physical correlation function factorize from the heavy ones. The same should be true for the matrix elements with light field insertions. Hence, the degenerate primary chiral vertex operator takes the form

$$V_{\Delta_+}(z) \equiv V_+(z), \quad V_+(z) := V_{\infty}^{\Delta'} \Delta_+ \tilde{\Delta}_0 (|\Delta_+\rangle \otimes \cdot) : \mathcal{V}_{\tilde{\Delta}} \rightarrow \mathcal{V}_{\Delta'}, \quad |\Delta_+\rangle \in \mathcal{V}_{\Delta_+}.$$

Now we can take the null field as a second level descendant of $V_+(z)$, namely

$$\chi_+(z) = \left(\hat{L}_{-2}(z) - \frac{3}{2(2\Delta_+ + 1)} \hat{L}_{-1}^2(z) \right) V_+(z), \quad \hat{L}_{-k}(z) := \frac{1}{2\pi i} \oint_{C_z} dw (w-z)^{1-k} T(w).$$

Moreover, we take the weights $\Delta', \tilde{\Delta}$ to fulfill the following fusion rules

$$\Delta(\sigma) := \frac{Q^2}{4} - \sigma^2 \quad \Rightarrow \quad \tilde{\Delta} = \Delta\left(\sigma - \frac{b}{4}\right), \quad \Delta' = \Delta\left(\sigma + \frac{b}{4}\right).$$

Hence, according to the Feigin-Fuchs theorem 1 we obtain the null vector decoupling equation

$$\langle \Delta', \Lambda^2 | \chi_+(z) | \tilde{\Delta}, \Lambda^2 \rangle = \langle \Delta', \Lambda^2 | \hat{L}_{-2}(z) V_+(z) | \tilde{\Delta}, \Lambda^2 \rangle + \frac{1}{b^2} \langle \Delta', \Lambda^2 | \hat{L}_{-1}^2(z) V_+(z) | \tilde{\Delta}, \Lambda^2 \rangle = 0,$$

which, after the use of Conformal Ward Identities assumes the form

$$\left[\frac{1}{b^2} z^2 \frac{\partial^2}{\partial z^2} - \frac{3z}{2} \frac{\partial}{\partial z} + \Lambda^2 \left(z + \frac{1}{z} \right) + \frac{\Lambda}{4} \frac{\partial}{\partial \Lambda} + \frac{\tilde{\Delta} + \Delta' - \Delta_+}{2} \right] \Psi(\Lambda, z) = 0, \quad (13a)$$

$$\Psi(\Lambda, z) := \langle \Delta', \Lambda^2 | V_+(z) | \tilde{\Delta}, \Lambda^2 \rangle. \quad (13b)$$

Making use of the definition of the Gaiotto states (9) we can expand the function $\Psi(\Lambda, z)$ and examine its structure with respect to z dependence

$$\begin{aligned}\Psi(\Lambda, z) &= z^{\Delta' - \Delta_+ - \tilde{\Delta}} \sum_{m, n \geq 0} \Lambda^{2(m+n)} z^{m-n} \sum_{\substack{|I|=m \\ |J|=n}} [G_{c, \Delta'}^m]^{(1^m)I} \langle \Delta', I | V_+(1) | \tilde{\Delta}, J \rangle [G_{c, \tilde{\Delta}}^n]^{J(1^n)} \\ &:= z^\kappa \Phi(\Lambda, z), \quad \kappa \equiv \Delta' - \Delta_+ - \tilde{\Delta}.\end{aligned}\quad (14)$$

Let us note that $\Phi(\Lambda, z)$ when separated into the ‘diagonal’ and ‘off-diagonal’ parts in m, n the former does not depend on z leaving the total z dependence to the latter. Indeed, for

$$\Phi(\Lambda, z) = \Phi^{(m=n)}(\Lambda) + \Phi^{(m \neq n)}(\Lambda, z), \quad (15)$$

we obtain

$$\begin{aligned}\Phi^{(m=n)}(\Lambda) &= \sum_{n \geq 0} \Lambda^{4n} \sum_{I \vdash n} [G_{c, \Delta'}^n]^{(1^n)I} \langle \Delta' | L_I V_+(1) L_{-I} | \tilde{\Delta} \rangle [G_{c, \tilde{\Delta}}^n]^{I(1^n)}, \\ \Phi^{(m \neq n)}(\Lambda, z) &= \sum_{\substack{m \neq n \\ m, n \geq 0}} \Lambda^{2(m+n)} z^{m-n} \sum_{\substack{I \vdash m \\ J \vdash n}} [G_{c, \Delta'}^m]^{(1^m)I} \langle \Delta' | L_I V_+(1) L_{-J} | \tilde{\Delta} \rangle [G_{c, \tilde{\Delta}}^n]^{J(1^n)}.\end{aligned}$$

Let us note that $\Phi(\Lambda, z)$ can also be written in the factor form, namely

$$\Psi(\Lambda, z) = z^\kappa \exp \left\{ \log \Phi^{(m=n)}(\Lambda) \right\} \left(1 + \frac{\Phi^{(m \neq n)}(\Lambda, z)}{\Phi^{(m=n)}(\Lambda)} \right) \equiv z^\kappa e^{\phi(\Lambda)} \zeta(\Lambda, z). \quad (16)$$

This observation is crucial for the ‘light’ field factorization phenomenon in the classical limit which we consider below.

3.2. Classical limit of Null Vector Decoupling Equation

The Null Vector Decoupling equation, after we express $\Psi(\Lambda, z)$ in terms of $\Phi(\Lambda, z)$ as in eq. (14), assumes the form (recall that $\Lambda = \hat{\Lambda}/\epsilon_1 b$)

$$\begin{aligned}\left[\frac{1}{b^2} z^2 \partial_z^2 + \left(\frac{2\kappa}{b^2} - \frac{3}{2} \right) z \partial_z + \frac{\hat{\Lambda}}{4} \partial_{\hat{\Lambda}} \right. \\ \left. + \frac{\kappa(\kappa-1)}{b^2} - \frac{3\kappa}{2} + \frac{\hat{\Lambda}^2}{b^2 \epsilon_1^2} \left(z + \frac{1}{z} \right) + \frac{\tilde{\Delta} + \Delta' - \Delta_+}{2} \right] \Phi \left(\frac{\hat{\Lambda}}{\epsilon_1 b}, z \right) = 0.\end{aligned}\quad (17)$$

The parameters in the above equations have the following behavior in the classical limit $b \rightarrow 0$:

$$\begin{aligned}\sigma = \xi/b \quad \Rightarrow \quad \Delta', \tilde{\Delta} \xrightarrow{b \rightarrow 0} \frac{1}{b^2} \delta \quad \tilde{\Delta} + \Delta' - \Delta_+ \xrightarrow{b \rightarrow 0} \frac{1}{b^2} 2\delta, \quad \text{where } \delta = \frac{1}{4} - \xi^2, \\ \kappa \xrightarrow{b \rightarrow 0} \frac{1}{2} - \xi, \quad \kappa(\kappa-1) \xrightarrow{b \rightarrow 0} -\left(\frac{1}{4} - \xi^2 \right) = -\delta \quad \Delta_+ \xrightarrow{b \rightarrow 0} -\frac{1}{2}.\end{aligned}\quad (18)$$

The function Φ also depends on b in its first argument. As we have already mentioned in the beginning of subsection 3.1 it is expected that the physical correlation function that has ‘light’ fields and ‘heavy’ fields, in the classical limit, factorizes into some function that depends on locations of the ‘light’ fields and exponentially divergent reminder of the ‘heavy’ fields. This factorization phenomenon, sometimes termed the “Zamolodchikov’s ansatz”, is believed to carry

over to case of purely chiral objects like the matrix element defining Ψ function in eq. (13b). Taking into account our earlier observation phrased in eq. (16) the chiral counterpart of the Zamolodchikov's ansatz can be formulated as follows

$$\Phi(\Lambda, z) = e^{\phi(\Lambda)} \zeta(\Lambda, z) = z^{-\kappa} \langle \Delta', \Lambda^2 | V_+(z) | \tilde{\Delta}, \Lambda^2 \rangle \xrightarrow{b \rightarrow 0} v(\hat{\Lambda}/\epsilon_1, z) \exp \left\{ \frac{1}{b^2} f_\delta(\hat{\Lambda}/\epsilon_1) \right\}, \quad (19a)$$

where

$$f_\delta(\hat{\Lambda}/\epsilon_1) = \lim_{b \rightarrow 0} b^2 \phi \left(\frac{\hat{\Lambda}}{\epsilon_1 b} \right), \quad v(\hat{\Lambda}/\epsilon_1, z) = \lim_{b \rightarrow 0} \zeta \left(\frac{\hat{\Lambda}}{\epsilon_1 b}, z \right). \quad (19b)$$

The factorization of matrix element in the form of eq. (19a) has been recently proved up to the leading order in b^{-1} in our paper [15]. Using the factorization ansatz (19b) and asymptotic form of parameters given in eq. (18) the Null Vector Decoupling equation (17) takes its classical form independent of b

$$\left[z^2 \partial_z^2 + 2 \left(\frac{1}{2} - \xi \right) z \partial_z + \frac{\hat{\Lambda}^2}{\epsilon_1^2} \left(z + \frac{1}{z} \right) + \frac{\hat{\Lambda}}{4} \partial_{\hat{\Lambda}} f_\delta(\hat{\Lambda}/\epsilon_1) \right] v(\hat{\Lambda}/\epsilon_1, z) = 0.$$

Substituting $v(\hat{\Lambda}/\epsilon_1, z) = z^\xi \psi(\hat{\Lambda}/\epsilon_1, z)$ and subsequently changing variable $z = e^{2ix}$ we arrive at the sought form of classical limit of the Null Vector Decoupling equation, that can be identified with the Mathieu equation (1)

$$\left[-\frac{d^2}{dx^2} + 8 \frac{\hat{\Lambda}^2}{\epsilon_1^2} \cos 2x + \hat{\Lambda} \partial_{\hat{\Lambda}} f_\delta(\hat{\Lambda}/\epsilon_1) - 4\xi^2 \right] \psi \left(\hat{\Lambda}/\epsilon_1, e^{2ix} \right) = 0. \quad (20)$$

This identification becomes clear when we relate the Mathieu parameters with the parameters of the above equation,

$$\lambda = -\hat{\Lambda} \partial_{\hat{\Lambda}} f_\delta \left(\hat{\Lambda}/\epsilon_1 \right) + 4\xi^2, \quad h = \pm \frac{2\hat{\Lambda}}{\epsilon_1}, \quad \xi = \frac{\nu}{2}.$$

Using coefficients of classical irregular pure gauge block expansion found by means of our *Mathematica* code and presented in eq. (12) one finds the coincidence with h expansion of the spectrum of Mathieu operator λ given in eq. (2),

$$\begin{aligned} \lambda &= -\hat{\Lambda} \partial_{\hat{\Lambda}} \left[\sum_{n=1} \left(\hat{\Lambda}/\epsilon_1 \right)^{4n} f_\delta^n \right] + 4\xi^2 \\ &= -\frac{4h^4}{16} f_{\frac{1}{4}-\frac{\nu^2}{4}}^1 - \frac{8h^8}{256} f_{\frac{1}{4}-\frac{\nu^2}{4}}^2 - \frac{12h^{12}}{4096} f_{\frac{1}{4}-\frac{\nu^2}{4}}^3 - \dots + 4 \left(\frac{\nu^2}{4} \right) \\ &= \nu^2 + \frac{h^4}{2(\nu^2-1)} + \frac{(5\nu^2+7)h^8}{32(\nu^2-4)(\nu^2-1)^3} + \frac{(9\nu^4+58\nu^2+29)h^{12}}{64(\nu^2-9)(\nu^2-4)(\nu^2-1)^5} + \dots \end{aligned}$$

3.3. Mathieu functions

The direct computations of the limit

$$v(\hat{\Lambda}/\epsilon_1, z) = \lim_{b \rightarrow 0} \left(1 + \frac{\Phi^{(m \neq n)}(\Lambda, z)}{\Phi^{(m=n)}(\Lambda)} \right), \quad z = e^{2ix},$$

by means of our *Mathematica* code confirmed its finiteness up to $(\hat{\Lambda}/\epsilon_1)^{12}$ order. The result of the computations up to $(\hat{\Lambda}/\epsilon_1)^6$ order reads

$$v(\hat{\Lambda}/\epsilon_1, e^{2ix}) = 1 + \frac{\hat{\Lambda}^2}{\epsilon_1^2} \left(\frac{e^{-2ix}}{2\xi - 1} - \frac{e^{2ix}}{2\xi + 1} \right) + \frac{\hat{\Lambda}^4}{\epsilon_1^4} \left(\frac{e^{4ix}}{4(\xi + 1)(2\xi + 1)} + \frac{e^{-4ix}}{4(\xi - 1)(2\xi - 1)} \right) \\
+ \frac{\hat{\Lambda}^6}{\epsilon_1^6} \left(\frac{e^{-2ix}(4\xi^2 - 8\xi + 7)}{4(\xi - 1)(2\xi - 1)^3(2\xi + 1)} - \frac{e^{2ix}(4\xi^2 + 8\xi + 7)}{4(\xi + 1)(2\xi - 1)(2\xi + 1)^3} \right. \\
\left. + \frac{e^{-6ix}}{12(\xi - 1)(2\xi - 3)(2\xi - 1)} - \frac{e^{6ix}}{12(\xi + 1)(2\xi + 1)(2\xi + 3)} \right) + \mathcal{O}\left(\frac{\hat{\Lambda}^8}{\epsilon_1^8}\right).$$

The relationship between v the Mathieu exponential me_ν is found to be

$$\psi(h/2, z) = e^{i\nu x} v(h/2, z) = \text{me}_\nu(x, h),$$

where

$$\text{me}_\nu(x, h) = e^{i\nu x} + \frac{h^2}{4} \left(\frac{e^{(\nu-2)ix}}{\nu - 1} - \frac{e^{(\nu+2)ix}}{\nu + 1} \right) + \frac{h^4}{32} \left(\frac{e^{(\nu+4)ix}}{(\nu + 1)(\nu + 2)} + \frac{e^{(\nu-4)ix}}{(\nu - 2)(\nu - 1)} \right) \\
+ \frac{h^6}{128} \left(\frac{(\nu^2 - 4\nu + 7)e^{(\nu-2)ix}}{(\nu - 2)(\nu - 1)^3(\nu + 1)} - \frac{(\nu^2 + 4\nu + 7)e^{(\nu+2)ix}}{(\nu - 1)(\nu + 1)^3(\nu + 2)} \right. \\
\left. + \frac{e^{(\nu-6)ix}}{3(\nu - 3)(\nu - 2)(\nu - 1)} - \frac{e^{(\nu+6)ix}}{3(\nu + 1)(\nu + 2)(\nu + 3)} \right) + \mathcal{O}(h^8). \quad (21)$$

Interestingly, the comparison of the above result with the one in literature (NIST Digital Library of Mathematical Functions <http://dlmf.nist.gov/28.15.E3>) reveals the discrepancy between the two formulas at h^2 order of h expansion of Mathieu exponential ($q \equiv h^2$),

$$\text{me}_\nu(z, h) = e^{i\nu z} + \frac{q}{4} \left(\frac{e^{i(\nu-2)z}}{\nu - 1} - \frac{e^{i(\nu+2)z}}{\nu + 1} \right) \\
+ \frac{q^2}{32} \left(\frac{e^{i(\nu+4)z}}{(\nu + 1)(\nu + 2)} + \frac{e^{i(\nu-4)z}}{(\nu - 1)(\nu - 2)} - \frac{2(\nu^2 + 1)}{(\nu^2 - 1)^2} e^{i\nu z} \right) + \mathcal{O}(q^4). \quad (22)$$

We believe that the above formula in eq. (22) taken from NIST Digital Library of Mathematical Functions is incorrect as it does not obey the formula for the Mathieu sinus function (<http://dlmf.nist.gov/28.12.iii>)

$$\text{se}_\nu(x, h^2) = \frac{i}{2} (\text{me}_\nu(x, h^2) - \text{me}_\nu(-x, h^2)).$$

4. Conclusions and outlook

The study presented above shows that the classical irregular conformal block is related to the spectrum of the Mathieu operator. Due to the discovery of the Gaiotto states it is possible to derive the Mathieu equation entirely within $2d$ CFT, without any reference to AGT relation. Moreover, taking into account the studies of WKB quantization of Seiber-Witten curves [8, 10] as it was mentioned in the Introduction we have also found a kind of classical AGT relation between the classical irregular conformal block and $2d$ twisted superpotential. This in turn constitutes the explicit test of the triple correspondence: $2d\text{CFT}/\mathcal{N} = 2$, $SU(2)$ gauge/2-particle QIS.

Furthermore, the quantum pure gauge irregular block and its classical counterpart studied here are merely an examples of much broader branch of possibilities to find a new equations along with their solutions as there are more irregular blocks for $N_f > 0$. As an example let us quote our last result for $N_f = 2$ case. The procedure analogous to the one described above one can find a generalized Mathieu equation,

$$\left[-\frac{d^2}{dx^2} + \frac{1}{2}h^2 \cos 4x + 4h\mu \cos 2x \right] \psi^2 = \lambda_2 \psi^2 ,$$

along with its spectrum λ_2 given by the formula

$$\lambda_2 = \nu^2 - 2h \frac{\partial}{\partial h} f_\delta^2\left(\frac{1}{2}h, \mu, \mu\right) ,$$

where f_δ^2 is $N_f = 2$ classical irregular block. This work is in progress.

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